ELECTRON INTERACTIONS: A FIELD THEORETIC GENERALIZATION OF THE GELL-MANN-BRUECKNER THEORY AND A CALCULATION OF EXCHANGE EFFECTS

Thesis by

Donald F. DuBois

In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1959

ABSTRACT

The method of Gell-Mann and Brueckner for treating electron interactions in a degenerate electron gas is generalized using the Feynman-Dyson techniques of field theory. A Feynman propagator is constructed for the effective interaction between electrons which takes into account the polarizability of the medium of unexcited particles in the Fermi sea. The well-known plasmon excitation appears as a singularity in this propagator. The plasmon is seen to be a correlated, resonant oscillation of the electron density field which is damped by transferring its energy to less correlated, multiple excitations. General expressions for the plasmon dispersion relation and for the plasmon level width are derived in terms of the polarizability of the many body medium.

The self-energies of the lowest states of the electron gas are discussed by using the adiabatic theorem. This enables us to derive an exact expression for the ground state energy in terms of the polarizability. Because of the degeneracy of the excited states of the non-interacting system, the adiabatic transforms of these states are not stationary states of the interacting system. However, as the momenta of the excited particles approach the Fermi momenta these states become asymptotically stationary. For states with only a few excited particles present an independent particle model is valid with the result that only the Feynman propagator for the physical one particle state is needed. This propagator, which is corrected for the virtual polarization of the medium by the particle, provides all the information concerning the energies and damping of the single particle states.

The second part of the paper is concerned with the detailed calculation of the effects of the interaction on the properties of an electron gas. The lowest order exchange correction to the plasmon energy is computed and found to be small in all cases of physical interest. However, the lowest order contributions to the plasmon damping are seen to modify the observed cut-off for plasmon excitation in electron energy loss experiments in a not negligible way. In applying the formalism to such experiments we also discuss the stopping power of an electron gas and derived the exact lowest order contribution to the single particle damping rate.

Using the self-energy method, the correction to the low temperature specific heat of an electron gas is computed exactly to one higher order in r_s (the interelectron spacing) beyond the calculation of Gell-Mann. It appears that the series in orders of r_s converges reasonably well only for $r_s < 2$. For $r_s < 0.8$ the specific heat is reduced from the value for non-interacting electrons while for $r_s > 0.8$ the specific heat is enhanced from this value. The change in sign appears to be a result of the Pauli Principle.

We conclude from these calculations that the procedure of expansion in orders of r_s gives useful results for values of $r_s < 2$. A formal calculation of the third order correction to the correlation energy is also carried out which will give a further clue concerning the convergence of the method if the integrations can be evaluated. For intermediate densities ($2 < r_s < 6$) the general perturbation approach may still be valid but a different approximation procedure for treating the polarization effects is needed.

ACKNOWLEDGEMENTS

The author wishes to express his thanks to Professor Murray Gell-Mann for suggesting this subject and for his constant encouragement throughout the course of this work. A number of stimulating discussions with Professor R. A. Ferrell and Dr. W. B. Karzas during the early stages of this work are also gratefully acknowledged.

The author is particularly thankful for the support given to him by a Rand Corporation Fellowship and an International Business Machine Corporation Fellowship.

TABLE OF CONTENTS

PART ONE

FIELD THEORETIC GENERALIZATION OF THE GELL-MANN-BRUECKNER METHOD

PART TWO

CALCULATION OF EXCHANGE CORRECTIONS TO THE PROPERTIES OF A DENSE ELECTRON GAS

APPENDIX

INTRODUCTION

Recently, Gell-Mann and Brueckner* [1] [2] introduced a new method for treating the problem of electron correlation effects in a degenerate gas of interacting electrons. They were able to compute the correlation energy and the correlation correction to the specific heat exactly, in the limit of high electron densities. The present work is concerned with a generalization of their original theory and an extension of calculations to electron densities approaching those found in actual metals.

G-B's method is based on the Feynman propagator method for summing up series of diagrams which correspond to terms in a perturbation expansion. Since it appears that they provide the most natural and most elegant expression of the theory, we will discuss the theory using the Feynman-Dyson techniques of field theory. Thus the first part of this paper is devoted to the application of these techniques to the non-relativistic, many Fermion problem. The G-B theory is the lowest order approximation, which we will call the pair approximation, to this generalized theory and higher order approximations are simply accounted for in our method. The role of the plasmon excitation, which appears naturally in the G-B theory, and the concept of single particle excitations will be considered in detail.

The second part of this paper will be devoted to the calculation of higher order corrections to the G-B theory using the propagator formalism. We will compute the lowest order correction, beyond the pair approximation, to the plasmon dispersion relation and we will derive an exact expression for lowest order contribution to the lifetime of

---

*Henceforth to be referred to as G-B.

the plasmon excitation.

The excitation of the many body medium by the scattering of incident electrons will be discussed along with the modification of the plasmon cut-off momentum due to the lifetime and dispersion corrections. Finally, the next highest correction to the low temperature specific heat of an electron gas will be computed. This extends the calculation of Gell-Mann [2] to the region of highest physical electron densities. In an appendix we will also present the formal calculation of the third order correction to the correlation energy. Unfortunately we were not able to evaluate all of the multiple integrals which occur here.

From these higher order calculations we can draw some conclusions concerning the convergence of these expansions and thus determine the range of validity of the high density calculations.

The problem of electron correlations has concerned physicists for a long time. Some twenty years ago Wigner [3] gave the first quantitative estimate of the correlation energy. The well known collective theory of Bohm and Pines [4] [5] [6] [7] described very well many of the qualitative aspects of the interacting electron gas such as the plasmon excitation and screening effects. However, the quantitative accuracy of this method seems doubtful in the light of recent exact calculations. Mäcke [8] proposed a method of summing up a certain series of perturbation terms to remove divergences, due to the long range of the Coulomb interaction, which otherwise appear. The G-B method enabled this program to be carried out by using Feynman propagator techniques.

The high density results of G-B for the correlation energy have subsequently been derived by other methods by Sawada et al. [9], Hubbard [10], and others. Hubbard [10] and Pines and Noziere [11] have given approximate interpolation formulae to connect the exact high density result with an exact low density calculation of Wigner [3]. The accuracy of these formulae is rather speculative until more is known about both the high density and low density limit.

The use of diagrammatical perturbation techniques has been quite useful in several recent studies of the many body problem. Goldstone [12] has applied the Dyson U matrix expansion to the problem of the ground state energy of a many Fermion system. He has derived the "linked cluster" expansion in terms of the two body scattering operator which was first given by Brueckner [13]. Hugenholtz has [14] applied a form of perturbation theory due to Van Hove to a treatment of the nuclear many body problem. To our knowledge no extensive application of the Feynman propagator techniques to the many Fermion problem has been published to date.

We will begin this paper with a discussion, in Chapter I, of the application of the Feynman-Dyson techniques to the evaluation of the S matrix for the many body system. A description of the states of the non-interacting system in terms of electrons excited from the Fermi sea and holes vacated therefrom will be used. The ground state, with no excited electrons or holes, is similar to the vacuum state except that the "passive," unexcited Fermions in the Fermi sea can still interact with each other and with particles excited from the sea. We will derive a set of rules for evaluating S matrix elements in terms of a single particle Feynman propagator for electrons and holes. The propagator for a density fluctuation is also derived.

In Chapter II we will apply this formalism to the interaction of two excited particles. Since these particles interact in the medium of the Fermi sea of unexcited particles, they can induce charge fluctuations in this medium. This polarization charge can then interact with one of the excited particles. The effective interaction between particles is shown to be screened by this polarization in such a way that it is screened at long distances so that the divergences associated with the long range Coulomb interaction no longer occur. A generalized dielectric constant is defined to describe the modified interaction. The calculation of the polarization is complicated by the fact that the electrons and holes excited from the sea can also interact with each other. This leads to so-called "local field corrections" to the polarization propagator derived in Chapter I.

The energies of the ground state and the low excited states of the interacting electron gas are discussed in Chapter III. By using the adiabatic theorem these states can be put into one to one correspondence with the eigenstates of the non-interacting system. The self energies of these states, due to the virtual polarization of the many body medium, are calculated using the Feynman techniques and the theory of the polarized interaction discussed in Chapter II. In this manner we will derive an exact expression for the ground state energy in terms of the generalized dielectric constant. Because of the degeneracy of the excited states of the non-interacting system, the adiabatic transforms of these states are not exactly stationary states of the interacting system. An electron can give up its energy in a real, energy conserving, excitation of the many body medium. Hence, states of a definite number of electrons and holes are not eigenstates of the complete Hamiltonian. However, states in which the electrons and holes are excited very close to the Fermi surface will be shown to be asymptotically stationary.

In Chapter III and mainly in Appendix B we will develop the theory of the damping and self energies of the low excited states using the Feynman-Dyson formalism. This theory is applicable to all many Fermion systems and the techniques are also applicable to the field theories of interacting elementary particles. We will also prove that an independent particle model is valid for the low excited states of a many Fermion system so that only the self energies and damping of single particle states need be computed.

In Chapter IV we discuss the well-known plasmon state which has no counterpart in the non-interacting system. This state develops adiabatically from a non-stationary "density fluctuation" state of the non-interacting system. The intimate connection of the plasmon with the propagator for the polarized interaction is discussed and the general formulae for the plasmon energy and level width are given. The plasmon state is seen to be a resonant, cooperative, excitation of the density field which, because of its structure, can decay by giving up its energy to various multiple excitations. Induced charge fluctuations in the many body medium are shown to be composed of the plasmon state plus a continuum of pair states. We will discuss the rules for the calculation of transition matrix elements when in the initial or final states, the many body medium is excited.

The second part of this work is concerned with detailed calculations of what we can call exchange corrections to the properties of the electron gas in the pair approximation. That is we will extend the calculations of G-B and others to lower electron densities, approaching the densities found in metals. We will follow G-B's procedure of expanding expressions in orders of the parameter r_s which is the ratio of the radius per electron to the Bohr radius.

Corrections to the plasmon dispersion relation are calculated in Chapter V to lowest order in r_s. The greater part of this chapter is concerned with a calculation of the exact lowest order contributions to plasmon decay. These are the two pair decay mode and the pair-plasmon decay mode. It is shown that for very long wavelengths, the plasmon is a well defined, long lived excitation which, for high densities, is adequately described by the pair approximation. For shorter wavelengths and lower densities the exchange or "local field" corrections to the plasmon become important and the plasmon is a less clearly defined state.

In Chapter VI we take up the intimately related problems of electron scattering by an electron gas and the lifetime of single particle excitations of the electron gas. First we will discuss the calculation of transition probabilities for various scattering processes including exchange contributions which have not been treated in previous work. We point out that in the generalized Bom approximation, the stopping power of an electron gas is given by the well-known Bethe formula because of the existence of sum rules which take on an especially simple form in our formalism. The effect of the finite lifetime of the plasmon on the observed plasmon cut off momentum in energy loss experiments is discussed in some detail. We find that for physical densities the cut-off is no longer well defined but that it certainly occurs for momenta smaller than the cut off computed from the pair approximation. This chapter is concluded with a calculation of the lifetime of single particle states. We find the well known result that single particle states are asymptotically stationary as their momenta approach the Fermi momentum.

The low temperature specific heat of a degenerate electron gas is computed in Chapter VII. The calculation of Gell-Mann is extended to electron densities near those found in metals. The calculation is based on the single particle model and the single particle self energies which were discussed in Chapter III. We find that the specific heat is enhanced from the non-interacting value [illegible] at physical densities.

The importance of this calculation is that it provides an example of the G-B expansion procedure to higher orders in r_s and thus also provides some information on the convergence of the series. The result shows that the G-B series corrected beyond the pair approximation probably converges reasonably well down to the highest densities found in metals. In Appendix D the third order calculation of the correlation energy is carried out formally. If the multiple integrals in this calculation can be evaluated the result will provide a further clue as to the convergence of the G-B series.

The method which we will apply in this work depends on the validity of the use of perturbation theory. The kinetic energy, in Rydbergs, of an electron at the Fermi surface is given by $1/\alpha^2 r_s^2$ where $\alpha = (4/9\pi)^{1/3}$ and r_s defined by $(4\pi r_s^3 a_o^3/3) = \rho^{-1}$ where ρ is the electron density. Thus the total kinetic energy of the system or the kinetic energy of a single electron near the Fermi surface is very large in the limit of high densities. The Coulomb interaction, on the other hand, is a relatively small perturbation on the motion of the electrons at high densities, being proportional to $1/r_s$ in lowest order. At least for the highest electron densities found in metals we expect that a perturbation expansion will be valid.

I. BASIC THEORY

A. ELECTRON-HOLE DESCRIPTION

We will adopt the formalism of the second quantization throughout this paper [15], [16]. The field equation for non-relativistic electrons moving in the field of the positive ion lattice is

$$\frac{\hbar^2}{2m}\nabla^2\psi_\alpha + V_I\psi_\alpha = \frac{\hbar}{i}\,\frac{\partial\psi_\alpha}{\partial t} \quad , \tag{1.1}$$

where $\psi_\alpha = \psi_\alpha(\vec{x},t)$ is the field operator which satisfies the well-known anti-commutation rules*

$$[\psi_\alpha^\dagger(\vec{x}',t),\ \psi_{\alpha'}(\vec{x}',t)]_+ = i\hbar\ \delta(\vec{x}-\vec{x}')\,\delta_{\alpha\alpha'}$$

$$[\psi_\alpha^\dagger(\vec{x},t),\ \psi_{\alpha'}^\dagger(\vec{x}',t)]_+ = [\psi_\alpha(\vec{x},t),\ \psi_{\alpha'}(\vec{x}',t)]_+ = 0 \quad . \tag{1.2}$$

The index α denotes the spin coordinate of the electron field. V_I is the (periodic) potential of the positive ion lattice. We will make the usual approximation of replacing the ions by a uniform background of positive charge which cancels the effect of the average electron charge density. Thus we approximate V_I by a constant (which is infinite!). The effect of V_I will be seen to cancel in all physical processes. The generalization of the formalism to be presented here to more realistic potentials is easily carried out in principle.**

---

* $\dagger$ denotes Hermitian conjugate.

** For many applications to metals the effect of the periodic potential of the lattice can be taken into account by replacing m by an effective mass m*.

The equations (1.1) possess the complete set of solutions (including degenerate spin variables)

$$u_{\vec{p},s}(\vec{x},t) = \frac{1}{V^{1/2}} e^{i\vec{p}\cdot\vec{x}} e^{-i\omega(p)t/\hbar} \chi_s \quad , \tag{1.3}$$

where

$$\omega(p) = V_I + \frac{p^2\hbar^2}{2m} \tag{1.4}$$

is the single electron energy, $\hbar\vec{p}$ is the momentum, V is the volume of the system of electrons, and χ_s is the spin eigenfunction $(s = \pm\frac{1}{2})$. We are assuming the usual periodic boundary conditions. $\psi_s(\vec{x},t)$ can be expanded in terms of this complete set

$$\psi_s(\vec{x},t) = \sum_{\vec{p}} a_{\vec{p},s}\,\eta(p - p_F)\,u_{\vec{p},s}(\vec{x},t) + b^+_{\vec{p},s}\,\eta(p_F - p)\,u_{-\vec{p},s}(\vec{x},t) \quad , \tag{1.5}$$

where $a_{\vec{p},s}$ is the destruction operator for an electron of momentum $\hbar\vec{p}$, spin s, while $b^+_{\vec{p},s}$ is the creation operator for a hole of momentum $-\hbar\vec{p}$, spin $-s$. These operators obey the usual anti-commutation relations which can be derived by substituting equation 1.5 in equation 1.2. The function $\eta(x)$ is defined by

$$\eta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases} \quad . \tag{1.6}$$

The association of creation and destruction operators with electrons and holes in this manner is possible since the operators in the usual description (i.e. electrons with all possible momenta and no holes), satisfy anti-commutation rules. In equation 1.5 the function $\eta(p - p_F)$

restricts electrons to momenta greater than the Fermi momentum, p_F, and $\eta(p_F - p)$ restricts holes to momenta less than p_F.

In this electron-hole description the ground state, Φ_o, of the non-interacting system is the state with no electrons and no holes. It is analogous to the vacuum state in relativistic field theory. Φ_o is not identical to the vacuum state however because of the so-called "passive particles" which fill the Fermi sea([12]. These are the particles with momenta less than p_F for which we have no explicit variables, but which have important interactions with the electrons and holes. We will discuss this in more detail below. Excited states of the system of a non-interacting Fermi gas are obtained from Φ_o by operating on it with creation operators for electrons and holes.

The electron-hole description given here is very similar to the electron-positron theory in quantum electrodynamics. Feynman's [17,18) (see also [15]) simplifying ideas can be used to great advantage in our considerations as well.

Following Feynman we construct a propagation kernel or propagator for electrons and holes. This propagator, $S_F(\vec{x}_2 - \vec{x}_1, t_2 - t_1)$, satisfies the inhomogeneous equation $(\vec{x}_{21} = \vec{x}_2 - \vec{x}_1, t_{21} = t_2 - t_1)$

$$\left| \frac{\hbar^2}{2m} \nabla_2^2 + V_I - i \frac{\partial}{\partial t_2} \right| S_F(\vec{x}_{21}, t_{21}) = i\, \delta(\vec{x}_{21})\, \delta(t_{21}) \quad . \quad (1.7)$$

The solution which satisfies the boundary conditions of this problem is*

---

* If we measure energies relative to the energy $\frac{1}{2m} p_F^2 + V_I$, then the holes will have negative energies and the electrons positive energies; just as in quantum electrodynamics.

$$S_F(\vec{x}_{21},t_{21}) = \begin{cases} \sum\limits_{\vec{p}>p_F} u_{\vec{p},s}(\vec{x}_2,t_2)\, u_{\vec{p},s}(\vec{x}_1,t_1) & t_2 > t_1 \\ \\ -\sum\limits_{\vec{p}<p_F} u_{\vec{p},s}(\vec{x}_2,t_2)\, u_{\vec{p},s}(\vec{x}_1,t_1) & t_2 < t_1 \end{cases} \quad . \tag{1.8}$$

We see that all components with energy less than the Fermi energy, $\omega = \omega(p_F)$, propagate backward in time while energies greater than $\omega(p_F)$ propagate forward in time. In the limit as $V \rightarrow \infty$ we can replace the sum over p by an integral in the usual way

$$S_F(\vec{x}_{21},t_{21}) = \frac{1}{(2\pi)^3}\int[d^3p(\eta(t_{21})\,\eta(p-p_F) - \eta(t_{12})\,\eta(p_F-p)]\; e^{i(\vec{p}\cdot\vec{x}-\omega(p)t/\hbar)} \quad . \tag{1.9}$$

We shall work almost exclusively in momentum-energy space so we will need a 4 dimensional Fourier transform of $S_F(\vec{x},t)$. In order to make the integral over t meaningful we add a convergence factor $e^{-\varepsilon|t|}$.* Then the transform is

$$\frac{-i\hbar}{(2\pi)^4} S_F(\vec{p},\omega) = \frac{1}{(2\pi)^4}\int d^3\vec{x} \int_{-\infty}^{\infty} dt\, S_F(\vec{x},t)\, e^{-\varepsilon|t|}\, e^{i\frac{\omega}{\hbar}t}\, e^{-i\vec{p}\cdot\vec{x}} \tag{1.10}$$

$$= \frac{-i\hbar}{(2\pi)^4}\left|\frac{\eta(p-p_F)}{\omega-\omega(p)+i\varepsilon} + \frac{\eta(p_F-p)}{\omega-\omega(p)-i\varepsilon}\right| \tag{1.11}$$

$$= \frac{-i\hbar}{(2\pi)^4}\left|\frac{P}{\omega-\omega(p)} - i\pi\,\delta(\omega-\omega(p))\,\varepsilon(p)\right| \quad . \tag{1.12}$$

---

* We shall see that this is equivalent to the assumption that the interaction between particles vanishes in the remote past $(t = -\infty)$ and in the distant future $(t = \infty)$. Another interpretation, which is perhaps more meaningful for this problem, is to regard $1/\varepsilon$ as the lifetime of the single particle state. We shall see in Chapter III that the one particle states of the interacting system do indeed have a finite lifetime.

where $\varepsilon(p) = \eta(p - p_F) - \eta(p_F - p)$. We see from equation 1.11 that the convergence factor defines the way in which we integrate around the poles of $S_F(\vec{p}, \omega)$ in the ω plane. In equation 1.12 P denotes the Cauchy principal value.

B. HAMILTONIAN AND S-MATRIX PERTURBATION THEORY

The total Hamiltonian for the system of N particles is

$$H_T = H_o + H_1 \tag{1.13}$$

with

$$H_o = \sum_{p<p_F} \omega(p) + \sum_{\substack{p>p_F \\ s}} \omega(p)\, a^+_{\vec{p},s}\, a_{\vec{p},s} - \sum_{\substack{p<p_F \\ s}} \omega(p)\, b^+_{\vec{p},s}\, b_{\vec{p},s} \tag{1.14}$$

where $\omega(p) = \hbar p^2/2m + V_I$.

If the interaction between electrons is taken to be the static Coulomb interaction then

$$H_1 = \frac{2\pi\hbar e^2}{V} \sum_q \rho_q \rho_{-q} \frac{1}{q^2} \quad , \tag{1.15}$$

where ρ_q is the Fourier component of the density operator, $\rho(\vec{x},t)$

$$\rho_q(t) = \int d^3x\, e^{i\vec{q}\cdot\vec{x}} \rho(\vec{x},t) \tag{1.16}$$

$$\rho(\vec{x},t) = \sum_s \psi^\dagger_s(\vec{x},t)\, \psi_s(\vec{x},t) \quad . \tag{1.17}$$

We can write ρ_q, using equation 1.5, as

$$\rho_q = \sum_{\vec{p},s} [a^+_{\vec{p}_1+\vec{q},s} \, \eta \, (|\vec{p}_1 + \vec{q}| - p_F) + b_{\vec{p}_1+\vec{q},s} \, \eta \, (p_F - |\vec{p}_1 + \vec{q}|)]$$

$$[a_{\vec{p},s} \, \eta \, (p - p_F) + b^+_{\vec{p},s} \, \eta \, (p_F - p)] \quad . \qquad (1.18)$$

Since the charge density operator $\rho(\vec{x},t)$ rather than the electron field operator $\psi_s(\vec{x},t)$ is the fundamental quantity occurring in the interaction we might expect that the density field will play the most important role in the interactions between electrons. In fact, we shall see that the effective interaction between electrons in the medium of the electron gas is modified in a fundamental way by the interaction with the density field.

Although the interaction (1.15) is time-independent it is very useful to introduce time variables and to use a time dependent perturbation theory. The physical reason for this is that the actual interaction between two electrons polarizes the surrounding medium of electrons and this induced charge modifies the interaction. Because of the inertia of the electrons, the effective potential of this polarized charge is time dependent; thus making the effective interaction between any two electrons time dependent. The quantum mechanical theory of this polarized interaction will be our main consideration in this paper.

We shall adopt the formalism of the S-matrix perturbation theory [15]. In the interaction representation the interaction Hamiltonian becomes time dependent

$$H_{1\varepsilon}(t) = e^{-iH_0 t/\hbar} \, H_1 \, e^{iH_0 t/\hbar} \, e^{-\varepsilon|t|} \quad . \qquad (1.19)$$

By adding the factor $e^{-\varepsilon|t|}$ we are making use of the adiabatic theorem. It is assumed that at $t = -\infty$ the interaction vanishes and is slowly turned on and finally turned off again to vanish at $t = \infty$. In the limit as $\varepsilon \rightarrow 0$ we do not expect this to affect the validity of our results.

The unitary time transformation operator $U_\varepsilon(t,t_o)$ is introduced which satisfies the equation

$$H_{1\varepsilon}(t) \cdot U_\varepsilon(t,t_o) = i\hbar \frac{\partial}{\partial t} U_\varepsilon(t,t_o) \qquad (1.20)$$

and the boundary condition $U_\varepsilon(t_o,t_o) = 1$. If $\Psi(t_o)$ is an interaction representation state vector at time t_o, then at time t

$$\Psi(t) = U_\varepsilon(t,t_o)\, \Psi(t_o) \quad . \qquad (1.21)$$

In particular at $t = -\infty$, when the interaction vanishes, if the system is in a "bare" state Φ of the non-interacting system, then at $t = 0$

$$\Psi(t{=}0) = U_\varepsilon(0, -\infty)\, \Phi \qquad (1.22)$$

we have a state, Ψ, of the interacting system. This method of generating states of the interacting system from those of the non-interacting system will be used throughout this paper.

The S-matrix is defined as

$$S = \lim_{\varepsilon \rightarrow 0} \lim_{\substack{t_1 \rightarrow \infty \\ t_2 \rightarrow -\infty}} U_\varepsilon(t_1,t_2) \quad . \qquad (1.23)$$

which, after taking the limits, can be written explicitly as

$$S = \sum_{n=0}^{\infty} \left(\frac{-i}{\hbar}\right)^n \frac{1}{n!} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n \, P[H(t_1) \cdots H(t_n)] \, . \qquad (1.24)$$

where P is Dyson's chronological operator which orders the operators in the brackets so that, left to right, they are in order of decreasing time variables.

The Feynman diagrams are obtained by applying in a well-known way [13] the algebra of the anti-commuting operators to the integrand in equation 1.24. We can establish a close correspondence with the diagrams in quantum electrodynamics by first rewriting the Hamiltonian H_1 in normal product form. (i.e. with all destruction operators to the right of all creation operators.) Note that in the electron-hole picture H_1 (see equations 1.15 and 1.18) is not in normal product form. Now if we write H_1 in normal product form and use the anti-commutation rules for the operators, we have in addition to normal products of four operators, normal products of two operators and also terms with no operators involved.

We can represent the possible interactions by Feynman graphs. The direction of increasing time is taken to be upward. A line in the direction of increasing time represents an electron $(p > p_F)$ while a line in the opposite direction represents a hole $(p < p_F)$ in agreement with the propagator, equation 1.8. The Coulomb interaction is taken to be a stationary or horizontal dotted line. Thus all interactions which involve four creation and destruction operators are represented by the possible topological forms of diagram 1a. Interactions involving only two operators arise from the commutation of creation and destruction operators for holes. Following Goldstone [12) we represent these interactions as in fig. 1b and 1c. The solid lines which do not propagate in time are taken to represent the "passive" unexcited particles in the Fermi sea. Finally we have terms which have no creation and destruction operators with the graphs 1d and 1e. We see then that the "passive particle" interactions arise as a result of the electron-hole description.

By expanding the S-matrix according to Wick's theorem [20] we can derive a set of rules for the construction of S-matrix elements from the diagrams. The rules in momentum-energy space are the most useful. One draws all topologically distinct diagrams leading upwards from the initial state to the final state. To construct the S-matrix element from a given diagram we have a product of :

1) A propagation function for each internal electron-hole line of momentum $\vec{p}$, energy ω :

$$\frac{-i\hbar}{(2\pi)^4} S_F(\vec{p}, \omega) = \frac{-i\hbar}{(2\pi)^4} \left[\frac{\eta(p - p_F)}{\omega - \omega(p) + i\varepsilon} + \frac{\eta(p_F - p)}{\omega - \omega(p) - i\varepsilon} \right].$$

2) A factor $\frac{i}{(2\pi)^4} \frac{4\pi e^2 \hbar}{q^2}$ for each Coulomb line of momentum q.

3) Delta functions $\frac{(2\pi)^4}{\hbar} \delta(\vec{p}_1 + \vec{p}_2 + \vec{q})\, \delta(\omega_1 + \omega_2 + u)\, \delta_{s_1 s_2}$ conserving energy, momentum, and spin at each vertex. [To apply this rule we take the Coulomb line as carrying a momentum q and <u>energy</u> u even though the Coulomb "propagator" (Rule 2) does not depend on u. This merely expresses conservation of energy in the interaction which always involves <u>two</u> vertices connected by a Coulomb line.]

4) (-1) for each closed loop.

5) $(-i/\hbar)^n$ for the n^{th} order coefficient in (1.24).

6) A factor $((2\pi)^{3/2})^{-1} \chi_s$ for each external electron-hole line of spin s.

We must also include the passive particle interactions. It will be shown in Chapter III that the interactions in fig. 1b contribute an infinite constant to the self energy of a particle which cancels the term V_I in equation 1.4. Thus we are justified in omitting this interaction and also the term V_I in the single particle energy from now on. The "exchange" interaction (fig. 1c) contributes a Coulomb factor (rule 2), the delta functions (rule 3) and a passive particle factor

$$-\frac{1}{(2\pi)^3}\,\eta\,(p_F - p) \qquad (1.25)$$

where p is the momentum of the passive particle.

The fact that $S_F(\vec{x}, t=0)$ as given by equation 1.9 is undefined enables us to <u>formally</u> include the passive particle factors in rule 1. We define

$$S_F(\vec{x},0) = -\frac{1}{(2\pi)^3}\int d^3p\ \ \eta\,(p_F - p)\ e^{i\vec{p}\cdot\vec{x}} \qquad (1.26)$$

which implies that

$$\int_{-\infty}^{\infty} d\,\omega\ S_F(\vec{p},\omega) = -\frac{1}{(2\pi)^3}\,\eta\,(p_F - p)\ . \qquad (1.27)$$

Then if we consider the passive particle line to represent a particle, with the propagator $S_F(\vec{x},t)$, which does not propagate in time we can use rule 1 for the passive particle factor. For all other cases we will deal with integrals over t so that the definition at the single point $t = 0$ is immaterial.

In the problem of a degenerate electron gas it is convenient to work in a special set of units in which momentum is measured in units of the Fermi momentum

$$p_F = \frac{\hbar}{\alpha\, r_o}\,, \qquad \alpha = \left(\frac{4}{9\pi}\right)^{1/3} = 0.520 \tag{1.28}$$

and energy is measured in units of the Fermi energy (times $\hbar^{-1}$) :

$$\frac{1}{\hbar}\,\frac{2}{\alpha^2 r_s^2}\ \text{Rydbergs} = \frac{2}{\alpha^2 r_s^2}\,\frac{e^4 m}{2\hbar^2}\,\frac{1}{\hbar}\ . \tag{1.29}$$

r_o is defined by

$$\frac{4\pi}{3}\, r_o^3 = \frac{V}{N} \tag{1.30}$$

where N is the total number of electrons. In these units the quantity

$$r_s = \frac{r_o}{r_b} \tag{1.31}$$

where r_b is the Bohr radius ($r_b = \hbar^2/me^2$). r_s replaces e^2 as the expansion parameter.

To transcribe the rules into these units the following transformations are useful :

$$\frac{V}{(2\pi)^3} \to \frac{3N}{8\pi} \tag{1.32}$$

$$\frac{V}{(2\pi)^4}\,\frac{4\pi e^2}{V}\,\frac{d^3q}{q^2} \to \frac{1}{2\pi}\,\frac{\alpha\, r_s}{2\pi^2}\,\frac{d^3q}{q^2} \tag{1.33}$$

$$\omega(p) = \frac{p^2}{2m} \to \frac{1}{2}\,p^2 \tag{1.34}$$

The right hand side expresses the left hand side of each equation in the new units. Therefore we merely modify the rules on page 17 to read :

2') A factor $\frac{i}{2\pi}\frac{\alpha\, r_s}{2\pi^2 q^2}$ for each Coulomb line

5') $(-i)^n$ for n^{th} order perturbation theory.

We should also point out that the factor $1/n!$ which occurs in equation 1.24 cancels exactly for all diagrams except those leading from ground state to ground state (the "vacuum fluctuation" diagrams). Except for the latter diagrams the expansion of the S-matrix always yields $n!$ identical terms.

C. DENSITY FLUCTUATIONS: PAIR PROPAGATION FUNCTION

As we mentioned above the density operator, $\rho(\vec{x},t)$ (equation 1.17), is very important in the quantum mechanical theory of the polarized interactions in the electron gas. We shall need a propagation function for charge density fluctuations for our later work. In this section we derive this propagation function and discuss its properties.

First let us note that the propagator for fluctuations in the electron-hole field as given by equations 1.8 and 1.9 can also be written as the matrix element

$$S_F(\vec{x}_{21},t_{21}) = (\Phi_o, T(\psi_\alpha(\vec{x}_2,t_2)\psi_\alpha^\dagger(\vec{x}_1,t_1))\,\Phi_o) \quad , \qquad (1.35)$$

where T is a time ordering operator which keeps the operator with the latest time variable on the left. The operation implied by T also takes into account the anti-commutation properties of the operators. For example

$$\begin{aligned} T(\psi_\alpha(\vec{x}_2,t_2)\psi_\alpha^\dagger(\vec{x}_1,t_1)) &= \psi_\alpha(\vec{x}_2,t_2)\psi_\alpha^\dagger(\vec{x}_1,t_1) && t_2 > t_1 \\ &= -\psi_\alpha^\dagger(\vec{x}_1,t_1)\psi_\alpha(\vec{x}_2,t_2) && t_2 < t_1 \ . \end{aligned} \qquad (1.36)$$

From this it seems reasonable to define the matrix element

$$-2Q^{o}(\vec{x}_{21},t_{21}) = (\Phi_{o}, T(\rho(\vec{x}_{2},t_{2})\,\rho(\vec{x}_{1},t_{1}))\,\Phi_{o}) \qquad (1.37)^{*}$$

as the propagator for density fluctuations in the electron field. This matrix element is easily evaluated to give

$$-Q^{o}(\vec{x}_{21},t_{21}) = \int d^{3}\vec{q} \int_{\substack{\vec{p}<1 \\ |\vec{p}+\vec{q}|>1}} d^{3}p \quad e^{-i|t_{21}|(\frac{q^{2}}{2}+\vec{q}\cdot\vec{p}_{1})}\, e^{i\vec{q}\cdot\vec{x}_{21}} \quad . \qquad (1.38)$$

The 4 dimensional transform of this is

$$-\frac{i}{(2\pi)^{4}}\,Q^{o}(\vec{q},\omega) = \frac{1}{(2\pi)^{4}}\int d^{3}\vec{x} \int_{-\infty}^{\infty} dt\; Q^{o}(\vec{x},t)\, e^{-\varepsilon|t|}\, e^{-i(\vec{q}\cdot\vec{x}-\omega t)}$$

$$= \frac{i}{(2\pi)^{4}} \int_{\substack{p<1 \\ |\vec{p}+\vec{q}|>1}} d^{3}p \left| \frac{1}{\omega-(q^{2}+\vec{q}\cdot\vec{p})+i\varepsilon} - \frac{1}{\omega+(q^{2}+\vec{q}\cdot\vec{p})-i\varepsilon} \right| \; . \qquad (1.39)$$

This propagator is essentially the amplitude for the diagram in fig. 2 which is the analogue of the vacuum polarization process in quantum electrodynamics. Here we have the virtual excitation and de-excitation of an electron-hole pair. The rules give for the amplitude of this diagram

$$(-i)^{2}\,(-1)\left[\frac{\alpha r_{s}}{2\pi^{2}q^{2}}\right]^{2} \frac{2}{(2\pi)^{2}} \int d^{3}\vec{p} \int_{-\infty}^{\infty} du\; S_{F}(\vec{p}+\vec{q},\, u+w)\; S_{F}(\vec{p},w) \; . \qquad (1.40)^{*}$$

Using equation 1.11 for $S_{F}(\vec{p},w)$ and performing the integration over w by contour integration we have

---

[*] Note that a factor of 2 appears here as a result of summing over the spin states.

$$+ \left[\frac{\alpha r_s}{2\pi^2 q^2}\right]^2 \frac{2}{(2\pi)}\, i \int d^3\vec{p}\ \left[\frac{\eta(|\vec{p}+\vec{q}| - 1)\ \eta(1-p)}{u - (\omega(p+q) - \omega(p)) + 2i\varepsilon} - \frac{\eta(p-1)\ \eta(1 - |\vec{p}+\vec{q}|)}{u + [\ \omega(p) - \omega(|\vec{p}+\vec{q}|)\ ) - 2i\varepsilon} \right] .$$

With $\omega(p) = \frac{1}{2} p^2$ and changing $\vec{p}$ to $-\vec{p}$ in the last term gives

$$- \left[\frac{\alpha r_s}{2\pi^2 q^2}\right]^2 \frac{2}{\ } \ \frac{i}{(2\pi)}\ Q^0(\vec{q},u) \ . \qquad (1.42)$$

Therefore we see that $Q^0(\vec{x},t)$ can also be interpreted as the amplitude for the propagation of an electron-hole pair. We shall call this propagator the pair propagator or polarization propagator.

Since we will use this propagator very often in the following work let us examine its analytic properties. The real and imaginary parts can be written

$$\mathrm{Re}\ Q^0(\vec{q},u) = \int_{\substack{p<1 \\ |\vec{p}+\vec{q}|>1}} d^3\vec{p}\ \frac{P}{\frac{q^2}{2} + \vec{q}\cdot\vec{p} - u} + \int_{\substack{p<1 \\ |\vec{p}+\vec{q}|>1}} d^3p\ \frac{P}{\frac{q^2}{2} + \vec{q}\cdot\vec{p} + u} \qquad (1.43)$$

$$\mathrm{Im}\ Q^0(\vec{q},u) = \pi \int_{\substack{p<1 \\ |\vec{p}+\vec{q}|>1}} d^3p\ \ (\frac{q^2}{2} + \vec{q}\cdot\vec{p} - u) + \pi \int_{\substack{p<1 \\ |\vec{p}+\vec{q}|>1}} d^3p\ \ (\frac{q^2}{2} + \vec{q}\cdot\vec{p} + u) . \qquad (1.44)$$

The restrictions, $|\vec{p} \pm \vec{q}| > 1$, on the region of integration in equation 1.43 which are a result of the Pauli Principle can be dropped.*

---

* The author expresses his thanks to Dr. R. A. Ferrell for pointing this out to him. The possibility of dropping the restrictions on other integrals will be discussed as we proceed.

If we violate the restrictions in the first integral (i.e. $\vec{p} + \vec{q} = \vec{p}_o$ where $p_o < 1$) the resulting contribution is exactly cancelled by violating the restrictions in the second integral $(\vec{p} + \vec{q} = -\vec{p}_o)$. Thus we may write

$$\mathrm{Re}\ Q^o(\vec{q},u) = \int_{p<1} d^3p \left[\frac{P}{(\frac{q^2}{2} + \vec{q}\cdot\vec{p}) - u} + \frac{P}{(\frac{q^2}{2} + \vec{q}\cdot\vec{p}) + u} \right] \tag{1.45}$$

which can be evaluated to give

$$\mathrm{Re}\ Q_o^o(q,u) = 2\pi \left[1 - \frac{1}{2q}\,[1 - (u - \frac{q}{2})^2]\, \ln \left| \frac{(u - q/2) + 1}{(u - q/2) - 1} \right| + \frac{1}{2q}\,[1 - (u + \frac{q}{2})^2]\, \ln \left| \frac{(u + q/2) + 1}{(u + q/2) - 1} \right| \right] . \tag{1.46}$$

The functions $\mathrm{Re}\ Q_o(q,qx)$ and $\mathrm{Im}\ Q_o(q,qx)$ arise frequently. Their power series expansions in q will be needed:

$$\mathrm{Re}\ Q_o(q,qx) = 4\pi[1 - \frac{x}{2} \ln \left|\frac{1 + x}{1 - x}\right|] + 0(q^2) \tag{1.47a}$$

$$= 4\pi\ R_o(x) + 0(q^2) \tag{1.47b}$$

and

$$\mathrm{Im}\ Q_o(q,qx) = 4\pi^2\ |x| + 0(q^2)\ . \tag{1.48}$$

The singularities of $Q_o(q,u)$ are of interest. $Q_o(q,u)$ has a discontinuity (or branch cut) along the real axis from $-q^2/2 - q$ to $q^2/2 + q$. The magnitude of the discontinuity is $2i\ \mathrm{Im}\ Q^o(q,u)$; the imaginary part being zero except along the discontinuity. The singularity arises from a continuous distribution of poles corresponding to the possible energies of (bare) pair states. (See the integrand of equation 1.39.)

Finally, we should point out the relation of $Q_o(q,u)$ to G-B's propagator $Q_q(u)$, (G-B equation (18)):

$$Q_o(q, iqu) = Q_q(u) .$$

$Q_q(u)$ has the power series expansion in q

$$Q_q(u) = 4\pi\, R(u) + q^2\, R_{(1)}(u) + q^4\, R_{(2)}(u) \tag{1.50}$$

where

$$R(u) = (1 - u \tan^{-1} \frac{1}{u}) \tag{1.51}$$

$$R_{(1)}(u) = \frac{2\pi}{(1 + u^2)^2} \quad . \tag{1.52}$$

II. THE EFFECTIVE INTERACTION; POLARIZATION OF THE MEDIUM

It is well-known that ordinary perturbation treatments of the Coulomb interactions between electrons fail because of the occurrence of unphysical "infra-red" divergences.* These divergences arise from the long range of the Coulomb interaction and the resulting singular behavior of the Coulomb potential.

Gell-Mann and Brueckner [1] have shown that these divergences can be removed within the framework of perturbation theory. Their method consists essentially of a formal summation of a certain family of Feynman diagrams to all orders in perturbation theory. The result of summing these apparently divergent terms is finite. The reader is urged to read their original paper for a brief and clear exposition of their treatment of the ground state problem using ordinary Rayleigh-Schrodinger perturbation theory.

A. DYNAMIC DIELECTRIC CONSTANT

The work of Lindhard [22)has clearly demonstrated the relation of the finite range of the effective interactions and the plasmon excitation to the concept of a generalized dielectric constant for the system of charges and fields. There is a close connection between the method of G-B (appropriately generalized) and the dielectric constant approach. Since we are neglecting the magnetic interactions we can expect only a comparison with Lindhard's longitudinal dielectric constant.

---

*For a review of the early work in this field see D. Pines [21].

To illustrate the idea consider the scattering of two electrons in the gas. The simplest diagram for the interaction is the scattering of the particles by the "bare" Coulomb interaction (Fig. 2a). Clearly it is unphysical to consider this process alone since we are neglecting the effect of the surrounding medium of unexcited particles. The scattering particles can also interact with the medium and excite density fluctuations in the medium. The resulting polarization charge (or induced charge) can then act on one of the particles via the Coulomb interaction.

In terms of graphs this means we should also consider the series of graphs, two members of which are represented by fig. 2, (b) and (c). Here the interaction excites and de-excites virtual pairs which in turn interact with the other particle. As we have seen these virtual pairs are equivalent to density fluctuations. More complicated processes can also occur since the virtual pairs can undergo complicated interactions before annihilating. For the moment let us restrict ourselves to the series of graphs in fig. 2. This is just the "pair-approximation" considered by G-B and Sawada et al. [9].

By formally summing these graphs we replace the Coulomb factor $(\alpha r_s/2\pi^2)\,(q^2)^{-1}$ by the series

$$\frac{\alpha r_s}{2\pi^2}\;\frac{1}{q^2}\left[\,1-\frac{\alpha r_s}{\pi^2 q^2}\,Q_o(q,\omega)+\left(\frac{\alpha r_s}{\pi^2 q^2}\right)^2 Q_o^2(q,\omega)-\cdots\,\right] \quad (2.1)$$

where q and ω are the energy and momentum transferred in the scattering and Q_o is the polarization or pair propagator discussed in I-C. Summing this series yields the effective interaction

$$\frac{\alpha r_s}{2\pi^2 q^2} \; \frac{1}{K_{r_s}(\vec{q},\omega)} = \frac{\alpha r_s^2}{2\pi^2} P_{r_s}(q,\omega) \tag{2.2}$$

where

$$\begin{aligned} K_{r_s}(q,\omega) &= 1 + \frac{\alpha r_s}{\pi^2 q^2} Q_o(q,\omega) \\ &= \frac{1}{q^2} P_{r_s}(q,\omega)^{-1} . \end{aligned} \tag{2.3}$$

We have, in effect, replaced the bare Coulomb interaction by an interaction which takes into account the polarization of the medium. It is a straightforward extension of the concepts of classical electrodynamics to regard $K^o_{r_s}(q,\omega)$ as a generalized or "dynamic" dielectric constant. Using equations 1.46 and 1.44 for $Q^o(q,\omega)$ it is easily verified that this is exactly Lindhard's definition of the longitudinal dielectric constant. Since $K_{r_s}(q,\omega)$ depends on q as well as ω we see that the space-time transform of this function is a variable function of space as well as time. This is to be expected since the polarization charge density depends upon the position in the gas where the interaction occurs.

Let us make a few general remarks on the properties of the effective interaction. First we note that one effect of the polarization of the medium is to make the effective interaction time-dependent. The finite propagation velocity arises from the inertia of the electrons which are excited from the sea and depends on the strength (e^2) of the Coulomb interaction. Secondly, we observe that the effective interaction does not diverge for small momentum transfers q. In the scattering of two electrons in the medium, for example, the energy transfer for small q is $\vec{q}\cdot\vec{p}$ where $\vec{p}$ is the original momentum of one of the scattered electrons. Thus as $q \to 0$

$$P^{o}_{r_s}(q, \omega) \to \frac{1}{q^2 + \frac{4\alpha r_s}{\pi} R^{o}(\vec{p} \cdot q)} \qquad (2.4)$$

(using 1.47 a, b and 1.48) which does not diverge. Thus the interaction has a finite range. The induced charge screens out the charge of the original electron at large distances.

The detailed properties of this effective interaction will be discussed in the remaining Chapters of this paper. Essentially all of the important effects of electron interactions on the properties of the electron gas arise from the modifying influence of the induced density fluctuations on the interaction.

B. GENERALIZED CALCULATION OF POLARIZATION CHARGE

In the "pair approximation" we have used only the lowest order contribution, $Q_o(q, \omega)$, to the polarization charge. To extend the theory we will now consider the corrections to $Q_o(q, \omega)$ due to virtual interactions of the pairs which are excited from the medium. We will first need a few definitions in order to make the discussion precise.

A "polarization diagram" (PD) is defined to be a diagram with two external Coulomb lines and no other external lines. The external Coulomb lines necessarily carry the same momentum q. A PD thus represents a virtual state which can absorb a momentum $\vec{q}$, propagate and finally give up this momentum. A "proper polarization diagram" (PPD) is a PD which cannot be broken up into two simpler PD's by cutting a single

Coulomb line. In fig. 3 diagrams a through h are PPD's whereas g is not a PPD.

With each PPD, i, we associate a propagator $Q_{r_s}^{(i)}(q,\omega)$; $Q_o(q,\omega)$ being the simplest example. The propagators $Q^{(1a)}$ and $Q^{(1b)}$ are also important for the applications in this paper and are discussed in Appendix A. Finally, the total polarization propagator is defined as the sum of all the propagators for PPD's:

$$Q_{r_s}(q,\omega) = Q_o(q,\omega) + Q_{r_s}^{(1a)}(q,\omega) + Q_{r_s}^{(1b)}(q,\omega) + \cdots$$
$$= \sum_{\substack{\text{all}\\ \text{PPD's}}} Q_{r_s}^{(i)}(q,\omega)\,. \tag{2.5}$$

The physical dielectric constant which includes the effect of all possible polarization processes is then

$$K_{r_s}(q,\omega) = 1 + \frac{\alpha r_s}{\pi^2 q^2} Q_{r_s}(q,\omega); \tag{2.6}$$

and the complete propagator for the effective interaction is

$$\frac{\alpha r_s}{2\pi^2} P_{r_s}(q,\omega) = \frac{\alpha r_s}{2\pi^2 q^2} K_{r_s}^{-1}(q,\omega) = \frac{\alpha r_s}{2\pi^2}\left[q^2 + \frac{\alpha r_s}{\pi^2} Q_{r_s}(q,\omega)\right]^{-1} \tag{2.7}$$

which includes all possible polarization diagrams which can be inserted into a single Coulomb line of momentum $\vec{q}$. $P_{r_s}(q,\omega)$ <u>is the effective interaction corrected to all orders of perturbation theory for polarization effects</u>.

In the actual calculation of the corrections $Q^{(i)}(q,\omega)$ to $Q_{r_s}(q,\omega)$ from the diagrams in fig. 3, one encounters divergences resulting from the use of the bare Coulomb interaction. This, of course, is corrected by using the polarized interaction, equation 2.2, in place of the Coulomb interaction and omitting all graphs with simple pair loops since they are included in equation 2.1. Thus we can regard the corrections $Q^{(i)}(q, \omega)$ as finite and well-defined functions of $\vec{q}$ and ω.

We can now amend the rules for the S-matrix expansion on page to include the effective interaction explicitly. The following amendments to the rules on pages 18 and 19 are to be made.

I.) In all graphs replace the Coulomb lines by the lines in fig. 3e for the effective interaction which now propagates in time. For each such line there is a factor $\frac{\alpha r_s}{2\pi^2 q^2} P_{r_s}(q, \omega) \frac{i}{2\pi}$ given by equation 2.7.

II.) Consider only those graphs which contain no polarization processes.

The first amendment includes all possible polarization processes in each interaction thus making the second amendment necessary so that no diagrams (in the original set) are counted twice. We shall see that for diagrams which lead from ground state to ground state these rules must be modified. This results from the fact that the factor $1/n!$ in the S-matrix expansion does not cancel exactly for these diagrams.

The amendment II can be considered to be a subsidiary condition which arises from the introduction of the effective interaction. This subsidiary condition is analogous to the well known subsidiary conditions in the Bohm-Pines approach, which arise from the introduction of plasmon coordinates. The fact that these conditions are not satisfied in their treatment of excited states has cast some doubt as to the quantitative validity of their results. In our case, however, the subsidiary conditions are trivially included for all states since we merely omit a certain class of Feynman diagrams.

These ideas can be formulated in a more elegant manner to exhibit closed expressions for $Q_{r_s}(q,\omega)$ and integral equations satisfied by it. However, such expressions are not useful for explicit calculations so we will not consider them here.

We will only briefly mention the annoying question of convergence. The series, equation 2.1, rigorously converges only for values of q greater than some number depending on r_s. We have however extended the expression 2.4 to $q = 0$. The justification for this step cannot be found in the perturbation theory alone. One can take the following view which is motivated by the physical ideas expressed above: The expansion of the S-matrix in powers of the bare Coulomb interaction is neither physically or mathematically legitimate. However through the Feynman diagrams we have kept track of all the terms of this illegal expansion. In summing up the polarization diagrams we are merely putting back together the functions which were illegally expanded. The method is consistent and unambiguous and therefore justified.

We should also point out that the calculation of the ground state energy (or correlation energy) by this method was rigorously verified in the pair approximation by Sawada et al. [9]. The relationship of our method to the dielectric method indicates the validity of the summation method.

The fact that the plasmon modes have not appeared in our results so far nor in G-B's original paper has caused some doubt and confusion as to the validity of the method. Sawada et al. [9] have shown that the plasmon contribution to the ground state energy is included in G-B's work. However, it seems to the author that their approach hides the real simplicity of the theory. In the following two chapters we will discuss the states of the electron gas and the intimate connection of the plasmon state with the propagator, $P_{r_s}(q,\omega)$, of the effective interaction will become apparent.

C. ANALYTIC PROPERTIES OF $Q_{r_s}(q,\omega)$.

For some of the general discussions in the next chapter we will need to understand the analytic properties of the propagator $Q_{r_s}(q,\omega)$. We have already discussed Q_o, the lowest order approximation to Q_{r_s} on page21 ; in Appendix A we also discuss $Q_{r_s}^{(1a)}$ and $Q_{r_s}^{(1b)}$ in detail. From these examples we can easily determine the properties of $Q_{r_s}^{(i)}$.

We have an integral (or sum) over all intermediate states of a product of energy denominators. These energy denominators are the difference of the energy of the intermediate state and the initial energy ω, plus or minus $i\varepsilon$. The propagators contain all possible time orderings by various combinations of energy denominators. Since, in general, a continuum of energies from 0 to ∞ is possible for intermediate states there is a branch cut or discontinuity from 0 to ∞, displaced below the real ω axis by $-i\varepsilon$. We see from the diagrams that all of these processes are symmetrical in time; thus there is a symmetrical cut along the negative ω axis displaced by $+i\varepsilon$.

Along the cut the propagators have an imaginary part. These arise from the $i\varepsilon$ in the energy denominators by virtue of the limiting relation

$$\lim_{\varepsilon \to 0} \frac{1}{\Delta E \mp i\varepsilon} = \frac{P}{\Delta E} \pm i\pi \delta(\Delta E)$$

where $\Delta E = E_f - E_i$; E_i being the energy of an intermediate state. Also from this relation we see that there is a discontinuity in the imaginary part on crossing the branch cut.

III. ENERGY STATES OF THE ELECTRON GAS

In this chapter we will discuss some of the low lying energy states of the interacting electron gas using the perturbation theory developed in Chapters I and II. These states can be put into one to one correspondence with the individual eigenstates Φ_p of H_o. In the following chapter we will discuss the plasmon state which is characteristic of the interacting system but has no counterpart in the non-interacting electron gas. The states Φ_p of the non-interacting system are described by enumerating the electrons, $\left\{ \vec{p}_i, s \right\}$ $(p_i > 1)$, and holes, $\left\{ \vec{p}_i', s' \right\}$ $(p_i' < 1)$, which are excited. These states are eigenstates of H_o with energy eigenvalues

$$\varepsilon_p = \varepsilon_o + \sum_{p_i > 1}^{\nu} \omega(p_i) - \sum_{p_i' < 1}^{\nu'} \omega(p_i') \qquad (3.1)$$

where ε_o is the Fermi energy $\varepsilon_o = \sum_{\text{all } p_i < 1} \omega(p_i)$. The ground state Φ_o is the state with no electrons or holes and has the eigenvalue ε.

Except for Φ_o, the states Φ_p are highly degenerate. This greatly complicates the problem of generating the eigenstates Ψ , of the interacting system, from the states Φ_p by perturbation theory.

The ground state energy, E_o, can be found quite readily since Φ_o is non-degenerate. We shall calculate the self energy of the ground state, Ψ_o, following a method due to Feynman. This will enable us to derive an exact formula for E_o in terms of the generalized dielectric constant.

Excited states present a much more difficult problem because of the degeneracies. In his paper on the specific heat of a degenerate electron gas [2], Gell-Mann presented a method for calculating the energies of low lying excited states. His more or less intuitive approach was based on the assumption that the eigenstates Ψ_p could be put in one to one correspondence with the states Φ_p and therefore would bear the same labels. That is, if one adiabatically turns on the interactions between electrons the states of the unperturbed system go over continuously into the states of the perturbed system; Ψ_p being the adiabatic transform of Φ_p.

In this way Gell-Mann establishes an independent particle model for the interacting system. It seems clear that the only consistent way to do this is by making such a correspondence with the independent particle states of the unperturbed system. Thus if $W(p)$ is the perturbed single particle energy, then the energy of a state which is the adiabatic transform of the unperturbed state with ν electrons and ν' holes is

$$E_p = E_o + \sum_{p_i > 1}^{\nu} W(p_i) - \sum_{p_i' < 1}^{\nu'} W(p_i') + O\left(\frac{\nu + \nu'}{N}\right) , \qquad (3.2)$$

where E_o is the perturbed ground state energy. The interaction between the excited particles gives a term of order $1/N$ for each pair of particles. Thus for states with relatively few excited particles $(\nu + \nu' \ll N)$ an independent particle model is expected to be a good approximation.

That an independent particle model must be valid in some sense is clear from the success such theories have had in predicting the electronic properties of solids.

To establish the validity of this model, however, one must reconcile it with two facts. The first, an experimental one, that the cross section for scattering an incident electron by the electrons in a metal is not zero. This means that the probability of being in the initial unperturbed state of one electron decreases with time. Clearly, the state describing this scattering process is not a stationary state of the many body system but the scattering process is a physical realization of the adiabatic transformation.

The second fact which must be reconciled with the independent particle model is a mathematical one. In ordinary stationary state perturbation theory, when one deals with a subset of degenerate zero order states, the degeneracy is first removed by picking linear combinations of the degenerate states which diagonalize the Hamiltonian. This method is clearly impossible for such a degenerate system as the present one. However, there is no unique way to make a correspondence between the true eigenstates and single degenerate unperturbed states.

This problem is a well-known one in quantum theory. The Weisskopf-Wigner problem of the atom interacting with the radiation field is the best example. Here the states which are the adiabatic transforms of the unperturbed states of the free atom are not stationary states. They are states with a finite (usually small) probability of decaying. The true eigenstates of the system are complicated linear combinations of the unperturbed states of atom plus radiation field. Clearly, the non-stationary states are the physically important states for this problem; the vast experimental knowledge of atomic spectra emphasizes this. The fact that the level widths are small makes the observation of discrete lines possible, but the widths cannot be zero; there must be a finite probability of decay. These quasi-stationary states are also important in statistical mechanics. A system of atoms reaches thermal equilibrium through the excitation or decay of these states; moreover the energies which appear in the formula for the equilibrium distribution are the energies of these meta-stable states.

The time dependent perturbation theory developed in Chapters I and II is well suited to this problem. We define the adiabatic transform of the unperturbed state, Φ_p, in the interaction representation as

$$\Psi_p(t) = U(t,t_o)\,\Phi_p \tag{3.3}$$

where the adiabatic U matrix is defined in equations 1.20 and 1.22. The initial time to will be taken to be $t_o = -\infty$ so we see that at $t = t_o$ the interaction vanishes. The boundary conditions are such that $\Psi_p(t_o) = \Phi_p$. In the Schrodinger representation the state $e^{-iH_o(t-t_o)}\,\Psi_p(t)$ satisfies the Schrodinger equation with the Hamiltonian $H_o + H_1$; we must investigate under what conditions it is a stationary state.

With the Feynman formalism we will calculate the self-energies of the states Φ_p. This is done by constructing a propagator for the state Φ_p, corrected for all virtual self energy processes. The condition that this propagator have a pole in the complex "energy" plane gives the dispersion relation for the state $\Psi_p(t)$. If the pole lies on the real energy axis then $\Psi_p(t)$ is a stationary state with the energy eigenvalue at the pole. If the pole has a small imaginary part, then $\Psi_p(t)$ can be interpreted to be a metastable state with a level width given by twice the imaginary part; the approximate energy of the state being the real part.

In Appendix B we will show that this method is essentially the same as the theory of radiation damping of Heitler and Ma [23) and that of F. Low [24]. The most important advantage of using the Feynman techniques is that the ground state self energy can be separated out of the self energy calculation for any state and treated as a separate problem.

B. GROUND STATE ENERGY

We now want to apply our perturbation theory to a calculation of the energy E_o of the ground state Ψ_o of the gas of interacting electrons. In the electron-hole scheme this is equivalent to the field theoretic problem of finding the energy of the true vacuum state of a system of interacting fields. This problem has been discussed by Gell-Mann and Low [25] for the general field theoretic case and by Goldstone [12] for the case of Fermions interacting via a static, finite range potential. We shall adopt a somewhat different method due to Feynman [17].

Because of conservation of momentum and energy, there exists no S-matrix element which connects the ground state Φ_o with any other unperturbed state. It follows that Φ_o is an eigenstate of S with eigenvalue C:

$$S\Phi_o = C\Phi_o \qquad (3.4)$$

Since S is unitary for a system admitting no bound states

$$|C|^2 = 1 \qquad \text{or} \qquad C = e^{-i\Gamma}; \qquad \Gamma \text{ real} . \tag{3.4a}$$

We shall see below that Γ is proportional to the (large) time interval of observation. The diagonal S-matrix element $(\Phi_o|S|\Phi_o) = e^{i\Gamma}$ can be interpreted as the amplitude that the system remains in the ground state, under the influence of the interaction, after a long time. The corresponding probability is independent of time which means that the state $\Psi_o(t) = U(t,t_o)\Phi_o$, which is the adiabatic transform of Φ_o, is stationary as $t \to \infty$ and $t_o \to -\infty$.* Thus in the Schrodinger representation

$$e^{-iH_o(t-t_o)}\,\Psi_o(t) = e^{-iE_o(t-t_o)}\,\Psi_o \tag{3.5}$$ **

and

$$(\Phi_o|e^{-iH_o(t-t_o)}\,\Psi_o(t)) = e^{-iE_o(t-t_o)}\,(\Phi_o|\Psi_o)$$

$$\to e^{-i\varepsilon_o(t-t_o)}\,(\Phi_o|S|\Phi_o)$$

$$= e^{-i\varepsilon_o(t-t_o)}\,e^{-i\Gamma}$$

Thus for a long time interval we can identify

$$\Gamma = (E_o - \varepsilon_o)(t - t_o) \tag{3.6}$$

and

$$(\Phi_o|\Psi_o) = 1 \tag{3.6a}$$

---

* This is only true for a large time interval $t - t_o$ so that energy is conserved. For shorter times there may be components off the energy shell. These incoherent transient components die out by destructive interference as $t - t_o \to \infty$. In Appendix B this is discussed in more detail.

** Here Ψ_o is the stationary state vector corresponding to $\Psi_o(t)$.

Knowing Γ we can compute the energy shift or self energy of the ground state from equation 3.6. The normalization of the eigenstate Ψ_o in equation 3.6a agrees with that found by Goldstone [12] when Ψ_o is generated by the "linked cluster" expansion.

Let us discuss the expansion of $(\Phi_o|S|\Phi_o)$ in terms diagrams using the rules on page 17, forgetting, for the moment, the amendments made on page 30 to take into account the effective interaction. We see that this S-matrix element is the sum of all diagrams leading from the ground state back to the ground state; i.e. all diagrams with no external lines. Examples of these are shown in fig. 4. The lowest order processes are 4-1a and 1b which involve only passive particles. Fig. 4-1a represents the average Coulomb energy of the system (which is infinite), and thus cancels the term in $\varepsilon_o = \sum_{p<1} \omega(p)$ arising from the interaction, V_I, with the positive background of charge. The diagram 4-1b represents the well-known "exchange" energy and will be considered below. The sum of these two terms is then $(\Phi_o, H_1\Phi_o)$.

Following Feynman we call the sum of all connected ground state diagrams $-iL$. By connected diagrams we mean diagrams which do not contain separate parts with no interconnecting lines. It then follows that

$$(\Phi_o|S|\Phi_o) = 1 - iL + \frac{(iL)^2}{2!} - \frac{(iL)^2}{3!} + \dots = e^{-iL}. \qquad (3.7)$$

From equation 3.6 we then have $(T = t - t_o)$

$$\Delta E_o = - \lim_{T\to\infty} \frac{L}{T} . \qquad (3.8)$$

Thus the self energy can be computed from the connected diagrams alone. It is easily shown that equation 3.8 is identical to the linked cluster expansion of Brueckner and of Goldstone. The advantage of equation 3.8

for our purposes is that it allows us to use the Feynman techniques in calculating ΔE_0.

Now let us consider the explicit form of $-L$. It is well-known, and indeed it follows trivially from the rules, that the amplitudes for all the connected ground state diagrams contain a factor $(2\pi)^4 \delta^{(3)}(0)\delta^{(1)}(0) = VT$. This results from the fact that these diagrams have no external lines and therefore as many δ functions arise from an application of rule (3) as there are corners in the diagram. Thus from equation 3.8 we see immediately that the ground state energy shift is proportional to the volume of the system. We shall see that the excited state self energies, measured relative to E_0, are independent of the volume provided the volume is large.

From fig. 4 we see that all the connected ground state diagrams are composed of closed chains of proper polarization diagrams, which we defined in Chapter II. Thus these processes represent the virtual polarization of the many body medium; the zero point motion of the system. [To make this identification with the polarization diagrams one must recognize that a subset of diagrams, of which Figs. 4-1b, 2b, 3e and 3f are members involves only one proper polarization diagram. Since zero momentum and energy are carried by these processes it is clear that the sum of all these single processes is proportional to

$$\int_0^{r_s} \frac{dr_s'}{r_s'} \frac{1}{2\pi i} \int d^3\vec{q} \int_{-\infty}^{\infty} d\omega \frac{\alpha r_s'}{\pi^2 q^2} Q_{r_s'}(q,\omega) \qquad (3.9)$$

i.e. that there is no propagation. [This is easily checked by applying the rules to these diagrams, except for fig. 4-1b. The diagram in 4-1b represents the well known exchange energy contribution. To include this in the formula 3.9 we must make use of the definition of $S_F(\vec{p}, t=0)$ which was introduced in 1.27 to include the passive particle effects. Thus

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \, Q_o(q,\omega) = Q_o(q, t=0) = \int d^3p \;\; S_F(\vec{p}+\vec{q}, t=0)\, S_F(\vec{p}, t=0)$$

$$\int d^3\vec{q} \quad \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} Q_o(q,\omega) \frac{1}{q^2} = \int d^3\vec{q} \int_{\substack{|\vec{p}+\vec{q}|<1 \\ \vec{p}<1}} d^3p \;\; \frac{1}{q^2} \qquad (3.9a)$$

Then (in Rydbergs)

$$N\, \varepsilon_{\text{exchange}} = - \frac{1}{\pi^2 \; \alpha r_s} \; \frac{3N}{8\pi} \; \frac{1}{2\pi i} \int d\omega \int d^3\vec{q}\; Q_o(q,\omega)\, \frac{1}{q^2}$$

$$= - \frac{4\pi^2}{\pi^2 \; \alpha r_s} \; \frac{3N}{8\pi} = - N \frac{.916}{r_s}$$

Thus these definitions are consistent. However it must be pointed out that they are introduced only as a convenience to simplify the formulas.]

We will now show that (in Rydbergs)

$$\Delta E_o = \frac{3N}{8\pi}\frac{1}{2}\frac{2}{\alpha^2 r_s^2} \int d^3\vec{q} \int \frac{d\omega}{2\pi i} \;\; \sum_{n=1}^{\infty} \int_0^{r_s} \left[- \frac{\alpha r_s'}{\pi^2 q^2}\, Q'_{r_s}(\vec{q},\omega)\right]^n \frac{dr_s'}{r_s'} \qquad (3.10)$$

where $Q_{r_s}(q,\omega)$ is the exact corrected polarization propagator defined in Chapter II. $Q_{r_s}(q,\omega)$ is the sum of all amplitudes for proper polarization processes. The factor $\frac{3N}{8\pi}$ results from the volume factor, $V = \frac{3N}{8\pi}$, in our special units and we have multiplied by $\frac{2}{\alpha^2 r_s^2}$ to express the energy in Rydbergs.

To understand the remaining factors let us first consider the pair approximation: $Q_{r_s}(q,\omega) = Q_o(q,\omega)$. The sum in equation 3.10 is then represented by the series of diagrams 4-2a, 3a, ... , the general n^{th} order diagram is shown in fig. 4. That the n^{th} order contribution is proportional to

$$V \int d^3q \int \frac{d\,\omega}{(2\pi i)} \left[- \frac{\alpha r_s}{\pi^2 q^2} Q_o(q,\omega)\right]^n$$

follows quickly from the rules. [Starting from the bottom of the diagram, if there are n-k loops on one side and k loops on the other side, conservation of energy and momentum at the lowest vertex gives

$$\left[\frac{-\alpha r_s}{\pi^2 q^2}\right]^n \left[Q_o(q,\omega)\right]^{n-k} \left[Q_o(-q, -\omega)\right]^k$$

Since $Q_o(q,\omega) = Q_o(-q, -\omega)$ the result follows after integrating over q, ω.] However, here the factor $1/n!$ which occurs in equation 1.24 does not cancel exactly.

The correct factor is found by applying the algebraic method of Wick to the reduction of the S-matrix. It follows from Wick's theorem that the correct factor is $k_n/n!$ where k_n is the number of distinct ways of connecting n points with a closed line, using each point once. k_n is the number of distinct contractions* from the n^{th} order process. It is easily seen that $k_n = n!/n$ since the n cyclical permutations of the n points in each arrangement merely repeat the same arrangement and are not distinct. Thus the correct factor is $1/n$.

---

* For the definition of contractions and their use in Wick's theorem see references [20] [15].

Now let us turn on the remaining interactions. It is clear from the discussion in Chapter II that the effect of including the complete Hamiltonian is to replace $Q_o(q,\omega)$, in equation (3.10), by $Q_{r_s}(q,\omega)$, the corrected polarization propagator. It can be shown directly that all the diagrams in $-L$ are given correctly. This can also be proved by using a formula used by Sawada [9] involving an integral over the coupling constant g^2.

$$E_o - \varepsilon_o = \int_0^{g^2} \frac{dg'^2}{g'^2} \langle \Psi_o | \underset{\sim}{H}_{1\,g^2}(0) | \Psi_o \rangle$$

where Ψ_o is the true ground state vector which, in the Heisenberg representation, can be written as

$$\Psi_o = \frac{U(0,-\infty)\,\Phi_o}{\langle \Phi_o | U(0,-\infty) | \Phi_o \rangle}$$

(see reference 15). $\underset{\sim}{H}_{1\,g^2}(t)$ is the interaction Hamiltonian in the Heisenberg representation, defined in terms of the interaction representation operator*

$$\underset{\sim}{H}_{1\,g^2}(t) = U^{-1}(t,0)\, H_{1\,g^2}(t)\, U(t,0).$$

In our units, we can write

$$E_o - \varepsilon_o = \frac{3N}{8\pi} \frac{2}{\alpha^2 r_s^2} \int_0^{r_s} \frac{dr'_s}{r'_s} \int \frac{d^3q}{(2\pi)^3} \frac{2\pi\alpha r_s}{q^2} \langle \Psi_o | \underset{\sim}{\rho}_{\vec{q}}(0)\, \underset{\sim}{\rho}_{-\vec{q}}(0) | \Psi_o \rangle \qquad (3.11)$$

where we have introduced equation 1.15 for H_1, replacing the summations by integrations.

Using what are now standard methods in field theory [15][25] it is readily shown (see also page 57) that the following relations hold:

$$\langle \Psi_o | T(\underset{\sim}{\rho}_{\vec{q}}(t_2)\, \underset{\sim}{\rho}_{-\vec{q}}(t_1) | \Psi_o \rangle$$

$$= \langle \Phi_o | T(s\,\rho_{\vec{q}}(t_2)\, \rho_{-\vec{q}}(t_1) | \Phi_o \rangle \qquad (3.11a)$$

$$= -2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} e^{-i\omega(t_2 - t_1)} \frac{Q_{r_s}(q,\omega)}{1 + (\alpha r_s/\pi^2 q^2)\, Q_{r_s}(q,\omega)}$$

---

* Here our notation makes the dependence of H_1 on g^2 explicit.

where the prime on the second expectation value indicates that in evaluating it in terms of Feynman diagrams, the disconnected, ground state diagrams are to be omitted. These are, in fact, cancelled by the denominator in the definition of Ψ_o (see [15]).

Using equation 3.11a in equation 3.11, we find*

$$E_o-\epsilon_o = -\frac{3N}{8\pi}\frac{1}{\alpha^2 r_s^2}\int_0^{r_s}\frac{dr'_s}{r'_s}\int d^3q\int_{-\infty}^{\infty}\frac{d\omega}{2\pi i}\frac{(\alpha r'_s/\pi^2q^2)\,Q_{r'_s}(q,\omega)}{1+(\alpha r'_s/\pi^2q^2)Q_{r'_s}(q,\omega)} \tag{3.12a}$$

or in terms of the dielectric constant $K_{r_s}(q,\ )$

$$= -\frac{3N}{8\pi}\frac{1}{\alpha^2 r_s^2}\int_0^{r_s}\frac{dr'_s}{r'_s}\int d^3q\int_{-\infty}^{\infty}d\omega\left[1-\frac{1}{K'_{r_s}(q,\omega)}\right]\frac{1}{2\pi i} \tag{3.12b}$$

We also obtain this expression by formally carrying out the summation over n in equation 3.10.

Since $Q_{r_s}(q,\omega) = O(\frac{1}{|\omega|^2})$ as $|\omega| \to \infty$ and since the singularities of $K_{r_s}(q,\omega)$ lie below or above the positive or negative Re ω axis, respectively,** we can deform the contour of integration to lie along the imaginary ω axis. With $\omega = iu$ we have then

$$E_o-\epsilon_o = -N\frac{3}{8\pi}\frac{1}{\alpha^2 r_s^2}\int_0^{r_s}\frac{dr'_s}{r'_s}\int d^3q\int_{-\infty}^{\infty}\frac{du}{2\pi}\left[1-\frac{1}{K'_{r_s}(q,iu)}\right] \tag{3.12c}$$

Now $Q_{r_s}(q,iu)$ is real for real values of u (in the limit as $\epsilon\to 0$) so that E_o is real, as it must be if it is an eigenvalue of H. That we can generate a real energy shift depends strongly on the fact that Φ_o is non-degenerate so that in all of the energy denominations one can let $\epsilon\to 0$ without encountering the contributions from any poles. This will not be the case for the excited states which we will consider.

The relation of equation 3.12b to G-B's equation (19) will be made clear in Appendix D where we formally compute the third order corrections to $E_o-\epsilon_o$.

---

* The appearance of the time order operator T in this expression does not cause any complications. This follows since, in equation 3.11, the expectation value is integrated over all value of $\vec{q}$.

** See section IIc and Chap. IV for discussion of the analytic properties of Q_{r_s} and K_{r_s}.

Some questions arise concerning how the Pauli Principle applies to the intermediate states of the linked cluster expansion. This was discussed in detail for the ground state problem by Goldstone. In ordinary Raleigh-Schrodinger perturbation expansions it is immaterial whether or not the exclusion principle is applied in intermediate states; terms which violate the exclusion principle always cancel in pairs. However in the linked cluster expansion where we have dropped unlinked terms this cancellation is not complete and it follows from the S-matrix expansion that we must include intermediate states which violate the exclusion principle. This is automatically done in using the Feynman propagators and diagrams. We shall discuss this again in the explicit calculation in Appendix C. The same sort of question arises in connection with our treatment of excited states so we shall have more to say about it in the next section.

C. EXCITED STATES

Our treatment of excited states will also involve the calculation of self energies of "bare" or unperturbed states, Φ_n. The perturbation method is reminiscent of the renormalization procedure in relativistic field theory [15]. We will compute the "physical" propagators for the states Φ_p, which are corrected for all virtual interactions. The singularities of these propagators provide the dispersion formulae for the excitation energies of the many electron system. If the unperturbed state, Φ_p, belongs to a set of degenerate states, this procedure does not lead to an eigenvalue of the complete Hamiltonian. In this case we can interpret the propagator as describing the evolution in time of a meta-stable state with a finite lifetime.

A formal discussion of this method will be given in Appendix B. In the remainder of this chapter we will discuss the application of the method to various states of the Fermi gas.

Let us consider calculation of the corrected propagator for the one particle state,

$$\Phi_p = \frac{a^{+}_{\vec{p},s}\,\Phi_o}{\sqrt{V}} \quad \left(\text{or} \quad \frac{b^{+}_{\vec{p},s}\,\Phi_o}{\sqrt{V}}\right)$$

of one electron (or hole). Such a state does not represent a physically realizable state of an electron gas since electrons and holes are created or annihilated in pairs. We shall show however that to a good approximation the members of a pair move independently of each other and the energy of the "physical" pair state is the sum of the corrected single electron-hole energies. In the extended system in which an external electron is scattered by the electron gas the one electron state is a physically realizable state.

Let us define a "self-energy graph" (SG) to be a graph leading from the initial state Φ_p and back to the same state. A "proper self energy graph" (PSG) is one in which the state Φ_p does not occur as an intermediate state in such a way that cutting the line (s) representing this state produces two self energy graphs of lower order. For example, (a) and (b) in fig. 6 are proper while (c) is not.

With each proper self energy graph, i, there is associated an amplitude, or "self energy part", $\sum^{(i)}(\vec{p},\omega)$, which depends on the momentum and energy variables of the external lines. For example for (a) in fig. 6 we have

$$-\frac{2\pi^2}{\alpha r_s^2}\sum^{(1)}(\vec{p},\omega) = \frac{1}{2\pi i}\int d^3\vec{q}\int_{-\infty}^{\infty} d\omega\; S_F(\vec{p}+\vec{q},\, u+\omega)\; P_{r_s}(\vec{q},u) \;. \qquad (3.13)$$

These amplitudes are defined in such a way that if an internal particle line is modified by inserting a PSG we replace the propagator $S_F(\vec{p},\omega)$ by

$$S_F(\vec{p},\omega) \quad \sum\nolimits^{(i)}_{r_s}(\vec{p},\omega)\; S_F(\vec{p},\omega) \;, \qquad (3.14)$$

Now the sum of all "proper self energy parts" we shall call $\sum_{r_s}(\vec{p},\omega)$

$$\sum\nolimits_{r_s}(\vec{p},\omega) = \sum\nolimits^{(1)}_{r_s}(\vec{p},\omega) + \sum\nolimits^{(2)}_{r_s}(\vec{p},\omega) + \dots \qquad (3.15)$$

For any state Φ_p the sum of all possible self corrections to the propagator for a single particle line is

$$\begin{aligned} S_F'(\vec{p},\omega) &= S_F(\vec{p},\omega) + S_F(\vec{p},\omega) \sum(\vec{p},\omega)\, S_F'(\vec{p},\omega) \\ &= S_F(\vec{p},\omega) + S_F(\vec{p},\omega) \sum(\vec{p},\omega)\, S_F(\vec{p},\omega) + S_F \sum S_F \sum S_F \\ &\qquad + \dots \end{aligned} \qquad (3.16)$$

or

$$S_F'(\vec{p},\omega) = [S_F(\vec{p},\omega)^{-1} - \sum\nolimits_{r_s}(\vec{p},\omega)]^{-1} \,. \qquad (3.17)$$

It is important to recognize that the sum of all single particle self energy processes has the effect of modifying the single particle propagator. The series of graphs which are formally summed in equation 3.16 is shown in fig. 6. The shaded bump represents the amplitude $\sum_{r_s}(\vec{p},\omega)$.

[The "forward scattering" self energy process (fig. 1b which results from the non-exchange interaction with the passive particles in the sea contributesan (infinite) negative constant to $\sum_{r_s}(\vec{p},\omega)$. This represents the interaction with the average electronic charge, i.e. the Coulomb energy. The term V_I in $\omega(p) = \frac{1}{2}p^2 + V_I$ which represents the interaction with the positive background charge cancels this constant. (see footnote on page 9)]

The physical picture of these self-energy effects is quite simple. An electron or hole moving through the medium of unexcited particles can polarize this medium in the manner discussed in Chapter II. The virtual polarization charge can then act back on the electron again through the Coulomb interaction. Unlike the self-energy processes in quantum electrodynamics, the electrons or holes can also undergo real, energy conserving, collisions of this type so that some of the intermediate states involved can lie on the energy shell.

The analytic properties of $S_F^{\prime}(p,\omega)$ will be important in some of our work. $\sum(p,\omega)$, and therefore $S_F^{\prime}(p,\omega)$, has a branch cut or discontinuity which lies along the real axis (displaced by $\pm i\epsilon$). This discontinuity arises from a continuous distribution of poles corresponding to the energies of the intermediate states involved. Along this discontinuity $\sum(p,\omega)$ has a non-zero imaginary part. In addition $S_F^{\prime}(p,\omega)$ has a pole corresponding to a zero in the denominator of equation 3.17 which occurs when

$$S_F^{-1}(p,Z_o) = \sum_{r_s}(p,Z_o) . \tag{3.18}$$

Clearly, the solution Z_o of this equation cannot lie on the real axis where $\mathrm{Im}\sum$ is non-zero. For $\omega \sim \frac{1}{2}$ we will show in Chapter VI that $\mathrm{Im}\sum(p,\omega)$ is proportional to $[\omega - \frac{1}{2} - O(r_s)]^2$ so that for some value of p the real solution $Z_o = \frac{1}{2} + O(r_s)$ is possible.

The corrected Feynman propagator $S_F'(p,\omega)$ yields all of the important physical information about the physical one particle state. As a guide, let us first examine the properties of the "bare" particle propagator $S_F(p,\omega)$. In Equation 1.12 $S_F(p,\omega)$ is divided into a real (or dispersive) part and an imaginary (or absorptive) part. The real part gives the virtual response of the electron field to a driving oscillation of frequency ω, while the imaginary part gives the resonant response for the frequency ω. Since the imaginary part contains a delta function of $\omega - \omega(p)$, the resonant response is very sharp, occurring only if $\omega = \omega(p)$. Hence the single particle energy may be defined as the frequency at which the imaginary part of $S_F(p,\omega)$ is at resonance.

The physical propagator determines the response of the interacting electron field to a driving source of frequency ω. In terms of real and imaginary parts we can write

$$S_F'(p,\ ) = \frac{[\omega-\omega(p) - \mathrm{Re}\sum(p,\omega)] - i[\mathrm{Im}\sum(p,\omega)]}{[\omega-\omega(p) - \mathrm{Re}\sum(p,\omega)]^2 + [\mathrm{Im}\sum(p,\omega)\ ]^2} \ . \qquad (3.19)$$

If we apply the same criterion for determining the energy W(p), of the physical electron state, we find that the frequency for resonance in the imaginary part is

$$W(p) = \omega(p) + \mathrm{Re}\sum_{r_s}(p,W(p)) \ , \qquad (3.20)$$

In general, $\mathrm{Im}\sum(p,\ W(p))$ is not zero so that the resonance is no longer sharp as was the case for $S_F(p,\omega)$. If, however, $\mathrm{Im}\sum(p,W(p))\ /\ W(p) \ll 1$ then the energy W(p) is well defined within a region measured by the level width which we define by*

$$\tfrac{1}{2}\Gamma(p) = \left| \mathrm{Im} \sum\nolimits_{r_s}(p,\ W(p)) \right| \; . \tag{3.21}$$

The physical single particle states are not exact eigenstates of the Hamiltonian H. Their energies are only defined within a range proportional to $\Gamma(p)$ which we shall see is the inverse lifetime of the single particle state. In order for these states to be well defined excitations of the many body system the condition $\Gamma/W \ll 1$ must be satisfied.

The finite lifetime of these states is a direct consequence of the degeneracy of the states Φ_n of the non-interacting system. This means that a single bare electron can be scattered by the many particle medium into another momentum state of lower energy. The energy difference is absorbed by the many body medium. Similarly, a hole can be scattered into a state of greater negative energy. In each case the final state which results from the scattering has energy equal to the initial state of a single created particle.

The number of final electron states of lower energy clearly approaches zero as the momentum p of the original electron approaches the Fermi momentum. Similarly, the number of hole states with greater negative energy approaches zero as the momentum of the original hole approaches the Fermi momentum. This means that $\Gamma(p) \to 0$ as p 1. In

---

[*] In our formalism, in which holes propagate backward in time, $\mathrm{Im}\sum(p,\ W(p))$ is positive for $p < 1$ and negative for $p > 1$. This is a direct result of the fact that the imaginary part of $S_F\ (p,\omega)$ differs in sign depending on whether $p < 1$ or $p > 1$. In Chapter VI this will be seen in detail.

Chapter VI we will derive the lowest order contributions to $\Gamma(p)$ and we will prove directly that $\Gamma(p) \propto (p - 1)^2$ for values of p near 1. Hence physical states representing an electron or hole excited very near the Fermi surface are very nearly stationary (i.e. meta-stable) states of the interacting system.

It should be clear that the energies W(p) of these meta-stable states are measured relative to the corrected ground state energy E_o. In our discussion so far we have omitted all the disconnected ground state diagrams which can occur along with these self energy diagrams. It is well known that the inclusion of these disconnected diagrams multiplies all matrix elements by a factor of e^{-iE_oT} (for large T) and thus shifts the energy by E_o.

The arguments in this section have been largely heuristic. In Appendix B we will derive these results in a more rigorous manner. We will also prove in Appendix B that an independent particle model can be formulated for the physical state which is the adiabatic transform of a state of several electrons and holes.

The idea here is simple. The self energy processes for a state of m particles can be divided into two classes:

a) Self energy processes involving only single particle lines with no interactions between lines. The result of these processes, we have seen, is to replace the bare single particle propagators S_F'. These diagrams take into account the interaction of a given excited particle with all of the unexcited particles in the Fermi sea. The excited particle polarizes the many body medium and the resulting induced charge acts back on the original particle. This polarization and reaction is incorporated in the effective interaction propagator $P_{r_s}(q,\omega)$.

b) Self energy processes involving interactions between particle lines. These processes measure the interaction energy between the n excited particles and give only a fraction m/N of the contribution from processes of type a). Thus for states with only a relatively few excited particles these interactions can be neglected.

It then follows that the energy of the physical state of ν electrons and ν' holes is given by equation 3.2. The level width or inverse lifetime of the many particle state is likewise shown to be

$$\Gamma_{\nu+\nu'} = \sum_{p_i > 1}^{\nu} \Gamma(p_i) + \sum_{p_i' > 1}^{\nu'} \Gamma(p_i') \qquad (3.22)$$

i.e. the level widths are additive.

Hence we conclude that only a knowledge of the corrected <u>single</u> particle propagator $S_F'(p,\omega)$ is required. For states of m excited particles we have shown then that an independent particle model holds to terms of order m/N, in agreement with Gell-Mann's assumption, equation 3.2. The significance of these states is clear only if the ratio of level width to energy is small. This requirement limits us to states where the excited particles have momenta close to the Fermi momentum ($p = 1$).

Now let us turn briefly to the question of how the Pauli Principle operates in intermediate states. As we mentioned above, if all the diagrams of a given order for a given S-matrix element are included it is immaterial whether or not we apply the exclusion principle to the intermediate states. For every diagram that violates the principle there is always an "exchange" diagram which also violates the principle and whose amplitude exactly cancels the first. In this discussion it is important to distinguish between the various time orderings of the interactions in the Feynman diagrams. The Feynman perturbation method automatically sums over all time orderings, some of which may violate the exclusion principle.

For example the self energy diagram in fig. 7a (in terms of the bare Coulomb interaction) has the time ordering shown which violates the PP. The "exchange" diagram contributes to the same order and also violates the PP. Both of these diagrams contribute the same amplitude but with opposite signs since they differ by one closed loop. Another pair of self energy diagrams which violate the PP and cancel in the same way is also shown in fig. 7

It is clear from these examples, and more complicated ones which the reader can construct, that intermediate states which violate the PP are included in our definition of the proper self energy processes. We have separated the ground state self energy processes from the other self energy processes and, as a result, the cancellation discussed above does not always occur.

D. ITERATIVE SOLUTION FOR W(p)

The solution W(p) of equation (3.20) can be generated by an iteration-perturbation method. A solution must, of course, be a perturbation expansion since $\sum_{r_s}$ itself is defined as a perturbation series, as in equation 3.15, in orders of the effective interaction. In addition equation 3.20 is a transcendental equation for W(p) which we can solve by iteration. This is clearly possible if perturbation theory is valid, for then the difference $W(p) - \omega(p)$ is a small number of order r_s. Thus we have the series of iterations:

$$\begin{aligned} 0)\quad & W = \omega \\ 1)\quad & W = \omega + \mathrm{Re}\sum(\omega) \\ 2)\quad & W = \omega + \mathrm{Re}\sum(\omega + \sum(\omega)\ ) \\ & \quad = \omega + \mathrm{Re}\sum(\omega) + \mathrm{Re}\sum{}'(\omega)\ \mathrm{Re}\sum(\omega) + \ldots \end{aligned} \tag{3.23}$$

where we have dropped the momentum index p for convenience. (Hence $\omega = \omega(p)$, $W = W(p)$) $\sum'(\omega)$ is defined as $\frac{d}{d\omega}\sum(\omega)$, etc.)

Terms involving derivatives such as $\mathrm{Re}\sum'(\omega)\ \mathrm{Re}\sum(\omega)$ in this series arise because the corrected energy W(p) should be used in $\mathrm{Re}\sum(W)$ as in equation 3.20. This is the same as saying that in proper self energy graphs the external lines carry the corrected energy. Likewise, the internal lines in these graphs can contain self energy parts which have the effect of replacing S_F by S_F'. If both of these effects are consistently treated together to any given order in r_s, no troubles arise from the repetition of energy denominators in processes such as figure 6C .

For example, the energy denominators which occur in the amplitude for figure 6c vanish as the momentum transferred by the scattering goes to zero. The effect of squaring the energy denominator gives rise to an extra power of the momentum transfer in the denominator. However if we consider the iteration term Re $\Sigma^{(1)\prime}(\omega) \cdot$ Re $\Sigma^{(1)}(\omega)$ along with this we find that there is a cancellation which removes this extra inverse power of the momentum transfer. This is evidently the simplest case of a general property which prevents divergences arising from the piling up of energy denominators.

A representation of the iteration terms by diagrams is useful for classifying these terms. It is not hard to show that all possible iteration terms are obtained from the proper self-energy graphs by inserting "interactions" (represented by an X) into the virtual electron-hole lines. These "interactions" carry no momentum and energy and contribute a factor of $+ \vec{p} \cdot \vec{q} \ \ \Sigma(P, \omega(p))$ if the momentum carried by the line is $\vec{p} - \vec{q}$.

IV. THE PLASMON AND THE POLARIZED INTERACTION

Some of the most important states of the interacting electron gas cannot be put into correspondence with single eigenstates of the non-interacting system. The states $\rho_q \Phi_o$, which are generated by operating on Φ_o with the density operator ρ_q (see equations 1.16 and 1.17), are not eigenstates of H_o. In fact the state

$$\rho_q \Phi_o = \sum_{\substack{|\vec{p}+\vec{q}|>1 \\ p<1 \\ s}} [a^+_{\vec{p}+\vec{q},s} \; b^+_{\vec{p},s} + a_{\vec{p}+\vec{q},s} \; b_{\vec{p},s}] \, \Phi_o \tag{4.1}$$

is a linear combination of one pair eigenstates with different energy eigenvalues. The propagator for this state of the non-interacting system was derived in Chapter I (equations 1.37 to 1.39) where it was pointed out that such a state represents a density fluctuation. It is clear from equation 1.37 that as $t_{21} \rightarrow \infty$ the probability that the non-interacting system remains in this state approaches zero.

An external charge moving through the gas is coupled to these states by virtue of the Coulomb interaction, i.e. these states are excited (either as real or virtual states) by a charged particle. The Coulomb interactions between the particles of the many-body medium alter these states to produce a well-defined excited state of the interacting system (at high densities) which is the well-known plasmon excitation.

If at t_1 $(= -\infty)$ the state $\rho_q \Phi_o$ is excited, the state at time t_2 which develops as a result of the interaction is

$$\Psi_q(t_2) = U(t_2,t_1) \rho_q \Phi_o \tag{4.2}$$

in the interaction representation. If we use the method discussed in Appendix B we can show that the "physical" propagator which results from all possible interactions leading from the state $\rho_q \Phi_o$ and back is given by the matrix element

$$(\Phi_o|T(S\,\rho(\vec{x}_2,t_2)\,\rho(\vec{x}_1,t_1))|\,\Phi_o) \qquad (4.3)$$

which reduces to equation 1.37 when $H_1 \to 0$ so that $S = U(\infty, -\infty) = 1$. In the limit as $t_2 \to \infty$, $t_1 \to -\infty$ we can write equation 4.3 as

$$\int \frac{d^3\vec{q}}{(2\pi)^3}\, e^{-i\vec{q}\cdot\vec{x}_{21}}\, (\rho_q(t_2)\,\Phi_o|S|\,\rho_q(t_1)\,\Phi_o)\ . \qquad (4.4)$$

Since in this limit the only non-zero S-matrix elements are between states of equal energy we can write the matrix element in equation 4.4 in the form

$$(\rho_q\,\Phi_o|e^{-iH_o t_{21}}\,U(t_2,t_1)|\,\rho_q\,\Phi_o) = (\rho_q\,\Phi_o|\Psi_q(t_2)). \qquad (4.5)$$

Thus if $\Psi_q(t_2)$ contains a stationary state component the propagator should be of the form $c\ e^{-iEt_{21}}$ as $t_{21} \to \infty$, where E is real and c is the projection of the stationary state on $\rho_q \Phi_o$.

If we decompose the matrix element in equation (4.4) using Wick's theorem we find that for large t_{21}

$$(\rho_q(t_2)\,\Phi_o|S|\,\rho_q(t_1)\,\Phi_o) = e^{-i\Delta E_o t_{21}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} e^{-i\omega t_{21}}\, [Q_{r_s}(q,\omega) - \frac{\alpha r_s}{\pi^2 q^2}\, Q^2_{r_s}(q,\omega) - (\frac{\alpha r_s}{\pi^2 q^2})^2\, Q^3_{r_s}(q,\omega) + \cdots] \qquad (4.6)$$

$$= e^{-i\Delta E_o t_{21}} \int_{-\infty}^{\infty} \frac{d}{2\pi i}\, e^{-i\omega t_{21}}\, \frac{Q_{r_s}(q,\omega)}{1 + \frac{\alpha r_s}{\pi^2 q^2}\, Q_{r_s}(q,\omega)}\ . \qquad (4.7)$$

The factor $e^{-iE_o t_{21}}$, where E_o is the corrected ground state energy, arises from separating out the disconnected ground state diagrams. We encountered the remaining series of diagrams, which are summed in equation 4.6, in Chapter II where we discussed the polarized interaction. If we write

$$P_{r_s}(q,\omega) = \frac{1}{q^2 + \frac{\alpha r_s}{\pi^2} Q_{r_s}(q,\omega)} = \frac{1}{q^2} - \frac{\alpha r_s}{\pi^2 q^2} \frac{Q_{r_s}(q,\omega)}{1 + \frac{\alpha r_s}{\pi^2 q^2} Q_{r_s}(q,\omega)} \qquad (4.8)$$

it is clear that the second term on the right is the effect of the total induced charge. From this equation the relation of equation 4.7 to the effective interaction is apparent.

The Coulomb interaction modifies the propagator Q for density fluctuations in two fundamentally different ways. First there are what we might call the "local field corrections" which change Q_o to Q_{r_s}. These were discussed in Chapter II in some detail. Secondly, there are renormalization effects which arise from the repetition of the interaction. These account for the denominator $1 + \frac{\alpha r_s}{\pi^2 q^2} Q(q,\omega)$ in equation 4.7 and are responsible for the plasmon excitation.

We can understand the introduction of the plasmon excitation by arguing heuristically in terms of diagrams. The corrected propagator for density fluctuations, equation 4.4 or 4.7 gives the response of the density field when a disturbance of momentum q and frequency ω is made, say by a test charge. In the pair approximation we know, from our discussion in Chapter II, that the disturbance can be described as the repeated excitation and de-exitation of pair states of momentum q. These pairs have a continuum of energies ranging from o to $\frac{1}{2}q^2 + q$. If the frequency ω of the test charge lies within this range of pair energies we see that it is possible for the disturbance to drastically alter its character by exciting a pair in a real, energy conserving, transition. However, if $\omega > \frac{1}{2} q^2 + q$ such a "decay" is not possible and the disturbance can only continue to excite and de-excite virtual pairs. Such a disturbance which continues to propagate without altering its characteristics corresponds to a resonant excitation of the medium. Its mathematical manifestation is that the denominator $1 + \frac{\alpha r_s}{\pi^2 q^2} Q_o(q,\omega)$ has a pole on the real ω axis.

Outside of the pair approximation the situation is complicated by the local field corrections to Q_{r_s}. Since multiple pair (and plasmon) excitations are now coupled into Q_{r_s} it becomes possible for a disturbance of frequency $\omega > \frac{1}{2} q^2 + q$ to decay into a state of two or more pairs. Hence the plasmon excitation will damp out at all frequencies. However for small enough values of r_s these higher order decay modes are weak enough so that the plasmon excitation is still well defined for $\omega > \frac{1}{2} q^2 + q$. We shall now make these statements precise by analyzing the mathematical properties of the propagator.

The integral over ω in equation 4.7 can be evaluated by contour integration. The polarization propagator $Q_{r_s}(q,\omega)$ has a branch discontinuity which runs from $\omega = o$ to $\omega = \infty$ displaced by $-i\varepsilon$ from the real ω axis and a discontinuity from $\omega = o$ to $\omega = -\infty$ displaced by $+i\varepsilon$ from the negative real axis (see figure 8). We discussed these singularities in Chapter II where we saw that on crossing the branch cut, $\mathrm{Im}Q_{r_s}(q,\omega)$ changes sign. The principal branch of this many valued function is defined by letting $\varepsilon \to o$ in the expressions for $Q_{r_s}(q,\omega)$ which are given by the Feynman rules. This gives the limiting value of the function on one side of the branch cut. The principal branch is thus defined by

$$\lim_{\delta \to o} \ \mathrm{Im}\, Q_{r_s}(q,x \pm i\delta) = \mp \gamma_{r_s}(q,x)$$

where $\gamma_{r_s}(q,x)$ is a positive function of x which is determined from the Feynman rules. For example, in the pair approximation where $Q_{r_s} = Q_o$ we have from equation (1.39)

$$\gamma_o(q,x) = \pi \int_{\substack{p<1 \\ |p+q|>1}} d^3p \ \delta(x - \tfrac{1}{2}\, q^2 - \vec{q}\cdot\vec{p}) \ ,$$

Next we analytically continue $Q_{r_s}(q,\omega)$ to complex values of $\omega = z$. In addition to the branch discontinuities of $Q_{r_s}(q,z)$ the integrand of equation 4.7 has α pole at $z = z_o$ where

$$1 + \frac{\alpha r_s}{\pi^2 q^2} \ Q_{r_s}(q,z_o) = o \ . \tag{4.9}$$

Since $\mathrm{Im}Q_{r_s}(q,z)$ is not zero, in general, for real z it is clear that the solution z_o of this equation does not lie on the real axis. It is readily shown, using the same arguments as given in Appendix B for the poles of the one electron propagators, that the pole at $z = z_o$ (or $z = \bar{z}_o$) does not lie on the principal branch of $Q_{r_s}(q,z)$ but instead lies on a neighboring Riemann sheet.

Hence, to evaluate equation 4.7 by contour integration we only enclose the branch lines when we deform the contour in the principal branch. For $t > o$ $(t < o)$ we close the contour in the lower (upper) half plane. In figure 8 the relationship of the contours and branch lines is shown explicitly. Since the only discontinuity in $Q_{r_s}(q,z)$ on crossing the branch is in the imaginary part we find using equations 4.7 and 2.6:*

$$\langle \rho_q \Phi_o | e^{-iHT} | \rho_q \Phi_o \rangle = e^{-iE_oT} \int_o^\infty \frac{d\omega}{2\pi i} e^{-i\omega T} \frac{2i\mathrm{Im}Q_{r_s}(q,\omega)}{|K_{r_s}(q,\omega)|^2} .$$

In terms of the real and imaginary parts of Q we have

$$\langle \rho_q \Phi_o | e^{-iHT} | \rho_q \Phi_o \rangle = e^{-iE_oT} \int_o^\infty$$

$$\frac{d\omega}{2\pi} \frac{2\mathrm{Im}Q(q,\omega)\ e^{-i\omega T}}{\left[1 + \frac{\alpha r_s}{\pi^2 q^2} \mathrm{Re}Q(q,\omega)\right]^2 + \left[\frac{\alpha r_s}{\pi^2 q^2} \mathrm{Im}Q(q,\omega)\right]^2} . \qquad (4.11)$$

The integrand of this equation becomes sharply peaked at $\omega = \Omega(q)$ where

$$1 + \frac{\alpha r_s}{\pi^2 q^2} \mathrm{Re}Q_{r_s}(q,\Omega(q) = o . \qquad (4.12)$$

---

*For the remainder of this section we will define $t_{21} = T$ and abbreviate the operator $e^{-iH_o t_{21}} U(t_2,t_1)$ by e^{-iHT}.

Let us define

$$(\omega^2 - \Omega^2(q) \quad B_{r_s}(q, \omega^2 - \Omega^2(q)$$

$$= q^2 + \frac{\alpha r_s}{\pi^2} \mathrm{ReQ}(q,\omega) \tag{4.13}$$

so that

$$B_{r_s}(q,o) = \frac{\alpha r_s}{\pi^2} \left[\frac{d}{d\omega^2} \mathrm{ReQ}_{r_s}(q,\omega) \right]_{\omega = \Omega(q)} \tag{4.14}$$

and further define

$$\tfrac{1}{2}\Gamma(q) = \frac{\alpha r_s}{\pi^2} \frac{1}{2\Omega(q)} \frac{1}{B(q,o)} \mathrm{ImQ}_{r_s}(q,\Omega(q)) \ . \tag{4.15}$$

Then if the conditions $\Gamma(q)\ T \ll 1$ and $\Gamma(q)/\Omega(q) \ll 1$, are satisfied we find that (for $T \to \infty$)

$$\langle \rho_q \Phi_o | e^{-iHT} | \rho_q \Phi_o \rangle \underset{T\to\infty}{=} e^{-i(E_o + \Omega(q))\ T} \ e^{-\frac{1}{2}\Gamma(q)T} \cdot \frac{q^2}{2\Omega(q)\ B(q,o)} \ . \tag{4.16}$$

If $\Gamma(q)$ is zero then it follows from the unitarity of the operator e^{-iHT} that there are no matrix elements of this operator connecting the state $\rho_q \Phi_o$ with any state orthogonal to it so we have

$$e^{-iHT} \rho_q \Phi_o = \frac{\langle \rho_q \Phi_o | e^{-iHT} | \rho_q \Phi_o \rangle}{\langle \rho_q \Phi_o | \rho_q \Phi_o \rangle} \rho_q \Phi_o$$

from which it further follows that $E_o + \Omega(q)$ is an eigenvalue of the

Hamiltonian H.

Thus we see from this development that the adiabatic transformation of the state $\rho_q \Phi_o$ generates an approximate eigenstate of the interacting system. This state or excitation is the well known plasmon excitation. It is a resonant oscillation of the density field. The nature of the density fluctuations are modified in a fundamental way by the interaction which introduces a mode of oscillation which has no counterpart in the non-interacting system.

Equation (4.12) is the generalized dispersion relation for the plasmon energy $\Omega(q)$, while equation (4.15) gives the level width of the plasmon state. The concept of a discrete state is possible only if $\Gamma / \Omega << 1$. This coupled with the condition that $\Gamma T << 1$ means that the plasmon decays only by a small amount while the energy time dependence of its state vector undergoes many oscillations.

Let us remark in passing that to lowest order in r_s the values of $\Omega(q)$ and $\frac{1}{2}\Gamma(q)$ can be found directly from the real and imaginary parts, respectively, of the complex number z_o at which the propagator has a pole (see equation 4.9).

The properties of the plasmon in the pair approximation have been studied extensively. If we approximate Q_{r_s} by Q_o (equations 1.39 to 1.48) we see that the dispersion relation, equation 4.12, is exactly the same as that discussed by Bohm-Pines [5], Ferrell [26], Sawada et al. [9] and others. Equation 4.12 can easily be solved in this case as a power series in q. The result is (see Ferrell [26]).

$$\Omega_o^2(q) = \Omega_p^2 \left[1 + \frac{12}{5} \frac{q^2}{\Omega_p^2} + \left(- \frac{6}{35} \frac{1}{\Omega_p^4} + \frac{1}{2} \frac{1}{\Omega_p^2} \right) q^4 + \ldots \right] \qquad (4.17)$$

where Ω_p is the classical plasma frequency, in our units

$$\Omega_p = \left(\frac{4\alpha r_s}{3} \right)^{1/2} .$$

From equation 1.44 we see that $\mathrm{Im} Q_o = o$ for $\omega > \frac{1}{2} q^2 + q$. Thus limiting energy corresponds to the maximum energy of a single (bare) pair of momentum q. If $\Omega(q) > \frac{1}{2} q^2 + q$ we see from equation 4.15 that the plasmon has an infinite lifetime, i.e. $\Gamma(q) = o$. Equation 4.17 shows that this is indeed the case for small enough q. However, as q increases the plasmon energy finally merges at $q = q_c$ with the upper limit of the pair continuum of energies. The value of q_c for a given value of r_s is found by solving equation 4.12 for q with $\omega = \frac{1}{2} q^2 + q$. This natural cut-off momentum was first discussed by Ferrell [26]. He has shown that for very small r_s

$$q_c \; \alpha \; r_s^{\frac{1}{2}} \left(\ln \frac{1}{r_s} \right)^{\frac{1}{2}} .$$

For values of $q > q_c$ the plasmon has a finite lifetime since it can decay by producing a single pair. From equations 4.15, 4.14, 1.45 and 1.44 we easily find that for $q > q_c$

$$\Gamma_o(q) = \left[P \int_{p<1} d^3p \; \frac{[q^2/2 + \vec{q}\cdot\vec{p}]}{\left[\Omega_o^2 - (q^2/2 + \vec{q}\cdot\vec{p})^2\right]^2} \right]^{-1} \text{times} \qquad (4.18)$$

$$\times \frac{4\pi^2}{q} \begin{cases} 1 & ; \text{ for } \frac{q}{2} + \frac{\Omega_0}{q} < 1 \\ \frac{\left\{1 - [q/2 - \Omega_0/q]^2\right\}}{2\Omega_0(q)} & ; \text{ for } \left| q/2 - \Omega_0/q \right| < 1 < q/2 + \Omega_0/q \end{cases} .$$

It is clear that for $\Omega_0 \ll \frac{1}{2} q^2 + q$, $\Gamma_0(q)$ approaches a value independent of r_s. The requirement that $\Gamma_0(q)/\Omega_0(q) \ll 1$ can only be satisfied for $\Omega_0 \approx \frac{1}{2} q^2 + q$, that is, very near the cut-off so that beyond the cut-off the plasmon state is no longer well defined. The plasmon is effectively absorbed by the continuum of pair states for $q > q_c$. The sharpness of this cut-off is a function of r_s. We will not analyze this further since we will see in Chapter VI that for physically realizable densities, this cut-off is masked by other effects.

In the general case Im Q_{r_s} is non-zero for all values of ω, since states of any number of pairs and plasmons are coupled into the density fluctuations. Hence the plasmon state is never a true stationary state. However, for small enough values of r_s we expect Im Q_{r_s} to be smaller for $\omega > \frac{1}{2} q^2 + q$, where only higher order multiple excitations contribute to Im Q_{r_s}, than for $\omega < \frac{1}{2} q^2 + q$ where single pair excitations also contribute. In Chapter V we will calculate the lowest order contribution to the plasmon lifetime for values of $q > q_c$, and in Chapter VI we will estimate the effect of the finite level width on the observed cut-off momenta in plasmon excitation by electron scattering. Also in Chapter V we will calculate the shift in the plasmon energy due to higher order local field corrections.

To complete this Chapter let us investigate some further properties of the propagator $P_{r_s}(q, \omega)$ for the polarized interaction in the light of what we have learned about the plasmon. From equations 4.8, 4.7, 4.10 and 2.6 we find that we can write

$$P_{r_s}(q,\omega) = \frac{1}{q^2} - \frac{1}{q^2}\int_o^\infty \frac{d\omega_1}{2\pi i}\,\frac{2\omega_1}{\omega^2-\omega_1^2}\,2i\ \mathrm{Im}\,\frac{1}{K_{r_s}(q,\omega_1)} \qquad (4.19)$$

where ω_1 is understood to have a negative imaginary part so that the Fourier transform over T is defined. In the pair approximation we can write this explicitly as

$$P_{r_s}(q,\omega) = \frac{1}{q^2} - \frac{\alpha r_s}{\pi^2 q^4}\int_{\substack{p<1\\ |\vec{p}+\vec{q}|>1}} d^3p\,\frac{2[\frac{q^2}{2}+\vec{q}\cdot\vec{p}]}{\omega^2-(\frac{q^2}{2}+\vec{q}\cdot\vec{p})^2}\,\frac{1}{\left|K(q,\frac{1}{2}q^2+\vec{q}\cdot\vec{p})\right|^2}$$

$$- \frac{1}{B_{r_s}(q,o)}\,\frac{\eta(q_c-q)}{\omega^2-\Omega_o^2(q)} \,. \qquad (4.20)$$

Here we have used equations 4.12 and 4.13 as well as equation 1.44 for $\mathrm{Im}\,Q_o(q,\omega)$. In the limit as $\varepsilon \to o$ the plasmon pole contributes a delta function for $q < q_c$ which accounts for the last term. $\Omega_o(q)$ is given by equation (4.17) and $B_{r_s}(q,o)$ is found from equation 4.14. In the pair approximation we have

$$B_{r_s}(q,o) = \frac{\alpha r_s}{\pi^2}\left[\frac{d}{d\omega^2}\,\mathrm{Re}\,Q_o(q,\omega)\right]_{\omega=\Omega_o(q)} \qquad (4.21a)$$

$$= \frac{\alpha r_s}{\pi^2}\,2P\int_{p<1} d^3p\,\frac{[q^2/2+\vec{q}\cdot\vec{p}]}{[\Omega_o^2(q)-(q^2/2+\vec{q}\cdot\vec{p})^2]^2} \qquad (4.21b$$

$$= \frac{q^2}{\Omega_p^2} - \frac{9}{5}\,\frac{q^4}{\Omega_p^4} + O\left(\frac{q^6}{\Omega_p^6}\right) . \qquad (4.21c)$$

The expansion in powers of $q^2/\Omega_p{}^2$ is valid only if $q \ll q_c$. As $q \to q_c$ $B_{r_s}(q,o)$ is a rapidly increasing function.

The interpretation of the three terms in equation 4.20 follows from our previous remarks. The first term is the bare Coulomb interaction. The second and third terms represent the interaction with the polarization charge; the coupling and propagation of the continuum of pair states and the plasmon state.

If we compare the second term in equation 4.20 with equations 1.40 and 1.42 we see that we can interpret the effect of the repeated interaction as modifying the coupling of the single pair excitation to charges. In the pair approximation only these modified pair excitations make up the continuum of states. Their effective coupling is

$$\frac{g^2(p,q)}{(2\pi)^3} = \frac{1}{2}\left(\frac{\alpha r_s}{\pi^2}\right)^2 \frac{1}{[q^2 + \frac{\alpha r_s}{\pi^2}\,\mathrm{Re}\,Q_o(q,\frac{1}{2}q^2+\vec{q}\cdot\vec{p})]^2 + [\frac{\alpha r_s}{\pi^2}\,\mathrm{Im}\,Q_o(q,\frac{1}{2}q^2+\vec{q}\cdot\vec{p})]^2} \tag{4.22}$$

instead of $\frac{1}{2}\left(\frac{\alpha r_s}{\pi^2}\right)^2 \frac{1}{q^4}$ as in equation 1.42.* In this approximation the pair propagator $e^{-i|t|(q^2/2+\vec{q}\cdot\vec{p})}$ is unchanged since there are no self energies for the one pair state. The modified interaction does not diverge for small q, instead we have from equation

$$\frac{g^2(p,q)}{(2\pi)^3} \xrightarrow[q\to 0]{} \frac{1}{2}\left(\frac{\alpha r_s}{\pi^2}\right)^2 \frac{1}{[q^2 + \frac{4\alpha r_s}{\pi}(1 - \frac{x}{2}\ln\left|\frac{1+x}{1-x}\right|)]^2 + [4\alpha r_s \frac{x}{2}]^2} \tag{4.23}$$

where $x = \hat{q}\cdot\hat{p}$. The interaction in the spacial representation is thus screened but the "screening length" depends on the states of the

---

*The factors of $(2\pi)^{-3}$ are chosen so that the usual Feynman rules apply.

interacting pair excitations (i.e. on p as well as q). Thus we have a refinement of the Bohm-Pines theory. The effect of the polarization of the medium is to introduce a screened interaction between particle-like excitations. Along with this a new type of excitation, the plasmon, is introduced and, furthermore, a second subsidiary condition is imposed. The new subsidiary condition is the restriction that only a single pair excitation can be coupled in this way to charges. The repeated annihilation and creation of pairs is included in the modified coupling constant. This subsidiary condition along with the subsidiary condition (II) on page 30 are probably related to Bohm-Pines subsidiary conditions. The exact connection, however, was not clear at the time of this writing.

In the present theory the cut off q_c for the plasmon state is introduced in a more natural and satisfactory way than in the Bohm-Pines approach. In the pair approximation the plasmon is a distinct eigenstate of the system when $q < q_c$. For $q > q_c$ the plasmon is no longer an eigenstate of the interacting system but can decay into the single pair states which form the continuum. The greater exactness of this method is at the expense of adding a more complicated screened Coulomb interaction between particle-like pair excitations. The subsidiary conditions in our approach are easily taken into account by the simple omission of certain Feynman diagrams.

<u>The real simplicity of the Gell-Mann-Brueckner method lies not in this separation into continuum and plasmon parts but, on the contrary, in the fact that both are included in the propagator</u> $P_{r_s}(q,\omega)$, as given by equation 2.3 . This greatly simplifies the calculation

of the properties of the interacting electron gas at high densities. A discussion of this separation is necessary, however, to show the connection of our approach and the Bohm-Pines theory.

In higher order approximations for $Q_{r_s}(q,\omega)$ more complicated states with various numbers of pairs are coupled in the continuum and the dispersion relation for the plasmon is altered. If self-energy effects are included to all orders, the pair states in the continuum have finite lifetimes. This points out the importance of the meta-stable pair states discussed in Chapter III; these are the states that are actually excited when a charged particle interacts with the electron gas. We will not discuss these higher approximations in a general manner but rather as corrections to the pair approximation.

Now let us consider the plasmon term in more detail. The plasmon, as defined by the pair approximation, we will call the "bare" plasmon. This is consistent with our previous definitions of "bare" particle states since this is the lowest order approximation in which the plasmon state occurs; in this approximation the plasmon has an infinite lifetime.

The division of the plasmon term into propagator and coupling constant is somewhat arbitrary. However, if the plasmon propagator is taken to be of the boson type

$$\frac{-i}{(2\pi)^4} D_F(q,\omega) = \frac{1}{(2\pi)^4} \frac{-i}{\omega^2 - \Omega^2(q)} \tag{4.24}$$

then the coupling of a bare plasmon with a charge Z is in our special units

$$\frac{g_p^2(q)}{(2\pi)^3} = Z^2 \frac{\alpha r_s}{2\pi^2} \frac{1}{\mathcal{B}_{r_s}(q,0)} \tag{4.25}$$

$$= Z^2 \frac{3}{8\pi} \frac{\Omega_p^4}{q^2} \left[1 + \frac{5}{9} \frac{q^2}{\Omega_p^2} + \ldots\right] \tag{4.25a}$$

for $q \quad q_c$. [The factor $-i/(2\pi)^4$ in equation 4.21 is chosen so that the usual Feynman rules apply: A factor i from rule 5' and a factor $(2\pi)^4$ from the delta functions at each vertex cancel this factor.]

For applications involving the virtual excitation of pairs and plasmons the separation of $P_{r_s}(q,\omega)$ into plasmon and continuum parts is neither physically meaningful or mathematically useful. However, if we consider the excitation of real pairs and plasmons the separation is useful. The amplitude for the excitation of a real pair state is the same as that for the excitation of a real pair state of the non-interacting system since in the pair approximation there are no self energies for the pairs. Only the coupling constant is changed to $g(p,q)$. The amplitude for the emission or absorption of a plasmon is evidently the same as that for a boson since it propagates like a boson. This amplitude is

$$\frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\Omega_o(q)}} . \tag{4.26}$$

In Appendix B the general relation of these amplitudes to the propagators is discussed. Simple amplitudes like equations 4.25 and 4.26 cannot be used in higher order approximations where the excitations have finite lifetimes. For most of our applications however, equations 4.25 and 4.26 will suffice.

PART TWO

CALCULATION OF EXCHANGE EFFECTS ON THE PROPERTIES OF AN ELECTRON GAS

INTRODUCTION

We will now apply the theory developed in Part I to the calculation of various properties of a gas of interacting electrons. In particular we will extend the calculations of G-B and others to lower densities. To do this we must consider "exchange" corrections to the results of the pair approximation.

Before proceeding with these calculations we should point out that our method of calculation will follow that of G-B. We will expand expressions in orders of r_s being careful to retain all terms of a given order. The expansions in orders of the polarized interaction, which we discussed in Part One, are much too complicated to carry out completely. In actual calculations we will simplify the expression for the polarized interaction to yield expansions exact only to some low order of r_s. This means that screening effects will only be treated exactly for small momentum transfers. Hence, those terms, which are infinite in an ordinary perturbation expansion, will now become finite functions of r_s because of the screening effects. Finite terms will remain unchanged.

V. PLASMON DISPERSION CORRECTIONS: LEVEL SHIFT AND LEVEL WIDTH

A. Self Energy of the Bare Plasmon

In the preceding chapter we discussed the properties of the "bare" plasmon which was defined by the pair approximation. In the high density limit ($r_s \to 0$) this approximation is, in fact, exact. However for the densities encountered in metals ($r_s > 1$) the validity of this approximation has not been demonstrated. In this chapter we will calculate the lowest order corrections to the plasmon dispersion formula, equation 4.9.

In the pair approximation the virtual pairs which make up the plasmon can only be excited or de-excited. In higher approximations the pairs can interact with each other or with the particles in the Fermi sea so that intermediate states with any number of pairs are possible. These higher order interactions will change the plasmon energy by an amount which we will call the self-energy of the "bare" plasmon and, since a plasmon can decay into states of two or more pairs, the plasmon will have a finite lifetime.

We will calculate the self-energy and the reciprocal lifetime to lowest order in the coupling αr_s so that, again, our results will only be valid for high densities. However, the size of these corrections is a measure of how fast the perturbation method converges.

The plasmon dispersion relation, equation 4.9, may also be written as*

---

* In this chapter we will use the fact, mentioned in Chapter IV, that in lowest order, the plasmon level width can be found directly from the real and imaginary parts of the complex value of w at which $P_{r_s}(q,w)$ has a pole (i.e. the value $\check{\Omega} \equiv z_o$ defined by equation 4.9.)

$$[\omega^2 - \Omega_o^2(q)] = -\frac{\alpha r_s}{\pi^2} \frac{Q_{r_s}(q,\omega) - Q_o(q,\omega)}{B_{r_s}(q,\omega^2 - \Omega_o^2)} \qquad (5.1)$$

where $\Omega_o(q)$ is the "bare" plasmon energy (equation 4.17) and $Br_s(q,\omega^2 - \Omega_o^2)$ is defined by equations 4.14 and 4.21. We can solve this by iteration; the first iteration gives

$$[\check{\Omega}_1^2(q) - \Omega_o^2(q)] = -\frac{\alpha r_s}{\pi^2} \frac{Q_{r_s}(q,\Omega_o) - Q_o(q,\Omega_o)}{Br_s(q,0)} \qquad (5.2)$$

or

$$\check{\Omega}_1(q) \cong \Omega_o(q) - \frac{\alpha r_s/\pi^2}{2\Omega_o(q)} \frac{[Q_{r_s}(q,\Omega_o) - Q_o(q,\Omega_o)]}{Br_s(q,0)} \ . \qquad (5.3)$$

Now $\check{\Omega}_1(q) = \Omega_1(q) + i\,\Gamma_1(q)/2$ and we can calculate the lowest order corrections Ω_1 and Γ_1 to the plasmon energy and level width from equation 5.3. We can also write equation 5.2 in terms of the electron-plasmon coupling of equation 4.25

$$[\check{\Omega}_1^2 - \Omega_o^2(q)] = -\frac{2g_p^2(q)}{(2\pi)^3}[Q_{r_s}(q,\Omega_o) - Q_o(q,\Omega_o)] \ . \qquad (5.4)$$

Let us first calculate the lowest order correction to the plasmon energy, i.e. the real part $\Omega_1(q) = \mathrm{Re}\ \check{\Omega}_1(q)$. This arises from the lowest order corrections to $Q_{r_s}(q,\Omega_o)$ which correspond to the diagrams (b) and (c) in figure 3. These corrections take into account the possibility that the virtual pairs, which make up the induced polarization charge, interact with each other or with the passive particles in the sea before de-exciting. In Appendix A we derive an expression for the propagator $Q^{(1)}_{r_s}(q,\omega)$ which takes these effects into account to lowest order in r_s. Since we will be interested in small values of q we can use equation A5 for $Q^{(1)}_{r_s}(q,\omega)$. Substituting in equation (5.3)

$$\Omega_1(q) - \Omega_o(q) = - \frac{\alpha r_s/\pi^2}{2\Omega_o(q)} \; \frac{Q^{(1)}_{r_s}(q,\Omega_o)}{B_{r_s}(q,0)} + \text{higher order terms in } r_s \tag{5.5}$$

expanding $Q^{(1)}_{r_s}(q,\Omega_o)$ (equation A5) in powers of q^2/Ω_p^2 we have

$$Q^{(1)}_{r_s}(q,\Omega_o) = (2\pi)^2 \; \frac{\alpha r_s}{4\pi^2} \; \frac{3q^4}{\Omega_p^4} \; \frac{4}{15} + 0(\frac{q^6}{\Omega_p^6}) \; . \tag{5.6}$$

Using this and equation 4.16b for $B_{r_s}(q,0)$ we get

$$\Omega_1(q) - \Omega_o(q) = - \frac{3}{40} \; q^2 \, \Omega_p \cong 0.037 r_s^{1/2} \, q^2 \; . \tag{5.7}$$

Thus the correction factor for the coefficient of the q^2 term in equation 4.11 for $\Omega_o(q)$ is

$$1 - \frac{1}{16} \Omega_p^2 = 1 - \frac{\alpha r_s}{12\pi} = 1 - 0.0138 r_s \; . \tag{5.8}$$

For $r_s = 2$ (nearly the value for A1) this amounts to a 2 percent correction to the dispersion coefficient. The coefficient of r_s in equation 5.8 is a factor of ten smaller than the approximate result found by Ferrell * [26] but the sign of the correction is the same. Our result also agrees

---

* Previous to our calculation Professor Ferrell pointed out to the author that an exact evaluation of this correction would give a smaller result (private communication).

qualitatively with an estimate of Pines [27], whose result was of the same magnitude as Ferrell's, and a result of Wolff's[28] which was not numerically evaluated.

The dispersion coefficient for the q^2 term has been measured experimentally by Watanabe [29], who studied the angular dependence of the characteristic energy loss,* $\hbar\Omega_p$, of 25-kev electrons passing through metal foils. In Table I we reproduce a table from Ferrell's paper which compares the theoretical values with experiments. The theoretical values, however, are calculated from our equation 5.8. Since our correction is much smaller than Ferrell's original correction, the agreement with experiment is slightly better. The disagreement is very likely due to the departure of the real metallic electrons from free electron behavior due to interactions with the positive ion lattice.

Although our result does not agree quantitatively with experiment it does give an indication of the convergence of the theory. For values of $r_s < 5$ the correction is less than 10 percent. If we assume that the series converges uniformly we expect, then, that the rest of the terms in the series, of order $\alpha^2 r_s^2$ and higher are negligible for these values of r_s. It seems safe to say that the high density limit, for this calculation, is accurate for the metals with highest electron densities. In Appendix A we discuss, briefly, the calculation of higher order corrections to $Q_{r_s}(q,\omega) - Q_o(q,\omega)$.

We can also derive this result in another way by considering the lowest order self energy processes for a bare plasmon. The irreducible Feynman diagrams would then be those in figure 9. The corrected propagator

---

* In the next chapter we shall consider the problem of electron scattering from metals.

for the plasmon then becomes

$$D_F'(\vec{q}, \omega) = \frac{1}{\omega^2 - \Omega_o^2(q) - \pi_{r_s}(q,\omega)} \tag{5.8}$$

where

$$\pi_{r_s}(q, \omega) = -\frac{2g_p^2(q)}{(2\pi)^3} Q_{r_s}^{(1)}(q,\omega) , \tag{5.9}$$

and the condition for the pole of the propagator $D_F'(\vec{q}, \omega)$ is identical to equation 5.1 in the approximation considered here. This serves as a check on the internal consistency of our formalism.

B. Plasmon Damping

The imaginary part of $\check{\Omega}_1(q)$, which is one half the level width, can also be computed from equation 5.3. However, the calculation of $\Gamma_1(q)$ for $q < q_c$ is much more difficult than the calculation of the lowest order energy shift $\Omega_1(q)$. As we point out in Appendix A, the amplitude $Q_{r_s}^{(1)}(q, \Omega_o)$ has a zero imaginary part to order r_s if $q < q_c$. This is because the plasmon cannot decay into a single pair if $q < q_c$ so we must go to order r_s^2 or higher in the amplitude $Q_{r_s}^{(1)}(q, \Omega_o)$, where there are intermediate states of two or more pairs. Thus to compute the imaginary part of $[Q_{r_s}(q, \Omega_o) - Q_o(q, \Omega_o)]$ exactly to order r_s^2 we must calculate the lowest order contribution from the amplitude $Q_{r_s}^{(2)}(q, \Omega_o)$ which is the sum of diagrams (d) to (f) in figure 3. These diagrams have two pair intermediate states and therefore contribute to the lowest order r_s^2 in the imaginary part. The analytic expression for

$Q_{r_s}^{(2)}(q, \Omega_o)$ is very complicated so we will proceed to evaluate $\Gamma_1(q)$ by an equivalent but more direct method.

In Appendix B we prove that the imaginary part of ω at the pole of the physical propagator is equal to one half (½) times the total probability of transition from the perturbed state, which is represented by the propagator, to all other perturbed states of (nearly) the same energy. This identification is valid only if $\Gamma/\Omega \ll 1$, which is the requirement that the state be well defined. To lowest order in r_s this transition probability is the same as the transition probability to all unperturbed states with the same energy as the unperturbed energy of the initial state. Thus we can compute $\Gamma_1(q)$ by calculating the transition probability for the lowest order process by which the plasmon can decay.

The lowest order process involving the plasmon-electron coupling is single pair production by a plasmon. As we have seen, energy can be conserved in this process only if $q > q_c$ and in this case the plasmon is absorbed in the continuum of single pair states. Thus the plasmon is not even an approximately independent mode for $q > q_c$ [26] [9]. For $q < q_c$ this decay mode does not exist for the free electron gas; in the pair approximation the plasmon is a stationary state for these momenta. For the electron densities found in metals higher order interactions are important and the plasmon can decay into two or more pairs for $q < q_c$. Double pair production is the lowest order process whereby the plasmon can dissipate its energy.

Two diagrams for this process are shown in figure 10. They represent two indistinguishable alternatives for the production of two pairs by an initial plasmon. Besides these two diagrams there are six more with the final electrons and holes interchanged in all possible permutations.

Since we are dealing with Fermions, interchanging electrons alone or holes alone gives a minus sign while interchanging both holes and both electrons gives a plus sign. (These signs, of course, follow from the S-matrix decomposition by Wick's theorem.)

The transition probability per unit time $\Gamma(q)$ for this process is given by the well-known formula

$$\Gamma(q) = \frac{1}{2\pi} \sum_{\substack{\text{final} \\ \text{states}}} |< 2 \text{ pairs}|M|1 \text{ plasmon} >|^2 \, \delta(\vec{p}_2 + \vec{p}_4 - \vec{p}_1 - \vec{p}_3 - \vec{q}) \, \delta\left[\tfrac{1}{2}(p_2^2 + p_4^2 - p_1^2 - p_3^2) - \Omega_o(q)\right] \quad (5.10)$$

where the matrix M is related to S by the equation

$$S - 1 = \delta(\vec{P}_i - \vec{P}_f)\,\delta(W_i - W_f)\,M \ . \qquad (5.11)$$

In this equation $\vec{P}_i$, W_i and $\vec{P}_f$, W_f are the initial and final total momenta and energies. The sum over final states is explicitly,

$$\sum_{\substack{\text{final} \\ \text{states}}} \rightarrow \sum_{S_1 S_2 S_3 S_4} \int_{p_1<1} d^3p_1 \int_{p_2>1} d^3p_2 \int_{p_3<1} d^3p_3 \int_{p_4>1} d^3p_4 \qquad (5.12)$$

where the sum over spin states is included. We can easily calculate the matrix element from the rules to get

$$< 2 \text{ pairs}|M|1 \text{ plasmon} > = (2\pi)^{-7/2} \frac{1}{\sqrt{2\Omega_o(q)}} \, g_p(q)\, 4\pi\alpha r_s \, i \qquad (5.13)$$

$$\left[\frac{(X_2^+X_1)(X_4^+X_3)}{(\vec{P}_4 - \vec{P}_3)^2}\, C(\vec{P}_1,\vec{P}_2) - \frac{(X_2^+X_3)(X_4^+X_1)}{(\vec{P}_4 - \vec{P}_1)^2}\, C(\vec{P}_3,\vec{P}_2) - \frac{(X_4^+X_1)(X_2^+X_3)}{(\vec{P}_2 - \vec{P}_3)^2}\, C(\vec{P}_1,\vec{P}_4) \right.$$

$$\left. + \frac{(X_4^+X_3)(X_2^+X_1)}{(\vec{P}_2 - \vec{P}_1)^2}\, C(\vec{P}_3,\vec{P}_4) \right]$$

where

$$C(\vec{P}_1,\vec{P}_2) = S_F(\vec{P}_1 + \vec{q},\, \omega(p_1) + \Omega_o(q)) + S_F(\vec{P}_2 - \vec{q},\, \omega(p_2) - \Omega_o(q))$$

$$= \frac{1}{\Omega_o - \Delta_{1q}} - \frac{1}{\Omega_o + \Delta_{2q}} \qquad (5.14)$$

with $\Delta_{1q} = \frac{1}{2} q^2 + \vec{q}\cdot\vec{P}_1$; $\Delta_{2q} = \frac{1}{2} q^2 - \vec{q}\cdot\vec{P}_2$. Since $\Omega_o(q) > \Delta_{1q}, \Delta_{2q}$ if $q < q_c$, we can obtain the expansion

$$C(\vec{P}_1,\vec{P}_2) = \frac{\vec{q}\cdot(\vec{P}_1 - \vec{P}_2)}{\Omega_p^2} + \frac{q^2}{\Omega_p^2} + \frac{(\vec{q}\cdot\vec{P}_1)^2 - (\vec{q}\cdot\vec{P}_2)^2}{\Omega_p^3} \qquad (5.15)$$

The factors of (2π) arise from $(2\pi)^{-3/2}$ for each ingoing or outgoing line, $(2\pi)^{-4}$ for both the internal electron-hole line and the Coulomb line, and $(2\pi)^4$ from the delta functions at each vertex. The other factors follow from the rules on pages 17 and 70 . All eight indistinguishable diagrams are included in the expression above.

In the expression above we have used the bare Coulomb interaction since conservation of momentum and energy prohibits zero momentum transfers in this interaction. This is just because for $q < q_c$ the plasmon cannot conserve energy and momentum in single pair production. Thus only short range collisions between electrons contribute to plasmon decay in agreement with arguments made by Bohm and Pines [6] [27]. The inclusion of polarization effects in this interaction will lead to higher order terms in r_s which we will neglect.

Fortunately, for high densities we do not have to evaluate Γ exactly since the "bare" plasmons are distinct only for small values of $q < q_c$. Thus we will evaluate equation 5.10 only to lowest order in q^2.

To do this we will use the expansion in equation 5.15 which is valid only if $\Omega_o(q)/(\frac{1}{2}q^2 + q) < 1$. Our results then will not be valid for values of q near the cut-off q_c. First, we integrate out the momentum conserving delta functions and introduce a vector variable $\vec{k}$ and define $\vec{P}_2 = \vec{P}_1 + \vec{k}$ and $\vec{P}_4 = \vec{P}_3 - \vec{k} + \vec{q}$. With these substitutions it is easily seen that the bracketed term in equation 5.13 is proportional to q^2. If we also replace $\vec{P}_3$ by $-\vec{P}_3$ we arrive at the expression

$$\Gamma(q) = \frac{1}{4}\frac{(3\pi^2)^3}{(2\pi)^8}\frac{\Omega_p^7}{q^2}\int d^3\vec{k} \int\limits_{\substack{p_1<1\\ |\vec{P}_1+\vec{k}|>1}} d^3p_1 \int\limits_{\substack{p_3<1\\ |\vec{P}_3+\vec{k}|>1}} d^3p_3\ \delta(k^2 + \vec{k}\cdot(\vec{P}_1+\vec{P}_3) - \Omega_p) \sum_{\text{spins}}$$

$$\frac{q^4}{\Omega_p^4}\left[\frac{(X_2^\dagger X_1)(X_4^\dagger X_3)}{k^2}\,[1 - 2(\vec{q}\cdot\vec{k})^2 - \frac{2}{\Omega_p}(\vec{q}\cdot\vec{k})(\vec{q}\cdot\bar{\vec{k}})] - \frac{(X_2^\dagger X_3)(X_4^\dagger X_1)}{\bar{k}^2}\,[1-2(\vec{q}\cdot\bar{\vec{k}})^2 - \frac{2}{\Omega_p}(\vec{q}\cdot\bar{\vec{k}})(\vec{q}\cdot\vec{k})]\right]^2 \tag{5.16}$$

where $\bar{\vec{k}} = \vec{k} + \vec{P}_1 + \vec{P}_3$. If we square the brackets, carry out the spin summations, and make use of the equivalence of $\vec{k}$ and $\bar{\vec{k}}$ to combine terms, the integrand becomes a difference of two terms. We can immediately carry out the integrations over $x = (\vec{q}\cdot\vec{k})$ and the azimuthal angle φ of $\vec{q}$ measured about $\vec{k}$.

$$\Gamma(q) = q^2 \Omega_p^3 \frac{27}{(2)^{10}} \frac{16\pi}{\pi^2} 2\int_0^\infty dk \int\limits_{\substack{p_1<1 \\ |\vec{P}_1+\vec{k}|>1}} d^3p_1 \int\limits_{\substack{p_3<1 \\ |\vec{P}_3+\vec{k}|>1}} d^3p_3 \; \delta(\vec{k}\cdot\vec{\bar{k}} - \Omega_p)$$

$$\frac{1}{k^2}\left[\frac{46}{15} + \frac{16}{15}\frac{k^2}{\Omega_p^2}\left(p_1^2 - (\vec{P}_1\cdot\vec{k})^2\right] - \text{terms of the form } \left(\frac{k}{\Omega_p}\right)^n . \qquad (5.17)$$

This expression for $\Gamma(q)$ is of the form $\Omega_p^3 f(\Omega_p)$ where $\Omega_p \propto r_s^{1/2}$. Higher order radiative corrections to two pair decay yield lowest order terms proportional to Ω_p^5 and three pair decay gives terms proportional to Ω_p^7. Since we are neglecting these higher order processes we will neglect terms in the expansion of $f(\Omega_p)$ of order Ω_p^2 and higher. This follows our general calculational procedure as discussed at the start of part II.

The integrals over p_1 and p_3 which are complicated by the Pauli Principle restrictions can be carried out according to a method outlined in appendix C. These integrals can be evaluated to terms proportional to Ω_p. From the results in equations C14 and C15 we see that the first term in brackets contributes terms of order 1 while the second or "exchange" term contributes terms of order Ω_p^2 (i.e. r_s). Since we are neglecting terms of order Ω_p^2 and higher so we will drop the exchange term. Thus

$$\Gamma(q) = q^2 \Omega_p^3 \frac{27\pi}{16} 2\int_0^\infty \frac{dk}{k^2}\left[\left(\frac{46}{15} + \frac{16}{15}\frac{k^2}{\Omega_p^2}\right) Z(\Omega_p, k)\right.$$

$$\left. - \frac{16}{15}\frac{k^2}{\Omega_p^2} W(\Omega_p, k) + O(\Omega_p^2)\right] \qquad (5.18)$$

where $Z(\Omega, k)$ and $W(\Omega, k)$ are given by equations C14 and C15 to first order in Ω . Finally, the integration over k can be carried out and we find

$$\Gamma_{(1)}(q) = q^2\Omega_p^2 \frac{3\pi}{40} [70 - 44 \ln 2 + 31\Omega_p + 0(\Omega_p^2)] \text{ .} \qquad (5.19)$$

This contribution to the level width is exact to lowest order in q^2 and to terms of order Ω_p.

Now the plasmon can also decay by exciting a pair plus another plasmon. The diagram for this process is also shown in figure 10. This contribution is of higher order in q^2 (i.e. q^4) since for small values of q the energy available to produce another plasmon is just at threshold. We compute $\Gamma_{(2)}(q)$ exactly to terms of order q^4. For this process we find, by steps similar to those leading to equation 5.16,

$$\Gamma_{(2)}(q) = \frac{1}{(2\pi)^5} \frac{q_p^2(q)}{2\Omega_0(q)} \, 2\int d^3k \int\limits_{\substack{p_1 < 1 \\ |\vec{p}_1 + \vec{q} - \vec{k}| > 1}} d^3p_1 \frac{g_p^2(k)}{2\Omega_0(k)} \qquad (5.20)$$

$$\times\, C^2(\vec{p}_1, \vec{p}_1 + \vec{q} - \vec{k})\, \delta[(\vec{q}-\vec{k})\cdot\vec{p}_1 + \frac{(\vec{q}-\vec{k})^2}{2} + \Omega_0(k) - \Omega_0(q)]$$

where $C^2(\vec{p}_1, \vec{p}_2)$ is again given by equation 5.15. This expression is easily evaluated for small q, where we can write (see equation 4.17)

$$\Omega_o(q) = \Omega_p + \frac{6}{5\Omega_p} q^2 + \ldots \quad .$$

It is easily seen from the kinematics that for small q, only small values of $|\vec{q} - \vec{k}|$ can contribute to the integrals. In this case the integration is easily carried out to give

$$\Gamma_{(2)}(q) = \frac{9\pi}{10} q^4 \Omega_p + 0(q^6, \Omega_p^2) . \qquad (5.21)$$

This simple result, being proportional to Ω_p is probably the exact contribution to the q^4 term to lowest order in r_s. Other contributions this order would come from the terms of order q^4 in $\Gamma_{(1)}(q)$, and perhaps from other higher order decay processes. It appears that these contributions would be of higher order in Ω_p, than the expression 5.21 for $\Gamma_{(2)}(q)$.

Combining equations 5.17 and 5.21 we then have the following expression for the plasmon level width $\Gamma(q)$, to lowest orders in Ω_p and q^2.

$$\Gamma(q) = 9.30\, q^2\Omega_p^2\, [1 + 0.785\Omega_p] + 2.83\, q^4\Omega_p + \ldots \quad .(5.22)$$

The lifetime of the plasmon is $1/\Gamma$ (in our units) which follows from the Uncertainty Principle. Very long wavelength plasmons have a very long lifetime. As the wavelength decreases (q increases) the level width increases until the condition, $\Gamma(q)/\Omega(q) \ll 1$, is no longer satisfied. For a given value of r_s this places a restriction on the range of momenta q for which the plasmon mode is well defined. At electron densities corresponding to those in actual metals this criterion limits q to values less than the cut off q_c found in the pair approximation. We shall defer a discussion of this to the next chapter where we will investigate the line shape in the characteristic energy loss experiments.

VI. EXCITATION OF THE MEDIUM

ELECTRON SCATTERING

In this chapter we will discuss the application of our methods to the problem of electron scattering from a degenerate electron gas. Experiments which measure the characteristic energy loss of fast electrons scattered from metal foils provide an excellent means of studying the excitations of the gas of conduction electrons in metals. With our formalism we can discuss this problem completely from a quantum mechanical standpoint. We will derive some of the basic formulas for the transition probabilities for this process.

In the lowest order approximation, the pair approximation, our results agree, apart from certain small exchange corrections, with the results of several authors [26] [30] [ ] who have used semi-classical or one electron quantum mechanical methods. In particular, Ferrell [26) using a self consistent single particle approach, has given the most complete calculation. His method and results can be shown to be completely equivalent to ours in the pair approximation. For this reason, and for economy of space and time, we will derive the general equations of our method and refer to Ferrell's work for the detailed results in the pair approximation.*

---

* An investigation of this problem in the pair approximation in the high density limit has also been carried out independently by Gell-Mann and R. Latter and W. Karzas of the Rand Corporation (private communication).

We shall consider in more detail the effect of exchange corrections to the results of the pair approximation. In particular, we find that the plasmon cut-off (q_c in pair approximation) is no longer sharp when exchange corrections to plasmon dispersion are included. We conclude from this preliminary investigation that the experimentally determined cut-off can depend strongly on the particular experimental details.

For a review of the theoretical and experimental work on this subject we refer the reader to a review article by Pines [27).

Our formalism must be modified somewhat to include external electrons. This is done by adding to our original Hamiltonian a term

$$H_e = \sum_{p_e, s_e} w(p_e)\, a^{+}_{\vec{p}_e, s_e}\; a_{\vec{p}_e, s_e} + \frac{2\pi \cancel{h} e^2}{V} \sum_q \rho_q \rho_{-qex} \frac{1}{q^2} \qquad 6.1$$

where $a^{+}_{\vec{p}_e, s_e}$ and $a_{\vec{p}_e, s_e}$ are creation and destruction operators for an external electron $\vec{p}_e, s_e$. ρ_{qe} is the Fourier component of the density operator for external electrons

$$\rho_{qe} = \sum_{\vec{p}_e, s_e} a^{+}_{\vec{p}_e, s_e}\; a_{\vec{p}_e, s_e} \qquad 6.2$$

and ρ_q is the charge density operator for the electron gas. The first term in equation 6.1 is the kinetic energy of the external particles while the second term is their energy of interaction with the electron gas.

The states Φ^e_n of the non-interacting system are states of various numbers of external electrons and excited pairs from the Fermi sea. The ground state Φ^e_o is the state with no external electrons or excited pairs. The one electron state is a physically realizable state of this extended system.

Since the added interaction term is of essentially the same form as the original interaction Hamiltonian it follows that the expansion of the S matrix for the new Hamiltonian will lead to the same types of Feynman diagrams and rules as discussed in Chapter I, with the exception that there are no incident hole lines. Note that a line representing an incident electron <u>can</u> jog backward in time since it can undergo exchange interactions with the electrons in the Fermi sea.

To calculate the transition probability for electron scattering by the medium we will make use of the dispersion relation, equation 3.21 for the one electron state.

A. Transition Probabilities for Electron Scattering From Self Energy Calculation

In Chapter III we discussed the construction of the physical propagator for an electron moving through the medium of the electron gas. We pointed out that the physical one electron state is not an eigenstate of the interacting system since (except for momenta near the Fermi momenta) there is a finite probability for the electron to be scattered by the medium. The possibility of decay of the one electron state manifests itself in the imaginary part, $\Gamma/2 = \text{Im}\, W(p_0)$, of the solution to the dispersion relation, equation 3.21. In Appendix B we prove that this imaginary part is indeed the total probability per unit time of transition from the one electron state. This definition of a probability of transition per second is only an approximate one which is valid if the lifetimes of the states involved are large.

The decay probability is easily understood in terms of the Feynman graphs for the proper self energy parts. All of these graphs lead from a one electron state to an intermediate state of one electron (or hole) and various numbers of lines representing the excitation of the medium. If any of these intermediate states lie on the energy shell of the original electron then the electron can decay into these states. The transition probability is the absolute square of an amplitude, $|\langle f|s|i\rangle|^2 = \langle i|s|f\rangle \langle f|s|i\rangle$ and thus the self energy graphs lead from the intermediate state f back to the initial state i. By cutting through the proper self energy graphs in all possible intermediate states we can deduce from the graphs which transition matrix elements are involved.

Let us examine some of the lowest order graphs. If we cut the simplest graph, a in figure 11 , we see that the only transition involved is to a state of one electron and one excitation of the medium. Note that the time orderings must be distinguished in these considerations; the other possible time ordering (graph b) of this lowest order process cannot contribute to the transition probability since energy cannot be conserved. Now the excitation of the medium, as we have seen, includes the plasmon mode and states of various numbers of pairs. Since the electrons which are excited from the medium are indistinguishable from the incident electron we expect exchange diagrams of the same order. To find these consider the graphs c and d which involve two excitations of the medium.

As shown in figure 11 , we can cut these graphs in a symmetrical way to get a two excitation intermediate state or, as in the time order of graph e, to get a single pair intermediate state. The latter gives an exchange contribution to the single excitation process. (The corresponding time order of the uncrossed diagram d does not conserve energy.) There are also intermediate states which arise from unsymmetrical cuts which correspond to cross terms in the square of the transition amplitude. The reader can continue these arguments to include higher order diagrams which will involve multiple excitation of the medium and exchange contributions to processes involving smaller numbers of excitations. For example graph f is an exchange contribution to the two pair intermediate state for the single excitation process. It arises from a self energy graph with three overlapping excitation lines.

If the respective lifetimes are long we expect approximate conservation of energy for the plasmon and pair components of the excitations. Thus for small momentum transfers the multiple plasmon processes will be separated in energy, by approximately, units of Ω_p. The continuum processes will extend over the whole spectrum so that we cannot distinguish between the contributions from various multiple excitations. We will only concern ourselves here with the single excitation process. Thus we must consider graphs a, e, f and in general all exchange process for the one excitation case. We will, of course, be calculating only to lowest orders in r_s for which we will need only the lowest order exchange processes.

In the pair approximation we could calculate the S matrix element for the transition directly using the amplitudes for emission and absorption of plasmons and modified pairs which we discussed in the last chapter. However, in high approximations where the plasmon has a finite lifetime these rules cannot be used.

B. Calculation of Transition Probabilities for Single Excitation.

We will now apply the ideas developed in section A to the calculation of the transition probability from the one electron state $\{\vec{p}_o, s\}$ to a state of one electron plus a single excitation of the medium. The contribution from graph a is

$$\frac{1}{2}\underset{e\to e+p}{\Gamma_o}(p_o) = -\mathrm{Im}\sum^{(1)}(p_o, w(p_o)) \qquad (6.3)$$

where $\sum^{(1)}$ is given by equation 3.13 and was discussed in Chapter III. (Here we have used the first iteration for the solution of equation 3.24.) Writing this explicitly

$$\frac{1}{2}\Gamma_{e\to e+p}(p_o) = +\frac{\alpha r_s}{2\pi^2}\,\mathrm{Im}\int d^3q \int_{-\infty}^{\infty}\frac{dw}{2\pi i}\, S_F(\vec{p}_o - \vec{q},\, w(p_o) - w)\, P_{r_s}(q,w) \qquad (6.4)$$

or, since $\int_{-\infty}^{\infty}\frac{dw}{2\pi i}\, s_F(\vec{p}_o - \vec{q},\, w(p_o) - w)\; 1/q^2$ is real, we can subtract this to get

$$\Gamma_{e\to e+p}(p_o) = -\left(\frac{\alpha r_s}{\pi^2}\right)^2 \mathrm{Im}\int \frac{d^2q}{q^2}\int_{-\infty}^{\infty}\frac{dw}{2\pi i}\, S_F(\vec{p}_o - \vec{q},\, w(p_o) - w)\; \frac{Q_{r_s}(q,w)}{q^2 + \frac{\alpha r_s}{\pi^2} Q(q,w)}\,. \qquad (6.5)$$

We can rotate the path of integration counter clockwise to the imaginary w axis, since $Q_{r_s}(q,w) \to (1/|w^2|)$ for large $|w|$, and we enclose only the contribution from the poles of the electron component of $S_F(\vec{p}_o - \vec{q}, w(p_o) - w)$ for $|\vec{p}_o - \vec{q}| < p_o$ (i.e. only these states can conserve energy). Since

$Q_{r_s}(q,w) = Q_{r_s}(q,-w)$ it is easily seen that the integral along the imaginary axis is real. Thus we are left with

$$\Gamma_{o\atop e\to e+p}(p_o) = -\left(\frac{\alpha r_s}{\pi^2}\right)^2 \text{Im}\int \frac{d^3q}{q^2}\, \frac{Q_{r_s}(q,\Delta)}{q^2 + \frac{\alpha r_s}{\pi^2} Q_{r_s}(q,\Delta)}\, \eta(|\vec{p}_o - \vec{q}| - 1) \quad (6.6)$$

$$= \left(\frac{\alpha r_s}{\pi^2}\right)^2 \int \frac{d^3q}{q^4}\, \frac{\text{Im}\, Q_{r_s}(q,\Delta)}{|K_{r_s}(q,\Delta)|^2}\, \eta(|\vec{p}_o - \vec{q}| - 1) \quad (6.6a)$$

$$= \left(\frac{\alpha r_s}{\pi^2}\right) \int \frac{d^3q}{q^2}\, \text{Im}\, \frac{1}{K_{r_s}(q,\Delta)}\, \eta(|\vec{p}_o - \vec{q}| - 1) \quad (6.6b)$$

where $\Delta = \frac{1}{2} p_o^2 - \frac{1}{2}(\vec{p}_o - \vec{q})^2 = -\frac{1}{2} q^2 + \vec{p}_o \cdot \vec{q}$ and $\Delta > 0$. From equation 6.6a, in which $K_{r_s}(q,\Delta)$ is the generalized dielectric constant defined in Chapter II, we see that this is exactly the same expression for the probability of transition per unit time as that derived by Hubbard using the semi-classical dielectric theory.* [To convert equation 6.4 to ordinary units we must multiply by $(2/\alpha^2 r_s^2)\,(e^4 m/\hbar^2)$.] However we see from our quantum mechanical treatment that this equation is not exact, although it turns out to be a very good approximation for many applications. Multiple excitations, which include multiple plasmon excitations, and exchange effects are neglected entirely by the classical theory. Clearly, if we are interested in a particular momentum transfer q, the transition probability $\Gamma_q(p_o)$ is obtained by dropping the integration over d^3q in equation (6.4).

---

* The semi-classical theory can be shown to be equivalent to our pair approximation.

Before discussing equations 6.4 in more detail let us derive the lowest order exchange correction for the one excitation process. From our discussion we see that this is

$$\frac{1}{2}\,\Gamma_{\substack{1x\\ e\to e+p}}(p_o) = -\mathrm{Im}\sum_{r_s}^{(2x)}(p_o, w(p_o)) \tag{6.7}$$

where $\sum_{r_s}^{(2)}$ is the amplitude for the proper self energy process in figure 6b. Evaluating this according to the rules we have

$$\begin{aligned}\frac{1}{2}\,\Gamma_{\substack{1x\\ e\to e+p}}(p_o) = -\left(\frac{\alpha r_s}{2\pi^2}\right)^2 \mathrm{Im}\int d^3q_1 \int d^3q_2 \int_{-\infty}^{\infty}\frac{dw_1}{2\pi i}\int_{-\infty}^{\infty}\frac{dw_2}{2\pi i}\\ S_F(\vec{p}_o - \vec{q}, w(p_o) - w_1)\, P_{r_s}(q_1, w_1) \qquad (6.8)\\ S_F(\vec{p}_o - \vec{q}_1 - \vec{q}_2, w(p_o) - w_1 - w_2)\, P_{r_s}(q_2, w_2)\\ S_F(\vec{p}_o - \vec{q}_2, w(p_o) - w_2) \quad .\end{aligned}$$

To simplify this we note that the exchange correction which we are after comes from the time ordering in figure 11e so we may replace the propagators in equation 6.8 by the propagators for electrons or holes according to the way they appear in this diagram. The integrations over w_1 and w_2 can then be carried out to give for a particular value of q

$$-\Gamma_{\substack{1x\\ e\to e+p}}(p_o, q) = \left(\frac{\alpha r_s}{2\pi^2}\right)^2 \int_{\substack{p_1<1\\ |\vec{p}_1+\vec{q}|>1}} d^3p_1 \;\frac{\delta\left(\frac{1}{2}q^2 + \vec{q}\,\vec{p}_1 - \Delta\right)}{q^2\,\bar{q}^2}\;\frac{1}{|K_{r_s}(q,\Delta)|\,|K_{r_s}(\bar{q},\Delta)|} \tag{6.9}$$

where $\vec{\bar{q}} = \vec{p}_1 + \vec{p}_o + \vec{q}$.

To see in detail why this is an exchange correction to $\Gamma_{\substack{o\\ e\to e+p}}$ let us write equation 6.6b in the pair approximation where only single pair states are coupled into the excitation of the continuum. Using equation 4.18 we have (omitting the integral over q)

$$\Gamma_{\substack{o\\ e\to e+p}}(p_o,q) = \left(\frac{\alpha r_s}{\pi^2}\right)^2 \frac{1}{q^4} \int_{\substack{p_1<1\\ |p_1+q|>1}} d^3p_1 \, \frac{\pi\, \delta(\frac{1}{2}q^2 + \vec{q}\,\vec{p}_1 - \Delta)}{|K_{r_s}(q,\Delta)|^2} \qquad (6.10)$$

$$+ \left(\frac{\alpha r_s}{\pi^2}\right) \frac{\pi}{2\Omega_o(q)} \, \frac{1}{B_{r_s}(q,0)} \, \delta(\Delta - \Omega_o(q)) \, \eta(q_c - q) \, .$$

In the pair approximation

$$|K_{r_s}(q,\Delta)|^2 = \left(1 + \frac{\alpha r_s}{\pi^2 q^2} \operatorname{Re} Q_o(q,\Delta)\right)^2 + \left(\frac{\alpha r_s}{\pi^2 q^2} \operatorname{Im} Q_o(q,\Delta)\right)^2 . \qquad (6.11)$$

Note that equations 6.10 and 6.11 could also have been derived using the rules for excitation of the medium in equations 4.20 to 4.24.

If we use the definitions in equations 4.20 and 4.22 for the plasmon and modified pair couplings we can write equations 6.9 and 6.10 as

$$\Gamma_{\substack{o\\ e\to e+p}}(p_o) = \frac{1}{(2\pi)^5} \int_{\substack{p_1<1\\ |p_1+q|>1}} d^3p_1 \; g^2(p_1,q) \; \delta(\tfrac{1}{2}q^2 + \vec{q}\cdot\vec{p}_1 - \Delta) \qquad (6.12)$$

$$+ \frac{1}{(2\pi)^2} \; g_p^2(q) \; \frac{\delta(\Delta - \Omega_o(q)) \; \eta(q_c - q)}{2\Omega_o(q)} \; .$$

$$\Gamma_{\substack{1x\\ e\to e+p}}(p_o) = -\frac{1}{2}\frac{1}{(2\pi)^5}\int_{\substack{p_1<1\\ |p_1+q|>1}} d^3p_1\; g(p_1,q)\; g(p_1,\bar{q})\; \delta\left(\tfrac{1}{2}q^2+\vec{q}\cdot\vec{p}_1-\Delta\right). \qquad (6.13)$$

In this form it is clear that equation 6.13 is an exchange correction to the first term in 6.12. It accounts for the fact that the incident electron and the electron excited in the modified pair are indistinguishable particles. Furthermore it is interesting to note that the second term in equation 6.12, which is the plasmon contribution, has no corresponding exchange correction; it is easily verified that equation 6.9 contains no delta function term at $\Delta = \Omega_o(q)$. Hence the plasmon behaves exactly like a boson.

C. Stopping Power of an Electron Gas

These transition probabilities can be applied to several problems. One is a problem which has concerned physicists for many years, the stopping power of an electron gas. It is well known that in such problems the form of the stopping power is given by the Bethe formula [31) [27) irrespective of the details of the interaction because of the existence of sum rules. Such a sum rule is known to exist for the problem at hand [31] [27]. They take on very simple forms in our formalism, namely, in terms of the generalized dielectric constant

$$\mathrm{Im}\int_0^\infty dw\; w\; K_{r_s}^{-1}(q,w) = -\,\mathrm{Im}\int_0^\infty dw\; w\; K_{r_s}(q,w) = -\frac{1}{2}\pi\Omega_p^2 . \qquad (6.14)$$

We prove this relation in Appendix E. In this equation $K_{r_s}(q,w)$ is the exact dielectric constant, corrected to all orders.

Thus, if only the transition probability $\Gamma_{\substack{o \\ e \to e+p}}$ is taken into account we can prove by the use of this sum rule that the stopping power is given by the usual Bethe formula. (Details of such a calculation can be found for example in Mott and Massey [32].) However, exchange effects and multiple excitation of plasmons are not taken into account. These effects are probably small. However, it lies beyond our present scope to consider them here in a quantitative manner.

D. Electron Scattering; Plasmon Cut-Off

Another application of equations 6.6 and 6.9 is the calculation of the angular distributions of fast electrons scattered through small angles by a metal foil. These are the so called characteristic energy loss experiments. Ferrell has analyzed this problem by methods equivalent to ours in the pair approximation. For very fast incident electrons only small momentum transfers q are allowed by energy conservation. In this case the exchange Coulomb denominator $\bar{q}^2 = (p_o + p_1 + q)^2$ is always large $(p_1 < 1)$ so that the exchange contribution is small and we will (as did Ferrell) neglect it. The angle of scattering essentially determines the momentum transfer q if p_o is large $(p_o >> q)$.

We will summarize Ferrell's results. For $q << q_c$ $(\theta << \theta_c)$ the plasmon contribution accounts for all of the sum rule so that the intensity of electrons scattered by exciting the continuum is very weak.

The plasmon continues to account for the lion's share of the sum rule except for q very near q_c where the plasmon coupling (equation 4.21) undergoes a sudden decrease to become zero at $q = q_c$. The continuum contribution, of course, correspondingly increases in intensity. For $q > q_c$ the plasmon contribution is not distinguishable from the continuum. Thus Ferrell finds that the experimentally observed cut off would be relatively sharp and would correspond very nearly to q_c (equation 4.18) Note that in the pair approximation the plasmon continues to contribute a delta function intensity distribution for $q < q_c$.

In higher order approximations we must consider two effects: 1) Multiple pair states are coupled into the excitation of the medium which shifts the plasmon energy and gives the plasmon a finite lifetime, i.e. the plasmon line is broadened. The continuum is also modified by the multiple pair contribution so that it extends over the whole energy range. 2) Multiple excitations of the medium, which include multiple plasmon excitations, also contribute to higher orders in r_s. The various plasmon lines are separated in energy by units of about Ω_p. We will fix our attention on the energy region near the first plasmon line.

As a preliminary estimate of the effect of these corrections we will consider the coupling of the multiple pair continuum only in energy region near the plasmon pole where its effect is to broaden the plasmon line. For the energy region away from the plasmon pole we will continue to use the pair approximation. Thus we write

$$\operatorname{Im}\frac{1}{q^2+\frac{\alpha r_s}{\pi^2}Q_{r_s}(q,\Delta)} = \operatorname{Im}\frac{1}{q^2+\frac{\alpha r_s}{\pi^2}Q_o(q,\Delta)} \quad \text{for} \quad \Delta < \frac{1}{2}q^2+q$$

$$= \operatorname{Im}\frac{1}{q^2+\frac{\alpha r_s}{\pi^2}Q_{r_s}(q,\Delta)} \quad \text{for} \quad \Delta > \frac{1}{2}q^2+q \,. \tag{6.15}$$

In the neighborhood of the plasmon energy $\Omega(q) = \Delta(q) + i\,\frac{1}{2}\Gamma(q)$, we can write (see equation 4.13)

$$\operatorname{Re}\left[q^2+\frac{\alpha r_s}{\pi^2}Q_{r_s}(q,w)\right] = [w^2-\Omega^2(q)]\;B_{r_s}(q,0) + O(w^2-\Omega^2)^2 \,. \tag{6.16}$$

Hence where this expansion is valid we can write for the plasmon contribution

$$\operatorname{Im}\frac{1}{q^2+\frac{\alpha r_s}{\pi^2}Q_{r_s}(q,\Delta)} \cong \frac{\Gamma/2}{[\Delta-\Omega(q)]^2+\frac{1}{4}\Gamma^2}\;\frac{1}{B_{r_s}(q,0)}\;\frac{1}{2\Omega(q)} \,. \tag{6.17}$$

In the limit as $\Gamma \to 0$ we see that this term reduces to the delta function in the pair approximation. When the expansion above is valid the plasmon contribution has the form of a broadened intensity distribution of half width $\Gamma(q)/2$.

For equation 6.16 to be valid we must have

$$(w^2-\Omega^2(q))\left\{\frac{\operatorname{Re}\frac{d^2}{d(w^2)^2}Q_{r_s}(q,w)}{\operatorname{Re}\frac{d}{d(w^2)}Q_{r_s}(q,w)}\right\}_{w=\Omega(q)} \ll 1 \,. \tag{6.18}$$

If q is near q_c, $(d/d(w^2))\, Q_{r_s}(q,w)|_{w=\Omega(q)}$ increases very rapidly (see equation 4.21) so that this condition will be satisfied only for very small values of $[w^2 - \Omega^2(q)]$.

The effect of this approximation then for values of $q > q_c$ is to leave the continuum term in equation 6.10 unaffected and to replace the plasmon delta function by the broadened distribution. This assumes that the effect of the multiple pair continuum is much weaker than the one pair continuum so that its principal effect is to spread out the otherwise singular plasmon delta function. This approximation has the advantage that the sum rule arguments made for the pair approximation still apply here since the integrated plasmon intensity is the same in both cases.

Now if we use the result derived in Chapter V for $\Gamma(q)$ (equation 5.22) we see that the width of the plasmon line increases as q increases. For sufficiently large values of r_s the plasmon can become so broadened as to be indistinguishable from the continuum, for values of $q < q_c$. A precise definition of this effective cut off q_m is not possible. However we may ask, for example, for the value of q at which the half width of the plasmon line becomes equal to the plasmon-continuum energy difference. That is for the solution q_m of

$$\frac{1}{2}\Gamma(q_m) = \Omega(q_m) - \frac{1}{2}q_m^2 - q_m \quad . \qquad (6.19)$$

If we use equations 4.12 and 4.15 for $\Omega(q)$ (this includes the exchange correction) and equation 5.22 for $\Gamma(q)$ we find that for $r_s = 2$

$$q_m = 0.33 \quad . \qquad (6.20)$$

We estimate the principal error in this to be due to our neglect of higher order terms in q^2 in our expression for $\Gamma(q)$. If these terms have the same effect as the corresponding terms in the expression for $\Omega(q)$ we estimate that the error is no more than 10 percent. This value of q_m is much smaller than the value $q_c = 0.73$ found by Ferrell for $r_s = 2$. Because of this the condition 6.18 is satisfied for $|w - \Omega(q_m)| \sim \Gamma(q_m)$ and our approximations seem to be valid.

However, for this value of q_m the plasmon still accounts for nearly all of the sum rule so that it would still be distinct. At $q = 0.6$ the plasmon line is still broader and accounts for a little more than half of the sum rule so that we can safely say that

$$0.4 < q_m < 0.6 \quad .$$

These broad limits on the effective cut off emphasize the fact that the cut off is not sharp at physical densities. In any case it is certainly lower than $q_c = 0.73$ for $r_s = 2$. This might account for the fact that different investigators have found rather different experimental values for q_m.

For example, Watanabe [29] has measured the cut off angle for the scattering of 25 kev electrons from aluminum ($r_s = 2.07$). He finds that the plasmon line fades into the continuum at 15 - 18 milliradians. The cut-off angle ϑ_c calculated from $q_c = 0.73$ in this case is 16.0 milliradians. G. Meyer [33] finds that ϑ_c is 11.1 - 14.9 milliradians for 30 kev electrons scattered from aluminum. For 30 kev electrons $\vartheta_c = 14.6$ milliradians. In both cases the uncertainties in the location of the cut-off are compatible with the line broadening discussed above. Precise measurements of Γ as a function of q have not been made.

We cannot, of course, expect precision agreement of this theory with experiment since we have neglected the effect of the positive ion lattice. In addition to the "second order" broadening effects studied in Chapter V, namely two pair decay and pair-plasmon decay, a plasmon in a metal can decay into a single pair which undergoes a band-band transition. Thus in actual metals the single pair continuum is no longer distinct from the plasmon line as is the case in a free electron gas for $q < q_c$. The damping due to this effect can be related to the optical constants n and k for the metal,* which however are not available experimentally in the necessary frequency range.

The experiments listed in the review by Pines [27] show that for Al, for example, the broadening of the plasmon line at $\vartheta = 0$ ($nq = 0$), where $\Gamma(q) = 0$, appears no greater than the width observed for electrons which have suffered no energy loss. Hence, the one pair decay process does not appear to be strong here and we expect that the "second order" decay modes are, indeed, primarily responsible for the broadening for values of q near the cut-off. In any case, since the various contributions to the level widths are additive, the results obtained above represent a lower limit to the level width and hence an upper limit to the effective cut-off.

The results of this Chapter and the previous one show that for physical electron densities the pair approximation gives an adequate representation of the plasmon only for values of $q \ll q_c$. For shorter wavelengths exchange effects become important; especially the damping associated with these effects.

---

*See Wolff [28]

E. DAMPING OF SINGLE PARTICLE STATES

Another important application of the equations derived in section B is for values of $P_o \sim 1$, that is, for one particle states excited very near the Fermi surface. The resulting expression for $\Gamma(P_o)$ can then be used to calculate the total transition probability for the scattering of a low energy incident electron <u>or</u> the level width (in appropriate units) of a low excited state, of the closed system of an electron gas, consisting of several electron-hole pairs. We proved in Chapter III and Appendix B that the level width of a state of several electrons and holes is the sum of the individual particle level widths, provided the number of excited particles is small compared to N.

Instead of evaluating $\Gamma(p_o)$ directly from equation 6.3 we will first evaluate Im $\sum(p_o,\omega)$ since its properties are of interest, especially in Appendix B. We can evaluate Im $\sum(p_o,\omega)$ by exactly the same steps as leading to equations 6.6 and we find

$$\mathrm{Im} \sum^{(1)} (p_o,\omega) = -\frac{1}{2}\left(\frac{\alpha r_s}{\pi^2}\right)^2 \int \frac{d^3q}{q^4} \, \frac{\mathrm{Im} Q_{r_s}(q,\Delta E)}{K_{r_s}(q,\Delta E)}$$

$$\times \left\{ \eta(|\vec{p}_o - \vec{q}| - 1)\,\eta(\Delta E) - \eta(1 - |\vec{p}_o - \vec{q}|)\,\eta(-\Delta E) \right\} \tag{6.20}$$

where $\Delta E = \omega - \omega(\vec{p}_o + \vec{q})$. Note that if $\omega = \omega(p)$ the second term in this equation only gives a contribution if $p_o < 1$. This term gives the level width of the one hole state.

This integral is very simply evaluated in the pair approximation if ΔE is small. We will restrict ourselves to values of $\omega \sim \frac{1}{2}$ so that only transitions with a small excitation energy ΔE are allowed by the restrictions on the integrand in equation 6.20. If we use equation 1.44 for $\text{Im}Q_o$, the integration over the solid angles of the vector q is simple and we find

$$\text{Im} \sum^{(1)} (p_o, \omega) = - \frac{2}{\pi} (\alpha r_s)^2 \frac{(\omega - \frac{1}{2})^2}{p} [\eta(\omega - \tfrac{1}{2}) - \eta(\tfrac{1}{2} - \omega)]$$

$$\int_0^2 \frac{dq}{q^4} (1 + O(q^2)) \left| K_{r_s} (q, \omega) \right|^{-2} \qquad (6.21)$$

$$+ \text{ higher order terms in } r_s \text{ and } (\omega - \tfrac{1}{2}) .$$

To evaluate this to lowest order in r_s we can neglect the terms of $O(q^2)$ indicated above and we find

$$\text{Im} \sum^{(1)} (p_o, \omega) = - \frac{\pi^2}{32p} \left(\frac{4\alpha r_s}{\pi} \right)^{1/2} (\omega - \tfrac{1}{2})^2 [\eta(\omega - \tfrac{1}{2}) - \eta(\tfrac{1}{2} - \omega)] .$$

$$(6.22)$$

The next order terms in r_s are found by evaluating $\text{Im} \sum^{(1)}$ more exactly to retain terms of this order and by evaluating the exchange contribution $\text{Im} \sum^{(2x)}$ to lowest order in r_s. Equation 6.22 is exact in the limit as $r_s \to 0$. For physical electron densities, however, the higher order corrections are important.

The single particle level width $\Gamma(p)$ is given by $2 \left| \text{Im} \sum (p, W(p)) \right|$ (see equation 3.21), where W(p) is the corrected single particle energy given by equation 3.20. To lowest order in r_s, $W(p) = \omega(p) = \frac{1}{2} p^2$, so that equation 6.22 predicts that $\Gamma(p) \alpha (p-1)^2$ or that particles excited very near the Fermi surface have a vanishingly small level width. This is generally true even if corrections to $\Gamma(p)$ are taken into account. In this case we must be consistent and also take into account self energy corrections in the energies of the final state. This means that S_F in equation 6.24 must be replaced by S_F', the corrected propagator.

To evaluate the expressions in this case we must also satisfy the condition $\Gamma(p)\, T \ll 1$ for all the one electron states involved. Only if this restriction is satisfied is the concept of a transition probability proportional to T valid (see Appendix B.) Now in the adiabatic formalism we are using, the quantity $1/\epsilon$ measures the time in which the interaction is "turned on." In the limit $T \rightarrow \infty$, $\epsilon \rightarrow 0$ we have $T \gtrsim 1/\epsilon$. Thus the condition $\Gamma(p) \ll \epsilon$ must be satisfied. This is equivalent to setting Im $\Sigma(p,\omega)$ equal to zero in all the one particle propagators occurring in Im $\Sigma(p,\omega)$ and then taking the limit as $\epsilon \rightarrow 0$. It is easy to see from the discussion of Chapter III that in this limiting process S_F' is of the same form as S_F except that $\omega(p)$ is replaced by $W(p)$, the corrected energy. Therefore equation 6.22 corrected for self energy effects can immediately be written as

$$\mathrm{Im} \sum(p_o,\omega) = -\frac{\pi^2}{32p^2} \left(\frac{4\alpha r_s}{\pi} \right)^{1/2} \frac{dW}{2dp} (\omega - W(1))^2 \, [\eta(\omega - W(1)) - \eta(W(1) - \omega)] \,, \qquad (6.23)$$

and the single particle level width is *

$$\Gamma(p) = 2 \left| \mathrm{Im} \textstyle\sum(p_o, W(p)) \right|$$

$$= \frac{\pi^2}{32p^2} \left(\frac{4\alpha r_s}{\pi} \right)^{1/2} (W(p) - W(1))^2 \frac{dW}{dp} \qquad (6.24)$$

$$= \frac{\pi^2}{32} \left(\frac{4\alpha r_s}{\pi} \right)^{1/2} (p-1)^2 \left(\frac{dW}{dp} \right)^2_{p=1} + O(r_s, (p-1)^4) .$$

This expression is exact as $r_s \rightarrow 0$ provided $(p - 1)^2$ is small.

Although this formula is exact only for small values of r_s, the vanishing of $\Gamma(p)$ as $(p - 1)^2$ in the limit as $p \rightarrow 1$ is a general property of many Fermion systems. Hugenholtz [34] has demonstrated this result for the nuclear medium. This results from the Exclusion Principle which causes the density of final states for the decay of an electron or hole vanishes as the momenta of these particles approaches the Fermi momentum. This was discussed in Chapter III.

Another general property of Im $\sum(p,\omega)$ is

$$\mathrm{Im} \textstyle\sum(p,\omega) \begin{cases} < 0 & \omega > W(1) \\ > 0 & \omega < W(1) \end{cases} . \qquad (6.25)$$

This is clearly the case in equation 6.23. It is also apparently true in general although we have not attempted a general proof. This

---

*This result has been independently derived in lowest order by Ferrell and Quinn (private communication).

property is a direct result of the electron-hole formalism in which electrons propagate forward in time and holes propagate backward in time. If this were not true the decay factors exp $-\Gamma t$ could be increasing exponentials which would be nonsense.

VII. LOW TEMPERATURE SPECIFIC HEAT OF A DEGENERATE ELECTRON GAS

In this chapter we will apply the general perturbation methods discussed in Chapter III to a calculation of the effect of the electron interactions on the specific heat of an electron gas at low temperatures. The lowest order corrections were first calculated correctly by Gell-Mann [2]. His result is exact in the limit as $r_s \to 0$. However, to apply the theory to the densities found in actual metals ($r_s > 1.8$) higher oder corrections are necessary to determine the convergence of the series.

We present there the calculation of the next term in the series which is considerably more complicated than the lowest order correction. Our result shows that the specific heat is enhanced from the value for non-interacting electrons in the region of highest physical densities. The change of sign apparent in the calculation of Gell-Mann at $r_s \sim 0.8$ still occurs when higher order corrections are included but the origin of the actual sign change is different.

To calculate the specific heat we will use the independent particle model formulated in Chapter III (and Appendix B). This model is valid for the low excited states where the number of excited particles is small compared with N. Fortunately, these independent particle states are asymptotically stationary if the particles have momenta near the Fermi momentum. Thus we expect that the results of Fermi-Dirac statistics apply when only the lowest states are important, i.e. at low temperatures. In this case the specific heat

$C(T_1 r_s)$ is proportional to the density of states at the edge of the Fermi sea which in turn is proportional to $[(dW/dp)\ p = 1]^{-1}$, where $W(p)$ is the single particle energy. In a forthcoming paper we will discuss the statistical mechanical problem in terms of the grand partition function of the system. Here we will show that the requirement for the ordinary Fermi-Dirac formula to hold is that the single particle level width $\Gamma(p)$ approach zero as $(p - 1)^2$. In Chapter VI we showed that this is indeed the case.

The primary importance of this calculation is that the specific heat is the property of the system for which the higher order corrections are most easily calculated. From this result we will gain some quantitative knowledge concerning the convergence of the Gell-Mann Brueckner type of expansion.

The iteration equations 3.23 will be used for the single particle energies $W(p)$. To calculate the specific heat at low temperatures we will use the well-known relation connecting $C(T,r_s)$ with the density of states at the edge of the Fermi sea

$$C(T,r_s) = m^{-1} \hbar^2 e^{-4} k^2 T \, 2[(\frac{dW}{dp})_{p=1}]^{-1} . \qquad (7.1)$$

We will need the derivatives of the single particle propagators to compute the derivative of the single particle energy. From equation 1.11 we have

$$[\frac{d}{dp} S_F(\vec{p}+\vec{q}, u+ w(\vec{p}))]_{p=1} = \vec{n}\cdot(\vec{n}+\vec{q}) \, \delta(|\vec{n}+\vec{q}|-1) \, [\frac{1}{u-w(|\vec{n}+\vec{q}|)+ w(1) + i\varepsilon} - \frac{1}{u-w(|\vec{n}+\vec{q}|) + w(1) - i\varepsilon}] + \vec{n}\cdot\vec{q}[\frac{1}{(u-w(|\vec{n}+\vec{q}|)+w(1) - i\varepsilon)^2} + \frac{1}{(u - w(|\vec{n}+\vec{q}|) + w(1) + i\varepsilon)^2}] . \qquad (7.2)$$

Since we have set $p = 1$, $(\vec{n} = \vec{P}/p)$ and because of the delta function, $\delta(|\vec{n}+\vec{q}| - 1)$ we may set $w(|\vec{n}+\vec{q}|) - w(1) = 0$, in the denominator of the first term. We also note that the second term in brackets is $S_F^2(\vec{n}+\vec{q}, u + w(1))$. Therefore we write

$$\frac{1}{(2\pi i)} [\frac{d}{dp} S_F(p+q,u + w(p))]_{p=1} = -\vec{n}\cdot(\vec{n}+\vec{q}) \, \delta(|\vec{n}+\vec{q}| - 1) \, \delta(u) + \frac{\vec{n}\cdot\vec{q}}{(2\pi i)} S_F^2(\vec{n}+\vec{q}, u + w(1)) . \qquad (7.3)$$

Similarly, it is easy to show that

$$\frac{1}{(2\pi i)} \left[\frac{d}{dp} S_F^2(\vec{p}+\vec{q}, u + w(p)\right]_{p=1}$$

$$= -\vec{n}\cdot(\vec{n}+\vec{q})\, \delta(|\vec{n}+\vec{q}| - 1)\, \delta'(u) + 2(\vec{n}\, \vec{q})\, \frac{1}{2\pi i} S_F^3(\vec{n}+\vec{q}), u + w(1)). \tag{7.4}$$

The first term on the right of equation 7.3 contains the factor $\delta(|\vec{n}+\vec{q}| - 1)\, \delta(u)$ which restricts this term to transitions $\vec{n} \rightarrow \vec{n}+\vec{q}$ on the Fermi surface which require no energy. This term, which is proportional to q^{-1}, is responsible for the fact that in ordinary perturbation theory (in terms of the bare Coulomb interaction) the expression for the specific heat is more divergent by one order than the corresponding expression for the correlation energy. The second term in equation 7.3 is the contribution from other transitions to within $(|\vec{n}+\vec{q}| < 1)$ or without $(|\vec{n}+\vec{q}| > 1)$ the Fermi sea. These transitions are less important since they do not add to the divergence at low momentum transfers.

1) The contribution to $W(p) - \omega(p)$ from the self energy process with one virtual excitation of the medium (see figure 6a) is *

$$W^{(1)}(p) = \frac{2}{\alpha^2 r_s^2} \left[\Sigma^{(1)}(p, \omega(p))\, (1 - \Sigma'^{(1)}(p, \omega(p))\right]$$

$$= -\frac{1}{\pi^2 \alpha r_s} \int d^3q \int_{-\infty}^{\infty} \frac{dw}{2\pi i} S_F(\vec{p}+\vec{q}, w + w(p))\, P_{r_s}(q, w)$$

$$\left[1 - \frac{\alpha r_s}{2\pi^2} \int d^3q_1 \int_{-\infty}^{\infty} \frac{dw_1}{2\pi i} S_F^2(\vec{p}+\vec{q}_1, w_1 + w(p)\, P_{r_s}(q_1, w)\right] + O(r_s) \tag{7.5}$$

---

[*] In this calculation we can replace $\mathrm{Re}\,\Sigma(p, \omega)$ by $\Sigma(p, \omega(p))$ since $\mathrm{Im}\,\Sigma \rightarrow (p-1)^2$. Thus in computing $d\Sigma / dp$ at $p = 1$ the imaginary part of Σ gives no contribution.

where we have multiplied by $2\alpha^{-2} r_s^{-2}$ to express the energy in Rydbergs and we have used equations 3.13 and 3.23. $\sum^{(1)}(p, w)$ is the amplitude for the one virtual excitation self energy process. (In this chapter we omit the subscript r_s.) We have included all terms which are of second order and lower in the effective interaction since we want to include all terms of order r_s^o. The term $-\sum'^{(1)} \sum^{(1)}$ arises from the iterative method of solution discussed in Chapter III. However since

$$- \frac{\alpha r_s}{2\pi^2} \int d^3q_1 \int_{-\infty}^{\infty} \frac{dw_1}{2\pi i} S_F^2(\vec{p}+\vec{q}, w_1 + w(p)) \frac{1}{q^2} = 0$$

we can subtract this from $\sum'^{(1)}$ and we find that

$$\sum'^{(1)}(\vec{p}, w(p)) = - \frac{1}{2}\left(\frac{\alpha r_s}{\pi^2}\right)^2 \int d^3q_1 \int_{-\infty}^{\infty} dw_1 \, S_F^2(\vec{p}+\vec{q}, w(p) + w_1) \times \frac{q^{-2} Q_{r_s}(q_1, w_1)}{q^2 + (\alpha r_s/\pi^2) Q_{r_s}(q, w_1)} ,$$

so that $2\alpha^{-2} r_s^2 \sum^{(1)} \sum^{(1)'}$ is proportional to r_s. It can be shown that this term contributes terms of order r_s to the specific heat. In fact if we include the contribution from the uncrossed, two virtual excitation process (figure 6d) we find a specific heat contribution of the form $r_s(a \ln r_s + K)$. The iteration diagrams and the diagrams with internal self energy parts should be considered together since certain cancellations always occur as we pointed out in Chapter III. In the present case if we considered these processes separately we would find leading terms proportional to $r_s^{1/2}$ which, in fact, cancel when the processes are considered together.

Thus to order r_s^o we can neglect the term $\sum'^{(1)}$ in equation 7.5 and the uncrossed two virtual excitation process as well. Using equations 7.3 and 7.4 we find that $[dW^{(1)}/dp]_{p=1}$ is a sum of two terms. The first is the contribution from transitions $\vec{n} \rightarrow \vec{n}+\vec{q}$ from one point to another on the Fermi surface (p = 1)

$$a\left(\frac{dW^{(1)}}{dp}\right)_{p=1} = \frac{1}{\pi^2 \alpha r_s} \int d^3q \; \vec{n}\cdot(\vec{n}+\vec{q}) \; \delta(|\vec{n}+\vec{q}| - 1) \; P_{r_s}(q,0) . \qquad (7.6a)$$

The second term is the contribution from transitions to states not on the Fermi surface.

$$b\left(\frac{dW^{(1)}}{dp}\right)_{p=1} = - \frac{1}{\pi^2 \; \alpha r_s} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \int d^3q \; \vec{n}\cdot\vec{q} \; S_F^2(\vec{n}+\vec{q}, \omega + w(1)) \; P_{r_s}(q,w) . \qquad (7.6b)$$

Let us consider each term separately:

a) Consider first the contribution from equation 7.6b. This arises from excitations on the surface of the Fermi sphere. If we put $\vec{n}' = \vec{n}+\vec{q}$, where n' is a unit vector by virtue of the delta functions, and integrate over all directions of the vector $\vec{n}'$ we get (with $x = \vec{n}\cdot\vec{n}'$)

$$a\left(\frac{dW^{(1)}}{dp}\right)_{p=1} = \frac{2\pi}{\pi^2 \; \alpha r_s} \int_{-1}^{1} \frac{x \; dx}{2(1 - x) + (\alpha r_s/\pi^2) \; Q_{r_s}(\sqrt{2(1-x)}, 0)}$$

$$= \frac{\pi}{\pi^2 \; \alpha r_s} \int_0^4 dz \; \frac{(1 - z/2)}{z + (\alpha r_s/\pi^2) \; Q_{r_s}(\sqrt{z} \, ,0)} . \qquad (7.7)$$

This integral clearly diverges logarithmically as $r_s \rightarrow 0$, since $Q_o(0,0) = 4\pi$. The effect of the induced polarization charge is to limit the range of the Coulomb interaction and makes the integral finite. If we define

$$\Delta Q_{r_s}(z^{1/2}, 0) = Q_{r_s}(z^{1/2}, 0) - [Q_{r_s}(0,0) + zQ'_{r_s}(0,0)], \quad (7.8)$$

where $Q'_{r_s}(q,0) = \frac{d}{d(q^2)} Q_{r_s}(q,0)$, we can write equation 7.7 as

$$a\left(\frac{dW^{(1)}}{dp}\right)_{p=1} = \frac{1}{\pi\alpha r_s}\int_0^4 dz$$

$$\frac{(1 - z/2)}{(1 + \frac{\alpha r_s}{\pi^2} Q'_{r_s}(0,0))\, a + \frac{\alpha r_s}{\pi^2} Q_{r_s}(0,0) + \frac{\alpha r_s}{\pi^2} \Delta Q_{r_s}(z^{1/2},0)} .$$

$$(7.9)$$

Writing

$$D_{r_s}(z) = (1 + \frac{\alpha r_s}{\pi^2} Q'_{r_s}(0,0))z + \frac{\alpha r_s}{\pi^2} Q_{r_s}(0,0) \quad (7.10)$$

we may recast equation 7.9 in the form

$$a\left(\frac{dW^{(1)}}{dp}\right)_{p=1} = \frac{1}{\pi\alpha r_s}\int_0^4 dz \frac{(1 - z/2)}{D_{r_s}(z)} - \frac{1}{\pi^3}\int_0^4 dz \frac{(1 - z/2)}{D^2_{r_s}(z)} \Delta Q_{r_s}(z^{1/2},0)$$

$$+ \frac{\alpha r_s}{\pi^5}\int_0^4 dz \frac{(1 - z/2)}{D^2_{r_s}(z)} \frac{[\Delta Q_{r_s}(z^{1/2}, 0)]^2}{[z + \frac{\alpha r_s}{\pi^2} Q_{r_s}(z^{1/2},0)]} . \quad (7.11)$$

The last term in equation 7.11 is easily seen to contribute terms of order r_s and higher so we will drop it. We can also simplify the second integral to terms of order r_s^o. Since

$$\Delta Q_{r_s}(z^{1/2}, 0) = \frac{1}{2} Q''_{r_s}(0,0)\, z^2 + O(z^3)$$

we can write to order r_s^o

$$a\left(\frac{dW^{(1)}}{dp}\right)_{p=1} = \frac{1}{\pi\alpha r_s} \int_0^4 dz\, \frac{(1 - z/2)}{D_{r_s}(z)} - \frac{1}{\pi^3} \int_0^4 dz\, \frac{(1 - z/2)}{D^2_{r_s}(z)}\, \Delta Q_o(z^{1/2},0)+0(r_s) \; . \tag{7.12}$$

If we consider each part separately we have

$$\frac{1}{\pi\alpha r_s} \int_0^4 dz\, \frac{(1 - z/2)}{D_{r_s}(z)} = \frac{1}{\pi\alpha r_s}\Bigg\{ \left(1 + \frac{\alpha r_s}{\pi^2} Q'_{r_s}(0,0)\right)^{-1} \left[1+ \frac{\alpha r_s}{2\pi^2} Q_{r_s} \left(1+ \frac{\alpha r_s}{\pi^2} Q'_{r_s}\right)^{-1}\right]$$

$$\ln\left|\frac{4\left(1 + \frac{\alpha r_s}{\pi^2} Q'_{r_s}\right) + \frac{\alpha r_s}{\pi^2} Q_{r_s}}{\frac{\alpha r_s}{\pi^2} Q_{r_s}}\right| - \frac{2}{\left(1 + \frac{\alpha r_s}{\pi^2} Q'_{r_s}\right)}\Bigg\} \tag{7.13}$$

$$\cong - \frac{1}{\pi\alpha r_s}\left[2 + \ln\left|\frac{\alpha r_s}{4\pi^2} Q_{r_s}(0,0)\right|\right] + \frac{1}{\pi^3}\Big[3Q'_{r_s}(0,0) + \frac{1}{4} Q_{r_s}(0,0)$$

$$+ \left(Q'_{r_s}(0,0) - \frac{1}{2} Q_{r_s}(0,0)\right) \ln\left|\frac{\alpha r_s}{4\pi^2} Q_{r_s}(0,0)\right| \Big] + 0(r_s) \; . \tag{7.13a}$$

Now

$$Q_{r_s}(0,0) = Q_o(0,0) + Q^{(1)}_{r_s}(0,0) + 0(r_s^2) \; .$$

Using equations 1.50 and 1.51 and the results of Appendix A for $Q^{(1)}_{r_s}$ we see that this is

$$Q_{r_s}(0,0) = 4\pi\left[1 + \frac{\alpha r_s}{\pi}\right] + 0(r_s^2) \tag{7.14}$$

and from equation 1.45 or 1.46

$$Q'_{r_s}(0,0) = - \frac{\pi}{3} \; . \tag{7.15}$$

Substituting this into equation (7.13a) we have

$$\frac{1}{\pi\alpha r_s} \int_0^4 dz \, \frac{(1 - z/2)}{D_{r_s}(z)} = - \frac{1}{\pi\alpha r_s} [2 + \ln \frac{\alpha r_s}{\pi}] - \frac{1}{\pi^2} [1 + \frac{7}{3} \ln \frac{\alpha r_s}{\pi}] + O(r_s) . \tag{7.16}$$

Next consider the second integral in equation 7.12. To evaluate this we can use equation 1.46 which gives

$$Q_o(q,0) = 2\pi[1 + (\frac{1}{2} - \frac{1}{4} q) \ln\left|\frac{2 + q}{2 - q}\right|] . \tag{7.17}$$

Expanding this and putting $q = z^{1/2}$ we have

$$\Delta Q_o(z,0) = 2\pi \sum_1^\infty (\frac{1}{2n + 3} - \frac{1}{2n + 1}) (\frac{z}{4})^{n+1} \tag{7.18}$$

so

$$- \frac{1}{\pi^3} \int_0^4 \frac{(1 - z/2)}{z^2} \Delta Q_o(z^{1/2},0) = \frac{1}{\pi^2} \sum_1^\infty \frac{1}{(2n + 1) \, (2n + 3)} [\frac{1}{n} - \frac{2}{n + 1}]$$

$$= \frac{-0.010}{\pi^2} = - 0.001 . \tag{7.19}$$

Adding equations 7.16 and 7.19 we have

$$a(\frac{dW^{(1)}}{dp})_{p=1} = - \frac{1}{\pi\alpha r_s} [2 + \ln \frac{\alpha r_s}{\pi}] - \frac{1}{\pi^2} [\frac{7}{3} \ln \frac{\alpha r_s}{\pi} + 1.01] + O(r_s) . \tag{7.20}$$

The leading terms of order r_s^{-1} were first calculated correctly by Gell-Mann[2].

b) Next consider the first term in equation 7.6b which is the contribution from excitations to states not on the Fermi surface. We will need the integral

$$\int_{-\infty}^{\infty} \frac{dw}{2\pi i} \, S_F^2(\vec{n}+\vec{q}, \, w + w(1)) \, P_{r_s}(q,w). \tag{7.21}$$

Since $\int_{-\infty}^{\infty} \frac{dw}{2\pi i} \; S_F^2(\vec{n}+\vec{q}, w + w(1)) \, q^{-2} \equiv 0$ we can subtract this from the right side of equation 7.27. Note that all of the poles of the propagator $S_F(\vec{n}+\vec{q}, w + w(1))$ lie below the positive Re w axis or above the negative Re u axis. Thus, since the integrand vanishes sufficiently fast as $|w| \to \infty$, we can rotate the path of integration counterclockwise to the imaginary w axis without crossing any singularities. Substituting $w = iqu$ we have

$$- \frac{\alpha r_s}{\pi^2 q^3} \int_{-\infty}^{\infty} \frac{du}{2\pi} \left[iu - \frac{q}{2} - \vec{q}\cdot\vec{n}\right]^{-2} \frac{Q_{r_s}(q, iqu)}{q^2 + (\alpha r_s/\pi^2)\, Q_{r_s}(q, iqu)} \; . \qquad (7.22)$$

Since $Q_{r_s}(q, iqu) = Q_{r_s}(q, -iqu)$ it is easy to see that this integral is real. Using this result we have

$$b\left(\frac{dW^{(1)}}{dp}\right)_{p=1} = \frac{-1}{\pi^4} \int_{-\infty}^{\infty} \frac{du}{2\pi} \int d^3q \, \frac{\vec{n}\cdot\vec{q}}{q^2} \left[iu - \frac{q}{2} - \vec{q}\cdot\vec{n}\right]^{-2} \frac{Q_{r_s}(q, iqu)}{q^2 + (\alpha r_s/\pi^2)\, Q_{r_s}(q, iqu)} \; . \qquad (7.23)$$

If we expand (with $x = \vec{n}\cdot\vec{q}$)

$$\left[iu - \frac{q}{2} - x\right]^{-2} = [iu - x]^{-2} + q[iu - x]^{-3} + O(q^2)$$

we note that, since the rest of the integrand is an odd function of x, the first term in this expansion does not contribute to the integral. The leading term in the integrand considered as a function of q is proportional to $1/q$ as $r_s \to 0$. Thus we must keep the screening term for small values of q. We can use equation 1.50 to approximate $Q_{r_s}(q, iqu)$ by $4\pi R(u)$. For higher order terms in q we can neglect the screening and still retain an expression exact to order r_s^o. Therefore, we have

$$b\left(\frac{dW^{(1)}}{dp}\right)_{p=1} = \frac{1}{\pi^4}\int_{-\infty}^{\infty} du \int_{-1}^{1} x\, dx \int_{o}^{1} q\, dq \, \frac{1}{(iu - x)^3}$$

$$\frac{4\pi\, R(u)}{q^2 + (4\alpha r_s/\pi)\, R(u)} + \Delta + O(r_s) \qquad (7.24)$$

where

$$\Delta = \lim_{\delta \to 0} \left\{ \frac{1}{\pi^4} \int_{-\infty}^{\infty} du \int_{-1}^{1} x\, dx \int_{\delta}^{\infty} \frac{dq}{q^2} \frac{1}{(iu - \frac{q}{2} - x)^2} Q_o(q, iqu) \right.$$

$$\left. - \frac{1}{\pi^4} \int_{-\infty}^{\infty} du \int_{-1}^{1} x\, dx \int_{\delta}^{1} \frac{dq}{q} \frac{1}{(iu - x)^3} 4\pi\, R(u) \right\} . \qquad (7.25)$$

We can perform the integral over q in equation 7.24 and we see that

$$b\left(\frac{dW}{dp}\right)_{p=1} = A \ln \frac{4\alpha r_s}{\pi} + B + \Delta \qquad (7.26)$$

where

$$A = \frac{-2}{\pi^3} \int_{-\infty}^{\infty} du \int_{-1}^{1} x\, dx\, [iu - x]^{-3} R(u)$$

$$= \frac{4}{\pi^2} \int_{o}^{1} x\, dx \int_{o}^{1} y\, dy\, (x + y)^{-3} = \frac{1}{\pi^2} \qquad (7.27a)$$

$$B = \frac{-2}{\pi^3} \int_{-\infty}^{\infty} du \int_{-1}^{1} x\, dx\, [iu - x]^{-3} \ln R(u) \qquad (7.27b)$$

$$= \frac{4}{\pi^3} \int_{-\infty}^{\infty} du\, (1 + u^2)^{-2} \ln R(u) = -0.199 .$$

To evaluate A we used an integral representation for $R(u)$ (see equations 1.50 and 1.45)

$$R(u) = -(\tfrac{1}{2}) \int_{-1}^{1} dy\, y\, [iu - y]^{-1} = [1 - u \tan^{-1} (u^{-1})].$$

B was evaluated by a simple numerical integration. In Appendix C we outline a method of integration for Δ which yields

$$\Delta = -0.0855. \tag{7.28}$$

Finally we add equations 7.20 and 7.26 to obtain the total contribution from the one virtual excitation process

$$\left(\frac{dW^{(1)}}{dp}\right)_{p=1} = -\frac{1}{\pi \alpha r_s} \left[2 + \ln \frac{\alpha r_s}{\pi}\right] - \frac{4}{3\pi^2} \ln \frac{\alpha r_s}{\pi} - 0.387$$

$$+ O(r_s). \tag{7.29}$$

We cannot yet draw any conclusions from this calculation since other self energy processes also contribute terms of this order.

2) Next we consider the irreducible self-energy graphs with two crossing virtual excitations (see figure 6b). The lowest order contribution to the self energy arises from the first term of the iterative expansion, equation 3.23

$$W^{(2a)}(p) = \sum_{r_s}^{(2a)} (\vec{p}, w(p)) = \frac{1}{2\pi^4} \int_{-\infty}^{\infty} \frac{dw_1}{2\pi i} \int_{-\infty}^{\infty} \frac{dw_2}{2\pi i} \int d^3q_1 \int d^3q_2 \tag{7.30}$$

$$S_F(\vec{p}+\vec{q}, w_1 + w(P))\, P_{r_s}(q, w_1)\, S_F(\vec{p}_1+\vec{q}_1+\vec{q}_2, w_1+w_2+w(p))\, P_{r_s}(q_2, w_2)$$

$$S_F(\vec{p}_1+\vec{q}_2, w_2 + w(p)).$$

From equation 7.3 we see that

$$\left(\frac{dW^{2a}}{dp}\right)_{p=1} = 2(K_{11} + K_{12}) + (K_{21} + K_{22}) \tag{7.31}$$

where

$$(K_{11} + K_{12}) = + \frac{1}{2\pi^4} \int_{-\infty}^{\infty} \frac{dw_1}{2\pi i} \int_{-\infty}^{\infty} \frac{dw_2}{2\pi i} \int d^3q_1 \int d^3q_2 \left[\frac{d}{dp} S(\vec{p}_1+\vec{q}_1, u_1+w(p))\right]_{p=1}$$

$$P_{r_s}(q,u_1)\, S(\vec{n}+\vec{q}_1+\vec{q}_2, w_1+w_2+w(1))\, P_{r_s}(q_2,w_2)\, S(\vec{n}+\vec{q}_2, w_2+w(1)) \tag{7.32}$$

$$(K_{21} + K_{22}) = + \frac{1}{2\pi^4} \int_{-\infty}^{\infty} \frac{dw_1}{2\pi i} \int_{-\infty}^{\infty} \frac{dw_2}{2\pi i} \int d^3q_1 \int d^3q_2\, S(\vec{n}+\vec{q}_1, w_1+w(1))$$

$$P_{r_s}(q,w_1) \left[\frac{d}{dp} S_F(\vec{p}+\vec{q}_1+\vec{q}_2, u_1+u_2+w(p))\right]_{p=1} P_{r_s}(q_2,w_2)\, S(\vec{n}+\vec{q}_2,w_2+w(1)). \tag{7.34}$$

K_{11} and K_{21} are the contributions from transitions on the Fermi surface (i.e. from the first term of equation 7.3) and K_{12} and K_{22} are the contributions from the transitions to states not on the Fermi surface (second term in equation 7.3). Let us consider each term separately.

a) Using equation 7.3 we can write K_{11} in the form

$$K_{11} = \frac{-1}{2\pi^4} \int_{-\infty}^{\infty} \frac{dw_1}{2\pi i} \int d^3q_1 \int d^3q_2 \frac{\vec{n}\cdot(\vec{n}+\vec{q}_1)\ \delta(|\vec{n}+\vec{q}_1| - 1)}{q_1^2 + (\alpha r_s/\pi^2)\, Q_{r_s}(q_1,0)} \tag{7.35}$$

$$S(\vec{n}+\vec{q}_1+\vec{q}_2, w_2+w(1))\, P_{r_s}(q_2,w_2)\, S(\vec{n}+\vec{q}_2,w_2+w(1)),$$

where we have integrated out the factor $\delta(w_1)$. We can rotate the path of integration over w_2 by +90° without crossing any poles. If we then substitute $\vec{q} \equiv \vec{q}_2$, $w_2 = iqu$ and $\vec{n}' \equiv \vec{n}+\vec{q}_1$ we can replace the integration over $\vec{q}_1$ by an integration over the directions of $\vec{n}'$ and we have

$$K_{11} = \frac{-1}{2\pi^4} \int d\Omega_{n'} \; \vec{n}\cdot\vec{n}' \frac{1}{(\vec{n}' - \vec{n})^2 + (\alpha r_s/\pi^2)\, Q_{r_s}(|\vec{n}' - \vec{n}|, 0)} \, I_{r_s}(\vec{n}\cdot\vec{n}') \tag{7.36}$$

where

$$I_{r_s}(\vec{n}\cdot\vec{n}') = \int_{-\infty}^{\infty} \frac{du}{2\pi} \int \frac{d^3q}{q} \frac{1}{[iu - \frac{1}{2} q_2 - \vec{n}\cdot\vec{q}]} \frac{1}{[iu - \frac{1}{2} q - \vec{n}'\cdot\vec{q}]}$$

$$\frac{1}{q^2 + (\alpha r_s/\pi^2)\, Q_{r_s}(q, iqu)} \; . \tag{7.37}$$

The last integral can be simplified to retain only terms of order r_s^o exactly. Since $Q_{r_s}(q, iqu) \to 4\pi\, R(u)$ (see equation 1.50) as $q \to 0$ we see that the integral diverges like $\ln r_s$ as $r_s \to 0$. Only the lowest order terms in q contribute to this divergence. For the higher order terms in q we can neglect the screening terms (i.e. set $r_s = 0$) since they contribute to order r_s^1 and higher. Thus to order r_s^o we can write

$$I_{r_s}(x) = \int_{-\infty}^{\infty} \frac{du}{2\pi} \int_{q<1} \frac{d^3q}{q} \frac{1}{[iu - \vec{n}\cdot\vec{q}]} \frac{1}{[iu - \vec{n}'\cdot\vec{q}]} \frac{1}{q^2 + (4\alpha r_s/\pi)\, R(u)}$$

$$+ \Delta(x) + O(r_s) \tag{7.38}$$

and

$$\Delta(x) = \lim_{\delta \to 0} \left\{ \int_{-\infty}^{\infty} \frac{du}{2\pi} \left\{ \int_{q>\delta} \frac{d^3q}{q^3} \frac{1}{[iu - \frac{1}{2} q - \vec{n}\cdot\vec{q}]} \frac{1}{[iu - \frac{1}{2} - \vec{n}'\cdot\vec{q}]} \right.\right.$$

$$\left.\left. - \int_{\delta<q<1} \frac{d^3q}{q^3} \frac{1}{[iu - \vec{n}.\vec{q}]} \frac{1}{[iu - \vec{n}.\vec{q}]} \right\} \right. . \tag{7.39}$$

Carrying out the integration over u in the last equation we have

$$\Delta(x) = \lim_{\beta \to 0} \left\{ \int_{q>\beta} d^3q \, \frac{1}{q^3} \, \frac{\eta(1 - |\vec{n} + \vec{q}|) - \eta(1 - |\vec{n}' + \vec{q}|)}{q \cdot (\vec{n} - \vec{n}')} - \int_{\beta<q<1} d^3q \, \frac{1}{q^3} \, \frac{\eta(-\vec{n}\cdot\vec{q}) - \eta(-\vec{n}'\cdot\vec{q})}{q\cdot(\vec{n} - \vec{n}')} \right\} \tag{7.40}$$

which can be evaluated to give

$$\Delta(x) = \frac{2\pi}{\sqrt{\frac{1}{2}(1-x)}} \left\{ \ln 2 \ln\left(1 + \sqrt{\tfrac{1}{2}(1-x)}\right) - \frac{5}{8}\left(\ln\left|1 - \tfrac{1}{2}(1-x)\right|\right)^2 - \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot\cdots\cdot(2n-2)(2n)^3} \left[\left(1 - \tfrac{1}{2}(1-x)\right)^n - 1\right] \right\} . \tag{7.41}$$

If we carry out the integration over q in equation 7.38 we find that $I_{r_s}(x)$ is of the form

$$I_{r_s}(x) = A(x) \ln \frac{4ar_s}{\pi} + B(x) + \Delta(x) + O(r_s) \tag{7.42}$$

where

$$A(x) = -\frac{1}{2} \int_{-\infty}^{\infty} \frac{du}{2\pi} \int d\Omega_{\vec{q}} \, \frac{1}{[iu - \vec{n}\cdot\hat{q}]} \, \frac{1}{[iu - \vec{n}'\cdot\hat{q}]} \tag{7.43}$$

$$= \frac{2\pi}{\sqrt{2(1-x)}} \ln \left| \frac{1 + \sqrt{\frac{1}{2}(1-x)}}{1 - \frac{1}{2}(1-x)} \right|$$

$$B(x) = -\frac{1}{2} \int_{-\infty}^{\infty} \frac{du}{2\pi} \int d\Omega_{\vec{q}} \, \frac{1}{[iu - \vec{n}\cdot\hat{q}]} \, \frac{1}{[iu - \vec{n}'\cdot\hat{q}]} \ln R(u) \tag{7.44}$$

$$= \frac{2\pi}{\sqrt{2(1-x)}} \int_{-\infty}^{\infty} \frac{du}{2\pi} \ln R(u) \, \frac{2}{\sqrt{1 + u^2 - \frac{1}{2}(1-x)}} \tan^{-1} \frac{\sqrt{\frac{1}{2}(1-x)}}{\sqrt{1+u^2 - \frac{1}{2}(1-x)}} .$$

The integral over $\vec{n}'$ in equation 7.36 reduces to an integral over $x = \vec{n}\cdot\vec{n}'$ $(\vec{n}' = \vec{n}+\vec{q}_1)$:

$$K_{11} = \frac{-1}{\pi^3} \int_{-1}^{1} \frac{x\,dx}{2(1-x) + (\alpha r_s/\pi^2)\, Q_{r_s}(\sqrt{2(1-x)},\, 0)}\, I_{r_s}(x) \,. \qquad (7.45)$$

We can again simplify this to terms of order r_s^0 to give

$$K_{11} = \frac{-1}{\pi^3} \int_{-1}^{1} \frac{x\,dx}{2(1-x) + 4\alpha r_s/\pi}\, I_{r_s}(1) + D_{r_s} + 0(r_s) \qquad (7.46)$$

where

$$D_{r_s} = \frac{-1}{\pi^3} \lim_{\epsilon\to 0} \int_{-1}^{1-\epsilon} \frac{x\,dx}{2(1-x)}\,[I_{r_s}(x) - I_{r_s}(1)] \,. \qquad (7.47)$$

If we carry out the integration over x in equation 7.46 we get

$$K_{11} = +\frac{1}{\pi^3}\,[1 + \frac{1}{2}\ln\frac{\alpha r_s}{\pi}]\, I_{r_s}(1) + D_{r_s} \qquad (7.48)$$

so we have

$$K_{11} = L(\ln\frac{\alpha r_s}{\pi})^2 + M\ln\frac{\alpha r_s}{\pi} + N \qquad (7.49)$$

where from equations 7.41 to 7.44

$$L = +\frac{1}{2\pi^3}\,A(1) = \frac{1}{2\pi^2} \qquad (7.50)$$

$$M = +\frac{1}{2\pi^3}\,[B(1) + \Delta(1) + 2A(1) + 2\pi\ln 2]$$

$$-\frac{1}{\pi^3} \lim_{\epsilon\to 0} \int_{-1}^{1-\epsilon} \frac{x\,dx}{2(1-x)}\,[A(x) - A(1)] \qquad (7.51)$$

$$N = +\frac{1}{\pi^3}\left[B(1) + \Delta(1) + 2\ln 2\right] - \frac{1}{\pi^3}\lim_{\epsilon\to 0}\int_{-1}^{1-\epsilon}\frac{x\,dx}{2(1-x)}\left[B(x)-B(1)+\Delta(x)-\Delta(1)\right]. \tag{7.52}$$

From equations 7.41 to 7.42 we have

$$A(1) = +\pi; \quad B(1) = -\pi\,(1.90)\,; \quad \Delta(1) = -2\pi\ln 2 \tag{7.53}$$

where $B(1)$ was evaluated by a numerical integration over u in equation 7.44. The integrals in equation 7.52 were evaluated by series expansions and numerical integration. Collecting the results we have:

$$\lim_{\epsilon\to 0}\int_{-1}^{1-\epsilon}\frac{x\,dx}{2(1-x)}\left[A(x) - A(1)\right] = \frac{\pi}{3}(2 - 3\ln 2) = +\frac{\pi}{3}(0.248) \tag{7.54}$$

$$\lim_{\epsilon\to 0}\int_{-1}^{1-\epsilon}\frac{x\,dx}{2(1-x)}\left[B(x) - B(1)\right] = -\int_{-\infty}^{\infty}du\,\ln R\left[\frac{2R+1}{1+u^2} - \left(\frac{R-1}{u}\right)^2\right] \tag{7.55}$$

$$= 0.594$$

$$\lim_{\epsilon\to 0}\int_{-1}^{1-\epsilon}\frac{x\,dx}{2(1-x)}\left[\Delta(x) - \Delta(1)\right] = \pi\left[2\ln 2 - 4 + \frac{\pi^2}{4}\right] = -0.471\,. \tag{7.55a}$$

Therefore

$$K_{21} = 0.051\left[\ln\frac{\alpha r_s}{\pi}\right]^2 + 0.013\left[\ln\frac{\alpha r_s}{\pi}\right] - 0.196\,. \tag{7.56}$$

b. Next we have the contribution K_{21} which arises when the central particle line (of momentum $\vec{n}+\vec{q}_1 + \vec{q}_2$) in diagram b of figure 6 represents a transition on the Fermi surface.

$$K_{21} = \frac{1}{2\pi^4}\int_{-\infty}^{\infty}\frac{dw_1}{2\pi i}\int_{-\infty}^{\infty}dw_2\, S(\vec{n}+\vec{q}_1, w_1 + w(1))\, P_{r_s}(q_1, w_1) \tag{7.57}$$

$$\vec{n}\cdot(\vec{n}+\vec{q}_1+\vec{q}_2)\,\delta(|\vec{n}+\vec{q}_1+\vec{q}_2|-1)\,\delta(w_1 + w_2)\, S(\vec{n}+\vec{q}_2, w_2 + w(1))\, P_{r_s}(q_2, w_2).$$

After integrating out the delta function with w_2 we can rotate the path of integration over w_1 by +90°. Then we have

$$K_{21} = \frac{1}{2\pi^4} \int_{-\infty}^{\infty} \frac{du}{2\pi} \int d\Omega_n \, \vec{n}\cdot\vec{n}' \int \frac{d^3q}{q} \, \frac{1}{iu - \frac{1}{2}q - \vec{q}\cdot\vec{n}} \, \frac{1}{iu + \frac{1}{2}q - \vec{q}\cdot\vec{n}'}$$

$$\frac{1}{q^2 + (\alpha r_s/\pi^2)\, Q_{r_s}(q, iqu)} \, \frac{1}{(\vec{n}' - \vec{n} - \vec{q})^2 + (\alpha r_s/\pi^2)\, Q_{r_s}(|\vec{n}'-\vec{n}-\vec{q}|, iqu)} \tag{7.58}$$

where we have made the substitutions $\vec{n}' = \vec{n}+\vec{q}, + \vec{q}_2$, where $|\vec{n}'| = 1$, and also $\vec{q}_1 = \vec{q}$; $w = iqu$. The integration over $\vec{q}_2$ is replaced by an integration over the directions of $\vec{n}'$ because of the delta function in the integrand of equation 7.57.

We can simplify this expression to order r_s^0 by noting that since the integral exists we can first integrate over $q = |\vec{q}|$, holding the other variables fixed. We cannot set $r_s = 0$ since we would introduce a divergence for small values of q, which is of the form $\alpha(\ln r_s)^2 + \quad \ln r_s + c$ as $r_s \to 0$. The form of this divergence is obtained by studying the form of the integrand for small q with $r_s \to 0$. If we set $q = 0$ in the last factor we find that the integration over $\vec{n}'$ is logarithmically divergent so we must include the r_s dependent screening term in the limit $q \to 0$. Thus we approximate

$$\lim_{q\to 0} Q_{r_s}(|\vec{n}' - \vec{n} - \vec{q}'|, iqu) \to 4\pi + O(|\vec{n}' - \vec{n}|^2) \tag{7.59}$$

for fixed $\vec{n}'$. The terms of order $|\vec{n}' - \vec{n}|^2$ will be seen to contribute only to higher orders in r_s.

The integrand, as a function of q, is thus of the form $(A \ln r_s + c) + qf(q,r_s)$ where $qf(q,r_s) \to 0$ as $q \to 0$. For the term independent of q we must retain the screening terms for small values of q in the first propagator factor. Thus again we can approximate $Q_{r_s}(q,iqu)$ by $4\pi R(u)$. We can neglect the screening terms for the q dependent part of the integrand since no divergence is introduced from the q integration. Thus we can write

$$K_{21} = \frac{1}{2\pi^4} \int_{-\infty}^{\infty} \frac{du}{2\pi} \int d\Omega\ \vec{n}\cdot\vec{n}' \int d^3q\, \frac{1}{q\cdot(\vec{n}-\vec{n}')} \left[\frac{1}{iu-\vec{n}\cdot\vec{q}} - \frac{1}{iu-\vec{n}'\cdot\vec{q}}\right]$$

$$\frac{1}{q_2^2 + (4\alpha r_s/\pi)\, R(u)}\ \frac{1}{(n'-n)^2 + (4\alpha r_s/\pi)} + E(r_s) + O(r_s) \qquad (7.60)$$

where

$$E(r_s) = \frac{1}{2\pi^4} \lim_{\delta \to 0} \int_{-\infty}^{\infty} \frac{du}{2\pi} \int_{-1}^{1} d\Omega\ \vec{n}'\cdot\vec{n} \left\{ \int_{\delta < q} \frac{d^3q}{q^2}\, \frac{1}{q^2 + \vec{q}\cdot(\vec{n}-\vec{n}')} \right.$$

$$\left[\frac{1}{iu - \frac{1}{2}q - \vec{n}\cdot\vec{q}} - \frac{1}{iu + \frac{1}{2}q - \vec{n}\cdot\vec{q}}\right] \frac{1}{(\vec{n}'-\vec{n}-\vec{q})^2 + 4\alpha r_s/\pi}$$

$$\left. - \int_{\delta \leq q < 1} \frac{d^3q}{q^2}\, \frac{1}{\vec{q}\cdot(\vec{n}-\vec{n}')} \left[\frac{1}{iu-\vec{n}\cdot\vec{q}} - \frac{1}{iu-\vec{n}'\cdot\vec{q}}\right] \frac{1}{(\vec{n}'-\vec{n})^2 + 4\alpha r_s/\pi} \right\}. \qquad (7.61)$$

The first term in equation 7.60 has already been computed in equations 7.38 to 7.56.

$$K_{21} = -L\left[\ln \frac{4\alpha r_s}{\pi}\right]^2 + M' \ln \frac{4\alpha r_s}{\pi} + N' + E(r_s) \qquad (7.62)$$

where

$$-L = -2^{-1}\pi^{-2} = -0.051$$

$$M' = -M + \frac{1}{2\pi^3}\,\Delta(1) = -0.084 \qquad (7.63)$$

$$N' = -N + \frac{1}{\pi^3}\,\Delta(1) - \frac{1}{\pi^3}\lim_{\varepsilon\to 0}\int_{-1}^{1-\varepsilon}\frac{x\,dx}{2(1-x)}\,[\Delta(x) - \Delta(1)] = 0.071$$

(see equations 7.50 to 7.56).

In writing the expression for $E(r_s)$ we have simplified the screening term $Q_{r_s}(\vec{n}' - \vec{n} - \vec{q}, iqu)$ by replacing it by 4π (see also equation 7.59). This is exact to order $r_s^{\,o}$ since the factor $(\vec{n}'-\vec{n}-\vec{q})^{-2}$ is "dangerous" only for small values of q, i.e. the integral over $\vec{n}^1$ behaves like lnq. We have also neglected terms of order $(\vec{m}^1 - \vec{n})^2$ which appear in equation 7.59 since they contribute to order $r_s^{\,1}$ and higher.

The integration over μ in the first term of equation 7.61 is trivial and results in a factor $[\eta(-\frac{1}{2}q^2 - \vec{n}\cdot\vec{q}) - \eta(\frac{1}{2}q^2 - \vec{n}\cdot\vec{q})]$. The remaining integrations are extremely tedious to carry out and we will merely quote the results here. We find (after a considerable amount of labor) that to lowest order in r_s

$$-E(r_s) = \frac{1}{\pi^2}\int_0^2 \frac{dk}{k^2 + \frac{4\alpha r_s}{\pi}}\,(1-k^2/2)\,J(k) + C_2 + C_3 \qquad (7.64)$$

where

$$C_2 = \frac{1}{2\pi^4}\int_{k<2}\frac{d^3k}{k^2}\,d\Omega_{\vec{n}^1}\,\frac{\eta(-\frac{1}{2}k^2 - \vec{k}\cdot\vec{n}) - \eta(\frac{1}{2}k^2 - \vec{k}\cdot\vec{n}^1)}{(\vec{k} + \hat{n} - \hat{n}^1)^2} \qquad (7.65a)$$

$$= \frac{1}{\pi^2}\,[8\ln 2 - \frac{\pi^2}{3}] \qquad (7.65b)$$

$$C_3 = \frac{1}{\pi^4} \int_{k>2} \frac{d^3k}{k^2} \int d\Omega_{\vec{n}^1} \; \vec{n}\cdot\vec{n}^1 \; \frac{1}{(\vec{k}+\vec{n}-\vec{n}^1)^2} \, \frac{1}{\vec{k}\cdot(\vec{k}+\vec{n}-\vec{n}^1)} \qquad (7.66a)$$

$$= -\frac{1}{\pi^2} \left[4\ln 2 - \frac{5\pi^2}{24} - \frac{1}{4}(\ln 2)^2 - \frac{1}{2} \sum_{n=1} \left(\frac{1}{2}\right)^n \frac{1}{n^2} \right] \qquad (7.66b)$$

and

$$J(k) = \frac{1}{k} \left\{ 2 \ln 2 \cdot \ln\left|\frac{2+k}{2-k}\right| - \frac{1}{2} \left[\ln\left|\frac{2+k}{2-k}\right|\right]^2 - 2 \sum_{n=1}^{\infty} \left[\left(\frac{k-2}{k+2}\right)^n - (-1)^n\right] \frac{1}{n^2} \right\}$$

$$-2 \int_{-1}^{1} \frac{dZ}{4+k^2(Z^2-1)} \ln|1+Z| \; . \qquad (7.67)$$

To obtain these expressions we have made the substitution $\vec{k} = \vec{q} + \vec{n} - \vec{n}^1$ in the integrand of the first term in equation (7.61) and have replaced the integration over d^3q by an integration over d^3k. The integrations over the solid angles of $\vec{k}$ and $\vec{n}^1$ can then be carried out by a long but straightforward computation. The second term in equation (7.61) was easily evaluated and the limit as $\beta \to 0$ was then taken.

On carrying out the integration over k in equation (7.64) we find to lowest order in r_s,

$$-E(r_s) = \frac{1}{\pi^2}(3\ln 2 - 1)\ln\frac{\alpha r_s}{\pi} - \frac{2}{\pi^2}(1-\ln 2) + C_1 + C_2 + C_3 \qquad (7.68)$$

where

$$C_1 = \frac{1}{\pi^2} \int_0^2 \frac{dk}{k} (1-k^2/2) [J(k) - J(0)] \tag{7.69}$$

$$= \frac{1}{\pi^2} [6(\ln 2-1) - \frac{\pi^2}{8} + \frac{7}{4} (\ln 2)^2 + \frac{7}{2} \sum_{n=1}^{\infty} (\frac{1}{2})^n \frac{1}{n^2}] .$$

Using equations 7.65b, 7.66b, 7.69 and 7.68 we have then

$$E(r_s) = \frac{1}{\pi^2} (1-3 \ln 2) \ln \frac{\alpha r_s}{\pi} + \frac{1}{\pi^2} \left(\frac{3}{2} (\ln 2)^2 + 3 \sum_{n=1}^{\infty} (\frac{1}{2})^n \frac{1}{n^2} - 4\right)$$

$$= -0.110 \ln \alpha r_s - 0.123 . \tag{7.70}$$

Combining this with equations 7.62 and 7.63 results in

$$K_{12} = -0.051 [\ln \frac{\alpha r_s}{\pi}]^2 - 0.193 [\ln \frac{\alpha r_s}{\pi}] - 0.052 . \tag{7.71}$$

We also have the term $2K_{12} + K_{22}$ which is the contribution from transitions to states not on the Fermi surface. By direct calculation it is straightforward to show that this term is of order r_s^1. We can also demonstrate this in terms of diagrams. From equation 7.3 we see that for these transitions the effect of differentiating the electron hole propagators with respect to p is to replace them by $\vec{n}\cdot\vec{q}$ times their square. In terms of diagrams this is formally equivalent to inserting into the electron-hole line an interaction of strength $\vec{n}\cdot\vec{q}$ (indicated by a circle) which does not carry any momentum or energy. (Note the similarity to the effect of differentiation with respect to w.)

The lowest order contribution to this process arises from the effect of the bare Coulomb interaction. (The integrals converge in this case.) In figure 13 we show the diagrams involved. All of the diagrams contribute the same energy denominators and matrix elements, but K_{22} and K_{12} differ by one internal hole line and thus differ in sign. There are two diagrams for K_{12} which are equal and are cancelled by the two equal terms in K_{22} (i.e. with $\vec{n}\cdot\vec{q}_1$ and $\vec{n}\cdot\vec{q}_2$) since $\vec{q}_1$ and $\vec{q}_2$ are equivalent. Thus all terms of order r_s^o are zero.

All the remaining self energy processes are of third order or higher in the number of virtual excitations and can be shown to contribute terms of order r_s and higher to the specific heat.

We have completed the calculation of all terms of order r_s^o. If we use equations 7.56 and 7.70 we finally obtain the total contribution from the two virtual excitation process; exact to terms of order r_s^o.

$$(dW^{2a}/dp)_{p=1} = 2K_{11} + K_{21}$$

$$= 0.051 \left[\ln \frac{\alpha r_s}{\pi}\right]^2 - 0.167 \left[\ln \frac{\alpha r_s}{\pi}\right] - 0.444 . \qquad (7.71)$$

If we add this to the contribution from the unperturbed energy, $(d\omega/dp)_{p=1} = 2\alpha^{-2} r_s^{-2}$, and the contribution, $(dW^{(1)}/dp)_{p=1}$, as given by equation 7.29 we then obtain an expression for $(dW/dp)_{p=1}$ exact to terms of order r_s^o.

$$(dW/dp)_{p=1} = \frac{2}{\alpha^2 r_s^2} - \frac{1}{\pi \alpha r_s}\left[2 + \ln \frac{\alpha r_s}{\pi}\right] + 0.051 \left[\ln \frac{\alpha r_s}{\pi}\right]^2 - 0.302 \left[\ln \frac{\alpha r_s}{\pi}\right]$$

$$- 0.831 + 0(r_s) . \qquad (7.72)$$

The correction to the specific heat C_F of the gas of interacting electrons is then

$$C_F/C = 1 - \frac{\alpha r_s}{2\pi} \ln \frac{\alpha r_s}{\pi} \left[1 + 2.98 \frac{\alpha r_s}{\pi} + 0(r_s^2)\right] - \frac{\alpha r_s}{\pi}\left[1 + 4.10 \frac{\alpha r_s}{\pi} + 0(r_s^2)\right]$$

$$+ 0.251 \left[\frac{\alpha r_s}{\pi} \ln \frac{\alpha r_s}{\pi}\right]^2 \qquad (7.73)$$

$$= 1 - 0.083\, r_s \ln r_s \left[1 + 0.798\, r_s + 0\,(r_s^2)\right]$$

$$- 0.017\, r_s \left[1 + 1.12\, r_s + 0(r_s^2) \right[\qquad (7.74)$$

$$+ 0.0070 \left[r_s \ln r_s\right]^2 + 0\,(r_s^3) .$$

The result in equation (7.74) is plotted as a function of r_s in figure 12 . Note that at for values of $r_s \lesssim 0.8$ the specific heat is reduced by the interactions while for $r_s \gtrsim 0.8$ the specific heat is enhanced. To understand this result it appears that the distinction between self energy transitions parallel to the Fermi surface ("surface" transitions) and transitions "perpendicular" to the Fermi surface is important. This distinction is a direct result of the Pauli Principle from which the two terms in equation 7.73 are seen to originate. That is, the discontinuity in the propagator $S_F(p,\omega)$ which distinguishes between electrons and holes is the origin of the "surface" transitions while the propagation of electrons or holes gives rise to the "perpendicular" transitions.

Now ${}_a(dW^{(1)}/dp)_{p=1}$ given by equation 7.20 and $(dW^{(2)}/dp)_{p=1}$ given by equation 7.71 both are contributions from "surface" transitions only. The sum of these terms is positive for values of r_s up to about $r_s = 1.5$. It is not hard to see from equation 7.7 that ${}_a(dW^{(1)}/dp)_{p=1}$ is always positive. The apparent change in sign in equation 7.20 at $r_s \sim 2$ is a result of not including higher order terms in this expansion. Thus the change of sign in Gell-Mann's lowest order calculation at $r_s \sim 0.8$ is not real although the corrected result also changes sign at this value of r_s. The sign change in equation 7.74 has a different origin.

The contribution ${}_b(dW^{(1)}/dp)_{p=1}$ (equation 7.29) arises from "perpendicular" transitions only, and can likewise be shown to be always negative. The change in sign of $[C_F/C-1]$ at $r_s \sim 0.8$ thus appears to result from a change in the relative importance of the two types of transitions. At very high densities only the surface transitions in the self energy process of figure 6a are important because, as we pointed out above, they contribute an extra inverse power of the momentum transfer. As r_s increases the "perpendicular" transitions from this process begin to contribute a negative amount, ${}_b(dW^{(1)}/dp)_{p=1}$, while the surface transitions from the process in figure 6b also become important. For $r_s < 0.8$ the "surface" transitions dominate while for $r_s > 0.8$ the "perpendicular" transitions dominate.

Thus it appears that for $r_s > 0.8$ and for densities approaching the highest physical electron densities the specific heat is enhanced from the value for non-interacting electrons.

The actual range of convergence of this series cannot be determined precisely without a knowledge of still higher order corrections. However, we can estimate this in several ways. First of all it is clear from a statement above that the series for ${}_a(dW^{(1)}/dp)_{p=1}$ cannot be valid for $r_s \sim 2$ since it incorrectly changes sign here. Also we can compare $(dW^{(2)}/dp)_{p=1}$ with ${}_a(dW^{(1)}/dp)_{p=1}$ and we find that their ratio is less than one for $r_s \lesssim 2$. It appears then that this series is quite accurate for $r_s < 1$ and predicts a change in sign at $r_s \sim 0.8$. For $1 < r_s < 2$ the series probably over estimates the enhancement by as much as 50% at $r_s = 2$. However the sign of the correction is probably correct in this density range.

This series gives us a further clue concerning the convergence of the perturbation approach. It appears that this method can give quantitative results for densities down to the highest densities found in metals. For intermediate densities it does not appear so hopeful. The time and labor necessary to compute the next higher order corrections are certainly not justified considering the uncertainties in our understanding of phonon and lattice effects. The electron-phononinteraction has been estimated by Ferrell [40] to give a 20% enhancement of the specific heat for sodium, and thus appears to be a stronger effect than the electron-electron interaction.

Because of these uncertainties and the rather limited accuracy in the experimental data it appears premature to undertake a detailed comparison with experiment at this time.

In addition to the specific heat, the single particle energies W(p) can be used to calculate other properties of the interacting electron gas. The width of the conduction band in metals as observed in X-ray emission spectra has also been calculated. Details of this calculation will be presented elsewhere.

VIII. CONCLUSIONS

In Part One of this paper we saw that the Gell-Mann-Brueckner theory finds a natural description in the Feynman-Dyson formalism of field theory. We developed a theory of the effective interaction between electrons which takes into account the polarization of the many body medium. One result of this polarization is to screen the effective interaction at large distances. This effective interaction is most easily described by a Feynman propagator which determines the response of the system to a test charge disturbance. In the pair approximation of G-B, only the simplest approximation to the polarization charge is made, i.e. the excitation of single pairs. One of the main problems in calculating higher order corrections to their calculations was the computation of higher order polarization effects. In Chapter III (and Appendix D) we derived an exact expression for the ground state energy which takes into account the virtual polarization of the medium.

In Chapter IV we saw that the most striking effect of the polarization process is the introduction of the plasmon mode which has no counterpart in the non-interacting system. This mode arises from the repeated excitation and de-excitation of the medium. In the pair approximation of G-B the plasmon is an eigenstate of the Hamiltonian for small momenta ($q < q_c$), since it cannot decay into single pair excitations. When higher order polarization effects are considered we find that the plasmon energy is shifted and the plasmon has a finite lifetime since it can decay into multiple excitations even for $q < q_c$.

In Chapter VI we derived the lowest order correction to the plasmon energy and found that it is small for $q < q_c$ even at physical densities. However, we also computed the damping of the plasmon which is of higher order in r_s than the energy correction since it arises from the two pair decay mode and the pair-plasmon decay mode. The damping from these processes is important at physical densities even for values of q appreciably less than q_c. Hence, the observed plasmon cut-off in the electron scattering experiments is less sharp and should actually be observed at somewhat lower values of q than predicted by the pair approximation. The fact that the damping from these processes is strong at physical densities leads one to expect that the plasmon level shift caused by these processes is not negligible. More work remains to be done on this matter.

We conclude from these calculations that the plasmon mode in a free electron gas is well defined for small values of $q \ll q_c$. For values of $q \lesssim q_c$ the plasmon becomes strongly damped. In addition the formulas for the plasmon energy dispersion (equations 4.17 with equation 5.7) and the plasmon level width (equation 5.22) are probably accurate only for densities down to the highest densities found in metals, say $r_s < 3$. We judge this from the size of the corrections which we have computed.

In Chapter III we investigated validity of an independent particle model for the interacting Fermi gas. We found that the lowest states with only a few excited electrons and holes can be well described by an independent particle model. For these states only single particle Feynman propagators are needed. These independent particle states are not exact eigenstates of the interacting system since electrons and holes can undergo real, energy conserving, "collisions" with the many body medium. The calculation in Chapter VI showed then the damping of these single particle states vanishes quite rapidly (as $(p - P_F)^2$) as the momenta of the particles approaches the Fermi momentum. Hence the lowest excited states are asymptotically stationary states.

The basic quantity in this independent particle model is the single particle propagator corrected for self energy effects. These effects arise from the virtual polarization of the medium by the particle and the subsequent reaction of the polarization charge on the particle. In Appendix B we proved that the energy shift and damping of the single particle states can be found directly from this propagator. This enabled us in Chapter VI to give a complete formal discussion of electron scattering from the electron gas, including exchange corrections.

In Chapter VII the independent particle model and the self-energy method was used to compute the low temperature specific heat of an electron gas. By a rather long calculation we extended the lowest order calculation of Gell-Mann to the next order in r_s. We found as r_s increases that for $r_s > 0.8$ the correction to the specific heat of a non-interacting gas changes from a reduction to an enhancement. This change of sign appears to be the result of the Pauli Principle. The series for the specific heat appears to converge only for values of $r_s \lesssim 2$ so that it applies only to the very highest electron densities found in metals. We find that for these densities the interactions increase the specific heat.

On the basis of these calculations we conclude that the procedure of expansion in orders of r_s is only successful for high densities extending down to the highest physical electron densities ($r_s \lesssim 2$). For intermediate densities ($2 < r_s < 5.6$) the method does not appear to converge rapidly enough to be useful. If the integrals in the third order correction to the correlation energy (Appendix D) can be evaluated we will have still another example of the convergence of this method.

If some simple approximation, valid at intermediate densities, to the dielectric constant $K_{r_s}(q,\omega)$ (or equivalently to $Q_{r_s}(q,\omega)$) can be found the expansion in orders of the effective interaction (not in orders of r_s) would still probably be useful. The possibility of this will be the subject of further research.

In conclusion let us say that through the efforts of many workers the problem of electron interactions is now well understood at least qualitatively. The method of Gell-Mann and Brueckner has led to quantitative results in the limit of high densities. Future development will lie in the intermediate density range and in the application of the theory to actual metals.

APPENDICES

APPENDIX A

HIGHER ORDER POLARIZATION PROPAGATORS

In Chapter II we discussed in a general manner the corrections to the irreducible polarization propagator $Q_{r_s}(q,\omega)$. In this appendix we shall derive the propagators for the lowest order corrections to $Q_{r_s}(q,\omega)$. These correspond to the diagrams b and c in figure 3 of the text. Applying our rules we have

$$-Q^{(1a)}(q,\omega) = \frac{\alpha r_s}{2\pi^2}\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2 \int d^3\vec{p}_1 \int d^3\vec{p}_3$$

$$S_F(\vec{P}_1+\vec{q},\ \omega_1+\omega)\ S_F(\vec{P}_1,\ \omega_1)\ P_{r_s}(\vec{P}_1-\vec{P}_2,\omega_1-\omega_2)$$

$$S_F(\vec{P}_2+\vec{q},\omega_2+\omega)\ S_F(\vec{P}_2,\omega_2) \tag{A1}$$

$$-Q^{(1b)}(q,\ \ ) = \frac{\alpha r_s}{2\pi^2}\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2 \int d^3\vec{P}_1 \int d^3 p_3$$

$$S_F(\vec{P}_1+\vec{q},\ \omega_1+\omega)\ S_F(\vec{P}_1,\omega_1)\ P_{r_s}(\vec{P}_1-\vec{P}_2,\ \omega_1-\omega_2)$$

$$S_F(\vec{P}_1+\vec{q},\omega_1+\omega)\ S_F(\vec{P}_2+\vec{q},\ \omega_2+\omega). \qquad \text{A2)}$$

The coefficients follow from the rules and the definitions in Chapter II. For the applications in this paper it is sufficient to calculate these propagators to lowest order in r_s. If we put $r_s = 0$ in the denominator of $P_r(\vec{P}_1-\vec{P}_2,\omega_1-\omega_2)$ (i.e. if we use the bare Coulomb interaction) then the integrals separately diverge due to the singular factor $(\vec{P}_1 - \vec{P}_2)^{-2}$. However, the divergences cancel in the sum $Q^{(1)} = Q^{(1a)} + Q^{(1b)}$. With this simplification we can carry out the integrations over ω_1 and ω_2 to get

$$Q^{(1)}_{r_s}(q,\omega) = \frac{\alpha r_s}{2\pi^2} \int d^3p_1 \int d^3p_2 \frac{1}{(\vec{P}_1 - \vec{P}_2)^2} \left[\frac{\eta(|\vec{P}_1+\vec{q}|-1)\,\eta(1-p_1)}{\omega - \Delta_1 + i\varepsilon}\right.$$

$$\left. - \frac{\eta(p_1-1)\;\eta(1-|\vec{P}_1+\vec{q}|)}{\omega - \Delta_1 - i\varepsilon}\right] \times$$

$$\left[\frac{\eta(|\vec{P}_2+\vec{q}|-1)\;\eta(1-p_2)}{\omega - \Delta_2 + i\varepsilon} - \frac{\eta(p_2-1)\;\eta(1-|\vec{P}_2+\vec{q}|)}{\omega - \Delta_2 - i\varepsilon}\right]$$

$$- \left[\frac{\eta(|\vec{P}_1+\vec{q}|-1)\;\eta(1-p_1)}{(\omega - \Delta_1 + i\varepsilon)^2} - \frac{\eta(p_1-1)\;\eta(1-|\vec{P}_1+\vec{q}|)}{(\omega - \Delta_1 - i\varepsilon)^2}\right] \times$$

$$[\,\eta(1-p_2) - \eta(1-|\vec{P}_2+\vec{q}|)] \qquad \text{(A3)}$$

where $\Delta_1 = q^2/2 + \vec{q}\cdot\vec{P}_1$ and $\Delta_2 = q^2/2 + \vec{q}\cdot\vec{P}_2$. [Note that we have used the definitions of equations 1.26 and 1.27 to perform one of the ω_2 integrations.] We can use an argument similar to that used to obtain equation 1.45 to show that the Pauli Principle restrictions can be dropped in real parts of these integrals. In fact, if $\vec{P}_1 + \vec{q} = \vec{P}_o$, where $p_o<1$, in the first term in the first brackets its contribution is exactly cancelled by the second term when $\vec{P}_1 = \vec{P}_o - \vec{q}$. Continuing this argument for the other terms in the integrand we can show that

$$\text{Re } Q^{(1)}_{r_s}(q,\omega) = \frac{\alpha r_s}{2\pi^2} \int d^3p_1 \int d^3p_2 \frac{1}{(\vec{P} - \vec{P}\ )^2} [\eta(1-p_1) - \eta(1-|\vec{P}_1+\vec{q}|)]$$

$$[\eta(1-p_2) - \eta(1-|\vec{P}_2+\vec{q}|)]$$

$$P\left[\frac{1}{\omega - q^2/2 - \vec{q}\cdot\vec{P}_1}\;\frac{1}{\omega - q^2/2 - \vec{q}\;\vec{P}_2} - \frac{1}{(\omega - q^2/2 - \vec{q}\cdot\vec{P}_1)^2}\right] \qquad \text{(A4)}$$

where P denotes the principal value. It is easy to see from equation A2 that $\text{Im}\, Q^{(1)}_{r_s}(q,\omega)$ is zero if $|\omega| > q^2/2 + q$ and that there is a discontinuity in $\text{Im}\, Q^{(1)}_{r_s}(q,\omega)$ along the real ω axis from $-q^2/2 - q$ to $q^2/2 + q$. Thus $Q^{(1)}_{r_s}(q,\omega)$ has the same branch cut as $Q_o(q,\omega)$. We will not have occasion to use $\text{Im}\, Q^{(1)}_{r_s}$ in any of our calculations so we will not give an explicit expression for it here.

For much of our work we will need this integral only in the limit of small q. Using the limiting relation

$$\eta\,(1 - |\vec{P} + \vec{q}|) \xrightarrow[q\to 0]{} \eta\,(1-p) - \vec{P}\cdot\vec{q}\;\delta\,(p-1) + O(q^2)$$

we find that

$$\text{Re}\, Q^{(1)}_{r_s}(q,\omega) \underset{q\to 0}{=} \frac{\alpha r_s}{4\pi^2}\,(2\pi)^2 \int_{-1}^{1} x_1\, dx_1 \int_{-1}^{1} x_2\, dx_2\, \frac{(x_1-x_2)}{|x_1-x_2|}$$

$$\frac{1}{(x_1 - \frac{\omega}{q})^2}\,\frac{1}{(x_2 - \frac{\omega}{q})}$$

$$= \frac{\alpha r_s}{4\pi^2}\,(2\pi)^2\, R^{(1)}(\frac{\omega}{q}) \tag{A5}$$

where $x_1 = (\vec{P}_1\cdot\vec{q})$ and $x_2 = (\vec{P}_2\cdot\vec{q})$. We shall also need

$$Q^{(1)}_{r_s}(0,0) = \text{Re}\, Q^{(1)}_{r_s}(0,0) = \frac{\alpha r_s}{4\pi^2}\,(2\pi)^2 \int_{-1}^{1} \frac{dx_1}{x_1} \int_{-1}^{1} dx_2\, \frac{(x_1 - x_2)}{|x_1 - x_2|} \tag{A6}$$

$$= (4\pi)^2\, \frac{\alpha r_s}{4\pi^2} = 4\alpha r_s\,. \tag{A6a}$$

To calculate the next order corrections to $Q_{r_s}(q,\omega)$ we would need to: i) Compute the terms of order r_s^2 in $Q_{r_s}^{(1)}(q,\omega)$ which we have neglected above by our use of the bare Coulomb interaction; ii) compute the lowest order terms in the amplitude $Q_{r_s}^{(2)}(q,\omega)$ which corresponds to diagrams d through f in figure 3 of the text. These diagrams involve two internal interactions and are therefore of lowest order r_s^2. Again we wish to emphasize that it would be misleading to include correlation effects in computing $Q_{r_s}^{(1)}(q,\omega)$ without including terms of the same order from $Q_{r_s}^{(2)}(q,\omega)$ and higher order amplitudes.

APPENDIX B

THE PROPAGATOR FORMALISM

In this appendix we will derive on more rigorous grounds some of the equations and concepts used in the text. In particular we will discuss: 1) The application of the Feynman-Dyson techniques to many-particle excited states. 2) The interpretation of the singularities of the physical propagators which lead to formulas for the energy shifts and lifetimes of excited states. 3) The relationship of our methods to other work of a similar nature. 4) The relationship between the physical propagator and the amplitude to emit or absorb the physical particle in a real process.

The basis of our method is the Feynman-Dyson expansion of the S matrix and the concept of a state of the interacting system which is the adiabatic transform of a state Φ_α of the non-interacting system. This is the state at t_2 which develops from the state Φ_α as the interaction is slowly turned on a $t_1 = -\infty$, or

$$\Psi_\alpha = U(t_2, t_1)\,\Phi_\alpha$$

in the interaction representation. In this way the "physical" state Ψ_α is defined in terms of a "bare" state Φ_α.

We will define the corrected propagator for a state of μ electrons (momenta and spins denoted by the single symbols $P_1, \dots, P_\mu$) and ν holes $(k_1, \dots, k_\nu)$ as the ground state expectation value*

$$G_{\mu\nu}(P_1, \dots, p \;; k_1 \cdots k \;, t_2 - t_1)$$

$$= (\Phi_o|P(S\, A_{\mu\nu}(t_2)\, A^+_{\mu\nu}(t_1)|\, \Phi_o) \qquad \text{B1}$$

where P is Dyson's time ordering operator and $A_{\mu\nu}(t)$ is the operator, in the interaction picture, which destroys the bare state $\Phi_{\mu\nu}$ of μ electrons and ν holes. That is

$$A_{\mu\nu}(t) = c_{\mu\nu}\, a_{P_1} \cdots a_P \;\; b_{k_1} \cdots b_k \qquad \text{B2}$$

$$\exp - it \left\{ \varepsilon_o + w(p_1) + w(p_2) + \cdots + w(p_\mu) - w(k_1) - w(k_2) - \cdots - w(k_\nu) \right\}$$

where $c_{\mu\nu}$ is an appropriate normalization factor, ε_o is the unperturbed ground state energy, and $w(p)$ is the unperturbed single particle energy. The order of the particle destruction operators in this expression is arbitrary. In equation B1, S denotes the S matrix $S = U(-T/2, T/2)$ where T is a large interval of time (see equation 1.23)

---

* This propagator is not, strictly speaking a Feynman type propagator. It is similar to the representation of many particle Feynman propagators in relativistic field theory [35][36]. Our definition differs from the latter in that we use the same time variable for all particles, which is not a relativistically invariant scheme. With this change our propagator is essentially the diagonal part of the Feynman propagators; i.e. the creation and destruction operators refer to the same state.

If we use equation 1.24 for the U matrix we have (using units where $\hbar = 1$)

$$G_{\mu\nu}(P_1, \ldots, P \ ; k_1 \ \cdots \ k \ ; t_2 - t_1) \qquad \text{B3}$$

$$= (\Phi_o | \sum_{n=o}^{\infty} (-i)^n \int_{-T/2}^{T/2} dt_1 \cdots \int_{-T/2}^{T/2} dt_n$$

$$P[H_1(t_1) \cdots H_1(t_n) A_{\mu\nu}(t_2) A^+_{\mu\nu}(t_1)] | \Phi_o) .$$

This matrix element can be evaluated in a perturbation series by using Wick's theorem [20] just as was done for the evaluation of the S matrix itself. It follows that $G_{\mu\nu}$ is the sum of all corrections which can be made to the uncorrected propagators for the state $\Phi_{\mu\nu}$. The uncorrected propagator is defined by equation B1 with $S = 1$. In terms of diagrams equation B3 is the sum of all diagrams leading from the virtual state $\Phi_{\mu\nu}$ at time t_1 and back to $\Phi_{\mu\nu}$ at time t_2. Ground state (vacuum fluctuation) diagrams can occur as well, but as we have seen these merely contribute a multiplicative factor, e^{-iE_oT} where E_o is the corrected ground state energy.

If we consider the time interval where $t_2 = T/2$, $t_1 = -T/2$ we can write the time ordered expression in equation B1 as

$$G \ = (\Phi_o | A_{\mu\nu}(T/2) \ U(T/2, -T/2) \ A_{\mu\nu}(-T/2) \ \Phi_o)$$

$$= (\Phi_{\mu\nu} | \exp - iT \left\{ \varepsilon_o + w(P_1) - \cdots - w(k \right\} U(T/2, -T/2) | \Phi_{\mu\nu}) \qquad \text{B4}$$

$$= (\Phi_{\mu\nu} | e^{-iH_oT} \ U(T/2, -T/2) | \ \Phi_{\mu\nu}) . \qquad \text{B5}$$

Thus, for a long time interval, $t_2 - t_1 = T$, the propagator $G_{\mu\nu}$ is the amplitude that the system which was in the state $\Phi_{\mu\nu}$ at $t_1 = -T/2$ remains in the state $\Phi_{\mu\nu}$ at time $t_2 = T/2$. The restriction to a large time interval is necessary if we use the rules in Chapter I to evaluate the U matrix. In these rules energy is conserved at each vertex of a diagram. The physical justification for this approximation is discussed in Chapter III.

Let us examine the diagrams which arise from the expansion of the many particle propagator, equation B3. As in the special case of two particles which we discussed in Section D of Chapter III, the diagrams can be divided into two groups: i) Those with no interactions between particle lines; ii) Those with interactions between particles. Consider a diagram of the latter class of nth order in the interaction. Since momentum and energy must be conserved at each vertex, the requirement that the final state be identical to the initial state determines at least one of the energy-momentum transfers in terms of all the others. That is, the momentum and energy of one of the virtual excitations are fixed which also means that one of the intermediate states is fixed. The contribution of a single intermediate state is proportional to $1/V$ where V is the volume of the system so we see that such diagrams can contribute terms of order $1/N$ or higher. For a state with $\mu+\nu$ excited particles there are $\mu+\nu$ possible diagrams which contribute terms of order $1/N$. Thus we see that we can neglect diagrams of the second type if $\mu+\nu/N \ll 1$. The remaining diagrams of type i involve only single particle self energy processes of the type discussed in Chapter III. If we neglect interactions between particle lines then the time variables $t_{a1}, t_{a2}, \ldots, t_{an}$ associated with the vertices along the ath particle line can be integrated over, independent of the remaining time variables associated with the other particles. Thus we are led to write

$$G_{\mu\nu}(P_1, \dots, P_\mu ; k_1, \dots, k_\nu ; t_2 - t_1)$$

$$= G_{oo}(t_2 - t_1)\, G_{1o}(P_1; 0; t_2 - t_1)\, G_{1o}(P_2; 0; t_2 - t_1)$$

$$\cdots G_{o1}(0, k_1; t_2 - t_1) \cdots G_{o1}(0, k_\nu; t_2 - t_1) + O\left(\frac{\mu+\nu}{N}\right) \qquad \text{B6}$$

where $G_{1o}(P_i, 0; t_2 - t_1)$ is the corrected propagator for a single electron state $\{\vec{P}_i, s_i\}$ and $G_{o1}(0, k_j; t_2 - t_1)$ is the corrected propagator for the single hole state $\{\vec{k}_j, s_j\}$. $G_{oo}(t_2 - t_1)$ is the propagator for the ground state which is, by equations B1 and B2 ,

$$G_{oo}(t_2 - t_1) = (\Phi_o | e^{-i\varepsilon_o(t_2 - t_1)}\, U(-T/2, T/2) | \Phi_o). \qquad \text{B7}$$

From section B of Chapter III we have the result for a large time interval $t_2 - t_1 = T$.

$$G_{oo}(T) = e^{-iE_o T} . \qquad \text{B8}$$

Now it is readily shown that the corrected Feynman type propagator $S'_F(\vec{x}, t)$ which we discussed in some detail in Chapter III is related to these propagators by

$$S'_F(\vec{P}, t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dw\, e^{-iwt}\, S'_F(\vec{P}, w)$$

$$= G_{1o}(\vec{P}; 0; t)\, \eta(p-1) + G_{o1}(0; \vec{P}; -t)\, \eta(1-p) . \qquad \text{B9}$$

That is, $S_F'(\vec{P},t)$ combines the electron and hole into one propagator and reverses the time for the hole. A compact definition of $S_F'(\vec{x}_{21},t_{21})$ is given by the ground state expectation $S_F'(\vec{x}_{21},t_{21}) = (\Phi_o|T(S\psi(\vec{x}_2,t_2)\overset{+}{\psi}(x_1,t_1)\,\Phi_o)$. The above results follow if $\psi(\vec{x},t)$ is decomposed into electron and hole parts.

Thus we find that for states where the relative number of excited particles $\mu+\nu/N$ is small we need only to study the properties of the single particle propagators $S_F'(\vec{x},t)$. Equation B6 then forms the basis for an independent particle formulation for the low excited states of an electron gas.

From Chapter III we know that

$$S_F'(\vec{x},t) = \frac{i}{(2\pi)^4}\int d^3p \int_{-\infty}^{\infty} dw \; \frac{e^{i(\vec{p}\cdot\vec{x}-wt)}}{w - w(p) - \sum(p,w)} \qquad \text{B10}$$

where the kernel $\sum(\vec{p},w)$ is the sum of all the amplitudes for proper self energy parts. In the complex w plane, $\sum(p,w)$ has a branch cut displaced by $-i\varepsilon$ below the real w axis from $w \cong 1/2^*$ $(= 1/2\ P_F^2/m$ in ordinary units) to $+\infty$ corresponding to a continuous distribution of poles which occur in the energy denominators for states of one hole plus excitations of the medium (i.e. modified pairs and plasmons as discussed in Chapter IV). Similarly there is a branch cut located above the real w axis by $+i\varepsilon$ from $w = 0$ to $w \overset{*}{\cong} 1/2$ corresponding to intermediate states of one hole plus excitations. Near $w \underset{*}{\cong} 1/2$ the density of intermediate states is of the order of $(w - 1/2)^2$ (see Chapter VI). It can also be shown that there is a discontinuity of $2i\ \mathrm{Im}\sum(p,w)$ on crossing the branch cut from below, and that for real w

---

* It was shown in Chapter VI that the dividing point of the branch line is at $w = W(1) = 1/2 + O(r_s)$. In the neighborhood of this point $\mathrm{Im}\sum(p,w) \propto (w - W(1))^2$.

$$\mathrm{Im}\,\Sigma(p,\omega)\begin{cases} < 0 & \omega > W(1) \sim 1/\lambda \\ > 0 & \omega < W(1) \sim 1/\lambda \end{cases} . \qquad \text{B11}$$

The difference in sign is, of course, due to the definition of the contour of integration which passes below the branch cut for $\omega > W(1)$ and above for $\omega < W(1)$.

In order to evaluate the integral B10 by contour integration it is necessary to have a precise understanding of the singularities and analytic properties of the analytic continuation of $\Sigma(p,\omega)$ to complex values of $\omega = Z$. For real Z this function is defined one side of the branch line by taking the limit as $\varepsilon \to 0$ in the Feynman propagators. If we analytically continue $\Sigma(p,Z)$ to complex Z we then define a branch (or Riemann sheet) of the multiple valued function $\Sigma(p,Z)$. We will call this the principal branch of $\Sigma(p,Z)$ which we can define more precisely by the equation

$$\lim_{\delta \to +0} \Sigma(x \pm i\delta) = \mathcal{E}(x) \mp \frac{1}{2}\, i\, \Gamma(x) \qquad \text{B12}$$

where x is real and $\Gamma(x)$ is a positive function of x. This definition is clearly equivalent to relation B11. [In this equation and those immediately to follow we will suppress the momentum index p. Thus $\omega \equiv \omega(p)$, $\Sigma(x) \equiv \Sigma(p,x)$.]

It is clear, then, that the zeros Z_0 of

$$f(Z) = Z - \omega - \Sigma(Z) \qquad \text{B13}$$

cannot lie on the real axis except if $Z_0 \approx 1/2$ (since $\Gamma(x) \propto (x-\frac{1}{2})^2$).

In general Z_o is complex. For every solution Z_o there is also a conjugate solution $\bar{Z}_o$. The relationship of the zeros Z_o to the principal plane of $\sum(p,Z)$ can be determined by expanding $\sum(p,Z)$ in a power series about $Z = \omega + i\delta$ where δ is infinitesimal. Within its radius of convergence this series provides an analytic continuation of $\sum(p,Z)$ across the branch line onto the adjacent branch (or Riemann sheet). If we then ask for the location of the zeros of $f(Z)$, (assuming that for sufficiently small r_s a zero lies within the circle of convergence) we find from equation B11

$$\operatorname{Im} Z_o \begin{cases} < 0 & ; \quad \delta > 0 \\ > 0 & \quad \delta < 0 \end{cases}$$

so that in each case to reach Z_o from the principal branch of $\sum(p,Z)$ we must analytically continue onto the adjacent branch. In this manner it can be shown that there are no zeros of $f(Z)$ lying on the principal branch of $\sum(p,Z)$.

Let us demonstrate this more explicitly to lowest order in r_s. Let us arbitrarily pick a zero, Z_o. Since $\sum$ is proportional to r_s we know that $Z_o = \omega + O(r_s)$ and we have to first order in r_s

$$Z_o = \omega + \sum(\omega) = \omega + \mathcal{E}(\omega) - \frac{i}{2}\Gamma(\omega).$$

Now from equation B13 we have to first order in r_s

$$\sum(Z_o) = \mathcal{E}(\omega) + \frac{i}{2}\Gamma(\omega)$$

if Z_o is assumed to lie on the principal plane. But then

$$f(Z_o) = Z_o - \omega - \sum(Z_o) = -i\Gamma(\omega)$$

to lowest order in r_s. Thus Z_o cannot be a zero of $f(Z_o)$ unless $\Gamma(\omega) \equiv 0$. In order to avoid a contradiction, then, Z_o cannot lie on the principal branch of $\sum(p,Z)$. Clearly the same holds true for $\bar{Z}_o$.

To evaluate equation B10 by contour integration we can deform the original contour, which lies in the principal plane, to encircle the branch lines without enclosing any poles (see figure 14). Since the discontinuity in $\sum(p,Z)$ is in the sign of the imaginary part, on crossing the discontinuity from below the total discontinuity is $2i\ \mathrm{Im}\sum(p,\omega) = i\Gamma(p,\omega)$. Hence, if we define $E(p,\omega) = \omega(p) + \mathrm{Re}\sum(p,\omega)$, we can write

$$S_F'(p,t_2 - t_1) \quad [\eta(p - 1)\eta(t_2 - t_1) - \eta(1 - p)\eta(t_1 - t_2)]$$

$$\frac{1}{2\pi} \int_{Cp} d\omega\, e^{-i\omega(t_2-t_1)} \frac{\Gamma(p,\omega)}{[\omega - E(p,\omega)]^2 + \frac{1}{4}\Gamma^2(p,\omega)}$$

where Cp is the real interval from W(1) to ∞ if $p > 1$ and Cp is the real interval from $-\infty$ to W(1) if $p < 1$.

Now $\Gamma(p,\omega) = O(r_s)$ while $E(p,\omega) = \omega(p) + O(r_s)$ so that if perturbation theory is valid we have $\Gamma/E \ll 1$. Therefore the integrand is sharply peaked at $\omega^* = E(p,\omega^*)$. As

$$t_2 - t_1 = T \rightarrow \infty \qquad \text{the integrand}$$

oscillates rapidly and the integral vanishes unless the width of the peak at $\omega = \omega^*$ is sharp enough to pick out less than one oscillation, i.e. if $\Gamma(p,\omega^*)\,T \ll 1$. In this case only the integrand in the neighborhood of ω^* contributes and we make very little error by replacing $E(p,\omega)$ by $E(p,\omega^*)$ and $\Gamma(p,\omega)$ by $\Gamma(p,\omega^*)$. The integral can then be evaluated to give $\exp - i\left\{E(p,\omega^*) - \frac{i}{2}\Gamma(p,\omega^*)\right\}T$.

Comparing with equation 1.9 for the free particle propagator we see that we can define $E(p,\omega^*) = W(p)$ as the energy of the physical one particle state and $\Gamma(p,w^*) = \Gamma(p)$ as the level width provided $\Gamma(p)\,T \ll 1$ and $\Gamma(p)/W(p) \ll 1$. The single particle energy is defined precisely by the equation

$$W(p) = \omega(p) + \mathrm{Re}\sum_{r_s}(p, W(p)) \qquad \text{B15a}$$

which, as we have seen in Chapter III, can be solved by iteration. The level width of the one electron state is given by

$$\frac{1}{2}\Gamma(p) = \lim_{\delta\to +0} \mathrm{Im} \sum_{r_s}(p,\ W(p) + i\delta)\,. \qquad \text{B15b}$$

Let us remark that to lowest orders in r_s, $W(p)$ and $\Gamma(p)$ can be found directly from the <u>pole</u> of the propagator $S_F'(p,\omega)$ which is the justification of our remarks in Chapter III. This is related to the fact that the discontinuity across the branch line is proportional to r_s. As the discontinuity vanishes the pole approaches the real axis and in the limit as $r_s \to 0$ the residue from the pole is the <u>only</u> contribution to the integral B10.

Hence in the limit of large $|t_2 - t_1| = T$

$$S_F'(P,T) \to e^{-iW(p)T} e^{-\frac{1}{2}\Gamma(p)|T|} [\eta(T)\,\eta(p-1) - \eta(-T)\,\eta(1-p)] \qquad \text{B16}$$

provided that

$$\frac{1}{2}\Gamma(p)\ |T| \ll 1. \qquad \text{B17}$$

This condition is necessary so that the physical one electron state is observable and therefore not lost in the background of transient excitations.

We can now use this result and equations B5, B6 and B9 to evaluate the amplitude for remaining in the state $\Phi_{\mu\nu}$. We have

$$(\Phi_{\mu\nu}|e^{-iH_oT}\ U(T/2, -T/2)|\ \Phi_{\mu\nu}) = e^{-iTE_{\mu\nu}}\ e^{-\frac{1}{2}\Gamma_{\mu\nu}T} \qquad \text{B18}$$

where

$$E_{\mu\nu} = E_o + \sum_{i=o}^{\mu} W(p_i) - \sum_{j=o}^{\nu} W(k_j) + 0(\frac{\mu+\nu}{N}) \qquad \text{B19}$$

$$\Gamma_{\mu\nu} = \sum_{i=o}^{\mu} \Gamma(p_i) + \sum_{j=o}^{\nu} \Gamma(k_j) + 0(\frac{\mu+\nu}{N}). \qquad \text{B20}$$

These equations describe an independent particle model for the gas of interacting electrons. It is clear that if $\Gamma \equiv 0$ that $E_{\mu\nu}$ is then an exact eigenvalue of the Hamiltonian $H_o + H_1$. As discussed in Chapter III, this is only rigorously true for the ground state. The conditions under which B18 is valid are

$$E_{\mu\nu}\, T \gg 1 \gg \frac{1}{2}\Gamma_{\mu\nu}\, T \qquad \text{B21}$$

and

$$\frac{E_{\mu\nu}}{\Gamma_{\mu\nu}} \gg 1 \,. \qquad \text{B22}$$

Physically, this means that we are interested in times much greater than the period of any of the lower lying states, while in this same time the physical state has not decayed appreciably. This presupposes that $E_{\mu\nu} \gg \Gamma_{\mu\nu}$ which is the requirement that the states of the interacting system be well defined. When this condition is not met, the concept of a state $\Psi_{\mu\nu}$ of the interacting system, which is the adiabatic transform of $\Phi_{\mu\nu}$, is not meaningful.

From equation B18 we can compute the probability of remaining in the state $\Phi_{\mu\nu}$ to be

$$|\langle \Phi_{\mu\nu}|\, e^{-iH_oT} S |\, \Phi_{\mu\nu}\rangle|^2 = e^{-\Gamma_{\mu\nu}T} = 1 - \Gamma_{\mu\nu}T \qquad \text{B23}$$

and since $S = U(T/2, -T/2)$ is unitary we have

$$\Gamma_{\mu\nu}\, T = \sum_{n\neq\mu\nu} |(\Phi_n|S|\, \Phi_{\mu\nu})|^2 \,. \qquad \text{B24}$$

So that if $\Gamma_{\mu\nu}\, T \ll 1$, $\Gamma_{\mu\nu}$ is given by the usual formula for the transition probability per unit time out of the state $\Phi_{\mu\nu}$. We can also compute $\Gamma_{\mu\nu}$ by computing $\Gamma(p)$ directly from $\sum(p;\, w)$. In this calculation the condition $\Gamma_i\, T \ll 1$ is equivalent to first taking the limit as $\Gamma \to 0$ in all of the propagators and then taking $T \to \infty$ and calculating with the Feynman rules. If this is done it can be shown that $\Gamma(p)$ is given by the usual formula for the transition probability per unit time out of the single particle state of momentum $\vec{P}$. This is demonstrated to lowest orders by specific calculations in the text.

These time dependent considerations suffer from the necessity of letting the time interval T become infinitely large. Another approach is to analyze the propagators, say $S_F'(x,t)$, in the Heisenberg representation (see [15]). Here one deals explicitly with the true eigenstates and eigenvalues of H which, of course, are not known to begin with. One can then show, for example, that if the bare single particle state is non-degenerate, the Fourier transformed propagator $S_F'(P,\omega)$, has a pole at $\omega = W(p)$ where $W(p) + E_0$ is a true eigenvalue of the complete Hamiltonian. If, on the other hand, the single particle state is degenerate, then $S_F'(P,\omega)$ has no true pole but instead has a branch discontinuity extending along the continuum of eigenvalues of H. If the degeneracy is small, i.e. if the degenerate states occupy only a small volume in phase space, then $S_F'(P,\omega)$ is strongly peaked at $\omega = W(p)$ where $W(p)$ is <u>approximately</u> an eigenvalue of H. In this case $W(p)$ is defined only to within a certain range $\Gamma(p)$ where $\Gamma(p)$ is proportional to the volume of phase space occupied by the degenerate states and to the range of energies into which these degenerate states are transformed by the interaction. We thus find that our time dependent approach is indeed correct.

These methods also apply to field theoretic problems of interacting elementary particles which one particle can decay weakly into one or several other particles. The properties of the single particle propagators provide a covariant definition of the lifetimes of these particles.

The self energy method is a more convenient way of actually computing the energies of excited states than the actual evaluation of the S matrix elements. This is because only proper self-energy diagrams need to be considered. This simplification is similar to the simplification of linked cluster perturbation expansions over ordinary Raleigh-Schrodinger perturbation theory.

A similar perturbation theory which provides a definition of the energy shift and lifetime of meta-stable states has been given by Hugenholtz [14]. He did not use the Feynman-Dyson theory but rather a form of perturbation theory developed by Van Hove [38]. This type of perturbation theory is not readily applicable to the electron gas problem where the Feynman propagator technique seems to be essential. The results of the theory presented above are also similar to the theory of radiation damping of Heitler and Ma [23]. Our method is even more closely related to the method of Low [24] who applied the Feynman-Dyson techniques to the theory of radiation damping. The concentration on the analytic properties of the physical propagator, however, is more closely related to the Hugenholtz method.

Before ending this appendix we should show how calculations can be carried out when these metastable states occur as initial or final states, i.e. as real processes. The applitude to emit a bare electron $(p > 1)$ or absorb a bare hole of spin S is

$$A_s^+(p,t) = X_s \, [\eta(p-1) - \eta(1-p)] \, e^{-it\,\omega(p)} \qquad \text{B28}$$

which we can also write as

$$A_s^+(p,t) = X_s \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} 2i \, \mathrm{Im} \, S_F(p,\omega) \, e^{-it\omega} \quad . \qquad \text{B29}$$

The interaction changes S_F into S_F' so that the corresponding amplitude for physical particles is

$$A_S^+ (p,t) = X_S \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{\Gamma(p,\omega)\, e^{-it\omega}}{[\omega - E(p,\omega)]^2 + \frac{1}{4}\, \Gamma(p,\omega)} \qquad \text{B30}$$

where we have used equations B10, and the definitions preceding equation B14. The amplitude to absorb an electron and emit a hole is the conjugate $A_S'(p,t)$. When these expressions are used in computing S matrix elements, the characteristic "intensity distribution" formulas familiar to all theories of line broadening appear.

APPENDIX C

EVALUATION OF INTEGRALS

Often in the theory of a degenerate electron gas integrals of the form

$$I(q) \equiv \int_{\substack{p<1 \\ |\vec{P}+\vec{q}|>1}} d^3p \; f(\vec{P},\vec{q}) \tag{C1}$$

where the integration is over all momenta within the Fermi sphere $(p=1)$ which can be excited outside the sphere by a momentum $\vec{q}$. If we introduce a parameter α (if $\alpha \leq 1$ all integrals which arise are finite) and define

$$J(q,\alpha) \equiv \int_{\substack{p<1 \\ |\vec{P}+\alpha\vec{q}|>1}} d^3p \; f(\vec{P},\vec{q}) \, . \tag{C2}$$

Then

$$\frac{dJ(q,\alpha)}{d\alpha} = \int_{p<1} d^3p \;\; \vec{q}\cdot(\vec{P}+\alpha\vec{q}) \; \delta(|\vec{P} + \alpha\vec{q}| - 1) \; f(\vec{P},\vec{q}) \, . \tag{C3}$$

Changing vector integration variables to $\vec{n} = \vec{P} + \alpha \vec{q}$ where $\vec{n}$ is a unit vector by virtue of the delta function and integrating over α we have

$$I(q) = J(q,1) = \int_0^1 d\alpha \int d\Omega_{\vec{n}} \; \vec{q}\cdot\vec{n} \; f(\vec{n} - \alpha\vec{q},\vec{q}) \; \eta(\vec{q}\cdot\vec{n} - \frac{q^2\alpha}{2}) \qquad \text{if} \quad q \leq 2 \tag{C4}$$

$$= q \int_0^{2\pi} d\phi \int_0^1 d\alpha \int_{q\alpha/2}^1 x \, dx \; f(\vec{n} - \alpha\vec{q}, \, \vec{q}) \tag{C5}$$

where $x = \vec{q}\cdot\vec{n}$ and ϕ is the polar angle measured about $\vec{q}$. In this form the restrictions on the limits of integration are simply incorporated in the integrations over the solid angle. For $q > 2$ the restriction

$|\vec{P} + \vec{q}| > 1$ is automatically satisfied if $p < 1$ so

$$I(q) = \int_{p<1} d^3p \; f(\vec{P},\vec{q}) \qquad \text{if} \quad q < 2 \, . \tag{C6}$$

1. Phase Space Integrals for Plasmon Decay

For the calculation of the plasmon lifetime the following integral is required (equation 5.17)

$$I(\Omega_p,k) = \int_{\substack{p_1<1 \\ |\vec{P}_1+\vec{k}|>1}} d^3p_1 \int_{\substack{p_2<1 \\ |\vec{P}_2+\vec{k}|>1}} d^3p_2 \quad \delta\,(k^2 + \vec{k}\cdot\vec{P}_1 + \vec{k}\cdot\vec{P}_2 - \Omega_p)$$

$$[a + b(p_1^2 - (\vec{P}_1\cdot\vec{k})^2] \, . \tag{C7}$$

This integral is essentially the volume of phase space available to the final states of two pairs. To evaluate these integrals with the Pauli Principle restrictions we can use the method outlined above. Using

$$\delta\,(k^2 + \vec{k}\cdot\vec{P}_1 + \vec{k}\cdot\vec{P}_2 - \Omega\,) = \frac{1}{2\pi k} \int_{-\infty}^{\infty} dt \; e^{it(k+\hat{\vec{k}}\cdot\vec{P}_1+\hat{\vec{k}}\cdot\vec{P}_2- \Omega/k)} \tag{C8}$$

we can write for $k \leq 2$

$$I(\Omega\,,k) = 2\pi k \int_{-\infty}^{\infty} dt \; e^{it(k- \Omega/k)}[(a + b)\, f_k^2(t) - bf_k(t)\, h_k(t)] \tag{C9}$$

where

$$f_k(t) = \int_0^1 d\alpha \int_{k\alpha/2}^1 x \, dx \; e^{it(x-k\alpha)} = \frac{1}{kt^2}\,(1 + \frac{i}{t})[e^{it(1-k)}-e^{it}] - \frac{1}{t^2}\, e^{-itk/2}$$

$$\tag{C10}$$

$$h_k(t) = \int_0^1 d\alpha \int_{k\alpha/2}^1 x^3\, dx\, e^{it(x-k\alpha)}$$

$$= \frac{1}{kt^2}\left(1 + \frac{3i}{t} - \frac{6}{t^2} - \frac{6i}{t^3}\right)\left[e^{it(1-k)} - e^{it}\right] - \frac{6}{t^4}\left(1 - \frac{k^2t^2}{24}\right) e^{-itk/2} . \tag{C11}$$

We will not need the integrals for $k > 2$ for small Ω . To evaluate the integral over t in equation C9 we note that the integrand is regular at $t = 0$ and we can deform the path of integration to go around a (small) semicircular indentation at $t = 0$ in the upper half plane. We call this contour **C** . It is then easily shown by contour integration that

$$\int_{-\infty\,C}^{\infty} dt\, e^{-ita(k)}\, t^{-n} = \frac{2\pi(-i)^n}{(n-1)!}\; a(k)^{n-1}\; \eta\,(a(k)) \tag{C12}$$

where $\eta\,(x)$ is the step function. Using this we can evaluate $I(\Omega\,,k)$ exactly. The exact expressions are rather long and since we are interested only in small values of Ω we will give only the expansions valid for small Ω . We have

$$\frac{1}{(2\pi)^2}\; I(\Omega\,,k) = (a + b)\; Z(\Omega\,,k) - bW(\Omega\,,k) \tag{C13}$$

where

$$Z(\Omega\,,k) = k \begin{cases} 0 & k < \Omega/2 \\ -\frac{1}{6}\frac{\Omega^3}{k^3} + \frac{\Omega}{k} - \frac{2}{3} + k\left(1 - \frac{1}{2}\frac{\Omega^2}{k^2}\right) & \frac{\Omega}{2} < k < \Omega \\ \frac{1}{6}\frac{\Omega^3}{k^3} + k\left(\frac{1}{2}\frac{\Omega^2}{k^2}\right) & \Omega < k \end{cases} \tag{C14}$$

$$W(\Omega\,,k) = k \begin{cases} 0 & k < \Omega/2 \\ -\frac{1}{20}\frac{\Omega^5}{k^5} + \frac{1}{2}\frac{\Omega^3}{k^3} - \frac{\Omega^2}{k^2} + \frac{\Omega}{k} - \frac{2}{5} & \\ \qquad + k\left(1 - \frac{2\Omega}{k} + \frac{3}{2}\frac{\Omega^2}{k^2} - \frac{1}{4}\frac{\Omega^4}{k^4}\right) & \Omega/2 < k < \Omega \\ \frac{1}{20}\frac{{}^{5}}{k^5} + k\left(\frac{1}{4}\frac{\Omega^4}{k^4}\right) & \Omega < k\,. \end{cases} \tag{C15}$$

Here we neglected terms of the order Ω^2 and higher. We see that for small values of Ω the leading terms arise from small values of k.

2. INTEGRALS ARISING IN SPECIFIC HEAT CALCULATION

a. For the evaluation of the specific heat we needed the subtraction integral, equation 7.25. The problem is to evaluate

$$H_\delta = \frac{2}{\pi^4} \int_{-\infty}^{\infty} \frac{du}{2\pi} \int_{-1}^{1} x\,dx \int_{\delta}^{\infty} \frac{dq}{q^3} \frac{1}{(iu - \frac{1}{2}q - x)^2} Q_q(u) \,. \qquad (C16)$$

The integration over u is easily accomplished using equation 1.43 for $Q_q(u) = Q_o(q, iqu)$. We then find

$$H_\delta = \frac{2}{\pi^3} \int_{\delta}^{2} \frac{dq}{q^3} \left[\int_{-q/2}^{1} dx\, x\, I_q^{(1)}(x) - \int_{q/2}^{1} dx\, x\, I_q^{(2)}(x) \right]$$

$$+ \int_{2}^{\infty} \frac{dq}{q^3} \int_{-1}^{1} dx\, x\, I_q^{(3)}(x) \qquad (C17)$$

where

$$I_q^{(1)} = \int_{\substack{p<1 \\ |\vec{p}+\vec{q}|>1}} d^3p \frac{1}{(q+x+\hat{q}\cdot\vec{p})^2} \qquad (C18)$$

$$I_q^{(2)} = \int_{\substack{p<1 \\ |\vec{p}+\vec{q}|>1}} d^3p \frac{1}{(x+\hat{q}\cdot\vec{p})^2} \qquad (C19)$$

$$I_q^{(3)} = \int_{p<1} d^3p \frac{1}{(q+x+\hat{q}\cdot\vec{p})^2} \,. \qquad (C20)$$

Using equation C5 for the integrals with the Pauli Principle restrictions we have (with $y = \hat{q}.\hat{p}$)

$$I_q^{(1)}(x) = 2\pi q \int_0^1 d\alpha \int_{\frac{q\alpha}{2}}^{1} y dy \, [q(1-\alpha) + x + y]^{-2}$$

$$= 2\pi q \, [\, \frac{x}{q} \ln \left| \frac{1+x+q}{1+x} \right| + \ln \left| \frac{1+x+q}{x+\frac{1}{2}q} \right| -1 \,] \tag{C21}$$

and similarly we find that

$$I_q^{(2)}(x) = - I_{-q}^{(1)}(x) \tag{C22}$$

Also

$$I_q^{(3)}(x) = 2\pi [\, (q + x) \ln \left| \frac{q+x+1}{q+x-1} \right| -2 \,] \tag{C23}$$

Using these results in equation C17 we can perform the integral over x which gives

$$H_\delta = \frac{4}{\pi^2} \int_\delta^2 \frac{dq}{q^2} \, [\, \frac{2}{3q} \ln (1 - \frac{1}{4} q^2) - \frac{1}{3} q + \frac{1}{12} q^2 \ln \left| \frac{2+q}{2-q} \right| \,] \tag{C24}$$

$$+ \frac{4}{\pi^2} \int_2^\infty \frac{dq}{q^3} \, [\, \frac{2}{3} \ln \left| \frac{q+2}{q-2} \right| + q \, (1 - \frac{1}{6} q^2) \ln \left| 1 - \frac{4}{q^2} \right| - \frac{2}{3} q \,]$$

Finally we carry out the integrations over q and obtain the result

$$H_\delta = \frac{4}{\pi^2} [\, \frac{1}{2} \ln \delta + \frac{3}{2} \ln 2 - \frac{5}{4} \,] + O(\delta) \tag{C25}$$

So that

$$= \frac{4}{\pi^2} [\, \frac{3}{2} \ln 2 - \frac{5}{4} \,] = -0.0855 \tag{C26}$$

b. As an example of the evaluation of the integrals arising from the exchange contribution to the specific heat we will discuss the integral in equation 7.55. We can write (from equations 7.36, 7.44)

$$\int_{-1}^{1-\beta} \frac{xdx}{2(1-x)} B(x) = \frac{1}{2\pi} \; \frac{1}{2} \int_{-\infty}^{\infty} \frac{du}{2\pi} \; \ln R(u) \quad d\Omega_{\vec{n}^{\prime}} \quad d\Omega_{\vec{q}} \qquad (C27)$$

$$\frac{1}{(\vec{n}-\vec{n}^{1})^{2}} \; \frac{1}{iu-\hat{q}\cdot\hat{n}} \; \frac{1}{iu-\hat{q}\cdot\hat{n}}$$

where $(\vec{n}-\vec{n}^{1})^{2} > 2$. The angular integrations over the azimuthal angle between $\vec{n}$ and $\vec{n}^{1}$ and over $y = (\hat{q}\cdot\vec{n}^{1})$ can easily be carried out to give

$$-\frac{1}{2} \int_{-\infty}^{\infty} \frac{du}{2\pi} \ln R \int_{-1+4\beta}^{1-4\beta} dx \, \frac{x}{u^{2}+x^{2}} \left[\ln \frac{1-x}{1+x} + \frac{2x}{1-x^{2}}\right] + \frac{4}{1+u^{2}} + O(\beta).$$

(C28)

Carrying out the integration over x gives

$$\frac{1}{2} \int_{-\infty}^{\infty} du \; \ln R \left\{ \frac{2}{1+u^{2}} (-\ln 2\beta) + \frac{4(1-u\tan^{-1}\frac{1}{u})}{1+u^{2}} - 2(\tan^{-1} \frac{1}{u})^{2} \right\}. \qquad (C29)$$

The integral with B(x) replaced by B(1) is found by taking the limit as $\vec{n} \rightarrow \vec{n}^{1}$ in the integrand of (C27) with the result

$$\int_{-1}^{1-\beta} \frac{x\,dx}{2(1-x)}\,B(1) = \int_{-\infty}^{\infty} du\,\frac{\ln R}{1+u^2}\left\{-\ln 2\beta - 2\right\} \quad . \qquad (C30)$$

Taking the difference of these integrals and the limit as $\beta \to 0$ we then obtain equation 7.55.

APPENDIX D

THIRD ORDER CORRELATION ENERGY

In this appendix we will briefly outline the steps necessary to compute the third order correction to the ground state self energy ΔE. Unfortunately some of the multiple integrals which result are too difficult to evaluate without the aid of an electronic computer. Thus we will present only a partial calculation. We present the calculation here as a specific demonstration that the theory can be consistently applied to higher orders with all divergences removed.

According to equation 3.12c the calculation of ΔE_o depends entirely on knowing the propagator $Q_{r_s}(q, iu)$. The expansion of $Q_{r_s}(q,iu)$ in terms of proper polarization diagrams is (omitting the argument (q,iu))

$$Q_{r_s} = Q_o + Q^{(1)}_{r_s} + Q^{(2)}_{r_s} + \cdots \qquad \text{D1}$$

where Q_o was discussed in Chapter I, section C and $Q^{(1)}$ was discussed in Appendix A. $Q^{(2)}_{r_s}$ is the propagator or amplitude for all proper polarization diagrams with two internal interactions.

We can expand equation 3.12c in orders of r_s in the following way (using equation B1)

$$\frac{1}{N}(E_o - \varepsilon_o) = -\frac{3}{8\pi^5}\left(\frac{\pi^2}{\alpha r_s}\right)^2 \int_{-\infty}^{\infty}\frac{du}{2\pi}\int d^3q \left\{ \ln\left(1 + \frac{\alpha r_s}{\pi^2 q^2} Q_o\right) \right.$$
$$+ \int_0^{r_s}\frac{dr'_s}{r'_s}\left[\left(\frac{\alpha r'_s}{\pi^2 q^2}\right)\frac{Q^{(1)}_{r'_s}}{1 + \frac{\alpha r_s}{\pi^2 q^2} Q_o} - \left(\frac{\alpha r'_s}{\pi^2 q^2}\right)^2 \frac{Q^{(1)}_{r'_s}\, Q_o}{\left(1 + \frac{\alpha r'_s}{\pi^2 q^2} Q_o\right)^2}\right.$$
$$\left.\left. + \left(\frac{\alpha r'_s}{\pi^2 q^2}\right)\frac{Q^{(2)}_{r'_s}}{1 + \frac{\alpha r'_s}{\pi^2 q^2} Q_o}\right]\right\} + O(r_s^2) \quad . \qquad \text{D2}$$

Let us take this term by term:

a.) The first term may be written as

$$-\frac{0.916}{r_s} - \frac{3}{8\pi^5}\left(\frac{\pi^2}{\alpha r_s}\right)^2 \int_{-\infty}^{\infty} \frac{du}{2\pi} \int d^2q\, q \left[\ln\left(1 + \frac{\alpha r_s}{\pi^2 q^2} Q_q(u)\right) - \frac{\alpha r_s}{\pi^2 q^2} Q_q(u)\right] \tag{D3}$$

where we have separated out the lowest order contribution which is the exchange energy (see equations 3.9 and 3.9a). We also have substituted qu for u and made use of equation 1.49 defining $Q_q(u)$. The second term in D3 is the contribution to the correlation energy per particle which was called ε' by G-B. They evaluated this expression to lowest order in r_s; we will evaluate it to the next highest order. To do this we make use of equation 1.50 which gives $Q_q(u)$ as a power series in q, i.e.

$$Q_q(u) = 4\pi\, R(u) + q^2 R_{(1)}(u) + \Delta_q(u)$$

which defines $\Delta_q(u)$. Using this expansion we can show that to order r_s we can write

$$\varepsilon' = -\frac{3}{8\pi^5}\left(\frac{\pi^2}{\alpha r_s}\right)^2 \int_{-\infty}^{\infty} \frac{du}{2\pi} \int d^3q\, q\left[\ln\left(1 + \frac{4\alpha r_s}{\pi q^2} R(u)\right) - \frac{4\alpha r_s}{\pi q^2} R(u) + \left(\frac{4\alpha r_s}{\pi}\right)^2 \alpha_q(u)\right]$$

$$- \frac{3}{8\pi^5}\, \alpha r_s\, 16 \int_{-\infty}^{\infty} \frac{du}{2\pi} \int \frac{d^3q}{q} \left[\frac{R^2(u)\, R_{(1)}(u)}{q^2 + \frac{4\alpha r_s}{\pi} R(u)} + \beta_q(u)\right] + O(r_s^2) \tag{D4}$$

where

$$-(4\pi)^2\, \alpha_q(u) = \frac{4\pi R}{q^2}\left(R_1 + q^{-2}\Delta\right) + \frac{1}{2}\left(R_1 + q^{-2}\Delta\right)^2 = \frac{1}{2q^4}\left[Q_q^2(u) - (4\pi R(u))^2\right] \tag{D5}$$

$$(4\pi)^2 \beta_q(u) = \frac{(4\pi R)^2}{q^4} \Delta + \frac{4\pi R}{q^2} (R_1 + q^{-2} \Delta)^2 + \frac{q^2}{3} (R_1 + q^{-2} \Delta)^3$$

$$= \frac{1}{3q^6} [Q_q^3(u) - (4\pi R(u))^3 - 3(4\pi R(u))^2 R_{(1)}(u)] . \quad D6$$

Now we perform the integrals over q for 0 to, say, 1 and correct to order r_s exactly. We find

$$\varepsilon' = \frac{3}{8\pi^5} \frac{\pi}{2} \frac{(4\pi)^2}{2\pi} \int_{-\infty}^{\infty} du\, R^2(u) [-1 + 2 \ln (\frac{4\alpha r_s}{\pi} R(u)) - \frac{1}{3} \frac{4\alpha r_s}{\pi} R(u)] + \delta$$

$$+ \frac{3\alpha r_s 4\pi}{2\pi^6} \int_{-\infty}^{\infty} du\, R^2(u)\, R_{(1)}(u) \ln(\frac{4\alpha r_s}{\pi} R(u)) + r_s\, \delta_1 \qquad D7$$

where

$$\delta = \frac{3}{\pi^3} \int_{-\infty}^{\infty} \frac{du}{2\pi} \int_0^1 q^2\, dq\, \alpha_q(u) + \frac{3}{\pi^3} \int_{-\infty}^{\infty} \frac{du}{2\pi} \int_1^{\infty} \frac{dq}{q} [Q_q(u)]^2 \qquad D8$$

and

$$\delta_1 = \frac{3\alpha 4\pi}{2\pi^6} \int_{-\infty}^{\infty} \frac{du}{2\pi} [\int_0^1 q^2\, dq\, \beta_q(u) + \frac{1}{3} \int_1^{\infty} \frac{dq}{q^3} [Q_q(u)]^3] \qquad D9$$

It is not difficult to see that $\delta^{(2)}$ is G-B's subtraction term which was evaluated by Pines to give $\delta^{(2)} = - 0.0508$. We can write equation D7 in the form

$$\varepsilon' = A \ln r_s + C + r_s (A_{11} \ln r_s + C_{11}) \qquad D10$$

where A and C were evaluated by G-B.

$$A = \frac{3}{\pi^3} \int_{-\infty}^{\infty} du\, R^2(u) = 0.0622 \qquad D11$$

$$C = \quad + A \quad \ln \frac{4\alpha}{\pi} + < \ln R>_{av} - \frac{1}{2} \quad = 0.050 \qquad \text{D12}$$

The coefficients of the higher order terms are

$$A_{11} = \frac{3\alpha}{2\pi^6} 4\pi \int_{-\infty}^{\infty} du \; R^2(u) \; R_{(1)}(u) = \frac{6\alpha}{\pi^4} (0.160) = \; 0.0052 \qquad \text{D13}$$

$$B_{11} = \frac{\alpha}{2\pi^5} 4\pi \int_{-\infty}^{\infty} du \; R^3(u) + \frac{3\alpha}{2\pi^6} 4\pi \int_{-\infty}^{\infty} du \; R^2(u) \; R_1(u) \ln R(u)$$

$$+ \delta_1 + 4\pi \frac{3\alpha}{2\pi^6} \ln \frac{4\alpha}{\pi} \qquad \text{D14}$$

b) The second and fourth term in equation D2 introduce the lowest order exchange effects. To terms proportional to r_s we can write their contributions as*

$$\varepsilon_{1x} = -\frac{3}{8\pi} \frac{1}{\alpha^2 r_s^2} \int_0^{r_s} \frac{dr_s'}{r_s'} \int d^3q \int \frac{du}{2\pi} \left[\frac{\alpha r_s'}{\pi^2 q^2} \left[Q_{r_s}^{(1,0)} \right.\right.$$

$$\left.\left. - 3 \quad \frac{\alpha r_s'}{\pi^2 q^2} \quad \frac{Q_{r_s}^{(1,0)} \; Q_o}{(1 + (\alpha r_s'/\pi^2 q^2) \; Q_o)} \right] + O(r_s^2) \qquad \text{D15}$$

where $Q_{r_s}^{(1,0)}$ is given explicitly in equation A5 of Appendix A. By carrying out the integrations over u and r_s' in the first term we find

$$\varepsilon_{1x} = \varepsilon_b^{(2)} + \frac{9}{8\pi^5} \frac{1}{r_s^2} \int_0^{r_s} dr_s' \; r_s' \int d^3q \int_{-\infty}^{\infty} \frac{du}{2\pi} \frac{Q_q(u) \; Q_o^{(10)}(q, \; iqu)}{q(q^2 + (\alpha r_s'/\pi^2) \; Q_q(u))} \qquad \text{D16}$$

---

* Note that

$$\int_0^{r_s} \frac{dr_s'}{r_s'} \int d^3q \int_{-\infty}^{\infty} \frac{du}{2\pi} \frac{r_s' \; Q_{r_s'}^{(1)}}{1 + (\alpha r_s'/\pi^2 q^2) Q_o} = \int_0^{r_s} \frac{dr_s'}{r_s'} \int d^3q \int_{-\infty}^{\infty} \frac{du}{2\pi} r_s'$$

$$\left[Q_{r_s}^{(10)} - \frac{2(\alpha r_s'/\pi^2 q^2) \; Q_{r_s'}^{(1,0)} \; Q_o}{(1 + (\alpha r_s'/\pi^2 q^2) \; Q_o)} \right] + O(r_s^2)$$

is proportional to the amplitude for diagram IIb in figure 4 if both Coulomb lines are replaced by the polarized interaction. The first term on the right corresponds to this diagram with only bare Coulomb interactions; the second term arises from the lowest order polarization corrections. The factor of 2 occurs since both interaction lines are equivalent. The fourth term in equation D2 also contributes such a term yielding the factor of 3 in equation D15.

where $\varepsilon_b^{(2)} = 0.046 \pm 0.02$ is G-B's second order exchange energy. To lowest order in r_s we need only include screening terms for small values of q. The result, exact to lowest order in r_s, is

$$\varepsilon_{1x} = \varepsilon_b^{(2)} + \frac{9\alpha}{8\pi^5} \int_0^{r_s} dr_s'(r_s')^2 \int_{-\infty}^{\infty} \frac{du}{2\pi} \int_{q<1} d^3q \, \frac{4\pi \, R(u) \, R^{(1)}(iu)}{q(q^2+(4\alpha r_s'/\pi)R)} + r_s \delta_{1x} \qquad \text{D17}$$

where

$$r_s \delta_{1x} = \lim_{\beta\to 0} \frac{9}{8\pi^5} \frac{1}{3} \int_{-\infty}^{\infty} \frac{du}{2\pi} \left\{ \int_{q>\beta} d^3q \, \frac{Q_q(u) \, Q_o^{(1)}(q, \, iqu)}{q^3} - \int_{1>q>\beta} d^3q \, \frac{4\pi \, R(u) \, \alpha r_s \, R^{(1)}(iu)}{q^3} \right\} \qquad \text{D18}$$

Performing the integration over q and r_s' in equation D17 we find

$$\varepsilon_{1x} = \varepsilon_b^{(2)} + r_s(A_{1x} \ln r_s + C_{1x}) + O(r_s^2) \qquad \text{D19}$$

where

$$A_{1x} = -\frac{3\alpha}{8\pi^5} \int_{-\infty}^{\infty} du \, R(u) \, R^{(1)}(iu) = -0.00045 \qquad \text{D20}$$

$$C_{1x} = -\frac{3\alpha}{8\pi^5} \left[\int_{-\infty}^{\infty} du \, R(u) \, R^{(1)}(iu) \ln R(u) \right] + A_{1x}\left[\ln \frac{4\alpha}{\pi} - \frac{1}{3}\right] + \delta_{1x}. \qquad \text{D21}$$

A_{1x} was evaluated analytically using equation A5 in appendix A for $R^{(1)}(iu)$.

c) Finally, we come to the third term in equation D2. When reduced to terms proportional to r_s, this becomes

$$\varepsilon_{2x} = -\frac{3}{8\pi^5} \left(\frac{\pi^2}{\alpha r_s}\right) \frac{1}{r_s} \int_0^{r_s} dr_s' \int_0^{\infty} \frac{du}{2\pi} \int d^3q \, \frac{1}{q^2} \, Q_{r_s}^{(2)}(q, \, iu) \qquad \text{D22}$$

We have not derived an expression for $Q_{r_s}^{(2)}(q, iu)$. However, we can avoid the necessity for this by noting that, according to the arguments leading up to equation 3.9 in the text, ε_{2x} is the sum of all third order perturbation diagrams which have three different Coulomb denominators. To order r_s we can drop the screening terms since the sum of these processes is still finite. Below we list the integrals which correspond to the third order diagrams in Figure 4 .

$$\varepsilon_{2x}^{(a)} = \frac{\alpha r_s}{\pi^2}\left(\frac{3}{8\pi^5}\right)\int d^3p_1\int d^3p_2\int\frac{d^3q_1}{q_1^2}\int\frac{d^2q_2}{q_2^2}\;\frac{1}{(\vec{q}_1+\vec{q}_2)^2} \qquad \text{D23}$$

$$\left\{\frac{R_1(\vec{p}_1,\vec{q}_1,\vec{q}_2)\,R_1(\vec{p}_2,\vec{q}_1,\vec{q}_2)+R_2(\vec{p}_1,\vec{q}_1,\vec{q}_2)\,R_2(\vec{p}_2,\vec{q}_1,\vec{q}_2)}{[q_1^2+\vec{q}\cdot(\vec{p}_1+\vec{p}_2)][(\vec{q}_1+\vec{q}_2)^2+(\vec{q}_1+\vec{q}_2)\cdot(\vec{p}_1+\vec{p}_2)]}\right.$$

$$\left.-\frac{2R_1(\vec{p}_1,\vec{q}_1,\vec{q}_2)\,R_2(\vec{p}_2,\vec{q}_1,\vec{q}_2)}{\vec{q}_1\cdot(\vec{p}_2-\vec{p}_1)(\vec{q}_1+\vec{q}_2)\cdot(\vec{p}_2-\vec{p}_1)}\right\}$$

$$\varepsilon_{2x}^{(b)} = -\frac{\alpha r_s}{2\pi^2}\left(\frac{3}{8\pi^5}\right)\int d^3p_1\int d^3p_2\int\frac{d^3q_1}{q_1^2}\int\frac{d^3q_2}{q_2^2}\;\frac{1}{(\vec{p}_1+\vec{p}_2+\vec{q}_1+\vec{q}_2)^2}$$

$$\left\{\begin{matrix}\text{first term}\\ \text{in}\\ \text{brackets}\\ \text{above}\end{matrix}\right\} \qquad \text{D24}$$

where

$$R_1(\vec{q}_1,\vec{q}_1,\vec{q}_2) \equiv \eta(1-p_1)\,\eta(|\vec{p}_1+\vec{q}_1|-1)\,\eta(|\vec{p}_1+\vec{q}_1+\vec{q}_2|-1)$$

$$R_2(\vec{p}_1,\vec{q}_1,\vec{q}_2) \equiv \eta(p_1-1)\,\eta(1-|\vec{p}_1+\vec{q}_1|)\,\eta(1-|\vec{p}_1+\vec{q}_1+\vec{q}_2|)$$

D25

$$\varepsilon_{2x}^{(c)}+\varepsilon_{2x}^{(d)} = -\frac{\alpha r_s}{2\pi^2}\left(\frac{3}{8\pi^5}\right)\int\limits_{\substack{p_1<1\\ |p_1+q|>1}} d^3p_1\int\limits_{\substack{p_2<1\\ |p_2+q|>1}} d^3p_2\int d^3p_3\int\frac{d^3q}{q^2}\;\frac{1}{(\vec{p}_1+\vec{p}_2+\vec{q})^2}\;\frac{1}{(\vec{p}_1-\vec{p}_3)^2}$$

$$\left\{\frac{\eta(1-|\vec{p}_3+\vec{q}|)-\eta(1-p_3)}{[q^2+\vec{q}\cdot(\vec{p}_1+\vec{p}_2)]^2}+\frac{\eta(|\vec{p}_3+\vec{q}|-1)\;\eta(1-p_3)}{[q^2+\vec{q}\cdot(\vec{p}_1+\vec{p}_3)][q^2+\vec{q}\cdot(\vec{p}_1+\vec{p}_2)]}\right\}$$

D26

These multiple integrals are much too difficult to evaluate without the aid of an electronic computer. With the restrictions shown on the domains of integration, these integrands are easily divided into positive and negative definite parts. Integrals of this type can be evaluated quite readily by Monte Carlo methods using an electronic computer. It is hoped that this will be undertaken in the near future.

If we call $\varepsilon_{2x} = \varepsilon_{2x}^{(a)} + \varepsilon_{2x}^{(b)} + \varepsilon_{2x}^{(c)} + \varepsilon_{2x}^{(d)} = r_s C_{22}$ then combining equations D10-14, D19-21, and D23-26 we find

$$\varepsilon' = 0.0622 \ln r_s - 0.096 + 0.0049|r_s \ln r_s + r_s(C_{11} + C_{12} + C_{22}) \qquad \text{D27}$$

where C_{11}, C_{12}, and C_{22} still await evaluation. We see from this result that an attempt to calculate the term proportional to r_s using ordinary third order perturbation theory would be disastrous, since the $r_s \ln r_s$ term cannot be expressed as a power series in r_s. This agrees with G-B's discussion of the form of the third order term.

The "linked cluster" expansion which we have used here (see page 41) is also an improvement on Rayleigh-Schrodinger (R-s) perturbation theory. In the latter theory we would have to include disconnected third order ground state diagrams which would further complicate the calculation. As we mentioned in Chapter III, our linked cluster expansion includes intermediate states which violate the exclusion principle. This is clear from the diagrams in figure 4 in which we have made no attempt to satisfy the exclusion principle. In R-S perturbation theory these diagrams are cancelled by similar disconnected diagrams. For instance, diagram a of figure 5 would be cancelled by disconnected diagram b for all states which violate the exclusion principle. The reader is referred to Goldstone's paper for more details concerning this matter.

APPENDIX E

GENERALIZED SUM RULE

Two sum rules involving oscillator strengths are well known in the theory of the stopping power of an electron gas [31] [27]. Here we wish to derive a generalized sum rule which reduces to the usual sum rules to lowest order in r_s (or e^2). The generalized sum rule is

$$\operatorname{Im} \int_0^\infty dw\, w \frac{1}{1 + \frac{\alpha r_s}{\pi^2 q^2} Q_{r_s}(q,w)} = -\frac{\alpha r_s}{\pi^2 q^2} \operatorname{Im} \int_0^\infty dw\, w\, Q_{r_s}(q,w) \tag{E1}$$

$$= -\frac{1}{2}\pi\Omega_p^2$$

or in terms of the dielectric constant

$$\operatorname{Im} \int_0^\infty dw \frac{w}{K_{r_s}(q,w)} = -\operatorname{Im} \int_0^\infty dw\, w\, K_{r_s}(q,w) = -\frac{1}{2}\pi\Omega_p^2 \quad . \tag{E2}$$

The derivation of this sum rule is very simple and depends only on the analytic properties of $Q_{r_s}(q,w)$ which we discussed in Chapter II. Since $Q_{r_s}(q,w) \to O(\frac{1}{|w|^2})$ as $|w| \to \infty$ and has no singularities in the first quadrant of the w plane, the path of integration for the integral

$$\int_0^\infty dw\, w \frac{Q^2_{r_s}(q,w)}{1 + \frac{\alpha r_s}{\pi^2 q^2} Q_{r_s}(q,w)} = \int_0^\infty du\, u \frac{Q^2_{r_s}(q, iu)}{1 + \frac{\alpha r_s}{\pi^2 q^2} Q_{r_s}(q, iu)} \tag{E3}$$

can be rotated counter clockwise to the imaginary axis. But since $Q_{r_s}(q, -iu) = Q_{r_s}(q, iu)$ this integral is real. Thus since

$$\mathrm{Im} \int_0^\infty dw \frac{w}{1 + \frac{\alpha r_s}{\pi^2 q^2} Q_{r_s}(q,w)}$$

$$= \mathrm{Im} \int_0^\infty dw\, w \left[- \frac{\alpha r_s}{\pi^2 q^2} Q_{r_s}(q,w) + \frac{\left(\frac{\alpha r_s}{\pi^2 q^2} Q_{r_s}(q,w)\right)^2}{1 + \frac{\alpha r_s}{\pi^2 q^2} Q_{r_s}(q,w)} \right] \qquad \text{(E4)}$$

the first result follows. The second result follows in exactly the same manner since $Q_{r_s}(q,w) = Q_o(q,w) + \Delta Q_{r_s}(q,w)$ where $\Delta Q_{r_s}(q,w) \to O(\frac{1}{|w|^3})$ as $|w| \to \infty$. We see this by studying the diagrams which contribute to $\Delta Q_{r_s}(q,w)$. The lowest order diagrams are shown in figure 3. The very lowest order contribution is $Q^{(1)}_{r_s}(q,w)$ which the explicit calculation in Appendix A shows to have this behavior. All other diagrams contribute a factor of three or more energy denominators (i.e. three or more intermediate states) and therefore behave as $1/|w^3|$ as $|w| \to \infty$. Thus

$$\frac{\alpha r_s}{\pi^2 q^2} \mathrm{Im} \int_0^\infty dw\, w\, Q_{r_s}(q,w) = \frac{\alpha r_s}{\pi^2 q^2} \mathrm{Im} \int_0^\infty dw\, w\, Q_o(q,w) \qquad \text{(E5)}$$

and we have from equation 1.44

$$\int_0^\infty dw\, w\, \mathrm{Im}\, Q_o(q,w) = \pi \int_{\substack{p<1 \\ |\vec{p}_1+q|>1}} d^3\vec{p}_1 \,(q^2/2 + \vec{q}\cdot\vec{p}_1) = \frac{2\pi^2}{3} q^2 . \qquad \text{(E6)}$$

The final result in equation E_1 follows using $\Omega_p^2 = 4\alpha r_s/3\pi$.

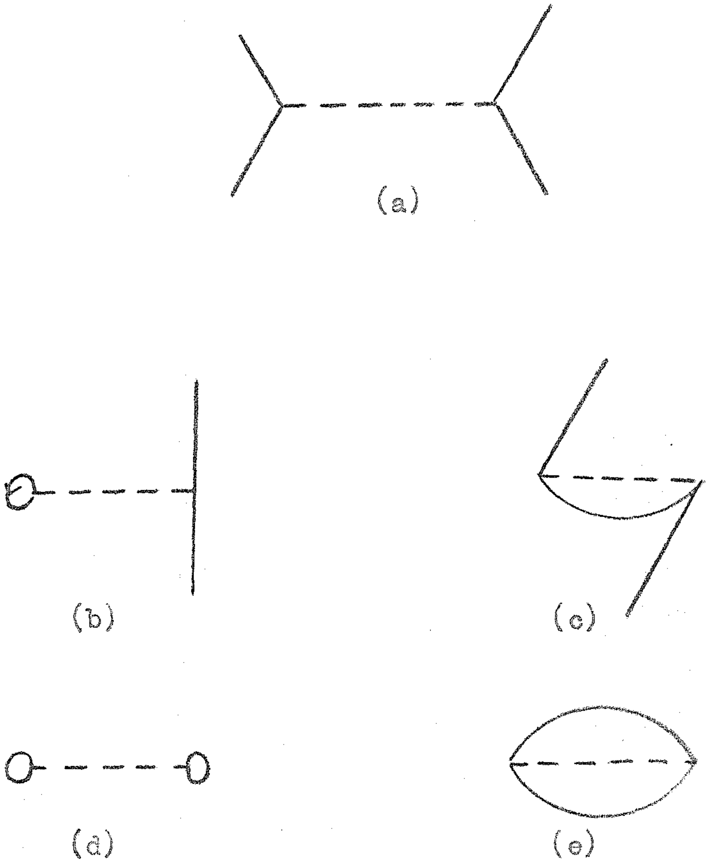


FIGURE 1

Feynman diagrams for the interactions in the electron-hole description: (a) scattering of excited particles, (b) forward scattering of an excited particle by a passive particle in the Fermi sea, (c) exchange scattering of an excited particle with a passive particle, (d) and (e) are the diagrams for the Coulomb and exchange energy, respectively.

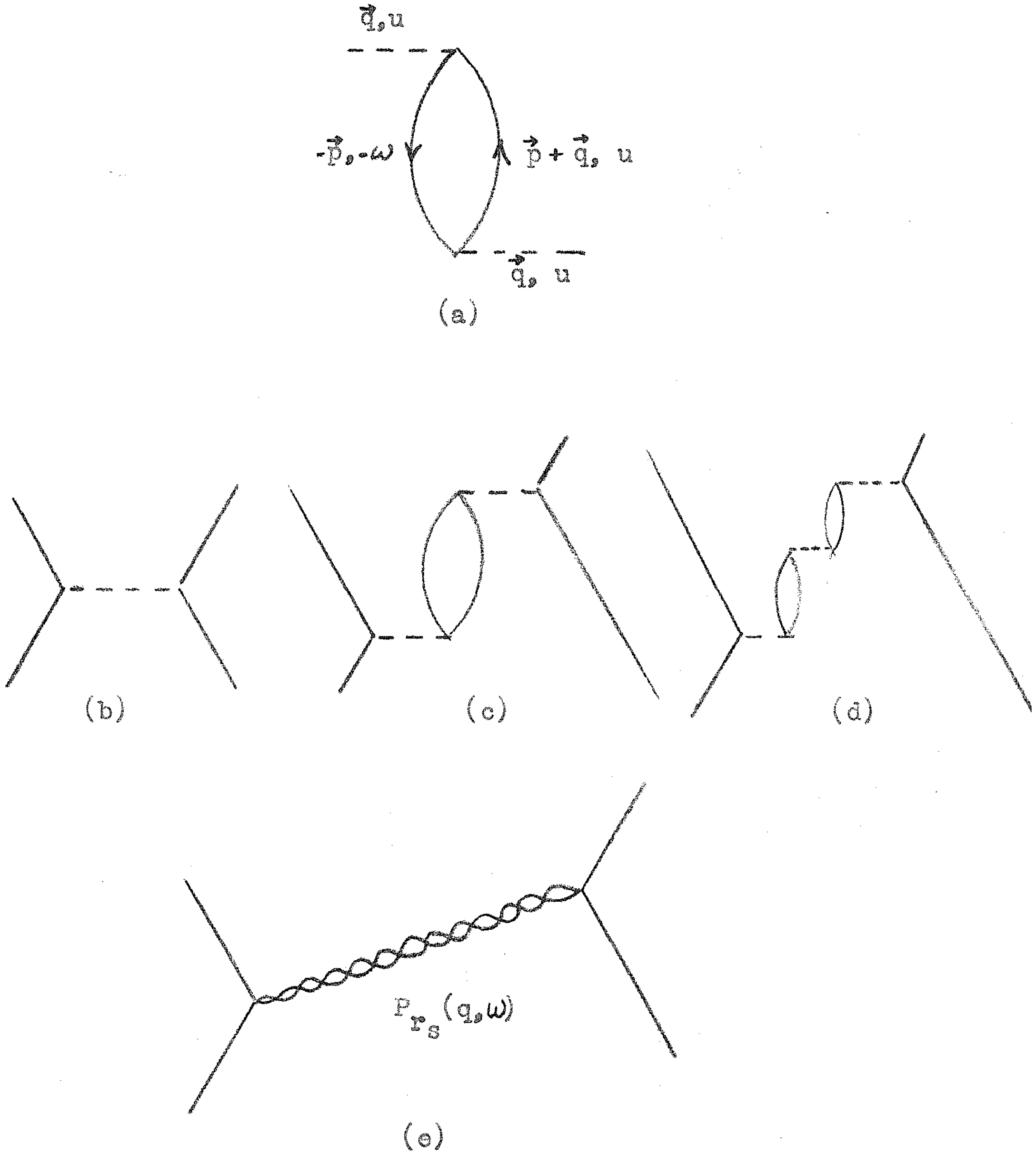


FIGURE 2

Diagrams for the polarization of the many body medium: (a) the simplest polarization process, excitation and de-excitation of an electron-hole pair, corresponding to the polarization propagator Q, (b) to (d) the series of diagrams for the interaction of two particles modified by the polarization of the medium, (e) diagram representing the effective interaction.

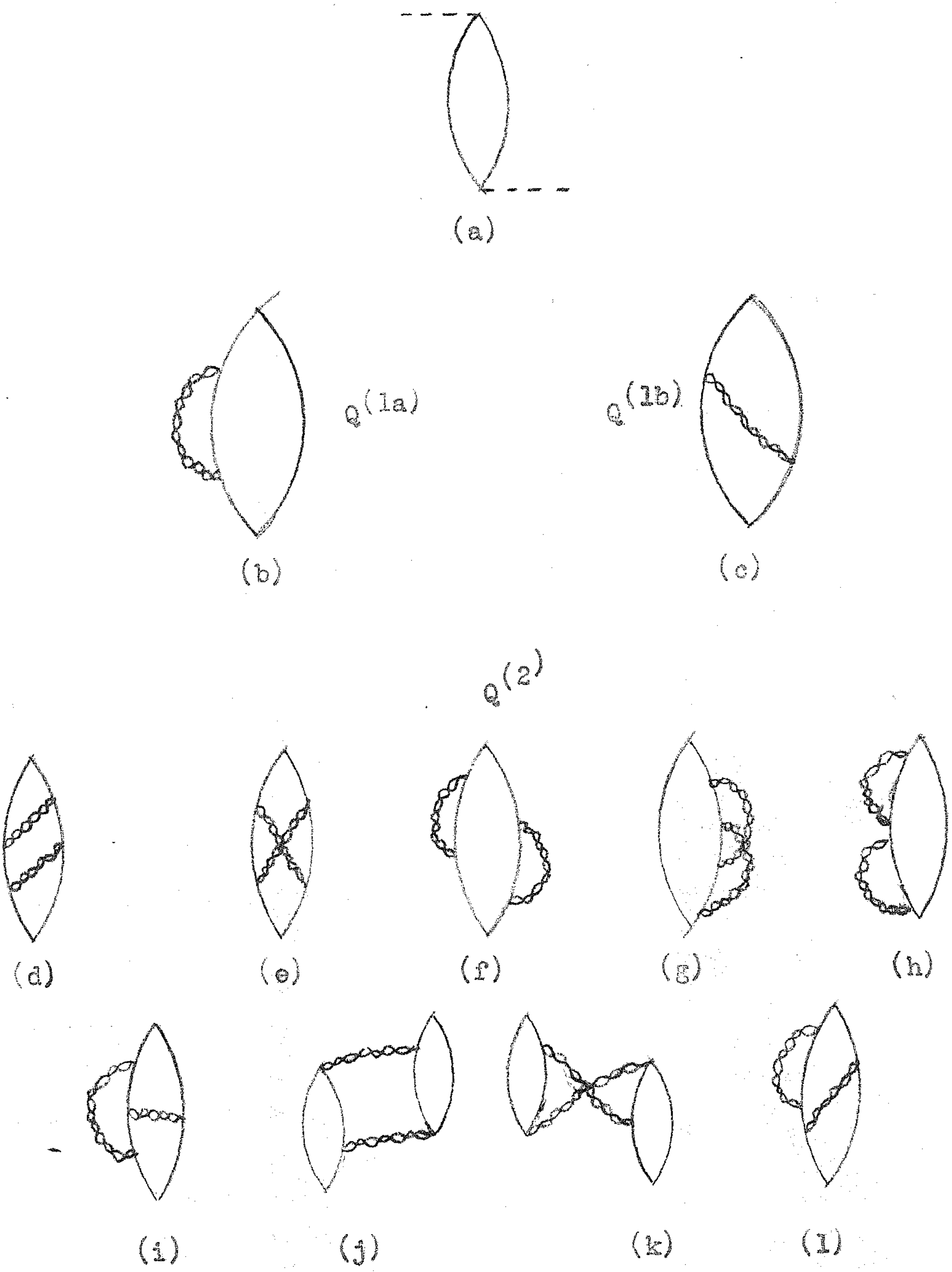


FIGURE 3

Local field corrections to the polarization propagators, (b) and (c), the lowest order corrections corresponding to $Q^{(1)} = Q^{(1a)} + Q^{(1b)}$, (d) through (l), the second order corrections corresponding to $Q^{(2)}$.

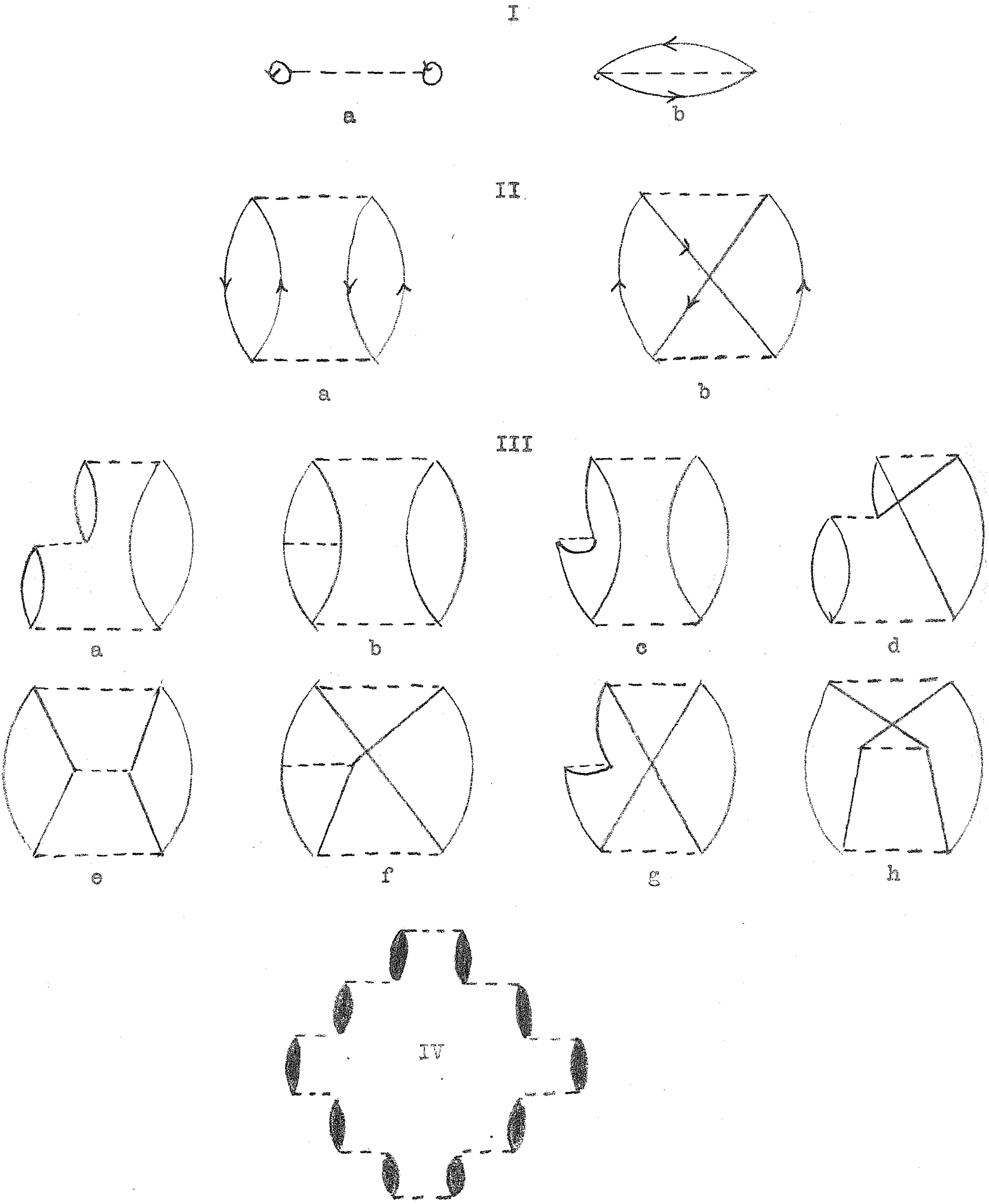


FIGURE 4

Ground state fluctuation diagrams:I(a) and (b), the Coulomb and exchange energy, II (a) and (b), the second order processes, III (a) through (h), the third order processes, IV- Schematic diagram for fluctuation with n polarization processes in a loop.

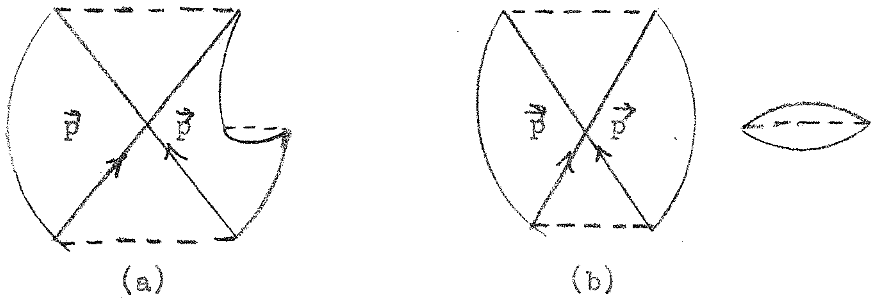


FIGURE 5

Cancellation of ground state diagrams which do not satisfy the exclusion principle. If diagram (a) violates the exclusion principle then diagram (b) exactly cancels it since they differ by one closed loop.

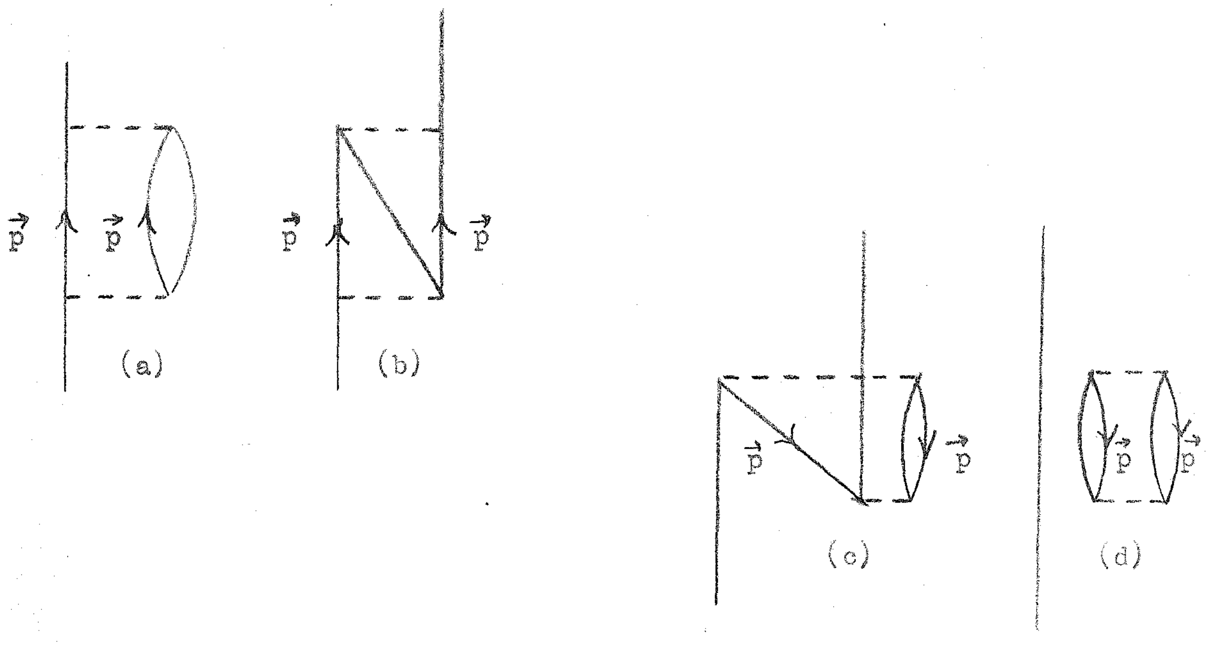


FIGURE 7

Exclusion principle cancellations for excited states: when (a) and (c) violate the exclusion principle, they are cancelled respectively by (b) and (d) when the latter also violate the principle. Note, however, that (d) is a disconnected diagram.

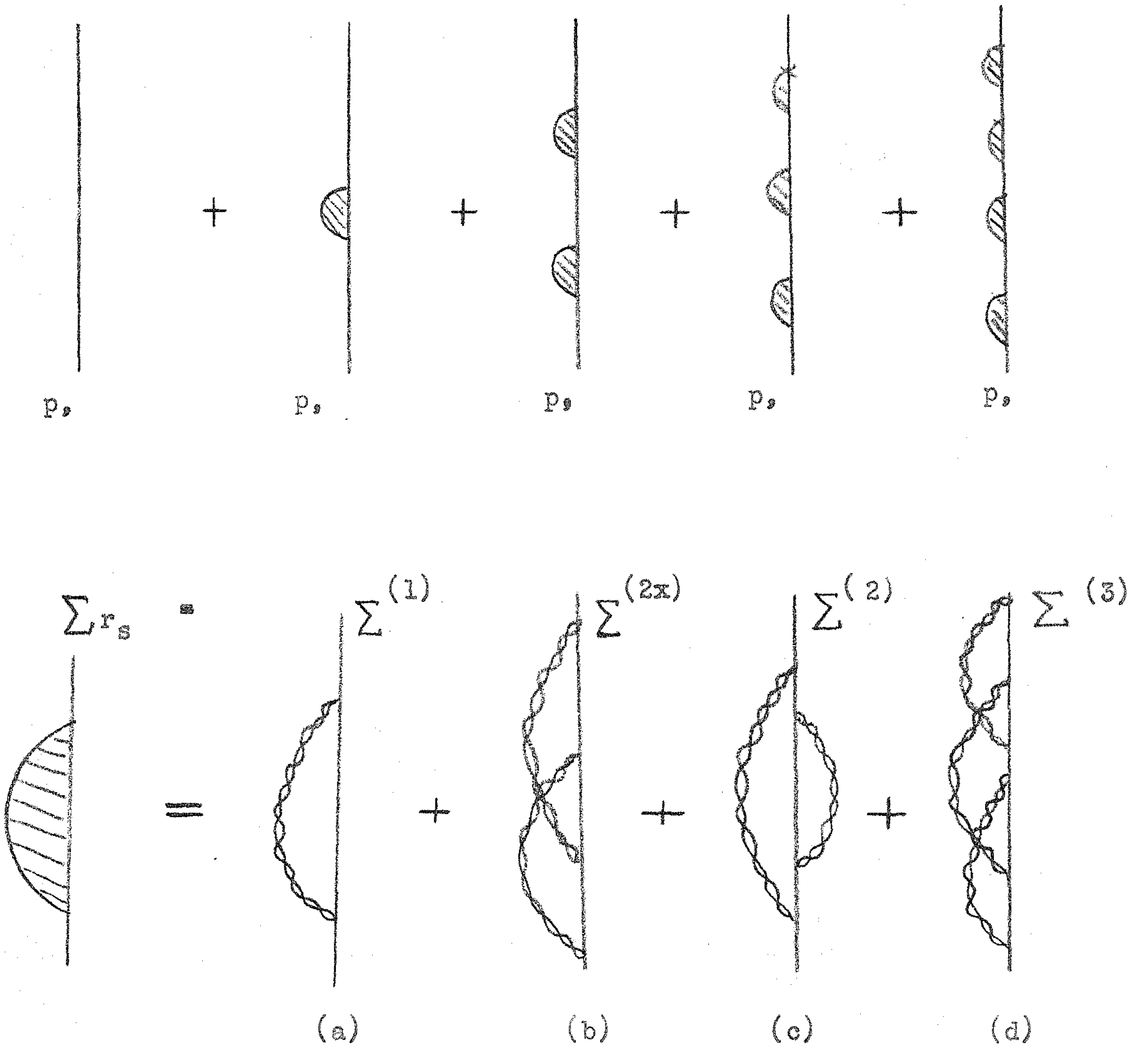


FIGURE 6

Single particle self energy processes: the series of diagrams which correct the single particle propagator. The shaded bump represents the sum of all proper self energy parts, the lowest orders of which are shown in (a) through (d).

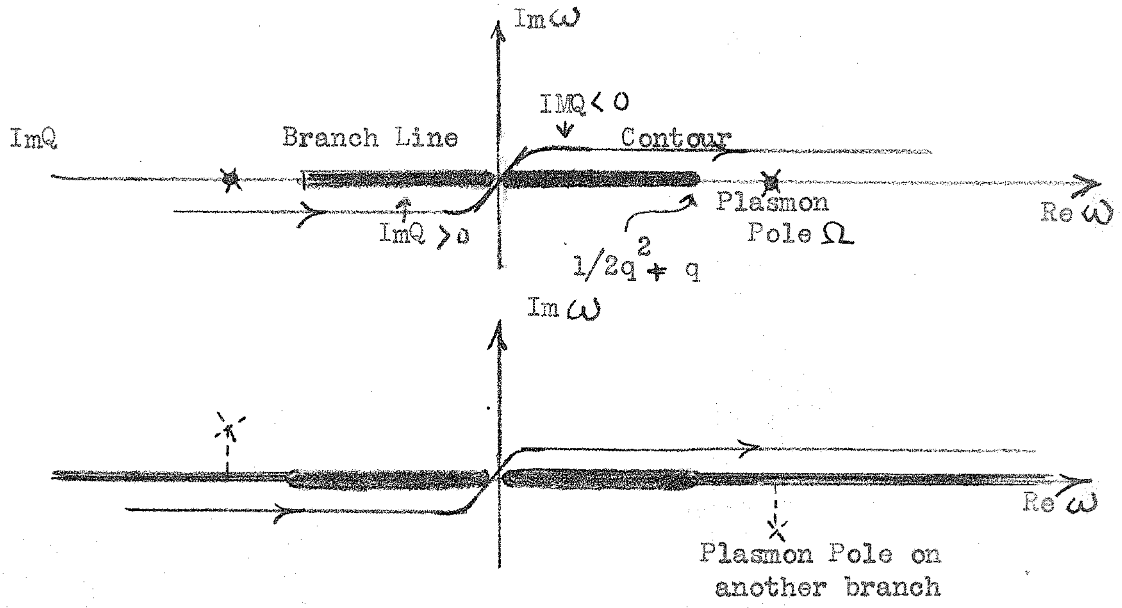


FIGURE 8

Singularities and Contours of Integration for $P_{r_s}(q,\omega) = q^{-2}K_{r_s}^{-1}(q,\omega)$. In the upper diagram these are shown for the pair approximation. The lower diagram is for the general case where the plasmon pole lies off the principal branch of $Q_{r_s}(q,\omega)$.

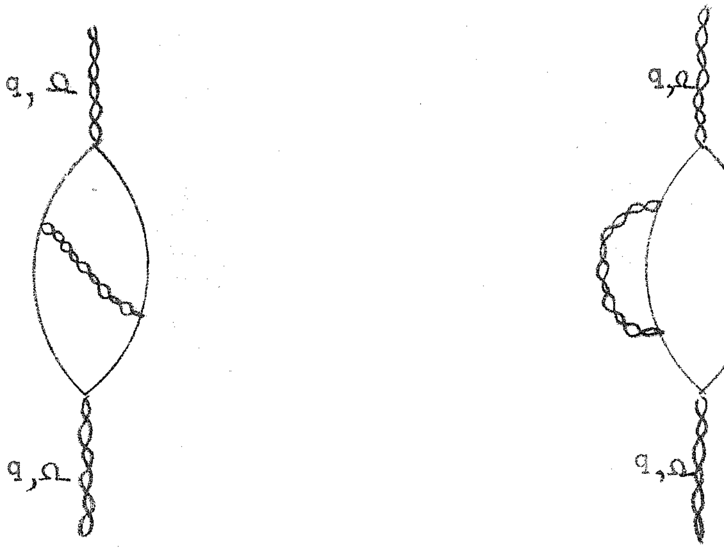


FIGURE 9

Lowest order proper self energy processes for the plasmon.

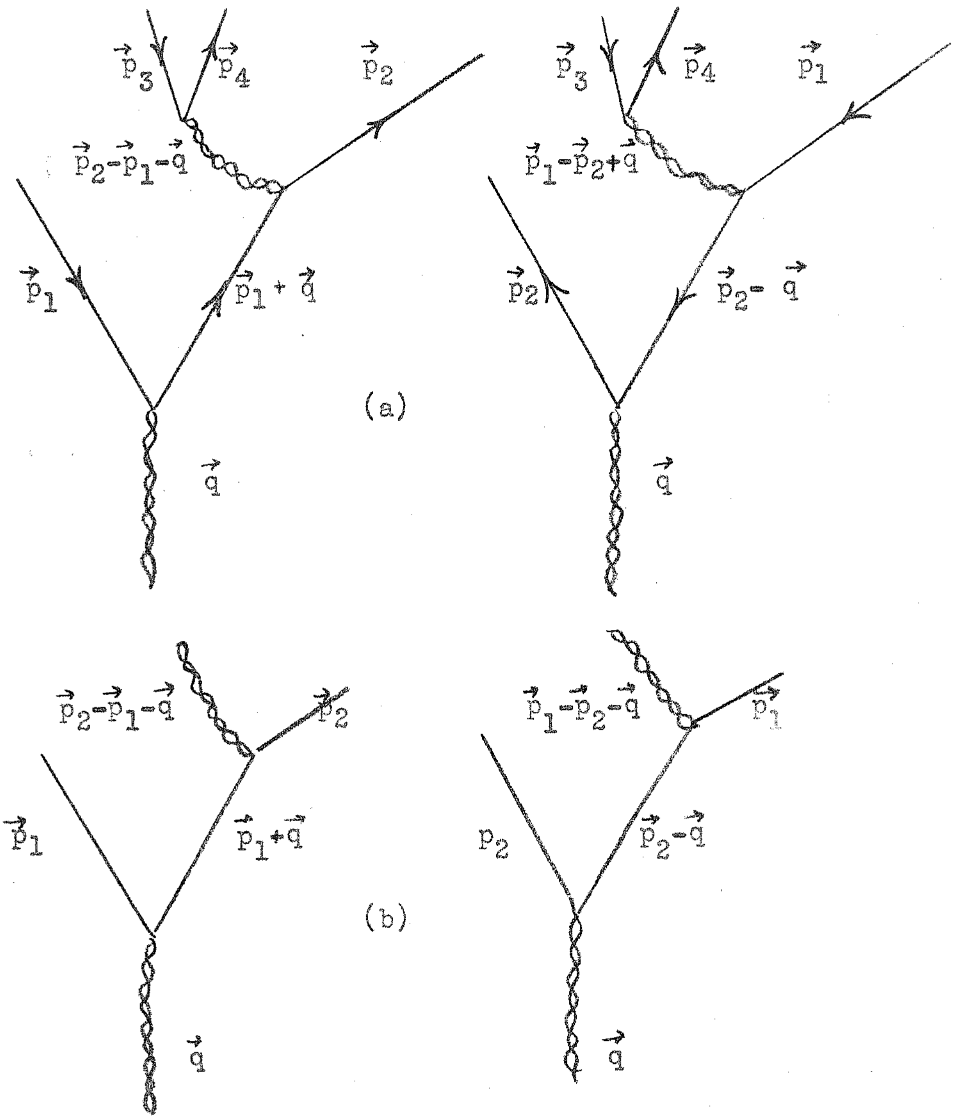


FIGURE 10

Plasmon decay: (a) two of the eight indistinguishable diagrams for the decay of a plasmon into two pairs, (b) the two diagrams for the decay of a plasmon into a pair plus another plasmon.

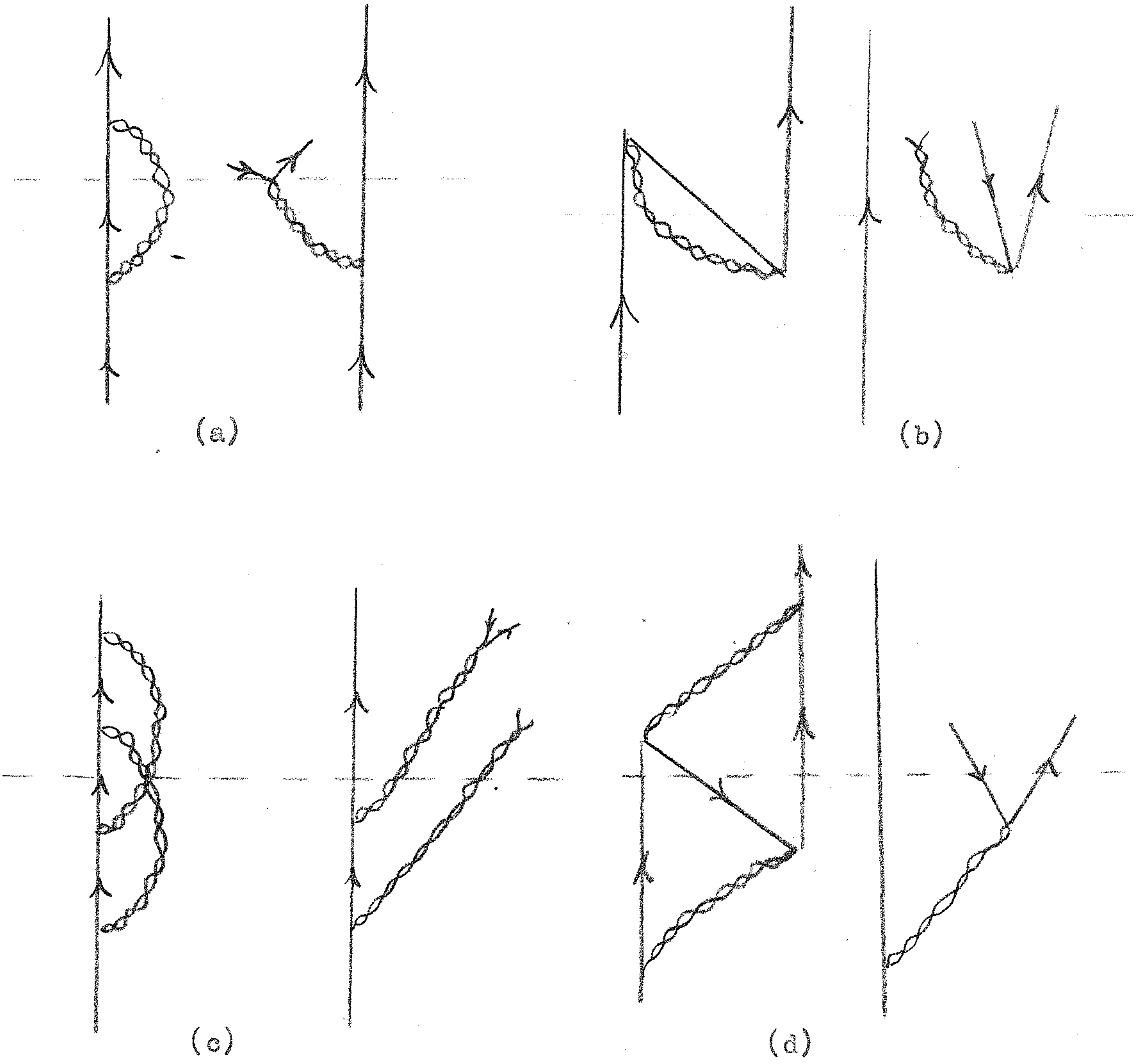


FIGURE 11

The process of diagram cutting to obtain transition probabilities from self energy parts: If the intermediate state lies on the energy shell of the particle, the imaginary part of the self-energy amplitude gives the transition probability for scattering into this intermediate state.

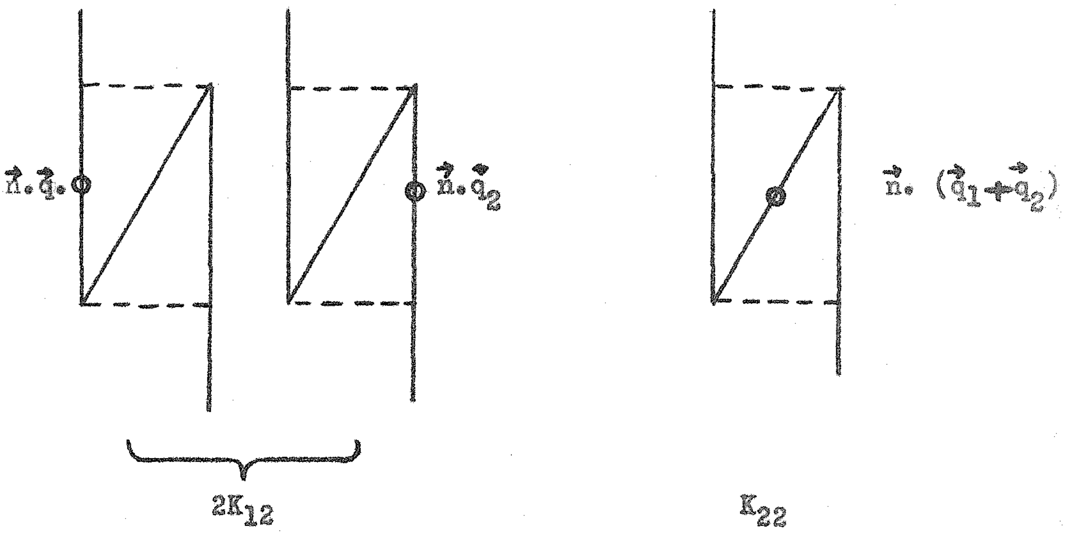
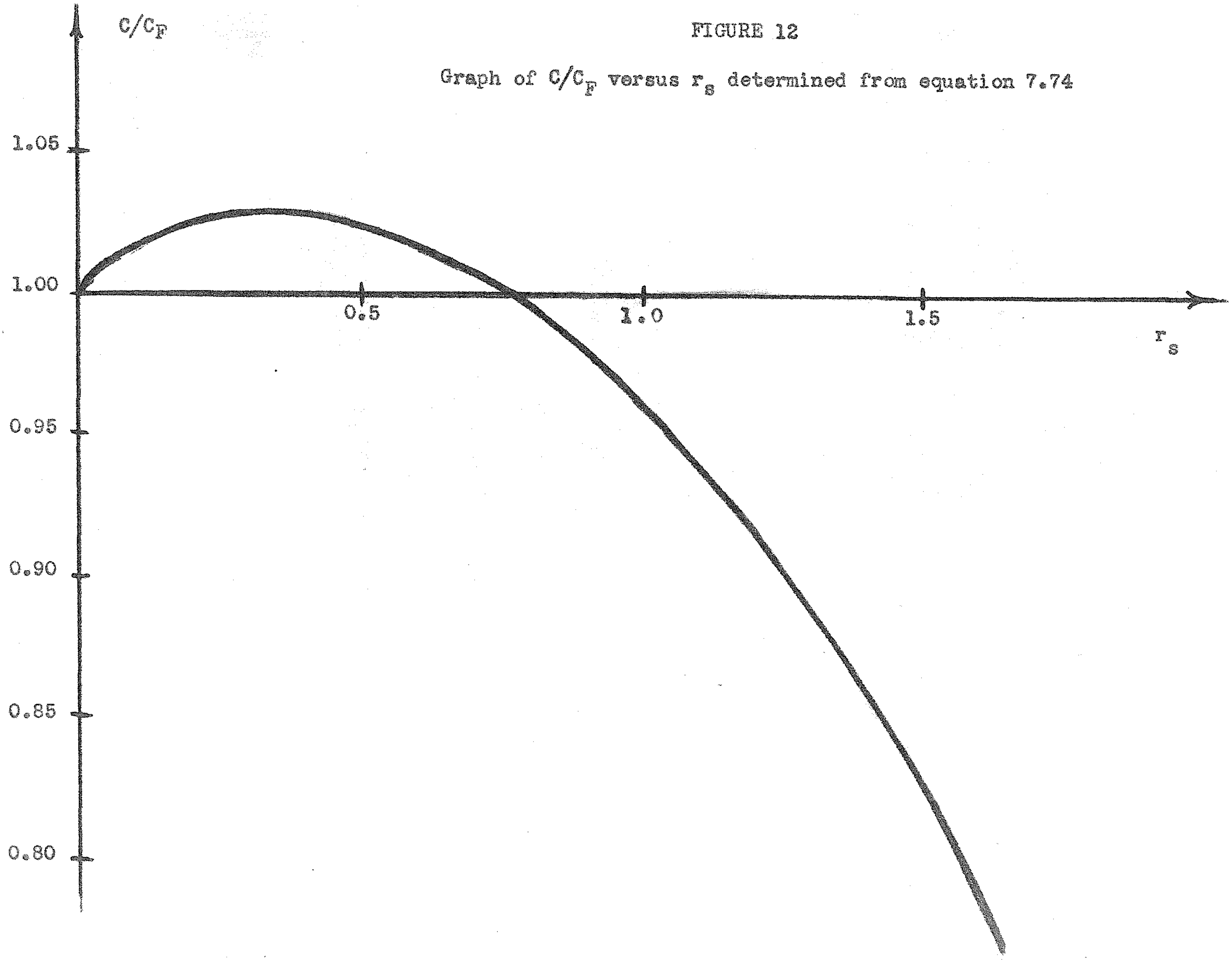


FIGURE 13

Diagrams for $2K_{12} + K_{22}$ showing the cancellation of these terms to lowest order in r_s. The effect of differentiating the propagators is to add an "interaction" $\vec{n}.\vec{q}$. into the particle lines as discussed in the text.

FIGURE 12

Graph of C/C_F versus r_s determined from equation 7.74



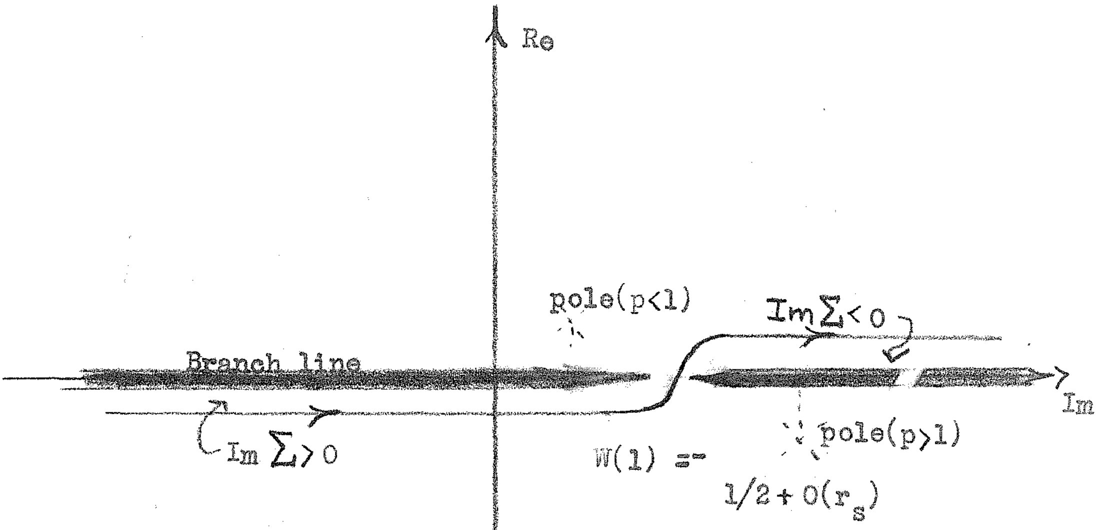


FIGURE 14

Relationship of the singularities and contours for the corrected Feynman propagators $S_F^{-1}(p,\omega)$.

TABLE 1

Theoretical and experimental values of the eigenloss and dispersion coefficients for various metals. The entries under exchange give the q^2 dispersion coefficient corrected by equation 5.8.

		Eigenloss		Dispersion Coefficient		
Metal	r_s	$\hbar\ _p$	Exp.	Pair approx.	Exchange	Experiment
Be	1.88	18.2	19.0	0.72	0.70	0.65
Al	2.07	15.7	15.0	0.65	0.63	0.74
Ge	2.08	15.6	16.5	0.65	0.63	1.22
Mg	2.65	10.9	10.5	0.57	0.49	0.81

REFERENCES

1. M. Gell-Mann and K. A. Brueckner, Phys. Rev. 106, 364 (1957).
2. M. Gell-Mann, Phys. Rev. 106, 367 (1957).
3. E. P. Wigner, Phys. Rev. 46, 1002 (1934); Trans. Faraday Soc., 34 678 (1938).
4. D. Bohm and D. Pines, Phys. Rev. 82, 625 (1951).
5. D. Bohm and D. Pines, Phys. Rev. 92, 609 (1953).
6. D. Bohm and D. Pines, Phys. Rev. 85, 332 (1952)
7. D. Pines, Phys. Rev. 92, 626 (1954).
8. W. Mäcke, Zeit. f. Naturforch, 5a, 192 (1950).
9. K. Sawada, Phys. Rev. 106, 372 (1957);
   K. Sawada, K. Brueckner, N. Fukuda, R. Brout, Phys. Rev. 108, 515 (1957).
10. J. Hubbard, Proc. Roy. Soc. A., 240, 539 (1957); Proc. Roy Soc. A. 243, 336 (1958).
11. P. Noziere and D. Pines, Phys. Rev. (to be published).
12. J. Goldstone, Proc. Roy. Soc. A, 239 (1957).
13. K. A. Brueckner, Phys. Rev. 100, 36 (1955).
14. N. M. Hugenholtz, Physica 23, 481 (1957).
15. S. Schweber, H. Bether, F. de Hoffmann, "Mesons and Fields," Row, Peterson and Co. (1955).
16. J. M. Jauch and F. Rohrlich, "The Theory of Photons and Electrons," Addison-Wesley (1955).
17. R. P. Feynman, Phys. Rev. 76, 749 (1949).
18. R. P. Feynman, Phys. Rev. 76, 769 (1949).
19. M. Gell-Mann and M. Goldberger, Phys. Rev. 91, 398 (1953).
20. G. C. Wick, Phys. Rev., 80, 268 (1950).

21. D. Pines, Revs. Modern Phys. 28, 184 (1956).

22. J. Lindhard, Kgl. Danske Videnskab. Selskab., Mat.-fys. Medd. 28, No. 8 (1954).

23. W. Heitler and S. T. Ma, Proc. Roy. Ir. Ac. 52, 109 (1949).

24. F. Low, Phys. Rev. 88, 53 (1952).

25. M. Gell-Mann and F. Low, Phys. Rev. 84, 350 (1951).

26. R. A. Ferrell, Phys. Rev. 107, 450 (1957).

27. D. Pines, Revs. Modern Phys. 28, 184 (1956).

28. R. A. Wolff, Phys. Rev. 92, 18 (1953).

29. H. Watanabe, J. Phys. Soc. Japan 11, 112 (1956).

30. J. Hubbard, Proc. Phys. Soc. (London) A68, 441 (1955).

31. R. A. Ferrell, Phys. Rev. 101, 554 (1956).

32. N. F. Mott and H. S. W. Massey, "Theory of Atomic Collisions" Oxford, Second Edition (1949), pp. 248-252.

33. G. Meyer, Z. Physik 148, 61 (1957).

34. N. M. Hugenholtz, Physica 23, 533 (1957).

35. H. Bethe and E. Salpeter, Phys. Rev. 84, 1232 (1951).

36. M. Gell-Mann and F. Low, Phys. Rev. 84, 350 (1951).

37. L. Van Hove, Physica 21, 901 (1955).