NONSPHERICAL PERTURBATIONS OF RELATIVISTIC GRAVITATIONAL COLLAPSE

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ABSTRACT

It is known that there can be no gravitational, electromagnetic, or scalar field perturbations (except angular momentum) of a Schwarzschild black hole. A gravitationally collapsing star with nonspherical perturbations must therefore radiate away its perturbations or halt its collapse. The results of computations in comoving coordinates are presented to show that the scalar field in a collapsing star neither disappears nor halts the collapse, as the star passes inside its gravitational radius.

On the star's surface, near the event horizon, the scalar field varies as $a_1 + a_2 \exp(-t/2M)$ due to time dilation. The dynamics of the field outside the star can be analyzed with a simple wave equation containing a spacetime-curvature induced potential. This potential is impenetrable to zero-frequency waves and thus a_1, the final value of the field on the stellar surface, is not manifested in the exterior; the field vanishes. The monopole perturbation falls off as t^{-2}; higher ℓ-poles fall off as $\ln t/t^{2\ell+3}$.

The analysis of scalar-field perturbations works as well for electromagnetic and gravitational perturbations and also for zero-rest-mass perturbation fields of arbitrary integer spin. All these perturbation fields obey wave equations with curvature potentials that differ little from one field to another. For all fields, radiatable multipoles ($\ell \geq$ spin of the field) fall off as $\ln t/t^{2\ell+3}$.

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I. INTRODUCTION

A. The Problem and Its History

A central role in relativistic astrophysics is played by the Schwarzschild geometry and by the line element:

$$ds^2 = (1 - 2M/r)\, dt^2 - (1 - 2M/r)^{-1}\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad . \quad (1)$$

(We use units in which $c = 1$, $G = 1$.) The most interesting and characteristic feature of this line element is its singular behavior at the gravitational radius $r = 2M$. On the one hand, we know that the $r = 2M$ surface does have very important properties; those of an event horizon and a trapped surface. But on the other hand, transforming this line element to a freely falling coordinate system reveals that there are no local pathologies at $r = 2M$; the geometry of spacetime is quite smooth there.

The most important astrophysical consequence of the properties of the $r = 2M$ surface is the inevitability of the catastrophic collapse of a star, once it is inside its gravitational radius. The absence of geometric pathologies at $r = 2M$ in the Schwarzschild geometry implies that no anomolously large forces should develop in the star to prevent it from falling inside its gravitational radius. This expectation has been fully confirmed by several calculations.[1,2]

Whether or not catastrophic collapse can be considered as a possible phenomenon for real astrophysical objects depends on the resolution of a recent controversy: Is our picture of gravitational collapse an idiosyncracy of perfect spherical symmetry? The correctness of our qualitative picture is supported by the argument[3,4] that initial aspherical perturbations of a body should remain small during collapse through the gravitational radius, since there are no strong tidal forces there. If the perturbations of the body remain small, then the perturbations of the geometry, and of the whole collapse process, should also remain small.

Because of the nature of the event horizon, we should than expect: (i) The gravitational field outside the event horizon should be asymptotically stationary[4] at large t. (ii) At large t, a distant observer "sees" the star as it is at the moment it crosses the event horizon. We expect, therefore, that the geometry left behind is a stationary geometry with aspherical perturbations. It has been shown, however, that such stationary perturbations cannot be well behaved at the event horizon and at spatial infinity. This indicates that, for our picture to be correct, the star must rid itself of all bumps before falling through $r = 2M$. But if that is true in all cases, there would have to be pathologically large forces at the event horizon, contrary to our expectations.

These difficulties have encouraged the viewpoint that the $r = 2M$ surface <u>does</u> have important <u>local</u> properties. Arguments have been given[5,6,7,8] to show that initially small perturbations become large without bound, stopping the collapse or destroying the event horizon. All these arguments have relied heavily on speculations regarding stationary solutions.

The opposite viewpoint was first championed by Doroshkevitch, Zel'dovich and Novikov[4] and by Novikov.[9] The most conclusive evidence that has been given for this viewpoint, that collapse with perturbations is qualitatively like collapse without them, in the work of de la Cruz, Chase and Israel.[10] They have numerically followed the electromagnetic and gravitational perturbations outside a perturbed collapsing thin shell. Their computations show that no singularity develops to halt the collapse and that the perturbations in the exterior fields die out at large times. It is the goal of the present work to analyze the evolution of perturbation fields in somewhat greater generality and to explain, in physical terms, how singularities are avoided.[11]

B. Outline and Summary

In this paper we use a first order perturbation analysis to see whether initially small asymmetries can greatly affect the collapse process. This approach is quite sufficient to resolve the problem. If, on the one hand, the perturbations grow without bound our results will be meaningless but we will be able to conclude that our present picture of gravitational collapse is wrong. If, on the other hand, the perturbations remain small, the approach is justified. Since the paradox of singular stationary perturbations occurs for first order perturbations, as well as in the full theory, then if the asymmetries do remain small, we shall be able to see how the paradox is avoided.

In principle the problem is the straightforward one of putting perturbations on a collapsing star just as, for example, Thorne and Campolattaro[12] put them on a static star. In practice the complications of coordinate system, gauge freedom, and many metric components to keep track of make such a problem discouragingly difficult.

There is good reason to suspect that the paradox is a result of properties of the event horizon, and that it should occur for many kinds of perturbations - not only gravitational perturbations. In fact, it is known that these same difficulties arise for electromagnetic perturbations,[13,14] for other integer-spin massless fields,[15] and for scalar fields.[16] In most of this paper we exploit the simplicity of a scalar field analogue.

Section II contains the formulation of such a massless scalar field analogue and shows that the static perturbations are singular. A modification of this scalar field to a Klein-Gordon field gives some interesting insights into the nature of the singularities.

Our investigation of the scalar field is divided into two main parts: (i) the "local problem", i.e. the study of the behavior of the scalar field in and near a star that collapses from an initially static configuration, containing a source for the scalar field; (ii) the evolution of the scalar field in the Schwarzschild exterior. In Section II the local problem is analyzed by a detailed calculation, using comoving coordinates, of a physically reasonable collapse situation. The resulting dynamic equations give no indication that $r = 2M$ has any special local significance, for the evolution of the field. Numerical integrations of those equations confirm this; the scalar field in the star remains finite as the star falls through its gravitational radius.

Section III deals with the second part of the problem, the field in the exterior and the resolution of the paradox. It is shown that a description of the dynamics using the Schwarzschild time t, and the r* coordinate of Regge and Wheeler,

$$r^* \equiv r + 2M \ln(r/2M - 1) + \text{constant}$$

leads to a simple picture of the propagation of scalar waves in the Schwarzschild geometry. In this picture the curvature of spacetime gives rise to a potential barrier which is transparent to high-frequency waves but impenetrable to those of zero frequency. It is precisely this impenetrability which gives rise to the paradox and which resolves it.

The resolution of the paradox is simply this: The field on the surface of the star can be considered a source for the field in the exterior. Due to time dilation between the surface and distant observers, the field on the surface must be asymptotically stationary in terms of Schwarzschild time.

The field on the surface then approaches some stationary final value, but this final value cannot be manifested in the exterior solution. The curvature potential prevents a distant observer from ever seeing it. For large time the exterior field is then sourceless and the field radiates itself away, vanishing at $t \to \infty$.

The simple nature of the process of the field radiating itself away is somewhat obscured by the complicated details of the curvature potential, so these ideas are presented first for a very idealized model barrier. This idealization permits exact calculations and results in an exterior field that vanishes exponentially in time, at large time. The real curvature potential inhibits a quick exponential fall off by backscattering outgoing waves. An analysis of this backscattering, reveals that the monopole perturbation fall off as t^{-2}, while higher ℓ-poles fall off as $\ln t/t^{2\ell+3}$.

The final justification of the scalar analogue is given in Section IV. Curvature-type potential equations have been derived by Regge and Wheeler[17] for odd-parity gravitational perturbations, and by Zerilli[18] for the even parity ones. The difference between these gravitational equations and our scalar equation in only in the details of the potential. In Section IV, the Regge-Wheeler and Zerilli equations are discussed and it is shown that their solutions at large times are precisely the same as those of the scalar field equation. In particular, radiateable gravitational multipoles avoid the singularities of the static solution by vanishing as $\ln t/t^{2\ell+3}$ just as their scalar counterparts do. The motivation for studying the scalar problem is, therefore, much greater than if the scalar field were only a plausible analogue.

Certain details of Section IV are left to an accompanying paper (hereafter referred to as Paper II). In that paper it is also shown that radiateable multipoles of any integer spin field satisfy a curvature-potential type equation, and fall off as $\ln t/t^{2l+3}$.

II. THE SCALAR ANALOGUE

A. The Paradox

The scalar analogue will consist of the following. We imagine a scalar field Φ, coupled to a scalar charge density j with some coupling constant κ, and obeying a wave equation:

$$\Phi^{;\nu}{}_{;\nu} = -\kappa j \ . \qquad (2)$$

There are other possible choices for the wave equation; at the end of this section we will consider others and see that our results are the same for any reasonable choice. The curvature of the geometry appears in the Christoffel symbols used to form the covariant derivatives in (2). We expect a contribution to the curvature due to the stress-energy of the scalar field such as

$$T_{\mu\nu} = \Phi_{,\mu}\Phi_{,\nu} - \tfrac{1}{2}g_{\mu\nu}\,\Phi_{,\alpha}\,\Phi^{,\alpha} \ .$$

The great advantage of studying a non-gravitational field is that we can ignore the contribution of the field energy to the geometry; throughout this section we use the unperturbed Schwarzschild geometry. This is easily justified since we can imagine a limiting process in which j is scaled by a small number ϵ, then $\Phi \sim \epsilon$ and perturbations of the geometry $\sim \epsilon^2$. Of course, it may be that Φ or its gradient (in non-pathological coordinates) become large without bound at some point, in which case we must abandon the perturbation scheme.

The situation we consider is that of a star whose matter is scalar charged. On an initial Cauchy hypersurface, on which the star is still outside its gravitational field, the star is a source of an exterior scalar field. The first, most critical, question we must ask is: Can the star collapse leaving an asymptotically static scalar field behind, or as in the case of gravitational and electromagnetic multipoles, must the scalar field either radiate away or greatly modify the collapse?

If the star collapses, it leaves behind the familiar Schwarzschild geometry described by line element (1). If a static[19] Φ field is left behind by the collapse, it must satisfy

$$-\Phi_{,\nu}{}^{;\nu} = (1-2M/r)\, d^2\Phi/d^2r + (2/r - 2M/r^2)\, d\Phi/dr - \ell(\ell+1)\,\Phi/r^2 = 0 \tag{3}$$

for an ℓ-pole. To analyze[20] this we introduce the convenient r^* radial coordinate of Regge and Wheeler,[17]

$$r^* = r + 2M\, \ell n(r/2M - 1) + \text{constant} \quad . \tag{4}$$

Note that $r \approx r^*$, for $r \gg M$, and that the event horizon $r = 2m$ is at $r^* = -\infty$. In terms of r^* derivatives, Eq. (3) becomes:

$$(1 - 2M/r)^{-1}\, d^2\Phi/d^2r^* + 2r^{-1}\, d\Phi/dr^* - \ell(\ell+1)\,\Phi/r^2 = 0 \quad . \tag{5}$$

The asymptotic solutions at large r^* are the usual flat-space forms,

$$\Phi \sim r^{*\ell} \quad \text{or} \quad \Phi \sim r^{*-(\ell+1)} \quad \text{at } r^* = +\infty \quad . \tag{6a}$$

And near the event horizon they are:

$$\Phi \sim r^* \quad \text{or} \quad \Phi \sim \text{constant} \quad \text{at } r^* = -\infty \quad . \tag{6b}$$

The solution $\bar{\phi} \sim r^*$ is unacceptable at $r^* = -\infty$. Specifically, the scalar field's stress-energy and its force on the charge carriers would be unbounded in a comoving frame. The solution $\bar{\phi} \sim r^{*\,\ell}$ at $r^* = +\infty$ is obviously pathological.

The question then is whether we can connect the well-behaved solutions at $r^* = -\infty$ and $r^* = +\infty$. If we are to connect a constant at one end to a decreasing solution at the other there must be a point of inflection ($d^2\bar{\phi}/dr^{*2} = 0$) at which the signs of $\bar{\phi}$ and $d\bar{\phi}/dr^*$ are opposite; this is clearly incompatible with (5). (This analysis is patterned after that of Vishveshwara.[21])

The monopole case, which is just as important here as the higher multipole cases, since the scalar field can radiate in an $\ell = 0$ mode, is somewhat special in that both solutions at $r^* = +\infty$ are well behaved. For this case, in fact, we have the simple exact solutions

$$\bar{\phi} = \ln(1 - 2M/r), \quad \text{or constant} \quad . \tag{7}$$

The solution $\bar{\phi}$ = constant is trivial and the $\ln(1 - 2M/r)$ solution has the expected $\bar{\phi} \sim r^*$ behavior at $r^* = -\infty$.[16]

If the paradox is indeed a manifestation of the special nature of the event horizon, the precise form of the wave equation should not be too important. In flat spacetime we consider the generalization of the free-field equation

$$\Box\bar{\phi} + C(x^\mu)\,\bar{\phi} = 0 \tag{8a}$$

where

$$\Box \equiv \partial_t^{\,2} - \nabla^2 \quad . \tag{8b}$$

In the absence of other fields, translational invariance demands that C be constant and we have the usual Klein-Gordon equation. When we make the usual replacement of ordinary partial derivatives by covariant derivatives we find that in (5) the coefficient of $\tilde{\Phi}$ is now

$$- \left[r^{-2}\, \ell(\ell + 1) + C(X^{\mu})\right] . \tag{9}$$

The emergence of the paradox depends on the sign of C. Usually C in the Klein-Gordon equation is taken to be M^2, where M is the mass of the particle mediating the field, in this case our derivation of the necessary singularities of the static solution is unaffected.

It is intriguing that imaginary mass particles have nonsingular static solutions, since imaginary masses are sometimes associated with faster-than-light motion. More specifically if $C(X^{\mu})$ is negative in some region of spacetime, high-frequency wave packets have group velocities greater than c in that region. It is just such faster-than-light effects that might be expected to rob the $r = 2M$ surface of its properties as a one-way membrane for information propagation.

One other point must be mentioned. In generalizing the wave equation from the laboratory to curved spacetime, it is possible that other curvature effects come in, in addition to the covariant derivatives. In particular, those who consider conformal invariance to be compelling would write the free-field equation as

$$\Phi_{;\nu}{}^{;\nu} + 1/6\, R\, \Phi = 0 \ .$$

Since $R = 0$ in the vacuum exterior, this modification is of no concern.[22]

B. The Local Problem

Our approach to the problem of the scalar field's evolution can conveniently be divided into two parts. In the first, the "local problem", the evolution of the field in the star and on its surface is followed, up to the point at which the surface passes through the event horizon and is causally disconnected from external observers. The results are then used as an input for the second part: the evolution of the exterior field. The local problem is also important because it resolves the question of whether perturbations remain small, and whether a first-order perturbation calculation is sufficient.

Some important work has already been done on the local problem for gravitational perturbations: The computer integrations by de la Cruz, Chase and Israel[10] and the analysis by Novikov.[9] In view of the uncertainty still surrounding the question of the behavior of fields at the event horizon, it was deemed useful to follow the evolution of the field in the local problem numerically, with a computer.

The problem is set up in a way that allows an unambiguous interpretation of the results. The background problem is the collapse of a momentarily static uniform pressureless star first described by Oppenheimer and Snyder.[23] On the initial $t = 0$ Cauchy hypersurface the Φ field is chosen to be static ($d\Phi/dt = 0$, $d^2\Phi/dt^2 = 0$) in the exterior; a stationary observer sees this field remain static until information about the collapse reaches him.

The Friedman line element,

$$ds^2 = d\tau^2 - a^2(\eta)\left[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \tag{10a}$$

with

$$a(\eta) = \tfrac{1}{2}a_o(1 + \cos\eta) \tag{10b}$$

$$\tau = \tfrac{1}{2}a_o(\eta + \sin\eta) \tag{10c}$$

describes the geometry of the interior of an Oppenheimer-Snyder star of density

$$\rho = 3a_o/8\pi a^3(\eta) \quad . \tag{11}$$

If the maximum χ (i.e., that for the stellar surface) is χ_o, then the mass and radius of the star's surface are:

$$M = \tfrac{1}{2}a_o \sin^3 \chi_o \tag{12}$$

$$r_{sf} = a(\eta) \sin \chi_o \quad . \tag{13}$$

At $\eta = 0$ the star is momentarily static and is about to begin its free-fall collapse.

The geometry outside the star is that of Schwarzschild, but we must avoid Schwarzschild's coordinates because of their poor description of the region $r = 2M$. Instead, we choose "comoving", i.e. "synchronous", coordinates. For a vacuum this means a system in which points with fixed spatial coordinate values move on timelike geodesics, and for which the time coordinate is the proper time along these geodesics.

The general comoving spherically symmetric, vacuum line element[24] is

$$ds^2 = dT^2 - \frac{(\partial r/\partial R)^2}{1 + 2E(R)}\, dR^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad , \tag{14a}$$

where $r(R,T)$ is derived as the solution of

$$\partial r/\partial T = -\sqrt{\frac{2M}{r} + 2E(R)} \quad . \tag{14b}$$

There are two arbitrary functions here, $E(R)$ and $r(R,T = 0)$, corresponding to our initial choice of velocity for our observers and to the initial scale for R. By choosing $r(R,T = 0) = R$, we give the R coordinates the physical interpretation of the initial radius (i.e., Schwarzschild radial coordinate) from which an observer starts falling. We must, of course, cut the geometry off at the surface of the star, which is initially $R = R_o = r_{initial} = a_o/2 \sin \chi_o$. We choose $E(R) = -M/R$ so that our observers are all initially static.

Since $\partial r/\partial T)_{R = R_{sf}} = 0$ and $\partial r/\partial \tau)_{r = r_{sf}} = 0$ the world line of the shells $r = r_{sf}$ and $R = R_{sf}$ are initially tangent and they are both geodesics. Thus, a consequence of our choice of $E(R)$ is that the star's surface always remains at $R = R_0$, and since both τ and T are proper time on this geodesic, the boundary between the interior and the exterior, we hereafter use only the symbol τ as the time coordinate in both regions.

The background coordinates are pictured in Fig. 1 where we use, as in calculations to follow, the specific choices $\chi_o = \pi/4$ and $a_o = 4\sqrt{2}\, M$, so that the star starts collapsing from $r_{sf} = 4M$. The function $r(R,\tau)$ is transcendental but is smooth, having no pathologies where the stellar surface crosses the event horizon. For our choice of $E(R)$, $r(R,0)$, χ_o, and a_o, it is approximately

$$r(R,\tau) \approx R\Big/\left(1 + .7\,(2M)\,t^2/R^3\right) \quad , \tag{15}$$

throughout the region of interest of the variables R, τ.

In the interior we must evaluate Eq. (2) in terms of the coordinates of line element (10). For an ℓ-pole field the result is

$$\left(a^2\,(\eta)\Phi_{,\eta}\right)_{,\eta}\Big/a^2(\eta) - (\sin^2\chi\,\Phi_{,\chi})_{,\chi}/\sin^2\chi \\ + \ell(\ell+1)\,\Phi/\sin^2\chi = -\,\kappa a^2(\eta)\,j \quad . \tag{16}$$

In the exterior ($R \geq R_{sf}$), in terms of the comoving coordinates of (14), Eq. (2) becomes:

$$(r^2\,r'\,\Phi_{,\tau})_{,\tau} - \sqrt{1 - 2M/R}\left(\left[r^2\sqrt{1 - 2M/R}\Big/r'\right]\Phi_{,R}\right)_{,R} \\ + r'\,\ell(\ell+1)\,\Phi = 0 \quad , \tag{17a}$$

where

$$r' = \partial r/\partial R \quad . \tag{17b}$$

The matching condition for $\tilde{\Phi}$ at the boundary $X = X_o$, $R = R_{sf}$ is that the derivative of $\tilde{\Phi}$, with respect to proper distance normal to the boundary is continuous and that $\tilde{\Phi}$ itself is continuous. (This can be shown by using Guass's theorem on a slab-like volume including the boundary.) It should be noticed that the system of Eqs. (16) and (17) and the matching condition in no way single out $r = 2M$ as a special surface. Viewing the local problem in these mathematical terms, we should be very surprised if a singularity develops there.

To solve the dynamical problem we also need to know the motion of the scalar charge carriers. This motion is freely specifiable since we are ignoring the forces due to the scalar field. A natural choice is to have each dust particle in the star carry a fixed charge ; this fixes the time dependence of j at any one value of X, θ, φ,:

$$j(\tau) \;\alpha\; \rho(\tau) \;\alpha\; a^{-3}(\eta) \quad .$$

The form of $j(\tau, X, \theta, \varphi)$ depends how the charge per particle varies from point to point in the star. We choose the radial dependence so that j vanishes smoothly at the surface, and we choose the angular dependence to be a spherical harmonic, e.g.,

$$j(\tau, X, \theta, \varphi) = \epsilon\left(a_o/a^3(\eta)\right) \; (1 - X^2/X_o{}^2) \; Y_l{}^m(\theta, \varphi) \quad . \tag{18}$$

Here ϵ is an expansion parameter which is chosen small enough so that we can ignore scalar field forces and stress energy. The coupling constant κ in (2) is taken to be $(2M)^2$ so that $\tilde{\Phi}$ is dimensionless.

It remains only to specify Φ and its time derivative initially. Outside the star we take Φ to be static, that is, a solution of (3) for $\Phi(R)$. These solutions characteristically go as $1/R^{\ell+1}$ at large R. In the interior, where there is not so natural a choice for Φ, we choose it such that $\Phi_{,\tau}$ and $\Phi_{,\tau\tau}$ vanish inside. The initial interior field will then be a superposition of the particular solution - depending on the scalar charge distribution - and the homogeneous solution of (16):

$$\Phi_{\text{interior}} = \Phi_{\text{particular}} + A\,\Phi_{\text{homogeneous}} \quad . \tag{19a}$$

The exterior field will be

$$\Phi_{\text{exterior}} = B\,\Phi_{\text{static}} \quad , \tag{19b}$$

and the constants A and B must be determined so as to make the initial data satisfy the junction conditions at the stellar surface.

The details of the calculation of the evolution of Φ are summarized in Table I and the results of numerical computations are shown in Fig. 2.

In both the $\ell = 0$ and $\ell = 3$ case, the field increases as the star grows smaller, but in neither case is there any strong local effect to distinguish the point at which the star's surface crosses the event horizon. At this moment of crossing (at $\eta = \pi/2$) both Φ and its derivatives (in comoving coordinates) are neither zero nor infinity. <u>The scalar perturbations do remain small</u>.

III. EVOLUTION OF THE SCALAR FIELD

A. The Curvature Potential; Characteristic Data

Having established that no catastrophic local phenomenon will interrupt the collapse, and having calculated data on the stellar surface, we now turn our attention to the evolution of the exterior field. In particular, we are interested in seeing how the asymptotic field at $t \to \infty$ avoids the singularity of a static solution. This will be accomplished by a radiative dispersal of the field.

The comoving coordinates which are so useful in studying the local problem are poorly suited to the radiation problem. To understand the nature of the radiation in the exterior we must use a reference system related to the static nature of the background - so our system must be stationary with respect to the Schwarzschild system. The line element is then (1) or if we use the r* coordinate defined in (4), the line element is

$$ds^2 = (1 - 2M/r)\,(dt^2 - dr^{*2}) - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad , \quad (20)$$

where $r(r^*)$ is a solution of the implicit equation (4). Note that the null radial lines in the geometry are $dr^* = \pm\, dt$.

Assuming that Φ has the angular dependence of a spherical harmonic, the equation governing its evolution is

$$\Phi_{;\nu}{}^{;\nu} = (1 - 2M/r)^{-1}(\Phi_{,tt} - \Phi_{,r^*r^*}) - 2r^{-1}\,\Phi_{r^*} + \ell(\ell+1)\, r^{-2}\,\Phi = 0 \quad . \quad (21)$$

If we introduce a modified field variable $\Psi = r\Phi$, then (21) takes the very simple form

$$\Psi_{,tt} - \Psi_{,r^*r^*} + F_\ell^{sc}(r^*)\,\Psi = 0 \quad , \quad (22a)$$

with

$$F_\ell^{sc}(r^*) = (1 - 2M/r)\left(2M/r^3 + \ell(\ell+1)/r^2\right) \quad . \quad (22b)$$

The function, $F_\ell^{sc}(r^*)$, which is very important to our analysis, is peaked strongly around small absolute values of r*. (See Fig. 3.) Its asymptotic forms are:

$$F_\ell^{sc}(r^*) \approx \begin{cases} \left.\begin{array}{ll} \dfrac{\ell(\ell+1)}{r^{*2}} + \dfrac{4M\,\ell(\ell+1)\,\ln(r^*/2M)}{r^{*2}}\,. & \text{if } \ell \neq 0 \\ 2M/r^{*3} & \text{if } \ell = 0 \end{array}\right\} r^* \gg M \\ \dfrac{\ell(\ell+1)+1}{(2M)^2}\exp\left\{(r^*/2M)+1\right\} \qquad r^* \ll -M \end{cases} \tag{23}$$

(Here the constant in (4) has been chosen such that $r^* = 0$ at $r = 4M$.)

The shape of $F_\ell^{sc}(r^*)$ is shown in Fig. 3.

The wave Eq. (22) can give us a simple picture of the nature of the radiation problem. If $F_\ell^{sc}(r^*)$ were simply the centrifugal barrier, $\ell(\ell+1)/r^{*2}$ in the region $r^* > 0$, then waves of the scalar field would propagate freely; there would be no gravitational effects on them. The existence of the region $r^* < 0$ and the fact that $F_\ell^{sc}(r^*)$ is <u>not</u> $\ell(\ell+1)/r^{*2}$ for $r^* > 0$ are due to the curvature of spacetime. To isolate curvature effects from the effects due to spherical coordinates (i.e., the centrifugal barrier), we can subtract $\ell(\ell+1)/r^{*2}$ from $F_\ell^{sc}(r^*)$ for r^* greater than, say, 20 M. The part of $F_\ell^{sc}(r^*)$ which remains and which is due to curvature we shall call the <u>curvature potential</u>.

The useful and interesting property of the curvature potential is that it is a very localized barrier to scalar waves. It can be thought of as a barrier between a flat space close zone adjacent to the stellar surface source, and a flat space distant zone (containing Joseph Weber and his scalar antenna) where we are most interested in the manifestations of this radiation.[25]

In this picture of the problem we will find the coordinates u and v, advanced and retarded time, to be useful. They are related to t and r^* by a 45° rotation:

$$u = t - r^* \qquad v = t + r^* \quad . \tag{24}$$

They are important in that they are null coordinates; if it were not for the scattering by the potential, information would propagate along u,v coordinate lines without distortion.

The specification of the problem is complete when we give the initial conditions that Ψ is static at $\tau = 0$ (and hence on the "first ray"), and when we put in the values of Ψ and its normal derivative on the stellar surface, from the previous section. The problem is pictured in this form in Fig. 4.

Consider the initial Cauchy data on the surface of the star. In the previous section we saw that Ψ and $\partial\Psi/\partial R$ are well-behaved functions of comoving coordinates from the onset of collapse to the passage through the event horizon. The fact that the variation of Ψ is bounded on a curve of finite length in comoving coordinates means that its variations on the curve of infinite length in Fig. 4 must be very small, asymptotically zero in fact at $u \to \infty$. Mathematically we can show that Ψ approaches its final value according to

$$\Psi \to a + b \exp\{- u/4M\} \qquad (25)$$

$$\text{as } t \to \infty \text{ (and } u \to \infty) \qquad \text{(a,b constants)} \quad .$$

This effect is in fact just the ordinary time dilation phenomenon between a falling frame and a static one and does not depend in any way on the surface falling in on a geodesic.

These asymptotic properties can be established most easily by using Kruskal coordinates[26]

$$U = - 4Me^{-u/4M} \qquad V = 4Me^{v/4M} \quad . \qquad (26)$$

Since U and V are well-behaved coordinates at r = 2M, the partial derivatives $\partial\Psi/\partial U$ and $\partial\Psi/\partial V$ should be finite. This implies that $\partial\Psi/\partial u$ must fall off sharply at the event horizon because

$$\partial\Psi/\partial u = (\partial\Psi/\partial U)\dot{}\exp\{-u/4M\} \quad , \tag{27}$$

and $u \to \infty$ at $r = 2M$. The advanced time v is finite at $r = 2M$ (for an ingoing world line) so that $\partial\Psi/\partial v$ is finite. If we picture the path of the surface through spacetime as $v(u)$ or $V(U)$, then near $r = 2M$ we have

$$\frac{dv}{du} = \frac{dV}{dU}\;\frac{dv/dV}{du/dU} \sim \exp\{-u/4M\} \quad . \tag{28}$$

The world line of the surface, therefore, appears <u>in our coordinates</u> to be almost an ingoing null line, and we conclude that:

$$d\Psi/du = \partial\Psi/\partial v\; dv/du + \partial\Psi/\partial u \sim \exp\{-u/4M\} \quad , \tag{29}$$

for large u. We shall see that the evolution of Ψ at large t depends only on the asymptotic, large u, behavior of Ψ on the surface.

For convenience, we will specify data on an ingoing null line rather than on the stellar surface. If the v distance between these two curves is δv at some u, then the error in $\Psi{,u}$ is approximately

$$\delta(\Psi,u) \approx (\Psi,u),v\;\delta v = -1/4\; F_\ell^{\,sc}(r^*)\;\Psi\delta v \;\alpha\; \exp\{-u/4M\}\;\delta v \quad . \tag{30}$$

So using the null line is justifiable in that it does not change the nature of the asymptotic data. (See Fig. 4.)

Summarizing then, we have reduced the physical problem to a mathematical problem in wave propagation, with data given on two characteristics:[27] the "first ray" $u = 0$, and the "stellar surface" $v = v_0$. The partial differential equation is (21) or equivalently

$$\Psi,uv + \tfrac{1}{4}\,F_\ell^{\,sc}(r^*)\;\Psi = 0 \quad . \tag{31}$$

The form of the characteristic data is

$$\Psi(u, v = v_0) \to a + b \exp\{-u/4M\} \tag{32a}$$

$$\text{at } u \gg M$$

and

$$\Psi(u = 0, v) \to \text{static solution} \to r^{*-\ell} \tag{32b}$$

$$\to v^{-\ell} \qquad \text{at } v \gg M \quad .$$

B. An Idealized Potential

Before going on to look closely at the manner in which the fields evolve, it is instructive to look at a very idealized analogue to our wave equation.

$$\Psi_{,tt} - \Psi_{,r^*r^*} + F_\ell(r^*)\,\Psi = 0 \tag{33a}$$

where

$$F_\ell(r^*) = \begin{cases} \ell(\ell+1)/r^{*2} & \text{for } r^* \geq 1 \\ 0 & \text{for } r^* < 1 \end{cases} \tag{33b}$$

The input data as before will be on characteristics: an exponentially damped fall-off at $v = v_0$, and first-ray data corresponding to an initially static solution. The extent to which we have eliminated some important physics with this idealization will become apparent presently.

Rather than dealing with a general ℓ we shall specialize to $\ell = 1$; the following calculation can be done in the same manner for any ℓ. It is interesting that even in this simple model equation, we have the "paradox". The static solutions are

$$\Psi = \begin{cases} c_1 + c_2 r^* & r^* < 1 \\ C_1/r^* + C_2 r^{*2} & r^* \geq 1 \quad . \end{cases} \tag{34}$$

We cannot match the two good solutions (with the usual conditions that Ψ and $\Psi_{,r^*}$ are continuous), so there can be no static solution that is well behaved at both $r^* = +\infty$ and $r^* = -\infty$.

We get perhaps the clearest picture of the nature of the paradox if we regard this problem as a purely mathematical problem in the propagation of waves, in one dimension, under the influence of a rather strange potential. Prior to the first ray, a distant observer sees a static field, the source of which is the charge in the star, or equivalently the field on the stellar surface. At $u \to \infty$ the stellar surface field again becomes static, and non-zero, so we might expect the distant observer to see a static non-vanishing field. This is impossible without singularities. In a sense then, this idealization is a reduction of the essence of the paradox to its simplest terms.

The advantage of the idealization is clear; the solutions in the two regions $r^* \gtrless 1$ can be written in very convenient forms depending on four arbitrary functions

$$\Psi = \begin{cases} \dfrac{df(v)}{dv} - \dfrac{f(v)}{r^*} + \dfrac{dg(u)}{du} + \dfrac{g(u)}{r^*} & r^* \geq 1 \\ \alpha(u) + \beta(v) & r^* < 1 \quad . \end{cases} \tag{35}$$

For further convenience we redefine for now u and v as

$$\begin{aligned} v &= t + r^* - 1 \\ u &= t - r^* + 1 \quad , \end{aligned} \tag{36}$$

so that $u = v$ when $r^* = 1$, and we use characteristic boundaries at $u = 0$ and $v = 0$.

At $u = 0$ we choose the condition $\Psi = 1/r^*$, while at $v = 0$ we choose

$$\Psi = 1 + (2k)^{-1} - (2k)^{-1}\, e^{-ku} \quad . \tag{37}$$

(The constants are chosen so that Ψ and $\Psi_{,u}$ are continuous at $t = 0$.) The solution to (33) with this input is

$$\Psi = -(2k)^{-1} e^{-ku} + (A/4k)\, e^{-kv} + kA^{\frac{1}{2}}\, e^{-\frac{1}{2}v} \cos(\tfrac{1}{2}v - \phi) \tag{38a}$$

$$\text{for } r^* \le 1$$

$$\Psi = -\tfrac{1}{2}kAe^{-ku} - kA^{\frac{1}{2}}\, e^{-\frac{1}{2}u} \sin(\tfrac{1}{2}u - \phi) + r^{*-1}\left\{\tfrac{1}{2}Ae^{-ku} + \sqrt{2}\, kA^{\frac{1}{2}}\, e^{-\frac{1}{2}u} \cos(\tfrac{1}{2}u - \phi - \pi/4)\right\} \tag{38b}$$

$$\text{for } r^* \ge 1$$

where

$$A = 1/(k^2 - k + \tfrac{1}{2}) \tag{39a}$$

$$\tan\phi = 1/2k - 1 \qquad 0 \le \phi \le 3\pi/4 \quad . \tag{39b}$$

The terms in Ψ that go as $\exp(-ku)$ represent the outgoing waves from the "stellar surface"; the $\exp(-kv)$ term represents reflected waves. The coefficients

$$T = \tfrac{1}{2}kA/(2k)^{-1} = k^2/(k^2 - k + \tfrac{1}{2}), \qquad R = (A/4k)/(2k)^{-1} = \tfrac{1}{2}/(k^2 - k + \tfrac{1}{2}) \quad ,$$

indicate the strength of the radiation respectively transmitted through, and reflected from the potential peak near $r^* = 1$. In the limit that the input waves have very high frequency

($k \to \infty$) the waves are transmitted completely with no reflection. But <u>in the low-frequency limit ($k \to 0$) they are completely reflected and there is no transmission</u>. The exponentials with frequencies $\omega = -1/2 \pm i/2$ are transients which are characteristic of the potential, and which enable the conditions at $r^* = 1$ to be satisfied.

The crucial thing to notice is that the solution falls off exponentially in time everywhere - i.e., at any r^* - thus avoiding the catastrophe of a static asymptotic solution. The paradox was founded on a belief that the value of Ψ at $v = 0$ and large u would penetrate the potential and show up at large r^*. We now see that the potential acts as a very effective barrier against zero frequency waves from the "surface of the star". It is this potential barrier that causes the paradox (there would be well-behaved static solutions if not for the potential) and that resolves it.

Although it seems likely that this vanishing of the field is really the essence of the resolution of our paradox, we shall go on to look into the details of the real problem. We shall in particular be concerned with the question: Does the solution to (22) vanish asymptotically at large t and, if so, how fast? We shall see that for more realistic potentials than those of the idealized example, the solution does not fall off exponentially but rather develops a power law [28] fall off asymptotically, which dominates the exponential fall off of the previous example. Nevertheless, the solution does fall off at large t.

The power law tail is caused by scattering of the radiation off the anomalous curvature part of the potential - i.e., from the fact that the real potentials do not have the convenient forms $\ell(\ell + 1)/r^{*2}$, but have higher order terms at large r^* also. The convenient forms of (35) correspond to unimpeded in- and outgoing waves. When we add the other curvature-induced parts of $F_\ell(r^*)$ we scatter these waves, effectively slowing the dilution of the field. Another viewpoint on this comes from the study of the spreading effect of potentials by Kundt and Newman.[29] In effect, they show that there is a zero measure of potential functions which give a nice separation of in- and outgoing waves as in (35). It seems that an exponential fall-off of Ψ (in the case of exponential fall off of surface data) is associated with these non-spreading potentials; our potentials - the anomalous parts of which come from the curvature of spacetime - will not be in this exalted class and we must expect scattering and other, slower fall-offs.

[Heuristically, we may argue for a non-exponential fall-off as follows: In the Kundt-Newman formalism we may formally write a solution for any potential in a form like (35). For the spreading potentials, however, an infinite number of derivatives of $f(v)$ and $g(u)$ are required. This gives rise to an infinite number of transient frequencies. (In our idealized case we had only $\omega = -1/2 \pm i/2$.) The sum of an infinite number of transient terms may be viewed as the Fourier integral of a function other than an exponential.]

We shall now investigate the solution for the case of the actual potential and we will concern ourselves chiefly with the asymptotic solution (large u, large t). A separate analysis is needed for the monopole case and for higher multipoles.

C. Monopole Fields

Since the scalar monopole can be radiated just as well as higher multipoles, there is no reason to expect its asymptotic solution to differ qualitatively from that of multipoles with $\ell > 0$. The great advantage in considering the monopole case is that the $\ell(\ell + 1)/r^2$ centrifugal barrier term vanishes and we can think of the total potential as localized near $r^* = 0$.

If $F_0^{sc}(r^*)$ vanished everywhere -- this would be the idealized potential for $\ell = 0$ -- then the solution to our wave equation, with the characteristic data of Eq. (25), would simply be

$$\Psi(u,v) = a + b \exp\{-u/4M\} \quad , \tag{40}$$

representing free propagation outward of the data on the stellar surface. Although this cannot be the total solution, this should be the behavior of the high-frequency components of the radiation $\left(e^{i\omega t} \text{ with } \omega \gg \sqrt{\max(F_0^{sc})} \approx 1/2M\right)$. This phenomenon has appeared in our simple example and is a well-known occurrence in quantum mechanics where an energetic wave train is little affected by a potential barrier of much lower energy. Equation (40) is then a first approximation to the behavior of the solution. In as much as it predicts a concentration of the waves near the first ray $u = 0$, this solution represents a wave front which will be the dominant solution near $u = 0$. The exact form of this wave front depends greatly on the details of the collapse; the crucial point here is that the wave front is exponentially damped.[30]

It is obvious that it is the low frequencies which are really involved in the paradox and in its resolution. These low frequencies make the greatest contribution (e.g., to a Fourier integral) at large times and so may very well lead to a modified asymptotic solution.

For now, we assume that F_0^{sc} is absolutely localized in some region $|r^*| < \beta M$. (It should be clear that the exponential tail of the potential at $r^* \to -\infty$ is ignorable. Later we must also justify ignoring the effects of the M/r^{*3} tail on the evolution of the asymptotic solution.) So now we have

$$\Psi_{,uv} = 0 \begin{Bmatrix} \text{in regions VI, IX} \\ \text{of Fig. 4} \end{Bmatrix} \tag{41}$$

From (41) and the data on the $v = v_0$ characteristic boundary (32a) it follows that the solution for Ψ at large u in region IX is

$$\Psi = b \exp\{-u/4M\} + f(v) \quad , \tag{42}$$

where f(v) is an, as yet, unspecified function. On the other characteristic boundary u = 0, we have

$$\Psi(u = 0,v) = \frac{r}{2M} \ln(1 - 2M/r) = -1 - \frac{2M}{v} + O\left(\frac{\ln(v/2M)}{v^2}\right) \quad . \tag{43a}$$

According to (7), Ψ in region VI must be

$$\Psi = -1 - \frac{2M}{v} + \ldots + g(u) \quad , \tag{43b}$$

where g(u) is a function we must determine along with f(v). Notice that f(v) and g(u) represent waves which propagate away from the potential in regions IX and VI respectively.

Now let us assume that the solution in the region $t \gg r^*$ is not an exponential in time - but rather something slower like a power law. (This will be justified in the results.) The solution in this region then can be written as

$$\Psi = \psi(t)\ \phi(r^*) \qquad \text{for } t \gg r^* \tag{44a}$$

with

$$\psi_{,t}/\psi \ll \phi_{,r^*}/\phi \quad . \tag{44b}$$

For convenience, let us use the symbols $\mathcal{U}, \mathcal{V}$ for $\partial\Psi/\partial u$ and $\partial\Psi/\partial v$ respectively. If A is a point in region VI and B is a point in region IX with the same v coordinate (see Fig. 4), then by (31) and (44):

$$\mathcal{V}_B - \mathcal{V}_A \approx +1/2\ \psi(t) \int_A^B F_0^{sc}(r^*)\ \phi(r^*)\ dr^* \quad . \tag{45}$$

Now $\mathcal{U} = 0$ (modulo an exponential fall off) in region IX so that there

$$\mathcal{V} \equiv \Psi_{,v} = \dot{\Psi}_{,t} \quad , \tag{46}$$

and also in region IX, for $t \gg r^*$,

$$\mathcal{V}(B) = \psi_{,v}|_B = \dot{\psi}(t)\ \phi(r^*) \quad . \tag{47}$$

Now if $\mathcal{V}(B)$ falls off in time as $\dot{\psi}(t)$ and hence faster than $\psi(t)$, it must be that $\mathcal{V}(A)$ cancels the integral in (45) - i.e., $\psi(t)$ must fall off like $\mathcal{V}$ for $t \gg r^*$. Furthermore, $\mathcal{U}_{,v} = \mathcal{V}_{,u}$ implies that

$$\mathcal{U}(D) - \mathcal{U}(C) \approx \mathcal{V}(B) - \mathcal{V}(A) \approx - \mathcal{V}(A) + O\big(\mathcal{V}'(A)\big) \tag{48}$$

at large t. Since $\mathcal{U}(C) = 0$ we conclude that $\mathcal{U}(D) \approx - \mathcal{V}(A)$, or the incoming and outgoing parts of the tail are equal in magnitude for $t \gg r^*$. This almost total reflection of the ingoing waves is another manifestation of the impenetrability of the barrier to low-frequency waves.

Now from (43b) we see that in region VI, $\mathcal{V}$ must be $2M/v^2$ so:

(i) $g(t)$ must fall off as $1/t^2$ for $t \gg r^*$.

(ii) From (48) and (43b), in region VI ,

$$\Psi = 2M\left(\frac{1}{u} - \frac{1}{v}\right) + \frac{\gamma}{v^2} \quad (\gamma \text{ some constant}) \quad . \tag{49}$$

(iii) In region IX, $\mathcal{V}$ must fall off as $1/t^3$ for $t \gg r^*$ so that

$$\mathcal{V} \sim \text{Order } (1/v^3)$$
$$\psi \sim \text{Order } (1/v^2) \quad . \tag{50}$$

In (49) we see that at any r^* if $t \gg r^*$, then $\Psi = (4Mr^* + \gamma)/t^2$, that is, Ψ fall off as $1/t^2$, and from (50) we see that this must be true for region IX also. Thus: a sufficiently long time after the wave front passes in region VI, or after the surface passes in region IX, the solution will fall off in time as $1/t^2$.

Before going on to discuss the meaning and implication of these results, we must justify having ignored the $2M/r^{*3}$ tail of the potential. It is clear that ignoring this tail in our analysis of the evolution amounts to assuming that in region VI, Ψ is transported unchanged (on a line of constant v) from the first ray to the edge of the potential barrier. We can calculate how much Ψ will change on this path for our solution, due to the $2M/r^{*3}$ tail of the potential:

$$\delta\Psi = \left\{\begin{matrix}\text{change in } \Psi \text{ in region I, on a line of constant v, due} \\ \text{to the tail of } F_0\text{: } \delta F_0 \sim 1/r^{*3}.\end{matrix}\right\}$$

$$= -1/4 \int_{u=0}^{r^* \approx \beta M} du\, \delta F_0\, \Psi$$

$$= -\frac{M}{2} \int_{u=0}^{u=v-\beta M} du \frac{1}{r^{*3}} \times \begin{cases} \Psi \,\alpha\, 1 \text{ near } u = 0 \\ \Psi \,\alpha\, M(1/u - 1/v) \quad \text{for } u \gg M \end{cases} \tag{51}$$

We can divide the integral into two parts: (i) the contribution at small u due to the wave front and (ii) the contribution, mostly near the barrier, due to the tail $2M(1/u - 1/v) + \gamma/v^2$ which we have calculated for Ψ. The first contribution is of order M/v^3 and we need give it no further consideration. The contribution of the tail is of order $M/\beta v^2$ and thus falls off at the same rate as Ψ, but we can make this error as small as we wish merely by making β sufficiently large.

The problem of monopole radiation just discussed was attacked numerically with an IBM 360/75 computer. The exact problem was investigated. That is, the exact potential $F_0^{sc}(r^*)$ as given in (22b) was used as well as the exact first ray solution $\Psi = r/2M \ln(1 - 2M/r)$. The results were in perfect agreement with the arguments presented above. Specifically Ψ was found to have a large wave front at small u, which gave way to an asymptotic solution for $t \gg r^*$, that did in fact go precisely as $2M(1/u - 1/v) + \gamma /v^2$ in region VI. In region IX the solution was found to be very accurately independent of u, and to go as $\Psi = \text{const.}/v^2$. Furthermore, the program kept track of $\mathcal{U}$ and $\mathcal{V}$ in region VI; from the first ray to quite small values of r^* it was found that δv as defined in (51) does fall off as $1/v^2$ but is always much smaller than $\mathcal{V}$ for $r^* > 20$ M or so. Results of these computations are presented in Figs. 5,6, and 7.

We can give a physical picture, in terms of scattering of waves, of the justification of ignoring the large r^* tail of the potential. Ignoring the potential tail really amounts to using the form $\mathcal{F}(u) + \mathcal{G}(v)$ for Ψ in region VI. We shall now show that an incoming wave scattered off the potential tail is much smaller than the incoming (primary) wave itself. In this calculation we take the tail $\delta F(r^*)$ to extend outward from $r^* = \beta M$, i.e.

$$\delta F(r^*) = 2M/r^{*3} \quad \beta M < r^* < \infty \tag{52}$$

If we have an incoming primary wave $\Psi_0 = \mathcal{G}(v)$ in region VI, we can calculate the scattering from δF (or the error in using $\Psi = \mathcal{G}(v)$ as a solution) by

$$\delta\Psi_{,uv} = -\tfrac{1}{4}\,\delta F \times \Psi_0 \tag{53}$$

and we will use the fact that at $r^* = \infty$ the form $\Psi_0 = \mathcal{G}(v)$ is a valid solution so that $\delta\Psi = 0$ there. Then

$$\delta\Psi,v = -\frac{1}{4}\int_{-\infty}^{u} \frac{2M}{r^{*3}}\vartheta(v)\,du = \left(-\frac{1}{4}\right)\frac{2M\vartheta(v)}{r^{*2}} \tag{54a}$$

and

$$\delta\Psi(u,v) - \delta\Psi(u,v=\infty) = \int_{\infty}^{v} \delta\Psi,_{v}\,dv \tag{54b}$$

$$\delta\Psi(u,v) = -\frac{1}{4}\int_{\infty}^{v} 2M\frac{\vartheta(v)}{r^{*2}}\,dv \quad .$$

If $\vartheta(v)$ falls off at large v, as it must, then an upper limit for the integral is

$$\delta\Psi < \left(-\frac{1}{4}\right) 2M\vartheta(v)\int_{\infty}^{v} \frac{dv}{r^{*2}} = \frac{M\vartheta(v)}{r^{*}} \quad . \tag{55}$$

So the effect of scattering by δF is a part of the solution that is down by $2M/r^{*}$ from the primary solution and at the matching region is, therefore, down by $1/\beta$. This verifies our earlier assertion that scattering effects fall off at large r^{*} and shows why the effects of the tail of the potential on evolution of the solution can be ignored.

D. Higher Multipoles

In the case of the monopole we saw that backscatter off the tail of the potential is unimportant; this is not true for $\ell > 0$ multipoles. If we ignore the anomalous tail of the potential and use

$$-4\Psi,_{uv} = \begin{cases} \Psi\,\ell(\ell+1)/r^{*2} & r^{*} > \beta M \\ 0 & r^{*} < -\beta M \\ F_{\ell}\Psi & -\beta M \le r^{*} \le \beta M \end{cases} \quad , \tag{56}$$

we shall see that we can solve for the large time asymptotic behavior of Ψ and, as in the monopole case, the tail of the waves will depend on the incoming radiation from the neighborhood of the first ray. In the monopole case we viewed the ingoing radiation as coming from the first ray [see (43)]. If this were the case for all ℓ modes, then the first ray data

$$\Psi = 1/r^{*\ell} + \text{const.} \times \ln r^*/r^{*\ell+1} + \dots \quad , \tag{57a}$$

would give us as the ingoing waves

$$\Psi = \text{const.} \times \ln(v)/v^{\ell+1} + \dots \quad . \tag{57b}$$

In the neighborhood of the first ray the dominant solution is that of the "primary waves", the exponentially damped wave packet that filters outward through the potential barrier from the collapsing star. These outgoing waves backscatter off the tail of the potential

$$\delta F_\ell^{\,sc} = \frac{\ell(\ell+1)}{r^{*3}} 4M \ln(r^*/2M) + \dots \quad , \tag{58}$$

giving ingoing radiation that augments that of (57b). These two contributions to the ingoing radiation are of the same order so we cannot now ignore backscatter off the potential tail. A careful calculation shows that the waves of (57b), which are due to the tail of the static solution, are precisely cancelled by those of (58), leaving a solution, on a line of constant $u \gg M$, of the form

$$\Psi = \text{const.} \times \ln(v)/v^{\ell+2} + \dots \quad , \tag{59}$$

independently of whether or not there was a static solution in the exterior before the arrival of the first ray. This very important fact is proved in the Appendix where we make a large-v expansion of the solution in the neighborhood of the first ray.

We shall now derive the simple relationship between the large-t asymptotic solution and the large-v ingoing waves from the wave front. To simplify the calculation we shall explicitly deal with the case $\ell = 1$; it will be obvious how to generalize to higher ℓ.

The basis of our approach is to use (56). That is: we disregard the scattering caused by the tail of the potential, except the scattering of the primary waves near the wavefront. This is justified by the more careful calculation in the Appendix in which we keep higher order terms due to the potential. With this simplification, we can write the solution as in (35), as

$$\Psi = \begin{cases} f'(v) - f(v)/r^* + g'(u) + g(u)/r^* & \text{in region VI} \quad (60a) \\ \alpha(u) + \gamma(v) & \text{in region IX} \quad . \quad (60b) \end{cases}$$

In the region $t \gg r^*$ - which we shall refer to, hereafter, as the asymptotic future region - the arguments of the previous section tell us we can again write the solution as

$$\Psi = \psi(t)\ \phi(r^*) \tag{61a}$$

$$\frac{\psi'}{\psi} \ll \frac{\phi'}{\phi} \quad . \tag{61b}$$

Now by matching the solutions (58) and (61) in the overlap of their regions of applicability, we will find that the asymptotic solution depend only on the form of $f(v)$. Since the time derivatives of Ψ can be ignored in the $t \gg r^*$ asymptotic future region according to (51b), we have

$$\Psi = \psi(t)\ \phi_{\text{static}}(r^*) \quad , \tag{62}$$

and for $|r^*| \gg M$, in this asymptotic region we have

$$\Psi = \psi(t)[a_6/r^* + b_6\, r^{*2} + \dots] \quad \text{in region VI} \tag{63a}$$

$$\Psi = \psi(t)[a_9 + b_9\, r^* + \dots] \quad \text{in region IX.} \tag{63b}$$

The expansion of (60b) for $t \gg \beta M$ gives us the solution for $r^* < -\beta M$ in the asymptotic region:

$$\Psi = \gamma(v) = \gamma(t + r^*) = \gamma(t) + r^*\gamma'(t) + \tfrac{1}{2}r^{*2}\gamma''(t) + \ldots \quad . \tag{64}$$

Expressions (63b) and (64) must agree. This gives us $b_9 = 0$ [or more precisely, $b_9 = O(a_9/t)$]. This means the solution for $\phi_{static}(r^*)$ is fixed, up to a scale factor given by a_9. In particular, both a_6 and b_6 will be nonzero in general.

The expansion of (60a) in the asymptotic region is

$$\begin{aligned}\Psi = r^{*-1}\Big(g(t) - f(t)\Big) + \tfrac{1}{2}\, r^*\Big(f(t) - g(t)\Big)'' \\ + \tfrac{1}{3}\, r^{*2}\Big(f(t) + g(t)\Big)''' + \tfrac{1}{8}\, r^{*3}\Big(f(t) - g(t)\Big)'''' + \ldots \quad .\end{aligned} \tag{65}$$

A comparison of (63) and (65) suggests that $\big(f(t) - g(t)\big)$ must be zero. Actually, we can only conclude that $\big(f(t) - g(t)\big)''$ falls off faster than $\big(f(t) + g(t)\big)$. Since $a_6 \neq 0$ in general, $\big(g(t) - f(t)\big)$ must fall off as $\big(f(t) + g(t)\big)'''$; this also guarantees that the higher powers of r^* fall off faster. We satisfy these requirements with

$$g(t) - f(t) = \mu f'''(t) \quad . \tag{66}$$

Observe that the near equivalence of g and f is the near equivalence of the ingoing and outgoing waves in the asymptotic solution in region IX. In other words, $g(t) \approx f(t)$ at large t tells us that almost all the ingoing waves from the first ray are reflected by the potential barrier. This is a phenomenon we noted for the monopole case and is closely related to the impenetrability of the barrier by zero frequency waves we saw for our idealized potential.

It now remains to be noticed that $\Psi(t)$ and $\gamma(t)$ must fall off as $f'''(t)$.

From (59) amd (60) we conclude that

$$f(v) = \text{const.} \times (\ell n\ v)/v^{\ell+1} + \ldots \quad . \tag{67a}$$

in general, and

$$f(v) = \text{const.} \times (\ell n\ v)/v^2 \tag{67b}$$

for the dipole case. (The constants here depend on details of the wavefront. Therefore, for a coordinate stationary observer at any r^*, for $t \gg r^*$, the dipole field falls off as $\ell n(t/2M)/t^5$.

We also, of course, now have the solutions elsewhere in regions VI and IX at large u, v - i.e., well away from the boundary characteristics $u = 0$ and $v = v_0$. The solution in region IX, for large v, is

$$\Psi = \gamma(v) = \text{const.} \times \ell n(v/4M)/v^5 \quad , \tag{68}$$

and in region VI for $u \gg M$ it is

$$\Psi = f'(v) + g'(u) + r^{*-1}\Big(g(u) - f(v)\Big)$$

$$= \text{const.} \times \left\{ -2\left[\frac{\ell n\ v}{v^3} + \frac{\ell n\ u}{u^3} + O\left(\frac{1}{v^3}, \frac{1}{u^3}\right)\right] + \frac{1}{r^*}\left[\frac{\ell n\ u}{u^2} - \frac{\ell n\ v}{v^2} + O\left(\frac{1}{u^2}, \frac{1}{u^3}\right)\right]\right\} . \tag{69}$$

[To give higher-order terms we would have to know f(v) in greater detail than that in (67b).]

It is simple to generalize the foregoing analysis to larger ℓ's. In the $t \gg r^*$ asymptotic region, the analogue of (63a) is:

$$\Psi = \psi(t)[a_6/r^{*\ell} + b_6 r^{*\ell+1} + \ldots] \quad . \tag{70}$$

And the equivalent of (60a) is

$$\Psi = f^{(\ell)}(v) - A_\ell^1 f^{(\ell-1)}(v)/r^* + \ldots + (-1)^\ell A_\ell^\ell f(v)/r^{*\ell}$$
$$+ g^{(\ell)}(u) + A_\ell^1 g^{(\ell-1)}(u)/r^* + \ldots + A_\ell^\ell g(u)/r^{*\ell} \tag{71a}$$

with

$$A_\ell^p = (\ell + p)!/2^p p!(\ell - p)! \quad . \tag{71b}$$

By expanding (71) for t >> r* and comparing with (70) we can easily show that $g(z) \approx (-1)^{\ell - 1} f(z)$ and that the ingoing and outgoing radiation parts are equal near the potential barrier. (This also follows from the argument leading to (48), which applies to any ℓ.) More precisely, we can show

$$g(z) = (-1)^{\ell-1} f(z) + \mu f^{(2\ell+1)}(z) \tag{72}$$

and that

$$\Psi(z) = \text{const.} \times f^{(2\ell+1)}(z) \quad . \tag{73}$$

The form (59) for the ingoing waves after the wavefront and the form (71), of the solution for general ℓ tell us that in region VI, after the passage of the wavefront

$$f(v) = \text{const.} \times \frac{\ell n\, v/4M}{v^2} + \mathcal{O}(1/v^2) \quad . \tag{74}$$

From (73) we have then: <u>For any ℓ, at any r*, for t >> r*, the field Ψ falls off as $\ell n\, t/t^{2\ell + 3}$</u>.

Let us now summarize the physics of the evolution of scalar field multipoles.

(i) Near the first ray (i.e., at small u) the solution is dominated by a wavefront: waves from the stellar surface that have passed through the potential barrier. These primary waves fall off exponentially in u since the variation of Ψ on the stellar surface is exponentially damped.

(ii) The wavefront of primary waves is backscattered by the tail of the potential and the "input" to the post-wavefront region is the ingoing radiation caused by this backscatter. For $\ell = 0$ the ingoing waves take the form $\Psi = -2M/v$ at large v. In the Appendix we see that for higher ℓ the ingoing waves go as a constant times $\ln(v)/v^{\ell+2}$, plus terms that fall off more quickly at large v.

(iii) This ingoing wave is reflected almost completely by the potential barrier near $r^* = 0$, resulting in equal amounts of destructively interfering ingoing and outgoing radiation in region VI, for large t.

(iv) In region IX the outgoing radiation, due to the stellar source, is exponentially damped and at large t, the solution is dominated by the ingoing radiation from region VI that does manage to penetrate the potential barrier. This will be of the form $\Psi = \text{const.} \times \ln v/v^{2\ell+3}$.

(v) In region VI, for $t \gg r^*$ the cancellation of in- and outgoing radiation leads to a Ψ that falls off as $\ln t/t^{2\ell+3}$. From (iv) this is the way in which Ψ falls off in region IX also.

(vi) Though we have started the collapse from a very relativistic static configuration, it is easy to see that our conclusions are independent of this. If the collapse starts from a radius $\gg M$, then the primary waves of (i) dominate for a longer time, but the qualitative evolution after the primary waves have passed is unchanged.

(vii) The locally measured stress-energy of the scalar field -- the energy that would influence a scalar field antenna -- contained in a spherical shell of radius δr is proportional to $(d\Psi/dt)^2\, \delta r$ at large r. The exponentially damped primary waves carry a total energy that is independent of the radius of the shell, but the contribution of the tail of Ψ, due to the scattering off the curvature potential, falls off with r. The tail then does not carry radiation energy per se; it only transports energy in the near zone.

This implies that the relativistic details of the late stages of collapse are not important in practical radiation calculations - unless the field happens to vary very quickly (in comoving coordinates) during the late stages. It should be emphasized that the resolution of the paradox is simply that the field vanishes as $t \to \infty$, avoiding a pathological static solution. The essence of this resolution was revealed in our study of the idealized potential: as $t \to \infty$, a distant observer does not "see" the star - at least, not its scalar field - as it was when it crossed the event horizon. Inasmuch as the phenomenon of the paradox and the decay of the field seem to depend on the background geometry, rather than on details of the field, we should be confident, even without the formal proofs of section IV and Paper II, that the physical ideas presented here do in fact resolve the paradoxes of gravitational and electromagnetic multipoles.

E. A Picture of the Decay of Ψ

In general, invariance of a problem under some group of transformations leads to a conserved quantity. For our radiation problem, the background space is independent of time and we can derive an energy-like conserved for the scalar field. This quantity can help us picture the decay of the field. The wave equation

$$\Psi_{,tt} - \Psi_{,r^*r^*} + F_\ell(r^*)\,\Psi = 0$$

leads us to define

$$\mathcal{H} \equiv \tfrac{1}{2}\,(\Psi_{,t})^2 + \tfrac{1}{2}\,(\Psi_{,r^*})^2 + \tfrac{1}{2}\,F_\ell\,\Psi^2 = (\Psi_{,u})^2 + (\Psi_{,v})^2 + \tfrac{1}{2}\,F_\ell\,\Psi^2 \qquad (75)$$

and

$$\mathcal{S} \equiv \Psi_{,t}\Psi_{,r^*} = (\Psi_{,v})^2 - (\Psi_{,u})^2 \quad , \qquad (76)$$

so that the equation of motion takes the form

$$\partial \mathcal{H}/\partial t = \partial \mathcal{S}/\partial r^* \quad . \tag{77}$$

We can interpret $\mathcal{H}$ as being like an energy density, $\mathcal{S}$ a sort of energy flux and (77) as a divergence equation. We take comfort in the fact that $\mathcal{H}$ is positive definite.

Let us apply this to the radiation problem with boundary values given on the null lines u = 0 and v = 0. On a spacelike hypersurface of constant t the total "energy" in the wave zone is

$$H \equiv \int_{r^* = -t}^{r^* = +t} dr^* \mathcal{H} \quad . \tag{78}$$

This total "energy" can only be changed by "energy" flowing across the boundaries

$$\begin{aligned} dH/dt &= \int_{r^*=-t}^{r^*=+t} dr^* \, \partial \mathcal{H}/\partial t + \mathcal{H}|_{r^*=+t} + \mathcal{H}|_{r^*=-t} \\ &= \int_{r^*=-t}^{r^*=+t} dr^* \, \partial \mathcal{S}/\partial r^* + \mathcal{H}|_{r^*=+t} + \mathcal{H}|_{r^*=-t} \\ &= [\mathcal{H} + \mathcal{S}]_{r^*=+t} + [\mathcal{H} - \mathcal{S}]_{r^*=-t} \\ &= [2\Psi^2_{,v} + \tfrac{1}{2}F_\ell \Psi^2]_{u=0} + [2\Psi^2_{,u} + \tfrac{1}{2}F_\ell \Psi^2]_{v=0} \quad . \end{aligned} \tag{79}$$

The two terms in the last line represent, respectively, the "energy" flowing across the first ray and across the stellar surface. If we consider what the asymptotic contributions at large t are, we find that the second term is exponentially small and hence negligible. The first term at most gives a contribution that falls off as t^{-2} so that $dH/dt \propto t^{-2}$ or less, which means that $H \propto a - b/t$ at large t.

Now we notice that $\mathcal{H}$ is a positive definite quantity and

$$H \gtrsim 1/2 \int F_l \Psi^2 \, dr^* \propto t(\bar{\Psi})^2 \qquad (80)$$

where $\bar{\Psi}$ is some sort of average value of Ψ on the hypersurface. This tells us that this average value of Ψ must fall off essentially as $t^{-\frac{1}{2}}$ or faster since H is essentially constant at large t.

If the information at t = 0 were dispersed by a very strong potential uniformly through the future light cone, Ψ would fall off precisely as $t^{-\frac{1}{2}}$; this is the familiar case of the so-called diffraction of waves studied by Lewis[32] and others. On the other hand, if there were no backscattering the waves would not spread at all so the integral for H would have a non-vanishing contribution in a spatial region independent of t, and Ψ would have a constant value on an ingoing or an outgoing characteristic.

For our problem neither limit applies. In a sense the high frequencies propagates on the characteristics and the low frequencies tend to spread, but the correct asymptotic solution demands a deeper analysis. While arguments based on the conserved flow cannot tell us just what sort of asymptotic solution will develop in the presence of our curvature potential, they do help in picturing the physics of the situation. One way of interpreting this picture of "energy" flow is to say the "final" value of Ψ on the surface of the star, as the surface crosses its gravitational radius, is ineffective in stopping the decay of the field.

In Paper II we shall deal with a complex field satisfying an equation like (22); the only things that must be changed to accommodate the complex field is that we define $\mathcal{K}$ and $\mathcal{S}$ as the real quantities:

$$\mathcal{K} \equiv \tfrac{1}{2}|\Psi_{,t}|^2 + \tfrac{1}{2}|\Psi_{,r*}|^2 + \tfrac{1}{2}F_\ell|\Psi|^2 \tag{81}$$

and

$$\mathcal{S} = \tfrac{1}{2}\left\{\Psi_{,t}\overline{\Psi}_{,r*} + \overline{\Psi}_{,t}\Psi_{,r*}\right\} \quad , \tag{82}$$

where the bar over the Ψ denotes the complex conjugate.

IV. GRAVITATIONAL PERTURBATIONS

The study of the scalar field is more than a plausible analogue; from the mathematics of the previous sections we can directly infer the dynamics of gravitational perturbations. In Paper II a unified view of all integer-spin massless field perturbations will be given with the aid of the null-tetrad formalism of Newman and Penrose.[33] Here we shall describe the physical nature of the fall-off of gravitational perturbations.

Although the mathematical description of gravitational perturbations is not greatly more difficult than that for other perturbations, the physical interpretation is complicated by gauge arbitrariness. Gravitational perturbations (e.g., perturbations in the Reimann tensor) are unavoidably mixed with perturbations in the background geometry. In physical terms, to give a value for a gravitational perturbation we must specify how it would be measured. Nevertheless, the physical nature of the fall-off of the perturbations is fairly clear.

The description of gravitational perturbations used here is essentially that of Regge and Wheeler,[17,34] (RW). This involves the use of vector and tensor spherical harmonics to separate the angular variables, and a convenient choice of gauge. In this RW gauge two functions of radius and time describe the odd-parity perturbations and three functions suffice to describe the even ones.

A. Odd Parity

We are not concerned with multipoles of $\ell < 2$. Such multipoles for the spin-2 gravitational field are nonradiateable. Specifically there can be no $\ell = 0$ odd-parity perturbation, and the $\ell = 1$ multipole has been fully investigated. As Vishveshwara[21] and independently Campolattaro and Thorne[35] have shown, the odd-parity dipole perturbation must be stationary (a consequence of the field equations) and corresponds to a small angular momentum in the star.

For quadrupole and higher multipole perturbations, Regge and Wheeler[17] found that the field equations lead to a wave equation similar to (22):

$$Q_{,tt} - Q_{,r^*r^*} + F_\ell^{op}(r^*)\, Q = 0 \quad . \tag{83a}$$

Here the curvature potential for odd-parity gravitational waves is

$$F_\ell^{op}(r^*) = (1 - 2M/r)\left\{\ell(\ell + 1)/r^2 - 6M/r^3\right\} \quad . \tag{83b}$$

The RW metric perturbations h_0 and h_1 can be derived from Q according to:

$$h_1 = rQ(1 - 2M/r)^{-1} \tag{84a}$$

$$h_{0,t} = (rQ)_{,r^*} \quad . \tag{84b}$$

The formal similarity of (22) and (83) is striking but to continue the analogy between the odd-parity gravitational perturbations and scalar perturbations we must ask whether Q, as a measure of the gravitational perturbations, is free from pathological coordinate effects. We shall see that it is not; Q vanishes at the event horizon even though locally measured perturbations are finite there.

Let us define

$$q(r,t) = \int_{\infty}^{t} Q(r,T)\, dT \quad . \tag{85}$$

In Paper II it is proven that q is measurable in the following sense: It is a linear combination of the components of the Reimann tensor referred to the orthonormal tetrad of a falling observer, and the coefficients in this linear combination are finite at $r = 2M$. This implies that on the stellar surface q and its proper time derivative are finite at the event horizon. Since $Q = q_{,t}$ then due to time dilation effects Q on the stellar surface will vanish as $(1 - 2M/r)$, when the surface crosses the event horizon at $t = \infty$.

If we now integrate (83a) over the time variable from ∞ to t, we find[36] that q must satisfy the same equation as Q,

$$q_{,tt} - q_{,r^*r^*} + F_{\ell}^{op}(r^*)\, q = 0 \quad . \tag{86}$$

The behavior of q on the stellar surface follows from the measurable nature of q. Since q and its proper time derivative are finite, the argument of (25) to (30) implies that for $u \gg M$ on the stellar surface,

$$q = q_0 + q_1 \exp\left\{- u/4M\right\} \quad . \tag{87}$$

The initial value problem for q also requires data on a line u = constant. If we choose the star and field outside it to be momentarily at rest, then q on the first ray signalling the onset of collapse must be the static solution of (86) which is well behaved at spatial infinity.

The structure of the initial value problems for Ψ and for q are then almost identical. The single exception is the difference of the potentials $F_\ell^{op}(r^*)$ and $F_\ell^{sc}(r^*)$. In our analysis of the evolution of scalar perturbations at large times the details of $F_\ell^{sc}(r^*)$ were not used. That analysis involved only (i) the exponential fall-off of the potential as $\exp\{r^*/4M\}$, as $r^* \to -\infty$, and (ii) the first two terms of F_ℓ^{sc} in a large r^* asympotic expansion. (See Eq. (23) for $\ell \neq 0$.) The $\ell(\ell + 1)\ r^{*-2}$ term is just the centrifugal barrier, and $4M\ \ell(\ell + 1)\ r^{*-3}\ \ln(r^*/2M)$ is the first perturbation term. It is this first perturbation term which is all-important in our calculations. The analysis in the Appendix shows that the dominant backscatter of the primary waves in the wavefront depends only on this term. The cancellation of ingoing and outgoing waves in the region near $r^* = 0$ (i.e. the reflection and transmission of the backscattered waves) is also independent both of higher order asymptotic terms in F_ℓ^{sc} and of the detailed structure of F_ℓ^{sc} near $r^* = 0$.

The potential $F_\ell^{op}(r^*)$ has the $\exp\{r^*/4M\}$ feature at $r^* \to -\infty$ and at large r^* it has precisely the same first two expansion terms as those of $F_\ell^{sc}(r^*)$. The analysis and results of Section III therefore apply immediately to q. The asymptotic evolution of q (for $\ell \geq 2$) is precisely the same as that of Ψ. In particular, at a fixed r, q falls off as $\ln t/t^{2\ell+3}$. This phenomenon of the field radiating itself away is, of course, the resolution of the paradox.

The evolution in time of the RW functions Q and h_1 can be found easily from (84). They fall off for large time at constant r, as:

$$h_1 \sim Q \sim \ln t/t^{2\ell+4} \quad . \tag{88}$$

The behavior of h_0 requires further comment. Equation (84b) implies

$$h_{0,t} = (rq_{,t})_{,r^*} \quad , \tag{89}$$

so that

$$h_0 = (r\dot{q})_{,r*} + b(r) \quad . \qquad (90)$$

According to (63) and (70), $(rq)_{,r*}$ has the same time dependence as q, at large t. Furthermore, b(r) must be zero or h_0 would be non-zero at large t, and this means there would be physical singularities[21] at r = 2M or at $r = \infty$. For h_0 then, at large t,

$$h_0 \sim \ell n\ t/t^{2\ell+3} \quad .$$

B. Even Parity

As in the odd-parity case, the properties of the nonradiateable $\ell < 2$ even-parity multipoles are well known. (i) By Birkhoff's theorem an $\ell = 0$ perturbation can only be a small static change in the mass. (ii) Even-parity dipole perturbations correspond to a coordinate displacement of the origin. Such displacements have no physical meaning and can always be annihilated by a guage transformation.[35] To analyze the $\ell \geq 2$ radiateable multipoles we need a wave equation like (22) or (86). Fortunately, Zerilli[18] has recently supplied such an equation. Zerilli's equation is in the context of the RW formalism and the RW gauge. Thus, we describe the even-parity perturbations by three functions: H, H_1, and K in the RW notation. Zerilli assumes perturbations to have exp(- ikt) time dependence. This does not suit our purposes here so while Zerilli replaces H_1 by $R \equiv H_1/k$, we define[37] it as

$$R(r,t) \equiv i \int_t^\infty H_1(r,T)\, dT \quad . \qquad (91)$$

Following Zerilli we define certain linear combinations of R and K:

$$\tilde{K} = a_1 K + a_2 R \qquad (92a)$$

$$\tilde{R} = a_3 K + a_4 R \quad , \qquad (92b)$$

where

$$a_1 = r^2/(\Lambda r + 3M) \qquad (93a)$$

$$a_2 = - i(r - 2M)/(\Lambda r + 3M) \qquad (93b)$$

$$a_3 = \left\{- \Lambda r^2 + 3\Lambda M r + 3M^2\right\}/(\Lambda r + 3M)^2 \qquad (93c)$$

$$a_4 = i(r - 2M)\left\{\Lambda(\Lambda + 1) r^2 + 3\Lambda M r + 6M^2\right\}/r^2(\Lambda r + 3M)^2 \qquad (93d)$$

$$\Lambda = \tfrac{1}{2}(\ell - 1) (\ell + 2) \quad . \qquad (93e)$$

(The third function, H, can be found from $\tilde{R}$ and $\tilde{K}$ by means of the field equations.[18]) With the definitions of (92) and (93), we can put the field equations in a very simple form:

$$\tilde{K}_{,r^*} = \tilde{R} \qquad (94a)$$

$$\tilde{R}_{,r^*} = F_\ell^{EP}(r^*)\ \tilde{K} + \tilde{K}_{,tt} \quad , \qquad (94b)$$

with

$$F_\ell^{EP}(r^*) = (1 - \frac{2M}{r})\ \frac{\left\{2\Lambda^2(\Lambda + 1)r^3 + 6\Lambda^2 M r^2 + 18\Lambda M^2 r + 18M^3\right\}}{r^3(\Lambda r + 3M)^2} \quad . \qquad (94c)$$

We can now combine (94a) and (94b) to get Zerilli's effective-potential equation,

$$\tilde{K}_{,tt} - \tilde{K}_{,r^*r^*} + F_\ell^{EP}(r^*)\ \tilde{K} = 0 \quad , \qquad (95)$$

an equation of the same form as (22) and (86).

In Paper II it is demonstrated that $\tilde{K}$ describes the even-parity perturbations with no pathological coordinate effects. That is, if locally measured gravitational perturbations are finite on the stellar surface during the passage through the event horizone, then $\tilde{K}$ and its proper time derivative on the stellar surface are also finite. From the argument of (25) to (30) it follows that for $u \gg M$ on the stellar surface,

$$\tilde{K} = K_0 + K_1\left\{\exp - u/4M\right\} \quad . \tag{96}$$

As we did for scalar and odd-parity waves, we may start the star and $\tilde{K}$ field from a momentarily static situation. The remaining input is then a static solution to (95) on the "first ray", u = constant. The initial value problems for Ψ, q, and $\tilde{K}$ are now quite similar. Furthermore, from (94c) $F_\ell^{EP}(r^*)$ has precisely the same first two asymptotic terms at $r^* = +\infty$ and at $r^* = -\infty$, as those given by (23) for $F_\ell^{sc}(r^*)$ if $\ell > 1$. The discussion for odd-parity waves shows that only these asymptotes of the potential are important to the analysis of Section III and we may therefore apply the results of Section III directly to $\tilde{K}$. That is, we conclude that the dynamics of $\tilde{K}$ far from the characteristic boundaries (the "stellar surface" and the "first ray") is the same as the dynamics of Ψ and of q. In particular, at a fixed r, $\tilde{K}$ falls off, at large t, as $\ell n\ t/t^{2\ell+3}$. From (94) we see that $\tilde{R}$ has this same asymptotic time dependence[38] and therefore by (92), K and R also fall off as $\ell n\ t/t^{2\ell+3}$. Using the field equations[17] we can show that H therefore dies out at this same rate. Since H_1 is a time derivative of R, it must fall off faster, as $\ell n\ t/t^{2\ell+4}$. We can conclude therefore that <u>at large times, even-parity gravitational perturbations vanish as $\ell n\ t/t^{2\ell+3}$,</u>

and it is this vanishing of the perturbations that resolves the paradox of the singularities.

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APPENDIX

The scalar potential of (22) (as well as the potentials of (83) and (94)) is characterized by an asymptotic form at large r^* dominated by a centrifugal barrier $\ell(\ell + 1)/r^{*2}$ augmented by a "tail": $4Mr^{*-3}\ \ell(\ell + 1)\ \ln(r^*/2M) + \mathcal{O}(M/r^{*3})$. In this appendix we shall consider the influence of this tail on the evolution of Ψ for $\ell > 0$, and in so doing we shall verify certain contentions of Section III. To simplify this discussion, throughout this appendix we shall replace the real potential, and its troublesome logarithmic term, by

$$F = \ell(\ell + 1)\ r^{*-2} + \epsilon\ r^{*-3} \quad . \tag{A-1}$$

This is for convenience only. It should be clear that the following calculations can be done as well for the real potential.

First we investigate the nature of the solution near the wavefront, to see why scattering is important there and to justify (59). Consider $\ell = 1$. The primary waves - outgoing waves which penetrate the potential barrier - are dominant near the first ray. By ignoring the potential tail we can calculate a first approximation to Ψ:

$$\underset{0}{\Psi} = g'(u) + g(u)/r^* \quad . \tag{A-2}$$

Using this we may calculate $\delta\Psi_{,v}$ in the manner of (45). We find $\delta\Psi_{,v} \sim \epsilon/v^3$, but this is just the order of $\Psi_{,v}$ in the wavefront if Ψ is static on the first ray. Clearly scattering of the wavefront cannot be ignored in the wavefront.

To investigate the wavefront more carefully, we expand Ψ (for general ℓ) in a series that includes backscatted waves:

$$\Psi = \sum_{p=0}^{\ell} r^{*-p} A_\ell^p \, g^{(\ell-p)}(u) + \sum_{p=0}^{\infty} B_\ell^p(r^*) \, g^{(\ell-p-1)}(u) \quad . \tag{A-3}$$

In this serie the functions $B_\ell^p(r^*)$ are not yet specified. The negative-order derivatives of $g(u)$ are to be interpreted as integrals:

$$g^{(-1)}(u) = \int^u g(u')\, du' \qquad g^{(-2)}(u) = \int^u g^{(-1)}(u')\, du' \tag{A-4}$$

and so forth. The constants of integration in (A-4) depend on our choice for Ψ on the first ray. For example, if we want $\Psi = 0$ prior to the first ray, we must choose the lower limits of the integrals to be at $u' = 0$ and we must choose $g(0) = 0$.

By putting (A-3) into the wave equation $\Psi_{,uv} = -\frac{1}{4} F \Psi$ with F given by (A-1) we get

$$B'^0_\ell = -\tfrac{1}{2} \epsilon A_\ell^0 / r^{*3} \tag{A-5}$$

and the recursion relation

$$2B'^{p+1}_\ell(r^*) = B''^p_\ell - \frac{\epsilon A_\ell^{p+1}}{r^{*p+4}} - B_\ell^p \left\{ \frac{\ell(\ell+1)}{r^{*2}} + \frac{\epsilon}{r^{*3}} \right\} \quad . \tag{A-6}$$

This leads to the B_ℓ^p terms having the behavior, at large r^*

$$B_\ell^p \sim \epsilon \, r^{*-(p+2)} \quad . \tag{A-7}$$

From this we see that we can choose the constants of integration in (A-4) to make the coefficients of r^{*-n} in (A-3) whatever we wish for $p > \ell + 1$. On the other hand, the coefficient of the $r^{*-(\ell+1)}$ term is fixed by $g(0)$ and hence by the coefficient of the $r^{*-\ell}$ term. The ratio of these two terms is always (for any ℓ) that of the static solution. We see therefore that we can represent a static solution on the first ray by (A-3) but a $\Psi(r^*)$ with a different ratio of coefficients of the (dominant) $r^{*-\ell}$ term requires the inclusion of ingoing waves at the first ray. Notice that $\ell = 0$ is an exception. In the monopole case the first perturbation term goes as ϵ/r^{*2}, so ingoing radiation is necessary to make Ψ static on the first ray.

It should be noted at this point that if there were no tail of the potential we could make Ψ vanish by specifying $g(u) = 0$ for $u \geq u_f$. But (A-3) and (A-4) show that we cannot make Ψ vanish by "turning off the source"; the backscattered waves persist. This is important because Newman and Penrose,[39] using their exactly conserved quantities, show that a static multipole field cannot change to a static spherical field in a finite time (as measured at spatial infinity). We therefore should expect to see such unavoidable backscattered waves.[40]

In general we do not expect to see $g(u)$ go strictly to zero at finite u, but we do know that the primary waves are exponentially damped, and that $g(u)$ and is (positive-order) derivatives must fall off as $\exp\{-u/4M\}$, becoming negligible for $u \gg M$. We are left then with only the negative-order derivatives in the expansion (A-3), so that the dominant term, at large r, is

$$B_\ell^\ell(r^*)\ f^{(-1)}(u) \sim (\epsilon/r^{*\ell+2}) \int g(u)\, du \quad . \tag{A-8}$$

We conclude: after the passage of the wave front the input to region VI goes as $v^{-(\ell+2)}$ aside from the exponentially damped tails of the outgoing radiation. This input and the other negative derivative terms in (A-3), depending as they do on the form of $g(u)$ and on ϵ, can be thought of as the ingoing backscatter of the primary waves.[41] If we do this calculation for the real potential in place of (A-1) we find that <u>at $u \gg M$ and at large r the</u>

form of Ψ is $\ln(v/2M)/v^{\ell+2}$, as we claimed in (59).

In the calculations of (59) to (74) we found that in the asymptotic ($t \gg r^*$) region Ψ falls off as $\ln t/t^{2\ell+3}$. This result depends on a delicate cancelling of in- and outgoing waves and we might worry that the potential tail destroys this balance. The following calculation is a justification for ignoring the potential tail in the evolution of Ψ in the asymptotic region. In the interest of brevity, most of the details are omitted but the basic idea does emerge that the in- and outgoing waves still cancel. [Another approach to this calculation, using Laplace transforms, will be published elsewhere by K.S. Thorne. Yet another method has recently been devised by Fackerell.[42]]

Let us expand the solution for Ψ, away from the wave front, in a power series. The in- and outgoing waves can be written, for a definite ℓ-pole, as a power series in r^*,

$$\Psi_{in} = \sum_{k=0}^{\infty} f_k(v)/r^{*k} \tag{A-9a}$$

$$\Psi_{out} = \sum_{k=0}^{\infty} g_k(u)/r^{*k} \quad . \tag{A-9b}$$

Here the functions f_k and g_k are a prior arbitrary for each k. By putting (A-9) into the wave equation with potential (A-1) we can derive recursion relations, e.g.

$$2(k+1)\, g'_{k+1} = [\ell(\ell+1) - k(k+1)]\, g_k + \epsilon\, g_{k-1} \quad . \tag{A-10}$$

We can solve this by iteration for $g_k(u)$ in terms of $g_0(u)$. We find

$$g_k^{(k)}(u) = c_0\, g_0(u) + c_1\, \epsilon\, g'_0(u) + c_2\, \epsilon^2\, g''_0(u) + \dots , \tag{A-11}$$

where the c_n are constants which depend on ℓ and k in a manner we shall not have to specify here. We can expand the ingoing waves the same way to get an equation like (A-11), with the same coefficients except for the signs.

In the asymptotic region ($t \gg r^*$) we can also expand the various functions of u and v, for example:

$$g_k(u) = g_k(t) - r^* g'_k(t) + \tfrac{1}{2} r^{*2} g''_k(t) - \ldots$$

$$f_k(v) = f_k(t) + r^* f'_k(t) + \tfrac{1}{2} r^{*2} f''_k(t) + \ldots \quad . \qquad \text{(A-12)}$$

From (A-11) and (A-12) we have that the terms in (A-9b) which fall off as r^{*n} must have the time dependence

$$g_k^{(k+n)}(t) = c_0\, g_0^{(n)}(t) + \left\{\begin{matrix}\text{terms which fall off} \\ \text{faster in t}\end{matrix}\right\} , \qquad \text{(A-13)}$$

and similarly for $f_k^{(k+n)}(t)$.

We have seen in Section III (see (62), (63)) that for $r^* \gg M$, in the asymptotic zone,

$$\Psi = \Psi(t)\,[a_6/r^{*\ell} + b_6 r^{*\ell+1} + \ldots \quad] \quad . \qquad \text{(A-14)}$$

The absence of a term in (A-14) which goes as $r^{*\ell}$ can be compatible with (A-9) and (A-13) only if

$$g_0(t) = \pm\, f_0(t) + \left\{\begin{matrix}\text{terms which fall off} \\ \text{faster in t}\end{matrix}\right\} \quad . \qquad \text{(A-15)}$$

In the same manner, the presence of the $r^{*(\ell+1)}$ term in (A-14) tell us that the dominant time behavior of Ψ must be $g_0^{(\ell+1)}(t) \approx f_0^{(\ell+1)}(t)$.

According to the argument following (A-8) we should have $\Psi \approx v^{-(\ell+2)}$ at small constant u, implying

$$f_0(v) = \text{const.} \times v^{-(\ell+2)} \quad . \tag{A-16}$$

and

$$\Psi(t) \sim f_0^{(\ell+1)}(t) \sim t^{-(2\ell+3)} \quad . \tag{A-17}$$

If we had used the "real"potential, rather than (A-1) , this result would be slightly different.

$$\Psi(t) \sim \ell n(t/2M)\ t^{-(2\ell+3)} \quad . \tag{A-18}$$

TABLE I

DETAILS OF THE LOCAL CALCULATION

Feature of the Problem	Region (see Fig. 1)	Equations
1. Geometry	Interior D	Friedmann line element. See Eq. (10)
	Exterior E, F	Schwarzschild geometry described in comoving coordinates. See Eq. (14)
2. Scalar charge density	Interior D	(i) $\ell = 0$ $\kappa j = -\left(a_0/a^3(\eta)\right)\ (\sqrt{2}\ 105/8)\cos\chi\cos^2 2\chi$ (This choice of j gives a simple particular solution for Φ in Eq. (16), at $\eta = 0$.) (ii) $\ell = 3$ $\kappa j = -\left(a_0/a^3(\eta)\right)\ (1 - \chi^2/\chi_0^2)\ Y_3{}^0(\theta,\varphi)$
	Exterior E, F	$j = 0$
3. Equations for the evolution of Φ	Exterior E, F	$\left(a^2(\eta)\ \Phi_{,\eta}\right)_{,\eta} - (\sin^2\chi\Phi_{,\chi})_{,\chi}/\sin^2\chi + \ell(\ell+1)\ \Phi/\sin^2\chi = \begin{cases} (\sqrt{2}\ 105/8)\left(a_0/a(\eta)\right)\cos\chi\cos^2 2\chi & \text{for } \ell = 0 \\ \left(a^0/a(\eta)\right)(1 - \chi^2/\chi_0^2)\ Y_3^0(\theta,\varphi) & \text{for } \ell = 3 \end{cases}$
	Dynamic Exterior E	Equation (16), with $j = 0$
	Static Exterior F	Φ remains static; it is a function of r only and is given by the initial conditions in 5 of this table.

TABLE I (continued)

Feature of the Problem	Region (see Fig. 1)	Equations
4. Junction conditions of the star's surface	Stellar surface C	$\bar{\Phi}$ is continuous and $\underset{\sim}{n} \cdot \underset{\sim}{\nabla}\bar{\Phi}$ is continuous i.e. $\bar{\Phi}_{,\chi}/a(\eta) = \bar{\Phi}_{,R}/2^{\frac{1}{2}}(\partial r/\partial R)$ [Here $\underset{\sim}{n}$ is a unit normal to the world line of the stellar surface.]
5. Initial conditions	Initial hypersurface (exterior) A	The initial field is static outside the star ($\bar{\Phi}_{,t} = 0$); hence $\bar{\Phi}$ is given by (i) $\ell = 0$ $\bar{\Phi} = B_0 \ell n(1 - 2M/R)$; $B_0 = 1$ (ii) $\ell = 3$ $\bar{\Phi} = B_3 \bar{\Phi}_{stat}$ where $\bar{\Phi}_{stat}$ is a solution to Eq. (3) and $\bar{\Phi}_{stat} \to 1/R^4$ at large R. $B_3 = - .0144$
	Initial hypersurface (interior) B	$\bar{\Phi}$ is a solution to the equations in 3 of this table at $\eta = 0$. We choose $\bar{\Phi}$ such that $\bar{\Phi}_{,\eta} = 0$ and $\bar{\Phi}_{,\eta\eta} = 0$ initially. Then (i) $\ell = 0$ $\bar{\Phi} = A_0 + \bar{\Phi}_p$ where $\bar{\Phi}_p = \sqrt{2}/8 \cos \chi \langle - 11 + \sin^2 2\chi \rangle$ $A_0 = 1 - \ell n\, 2$ (ii) $\ell = 3$ $\bar{\Phi} = A_3 \bar{\Phi}_h + \bar{\Phi}_p$ where $\bar{\Phi}_p$ is the particular solution of

TABLE I (continued)

Feature of the Problem	Region (see Fig. 1)	Equations
		$(\sin^2\chi\Phi_{,\chi})_{,\chi}/\sin^2\chi - 12\,\Phi/\sin^2\chi$ $= Y_3^0(\vartheta,\varphi)\,(1 - \chi^2/\chi_0^2)$ which goes as $(\chi^2/6)\,Y_3^0$ near $\chi = 0$; Φ_h is the homogeneous solution that goes as $\chi^3\,Y_3^0$ near $\chi = 0$. $A_3 \approx .394$
6. Results for Φ field	D, E, F	See Fig. 2 for numerical results.
7. Numerical constants		χ_0 is chosen as $\pi/4$ for convenience. The star collapses from initial radius $r = 4M$, so that $a_0 = 4\sqrt{2}\,M$. The surface passes through its event horizon at $\eta = \pi/2$, $\tau/2M = \sqrt{2}\,(1 + \pi/2) \approx 3.64$.

TABLE II

REGIONS OF THE RADIATION PROBLEM

Region (see Fig. 4)	Description
I	The initial Cauchy hypersurface t = 0, outside the star. On this hypersurface Φ is chosen to be static. For $\ell = 0$, $\Phi = \text{const.} \times \ln(1 - 2M/R)$. For $\ell \neq 0$, Φ is a solution to Eq. (3).
II	The first ray, u = 0. This first outgoing scalar ray carries information to the exterior that the star has begun to collapse.
III	The static region. This region has not yet received information that the star has begun to collapse. See region F of Fig. 1.
IV	The wavefront. Most of the high-frequency radiation from the stellar source moves on outgoing null lines, is affected only slightly by the potential, and is contained in a wavefront of extent $\Delta u \sim M$.
V	The potential barrier. This region near $r^* \approx 0$ is the domain in which $F_\ell(r^*)$ is large. (See Fig. 3.)
VI	The distant wave zone. This is the spacetime region far from the star and subsequent to the first ray. It is where scalar (and other) radiation would be detected by antennae.
VII	The world line of the surface of the star. The data for Φ, and its derivative normal to the surface, on VII are a result of the computations of Section IIB. (See also region C of Fig. 1)
VIII	The "stellar surface" $v = v_0$. This is a null line approximating the world line of the stellar surface. (see Eq. (30))
IX	The near wave region. The vacuum exterior near the stellar surface. The field here obeys $\Psi_{,tt} - \Psi_{,r^*r^*} \approx 0$.
X	The stellar interior. The dynamics of this region affects the star's exterior only via the data it creates on the stellar surface VII.

FOOTNOTES

1. J.R. Oppenheimer and H. Snyder, Phys. Rev. **56**, 455 (1939).
2. M.M. May and R.H. White, Phys. Rev. **141**, 1232 (1966).
3. P.G. Bergman, Phys. Rev. Lett. **12**, 139 (1964).
4. A.G. Doroshkevitch, Ya.B. Zel'dovich, and I.D. Novikov, Zh. Eksp. Teor. Fiz. **49**, 170 (1965); English translation in Sov. Phys.-JETP **22**, 122 (1966). (Note that this paper incorrectly predicts a t^{-1} fall off of the gravitational quadrupole perturbation.)
5. W. Israel, Nature **216**, 148 (1967).
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8. A.I. Janis, E.T. Newman, and J. Winicour, Phys. Rev. Lett. **20**, 878 (1968).
9. I.D. Novikov, Zh. Eksp. Teor. Fiz. **57**, 949 (1969). (Englished translation in Sov. Phys.-JETP **30**, 518.) Novikov has demonstrated that on a Cauchy hypersurface with the stellar surface at the event horizon, the general initial value data are nonsingular. This complements the proof of Ref. 4.
10. V. de la Cruz, J.E. Chase, and W. Israel, Phys. Rev. Lett. **24**, 423 (1970).
11. A.Z. Patashinski and A. A. Harkov, Preprint No. 317 of the Institute of Nuclear Physics of the Siberian Branch of the U.S.S.R. Academy of Sciences, Novosibirsk (1969). (Also JETP, in press.) This work has a somewhat similar, though incomplete, viewpoint as the present paper. The author received this preprint when near the completion of his research on the problem.
12. K.S. Thorne and A. Campolattaro, Astrophys. J. **149**, 591 (1967).
13. V.L. Ginzburg and L. Ozernoy, Zh. Eksp. Teor. Fiz. **47**, 1030 (1964). English translation in Sov. Phys.-JETP **20**, 689 (1965).

14. W. Israel, Commun. Math. Phys. **8**, 245 (1968).

15. R.H. Price, accompanying paper, Phys. Rev.

16. J.E. Chase, paper to be published. Chase generalizes our conclusions about the impossibility of a nonsingular static scalar field.

17. T. Regge and J.A. Wheeler, Phys. Rev. **108**, 1063 (1957).

18. F.J. Zerilli, Phys. Rev. Lett. **24**, 737 (1970).

19. Static, here and throughout, means independent of the time coordinate geared to the time-like Killing field, i.e. Schwarzschild time.

20. Actually the solutions are confluent hypergeometric functions.

21. C.V. Vishveshwara, Technical Report No. 778 (University of Maryland, Department of Physics and Astronomy, College Park, Maryland, 1968).

22. It is, of course, possible that Φ couples to the Riemann tensor (just as the electromagnetic vector potential couples to the Ricci tensor). It is difficult to see how this coupling could be achieved without introducing higher derivatives or violating the equivalence principle. Such questions are further discussed in Paper II.

23. The description used here is primarily due to D.L. Beckedorff and C.W. Misner, unpublished paper (1962) and D.L. Beckedorff unpublished A.B. Senior thesis, Princeton University (1962). See also the reference in footnote 26.

24. C.G. Callan, unpublished Ph.D. Thesis, Princeton University, (1964).

25. T. Regge and J.A. Wheeler (see footnote 17) also used the concept of an effective potential in their analysis of the stability of the Schwarzschild geometry against odd-parity gravitational perturbations.

26. Our notation for u, v and for U, V is that of C.W. Misner, "Gravitational Collapse" in <u>Astrophysics and General Relativity, Vol. 1</u>, ed. M. Chretien, S. Deser, and J. Goldstein (Gordon and Breach, New York, 1969).

27. Characteristic data require only the specification of Ψ, not of its normal derivatives.

28. Actually the fields (except for $\ell = 0$) fall off as a logarithm of t, divided by a power of t. For want of a better expression we shall call it a power law.

29. W. Kundt and E.T. Newman, J. Math. Phys. 9, 2193 (1968).

30. This exponentially damped wavefront is the total solution proposed by Patashinski and Harkov. (See footnote 11.)

31. We could get the same results using a δ-function potential and Fourier integrals. This would be less instructive and harder to justify.

32. R.M. Lewis, in Asymptotic Solutions of Differential Equations and Their Application, C. Wilcox, ed. (John Wiley, New York, 1964).

33. E.T. Newman and R. Penrose, J. Math. Phys. 3, 566 (1962). Also see A.I. Janis and E.T. Newman, J. Math. Phys. 6, 902 (1965).

34. The field equations as given by Regge and Wheeler contain errors. For the corrected equations see Thorne and Campolattaro (footnote 12) or Vishveshwara (footnote 21).

35. A. Campolattaro and K.S. Thorne, Astrophys. J. 159, 847 (1970).

36. In deriving (86) we must use the fact $\partial Q/\partial t \to 0$ as $t \to \infty$. This follows from the argument (see the discussion in Section I or the reference in footnote 4) that the field must be static at large t.

37. The choice of ∞ as the upper limit on the integral allows us to conclude from the field equations that $d/dr\{iR(1 - 2M/r)\} = H + K$. This is necessary in Zerilli's derivation.

38. Equation (63b) with $b_9 = 0$ $\left(\text{or } \ddot{K} \approx \gamma(v)\right)$ would seem to imply that in region IX, $\tilde{R} = \tilde{K}_{,r*} \sim \ell n\ t/t^{2\ell+4}$. This would be incompatible with (94b). Throughout Section III we have assumed $F_\ell = 0$ in region IX. Thus we

have ignored the $\Psi(t)\exp(r^*/2M)$ term in (63b). This term makes a negligible difference for Ψ, q, or $\tilde{K}$ in region IX, but it gives the dominant asymptotic time behavior for $\Psi_{,r^*}$, $q_{,r^*}$, and $\tilde{K}_{,r^*}$.

39. E.T. Newman and R. Penrose, Proc. Roy. Soc. A. 305, 175 (1968).

40. The necessity of backscatter is discussed by Kundt and Newman (see footnote 29). They show that almost every wave equation leads to a backscattered tail.

41. The interpretation of these integral terms as backscatter is reasonable because they depend on data spread out over a section of the past light cone. Outgoing waves depend only on data at a fixed u.

42. E.D. Fackerell, Astrophys. J., to be published.

FIGURE CAPTIONS

Fig. 1. The "local problem" pictured in the comoving coordinates τ, R, and χ. At $\tau = \tau_0 = 2\sqrt{2}\,(1 + \pi/2)M$ the stellar surface passes through the event horizon. The details of the local calculation are explained in Table I.

Fig. 2. The results of computer integrations for the evolution of $\bar{\Phi}$ for $\ell = 0$ and $\ell = 3$. In the $\ell = 3$ case the great variation of $\bar{\Phi}$ from $\eta = 0$ to $\eta = \pi/2$ necessitates plotting the logarithm of $\bar{\Phi}$. In both the $\ell = 0$ and $\ell = 3$ plots, the scale of coordinates has been chosen to make the curves for $\eta = 0$ appear smooth. For later times the radial derivatives appear discontinuous at the stellar surface. This is wholly due to the change in time of the radial coordinates. The derivative of $\bar{\Phi}$ with respect to proper radial distance is always continuous. For curves a and c: $\eta = 0$, $\tau = 0$, and the values of $\bar{\Phi}$ are the initial, static values. For curve b: $\eta = 3\pi/8$, $\tau = 2.973$, and $\bar{\Phi}$ is static for $R > 4.344$. For curve d: $\eta = \pi/4$, $\tau = 2.111$, and $\bar{\Phi}$ is static for $R > 3.635$. For curves c and e: $\eta = \pi/2$, $\tau = 3.636$, and $\bar{\Phi}$ is static for $R > 4.900$. For a description of initial data and further details, see Table I and the text.

Fig. 3. The appearance of the peak of $F_\ell^{sc}(r^*)$ for $\ell = 0$ and $\ell = 1$. Here the constant in Eq. (4) has been chosen so that $r^* = r - 4M + 2M \times \ln(r/2M - 1)$, and $r^* = 0$ at $r = 4M$, which is the radius from which the star starts to collapse in the calculations of Section III. Note that F_ℓ^{sc} is sharply peaked in the neighborhood of $r^* = 0$. In fact, the peak occurs at $r = 8M/3$ for $\ell = 0$ and for $\ell \neq 0$ at $r_{peak} = 2M\left\{3(L - 1) + \sqrt{9(L - 1)^2 + 32L}\right\}/4L$ where $L \equiv \ell(\ell + 1)$. For $\ell = 1$,

$r_{peak} = 2.88M$; for $\ell = 2$, $r_{peak} = 2.95M$; for $\ell = 3$, $r_{peak} = 2.97M$; for $\ell \to \infty$, $r_{peak} \to 3M$.

Fig. 4. The "radiation problem" pictured in r, t or u, v coordinates. For explanations and descriptions of features of this diagram, see Table II.

Fig. 5. The results of computer integrations of the asymptotic fall off of Ψ for a coordinate stationary observer in the case $\ell = 0$. The slopes of the log Ψ vs. log t curves all approach a slope of -2 at large t, verifying the t^{-2} fall off derived in the text. The computations for these curves use $R_{surf.}(t = 0) = 4M$, and $\Psi = (r/2M)\, \ell n(1 - 2M/r)$ on the first ray. The "surface" data at $v = 0$ were taken to be $a + b \exp - u/4M$, with a and b chosen so that Ψ on the "surface" and on the first ray matches smoothly at $t = 0$. [As in Fig. 3, r* is defined as zero at $r = 4M$.] The dashed lines in the circled insert depict the points for which Ψ is plotted in the three curves.

Fig. 6. The results of computer integrations for the behavior of Ψ in region VI of Fig. 4, along a line of constant v. The "corrected" value of Ψ is defined: $\Psi_{COR}(u) \equiv \Psi(u) + [1 - \Psi(u = 0)] \approx \Psi(u) + 2M/v$. According to the analysis in the text Ψ_{COR} should be approximately $2M/u$ except very near the wavefront or the potential barrier. The computer results verify this. The dashed line in the circled insert depicts the points for which values of Ψ_{COR} are plotted here. Note that $\Psi_{COR} \simeq 2M/u$ even for $r^* = 0$. For further discussion, see the text (especially Eq. (49)).

Fig. 7. The justification for localizing the curvature potential in the monopole case. The computer integrations show that $\mathcal{V} \equiv \Psi_{,v}$ on the first ray falls off as v^{-2}. Also plotted here is $\delta\mathcal{V}$: the change in $\mathcal{V}$, on a line of constant v, between the first ray and $r^* = 20M$ [i.e. the change in $\mathcal{V}$ along one of the dotted lines in the circled insert]. The plot of $\delta\mathcal{V}$ as a function of v [i.e. as a function of which dotted line in the insert is used] shows that $\delta\mathcal{V} \propto 1/v^2$. Though $\delta\mathcal{V}$ falls off at the same rate as $\mathcal{V}$, it is only 10% as large. For further discussion, see the text (especially Eq. (51)).

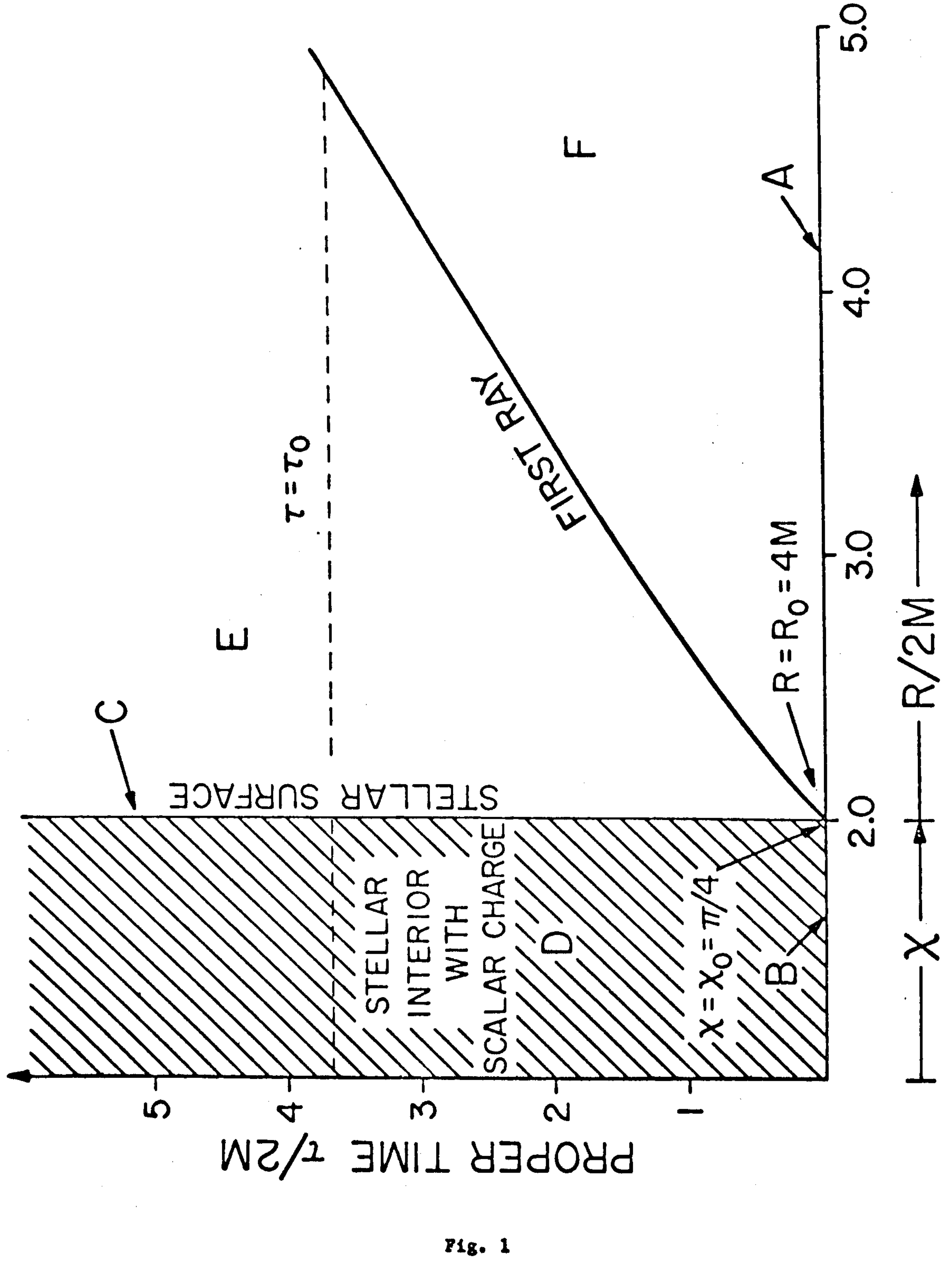

PROPER TIME τ/2M
1
2
3
4
5
R/2M
2.0
3.0
4.0
5.0
X
STELLAR SURFACE
STELLAR
INTERIOR
WITH
SCALAR CHARGE
D
E
F
A
B
C
FIRST RAY
τ = τ0
R = R0 = 4M
X = X0 = π/4

Fig. 1

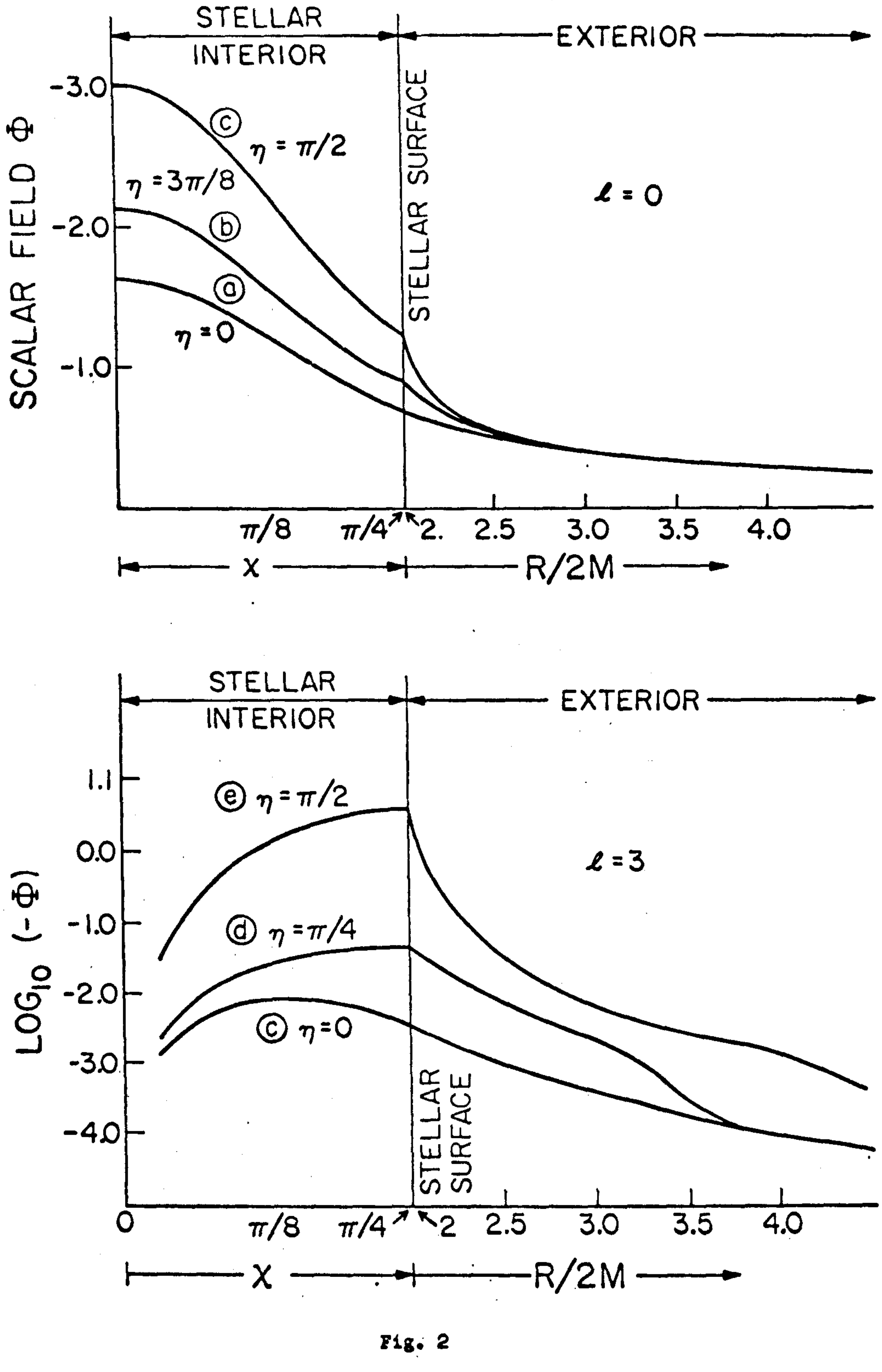
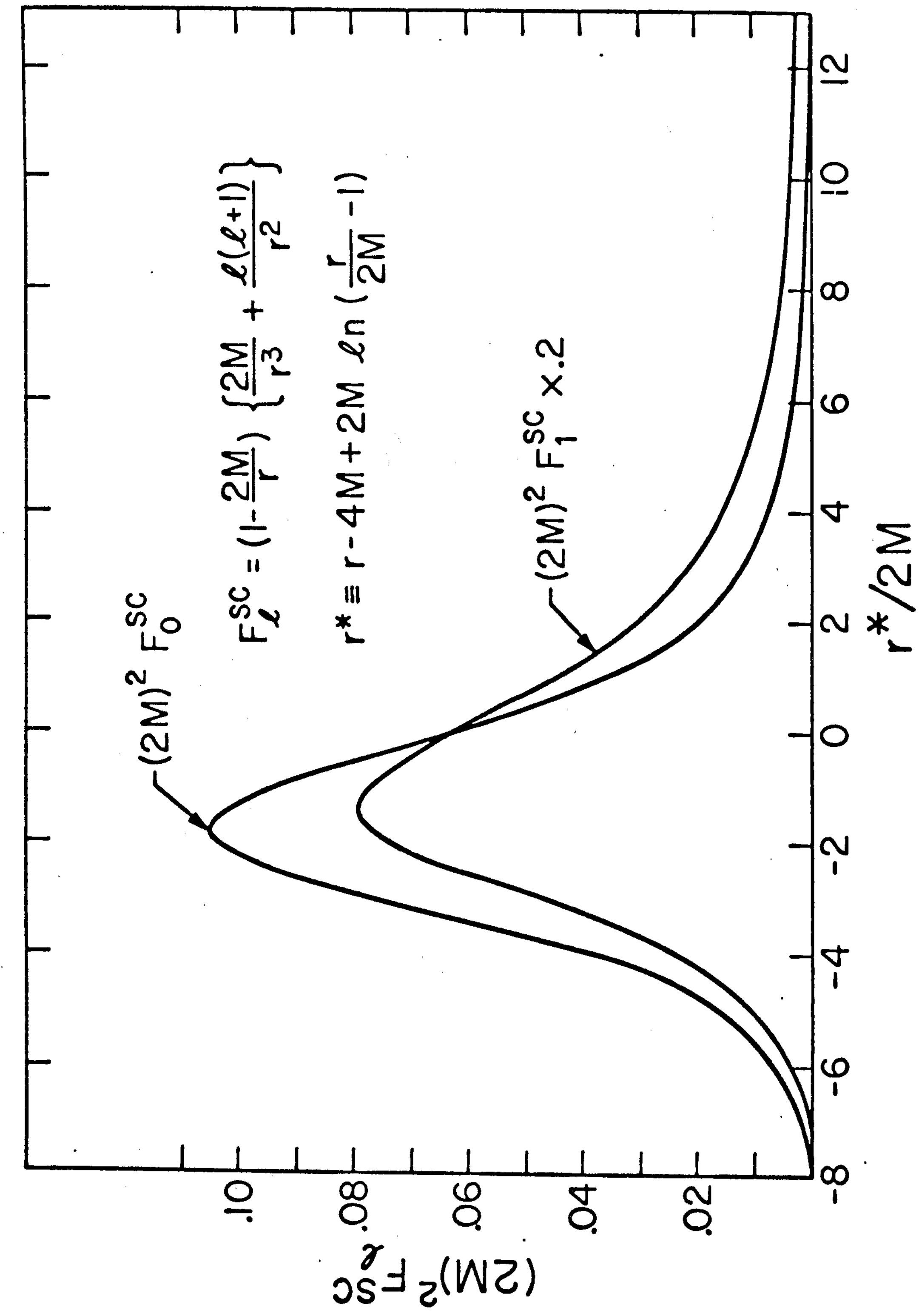

STELLAR INTERIOR
EXTERIOR
SCALAR FIELD Φ
-3.0
-2.0
-1.0
ⓒ η = π/2
η = 3π/8
ⓑ
ⓐ
η = 0
STELLAR SURFACE
ℓ = 0
π/8
π/4
2.
2.5
3.0
3.5
4.0
X
R/2M
STELLAR INTERIOR
EXTERIOR
LOG10 (-Φ)
1.1
0.0
-1.0
-2.0
-3.0
-4.0
ⓔ η = π/2
ⓓ η = π/4
ⓒ η = 0
ℓ = 3
STELLAR SURFACE
0
π/8
π/4
2
2.5
3.0
3.5
4.0
X
R/2M

Fig. 2



$(2M)^2 F_0^{SC}$
$F_\ell^{SC} = (1-\frac{2M}{r})\left\{\frac{2M}{r^3} + \frac{\ell(\ell+1)}{r^2}\right\}$
$r^* \equiv r-4M+2M\,\ell n\,(\frac{r}{2M}-1)$
$(2M)^2 F_1^{SC} \times .2$
$(2M)^2 F_\ell^{SC}$
.10
.08
.06
.04
.02
-8
-6
-4
-2
0
2
4
6
8
10
12
$r^*/2M$

Fig. 3

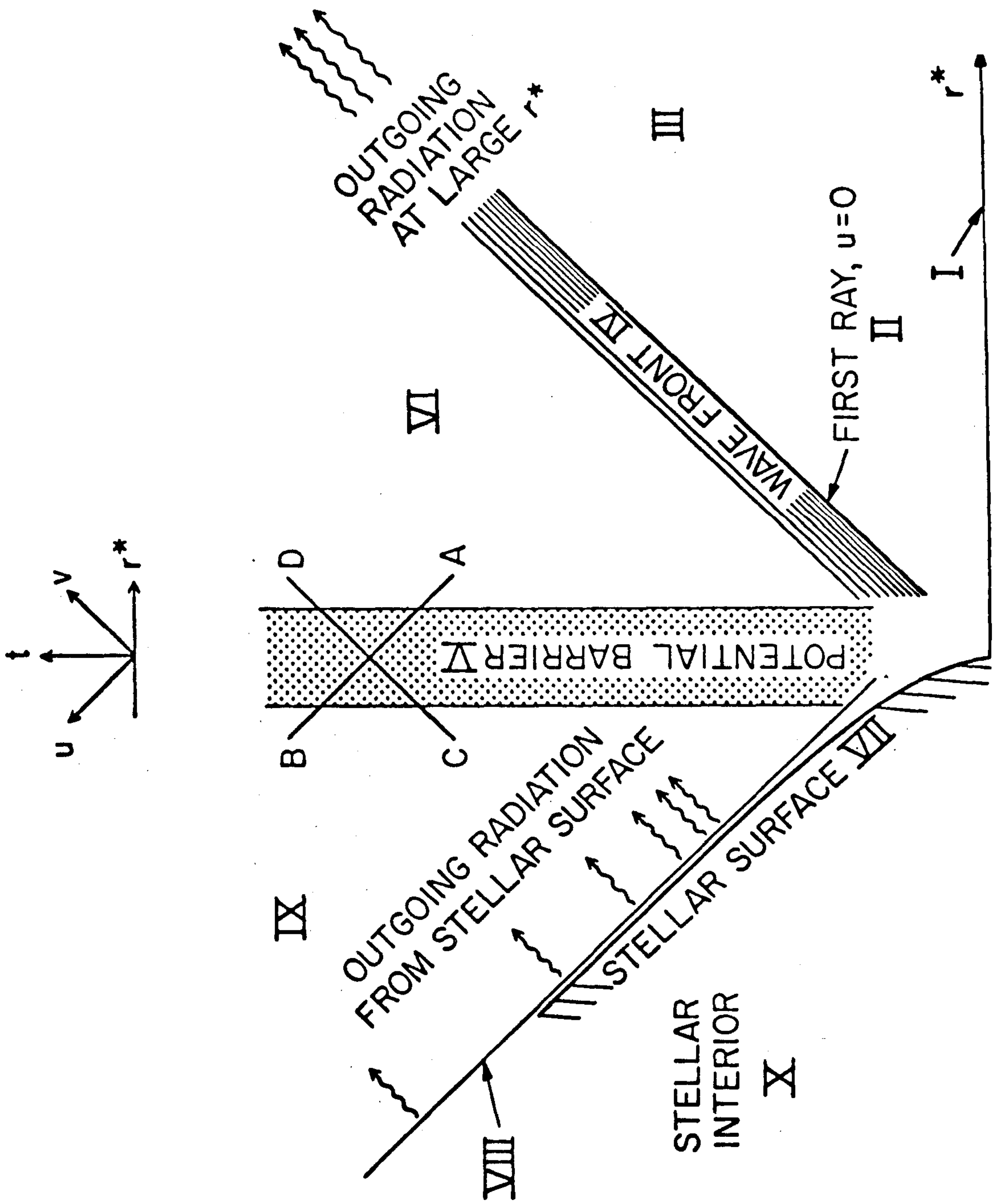

OUTGOING RADIATION AT LARGE r*
III
FIRST RAY, u=0
II
I
r*
WAVE FRONT IV
VI
POTENTIAL BARRIER V
A
B
C
D
u
t
v
r*
OUTGOING RADIATION FROM STELLAR SURFACE
STELLAR SURFACE VII
IX
VIII
STELLAR INTERIOR
X

Fig. 4

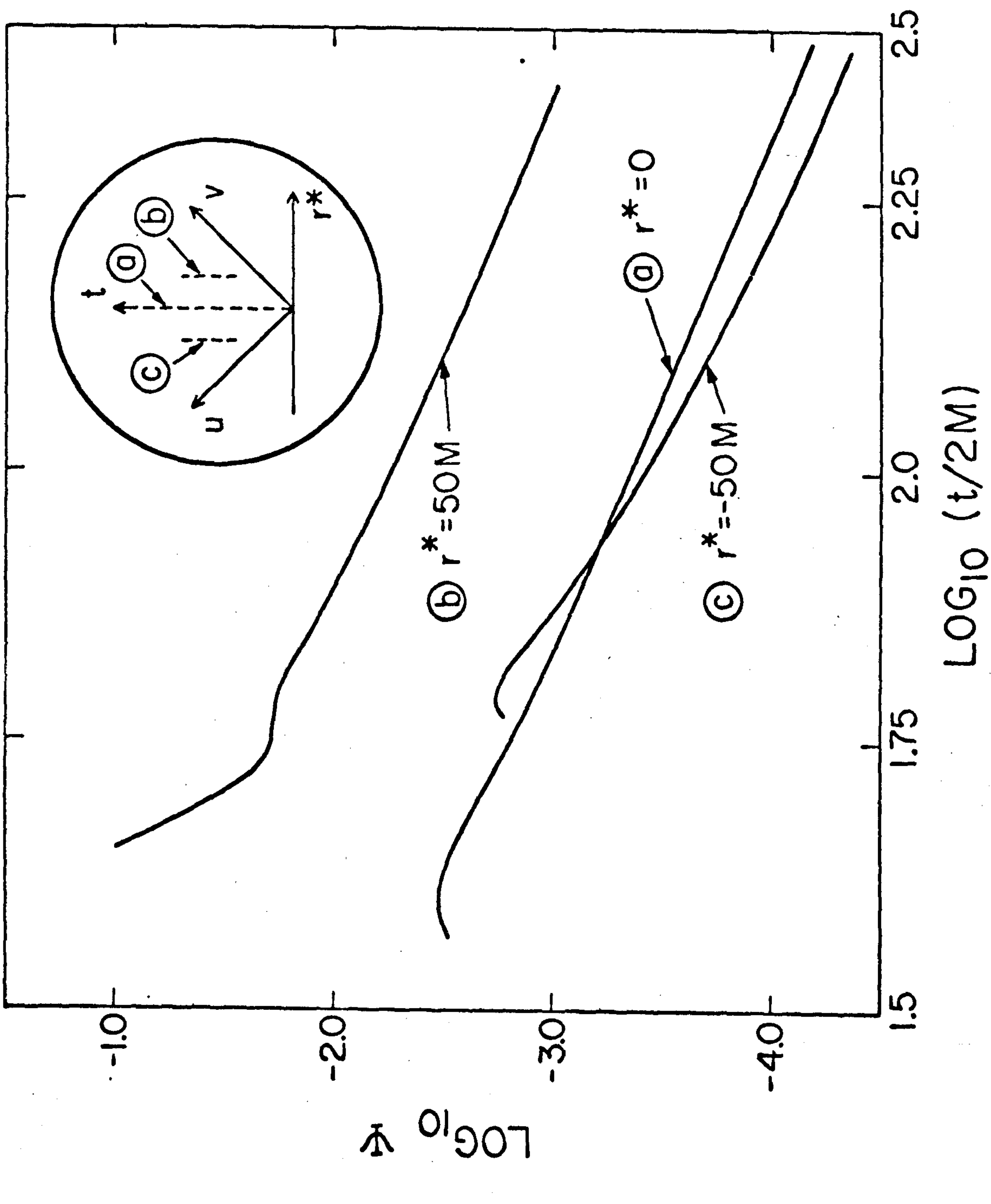

1.5
1.75
2.0
2.25
2.5
LOG10 (t/2M)
-1.0
-2.0
-3.0
-4.0
LOG10 Ψ
ⓐ r*=0
ⓑ r*=50M
ⓒ r*=-50M
t
u
v
r*
a
b
c

Fig. 5

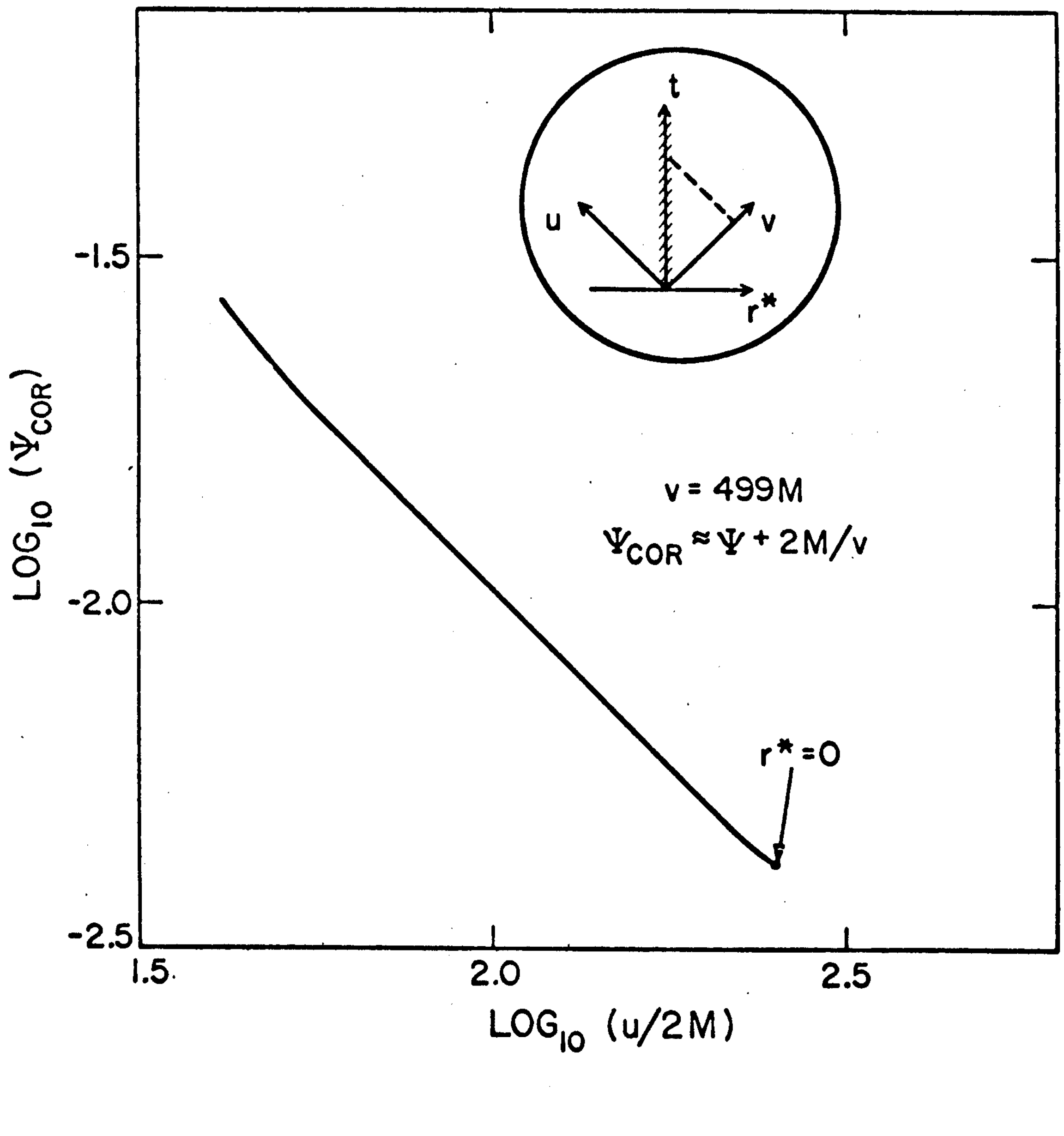
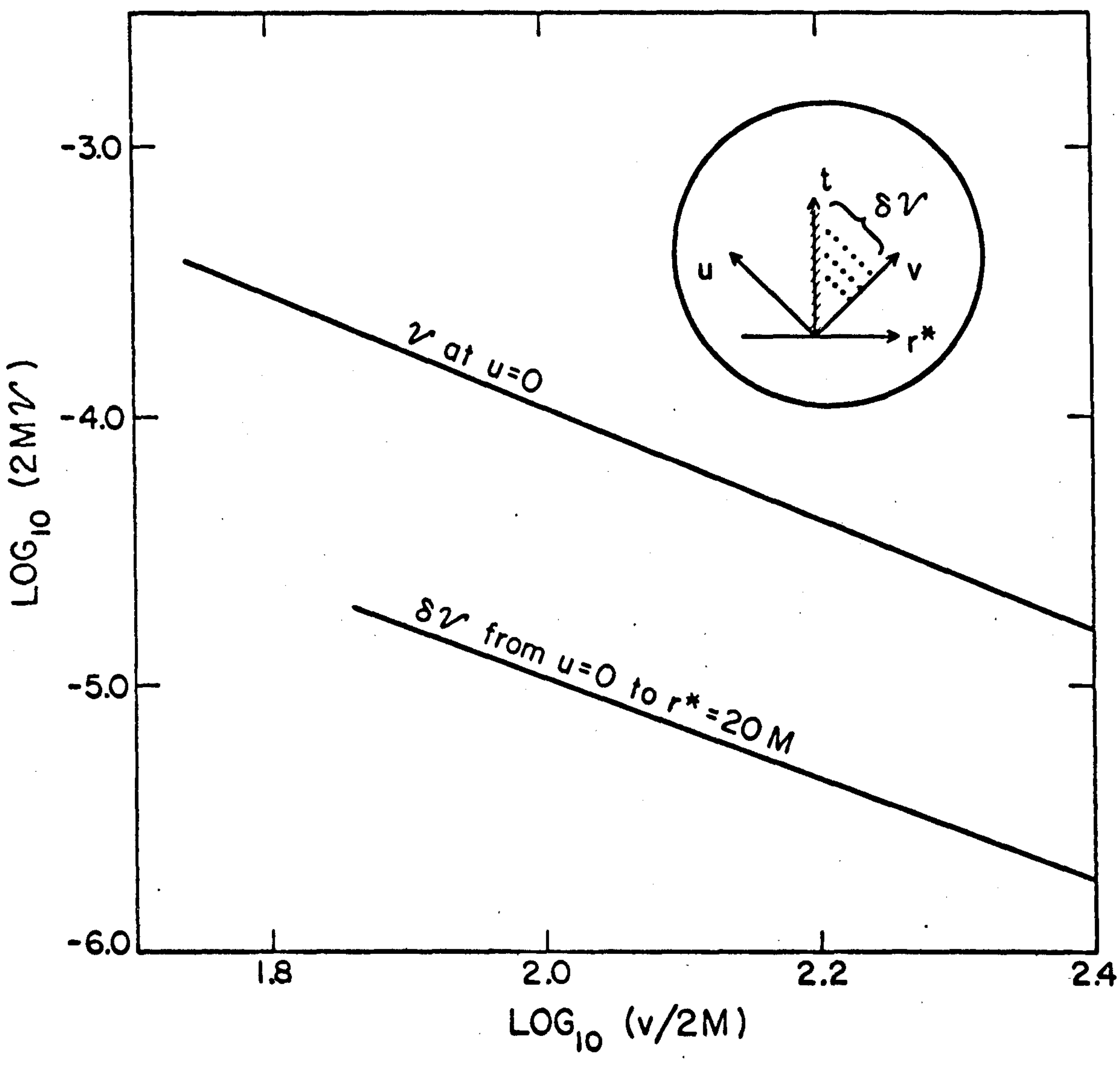


Fig. 6



-3.0
-4.0
-5.0
-6.0
1.8
2.0
2.2
2.4
LOG10 (v/2M)
LOG10 (2M𝒱)
𝒱 at u=0
δ𝒱 from u=0 to r* =20 M
t
δ𝒱
u
v
r*

Fig. 7

I. INTRODUCTION AND SUMMARY

In the preceding paper (Paper I) an analysis is provided of the evolution of scalar and gravitational perturbation fields, during relativistic gravitational collapse. It is concluded that radiatable multipoles of these fields must vanish at large times. Recent work[1,2] has indicated that this is also true for electromagnetic and neutrino multipoles. In the present paper it is shown that this property of radiatable multipoles is a rather general feature of the Schwarzschild geometry. In particular, the mathematical analysis in Paper I can be used for all integer-spin zero-rest-mass perturbation fields, and the physical picture offered in Paper I is quite generally applicable.

The principles of the construction of general-spin field theories in flat spacetime have long been known.[3,4] They follow from the very general requirement that physical fields should generate irreducible representations of the Lorentz group. Such representations are handled very conveniently in the spinor formalism, in which it is straightforward to write down the field theory for any integer or half-integer spin. The mysteries of spinors may be circumvented by the use of the Newman-Penrose[5] (NP) null-tetrad formalism, which is equivalent in most ways to the spinor (or dyad) formalism for integer-spin fields. Except in Sec. IIIA, where the spinor formalism is necessary, the NP formalism and notation will be used throughout this paper. (The reader can avoid spinors in this paper by accepting Eqs. (46) and (47), and ignoring their derivations.)

It is true that the NP formalism cannot handle odd half-integer spin fields, but in any case such fields require quantum mechanical considerations and a more delicate analysis. Hartle[2] has used such delicacy in analyzing the neutrino field outside Schwarzschild and Kerr black holes. He has shown that no static neutrino field can exist outside black holes, so that neutrino fields must also be radiated away during collapse.

The usual way of extending flat-spacetime field theories to curved spacetime is by "minimal coupling". (Replace partial derivatives everywhere by covariant derivatives). This extension gives the correct equations for electromagnetic fields and for neutrino fields -- i.e., the cases spin = 1 and $\frac{1}{2}$ -- but for higher spin fields the resultant equations are inconsistent. (A tantalizing point: Does nature, like physicists, have difficulty with higher spin massless fields?)

Consistent field equations for higher spin fields are not known. Lacking them we will use the equations that result from minimal coupling to analyze higher spin fields. Physical considerations presented later indicate, however, that modifying the minimal coupling scheme to achieve self-consistency should not change those features of the field equations which are crucial to our analysis.

We shall also assume that the fields are very weak, and our analysis shall be confined to first-order perturbations. The stress-energy of these perturbation fields is second order in the size of the field, so we shall ignore the influence of the fields on the curvature of spacetime.

It will be seen that all massless, integer-spin field theories have a remarkably similar appearance in the NP formalism, and that, for a particular multipole the NP "spin-weighted spherical harmonics" give rise to a separation of angular variables in a manner that is considerably more convenient than that of scalar, vector, and tensor spherical harmonics.

From these simple, similar-appearing sets of field equations, it will be shown that: (i) All radiatable multipoles ($\ell \geq$ spin of field) give rise to a "static paradox"; there is no static solution well behaved at the event horizon and at spatial infinity. (ii) For all radiatable multipoles the paradox is resolved by these multipoles necessarily disappearing at large time. By invoking the results of Paper I we are led to the conclusion that (iii) during nearly spherical gravitational collapse, anything that can be radiated, will be radiated, and in fact all measurable manifestations of these multipoles will die out as $\ln t/t^{2l+3}$. (Bernard F. Schutz[6] has emphasized the truth of the converse: anything that is radiated, can be radiated.)

As explained in Paper I, the problem of gravitational perturbations is more subtle than that of nongravitational ones. In the NP formalism, the Bianchi identities have an appearance very similar to the field equations for a nongravitational, spin-2, zero-mass field. This may be exploited to develop an approach to gravitational perturbations quite different from the usual method of Regge and Wheeler.[7] Such an approach, especially in the odd-parity case, may well be more useful for certain problems than other perturbation techniques.

This paper is organized as follows. Section II is a brief outline of some mathematical concepts involving spinors and their connection to the NP formalism. This section is intended to define our notation and provide references, rather than to develop the mathematics.

Section III deals with nongravitational integer-spin perturbations. Field equations are derived (assuming minimal coupling) and are specialized to the Schwarzschild geometry. The NP formalism is shown to be very convenient for discussing parity, separating angular variables, and analyzing event-horizon behavior. The evolution of these fields is then analyzed using the results of Paper I. (In this section and in Sec. IV familiarity with Paper I is essential.)

In Sec. IV the NP formalism for gravitational perturbations is used to supply some missing details of Paper I. Most of the tedious mathematical particulars of Sec. IV have been relegated to Appendices B through F.

II. MATHEMATICAL PRELIMINARIES

A. Dyad and Null Tetrad Formalisms

1. The Newman-Penrose Formalism[5]

The null tetrad. Newman and Penrose define a null tetrad everywhere in spacetime, with legs $\underset{\sim}{l}$, $\underset{\sim}{n}$, $\underset{\sim}{m}$, $\underset{\sim}{m}^*$. Here $\underset{\sim}{l}$ and $\underset{\sim}{n}$ are real vectors, $\underset{\sim}{m}$ is complex, and $\underset{\sim}{m}^*$ is its complex conjugate. They are required to satisfy

$$\begin{aligned} \underset{\sim}{l}\cdot\underset{\sim}{n} &= 1 \\ \underset{\sim}{m}\cdot\underset{\sim}{m}^* &= -1 \end{aligned} \qquad (1)$$

and to have all other inner products vanish. Tensors are projected on the tetrad legs, and the resulting scalars are used in place of tensor components.

Spin coefficients. The properties of the affine connection are manifested not in Christoffel symbols but rather in the (projected) gradients of the tetrad, e.g., $l_{\mu;\nu}l^{\mu}n^{\nu}$. In the NP formalism these quantities, which are not all independent, are combined into twelve complex, independent linear combinations called the spin coefficients:

$$\alpha = \tfrac{1}{2}(l_{\mu;\nu}n^{\mu}m^{*\nu} - m_{\mu;\nu}m^{*\mu}m^{*\nu}) \qquad (2a)$$

$$\beta = \tfrac{1}{2}(l_{\mu;\nu}n^{\mu}m^{\nu} - m_{\mu;\nu}m^{*\mu}m^{\nu}) \qquad (2b)$$

$$\gamma = \tfrac{1}{2}(l_{\mu;\nu}n^{\mu}n^{\nu} - m_{\mu;\nu}m^{*\mu}n^{\nu}) \qquad (2c)$$

$$\epsilon = \tfrac{1}{2}(\ell_{\mu;\nu}n^{\mu}\ell^{\nu} - m_{\mu;\nu}m^{*\mu}\ell^{\nu}) \tag{2d}$$

$$\kappa = \ell_{\mu;\nu}m^{\mu}\ell^{\nu} \tag{2e}$$

$$\pi = -n_{\mu;\nu}m^{*\mu}\ell^{\nu} \tag{2f}$$

$$\rho = \ell_{\mu;\nu}m^{\mu}m^{*\nu} \tag{2g}$$

$$\lambda = -n_{\mu;\nu}m^{*\mu}m^{*\nu} \tag{2h}$$

$$\sigma = \ell_{\mu;\nu}m^{\mu}m^{\nu} \tag{2i}$$

$$\nu = -n_{\mu;\nu}m^{*\mu}n^{\nu} \tag{2j}$$

$$\mu = -n_{\mu;\nu}m^{*\mu}m^{\nu} \tag{2k}$$

$$\tau = \ell_{\mu;\nu}m^{\mu}n^{\nu} \tag{2ℓ}$$

Special names are also given to projected first-order differential operators:

$$D = {}_{;\mu}\ell^{\mu} \tag{3a}$$

$$\Delta = {}_{;\mu}n^{\mu} \tag{3b}$$

$$\delta = {}_{;\mu}m^{\mu} \tag{3c}$$

$$\delta^{*} = {}_{;\mu}m^{*\mu} \tag{3d}$$

The Weyl tensor. The ten independent components of the Weyl tensor $C_{\alpha\beta\gamma\delta}$ are represented, in the NP formalism by the five complex quantities,

$$\Psi_2 = -C_{\alpha\beta\gamma\delta}\,\ell^{\alpha}m^{\beta}\ell^{\gamma}m^{\delta} \tag{4a}$$

$$\Psi_1 = -C_{\alpha\beta\gamma\delta}\,\ell^{\alpha}n^{\beta}\ell^{\gamma}m^{\delta} \tag{4b}$$

$$\Psi_0 = -\tfrac{1}{2} C_{\alpha\beta\gamma\delta} (\ell^\alpha n^\beta \ell^\gamma n^\delta + \ell^\alpha n^\beta m^\gamma m^{*\delta}) \qquad (4c)$$

$$\Psi_{-1} = -C_{\alpha\beta\gamma\delta} \ell^\alpha n^\beta n^\gamma m^{*\delta} \qquad (4d)$$

$$\Psi_{-2} = -C_{\alpha\beta\gamma\delta} n^\alpha m^{*\beta} n^\gamma m^{*\delta} . \qquad (4e)$$

It is very important to note that the subscript conventions of Eqs. (4) differ (for reasons which will become clear) from those of other authors.

This Paper	Elsewhere
Ψ_{-2}	Ψ_4
Ψ_{-1}	Ψ_3
Ψ_0	Ψ_2
Ψ_{+1}	Ψ_1
Ψ_{+2}	Ψ_0

2. Spinors[8]

A 1-spinor ζ^A is a vector in a two dimensional ($A = 0, 1$) complex vector space. The spinor transformations are taken to be unimodular 2×2 matrices, in order to take advantage of the homomorphism between SL(2,C) and the Lorentz group. Thus a correspondence can be set up between a Lorentz transformation $\mathcal{L}$, and an element L of SL(2,C). Four different types of 1-spinors are distinguished by position of the index, and whether it is dotted. If L corresponds to $\mathcal{L}$ then: (i) ξ^A is transformed by L. (ii) ξ_A is transformed by L^{-1}. (iii) $\xi^{\dot{A}}$ is transformed by L^*, the complex conjugate of L. (iv) $\xi_{\dot{A}}$ is transformed by $(L^{-1})^*$. Higher rank spinors are indexed quantities -- e.g., $\zeta^{AB}{}_{\dot{C}}$ -- that transform the same way as outer products of 1-spinors.

The antisymmetric spinors ϵ^{AB} and ϵ_{AB} are used to raise and lower indices

$$\zeta^A = \epsilon^{AB}\zeta_B \qquad \zeta_B = \zeta^A\epsilon_{AB} \tag{5}$$

and to form a scalar product $\epsilon_{AB}\zeta^A\xi^B$.

A set of four independent Hermitean matrices such as

$$\sigma^t_{A\dot{X}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad \sigma^x_{A\dot{X}} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad \sigma^y_{A\dot{X}} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \quad \sigma^z_{A\dot{X}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{6}$$

can be used to set up an equivalence between tensors and spinors:

$$T^{\mu\nu\cdots} = \sigma^{\mu}_{A\dot{X}}\,\sigma^{\nu}_{B\dot{Y}}\cdots\xi^{A\dot{X}B\dot{Y}\cdots} \quad . \tag{7}$$

An affine connection can be specified for spinor fields such that

$$\epsilon_{AB;\mu} = \sigma^{\nu}_{A\dot{X};\mu} = 0 \tag{8a}$$

and

$$(\xi_{A;\mu})^* = (\zeta_A)^*_{;\mu} \quad . \tag{8b}$$

In this way the operations of raising and lowering indices, complex conjugation, and taking spinor equivalent of tensors [Eq. (7)] commute with covariant differentiation.

3. The Dyad Formalism[9]

The dyad. In analogy to defining a tetrad as a reference basis for tensors one defines a dyad as a reference basis for spinors: two fields of 1-spinors o and ι with the "normalization"

$$o_A\,\iota^A = -\,o^A\,\iota_A = 1. \tag{9}$$

The symbol $\zeta_{\underline{a}}$ is used to denote either dyad leg:

$$\zeta_{\underline{0}} = o \qquad \zeta_{\underline{1}} = \iota$$
$$\zeta^{*}_{\underline{\dot{0}}} = o^{*} \qquad \zeta^{*}_{\underline{\dot{1}}} = \iota^{*} . \tag{10}$$

Spinors are projected onto the dyad to form dyad components, which are scalars with respect to transformations in spinor space:

$$\tau_{\underline{a}\,\underline{b}} = \tau_{CD}\, \zeta^{C}{}_{\underline{a}}\, \zeta^{D}{}_{\underline{b}} . \tag{11}$$

(Note: underlined indices $\underline{a}$, $\underline{b}$, ... always denote dyad indices.)

The affinity. The affine connection in the dyad formalism, is stated in terms of the dyad-projected covariant derivatives of the dyad legs. These are given the symbols

$$\Gamma_{\underline{a}\,\underline{b}\,\underline{c}\,\underline{\dot{d}}} \equiv \zeta_{\underline{a}X;\underline{c}\,\underline{\dot{d}}}\, \zeta^{X}{}_{\underline{b}} \tag{12a}$$

where

$$;_{\underline{c}\,\underline{\dot{d}}} \equiv\, ;_{\mu}\, \sigma^{\mu}_{A\dot{B}}\, \zeta^{A}{}_{\underline{c}}\, \zeta^{*\dot{B}}_{\underline{\dot{d}}} . \tag{12b}$$

These Γ's can be shown to be symmetric on the first two indices $\underline{a}$, $\underline{b}$.

The dyad-NP tetrad correspondence.[5] As NP have shown, a close relationship exists between the dyad and null tetrad formalisms. In particular, if we choose the dyad at every point in space in some way, then the correspondence

$$\ell^{\mu} = \sigma^{\mu}{}_{A\dot{B}}\, o^{A}\, o^{*\dot{B}} \qquad n^{\mu} = \sigma^{\mu}{}_{A\dot{B}}\, \iota^{A}\, \iota^{*\dot{B}}$$
$$m^{\mu} = \sigma^{\mu}{}_{A\dot{B}}\, o^{A}\, \iota^{*\dot{B}} , \tag{13}$$

leads to a well-defined null tetrad satisfying Eq. (1). Conversely specifying the null tetrad is equivalent to specifying a dyad (aside from the possibility of changing the sign of o and ι).

With the correspondence (13) the spin coefficients (2) are related to the gradients of the dyad field (12) in a very simple way. (See Table I.) The derivative operators of Eq. (3) can also be simply expressed now in the dyad language:

$$D = {}_{;\underline{0}\underline{\dot{0}}} \qquad \Delta = {}_{;\underline{1}\underline{\dot{1}}} \qquad \delta = {}_{;\underline{0}\underline{\dot{1}}} \qquad \delta^* = {}_{;\underline{1}\underline{\dot{0}}} \quad . \tag{14}$$

4. Tetrad and Dyad Rotations

At a point there are six degrees of freedom in the way the null tetrad may be rotated while still satisfying the relations (1). Of greatest interest for the present work are the two degrees of freedom in the transformations

$$\tilde{l}^{\mu} = \lambda l^{\mu} \tag{15a}$$

$$\tilde{n}^{\mu} = \lambda^{-1} n^{\mu} \tag{15b}$$

$$\tilde{m}^{\mu} = e^{i\varphi} m^{\mu} \ . \tag{15c}$$

Here λ and φ must be real functions if Eq. (1) is to remain true. If a NP scalar transforms under Eqs. (15) as

$$\tilde{T} = \lambda^{C} e^{iS\varphi} T$$

then T is said to have conformal weight C and spin weight S. For instance the NP scalar $T_{\alpha\beta\gamma} n^{\alpha} m^{\beta} m^{\gamma}$ has conformal weight -1 and spin weight +2. (Notice that the subscript p on the Weyl tensor component ψ_p of Eq. (4)

indicates both its spin weight and its conformal weight.)

Conformal and spin rotations of the null tetrad induce the following transformation on the equivalent dyad

$$\tilde{o} = \Lambda^{\frac{1}{2}} e^{i\varphi/2} o$$
$$\tilde{\iota} = \Lambda^{-\frac{1}{2}} e^{-i\varphi/2} \iota . \tag{16}$$

From Eq. (11) it follows that for a dyad scalar $T_{\underline{A}\underline{B}\underline{\dot{C}}}\cdots$

$$2 \times \text{Conformal Weight} = (\text{No. of } \underline{0} \text{ s}) + (\text{No. of } \underline{\dot{0}} \text{ s}) - (\text{No. of } \underline{1} \text{ s}) - (\text{No. of } \underline{\dot{1}} \text{ s}) \tag{17a}$$

$$2 \times \text{Spin Weight} = (\text{No. of } \underline{0} \text{ s}) + (\text{No. of } \underline{\dot{1}} \text{ s}) - \text{No. of } \underline{1} \text{ s}) - (\text{No. of } \underline{\dot{0}} \text{ s}) . \tag{17b}$$

5. Newman-Penrose Special System[10]

For many purposes it is useful to choose the null tetrad and the coordinates of the spacetime with certain special geometrically defined restrictions. For spherical geometry this means (i) using the coordinates u and r, retarded time and Schwarzschild radial coordinate, (ii) choosing $\ell_\nu = u_{,\nu}$ so that $\underset{\sim}{\ell}$ is pointing in the outgoing null direction and is parallel transported along itself, and (iii) choosing $\underset{\sim}{n}$ and $\underset{\sim}{m}$ to be parallel transported in the $\underset{\sim}{\ell}$ direction. (For details and a general definition of the special system, see NP.) The NP special system has the following simplifications

$$\kappa = \pi = \epsilon = 0 \qquad \rho = \rho^* \qquad \tau = \alpha^* + \beta, \tag{18}$$

and, with $x^0 = u$ and $x^1 = r$,

$$g^{01} = 1 \qquad g^{00} = g^{02} = g^{03} = 0. \tag{19}$$

If the components of $\underset{\sim}{m}$ and $\underset{\sim}{n}$ are expressed as

$$m^\mu = \omega\delta_2{}^\mu + \xi^i\,\delta_i{}^\mu \qquad (i = 2,3) \tag{20a}$$

$$n^\mu = \delta_1{}^\mu + U\delta_2{}^\mu + X^i\delta_i{}^\mu\,, \tag{20b}$$

then from Eq. (1) the metric must be

$$g^{00} = 0 \qquad g^{12} = 1$$

$$g^{22} = 2\,(U - \omega\omega^*) \tag{21a}$$

$$g^{2i} = X^i - (\xi^i\omega^* + \xi^{*i}\omega) \tag{21b}$$

$$g^{ij} = -\,(\xi^i\xi^{*j} + \xi^{*i}\xi^j). \tag{21c}$$

With Eqs. (20) and (21) one can derive differential relations between the metric functions U, ω, ξ^i, X^i and the spin coefficients [Eqs. (6.10) of NP].

6. Gravitational Equations

The physical manifestations of spacetime curvature involve the Riemann tensor. (In a vacuum the Weyl tensor is equal to the Riemann tensor.) In the NP formalism the Bianchi identities are first-order equations for the null-tetrad projections of the Weyl tensor. These equations are similar in appearance to the field equations for a spin-2 nongravitational field. In the NP special system the equations are

$$D\Psi_1 - 4\rho\Psi_1 = (\delta^* - 4\alpha)\,\Psi_2 \tag{22a}$$

$$D\Psi_0 - 3\rho\Psi_0 = (\delta^* - 2\alpha)\Psi_1 - \lambda\Psi_2 \tag{22b}$$

$$D\Psi_{-1} - 2\rho\Psi_{-1} = \delta^*\Psi_0 - 2\lambda\Psi_1 \tag{22c}$$

$$D\Psi_{-2} - \rho\Psi_{-2} = (\delta^* + 2\alpha)\Psi_{-1} - 3\lambda\Psi_0 \tag{22d}$$

$$\Delta\Psi_2 + [\mu - 4\gamma]\Psi_2 = (\delta - 4\tau - 2\beta)\Psi_1 + 3\sigma\Psi_0 \tag{22e}$$

$$\Delta\Psi_1 + 2[\mu - \gamma]\Psi_1 = \delta\Psi_0 - 3\tau\Psi_0 + \nu\Psi_2 + 2\sigma\Psi_{-1} \tag{22f}$$

$$\Delta\Psi_0 + 3\mu\Psi_0 = (\delta + 2\beta - 2\tau)\Psi_{-1} + 2\nu\Psi_1 + \sigma\Psi_{-2} \tag{22g}$$

$$\Delta\Psi_{-1} + [2\gamma + 4\mu]\Psi_{-1} = (\delta - \tau + 4\beta)\Psi_{-2} + 3\nu\Psi_0 . \tag{22h}$$

Other differential relations relating the Ψ's to derivatives of the spin coefficients can be derived [Eqs. (6.11) of NP]. These are the NP version of the formula for the Weyl tensor in terms of derivatives of the Christoffel symbols.

B. Spherical Geometry

1. Null Tetrad

For the Schwarzschild geometry we shall use Schwarzschild coordinates t, r, θ, φ, and also u and v, retarded and advanced time,

$$u = t - r^* \tag{23a}$$

$$v = t + r^* , \tag{23b}$$

where

$$r^* = r + 2M\,\ln(r/2M - 1). \tag{24}$$

The t, r, Θ, φ components of the NP special tetrad are

$$\ell^{\mu} = [(1-2M/r)^{-1}, 1, 0, 0] \tag{25a}$$

$$n^{\mu} = \tfrac{1}{2}[1, -(1-2M/r), 0, 0] \tag{25b}$$

$$m^{\mu} = 1/\sqrt{2}\,[0, 0, 1/r, i/(r\sin\Theta)]. \tag{25c}$$

For this tetrad field the nonvanishing spin coefficients are

$$\begin{aligned} \rho &= -1/r & \beta &= -\alpha = \cot\Theta/2\sqrt{2}\,r \\ \gamma &= M/2r^2 & \mu &= -(1-2M/r)/2r, \end{aligned} \tag{26}$$

and the differential operators of Eq. (3) are

$$D = 2(1-2M/r)^{-1}\,\partial_v \tag{27a}$$

$$\Delta = \partial_u \tag{27b}$$

$$\delta = (\sqrt{2}\,r)^{-1}[\partial_\Theta + (i/\sin\Theta)\,\partial_\varphi]\,. \tag{27c}$$

The Weyl tensor has only one nonvanishing NP component

$$\Psi_0 = -M/r^3. \tag{28}$$

With Eq. (25) the functions defined in Eq. (20) can be evaluated as

$$\begin{aligned} &\omega = X^i = 0 \\ &U = -\tfrac{1}{2}(1-2M/r) \\ &\xi^{\Theta} = 1/\sqrt{2}\,r \qquad \xi^{\varphi} = i/\sqrt{2}\,r\sin\Theta. \end{aligned} \tag{29}$$

2. Spin-Weighted Functions[11]

In calculations on a spherical background it is customary to expand scalars in spherical harmonics to achieve a separation into nonmixing multipoles. This does not work for NP scalars. The underlying reason is that the $\underset{\sim}{m}$ and $\underset{\sim}{m}^*$ vectors destroy the spherical symmetry under coordinate rotation. The "background" in the NP formalism is, however, spherically symmetric under a coordinate rotation coupled to a spin weight transformation [Eq. (15)]. The way a NP scalar transforms under a coordinate rotation clearly depends on its spin weight.

A very elegant technique for separating angular variables in NP equations has been given by Newman and Penrose and others.[11] One expands a spin-weight-S scalar in spin-weight-S spherical harmonics. The spin-weight-zero spherical harmonics are just the ordinary $Y^{\ell}{}_{m}(\theta,\varphi)$, and spherical harmonics with spin weight +1 and -1, are proportional to $rm^{\mu}\partial_{\mu}Y^{\ell}{}_{m}(\theta,\varphi)$ and $rm^{*\mu}\partial_{\mu}Y^{\ell}{}_{m}(\theta,\varphi)$ respectively. Spherical harmonics of any spin weight may be constructed with the spin-weighted-raising operator $\eth$ ("edth") and the lowering operator $\bar{\eth}$. Acting on a scalar of spin weight S they are

$$\eth = -\sqrt{2}\,r\,(m^{\mu}\partial_{\mu} - S\cot\theta/\sqrt{2}\,r) = -[\partial_{\theta} - S\cot\theta + (i/\sin\theta)\,\partial_{\varphi}] \tag{30a}$$

$$\bar{\eth} = -\sqrt{2}\,r\,(m^{*\mu}\partial_{\mu} + S\cot\theta/\sqrt{2}\,r) = -[\partial_{\theta} + S\cot\theta - (i/\sin\theta)\,\partial_{\varphi}] \tag{30b}$$

The normalization for a spin-weight-S spherical harmonic is

$$_{S}Y^{\ell}{}_{m}(\theta,\varphi) = \begin{cases} [(\ell-S)!/(\ell+S)!]^{\frac{1}{2}}\,\eth^{S}Y^{\ell}{}_{m}(\theta,\varphi) & 0 \le S \le \ell \\ [(\ell+S)!/(\ell-S)!]^{\frac{1}{2}}\,(-1)^{S}\,\bar{\eth}^{-S}\,Y^{\ell}{}_{m}(\theta,\varphi) & -\ell \le S \le 0. \end{cases} \tag{31}$$

Notice that, in terms of δ and β for our tetrad (20),

$$\eth = -\sqrt{2}\, r\, (\delta - 2S\beta) \tag{32a}$$

$$\bar{\eth} = -\sqrt{2}\, r\, (\delta^{*} + 2S\beta). \tag{32b}$$

3. Despun Quantities

If Φ is a quantity of spin-weight $S < 0$ then the operator

$$(-2^{-\frac{1}{2}}\eth)^{-S} = r^{-S}(\delta + 2\beta)(\delta + 4\beta) \cdots (\delta - 2S\beta) \ , \qquad S < 0 \tag{33}$$

raises its spin weight to zero. If $S > 0$ the reduction to spin-weight zero can be accomplished by

$$(-2^{-\frac{1}{2}}\bar{\eth})^{S} = r^{S}\,(\delta^{*} + 2\beta)(\delta^{*} + 4\beta) \cdots (\delta^{*} + 2S\beta), \qquad S > 0 \tag{34}$$

In the remainder of this paper, the spin-weight-zero quantities that result from the application of Eq. (33) or Eq. (34) will be called "despun" and denoted by a caret. Thus the despun form of the spin-weighted-S quantity Φ is

$$\hat{\Phi} = \begin{cases} (-2^{-\frac{1}{2}}\eth)^{-S}\Phi & \text{if} \quad S < 0 \\ (-2^{-\frac{1}{2}}\bar{\eth})^{S}\Phi & \text{if} \quad S > 0 \end{cases} \tag{35}$$

(Note that $\hat{\Phi} = \Phi$, if Φ is spin-weight-zero.) When equations for a linear problem on a spherical background are written in despun form, all perturbation fields can be expanded in the ${}_0Y^{\ell}{}_m$'s, the ordinary spherical harmonics. There is no loss of information involved in "despinning" a quantity. If Φ is of spin-weight S, it can be reconstructed from its despun form according to

$$\Phi = \begin{cases} [(\ell-S)!/(\ell+S)!]\,(2^{\frac{1}{2}}\eth)^{S}\,\hat{\Phi} & \text{if} \quad S > 0 \\ [(\ell+S)!/(\ell-S)!]\,(2^{\frac{1}{2}}\bar{\eth})^{-S}\,\hat{\Phi} & \text{if} \quad S < 0. \end{cases} \tag{36}$$

Equation (36) follows from Eq. (31) and from the useful relationships,

$$\bar{\eth}\eth\,{}_{S}Y_{\ell m} = -(\ell-S)(\ell+S+1)\,{}_{S}Y^{\ell}{}_{m} \tag{37a}$$

$$\eth\bar{\eth}\,{}_{S}Y_{\ell m} = -(\ell+S)(\ell-S+1)\,{}_{S}Y^{\ell}{}_{m} \tag{37b}$$

III. NONGRAVITATIONAL, INTEGER-SPIN ZERO-REST-MASS FIELDS

A. General Field Equations in the Schwarzschild Geometry

1. Field Equations in Flat Spacetime

A physical field should generate an irreducible representation of its symmetry group for reasons that are familiar from quantum mechanics. Our starting point in studying general-spin[12] fields is the set of representations of SL(2,C), which is the universal covering group of the proper (time-direction preserving, orientation preserving) Lorentz group $\mathcal{L}_{++}$. It is well known that a spin-s representation of $\mathcal{L}_{++}$ is generated by a totally symmetric spinor $\Phi_{AB\cdots K}$ with 2s indices. A spin s field is thus described by the $2s+1$ independent complex components of $\Phi_{AB\cdots K}$.

Several different possible sets of equations for these Φ's are given in the literature[3] for the case in which Φ describes a zero-rest-mass field, but Penrose[4] has shown that one of these sets is to be preferred:

$$\Phi_{AB\cdots K,}{}^{A}{}_{\dot{Z}} = 0 \tag{38}$$

or equivalently

$$\Phi_{AB\cdots K,Y\dot{Z}} = \dot{\Phi}_{YB\cdots K,A\dot{Z}} \quad . \tag{39}$$

(Recall that $_{,Y\dot{Z}} \equiv \sigma^{\mu}_{Y\dot{Z}}\, \partial\mu$.) Penrose has shown that the alternate field equations are "potential" representations in that they admit solutions of lower spin corresponding to gauge arbitrariness. Equation (39) on the other hand, is devoid of such a gauge group; hence the quantities $\Phi_{AB\ldots K}$ in Eq. (39) should represent the actual physical manifestations of the field. This distinction will be important.

2. Field Equations in Curved Spacetime

Minimal coupling. Equations (39) describe a field in flat spacetime with inertial coordinates. We will now argue that Eq. (39) must also be true locally -- that is, in local inertial coordinates -- in curved spacetime. Suppose there were extra terms in Eq. (39) due to the effects of spacetime curvature. Since (39) consists of first-order differential equations, the curvature would be first-order in the size of our frame so that curvature effects would not decrease with the size of our frame. We have argued that $\Phi_{AB\ldots K}$ represents a measurable physical quantity (versus a potential) so a first-order curvature effect may be interpreted as a violation of the strong equivalence principle. This matter will be discussed further; for now we will take (39) to be locally correct.

If (39) is true locally in local Minkowski coordinates, then covariance demands that the field equations are

$$\Phi_{AB\cdots K;Y\dot{Z}} = \Phi_{YB\cdots K;A\dot{Z}} \tag{40}$$

in general coordinates. The extension of the flat spacetime equations (39) to the curved spacetime form (40) is known as minimal coupling.

The field equations in dyad language. Equation (40) can be "dyadified" by the method of Eq. (11). The left side of (40) then becomes

$$\begin{aligned}
&\Phi_{AB\ldots K;Y\dot{Z}}\, \zeta^{Y}{}_{\underline{y}}\, \zeta^{*\dot{Z}}{}_{\underline{\dot{z}}}\, \zeta^{A}{}_{\underline{a}}\, \zeta^{B}{}_{\underline{b}} \cdots \zeta^{K}{}_{\underline{k}} \\
&= (\Phi_{AB\ldots K}\, \zeta^{A}{}_{a} \cdots \zeta^{K}{}_{\underline{k}})_{;\underline{y}\,\underline{\dot{z}}} - \Phi_{AB\ldots K}\, (\zeta^{A}{}_{\underline{a}} \cdots \zeta^{K}{}_{\underline{k}})_{;\underline{y}\,\underline{\dot{z}}} \qquad (41) \\
&= \Phi_{\underline{a}\,\underline{b}\cdots\underline{k};\underline{y}\,\underline{\dot{z}}} + \Phi^{A}{}_{\underline{b}\ldots\underline{k}}\, \zeta_{\underline{a}A;\underline{y}\,\underline{\dot{z}}} + \cdots + \Phi_{\underline{a}}{}^{B}{}_{\underline{c}\,\underline{d}}\, \zeta_{\underline{b}B;\underline{y}\,\underline{\dot{z}}}
\end{aligned}$$

If the right side of Eq. (40) is similarly "dyadified" and relations like

$$\begin{aligned}
\Phi^{A}{}_{\underline{b}\ldots\underline{k}}\, \zeta_{aA;y\dot{z}} &= \zeta^{A}{}_{\underline{e}}\, \Phi^{\underline{e}}{}_{\underline{b}\cdots\underline{k}}\, \zeta_{\underline{a}A;\underline{y}\,\underline{\dot{z}}} \\
&= \epsilon^{\underline{e}\,\underline{f}}\, \Phi_{\underline{f}\,\underline{b}\cdots\underline{k}}\, \Gamma_{\underline{a}\,\underline{e}\,\underline{y}\,\underline{\dot{z}}}
\end{aligned} \qquad (42)$$

are used, then Eq. (40) becomes

$$\begin{aligned}
&\Phi_{\underline{a}\,\underline{b}\cdots\underline{k};\underline{y}\,\underline{\dot{z}}} + \epsilon^{\underline{e}\,\underline{f}}\Phi_{\underline{f}\,\underline{b}\cdots\underline{k}}\, \Gamma_{\underline{a}\,\underline{e}\,\underline{y}\,\underline{\dot{z}}} + \cdots + \epsilon^{\underline{e}\,\underline{f}}\Phi_{\underline{a}\,\underline{b}\cdots\underline{f}}\, \Gamma_{\underline{k}\,\underline{e}\,\underline{y}\,\underline{\dot{z}}} \\
&= \Phi_{\underline{y}\,\underline{b}\cdots\underline{k};\underline{a}\,\underline{\dot{z}}} + \epsilon^{\underline{e}\,\underline{f}}\Phi_{\underline{f}\,\underline{b}\cdots\underline{k}}\, \Gamma_{\underline{y}\,\underline{e}\,\underline{a}\,\underline{\dot{z}}} + \cdots + \epsilon^{\underline{e}\,\underline{f}}\Phi_{\underline{y}\,\underline{b}\cdots\underline{f}}\, \Gamma_{\underline{k}\,\underline{e}\,\underline{a}\,\underline{\dot{z}}}\ .
\end{aligned} \qquad (43)$$

This equation generates the field equations for the $2s+1$ field variables $\Phi_{\underline{a}\,\underline{b}\cdots\underline{k}}$. In Eq. (43) $\underline{\dot{z}}$ can be $\underline{\dot{0}}$ or $\underline{\dot{1}}$ but $\underline{a},\underline{y}$ must be $\underline{0},\underline{1}$, or equivalently $\underline{1},\underline{0}$, for a nonvacuous equation. Since there are $2s$ inequivalent ways of picking $\underline{b}\cdots\underline{k}$, Eq. (43) generates $4s$ equations.

Because of the total symmetry on indices, a Φ field component for a spin s field is completely characterized by N, the number of $\underline{0}$'s, or M, the number of $\underline{1}$'s, or $p = \frac{1}{2}(N-M)$. According to Eq. (17), p then indicates the spin weight and the conformal weight of the field component; this will be exploited by using p as the index, e.g., for a spin-3 field

$$\Phi_{\underline{0}\,\underline{1}\,\underline{0}\,\underline{0}\,\underline{0}\,\underline{0}} \equiv \Phi_2. \tag{44}$$

The field equations with spin coefficients. In each side of Eq. (43) the Γ's can have only four different forms. The left side of (43) can be reduced to

$$\begin{aligned} \Phi_{p;\underline{y}\dot{\underline{z}}} + N \Big\{ \Phi_{p-1}\, \Gamma_{\underline{0}\,\underline{0}\,\underline{y}\,\dot{\underline{z}}} - \Phi_p\, \Gamma_{\underline{1}\,\underline{0}\,\underline{y}\,\dot{\underline{z}}} \Big\} \\ + M \Big\{ \Phi_p\, \Gamma_{\underline{0}\,\underline{1}\,\underline{y}\,\dot{\underline{z}}} - \Phi_{p+1}\, \Gamma_{\underline{1}\,\underline{1}\,\underline{y}\,\dot{\underline{z}}} \Big\} \end{aligned} \tag{45}$$

and similarly for the right side. For a given p there are only two different nonvacuous equations in Eq. (43) according to whether $\dot{\underline{z}}$ is $\dot{\underline{0}}$ or $\dot{\underline{1}}$. These equations can be written in terms of the spin coefficients (12) and the differential operators (14). For example, setting $\dot{\underline{z}} = \dot{\underline{0}}$ leads to the 2s equations:

$$\begin{aligned} & D\Phi_{p-1} + (s+p-1)\Big\{\kappa\Phi_{p-2} - \epsilon\Phi_{p-1}\Big\} + (s-p+1)\Big\{\epsilon\Phi_{p-1} - \pi\Phi_p\Big\} \\ & = \delta^*\Phi_p + (s+p)\Big\{\rho\,\Phi_{p-1} - \alpha\Phi_p\Big\} + (s-p)\Big\{\alpha\Phi_p - \lambda\Phi_{p+1}\Big\} \end{aligned} \tag{46}$$

for $-s+1 \le p \le s$. Here N and M have been eliminated in favor of $p = \frac{1}{2}(N-M)$ and $s = \frac{1}{2}(N+M$. Setting $\dot{\underline{z}} = \dot{\underline{1}}$ leads to the other 2s equations

$$\begin{aligned} & \Delta\Phi_{p+1} + (s+p+1)\Big\{\tau\Phi_p - \gamma\Phi_{p+1}\Big\} + (s-p-1)\Big\{\gamma\Phi_{p+1} - \nu\Phi_{p+2}\Big\} \\ & = \delta\,\Phi_p + (s+p)\Big\{\sigma\Phi_{p-1} - \beta\Phi_p\Big\} + (s-p)\Big\{\beta\Phi_p - \mu\Phi_{p+1}\Big\}, \end{aligned} \tag{47}$$

for $-s \le p \le s-1$.

All specifically dyadic notation has been eliminated; from this point on we can forget that the derivation of Eqs. (46) and (47) ever involved spinors. We can view the Φ_p now not as dyadic projections of spinor fields, but as NP tetrad projections of the tensor which describes our spin s field.[13] In the remainder of this paper this viewpoint and the language of null tetrads will be used almost exclusively.

3. Field Equations in the Schwarzschild Geometry

In our study of the dynamic evolution of spin-s fields outside a collapsing star, we shall treat the fields as first-order perturbations -- that is, we can ignore their effect (second-order) on the spacetime geometry. The geometry then remains the Schwarzschild geometry [See Eqs. (26), (27)] and (46) and (47) become

$$D\,[r^{s+p}\,\Phi_{p-1}] = (\delta^* + 2p\beta)\,\Phi_p\,r^{s+p} \tag{48a}$$

$$\Delta\,[(1-2M/r)^{p+1}\,r^{s-p}\,\Phi_{p+1}] = (\delta - 2p\beta)\,\Phi_p\,r^{s-p}\,(1-2M/r)^{p+1}. \tag{48b}$$

A great simplification now results if the equations (48) are "despun" according to the prescription of Sec. IIB. [See Eqs. (35) and (37).]

In despun form the field equations for the spin-s field are the 4s equations:

$$2(1-2M/r)^{-1}\partial_v(r^{p+s}\,\hat{\Phi}_{p-1}) = \begin{cases} r^{p+s-1}\,\hat{\Phi}_p & \text{for } p-1 \ge 0 \\ -\frac{1}{2}(\ell+p)(\ell-p+1)r^{p+s-1}\,\hat{\Phi}_p & \text{for } p-1 < 0 \end{cases} \quad \text{for } -s+1 \le p \le s \tag{49a}$$

$$(1-2M/r)^{-(p+1)}\partial_u[r^{s-p}(1-2M/r)^{p+1}\hat{\Phi}_{p+1}]$$

$$=\left\{\begin{array}{ll} r^{s-(p+1)}\hat{\Phi}_p & \text{for } p<0 \\ -\frac{1}{2}r^{s-(p+1)}(\ell-p)(\ell+p+1)\hat{\Phi}_p & \text{for } p+1>0 \end{array}\right\} \text{ for } -s\le p\le s-1 \tag{49b}$$

where Eq. (27) has been used.

B. Properties of the Φ Fields in the Schwarzschild Geometry

1. Parity and Reality

A multipole of a physical quantity is said to be of even or odd parity according to the way the quantity changes under the transformation $(\theta,\varphi)\rightarrow(\pi-\theta,-\varphi)$. The quantity is even parity if it changes by a factor $(-1)^{\ell}$, it is odd parity if the factor is $(-1)^{\ell+1}$. Consider the field component Φ_0. For a multipole, $\Phi_0 \propto Y^{\ell}{}_m(\theta,\varphi)$ and our first impulse is to say Φ_0 has even parity. But the tetrad legs have transformation properties under a parity inversion, so the physical quantities represented by Φ_0 need not be of purely even parity.

Under a parity inversion ℓ and n are unchanged according to Eq. (25) but $m \rightarrow m^*$. Since the complex nature of Φ_0 comes entirely from m and m^*, a parity inversion of Φ_0 amounts to (i) a parity inversion of the physical field that Φ_0 represents, and (ii) complex conjugation of Φ_0. This leads to a very interesting observation: the real part of Φ_0 represents an even-parity physical quantity, and its imaginary part represents an odd-parity physical quantity.

We could make similar arguments for the higher spin-weight quantities but it is simpler to use despun quantities. The same argument we used for Φ_0 applies to any despun field $\hat{\Phi}_p$: The complex nature of $\hat{\Phi}_p$ comes only from $\underset{\sim}{m}$ and $\underset{\sim}{m}^*$, and $\hat{\Phi}_p \sim Y^{\ell}{}_m(\Theta,\varphi)$. It follows that the even-real and odd-imaginary correlation holds for all $\hat{\Phi}_p$.

The usefulness of the parity categorization follows from the well-known nonmixing property of definite parity modes. In the language of the NP formalism this has a very simple expression: In the field equations (49), the coefficients -- which come from the spin coefficients -- are real, so the real and imaginary parts of the ϕ's do not mix.

Customarily, and in particular in the Regge-Wheeler[7] formalism, one uses a necessarily complex mode, e.g., $f(r,t)\, Y^{\ell}{}_m(\Theta,\varphi)$, to represent a multipole of a real physical quantity. In the arguments of Appendix C we shall consider such a mode to be a real quantity; this leads to no problems in practice.

2. Conformal Weight and Behavior at the Event Horizon

The usual approach to avoiding the odious features of the Schwarzschild coordinates near $r = 2M$ is to transform to Kruskal coordinates or one of the freely-falling coordinate systems now on the market, and to suffer the consequences. The NP formalism provides a simpler means through the concept of conformal weight.

The canonical local observer. Consider a radially moving observer who has a local (orthonormal -- not NP) tetrad with time and radial legs $\underset{\sim}{e}_T$ and $\underset{\sim}{e}_R$. He could construct the two real null legs of a local NP tetrad as

$$\underset{\sim}{L} = 2^{-\frac{1}{2}}\,(\underset{\sim}{e}_T + \underset{\sim}{e}_R) \qquad\qquad \underset{\sim}{N} = 2^{-\frac{1}{2}}\,(\underset{\sim}{e}_T - \underset{\sim}{e}_R). \tag{50}$$

Of course he is justified in constructing them as any <u>finite</u> conformal rotation of the above two:

$$\underset{\sim}{L} \rightarrow \Lambda \underset{\sim}{L} \qquad \underset{\sim}{N} \rightarrow \Lambda^{-1} \underset{\sim}{N}.$$

With this understanding let us accept Eq. (50) and its inverse

$$\underset{\sim}{e}_T = 2^{-\frac{1}{2}} (\underset{\sim}{L} + \underset{\sim}{N}) \qquad \underset{\sim}{e}_R = 2^{-\frac{1}{2}} (\underset{\sim}{L} - \underset{\sim}{N}), \tag{51}$$

as defining a canonical radially moving observer $\mathcal{O}$ associated with any NP null tetrad, and conversely. If there is a second observer $\mathcal{O}'$ moving radially outward with proper velocity β as measured by $\mathcal{O}$, then the orthonormal legs $(\underset{\sim}{e}'_T, \underset{\sim}{e}'_R)$ are related to $(\underset{\sim}{e}_T, \underset{\sim}{e}_R)$ by a Lorentz transformation. The canonically associated NP tetrads are related by a conformal rotation

$$\underset{\sim}{L}' = \left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}} \underset{\sim}{L} \qquad \underset{\sim}{N}' = \left(\frac{1-\beta}{1+\beta}\right)^{\frac{1}{2}} \underset{\sim}{N}. \tag{52}$$

<u>The relation of freely-falling and "special" tetrads.</u> If $\underset{\sim}{e}_t$ and $\underset{\sim}{e}_r$ are the basis vectors in the Schwarzschild coordinates then a coordinate stationary observer (cso) has an <u>orthonormal</u> tetrad with time and radial legs

$$\underset{\sim}{e}_{[t]} = (1 - 2M/r)^{-\frac{1}{2}} \underset{\sim}{e}_t \qquad \underset{\sim}{e}_{[r]} = (1 - 2M/r)^{\frac{1}{2}} \underset{\sim}{e}_r . \tag{53}$$

By Eq. (50) then, the $[t, r, \theta, \varphi]$ contravariant components of $\underset{\sim}{L}_{cso}$, $\underset{\sim}{N}_{cso}$ are

$$\underset{\sim}{L}_{cso} = 2^{-\frac{1}{2}} [(1 - 2M/r)^{-\frac{1}{2}}, (1 - 2M/r)^{\frac{1}{2}}, 0, 0] \tag{54a}$$

$$\underset{\sim}{N}_{cso} = 2^{-\frac{1}{2}} [(1 - 2M/r)^{-\frac{1}{2}}, - (1 - 2M/r)^{\frac{1}{2}}, 0, 0]. \tag{54b}$$

A freely-falling observer (FFO) who starts from rest at spatial infinity, falls in with a proper velocity $\beta = -(2M/r)^{\frac{1}{2}}$ as measured by a coordinate stationary observer. From Eq. (52) it fóllows that

$$\underset{\sim}{L}_{FFO} = 2^{-\frac{1}{2}}\left\{[1+\sqrt{(2M/r)}]^{-1},\ [1-\sqrt{(2M/r)}],0,0\right\} \qquad (55a)$$

$$\underset{\sim}{N}_{FFO} = 2^{-\frac{1}{2}}\left\{[1-\sqrt{(2M/r)}]^{-1},\ -[1+\sqrt{(2M/r)}],0,0\right\}. \qquad (55b)$$

The special NP tetrad with $\underset{\sim}{\ell}$, $\underset{\sim}{n}$ defined by Eq. (25) is related to the above tetrad by

$$\underset{\sim}{\ell} = \Lambda\, \underset{\sim}{L}_{FFO} \qquad \underset{\sim}{n} = \Lambda^{-1}\, \underset{\sim}{N}_{FFO} \qquad (56a)$$

where

$$\Lambda = 2^{\frac{1}{2}}\,[1+\sqrt{(2M/r)}]/(1-2M/r)\ . \qquad (56b)$$

If a physical quantity is nonspecial at the event horizon -- i.e., neither 0 nor ∞ as measured by a freely-falling observer -- then NP scalars constructed from this physical quantity and from $\underset{\sim}{L}_{FFO}$ and $\underset{\sim}{N}_{FFO}$ will be finite there. By the definition of conformal weight then, <u>any of our NP scalars ψ of conformal weight C, representing a physical quantity that is nonspecial at the gravitational radius will have the dominant behavior there:</u>

$$\psi \sim (1-2M/r)^{-C}. \qquad (57)$$

In particular, if the integer-spin, zero-rest-mass fields of Sec. IIIA are well behaved at the event horizon, then our description of them in terms of the Φ_p fields will have the apparent pathologies at $r = 2M$,

$$\Phi_p \sim (1-2M/r)^{-p}. \qquad (58)$$

C. An Illustration: Electromagnetism

1. Representation of the Field

The familiar spin-1 case, electromagnetism, is a concrete illustration for the points made in Sects. IIIA and IIIB. The electromagnetic field tensor $F_{\mu\nu}$ has as its spinor equivalent [see Eq. (6)] the Hermitean spinor

$$F_{A\dot{X}B\dot{Y}} = \sigma^{\mu}_{A\dot{X}}\, \sigma^{\nu}_{B\dot{Y}}\, F_{\mu\nu}. \tag{59}$$

A symmetric 2-spinor may be defined, containing the same information as $F_{A\dot{X}B\dot{Y}}$:

$$\phi_{AB} = \tfrac{1}{2}\, \epsilon^{\dot{X}\dot{Y}}\, F_{A\dot{X}B\dot{Y}} \tag{60a}$$

$$F_{A\dot{X}B\dot{Y}} = \tfrac{1}{2}\, (\epsilon_{AB}\, \phi^{*}_{\dot{X}\dot{Y}} + \epsilon_{\dot{X}\dot{Y}}\, \phi_{AB}). \tag{60b}$$

If a dyad basis is defined, then the symmetric dyad components $\phi_{\underline{a}\,\underline{b}}$ are the sort of objects discussed in Sec. IIIA. From Eqs. (59), (60), (11), and (13), these dyad scalars may be expressed in terms of NP scalars,

$$\phi_{\underline{0}\,\underline{0}} = \phi_{+1} = F_{\mu\nu} \ell^{\mu} m^{\nu} \tag{61a}$$

$$\phi_{\underline{0}\,\underline{1}} = \phi_{0} = \tfrac{1}{2}\, F_{\mu\nu} (\ell^{\mu} n^{\nu} - m^{\mu} m^{*\nu}) \tag{61b}$$

$$\phi_{\underline{1}\,\underline{1}} = \phi_{-1} = F_{\mu\nu}\, m^{*\mu} n^{\nu}. \tag{61c}$$

It is easy to see from Eqs. (61) and (15a) that the subscript of the ϕ_p does denote both conformal weight and spin weight, as claimed.

With the use of Eqs. (25) and (62) the ϕ_p can be expressed in terms of the components of the $\underset{\sim}{E}$ and $\underset{\sim}{B}$ fields as measured in the orthonormal tetrad of an observer stationary in the Schwarzschild coordinates,

$$\Phi_{+1} = 2^{-\frac{1}{2}}(1-2M/r)^{-\frac{1}{2}}\left\{(E^{[\theta]} - B^{[\varphi]} + i(E^{[\varphi]} + B^{[\theta]})\right\} \tag{62a}$$

$$\Phi_0 = -\tfrac{1}{2}\left\{E^{[r]} + iB^{[r]}\right\} \tag{62b}$$

$$\Phi_{-1} = -2^{-\frac{1}{2}}(1-2M/r)^{\frac{1}{2}}\left\{(E^{[\theta]} + B^{[\varphi]}) - i(E^{[\varphi]} - B^{[\theta]})\right\} \ . \tag{62c}$$

The brackets [ ] on indices denote components with respect to the orthonormal basis $\underset{\sim}{e}_{[t]}, \underset{\sim}{e}_{[r]}, \underset{\sim}{e}_{[\theta]}, \underset{\sim}{e}_{[\varphi]}$.

2. Behavior at the Event Horizon

The points established in Sec. IIIB are nicely exhibited in Eq. (62). First consider conformal weights and transformation properties. In the transformation of $\underset{\sim}{E}$ and $\underset{\sim}{B}$ between relatively moving frames, the components parallel to the relative velocity are invariant. Thus $E^{[r]}$ and $B^{[r]}$ are the same for the coordinate-stationary and radially falling observers. The field Φ_0 should therefore be finite (and in general nonzero) at the event horizon, which is what is predicted by Eq. (58) for the conformal-weight-zero field component.

The transformation law for perpendicular components is

$$\tilde{\underset{\sim}{B}}_\perp = \frac{1}{(1-\beta^2)^{\frac{1}{2}}}(\underset{\sim}{B}_\perp - \underset{\sim}{\beta}\times\underset{\sim}{E}), \qquad \tilde{\underset{\sim}{E}}_\perp = \frac{1}{(1-\beta^2)^{\frac{1}{2}}}(\underset{\sim}{E}_\perp + \underset{\sim}{\beta}\times\underset{\sim}{B})$$

where $\underset{\sim}{\beta}$ is the velocity of the frame $\tilde{\mathcal{S}}$ as measured in $\mathcal{S}$. If $\tilde{\mathcal{S}}$ is freely falling inward from rest at spatial infinity then $\beta^{[r]} = -(2M/r)^{\frac{1}{2}}$ and

$$\tilde{B}^{[\theta]} + \tilde{E}^{[\varphi]} = \frac{(1-2M/r)^{\frac{1}{2}}}{1+(2M/r)^{\frac{1}{2}}}\left\{B^{[\theta]} + E^{[\varphi]}\right\} \tag{63a}$$

$$\tilde{B}^{[\theta]} - \tilde{E}^{[\varphi]} = \frac{1+(2M/r)^{\frac{1}{2}}}{(1-2M/r)^{\frac{1}{2}}}\left\{B^{[\theta]} - E^{[\varphi]}\right\} \tag{63b}$$

$$\tilde{B}^{[\varphi]} + \tilde{E}^{[\theta]} = \frac{1+(2M/r)^{\frac{1}{2}}}{(1-2M/r)^{\frac{1}{2}}}\left\{B^{[\varphi]} + E^{[\theta]}\right\} \tag{63c}$$

$$\tilde{B}^{[\varphi]} - \tilde{E}^{[\theta]} = \frac{(1-2M/r)^{\frac{1}{2}}}{1+(2M/r)^{\frac{1}{2}}}\left\{B^{[\varphi]} - E^{[\theta]}\right\} \tag{63d}$$

Since $\tilde{\underset{\sim}{E}}$ and $\tilde{\underset{\sim}{B}}$ are measured by a freely-falling observer, the expressions on the left in Eqs. (63) are finite if the field itself is well behaved at the event horizon. This observation and Eqs. (62) imply that, $\Phi_p \sim (1-2M/r)^{-p}$ near $r = 2M$, in agreement with Eq. (58).

3. Parity

In the parlance of electromagnetic radiation theory, "electric" or even-parity modes are the multipole modes with radial components of $\underset{\sim}{E}$; the "magnetic" or odd-parity modes have radial $\underset{\sim}{B}$ components. According to (62b) then, the real and imaginary parts of Φ_0 correspond respectively to the even- and odd-parity modes.

A discussion of the parity of tangential components is simplest with the formalism of Regge and Wheeler[7] for vector spherical harmonics. In their notation the tangential ($j = \theta,\varphi$) components are

$$E^{[j]} = \mathcal{E}_{EP}\, Y^{\ell}{}_{m[j]}(\theta,\varphi) \qquad B^{[j]} = \mathcal{B}_{EP}\, \Phi^{\ell}{}_{m[j]}(\theta,\varphi) \quad \begin{cases}\text{Even}\\ \text{Parity}\end{cases} \tag{64a}$$

$$E^{[j]} = \mathcal{E}_{OP}\, \Phi^{\ell}{}_{m[j]}(\theta,\varphi) \qquad B^{[j]} = \mathcal{B}_{OP}\, Y^{\ell}{}_{m[j]}(\theta,\varphi) \quad \begin{cases}\text{Odd}\\ \text{Parity}\ .\end{cases} \tag{64b}$$

If these expressions are used in Eq. (53) one finds, for example

$$\hat{\Phi}_{+1} = -\tfrac{1}{2}(1-2M/r)^{-\frac{1}{2}}[\ell(\ell+1)]\times\Big\{(\varepsilon_{EP}-\beta_{EP}) + i(\varepsilon_{OP}+\beta_{OP})\Big\} Y^{\ell}{}_{m}(\theta,\varphi). \tag{65}$$

This shows the relation of parity and reality. (Note that we must interpret $\varepsilon_{EP} Y^{\ell}{}_{m}$, for example, as real in the same spirit in which we made the expansions (64) for the real quantities $E^{[j]}$ and $B^{[j]}$.)

4. Field Equations

The source-free field equations of electromagnetism are Maxwell's equations with $J^{u} = 0$,

$$F^{\mu\nu}{}_{;\nu} = 0 \tag{66a}$$

$$\bar{F}^{\mu\nu}{}_{;\nu} = 0$$

Here $\bar{F}^{\mu\nu}$ is the dual to the tensor $F^{\mu\nu}$. With Eqs. (59), (60), and the formalism outlined in Sec. II, the field equations (66) may be written in spinor form,

$$\Phi_{AB;\dot{X}}{}^{A} = 0. \tag{67}$$

The field equations (66) can be described (without resorting to the spinor formalism) in terms of the Φ_p introduced in Eqs. (61). If this is done and the spin coefficient notation is used, the result is precisely Eqs. (46) and (47) for the case $s = 1$. If the special tetrad (25) is used, along with the associated spin coefficients (26), the despun field equations are simply

$$D(r\,\hat{\Phi}_{-1}) = -\tfrac{1}{2}\,\ell(\ell+1)\,\hat{\Phi}_{0} \tag{68a}$$

$$D(r^2 \hat{\Phi}_0) = r \hat{\Phi}_1 \tag{68b}$$

$$\Delta(r^2 \hat{\Phi}_0) = r \hat{\Phi}_{-1} \tag{68c}$$

$$\Delta[(1-2M/r)\, r \hat{\Phi}_1] = -\tfrac{1}{2}\, \ell(\ell+1)(1-2M/r)\, \hat{\Phi}_0 . \tag{68d}$$

D. Evolution of the Fields

In Paper I the problem of the evolution of a scalar field outside a collapsing scalar-charged star is analyzed. Here the equivalent question for fields of other integer spins is asked. Specifically: If a star contains a source of a spin-s field, how will the field outside the star evolve as the star collapses? It will be seen that the solution to this problem can be inferred directly from the analysis in Paper I.

1. A Wave Equation for $\hat{\Phi}_0$

The quantity central to our analysis of any integer-spin field is the spin-weight-zero, conformal-weight-zero, field component $\hat{\Phi}_0$. Two of the equations in (49) which involve Φ_0 are:

$$\partial_v(r^s \hat{\Phi}_{-1}) = -\tfrac{1}{4}\, r^{s-1}\, \ell(\ell+1)\, \hat{\Phi}_0\, (1-2M/r) \tag{69a}$$

$$\partial_u(r^{s+1} \hat{\Phi}_0) = r^s \hat{\Phi}_{-1} \quad . \tag{69b}$$

These can be combined into a second order equation for $\hat{\Phi}_0$:

$$\partial_v \partial_u (r^{s+1} \hat{\Phi}_0) = -\tfrac{1}{4}\, r^{s-1}\, \ell(\ell+1)(1-2M/r)\, \hat{\Phi}_0 \; , \tag{70}$$

or

$$(r^{s+1}\,\hat{\Phi}_0)_{,tt} - (r^{s+1}\,\hat{\Phi}_0)_{,r^*r^*} + F_\ell(r^*)\, r^{s+1}\,\hat{\Phi}_0 = 0 \tag{71a}$$

where

$$F_\ell(r^*) \equiv (1-2M/r)\,\ell(\ell+1)/r^2. \tag{71b}$$

2. Singularities of the Radiatable Multipoles

It is appropriate to re-emphasize here two important properties of $\hat{\Phi}_0$:

(i) According to the arguments of Penrose (see Sec. IIIA), the Φ's are physically measurable quantities, by contrast with potentials. Example: for $s = 1$, the Φ's are algebraically related to the field tensor $F_{\mu\nu}$ rather than the vector potential A_μ. In particular $\hat{\Phi}_0$ is the linear combination of the radial $\underset{\sim}{E}$ and $\underset{\sim}{B}$ components given by Eq. (62b).

(ii) Since $\hat{\Phi}_0$ has zero conformal weight, then according to the arguments in Sec. IIB, $\hat{\Phi}_0$ is finite at $r = 2M$ if the physical field it represents is finite there.

If $\hat{\Phi}_0$ is static it must satisfy

$$(r^{s+1}\,\hat{\Phi}_0)_{,r^*r^*} - F_\ell(r^*)\, r^{s+1}\,\hat{\Phi}_0 = 0 \tag{72}$$

The asymptotes of the solution are:

$$\hat{\Phi}_0 \to r^{*\ell-s} \quad \text{or} \quad \to r^{*-\ell-s-1} \quad \text{at } r^* = +\infty \tag{73a}$$

$$\hat{\Phi}_0 \to r^* \quad \text{or} \quad \to \text{constant} \quad \text{at } r^* = -\infty \tag{73b}$$

Because $\hat{\Phi}_0$ must be finite at the event horizon, it cannot go as r^* at $r^* = -\infty$. Furthermore, if the multipole is radiatable -- that is, if $\ell \geq s$ -- only the solution $r^{*-\ell-s-1}$ is acceptable at $r^* = +\infty$. No solution of Eq. (72) can connect the two acceptable asymptotes (see the argument of Eq. (6) in Paper I). Thus, for a radiatable multipole of any integer-spin, zero-rest-mass perturbation field there is no static solution that is well behaved at the event horizon and at spatial infinity. This is the familiar paradox of Paper I. The resolution of this paradox is also that of Paper I. (For nonradiatable multipoles -- that is $\ell < s$ -- the well behaved asymptotic solutions can be joined, so there is no paradox.)

3. Resolution of the Paradox; Dynamics of $\hat{\Phi}_0$

The potential F_ℓ. Equation (71) is very similar to the equation governing the dynamics of a scalar field Ψ^{sc} [Eq. (22) of Paper I]

$$\Psi^{sc}_{,tt} - \Psi^{sc}_{,r^*r^*} + F_\ell^{\,sc}(r^*)\,\Psi^{sc} = 0 \tag{74a}$$

where

$$F_\ell^{\,sc}(r^*) = (1-2M/r)[2M/r^3 + \ell(\ell+1)/r^2]. \tag{74b}$$

The difference between Eqs. (71) and (74) is only in the details of the potentials F_ℓ and $F_\ell^{\,sc}$. Furthermore, the potentials have the same asymptotic forms at $r^* = +\infty$. Namely, the asymptotic of F_ℓ, for $\ell \neq 0$, are

$$\begin{aligned} F_\ell(r^*) &\approx \ell(\ell+1)/r^{*2} + 4M\,\ell(\ell+1)\,r^{*-3}\,\ell n(r^*/2M) \\ &\qquad + O(r^{*-3}) \qquad\qquad r^* \gg M \\ &\approx \ell(\ell+1)(2M)^{-2}\exp\{r^*/2M\} \qquad r^* \ll -M. \end{aligned} \tag{75}$$

The boundary value problem for $\hat{\Phi}_0$. Since $\hat{\Phi}_0$ is measurable and has no pathological coordinate effects ar $r = 2M$, it should be finite on the surface of the collapsing star during the passage through the event horizon. According to the argument of Sec. IIIA of Paper I, this means that the data for $\hat{\Phi}_0$ on the stellar surface must have the asymptotic form

$$\hat{\Phi}_0 \rightarrow a + b \exp[-u/4M] \tag{76}$$

as $u \rightarrow \infty$. If the star begins momentarily stationary with $\hat{\Phi}_0$ static outside it then the other boundary data can best be given on the first ray, $u = u_0$. The form of $\hat{\Phi}_0$ on $u = u_0$ is that solution of Eq. (72) which is well behaved at $r^* = \infty$.

Comparison with the scalar problem. In Paper I it is shown that the details [other than those given in Eq. (75)] of the potential do not make a difference to the asymptotic ($t \gg |r^*|$) evolution of the scalar field. The boundary value problem for the function $r^{s+1}\,\hat{\Phi}_0$ is therefore equivalent to the scalar field problem, insofar as the asymptotic evolution is concerned. (See especially the discussion in Sec. IVA of Paper I, comparing the boundary value problems of the scalar field and of the odd-parity gravitational perturbations.) The analysis and the results of Sec. III of Paper I regarding asymptotic evolution must apply then to $r^{s+1}\,\hat{\Phi}_0$. In particular, it follows that if Φ_0 is the spin-weight-zero NP component of a zero-rest-mass perturbation field of any integer spin s, and Φ_0 describes a radiatable multipole (one for which $\ell \geq s$), then at a constant radius, Φ_0 dies out as $\ln t/t^{2\ell+3}$ as $t \rightarrow \infty$. (For further details of the asymptotic solution, see Paper I.)

4. Evolution of ϕ_p

The structure of the field equations (49) guarantees that all the $\hat{\Phi}_p$ can be calculated, once $\hat{\Phi}_p$ is known. For example, if $p = 1$ in Eq. (49a) and $p = -1$ in (49b), $\hat{\Phi}_1$ and $\hat{\Phi}_{-1}$ can be calculated from

$$2(1-2M/r)^{-1}\,\partial_v(r^{s+1}\,\hat{\Phi}_0) = r^s\,\hat{\Phi}_1 \tag{77a}$$

$$\partial_u(r^{s+1}\,\hat{\Phi}_0) = r^s\,\hat{\Phi}_{-1}\,. \tag{77b}$$

It should be noticed that the calculation of $\hat{\Phi}_1$ and $\hat{\Phi}_{-1}$ introduces no new integration constants; $\hat{\Phi}_1$ and $\hat{\Phi}_{-1}$ are determined unambiguously. With the use of the other equations of Eq. (49), $\hat{\Phi}_2$ can be calculated unambiguously from $\hat{\Phi}_1$, and so forth until all the $\hat{\Phi}_p$ are known. Thus everything about the field is known, once $\hat{\Phi}_0$ is known.

The details of the calculation of the $\hat{\Phi}_p$ are outlined in Appendix A. Some results of these calculations are:

(i) <u>In the asymptotic future ($t \gg r^*$) all the ϕ_p vanish as $\ln t/t^{2l+3}$ at a constant r.</u>

(ii) In the region for which $t \gg r^*$ and $r^* \gg M$, all the ϕ_p have the same dependence on r^* and t,

$$\hat{\Phi}_p \sim r^{*l-s}\,\ln t/t^{2l+3}. \tag{78}$$

(iii) In the region for which $t \gg |r^*|$ and $r^* \ll -M$, the $\hat{\Phi}_p$ have the form

$$\hat{\Phi}_p \sim \text{const.} \times \ln t/t^{2l+3} \qquad \text{if}\quad p \geq 0 \tag{79a}$$

$$\hat{\Phi}_p \sim \exp\{-pr^*/2M\}\,\ln t/t^{2l+3} \qquad \text{if}\quad p < 0. \tag{79b}$$

It is interesting that all the $\hat{\Phi}_p$ have the same form, according to Eq. (78). According to flat spacetime calculations,[14] for outgoing radiation the Φ_p with negative p are largest at large r. For ingoing radiation the Φ_p with positive p are dominant. Equation (78) then heuristically agrees with the analysis in Paper I which shows that ingoing and outgoing radiation have intensity in the region where $t \gg r^*$ and $r^* \gg M$.

The results in Eqs. (79) have a similar heuristic interpretation. The $\hat{\Phi}_p$ with negative p are extremely small because near the horizon ($r^* \ll -M$) at late times ($t \gg M$) there should be negligible outgoing radiation.

5. Conserved Quantities; Nonradiatable Multipoles

Do the nonradiatable ($l < s$) multipoles also vanish? In flat spacetime [$M = 0$ and $r = r^*$ in Eq. (49)] it is simple to construct a set of conserved quantities. These are spherical surface integrals (t and r constant) which involve only the nonradiatable multipoles of the field. The field equations guarantee that these integrals are independent of both r and t. Familiar examples are the monopole in electromagnetism and the monopole and dipole in linearized General Relativity.

For a perturbation field in curved spacetime these integrals are not independent of r, but if they are evaluated at future null infinity ($v = 0$) they are independent of u. So in curved spacetime there is also a conserved quantity corresponding to each nonradiatable multipole. It follows that nonradiatable multipole perturbations which are initially nonzero cannot vanish in the exterior of a collapsing star.

E. Difficulties of Higher Spin Theories

As the caveat in the introduction warns, there are difficulties with zero-rest-mass perturbation fields. Except for the case $s = 1$ (electromagnetism) or $s = \frac{1}{2}$ (neutrinos), the field equations in (39) or (43) are inconsistent. It follows that (49) is also in general an inconsistent set of equations. (This inconsistency does not occur in the field equations for the $s = 2$ gravitational perturbations, but gravitational perturbation cannot be described by the formalism of Sec. IIIA.)

It would seem that there are three possibilities. (i) The geometry of spacetime must be restricted to those cases for which the higher spin field equations are consistent. (ii) Integer-spin, zero-rest-mass fields are impossible, except for electromagnetism. (iii) Equation (39) is not correct; "minimal coupling" -- i.e., replacing a comma by a semicolon -- does not give the correct generalization of a flat spacetime field theory to curved spacetime.

Only the third possibility appears to be sensible. This requires that in Sec. IIIA the argument for minimal coupling based on the strong equivalence principle is incorrect; the integer-spin fields must couple in some fashion to the Riemann tensor, except for the spin-0 and spin-1 cases. Although the corrected, consistent field equations are not known, one can make reasonable speculations about the nature of these corrected equations in the Schwarzschild geometry. Of greatest concern is the way in which (71) is altered by coupling of the field to the Riemann tensor. The simplest case would be for the corrected equations (49) to lead to a wave equation like (71) for some linear combination ψ of the $\hat{\phi}_p$:

$$\psi_{,tt} - \psi_{,r^*r^*} + \mathcal{F}_l(r^*)\psi = 0. \tag{80}$$

This seems likely on physical grounds since the very different cases of scalar, electromagnetic, even-parity and odd-parity gravitational, perturbations all lead to such an equation.

In Eq. (80), the potential $\mathcal{F}_\ell(r^*)$ would have the form

$$\mathcal{F}_\ell(r^*) = \left(1 - \frac{2M}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} + \mathcal{R}(r)\right] . \tag{81}$$

Here $\mathcal{R}(r)$ is some correction term due to the Riemann-tensor coupling. Physically one should expect this correction to behave like M/r^3 as $r^* \to \infty$ and as a constant as $r^* \to -\infty$, for the following reasons.

(i) At large r the effects of curvature -- e.g., the Riemann tensor components -- fall off as M/r^3. In terms of the NP formalism for the Weyl tensor, this fact is evident in Eq. (28).

(ii) At $r^* \to -\infty$ the effects of the curvature and of the centrifugal potential are finite. The $(1 - 2M/r)$ factor in $\mathcal{F}_\ell$ would come -- as always -- from spreading these finite effects over the infinite r^* coordinate distance corresponding to $r \approx 2M$.

If these conclusions prove to be true then the principle results of Sec. IIID will be true for these fields. (The details of Secs. IIID4 and IIID5 might apply only to the electromagnetic case, for which minimal coupling is correct.)

IV. GRAVITATIONAL PERTURBATIONS

The resolution of the paradox of radiatable gravitational multipoles has been the principal motivation for this work and for Paper I. In Paper I, gravitational perturbations were analyzed in the context of the Regge-Wheeler[7] (RW) formalism. Certain technical details of that analysis were omitted.

These details will now be presented in the NP framework.

An analysis of gravitational perturbations in the NP framework is suggested by the fact that the Bianchi identities (22) bear a striking resemblance to the field equations (46) and (47) for a nongravitational spin-2 field. For our problem of nonspherical perturbations there is, however, an important difference. Gravitational perturbations are perturbations of the spacetime geometry itself. Consequently the Schwarzschild geometry spin coefficients (26) cannot be used; first-order perturbations of the spin coefficients must be considered.

In Appendix B the quantities used to describe spacetime curvature are expanded about their Schwarzschild values. Perturbations (denoted by a subscript B) in the Weyl tensor components, spin coefficients, NP metric functions, are defined by [see Eqs. (2), (4), (20)]

$$\Psi_p = \Psi_{pA} + \Psi_{pB}$$

$$\mu = \mu_A + \mu_B$$

$$\omega = \omega_A + \omega_B$$

and so forth. Here the values with subscript A are the Schwarzschild values as given by Eqs. (26), (28), and (29). The first thing to notice is the connection between parity and reality for the despun first-order quantities.[15] Just as the nongravitational fields, the real parts of despun quantities represent even parity modes of physical quantities; imaginary parts represent odd-parity modes. This correspondence of parity and reality for the $\hat{\Psi}_p$ follows from a slight modification of the argument given in Sec. III for the $\hat{\Phi}_p$. The correspondence then follows for the spin coefficients, $\hat{\mu}$, $\hat{\sigma}$, etc., and the metric functions $\hat{U}$, $\hat{\omega}$, etc., from the reality of the coefficients in the equations of Appendix B. The interrelation of parity and reality is particularly clear in Appendix B, where a comparison is made between perturbations in the NP formalism, and in the RW formalism.

A. Odd-Parity Perturbations

1. A New Derivation of the Regge-Wheeler Wave Equation

The NP formalism for gravitational perturbations leads to the RW wave equation in a fairly straightforward fashion. Also the physical interpretation of the RW odd-parity wave equation is facilitated in the NP formalism.

Equation (B6g) involves perturbations (denoted by a subscript B) of the Weyl tensor, a spin coefficient, and a metric function:

$$\Delta\{\hat{\Psi}_{0_B} r^3\} = r^2 \hat{\Psi}_{-1} + 3M\,(\hat{\mu}_B - \hat{U}_B/r).$$

Operating on this equation with D, and using (B6c) for $Dr^2\hat{\Psi}_{-1}$, yields

$$D\{\Delta\hat{\Psi}_{0_B} r^3\} = -\tfrac{1}{2} r\, \ell(\ell+1)\, \hat{\Psi}_{0_B} + 3M\,\{\hat{\omega}^*/r^2 + D(\hat{\mu}_B - \hat{U}_B/r)\}\ . \tag{82}$$

The imaginary part of this equation is a relation for odd-parity quantities. Furthermore: (i) By definition [see Eqs. (20)] U is real. (ii) From Eqs. (B5f) and (B4g) therefore

$$\mathrm{Im}(D\hat{\mu}_B) = \mathrm{Im}(\hat{\Psi}_0 - \hat{\omega}^*/r^2)\ . \tag{83}$$

Equation (82) then becomes

$$D\,\Delta\,\mathrm{Im}(\hat{\Psi}_{0_B} r^3) = \{-\tfrac{1}{2}\ell(\ell+1)/r^2 + 3M/r^3\}\ \mathrm{Im}(\hat{\Psi}_{0_B} r^3), \tag{84}$$

or

$$\mathrm{Im}(\hat{\Psi}_{0_B} r^3)_{,tt} - \mathrm{Im}(\hat{\Psi}_{0_B} r^3)_{,r^*r^*} + F^{OPG}(r^*)\ \mathrm{Im}(\hat{\Psi}_{0_B} r^3) = 0 \qquad (85a)$$

where

$$F^{OP} = (1 - 2M/r)\ \{\ell(\ell+1)/r^2 - 6M/r^3\}. \qquad (85b)$$

This is a curvature-potential wave equation of the same type as the scalar field equation which was analyzed in Paper I. Interestingly, the potential F^{OP} is precisely the same as the potential in the RW equation for odd-parity waves. (See Sec. IVB of Paper I.)

2. The "Measurability" of $\hat{\Psi}_{0_B}$

It is now necessary to establish that $\hat{\Psi}_{0_B}$ is "measurable" -- that is, that Ψ_{0_B} is a quantity that can be measured by a falling observer. The Weyl tensor is certainly measurable in terms of the real physical effects associated with geodesic deviation. Therefore, if the NP tetrad at a point is known, $\hat{\Psi}_0$ and hence $\hat{\Psi}_{0_B}$ can be calculated. Three problems must be considered, however, before one can say $\hat{\Psi}_{0_B}$ is measurable: (i) coordinate effects at the event horizon, (ii) gauge arbitrariness, and (iii) first order uncertainty in the null tetrad.

It might be ˜eared that the choice of null tetrad means that $\hat{\Psi}_{0_B}$ at the event horizon is the value associated with a nonphysical observer -- e.g., an observer stationary at r = 2M. But Ψ_0 is conformal-weight zero so coordinate effects cannot be of any importance. Similarly the following argument shows that gauge arbitrariness is of no importance.

If

$$x^{\mu'} = x^{\mu} + \eta^{\mu} \qquad (86a)$$

then

$$\hat{\Psi}'_{0_B} = \hat{\Psi}_{0_B} - \eta^{\mu}\,\hat{\Psi}_{0,\mu}$$
$$= \hat{\Psi}_{0_B} \cdot \frac{3M}{r^4}\,\eta^{r} \;. \qquad (86b)$$

Since η^r is real, $\mathrm{Im}(\hat{\Psi}_{0_B})$ is gauge invariant.[16]

The geometry is now slightly deformed from the Schwarzschild geometry, so (25) no longer specifies the null tetrad precisely. In Appendix E it is shown that this also is of no importance to the "measurability" of $\hat{\Psi}_{0_B}$. There is no first order change in $\hat{\Psi}_0$ for small rotations of the tetrad.

Since $\mathrm{Im}(\hat{\Psi}_{0_B})$ is measurable, it must be finite on the surface of the star during the passage through the event horizon, and it must be bounded at $r = +\infty$. With these constraints the analysis of (85) is identical to that for the odd-parity gravitational perturbation q of Paper I. The result, as usual, is that $\mathrm{Im}(\hat{\Psi}_{0_B})$ falls off as $\ln t/t^{2\ell+3}$ at large time. The odd-parity perturbations have already been studied in Paper I in the RW formalism, but the analysis of the Ψ_p can be completed within the NP framework. This approach is outlined in Appendix F.

3. The Relation of $\mathrm{Im}(\hat{\Psi}_{0_B})$ and Q

Regge and Wheeler use a metric perturbation Q to describe odd-parity perturbations of the geometry. Though Q and $\mathrm{Im}(\hat{\Psi}_{0_B})$ arise in quite different contexts, they are quite closely related, so it is not a coincidence that the RW odd-parity wave equation has precisely the same potential as (85).

In Appendix D, $\hat{\Psi}_{0_B}$ is calculated in the RW gauge for odd-parity perturbations. Since $\mathrm{Im}(\hat{\Psi}_{0_B})$ is gauge invariant, the result in Appendix D is also $\mathrm{Im}(\hat{\Psi}_{0_B})$ in the NP gauge. In terms of the RW metric perturbations $h_0(r,t)$

and $h_1(r,t)$, that result is

$$\mathrm{Im}(\hat{\Psi}_{0_B}) = \tfrac{1}{4}\, \ell(\ell+1)\ \{(h_0/r^2)_{,r} - (h_1/r^2)_{,t}\}\ . \tag{87}$$

The field equations for odd-parity perturbations[17] can be combined to give

$$\{(h_0/r^2)_{,r} - (h_1/r^2)_{,t}\}_{,t} = (1 - 2M/r)(\ell-1)(\ell+2)\ h_1/r^4. \tag{88}$$

Since

$$Q \equiv (1 - 2M/r)\ h_1/r\ , \tag{89}$$

the relation of Q and $\mathrm{Im}(\hat{\Psi}_{0_B})$ is

$$(\mathrm{Im}\,\hat{\Psi}_{0_B} r^3)_{,t} = \frac{(\ell+2)!}{4(\ell-2)!}\, Q\ ; \tag{90}$$

so it is no surprise that Q also satisfies (85).

The function $q(r,t)$ introduced in Paper I is, in fact

$$q = \frac{4(\ell-2)!}{(\ell+2)!}\,(\mathrm{Im}\,\hat{\Psi}_{0_B} r^3)_{,t}\ . \tag{91}$$

This follows from

$$q_{,t} = Q$$

[There can be no integration constant in (91) since both $\mathrm{Im}(\hat{\Psi}_{0_B})$ and q vanish for all r (greater than 2M) as $t \to \infty$.]

B. Even-Parity Perturbations

As in the RW formalism, the analysis of the even-parity perturbations is considerably more difficult than that for the odd-parity ones. For the even waves, it has not yet been possible to derive an equation like (85) from the perturbed NP equations. Fortunately Zerilli[18] has supplied such a wave equation in the framework of the RW formalism. Zerilli's equation and the asymptotic evolution of its solution are discussed in Paper I. One detail, however, is missing from that analysis. A proof that $\tilde{K}$, Zerilli's field variable, is finite on the stellar surface, during its passage through the event horizon, is necessary to vindicate the analysis in Paper I.

The proof to be given here will not involve showing that $\tilde{K}$ is "measurable" in the sense that $\text{Im}(\hat{\Psi}_{0_B})$ is, but the proof does make use of some of the properties of the Ψ_p and illustrates the way in which the NP formalism may be useful for gravitational perturbations.

For convenience the proof will be given for

$$Z \equiv (\Lambda + 3M/r)\,\tilde{K} \tag{92}$$

with

$$\Lambda \equiv \tfrac{1}{2}\,(\ell-1)(\ell+2),$$

rather than for $\tilde{K}$. The RW even-parity field equations[17] give

$$Z_{,t} = rK_{,t} - (1-2M/r)\;H_1 \tag{93a}$$

$$Z_{,r^*} = rK_{,r^*} - (1-2M/r)\;H\;, \tag{93b}$$

where H, H_1, and K are the metric perturbations in the RW notation. Equations (93a) and (93b) can be added and subtracted to give DZ and ΔZ.

In Appendix C the relation of NP and RW quantities is worked out and DZ and ΔZ are expressed entirely in terms of NP quantities [Eqs. (C10)]. The event-horizon behavior of the NP quantities can be deduced from the equations of Appendix B and the fact that $\Psi_p \sim (1-2M/r)^{-p}$ near r = 2M. [See Eq. (C11).] With these results, the event-horizon behavior of DZ and

ΔZ are seen to be

$$(1 - 2M/r)\ DZ \sim \partial_v Z \sim 1 \tag{94a}$$

$$\Delta Z = \partial_u Z \sim (1 - 2M/r)\ . \tag{94b}$$

This result (94) means that as the stellar surface passes through the event horizon, Z is finite and its proper-time derivative is finite. These are precisely the conditions needed to justify the analysis in Paper I.

APPENDIX A: EVALUATION OF THE Φ_p FIELDS

It follows from (70) of Paper I that in "Region VI" ($t \gg r^*$ and $r^* \gg M$) $\hat{\Phi}_0$ can be approximated by

$$r^{s+1}\,\hat{\Phi}_0 = \varphi(t)\,[a_6/r^{*\ell} + b_6\,r^{*\ell+1} + \cdots\cdots] \,, \tag{A1}$$

where

$$\varphi(t) = \ln t/t^{2\ell+3}.$$

Equation (77) of the present paper gives a prescription for calculating $\hat{\Phi}_1$ and $\hat{\Phi}_{-1}$. Putting (A1) in (77) yields

$$r^s\,\hat{\Phi}_1 = (1-2M/r)\,\varphi(t)\,[-\,\ell a_6/r^{*\ell+1} + (\ell+1)\,b_6\,r^{*\ell}] \tag{A2a}$$

$$r^s\,\hat{\Phi}_{-1} = \tfrac{1}{2}\,\varphi(t)\,[\ell a_6/r^{*\ell+1} - (\ell+1)\,b_6\,r^{*\ell}]\,. \tag{A2b}$$

In (A2) terms that fall off as $\dot{\varphi}(t)$ have been ignored.

The field equations (49) can next be used to solve for $\hat{\Phi}_2$, $\hat{\Phi}_{-2}$ and so forth. This calculation involves taking the derivatives ∂_u and ∂_v. According to the arguments of Sec. III of Paper I, the derivative ∂_t can be ignored by comparison with ∂_{r^*}, in the asymptotic region $t \gg r^*$. Equation (49) then leads to $\hat{\Phi}_p$ fields all of which have the same dependence on time at large t,

$$\hat{\Phi}_p(t) \sim \varphi(t). \tag{A3}$$

Since $r^* \gg M$, Eq. (A2) can be approximated by

$$r^s\,\hat{\Phi}_1 = \varphi(t)\,(\ell+1)\,b_6\,r^{*\ell} \tag{A4a}$$

$$r^s\,\hat{\Phi}_{-1} = -\,\tfrac{1}{2}\,\varphi(t)\,(\ell+1)\,b_6\,r^{*\ell}. \tag{A4b}$$

In this region the field equations (49) are

$$2\partial_v(r^{s+p}\,\hat{\Phi}_{p-1}) = r^{s+p-1}\,\hat{\Phi}_p \qquad p \geq 1 \qquad \text{(A5a)}$$

$$\partial_u(r^{s-p}\,\hat{\Phi}_{p+1}) = r^{s-p-1}\,\hat{\Phi}_p \qquad p \leq -1. \qquad \text{(A5b)}$$

If only the dominant term in r^* is kept, and terms which fall off as $\dot{\varphi}(t)$ or faster are ignored, the solutions to (A5) are

$$\hat{\Phi}_p = \varphi(t)\,\frac{(\ell+p)!}{\ell!}\,b_6\,r^{*\ell-s} \qquad p \geq 0 \qquad \text{(A6a)}$$

$$\hat{\Phi}_p = \varphi(t)\,(-\tfrac{1}{2})^p\,\frac{(\ell-p)!}{\ell!}\,b_6\,r^{*\ell-s} \qquad p \leq 0. \qquad \text{(A6b)}$$

In "Region IX" ($t \gg r^*$ and $r^* \ll -M$) the analysis in Paper I implies

$$\hat{\Phi}_0 = \varphi(t)\,[a_9 + c_9\,\exp\{r^*/2M\} + \cdots]. \qquad \text{(A7)}$$

(The term $\exp\{r^*/2M\}$ does not appear in (63b) of Paper I. This term is unimportant for the asymptotic time behavior of Ψ^{sc} or $\hat{\Phi}_0$, but it is crucial for the time dependence of $\partial_v\,\hat{\Phi}_0$ and $\partial_u\,\hat{\Phi}_0$. It is discussed in Paper I in connection with odd-parity gravitational perturbations.)

Equation (A7) can now be used in (77). It should be noted that for $r^* \ll -M$,

$$D = 2(1-2M/r)^{-1}\,\partial_v \sim \exp\{-r^*/2M\}\,\partial_{r^*}$$

$$\Delta = \partial_u \sim -\tfrac{1}{2}\,\partial_{r^*}\,.$$

The results are

$$\hat{\Phi}_1 \sim \varphi(t) \qquad \text{(A8a)}$$

$$\hat{\Phi}_{-1} \sim \varphi(t)\ \exp\{r^*/2M\}. \qquad \text{(A8b)}$$

Repeated application of Eq. (49) in Region IX gives the results presented in Eq. (79).

APPENDIX B: EXPANSION ABOUT THE SCHWARZSCHILD GEOMETRY

The equations relating metric components, spin coefficients, and Weyl tensor components in the NP formalism are given in the 1962 NP paper.[5] We shall use the notation (NP 6.10), for example to cite a numbered equation in that paper. For the geometrically defined "special" coordinate system and "special" tetrad the pertinent equations are (NP 6.1) through (NP 6.12). See also Eqs. (18) to (22) of the present paper.

We shall now expand all quantities in these equations about their Schwarzschild values, and find equations for the first-order perturbations. Where necessary to avoid confusion, Schwarzschild values will be denoted by a subscript A and first order perturbations by a subscript B. For example, the spin-coefficient μ is expanded as

$$\mu = \mu_A + \mu_B.$$

Here $\mu_A = -(1-2M/r)/2r$, the Schwarzschild-geometry value. Note that the use of a "special" system guarantees

$$\kappa_B = \pi_B = 0 \qquad \rho_B = \rho_B^* \qquad \tau_B = \alpha_B^* + \beta_B.$$

The metric equations (NP 6.10) relate the components of the metric tensor [see Eq. (18)] to the spin coefficients. The first-order perturbations in these equations are

$$r^{-1}\, Dr\xi_B^{\,i} = \rho_B \xi_A^{\,i} + \sigma \xi_A^{*i} \tag{B1a}$$

$$D\,\omega r = -\,r\tau \tag{B1b}$$

$$D\,X^i = \tau\xi_A^{*i} + \tau^*\xi_A^{\,i} \tag{B1c}$$

$$D U_B = -(\gamma_B + \gamma_B^*) \quad \text{(B1d)}$$

$$\delta X^i_B - X^j \partial_j \xi^i_A - r^{-1} \Delta \xi^i_B r + r^{-1} U_B \xi^i_A = (\mu + \gamma^* - \gamma)_B \xi^i_A + {}^* \xi^{*i}_A \quad \text{(B1e)}$$

$$(\delta + 2\beta)\, \xi^{*i}_B - \xi^{*i}_B \partial_j \xi^i_A - (\delta^* + 2\beta)\, \xi^i_B + \xi^j_B \partial_j \xi^{*i}_A$$
$$+ \omega r^{-1} \xi^{*i}_A + \omega^* r^{-1} \xi^i_A = (\beta^* - \alpha)_B \xi^i_A + (\alpha^* - \beta)_B \xi^{*i}_A \quad \text{(B1f)}$$

$$(\delta + 2\beta)\omega^* - (\delta^* + 2\beta)\omega = \mu - \mu^* \quad \text{(B1g)}$$

$$\delta U_B - r^{-1}(1 - 2M/r)\, \Delta\{\omega r/(1 - 2M/r)\} = -\nu^*. \quad \text{(B1h)}$$

In these equations the symbols D, Δ, δ, δ^* for the differential operators are taken to be the unperturbed forms as given in Eq. (27).

To use these equations most conveniently we seek combinations which are of definite spin weight. Let us define

$$\ell_B{}^\mu = 0 \quad \text{(B2a)}$$

$$n_B{}^\mu = U_B \delta_2{}^\mu + X^i \delta_i{}^\mu \qquad (i = 2,3) \quad \text{(B2b)}$$

$$m_B{}^\mu = \omega \delta_2{}^\mu + \xi^i_B \delta_i{}^\mu\,, \quad \text{(B2c)}$$

To be the perturbations of the NP special tetrad from the Schwarzschild value. Now we notice that $\underset{\sim}{n}_A \cdot \underset{\sim}{n}_B$, $\underset{\sim}{m}_B \cdot \underset{\sim}{n}_A$, $\underset{\sim}{m}_A \cdot \underset{\sim}{n}_B$, $\underset{\sim}{m}^*{}_A \cdot \underset{\sim}{m}_B$, and $\underset{\sim}{m}_A \cdot \underset{\sim}{m}_B$ are first order quantities which under coordinate rotation, transform with spin weight 0, 1, 1, 0, 2, respectively. This enables us to form the definite-spin-weight combinations of the metric perturbations,

$$U_B = \underset{\sim}{n}_A \cdot \underset{\sim}{n}_B \qquad \text{(spin-weight 0)} \qquad \text{(B3a)}$$

$$\omega = \underset{\sim}{m}_B \cdot \underset{\sim}{n}_A \qquad \text{(spin-weight 1)} \qquad \text{(B3b)}$$

$$X_1 \equiv X^\Theta + i \sin\Theta X^\varphi = \sqrt{2}\, r^{-1}\, \underset{\sim}{m}_A \cdot \underset{\sim}{n}_B \qquad \text{(spin-weight 1)} \qquad \text{(B3c)}$$

$$\Xi_0 \equiv \xi_B{}^\Theta - i \sin\Theta \xi_B{}^\varphi = \sqrt{2}\, r^{-1}\, \underset{\sim}{m}^*{}_A \cdot \underset{\sim}{m}_B \qquad \text{(spin-weight 0)} \qquad \text{(B3d)}$$

$$\Xi_2 \equiv \xi_B{}^\Theta + i \sin\Theta \xi_B{}^\varphi = \sqrt{2}\, r^{-1}\, \underset{\sim}{m}_A \cdot \underset{\sim}{m}_B \qquad \text{(spin-weight 2)} \qquad \text{(B3e)}$$

With these definitions to guide us, we can make definite-spin-weight combinations of equations (B1). (In doing this we shall need to know the spin weights of the spin-coefficient perturbations. These and the conformal weights for the nonvanishing spin-coefficient perturbations are given in Table II.) For an ℓ-pole, the definite-spin-weight combinations of (B1), in despun form, are

$$Dr\hat{\Xi}_2 = \sqrt{2}\, \hat{\sigma} \qquad \text{(B4a)}$$

$$Dr\hat{\Xi}_0 = \sqrt{2}\, \hat{\rho}_B \qquad \text{(B4b)}$$

$$Dr\hat{\omega} = -r\hat{\tau} \qquad \text{(B4c)}$$

$$DX_1 = \sqrt{2}\, r^{-1}\, \hat{\tau} \qquad \text{(B4d)}$$

$$D\hat{U}_B = -2\, \mathrm{Re}(\gamma_B) \qquad \text{(B4e)}$$

$$-\tfrac{1}{2}r^{-1}\,(\ell-1)(\ell+2)\;\hat{X}_1 - r^{-1}\Delta r\hat{\Xi}_2 = \sqrt{2}\;r^{-1}\;\hat{\lambda}^* \tag{B4f}$$

$$\mathrm{Im}(\hat{\omega}) = -\;\mathrm{Im}(r\hat{\mu}_B) \tag{B4g}$$

$$\tfrac{1}{2}r^{-1}\;\ell(\ell+1)\;\hat{U}_B + r^{-1}(1-2M/r)\;\Delta\;\{\hat{\omega}r/(1-2M/r)\} = \hat{\nu}^* \tag{B4h}$$

In the same spirit we can expand the quantities in Eqs. (NP 6.11) and get certain useful definite-spin-weight equations for the perturbations:

$$D\{\hat{\rho}_B\,r^2\} = 0 \tag{B5a}$$

$$D\{\hat{\sigma}\;r^2\} = r^2\,\hat{\Psi}_2 \tag{B5b}$$

$$D\{\hat{\tau}\;r\} = r\,\hat{\Psi}_1 \tag{B5c}$$

$$D\{\mathrm{Re}\;\hat{\gamma}_B\} = \mathrm{Re}\;\hat{\Psi}_0 \tag{B5d}$$

$$D\{\hat{\lambda}\;r\} = -\tfrac{1}{2}(1-2M/r)\;\hat{\sigma}^* \tag{B5e}$$

$$D\{\hat{\mu}_B\,r\} = r\,\hat{\Psi}_0 \tag{B5f}$$

$$D\,\hat{\nu} = -\tfrac{1}{2}(1-2M/r)\;\hat{\tau}^* + \hat{\Psi}_{-1} \tag{B5g}$$

$$r^{-2}\,(1-2M/r)\;\Delta\{\hat{\lambda}\;r^2(1-2M/r)^{-1}\} + (2r)^{-1}\;(\ell-1)(\ell+2)\,\hat{\nu} = -\hat{\Psi}_{-2} \tag{B5h}$$

$$-\tfrac{1}{2}\ell(\ell+1)\;\hat{\rho}_B + r^{-1}\;\hat{\omega} - \hat{\sigma} + \hat{\tau} = -r\,\hat{\Psi}_1 \tag{B5i}$$

$$\mathrm{Im}(\hat{\tau}) = \mathrm{Im}(\hat{\mu}_B) + r\;\mathrm{Im}(\hat{\Psi}_{0_B}) \tag{B5j}$$

$$\hat{\lambda} + \tfrac{1}{2}\;\ell(\ell+1)\hat{\mu}_B - (2r)^{-1}\;(1-4M/r)\hat{\omega}^* = -\tfrac{1}{2}(1-2M/r)\hat{\tau}^* - r\;\hat{\Psi}_{-1} \tag{B5k}$$

$$\cdot\,(2r)^{-1}\,\hat{U}_B(1-4M/r) + \hat{\nu} - r^{-1}\,(1-2M/r)\,\Delta\{\hat{\mu}r^2/(1-2M/r)\} = -(1-2M/r)\;\mathrm{Re}\hat{\gamma}_B \tag{B5ℓ}$$

$$-\tfrac{1}{2}(\ell-1)(\ell+2)\hat{\tau} - (1-2M/r)^{-1}\Delta\{\hat{\sigma}\, r(1-2M/r)\} = -\hat{\lambda}^* \qquad \text{(B5m)}$$

$$(1-2M/r)^{-1}\Delta\{\hat{\rho}_B \langle 1-2M/r)r\} + r^{-1}\hat{U}_B - \hat{\tau} = -2\mathrm{Re}\,\hat{\gamma}_B + \hat{\mu}^*{}_B - r\,\hat{\Psi}_{0_B} \qquad \text{(B5n)}$$

$$r^{-1}\Delta\{\hat{\tau}^* r^2\} + \ell(\ell+1)\,\mathrm{Re}(\hat{\gamma}_B) + 2\hat{\omega}^* M/r^2 = -\hat{\nu} - r\,\hat{\Psi}_{-1}. \qquad \text{(B5o)}$$

The perturbations of the Bianchi identities [Eq. (22) or (NP 5.12)] have definite spin weight. In despun form for an ℓ-pole they are:

$$D\{\hat{\Psi}_1 r^4\} = \hat{\Psi}_2 r^3 \qquad \text{(B6a)}$$

$$D\{\hat{\Psi}_{0_B} r^3\} = r^2\hat{\Psi}_1 - 3\hat{\rho}_B M \qquad \text{(B6b)}$$

$$D\{\hat{\Psi}_{-1} r^2\} = -\tfrac{1}{2} r\,\ell(\ell+1)\,\hat{\Psi}_{0_B} + 3\hat{\omega}^* M r^{-2} \qquad \text{(B6c)}$$

$$D\{\hat{\Psi}_{-2}\, r\} = -\tfrac{1}{2}(\ell-1)(\ell+2)\,\hat{\Psi}_{-1} + 3\hat{\lambda} M r^{-2} \qquad \text{(B6d)}$$

$$\langle 1-2M/r\rangle^{-2}\Delta\{\hat{\Psi}_2\, r(1-2M/r)^2\} = -\tfrac{1}{2}(\ell-1)(\ell+2)\,\hat{\Psi}_1 - 3\hat{\sigma} M r^{-2} \qquad \text{(B6e)}$$

$$(1-2M/r)^{-1}\Delta\{\hat{\Psi}_1 r^2(1-2M/r)\} = -\tfrac{1}{2} r\,\ell(\ell+1)\,\hat{\Psi}_{0_B} + 3M r^{-1}(\hat{\tau} + r^{-1}\hat{\omega}) \qquad \text{(B6f)}$$

$$\Delta\{\hat{\Psi}_{0_B} r^3\} = r^2\,\hat{\Psi}_{-1} + 3M(\hat{\mu}_B - U_B\, r^{-1}) \qquad \text{(B6g)}$$

$$(1-2M/r)\,\Delta\,\hat{\Psi}_{-1}\{r^4 (1-2M/r)^{-1}\} = r^3\,\hat{\Psi}_{-2} - 3\hat{\nu} M r \qquad \text{(B6h)}$$

APPENDIX C: THE REGGE-WHEELER GAUGE AND GAUGE TRANSFORMATIONS

In the Regge-Wheeler (RW) formalism,[7] the metric perturbations are analyzed into scalar vector and tensor harmonics. Ten functions of r and t, three for odd parity and seven for even parity, suffice to describe the metric. They can be defined by

$$g^{\mu\nu} = g_A^{\mu\nu} + g_B^{\mu\nu} \tag{C1a}$$

$$-(1-2M/r)\, g_B^{tt} = H_0(r,t)\, Y^{\ell}{}_m \tag{C1b}$$

$$-(1-2M/r)^{-1}\, g_B^{rr} = H_2(r,t)\, Y^{\ell}{}_m \tag{C1c}$$

$$g_B^{rt} = H_1(r,t)\, Y^{\ell}{}_m \tag{C1d}$$

$$-r^2\,(1-2M/r)^{-1}\, g_B^{ri} = h_r{}^{even}(r,t)\, \psi^i_{\ell m} + h_r{}^{odd}(r,t)\, \varphi^i_{\ell m} \tag{C1e}$$

$$r^2\,\langle 1-2M/r\rangle\, g_B^{ti} = h_0{}^{even}(r,t)\, \psi^i_{\ell m} + h_0{}^{odd}(r,t)\, \varphi^i_{\ell m} \tag{C1f}$$

$$-r^4\, g_B^{ij} = h_2{}^{odd}(r,t)\chi^{ij}_{\ell m} + r^2K(r,t)\phi^{ij}_{\ell m} + r^2G(r,t)\Psi^{ij}_{\ell m} \tag{C1g}$$

(Here i,j range over Θ, φ, and ψ^i, φ^i, ϕ^{ij}, Ψ^{ij}, χ^{ij} are components of the vector and tensor spherical harmonics; they are related to those given by Thorne and Campolattaro[17] but their indices are raised by the 2-sphere metric $\gamma^{\Theta\Theta} = 1$, $\gamma^{\Theta\varphi} = 0$, $\gamma^{\varphi\varphi} = \sin^2\Theta$.) In the RW gauge, the four functions associated with gauge freedom are chosen such that $h_2{}^{odd}$, $h_0{}^{even}$, $h_r{}^{even}$, and G are made to vanish.

Although g_B^{tt}, g_B^{rr}, and g_B^{tr} have definite spin seight (zero) the other perturbation components do not. We can, however, easily form definite-spin-weight combinations (subscripts denote spin weight):

$$G_1^t \equiv 2 r m_i g_B^{ti} \tag{C2a}$$

$$G_1^r \equiv 2 r m_i g_B^{ri} \tag{C2b}$$

$$G_0 \equiv 2r^2 g_B^{ij} m_i m^*_j \tag{C2c}$$

$$G_2 \equiv 2r^2 g_B^{ij} m_i m_j \tag{C2d}$$

The three complex functions G_1^t, G_1^r, G_2 along with the four real functions G_0, g_B^{tt}, g_B^{tr}, g_B^{rr} are our new description of metric perturbations.

In despun form, in terms of the RW functions, the perturbations are:

$$\hat{g}_B^{tt} = -\hat{H}_0 \langle 1 - 2M/r \rangle^{-1} \tag{C3a}$$

$$\hat{g}_B^{tr} = \hat{H}_1 \tag{C3b}$$

$$\hat{g}_B^{rr} = -\hat{H}_2 \langle 1 - 2M/r \rangle \tag{C3c}$$

$$\hat{G}_1^r = -(1 - 2M/r)\, \ell(\ell+1) \quad \{\hat{h}_r^{\,even} + i\, \hat{h}_r^{\,odd}\} \tag{C3d}$$

$$\hat{G}_1^t = (1 - 2M/r)^{-1}\, \ell(\ell+1) \quad \{\hat{h}_0^{\,even} + i\, \hat{h}_0^{\,odd}\} \tag{C3e}$$

$$\hat{G}_0 = 2r^2 \hat{K} + r^2\, \ell(\ell+1)\, \hat{G} \tag{C3f}$$

$$\hat{G}_2 = -\tfrac{1}{2}\left[r^2\hat{G} + i\,\hat{h}_2^{\,odd}\right] \ell(\ell+1)(\ell-1)(\ell+2). \tag{C3g}$$

(Note: $\hat{h}_0^{\,even}$ denotes $h_0^{\,even}\, Y^{\ell}{}_m(\theta,\varphi)$, and so forth.)
We can immediately see that real and imaginary parts of the despun perturbations correspond respectively to even and odd parity, as they must.

The metric perturbations in the NP formalism are given in terms of the functions U_B, X^i_B, ξ^i_B, ω_B. If we transform perturbations of Eq. (21) from u,r to t,r coordinates for the background and take the definite-spin-weight combinations according to (C2), we get

$$\langle 1 - 2M/r\rangle\, \hat{g}^{tt}_B = \hat{g}^{tr}_B = \langle 1 - 2M/r\rangle^{-1}\, \hat{g}^{rr}_B = 2\,\langle 1 - 2M/r\rangle^{-1}\, \hat{U}_B \tag{C4a}$$

$$\hat{G}^r_1 = 2r\hat{\omega} - \sqrt{2}\, r^2\, \hat{X}_1 = (1 - 2M/r)\, G^t_1 \tag{C4b}$$

$$\hat{G}_0 = -r^3\; 2\sqrt{2}\,\mathrm{Re}(\hat{\Xi}_0) \tag{C4c}$$

$$\hat{G}_2 = -r^3\; 2\sqrt{2}\, \hat{\Xi}_2\;. \tag{C4d}$$

Notice that we have a description in terms of only six functions; this is because the choice of (geometrically defined) NP special coordinates is tantamount to a gauge choice.

An infinitesmal coordinate transformation

$$x^{\mu'} = x^{\mu} + \eta^{\mu},$$

induces a gauge transformation of the metric perturbations

$$g'^{\mu\nu}_B = g^{\mu\nu}_B - \eta^{\mu}{}_{,\alpha}\, g^{\alpha\nu}_A - \eta^{\nu}{}_{,\alpha}\, g^{\alpha\mu}_A + \eta^{\alpha}\, g^{\mu\nu}_A{}_{,\alpha}\;. \tag{C5}$$

As usual we look for quantities of definite spin weight. The quantities η^r and η^t have zero spin weight and the combination

$$\eta_1 = -\sqrt{2}\, r^{-1} m_{A\,i}\, \eta^i = \eta^\theta + i \sin\theta\, \eta^\varphi \tag{C6}$$

has spin weight 1. The despun form of the gauge transformation equations (C5) are

$$\hat{g}'^{rr}_B = \hat{g}^{rr}_B + 2\,(1-2M/r)\,\hat{\eta}^r{}_{,r} - 2Mr^{-2}\,\hat{\eta}^r \tag{C7a}$$

$$\hat{g}'^{tt}_B = \hat{g}^{tt}_B - 2\,\langle 1-2M/r\rangle^{-1}\,\hat{\eta}^t{}_{,t} - \langle 1-2M/r\rangle^{-2}\, 2Mr^{-2}\,\hat{\eta}^r \tag{C7b}$$

$$\hat{g}'^{tr}_B = \hat{g}^{tr}_B - \langle 1-2M/r\rangle^{-1}\,\hat{\eta}^r{}_{,t} + \langle 1-2M/r\rangle\,\hat{\eta}^t{}_{,r} \tag{C7c}$$

$$\hat{G}'^r_1 = \hat{G}^r_1 + \ell(\ell+1)\,\hat{\eta}^r - \sqrt{2}\, r^2\,(1-2M/r)\,\hat{\eta}_{1,r} \tag{C7d}$$

$$\hat{G}'^t_1 = \hat{G}^t_1 + \ell(\ell+1)\,\hat{\eta}^t + \sqrt{2}\, r^2\,(1-2M/r)^{-1}\,\hat{\eta}_{1,t} \tag{C7e}$$

$$\hat{G}'_0 = \hat{G}_0 + 4r\,\hat{\eta} + 2\sqrt{2}\, r^2\, \mathrm{Re}(\hat{\eta}_1) \tag{C7f}$$

$$\hat{G}'_2 = \hat{G}_2 - \sqrt{2}\, r^2\,(\ell-1)(\ell+2)\,\hat{\eta}_1\,. \tag{C7g}$$

Finally, from (C3) and (C7) we can get relations between RW and NP metric perturbations and the components η^μ, of the gauge transformation between them:

$$2\hat{U}_B = -H\,\langle 1-2M/r\rangle + 2\,\langle 1-2M/r\rangle\,\hat{\eta}^r{}_{,r} - 2Mr^{-2}\,\hat{\eta}^r \tag{C8a}$$

$$2\hat{U}_B = -H\,\langle 1-2M/r\rangle - 2\,\langle 1-2M/r\rangle\,\hat{\eta}^t{}_{,t} - 2Mr^{-2}\,\hat{\eta}^r \tag{C8b}$$

$$2\hat{U}_B \langle 1-2M/r\rangle^{-1} = H_1 - \langle 1-2M/r\rangle^{-1}\,\hat{\eta}^r{}_{,t} + \langle 1-2M/r\rangle\,\hat{\eta}^t{}_{,r} \tag{C8c}$$

$$2r\hat{\omega} - \sqrt{2}\,r^2\hat{X}_1 = -(1-2M/r)\,\ell(\ell+1)\,i\,\hat{h}_r{}^{odd} + \ell(\ell+1)\hat{\eta}^r - \sqrt{2}\,r^2(1-2M/r)\hat{\eta}_{1,r} \tag{C8d}$$

$$2r\hat{\omega} - \sqrt{2}\,r^2\hat{X}_1 = \ell(\ell+1)\,i\,h_0{}^{odd} + \ell(\ell+1)(1-2M/r)\hat{\eta}^t + \sqrt{2}\,r^2\hat{\eta}_{1,t} \tag{C8e}$$

$$-r^3\,2\sqrt{2}\,\mathrm{Re}\,\{\hat{\Xi}_0\} = 2r^2\hat{K} + 4r\,\hat{\eta}^r + 2\sqrt{2}\,r^2\,\mathrm{Re}\,\{\hat{\eta}_1\} \tag{C8f}$$

$$\hat{\Xi}_2 = (2r)^{-1}(\ell-1)(\ell+2)\,\hat{\eta}_1\,. \tag{C8g}$$

In Sec. IVB we analyze even-parity perturbations with the use of a metric perturbation function Z defined by Eq. (92). According to (93) the D and Δ derivatives of Z are

$$DZ = r\,DK - \{H_1 + H\} \tag{C9a}$$

$$\Delta Z = r\,\Delta K + \tfrac{1}{2}(1-2M/r)\,\{H - H_1\}. \tag{C9b}$$

For purely even-parity perturbations, Eqs. (C8) allow us to express $\hat{K}$, $\hat{H}_1 + \hat{H}$, and $\hat{H}_1 - \hat{H}$, in terms of NP quantities:

$$\frac{(\ell+2)!}{(\ell-2)!}(\hat{H}_1 + \hat{H}) = 2\sqrt{2}\,(1-2M/r)\,D\left\{r^2 D\left[r\hat{\Xi}_2\right]\right\} \tag{C10a}$$

$$\hat{H}_1 - \hat{H} = 4(1-2M/r)^{-1}\,\hat{U}_B - 4[\ell(\ell+1)]^{-1}\,\Delta\left\{(1-2M/r)^{-1}\times\left[\hat{\Gamma} - 2\sqrt{2}\,r^2\,\Delta(r\hat{\Xi}_2)/(\ell-1)(\ell+2)\right]\right\} \tag{C10b}$$

$$\hat{K} = -\sqrt{2}\, r\hat{\Xi}_0 - 2[r\,\ell(\ell+1)]^{-1}\left\{\hat{\Gamma} + 2\sqrt{2}\, r^2 (r\hat{\Xi}_2)_{,r*}/(\ell-1)(\ell+2)\right\} - 2\sqrt{2}\, r\hat{\Xi}_2/(\ell-1)(\ell+2) \tag{C10c}$$

where

$$\hat{\Gamma} \equiv 2r\hat{\omega} - \sqrt{2}\, r^2 \hat{X}_1. \tag{C10d}$$

We can infer the event-horizon behavior of the spin-coefficient perturbations, and of the NP metric perturbations, from the known event-horizon behavior of the perturbations in the Ψ's. [See Eqs. (B4), (B5), (B6).] The quantities important to our analysis here, behave as

$$\hat{\Xi}_2 \sim \hat{\Xi}_0 \sim 1 \qquad \hat{\omega} \sim \hat{X}_1 \sim (1 - 2M/r)$$

$$\hat{U}_B \sim (1 - 2M/r)^2$$

near the event horizon. We can conclude from (C10) and from the properties of D and Δ that

$$H + H_1 \sim (1 - 2M/r)^{-1} \qquad H_1 - H \sim (1 - 2M/r)$$

$$K \sim 1.$$

These results, and (C9) predict that near the event horizon:

$$DZ \sim (1 - 2M/r)^{-1} \qquad \Delta Z \sim (1 - 2M/r).$$

APPENDIX D: CALCULATION OF $\hat{\Psi}_{0_B}$ FOR ODD PARITY

We know that Ψ_{0_B} must be purely imaginary for odd-parity perturbations. From Eq. (4c) the perturbation of Ψ_0 can be written (in vacuum),

$$- 2\Psi_0 = \underbrace{R_{A\alpha\beta\gamma\delta}\,(\ell^\alpha n^\beta \ell^\gamma n^\delta)_B}_{\text{I}} + \underbrace{R_{A\alpha\beta\gamma\delta}\,(\ell^\alpha n^\beta m^\gamma m^{*\delta})_B}_{\text{II}}$$

$$+ \underbrace{R_{B\alpha\beta\gamma\delta}\,(\ell^\alpha n^\beta \ell^\gamma n^\delta)_A}_{\text{III}} + \underbrace{R_{B\alpha\beta\gamma\delta}\,(\ell^\alpha n^\beta m^\gamma m^{*\delta})_A}_{\text{IV}} \qquad \text{(D1)}$$

The unperturbed Riemann, or Weyl, tensor has components

$$R_{\theta r\theta r} = \sin^{-2}\theta\, R_{\varphi r\varphi r} = M/r\,(1 - 2M/r) \qquad \text{(D2a)}$$

$$R_{\theta t\theta t} = \sin^{-2}\theta\, R_{\varphi t\varphi t} = -M/r(1 - 2M/r) \qquad \text{(D2b)}$$

$$R_{rtrt} = 2M/r^3 \qquad \text{(D2c)}$$

$$R_{\varphi\theta\varphi\theta} = 2Mr\sin^2\theta, \qquad \text{(D2d)}$$

and all others vanish, except those related to the above by symmetry.

Of course the expressions like $(\ell^\alpha n^\beta \ell^\gamma n^\delta)_B$ must be expanded into first order perturbations, e.g.,

$$(\ell^\alpha n^\beta \ell^\gamma n^\delta)_B = \ell_A{}^\alpha n_A{}^\beta \ell_A{}^\gamma n_B{}^\delta + \cdots \quad .$$

In term I three of the indices $\alpha\beta\gamma\delta$ will always be t or r so by (D2) the perturbation terms will involve $n_B{}^r$, $n_B{}^t$, $\ell_B{}^r$, or $\ell_B{}^t$, all of which vanish for odd parity, so term I vanishes. In term II there is no combination of the three unperturbed tetrad vectors that couples to any of the nonvanishing components in (D2). Term III is clearly real so it must vanish for odd parity perturbations.

We are left with only term IV. Taking the imaginary part and using the known values of the unperturbed NP tetrad, we find

$$\mathrm{Im}(\mathrm{IV}) = -R_{Brt\theta\varphi}/r^2 \sin\theta. \tag{D3}$$

We can calculate $R_{Brt\theta\varphi}$ from[19]

$$\begin{aligned} R_{B\alpha\beta\gamma\delta} = \tfrac{1}{2}\Big\{ & h_{\beta\gamma;\alpha\delta} + h_{\alpha\delta;\beta\gamma} - h_{\alpha\gamma;\beta\delta} - h_{\beta\delta;\alpha\gamma} \\ & + R_{A\alpha\sigma\gamma\delta}\, h^{\sigma}{}_{\beta} + R_{A\sigma\beta\gamma\delta}\, h^{\sigma}{}_{\alpha}\Big\}. \end{aligned} \tag{D4}$$

Since $\mathrm{Im}(\psi_0)$ is gauge invariant we can evaluate this in the RW gauge, and we find (in RW notation)

$$R_{Brt\theta\varphi}/r^2 \sin\theta = \tfrac{1}{2}\, l(l+1)\left\{\left(\frac{h_0}{r^2}\right)_{,r} - \left(\frac{h_1}{r^2}\right)_{,t}\right\} Y^l{}_m(\theta,\varphi), \tag{D5}$$

so that

$$\mathrm{Im}(\hat{\psi}_{0_B}) = \tfrac{1}{4}\, l(l+1)\left\{(h_0/r^2)_{,r} - (h_1/r^2)_{,t}\right\}. \tag{D6}$$

APPENDIX E: INFINITESIMAL TETRAD ROTATIONS; TRANSFORMATION OF THE Ψ_p

Totations of the tetrad that preserve the basic relation (.1) are of three types (a,b complex; Λ,φ real)[20]:

$$
\begin{aligned}
1)\quad & \underset{\sim}{\ell} \to \underset{\sim}{\ell} \\
& \underset{\sim}{m} \to \underset{\sim}{m} + a\,\underset{\sim}{\ell} \\
& \underset{\sim}{n} \to \underset{\sim}{n} + a\,\underset{\sim}{m}^* + a^*\,\underset{\sim}{m} + a\,a^*\,\underset{\sim}{\ell}\,,
\end{aligned}
\tag{E1}
$$

$$
\begin{aligned}
2)\quad & \underset{\sim}{n} \to \underset{\sim}{n} \\
& \underset{\sim}{m} \to \underset{\sim}{m} + b\,\underset{\sim}{n} \\
& \underset{\sim}{\ell} \to \underset{\sim}{\ell} + b\,\underset{\sim}{m}^* + b^*\,\underset{\sim}{m} + b\,b^*\,\underset{\sim}{m}
\end{aligned}
\tag{E2}
$$

$$
\begin{aligned}
3)\quad & \underset{\sim}{\ell} \to \Lambda\,\underset{\sim}{\ell} \\
& \underset{\sim}{n} \to \Lambda^{-1}\,\underset{\sim}{n} \\
& \underset{\sim}{m} \to e^{i\varphi}\,\underset{\sim}{m}
\end{aligned}
\tag{E3}
$$

For <u>small</u> tetrad rotations, the Ψ_p change values to <u>first order</u> in a, b, $(\Lambda - 1)$, φ. If, as in our problem, Ψ_0 has the only nonvanishing zero order value, these induced "gauge rotations" are, for the three types of tetrad rotations:

$$1)\quad \Psi_{-1} \to \Psi_{-1} + 3\,a^*\,\Psi_{0_A} \tag{E4}$$

$$2)\quad \Psi_1 \to \Psi_1 + 3\,b\,\Psi_{0_A} \tag{E5}$$

$$3)\quad \Psi_p \to \Lambda^p\,e^{ip\varphi}\,\Psi_p \quad (\therefore \text{ no first order changes}). \tag{E6}$$

APPENDIX F: THE ODD PARITY Ψ_p

Here we will outline one method for calculating all the $\dot{\hat{\Psi}}_p$ entirely in the NP framework, once $\hat{\Psi}_0$ is known. We will assume that we are dealing with a purely odd-parity perturbation, so that we are calculating purely imaginary parts of the despun NP quantities.

(i) Since $Im(\hat{\rho}_B) = 0$, we can immediately find $\hat{\Psi}_1$ from (B6b) and $\hat{\Psi}_2$ from (B6a).

(ii) From (B6f) we can calculate $\hat{\tau} + r^{-1}\hat{\omega}$.

(iii) The imaginary part of (B5n) gives us $\hat{\tau} + \hat{\mu}_B = r\,\hat{\Psi}_{0_B}$.

(iv) From the above and (B4g) we can calculate $\hat{\omega}$, $\hat{\tau}$, and $\hat{\mu}_B$.

(v) Equation (B6g) then gives us $\hat{\Psi}_{-1}$.

(vi) Equation (B5o) can be used to calculate $\hat{\nu}$ and then (B6h) gives us $\hat{\Psi}_{-2}$.

The asymptotic time dependence of all the perturbations follows the familiar $\ln t/t^{2l+3}$ fall off.

TABLE I.[a] Spin Coefficients

$\Gamma_{\underline{a}\,\underline{b}\,\underline{c}\,\underline{\dot{d}}}$			
$\underline{a}\,\underline{b}$ \ $\underline{c}\,\underline{\dot{d}}$	$\underline{0}\,\underline{0}$	$\underline{0}\,\underline{1}$ or $\underline{1}\,\underline{0}$	$\underline{1}\,\underline{1}$
$\underline{0}\,\underline{\dot{0}}$	κ	ϵ	π
$\underline{1}\,\underline{\dot{0}}$	ρ	α	λ
$\underline{0}\,\underline{\dot{1}}$	σ	β	μ
$\underline{1}\,\underline{\dot{1}}$	τ	γ	ν

[a] This table appears in the 1962 NP paper.

TABLE II.[a] Spin Weights and Conformal Weight of Spin-Coefficient Perturbations

Spin Coefficient	Spin Weight	Conformal Weight	Spin Coefficient	Spin Weight	Conformal Weight
ρ_B	0	1	λ_B	-2	-1
σ_B	2	1	μ_B	0	-1
α_B	n.d.[b]	n.d.	ν_B	-1	-2
β_B	n.d.	n.d.	γ_B	n.d.	n.d.
τ_B	1	0	$\mathrm{Re}(\gamma_B)$	0	n.d.

[a] The transformation properties of the spin coefficients are given by W. Kinnersley, unpublished Ph.D. thesis, California Institute of Technology (1967).

[b] n.d. indicates that the quantity does not have a definite spin or conformal weight.

FOOTNOTES

1. V. de la Cruz, J. E. Chase, and W. Israel, Phys. Rev. Letters **24**, 423 (1970).
2. J. B. Hartle, to be published.
3. The problem of general covariant field theories dates back to P.A.M. Dirac, Proc. Roy. Soc. **A155**, 447 (1936). A rather complete development is given in L. Gårding, Proc. Cambridge Phil. Soc. **41**, 49 (1945). See also V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. U.S. **34**, 211 (1948). Other references are given in these papers.
4. R. Penrose, Proc. Roy. Soc. **A284**, 159 (1965).
5. E. Newman and R. Penrose, J. Math. Phys. **3**, 566 (1962).
6. Unpublished private communication; see also J. Smith, Los Angeles Times, Part IV, p. 1, August 27, 1970.
7. T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063 (1957).
8. A good introduction to spinors is given by F. A. E. Pirani in _Lectures on General Relativity, Brandeis Summer Institute in Theoretical Physics, Vol. I_ (Prentice-Hall, Englewood Cliffs, 1964).
9. There is a brief introduction to the dyad formalism in the Newman-Penrose paper (footnote 5). Their notation will be followed, for the most part.
10. Newman and Penrose refer to a special coordinate system, but for our purposes we can consider it as a special null tetrad field.
11. E. T. Newman and R. Penrose, J. Math. Phys. **7**, 863 (1966). See also J. N. Goldberg et al., J. Math. Phys. **8**, 2155 (1967).
12. "Spin" refers to the order of a representation and should not be confused with spin weight. In this paper spin will be denoted by lower-case s and spin weight by capital S.

13. If, at the outset, tensors had been used to describe spin-s fields (as in Sec. IIIC, for electromagnetism), the spinor and dyad formalisms could have been avoided altogether. The description in purely tensor terms is much less convenient then the spinor description.

14. See, for example, A. I. Janis and E. T. Newman, J. Math. Phys. 6, 902 (1965).

15. Except for α_B, β_B, and γ_B which do not have definite spin weight. (See Table II.)

16. It can also be argued that the NP special tetrad is geometrically defined (except for rotations in θ and φ).

17. The field equations as given by Regge and Wheeler contain errors. The corrected equations can be found in K. S. Thorne and A. Campolattaro, Astrophys. J. 149, 591 (1967); ibid., 152, 673 (1968).

18. F. Zerilli, to be published.

19. R. A. Isaacson, Phys. Rev. 166, 1263 (1968).

20. W. Kinnersley, unpublished Ph.D. thesis, California Institute of Technology (1968).