ECONOMETRICS OF DUOPOLISTIC GAMES IN PRICES AND ADVERTISING:

The Case of The U.S. Soft Drink Industry

Thesis by

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DEDICATION

To Djamila and our parents

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ABSTRACT

This thesis pursues the double purpose of measuring, and improving the tools of measurement of, the economic impact of both advertising and pricing decisions by firms in duopolistic industries. In seeking to obtain efficient statistical estimates of the effect of these variables on market demands, we specify various structural economic models from which we derive estimable systems of simultaneous equations. The essential hypothesis is that, at any given period, observations on the variables of these simultaneous-equation econometric models have arisen as the equilibrium outcomes of some specified games of competition between firms.

This work illustrates a new methodology that combines game theoretic considerations and modern econometric and statistical tools. Our empirical findings have, indeed, demonstrated how fruitful and promising such a combination is.

The analysis of data on the U.S. soft drink industry by means of the framework developed in this study produces two types of results. First, we obtain more accurate estimates of the economic impact of advertising, a highly strategic and instrumental variable for firms, than those obtained so far with available techniques. We utilize full information maximum likelihood methods to estimate simultaneous-equation econometric models of the U.S. soft drink industry, each of which incorporates information about a specific form of competition between firms. Second, using recent econometric techniques, we perform some statistical tests which enable us to discriminate among the different models. We are, therefore, in a position of determining which of the various formal representations of the industrial organization of such a sector is most compatible with the available data.

TABLE OF CONTENTS

LIST OF TABLES AND FIGURES

INTRODUCTION

Starting with the break-through work of Augustin Cournot (1838), economists have long been interested in understanding and modelling the behavior of firms in industries which do not meet the conditions of perfect competition. In his model, which is intended to describe an oligopolistic industry, Cournot introduced his concept of market equilibrium, which has been generalized and integrated in modern game theory by John Nash (1951) as the non-cooperative equilibrium. While in the original work the firm's key decision variable is quantity of output, the Cournot model has been adapted to the case where price is the firm's decision variable. An important piece of work along these lines has been achieved by Joseph Bertrand (1883). A large literature then followed on oligopoly theory, based on behavioral assumptions of the Cournot/Bertrand type. In order to construct formal models intended to describe an oligopolistic industry, economic theorists have widely assumed that firms set output or price in a Cournot/Bertrand fashion. Namely, in the decision-making process each firm assumes that its decision will have no effect on its rival's actions. Recently, however, researchers have come to question such an assumption.[1] Indeed, as Scherer (1970) notes, "Economists have come to believe that the Cournot assumption is quite unrealistic when applied to pricing decisions involving only a few firms." One good reason is that, in a duopolistic industry, unless a firm is "myopic," it is able to match price cuts by its rival firm almost instantaneously.

In recent years, economists and in particular researchers in industrial organization, being aware of unappealing features of price competition in oligopoly, have focused attention on nonprice rivalry among firms. One such avenue of study revolves around the idea that advertising constitutes a potential scope within which oligopolistic firms could compete. As

Scherer (1970) points out, "...businessmen may seek an outlet for their aggressive instincts on non-price dimensions such as advertising, where the drawbacks are not so obvious." Moreover, in the case of advertising competition, the Cournot assumption has been judged to be more realistic. Whereas a firm can react almost immediately to a price decision by its competitor, it could take a substantial amount of time for a retaliatory advertising campaign to be set. Further, there is some sort of uncertainty for the firm as to the real effect of an advertising effort. Indeed, the impact of any promotional campaign depends not only on the spending assigned to it, but also on the quality of the "appeal." For an analytical treatment of oligopoly with advertising competition (under Cournot/Bertrand type behavior on the part of the firms), see Schmalensee (1976). See also Friedman (1983).

The issue of the economic impact of advertising of advertising has attracted considerable interest among researchers. See the extensive literature that comprises the works of Schmalensee (1972), Metwally (1976), Lambin (1976), Friedland (1977), Martin (1979), Arterburn and Woodbury (1981), Pagoulatos and Sorensen (1981), Netter (1982), Arndt and Simon (1983), Albion (1983), Buxton et al.(1984), Kardasz and Stolerry (1984), Gomes (1986), and Schroeter et al.(1987) among others. For an interesting review of the traditional literature on the subject, see Comanor and Wilson (1979). While from a theoretical point of view no clear-cut answer has been provided, the debate over the effect of advertising has generated an impressive set of empirical studies with conflicting results. For example, one school of thought based on a study by Bain (1959) claims, on statistical grounds, that industrial concentration and advertising (ratio to sales) are positively correlated. Thus, firms which are heavily advertising are enjoying monopoly profits. On the other hand, stemming from a study published by Telser (1964) a number of empirical studies found support for the hypothesis that such a correlation is not substantial. See the debate between Mann et al. (1967, 1969) and

Ekelund and Maurice (1969).

Further work has focused on the direct investigation of the relationship between profit rates and advertising. In this context two points are worthwhile noting in the literature. First, earlier studies have failed to consider the capital investment nature of advertising. Then, much discussion has taken place concerning the duration of cumulative advertising impact (see Hirschey (1982)). Empirical studies have led economists to believe that advertising yields a short-term effect of one year or less. See Clarke (1976).[2] The next step was to measure this effect using econometric methods. Single-equation as well as simultaneous-equation models have been suggested in order to capture the quantitative effect of advertising (market structure) on profits (economic performance).

The goal of this thesis is to make a contribution to these ongoing fields of research and to emphasize the relationship between empirical practices and theoretical developments in those fields. Our starting point is to investigate directions for improving econometric means of measuring the economic impact of both advertising and pricing decisions by firms, in the context of duopolistic industries. We specify various structural economic models of the Cournot type for a duopoly, from which we derive estimable systems of simultaneous equations. At any given period, observations on the variables of these simultaneous-equation models are assumed to have arisen as the equilibrium outcomes of some specified games of competition between firms. These models are used to analyze empirical data on the U.S. soft drink industry. Two types of outputs result from such an analysis. First, as more information about behavior of firms is incorporated in these models, we obtain more accurate estimates than those obtained so far with available techniques. Second, using recently developed econometric methods, we perform some statistical model selection tests. Thus, our work suggests a statistical approach to industrial organization in that we seek to determine which of the various formal representations of the U.S. soft drink sector is most compatible with the empirical data.

The plan of the dissertation is as follows. In chapter 1, we analyze a set of duopolistic continuous games and formulate simultaneous-equation models for these games. Though these models are nonlinear and nonnested in each other, we show that they are all nested in a general linear model. In chapter 2, we describe a duopolistic discrete game and derive empirical simultaneous-equation models for qualitative variables. The last chapter contains the empirical part of the thesis. In this chapter, we specify and estimate, by standard regression methods and maximum likelihood techniques, the models developed in chapter 1. We use data on the market for the regular carbonated drink in the U.S. soft drink industry. We then perform some specification tests and some tests for nonnested hypotheses which allow us to determine which of the different models is most adequate in view of the available data. Finally, we discuss the empirical findings and give some general thoughts and directions for future research.

CHAPTER 1. ECONOMETRIC ANALYSIS OF DUOPOLISTIC CONTINUOUS GAMES IN PRICE AND ADVERTISING

1. INTRODUCTION

The principal purpose of this chapter is to develop econometric models for measuring the effect of advertising on market demand in the context of a duopoly. In such industries and in particular when the good produced is homogenous, firms often find it more lucrative to attract additional market share through nonprice competition. Advertising constitutes an important tool in that direction. Thus, in such an economic context, firms can be viewed as being involved in a noncooperative game, where a strategy for a representative firm is a vector expressing levels of advertising expenditures and prices. The novelty of the analysis conducted in this chapter is that it recognizes explicitly this gaming nature in the econometric modelling of observed data.

While a large number of empirical studies have analyzed advertising and its impact on market demand, they have frequently failed to explicitly model the process by which price and advertising levels are determined. See Quandt (1964). The analysis presented in this chapter attempts to fill this void. Unlike previous empirical studies, we shall take into account, in the econometric modelling process, the different forms of duopolistic competition in a given industry. More specifically, we shall derive econometric models under the assumption that the data are the equilibrium outcomes of various games in price and advertising. Such an approach presents two advantages as it will be seen in subsequent chapters. First, using recent econometric techniques, it allows us to identify the game which is the most adequate in view of the available data. Second, given the most adequate model, it provides more accurate estimates of the effect of advertising on market demand than those which have been obtained so far in the literature.

Along the lines of Bjorn and Vuong (1984, 1985) and Breshnahan and Reiss (1986), the data are assumed to be generated as the equilibrium outcome of one of the particular games which we shall describe below. We assume, for reasons previously discussed, that firms compete not only through prices, but also through advertising. That is, the strategies available to the firms are price and advertising levels. Thus, the import of this study is that it illustrates how econometric models can be derived from an economic structure characterized by firms involved in games of competition in price and advertising.

In section 2, we outline the basic features of the duopoly under study and describe six games between firms. In the first game, firms set price and advertising levels simultaneously. In the second game, firms choose price in a first stage and advertising in a second stage. In the fourth game, one firm sets price in a first stage, while its advertising and the other firm's price and advertising levels are chosen in a second stage. The third and fifth games are respectively identical to the second and fourth games with the role of advertising and price reversed. In the sixth game, one firm acts as a leader in choosing price and advertising levels. Noncooperative Nash equilibrium is used in the first five games and subgame perfection (see Selten (1975)) of this equilibrium is required in order to sustain credibility of commitments in the two-stage games. Finally, the usual Stackelberg equilibrium notion is used to solve the sixth game.

Section 3 contains the derivation of the econometric models associated with each of the six games discussed in the previous section. Although none of the models is nested in another, it turns out that each of them is equivalent to a general linear simultaneous-equation model with a specific set of nonlinear constraints on the parameters. This observation suggests that one can solve the identification problem in a more general framework (the general model), rather than tackling it within the context of each specific model. This task is carried out in this section.

2. THEORETICAL DUOPOLISTIC CONTINUOUS GAMES

The discussion which follows stands along the lines of an approach presented in Economides (1986). We analyze a series of six games in the context of a simple duopoly where advertising and price are the firms' decision variables.

We consider an industry largely dominated by two firms, firm 1 and firm 2. These firms compete through price and advertising levels. The market conditions are summarized by the following demands faced by the firms:

$$q_i = \gamma_{i0} + \alpha_{ii} p_i + \alpha_{ij} p_j + \gamma_{ii} A_i^{1/2} + \gamma_{ij} A_j^{1/2} \tag{2.1}$$

$i, j = 1,2,\ i \neq j$ where, q represents quantity, p is price, A is advertising expenditure and, the α's and the γ's are unknown parameters. This specification of the demand functions shows diminishing returns for advertising and allows for different types of economic environments. For instance, when $\alpha_{ii} = \gamma_{ij} = -1$ and $\alpha_{ij} = \gamma_{ii} = 1,\ i, j = 1,2,\ i \neq j$, the environment is very competitive. See Economides (1986). In this chapter, we choose to let the data in hand decide on the kind of environment in which the firms evolve. Thus, we do not impose any a priory constraints on these unknown parameters.

The technology characterizing the industry in which the firms operate is described by the following cost functions:

$$C_i(q_i) = c_i q_i \tag{2.2}$$

$i = 1,2$ where, c_i represents constant marginal cost of firm i. Firm $i's$ profit function is thus written as:

$$\Pi_i = (p_i - c_i)(\gamma_{i0} + \alpha_{ii} p_i + \alpha_{ij} p_j + \gamma_{ii} A_i^{1/2} + \gamma_{ij} A_j^{1/2}) - A_i \tag{2.3}$$

$i, j = 1,2,\ i \neq j$.

In five of the six games discussed, we make use of the noncooperative Nash equilibrium solution concept. In the first game, firms set price and advertising levels simultaneously. In the second game, firms choose simultaneously their price levels in a first stage, while advertising level choices are made simultaneously in a second stage. In the third game, firms commit to choose first advertising in a first stage and then set prices in the last stage. In the fourth game, one of the firms has the ability to choose its price in a first stage, while its advertising and the other firm's advertising and price levels are chosen simultaneously in a second stage. In the fifth game, one firm chooses its advertising level in a first stage, while its price as well as the other firm's price and advertising levels are set in the last stage. In order to sustain credibility in the two-stage games we require that Nash equilibrium be prescribed at each stage. Thus, the notion of subgame perfect Nash equilibrium is used in those cases. Finally, we describe a sixth game in which one of the firms acts as a leader in setting price and advertising levels. In this latter game, the usual Stackelberg equilibrium solution concept is utilized.[3]

The remainder of this section characterizes in a set of propositions (propositions 1-6), the (unique) equilibrium of each of the games previously described. As our principal concern in this chapter is to formulate estimable structural models for these games, we should in principle solve for the equilibrium values of the decision variables. Given these propositions, computation of the equilibrium values of these games is a straightforward but tedious exercise. The econometric approach adopted in this chapter does not, however, necessitate such an endeavor. Indeed (see section 3), we use the equations which characterize those values to generate structural econometric models.[4] We now present the first-order conditions which characterize such equilibrium price-advertising pairs for each of the games described above.

Proposition 1. The unique Nash equilibrium $(p_1^*, A_1^{\frac{1}{2}*}, p_2^*, A_2^{\frac{1}{2}*})$ of the one-stage simultaneous game is given by the solution of the following system of equations:

$$\alpha_{11}(p_1^* - c_1) + \gamma_{10} + \alpha_{11}p_1^* + \alpha_{12}p_2^* + \gamma_{11}A_1^{\frac{1}{2}*} + \gamma_{12}A_2^{\frac{1}{2}*} = 0 \tag{2.4}$$

$$\alpha_{22}(p_2^* - c_2) + \gamma_{20} + \alpha_{21}p_1^* + \alpha_{22}p_2^* + \gamma_{21}A_1^{\frac{1}{2}*} + \gamma_{22}A_2^{\frac{1}{2}*} = 0 \tag{2.5}$$

$$\tfrac{1}{2}\gamma_{11}(p_1^* - c_1)A_1^{-\frac{1}{2}*} - 1 = 0 \tag{2.6}$$

$$\tfrac{1}{2}\gamma_{22}(p_2^* - c_2)A_2^{-\frac{1}{2}*} - 1 = 0. \tag{2.7}$$

Proof. In this game firms make their price and advertising decisions simultaneously in one stage. The Nash equilibrium price-advertising pairs are found by solving the profit maximization problems of the firms with respect to the decision variables $p_1, A_1^{\frac{1}{2}}, p_2, A_2^{\frac{1}{2}}$. The first-order conditions are given by:

$$\frac{\partial \Pi_i}{\partial p_i} = 0 \;\; ; \;\; \frac{\partial \Pi_i}{\partial A_i} = 0 \tag{2.8}$$

$i = 1,2$ where Π_i is given by equation (2.3) in the text. Expanding, we obtain equations (2.4)-(2.7) displayed in the proposition.

Q.E.D.

Proposition 2. The unique Nash equilibrium $(p_1^*, A_1^{\frac{1}{2}*}, p_2^*, A_2^{\frac{1}{2}*})$ of the two-stage simultaneous game, where price decisions are made in the first stage, is given by the solution of the system of equations (2.4)-(2.7) displayed in proposition 1.

Proof. In the second-stage subgame in which firms set their advertising levels, first-order conditions in these decisions variables (equations (2.6) and (2.7)) are solved to obtain:

$$A_i^{1/2*} = 1/2\gamma_{ii}(p_i - c_i) \tag{2.9}$$

$i = 1,2$. Substituting back into the profit functions, we find the equilibrium profits of the subgame in advertising. These functions are the payoffs of the first-stage subgame in prices and are expressed as:

$$\Pi_i = (p_i - c_i)\left[\gamma_{i0} + \alpha_{ii}p_i + \alpha_{ij}p_j + 1/2\gamma_{ii}^2(p_i - c_i) + 1/2\gamma_{ij}\gamma_{jj}(p_j - c_j)\right] - 1/4\gamma_{ii}^2(p_i - c_i)^2 \tag{2.10}$$

$i, j = 1,2$, $i \neq j$. The first-order conditions of this game in prices are given by:

$$(p_i - c_i)(\alpha_{ii} + 1/2\gamma_{ii}^2) + \gamma_{i0} + \alpha_{ii}p_i + \alpha_{ij}p_j + 1/2\gamma_{ii}^2(p_i - c_i) + 1/2\gamma_{ij}\gamma_{jj}(p_j - c_j) - 1/2\gamma_{ii}^2(p_i - c_i) = 0 \tag{2.11}$$

$i, j = 1,2$, $i \neq j$. Regrouping terms and using (2.9) above, we see that these first-order conditions are the same as equations (2.4) and (2.5) given in proposition 1.

Q.E.D.

Proposition 3. The unique Nash equilibrium $(p_1^*, A_1^{1/2*}, p_2^*, A_2^{1/2*})$ of the two-stage simultaneous game, where advertising decisions are made in the first stage, is given by the solution of the following system of equations:

$$\alpha_{11}(p_1^* - c_1) + \gamma_{10} + \alpha_{11}p_1^* + \alpha_{12}p_2^* + \gamma_{11}A_1^{1/2*} + \gamma_{12}A_2^{1/2*} = 0 \tag{2.12}$$

$$\alpha_{22}(p_2^* - c_2) + \gamma_{20} + \alpha_{21}p_1^* + \alpha_{22}p_2^* + \gamma_{21}A_1^{1/2*} + \gamma_{22}A_2^{1/2*} = 0 \tag{2.13}$$

$$1/2\left[\gamma_{11} + \alpha_{12}\frac{\alpha_{21}\gamma_{11} - 2\alpha_{11}\gamma_{21}}{4\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}\right](p_1^* - c_1) - A_1^{1/2*} = 0 \tag{2.14}$$

$$\tfrac{1}{2}\left[\gamma_{22}+\alpha_{21}\frac{\alpha_{12}\gamma_{22}-2\alpha_{22}\gamma_{12}}{4\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}}\right](p_2^*-c_2)-A_2^{\frac{1}{2}*}=0. \tag{2.15}$$

Proof. This game is merely the counterpart of the game discussed in the previous proposition where the roles of price variables and advertising variables are interchanged. Solving the first-order conditions of the second-stage subgame in prices (equations (2.12) and (2.13)) yields:

$$p_i=\left[\frac{-2\alpha_{jj}\gamma_{i0}+\alpha_{ij}\gamma_{j0}+2\alpha_{ii}\alpha_{jj}c_i-\alpha_{ij}\alpha_{jj}c_j+(\alpha_{ij}\gamma_{ji}-2\alpha_{jj}\gamma_{ii})A_1^{\frac{1}{2}}+(\alpha_{ij}\gamma_{jj}-2\alpha_{jj}\gamma_{ij})A_2^{\frac{1}{2}}}{4\alpha_{ii}\alpha_{jj}-\alpha_{ij}\alpha_{ji}}\right] \tag{2.16}$$

$i,j=1,2,\ i\neq j$. The first-order conditions of the first-stage subgame in advertising are given by:

$$\frac{\partial p_i}{\partial A_i}(\gamma_{i0}+\alpha_{ii}p_i+\alpha_{ij}p_j+\gamma_{ii}A_i^{\frac{1}{2}}+\gamma_{ij}A_j^{\frac{1}{2}})+(p_i-c_i)\left[\alpha_{ii}\frac{\partial p_i}{\partial A_i}+\alpha_{ij}\frac{\partial p_j}{\partial A_i}+\tfrac{1}{2}\gamma_{ii}A_i^{-\frac{1}{2}}\right]-1=0. \tag{2.17}$$

$i,j=1,2,\ i\neq j$. Substituting equations (2.12) and (2.13) and using equation (2.16), we see that these first-order conditions become equations (2.14) and (2.15) given in the proposition.

Q.E.D.

Proposition 4. The unique Nash equilibrium $(p_1^*,\ A_1^{\frac{1}{2}*},\ p_2^*,\ A_2^{\frac{1}{2}*})$ of the two-stage simultaneous game, where firm 1's price decision is made in the first stage, is given by the solution of the following system of equations:

$$\gamma_{10}+\alpha_{11}p_1^*+\alpha_{12}p_2^*+\gamma_{11}A_1^{\frac{1}{2}*}+\gamma_{12}A_2^{\frac{1}{2}*}+\left[\alpha_{11}-\frac{(\alpha_{12}+\frac{1}{2}\gamma_{12}\gamma_{22})(\alpha_{21}+\frac{1}{2}\gamma_{11}\gamma_{21})}{(2\alpha_{22}+\frac{1}{2}\gamma_{22}^2)}\right](p_1^*-c_1)=0 \tag{2.18}$$

$$\alpha_{22}(p_2^*-c_2)+\gamma_{20}+\alpha_{21}p_1^*+\alpha_{22}p_2^*+\gamma_{21}A_1^{\frac{1}{2}*}+\gamma_{22}A_2^{\frac{1}{2}*}=0 \tag{2.19}$$

$$\tfrac{1}{2}\gamma_{11}(p_1^* - c_1)A_1^{-\frac{1}{2}*} - 1 = 0 \tag{2.20}$$

$$\tfrac{1}{2}\gamma_{22}(p_2^* - c_2)A_2^{-\frac{1}{2}*} - 1 = 0. \tag{2.21}$$

Proof. In this game firm 1 has the opportunity to set its price in a first stage while its avertising decision as well as firm 2's advertising and price decisions are made in the last stage. The first-order conditions of the second-stage subgame are given by equations (2.19), (2.20), and (2.21) in the text. These in turn are solved for p_2, $A_1^{\frac{1}{2}}$, and $A_2^{\frac{1}{2}}$ in terms of p_1, to yield:

$$p_2 = \frac{1}{2\alpha_{22} + \frac{1}{2}\gamma_{22}^2}\Big[-\gamma_{20} + \tfrac{1}{2}\gamma_{11}\gamma_{21}c_1 + (\alpha_{22} + \tfrac{1}{2}\gamma_{22}^2)c_2 - (\alpha_{21} + \tfrac{1}{2}\gamma_{11}\gamma_{21})p_1\Big] \tag{2.22}$$

$$A_1^{\frac{1}{2}} = \tfrac{1}{2}\gamma_{11}(p_1 - c_1) \tag{2.23}$$

$$A_2^{\frac{1}{2}} = \frac{\frac{1}{2}\gamma_{22}}{2\alpha_{22} + \frac{1}{2}\gamma_{22}^2}\Big[-\gamma_{20} + \tfrac{1}{2}\gamma_{11}\gamma_{21}c_1 - \alpha_{22}c_2 - (\alpha_{21} + \tfrac{1}{2}\gamma_{11}\gamma_{21})p_1\Big]. \tag{2.24}$$

The first-order condition of the first-stage game where firm 1 has simply to maximize its payoff is given by:

$$\gamma_{10} + \alpha_{11}p_1 + \alpha_{12}p_2 + \gamma_{11}A_1^{\frac{1}{2}} + \gamma_{12}A_2^{\frac{1}{2}} + (p_1 - c_1)\left[\alpha_{11} + \alpha_{12}\frac{\partial p_2}{\partial p_1} + \gamma_{11}\frac{\partial A_1^{\frac{1}{2}}}{\partial p_1} + \gamma_{12}\frac{\partial A_2^{\frac{1}{2}}}{\partial p_1}\right] - \frac{\partial A_1}{\partial p_1} = 0. \tag{2.25}$$

Now, it suffices to use equations (2.20) of the proposition, (2.22), (2.23), and (2.24) above, to see that this first-order condition is equivalent to equation (2.18).

Q.E.D.

Proposition 5. The unique Nash equilibrium $(p_1^*, A_1^{\frac{1}{2}*}, p_2^*, A_2^{\frac{1}{2}*})$ of the two-stage simultaneous game, where firm 1's advertising decision is made in the first stage, is given by the solution of the following system of equations:

$$\alpha_{11}(p_1^* - c_1) + \gamma_{10} + \alpha_{11}p_1^* + \alpha_{12}p_2^* + \gamma_{11}A_1^{1/2*} + \gamma_{12}A_2^{1/2*} = 0 \tag{2.26}$$

$$\alpha_{22}(p_2^* - c_2) + \gamma_{20} + \alpha_{21}p_1^* + \alpha_{22}p_2^* + \gamma_{21}A_1^{1/2*} + \gamma_{22}A_2^{1/2*} = 0 \tag{2.27}$$

$$1/2\left[\gamma_{11} + \frac{(\alpha_{12} + 1/2\gamma_{12}\gamma_{22})(\alpha_{21}\gamma_{11} - 2\alpha_{11}\gamma_{21})}{2\alpha_{11}(2\alpha_{22} + 1/2\gamma_{21}\gamma_{22}) - \alpha_{21}(\alpha_{12} + 1/2\gamma_{12}\gamma_{22})}\right](p_1^* - c_1) - A_1^{1/2*} = 0 \tag{2.28}$$

$$1/2\,\gamma_{22}(p_2^* - c_2)A_2^{-1/2*} - 1 = 0. \tag{2.29}$$

Proof. The game analyzed here is of similar nature as the one described in the previous proposition. Firm 1 precommits itself to choose its optimal advertising level in the first stage, while its price level is set in the last stage together with the optimal price-advertising pair of firm 2. First-order conditions of the second-stage subgame, given by equations (2.26), (2.27) and (2.29), are solved in terms of $A_1^{1/2}$. This yields:

$$p_1 = a_1\gamma_{10} + b_1c_1 + d_1c_2 + e_1A_1^{1/2} \tag{2.30}$$

$$p_2 = a_2\gamma_{10} + b_2\gamma_{20} + d_2c_1 + e_2c_2 + fA_1^{1/2} \tag{2.31}$$

$$A_2^{1/2} = 1/2\gamma_{22}\left[a_2\gamma_{10} + b_2\gamma_{20} + d_2c_1 + e_2c_2 + fA_1^{1/2}\right] \tag{2.32}$$

where, a_i, b_i, d_i, e_i, $i = 1,2$ and f are defined by:

$$\Delta \equiv 2\alpha_{11}(2\alpha_{22} + 1/2\gamma_{21}\gamma_{22}) - \alpha_{21}(\alpha_{12} + 1/2\gamma_{12}\gamma_{22})$$

$$a_1 \equiv -\frac{(2\alpha_{22} + 1/2\gamma_{21}\gamma_{22})}{\Delta} \;;\; a_2 \equiv \frac{\alpha_{21}}{\Delta}$$

$$b_1 \equiv \alpha_{11}\frac{(2\alpha_{22} + 1/2\gamma_{21}\gamma_{22})}{\Delta} \;;\; b_2 \equiv -\frac{2\alpha_{11}}{\Delta}$$

$$d_1 \equiv \left[\frac{\tfrac{1}{2}\gamma_{12}\gamma_{22}(2\alpha_{22} + \tfrac{1}{2}\gamma_{21}\gamma_{22}) - (\alpha_{12} + \tfrac{1}{2}\gamma_{12}\gamma_{22})(\alpha_{22} + \tfrac{1}{2}\gamma_{21}\gamma_{22})}{\Delta} \right] \; ; \; d_2 \equiv -\frac{\alpha_{11}\alpha_{21}}{\Delta} \tag{2.33}$$

$$e_1 \equiv \left[\frac{\gamma_{21}(\alpha_{12} + \tfrac{1}{2}\gamma_{12}\gamma_{22}) - \gamma_{11}(2\alpha_{22} + \tfrac{1}{2}\gamma_{21}\gamma_{22})}{\Delta} \right]$$

$$e_2 \equiv \left[\frac{2\alpha_{11}(\alpha_{22} + \tfrac{1}{2}\gamma_{21}\gamma_{22}) - \tfrac{1}{2}\alpha_{21}\gamma_{12}\gamma_{22})}{\Delta} \right]$$

$$f \equiv \frac{(\alpha_{21}\gamma_{11} - 2\alpha_{11}\gamma_{21})}{\Delta}.$$

Substituting back (2.30), (2.31) and (2.32) into firm 1's profit function, we find this firm's payoff function for the first-stage subgame in advertising. The first-order condition of this subgame is given by:

$$\tfrac{1}{2}e_1 A_1^{-1/2}\left[\gamma_{10} + \alpha_{11}p_1 + \alpha_{12}p_2 + \gamma_{11}A_1^{1/2} + \gamma_{12}A_2^{1/2}\right] + (p_1 - c_1)\left[\tfrac{1}{2}\alpha_{11}e_1 A_1^{-1/2} + \tfrac{1}{2}\alpha_{12}f A_1^{-1/2} + \tfrac{1}{2}\gamma_{11}A_1^{-1/2} + \tfrac{1}{4}\gamma_{12}\gamma_{22}f A_1^{-1/2}\right] - 1 = 0. \tag{2.34}$$

Now, it suffices to use (2.26), substitute back for the expressions of e_1 and f given in (2.33) and regroup terms, to see that this first-order condition can be written in the form of equation (2.28) displayed in the proposition.

Q.E.D.

Proposition 6. The unique Stackelberg equilibrium $(p_1^*, A_1^{1/2*}, p_2^*, A_2^{1/2*})$ of the duopolistic game in price and advertising where firm 1 is the leader is given by the solution of the following system of equations:

$$\gamma_{10} + \alpha_{11}p_1^* + \alpha_{12}p_2^* + \gamma_{11}A_1^{\frac{1}{2}*} + \gamma_{12}A_2^{\frac{1}{2}*} + \left[\alpha_{11} - \alpha_{21}\frac{\alpha_{12} + \frac{1}{2}\gamma_{12}\gamma_{22}}{2\alpha_{22} + \frac{1}{2}\gamma_{22}^2}\right](p_1^* - c_1) = 0 \tag{2.35}$$

$$\alpha_{22}(p_2^* - c_2) + \gamma_{20} + \alpha_{21}p_1^* + \alpha_{22}p_2^* + \gamma_{21}A_1^{\frac{1}{2}*} + \gamma_{22}A_2^{\frac{1}{2}*} = 0 \tag{2.36}$$

$$\tfrac{1}{2}\left[\gamma_{11} - \gamma_{21}\frac{\alpha_{12} + \frac{1}{2}\gamma_{12}\gamma_{22}}{2\alpha_{22} + \frac{1}{2}\gamma_{22}^2}\right](p_1^* - c_1) - A_1^{\frac{1}{2}*} = 0 \tag{2.37}$$

$$\tfrac{1}{2}\gamma_{22}(p_2^* - c_2)A_2^{-\frac{1}{2}*} - 1 = 0. \tag{2.38}$$

Proof. In this game, firm 1, which is assumed to be the leader, solves for its optimal advertising and price levels, taking into account the optimizing behavior of firm 2, the follower. The first-order conditions of the follower's problem are given by (2.36) and (2.38) in the text. These are solved for p_2 and $A_2^{\frac{1}{2}}$ in terms of p_1 and $A_1^{\frac{1}{2}}$ to yield:

$$p_2 = a_1\gamma_{20} + b_1c_2 + d_1p_1 + e_1A_1^{\frac{1}{2}} \tag{2.39}$$

$$A_2^{\frac{1}{2}} = \tfrac{1}{2}\left[a_1\gamma_{20} + (b_1 - 1)c_2 + d_1p_1 + e_1A_1^{\frac{1}{2}}\right] \tag{2.40}$$

where,

$$a_1 \equiv \frac{1}{2\alpha_{22} + \frac{1}{2}\gamma_{22}^2} \; ; \; \Delta \equiv 2\alpha_{22} + \tfrac{1}{2}\gamma_{22}^2$$

$$b_1 \equiv \frac{\alpha_{22} + \frac{1}{2}\gamma_{22}^2}{\Delta} \; ; \; d_1 \equiv -\frac{\alpha_{21}}{\Delta} \; ; \; e_1 \equiv -\frac{\gamma_{21}}{\Delta}. \tag{2.41}$$

Next, we substitute back (2.39) and (2.40) into firm 1's profit function, and calculate the first-order conditions of its maximization problem which turn out to be equations (2.35) and (2.37) given in the proposition.

Q.E.D.

3. ECONOMETRIC MODELS OF DUOPOLISTIC CONTINUOUS GAMES

In the previous section we have derived the first-order conditions which characterize the price-advertising equilibrium values $(p_1^*, A_1^{1/2*}, p_2^*, A_2^{1/2*})$ of each of the duopolistic games discussed. We now turn to the econometric formulation of these equilibria. We make the following assumptions on the constant terms γ_{i0} in the firms' demand equations and on the firms' (real) average costs c_i:

Assumption 1. Let $\gamma_{i0} = z_i'\beta_i$ and $c_i = x_i'\eta_i$ where, for $i = 1,2$, z_i' and x_i' are vectors of exogenous variables, and β_i and η_i are respectively $(h_{z_i} \times 1)$ and $(h_{x_i} \times 1)$ vectors of unknown parameters.

The next assumption provides a way of partitioning the vectors of exogenous variables.

Assumption 2. The vectors of exogenous variables are partitioned as follows:

$$z_i' = \left[m', z', \bar{z}_i' \right] \tag{3.1}$$

$$x_i' = \left[m', x', \bar{x}_i' \right] \tag{3.2}$$

where, the different components have dimensions h_m, h_z, h_x, $h_{\bar{z}_i}$ and $h_{\bar{x}_i}$, $i = 1,2$.

The vector m incorporates variables which are common to the z_i's and x_i's such as the constant term. The vector z contains variables which are common to the z_i's only. The vector x contains variables which are common to the x_i's only. Finally, the vectors $\bar{z}_i$ and $\bar{x}_i$, $i = 1,2,$

are vectors of exogenous variables specific to z_i and x_i, respectively.

Consider the following linear simultaneous-equation model, called the "**general**" model:

$$q_{1t} - \alpha_{11}p_{1t} - \alpha_{12}p_{2t} - \gamma_{11}A_{1t}^{1/2} - \gamma_{12}A_{2t}^{1/2} - z_{1t}'\beta_1 = u_{1t} \qquad (3.3)$$

$$q_{2t} - \alpha_{21}p_{1t} - \alpha_{22}p_{2t} - \gamma_{21}A_{1t}^{1/2} - \gamma_{22}A_{2t}^{1/2} - z_{2t}'\beta_2 = u_{2t} \qquad (3.4)$$

$$q_{1t} + \lambda_1 p_{1t} - x_{1t}'\delta_1 = u_{3t} \qquad (3.5)$$

$$q_{2t} + \lambda_2 p_{2t} - x_{2t}'\delta_2 = u_{4t} \qquad (3.6)$$

$$\mu_1 p_{1t} + A_{1t}^{1/2} - x_{1t}'\delta_3 = u_{5t} \qquad (3.7)$$

$$\mu_2 p_{2t} + A_{2t}^{1/2} - x_{2t}'\delta_4 = u_{6t} \qquad (3.8)$$

$t = 1,2,\ldots,T$ where, the δ_i's are vectors of parameters of the same dimension as the η_i's defined in assumption 1 and, $cov(u_{1t}, u_{2t}, u_{3t}, u_{4t}, u_{5t}, u_{6t})' = \Sigma$.[5]

Propositions 7 through 11 below describe five simultaneous-equation models. Each model embodies behavioral equations which are essentially the mathematical characterization of the equilibrium of one of the games discussed in section 2. We observe that these models are all nested in the general linear simultaneous-equation model.

Proposition 7. The one-stage simultaneous game and the two-stage simultaneous game where price decisions are made in the first stage, can be represented by a simultaneous-equation model given by the general model on which one imposes the following restrictions on the parameters:

$$\lambda_1 = \alpha_{11} \ ; \ \delta_1 = \alpha_{11}\eta_1 \tag{3.9}$$

$$\lambda_2 = \alpha_{22} \ ; \ \delta_2 = \alpha_{22}\eta_2 \tag{3.10}$$

$$\mu_1 = -\tfrac{1}{2}\gamma_{11} \ ; \ \delta_3 = -\tfrac{1}{2}\gamma_{11}\eta_1 \tag{3.11}$$

$$\mu_2 = -\tfrac{1}{2}\gamma_{22} \ ; \ \delta_4 = -\tfrac{1}{2}\gamma_{22}\eta_2. \tag{3.12}$$

Proof. The first two equations of the general linear simultaneuos-equation model given in (3.3) and (3.4), common to all of the six models, are merely the demands faced by the firms in which we substitute for the expression of γ_{i0}, $i = 1,2$ given by assumption 1, and incorporate the error terms u_1 and u_2. Again, using assumption 1, equations (2.4)-(2.7), which characterize the equilibrium of the one-stage game and the two-stage game where price decisions are made in the first stage, can be rewritten as:

$$q_i^* + \alpha_{ii}p_i^* - x_i'\alpha_{ii}\eta_i = 0 \ ; \ -\tfrac{1}{2}\gamma_{ii}p_i^* + A_i^{\frac{1}{2}*} + x_i'\tfrac{1}{2}\gamma_{ii}\eta_i = 0 \tag{3.13}$$

$i = 1,2$. Next, we introduce the error terms u_i, $i = 3,4,5,6$ into these equations and observe that they are equivalent to the last four equations of the general model (equations (3.5)-(3.8) above), provided that one imposes the restrictions (3.9)-(3.12) on the parameters.

Q.E.D.

Proposition 8. The two-stage simultaneous game, where advertising decisions are made in the first stage, can be represented by a simultaneous-equation model given by the general model on which one imposes the following restrictions on the parameters:

$$\lambda_1 = \alpha_{11} \ ; \ \delta_1 = \alpha_{11}\eta_1 \tag{3.14}$$

$$\lambda_2 = \alpha_{22} \;\; ; \;\; \delta_2 = \alpha_{22}\eta_2 \tag{3.15}$$

$$\mu_1 = -\tfrac{1}{2}\left[\gamma_{11} + \alpha_{12}\frac{\alpha_{21}\gamma_{11} - 2\alpha_{11}\gamma_{21}}{4\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}\right] \;\; ; \;\; \delta_3 = -\tfrac{1}{2}\left[\gamma_{11} + \alpha_{12}\frac{\alpha_{21}\gamma_{11} - 2\alpha_{11}\gamma_{21}}{4\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}\right]\eta_1 \tag{3.16}$$

$$\mu_2 = -\tfrac{1}{2}\left[\gamma_{22} + \alpha_{21}\frac{\alpha_{12}\gamma_{22} - 2\alpha_{22}\gamma_{12}}{4\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}\right] \;\; ; \;\; \delta_4 = -\tfrac{1}{2}\left[\gamma_{22} + \alpha_{21}\frac{\alpha_{12}\gamma_{22} - 2\alpha_{22}\gamma_{12}}{4\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}\right]\eta_2. \tag{3.17}$$

Proof. Using assumption 1, equations (2.12)-(2.15) which define the equilibrium of this game can be rewritten as:

$$q_i^* + \alpha_{ii}p_i^* - x_i'\alpha_{ii}\eta_i = 0 \tag{3.18}$$

$$-\tfrac{1}{2}\left[\gamma_{ii} + \alpha_{ij}\frac{\alpha_{ji}\gamma_{ii} - 2\alpha_{ii}\gamma_{ji}}{4\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}\right]p_i^* + A_i^{\frac{1}{2}*} + x_i'\tfrac{1}{2}\left[\gamma_{ii} + \alpha_{ij}\frac{\alpha_{ji}\gamma_{ii} - 2\alpha_{ii}\gamma_{ji}}{4\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}\right]\eta_i = 0 \tag{3.19}$$

$i = 1,2$. Introducing the error components, we notice that these equations are similar to the last four equations of the general model, on which one imposes the nonlinear restrictions (3.14)-(3.17) given in the proposition, on the parameters.

Q.E.D.

Proposition 9. The two-stage simultaneous game, where firm 1's price decision is made in the first stage, can be represented by a simultaneous-equation model given by the general model on which one imposes the following restrictions on the parameters:

$$\lambda_1 = \left[\alpha_{11} - \frac{(\alpha_{12} + \tfrac{1}{2}\gamma_{12}\gamma_{22})(\alpha_{21} + \tfrac{1}{2}\gamma_{11}\gamma_{21})}{(2\alpha_{22} + \tfrac{1}{2}\gamma_{22}^2)}\right] \;\; ;$$

$$\delta_1 = \left[\alpha_{11} - \frac{(\alpha_{12} + \frac{1}{2}\gamma_{12}\gamma_{22})(\alpha_{21} + \frac{1}{2}\gamma_{11}\gamma_{21})}{(2\alpha_{22} + \frac{1}{2}\gamma_{22}^2)} \right] \eta_1 \tag{3.20}$$

$$\lambda_2 = \alpha_{22} \ ; \ \delta_2 = \alpha_{22}\eta_2 \tag{3.21}$$

$$\mu_1 = -\frac{1}{2}\gamma_{11} \ ; \ \delta_3 = -\frac{1}{2}\gamma_{11}\eta_1 \tag{3.22}$$

$$\mu_2 = -\frac{1}{2}\gamma_{22} \ ; \ \delta_4 = -\frac{1}{2}\gamma_{22}\eta_2. \tag{3.23}$$

Proof. Making use of assumption 1 we rewrite the equations which define the equilibrium of this game (equations (2.18)-(2.21) in the text) as:

$$q_1^* + \left[\alpha_{11} - \frac{(\alpha_{12} + \frac{1}{2}\gamma_{12}\gamma_{22})(\alpha_{21} + \frac{1}{2}\gamma_{11}\gamma_{21})}{(2\alpha_{22} + \frac{1}{2}\gamma_{22}^2)} \right] p_1^* - x_1' \left[\alpha_{11} - \frac{(\alpha_{12} + \frac{1}{2}\gamma_{12}\gamma_{22})(\alpha_{21} + \frac{1}{2}\gamma_{11}\gamma_{21})}{(2\alpha_{22} + \frac{1}{2}\gamma_{22}^2)} \right] \eta_1 = 0 \tag{3.24}$$

$$q_2^* + \alpha_{22}p_2^* - x_2'\alpha_{22}\eta_2 = 0 \tag{3.25}$$

$$-\frac{1}{2}\gamma_{ii}p_i^* + A_i^{\frac{1}{2}*} + x_i'\frac{1}{2}\gamma_{ii}\eta_i = 0 \tag{3.26}$$

$i = 1,2$. Adding the error terms, we see that these equations are obtained from the last four equations of the general model by imposing the nonlinear restrictions given by (3.20)-(3.23) in the proposition.

Q.E.D.

Proposition 10. The two-stage simultaneous game, where firm 1's advertising decision is made in the first stage, can be represented by a simultaneous-equation model given by the general model on which one imposes the following restrictions on the parameters:

$$\lambda_1 = \alpha_{11} \ ; \ \delta_1 = \alpha_{11}\eta_1 \tag{3.27}$$

$$\lambda_2 = \alpha_{22} \ ; \ \delta_2 = \alpha_{22}\eta_2 \tag{3.28}$$

$$\mu_1 = -\tfrac{1}{2}\left[\gamma_{11} + \frac{(\alpha_{12} + \frac{1}{2}\gamma_{12}\gamma_{22})(\alpha_{21}\gamma_{11} - 2\alpha_{11}\gamma_{21})}{2\alpha_{11}(2\alpha_{22} + \frac{1}{2}\gamma_{21}\gamma_{22}) - \alpha_{21}(\alpha_{12} + \frac{1}{2}\gamma_{12}\gamma_{22})}\right] \ ;$$

$$\delta_3 = -\tfrac{1}{2}\left[\gamma_{11} + \frac{(\alpha_{12} + \frac{1}{2}\gamma_{12}\gamma_{22})(\alpha_{21}\gamma_{11} - 2\alpha_{11}\gamma_{21})}{2\alpha_{11}(2\alpha_{22} + \frac{1}{2}\gamma_{21}\gamma_{22}) - \alpha_{21}(\alpha_{12} + \frac{1}{2}\gamma_{12}\gamma_{22})}\right]\eta_1 \tag{3.29}$$

$$\mu_2 = -\tfrac{1}{2}\gamma_{22} \ ; \ \delta_4 = -\tfrac{1}{2}\gamma_{22}\eta_2. \tag{3.30}$$

Proof. Equations (2.26)-(2.29) which characterize the equilibrium of this game are rewritten in the following form:

$$q_i^* + \alpha_{ii}p_i^* - x_i'\alpha_{ii}\eta_i = 0,\ i = 1,2 \tag{3.31}$$

$$-\tfrac{1}{2}\left[\gamma_{11} + \frac{(\alpha_{12} + \frac{1}{2}\gamma_{12}\gamma_{22})(\alpha_{21}\gamma_{11} - 2\alpha_{11}\gamma_{21})}{2\alpha_{11}(2\alpha_{22} + \frac{1}{2}\gamma_{21}\gamma_{22}) - \alpha_{21}(\alpha_{12} + \frac{1}{2}\gamma_{12}\gamma_{22})}\right]p_1^* + A_1^{\frac{1}{2}*}$$
$$+ x_1'\tfrac{1}{2}\left[\gamma_{11} + \frac{(\alpha_{12} + \frac{1}{2}\gamma_{12}\gamma_{22})(\alpha_{21}\gamma_{11} - 2\alpha_{11}\gamma_{21})}{2\alpha_{11}(2\alpha_{22} + \frac{1}{2}\gamma_{21}\gamma_{22}) - \alpha_{21}(\alpha_{12} + \frac{1}{2}\gamma_{12}\gamma_{22})}\right]\eta_1 \tag{3.32}$$

$$-\tfrac{1}{2}\gamma_{22}p_2^* + A_2^{\frac{1}{2}*} + x_2'\tfrac{1}{2}\gamma_{22}\eta_2 = 0. \tag{3.33}$$

Incorporating the error terms into these equations, we see that they are equivalent to the last four equations of the general model given that the nonlinear restrictions (3.27)-(3.30) are imposed on the parameters.

Q.E.D.

Proposition 11. The Stackelberg duopolistic game in price and advertising, where firm 1 is the leader, can be represented by a simultaneous-equation model given by the general model on which one imposes the following restrictions on the parameters:

$$\lambda_1 = \alpha_{11} - \alpha_{21}\,\frac{\alpha_{12} + \tfrac{1}{2}\gamma_{12}\gamma_{22}}{2\alpha_{22} + \tfrac{1}{2}\gamma_{22}^2}\ ;\ \delta_1 = \left[\alpha_{11} - \alpha_{21}\,\frac{\alpha_{12} + \tfrac{1}{2}\gamma_{12}\gamma_{22}}{2\alpha_{22} + \tfrac{1}{2}\gamma_{22}^2}\right]\eta_1 \tag{3.34}$$

$$\lambda_2 = \alpha_{22}\ ;\ \delta_2 = \alpha_{22}\eta_2 \tag{3.35}$$

$$\mu_1 = -\tfrac{1}{2}\left[\gamma_{11} - \gamma_{21}\,\frac{\alpha_{12} + \tfrac{1}{2}\gamma_{12}\gamma_{22}}{2\alpha_{22} + \tfrac{1}{2}\gamma_{22}^2}\right]\ ;\ \delta_3 = -\tfrac{1}{2}\left[\gamma_{11} - \gamma_{21}\,\frac{\alpha_{12} + \tfrac{1}{2}\gamma_{12}\gamma_{22}}{2\alpha_{22} + \tfrac{1}{2}\gamma_{22}^2}\right]\eta_1 \tag{3.36}$$

$$\mu_2 = -\tfrac{1}{2}\gamma_{22}\ ;\ \delta_4 = -\tfrac{1}{2}\gamma_{22}\eta_2. \tag{3.37}$$

Proof. Using assumption 1 we rewrite equations (2.35)-(2.38), which define the equilibrium of this game, as follows:

$$q_1^* + \left[\alpha_{11} - \alpha_{21}\,\frac{\alpha_{12} + \tfrac{1}{2}\gamma_{12}\gamma_{22}}{2\alpha_{22} + \tfrac{1}{2}\gamma_{22}^2}\right]p_1^* - x_1'\left[\alpha_{11} - \alpha_{21}\,\frac{\alpha_{12} + \tfrac{1}{2}\gamma_{12}\gamma_{22}}{2\alpha_{22} + \tfrac{1}{2}\gamma_{22}^2}\right]\eta_1 = 0 \tag{3.38}$$

$$q_2^* + \alpha_{22}p_2^* - x_2'\alpha_{22}\eta_2 = 0 \tag{3.39}$$

$$-\tfrac{1}{2}\left[\gamma_{11} - \gamma_{21}\,\frac{\alpha_{12} + \tfrac{1}{2}\gamma_{12}\gamma_{22}}{2\alpha_{22} + \tfrac{1}{2}\gamma_{22}^2}\right]p_1^* + A_1^{1/2*} + x_1'\,\tfrac{1}{2}\left[\gamma_{11} - \gamma_{21}\,\frac{\alpha_{12} + \tfrac{1}{2}\gamma_{12}\gamma_{22}}{2\alpha_{22} + \tfrac{1}{2}\gamma_{22}^2}\right]\eta_1 = 0 \tag{3.40}$$

$$-\tfrac{1}{2}\gamma_{22}p_2^* + A_2^{1/2*} + x_2'\,\tfrac{1}{2}\gamma_{22}\eta_2 = 0. \tag{3.41}$$

After adding the error terms, we see that these equations are obtained from the last four equations of the general model by imposing the nonlinear restrictions (3.34)-(3.37) given in the proposition on the parameters.

Q.E.D.

Next, we turn to the identification of these econometric models. As demonstrated earlier, all of the five models which represent the various games are nested in the general model. Consequently, we study the identification of this larger model. Indeed, identification of the general model will necessarily imply identification of each of the specific models.

The next proposition establishes necessary conditions for identification of the general model.

Proposition 12. The necessary conditions for identification of the general linear simultaneous-equation model are given by:

$$h_x + h_{\bar{z}_2} + h_{\bar{x}_1} + h_{\bar{x}_2} \geqq 4 \tag{3.42}$$

$$h_x + h_{\bar{z}_1} + h_{\bar{x}_1} + h_{\bar{x}_2} \geqq 4 \tag{3.43}$$

$$h_z + h_{\bar{z}_1} + h_{\bar{z}_2} + h_{\bar{x}_2} \geqq 1 \tag{3.44}$$

$$h_z + h_{\bar{z}_1} + h_{\bar{z}_2} + h_{\bar{x}_1} \geqq 1. \tag{3.45}$$

Proof. The general linear simultaneous-equation model, given by equations (3.3)-(3.8) above, can be written in the standard form as follows:

$$\Gamma y_t + B x_t = u_t \tag{3.46}$$

$t = 1,2,\ldots,T$ where,

$$\Gamma = \begin{bmatrix} 1 & 0 & -\alpha_{11} & -\alpha_{12} & -\gamma_{11} & -\gamma_{12} \\ 0 & 1 & -\alpha_{21} & -\alpha_{22} & -\gamma_{21} & -\gamma_{22} \\ 1 & 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mu_2 & 0 & 1 \end{bmatrix} \tag{3.47}$$

$$B = \begin{bmatrix} -\beta'_{m1} & -\beta'_{z1} & 0 & -\beta'_{\bar{z}_1 1} & 0 & 0 & 0 \\ -\beta'_{m2} & -\beta'_{z2} & 0 & 0 & -\beta'_{\bar{z}_2 2} & 0 & 0 \\ -\delta'_{m1} & 0 & -\delta'_{x1} & 0 & 0 & -\delta'_{\bar{x}_1 1} & 0 \\ -\delta'_{m2} & 0 & -\delta'_{x2} & 0 & 0 & 0 & -\delta'_{\bar{x}_2 2} \\ -\delta'_{m3} & 0 & -\delta'_{x3} & 0 & 0 & -\delta'_{\bar{x}_1 3} & 0 \\ -\delta'_{m4} & 0 & -\delta'_{x4} & 0 & 0 & 0 & -\delta'_{\bar{x}_2 4} \end{bmatrix} \tag{3.48}$$

$$y_t = \begin{bmatrix} q_{1t} \\ q_{2t} \\ p_{1t} \\ p_{2t} \\ A_{1t}^{1/2} \\ A_{2t}^{1/2} \end{bmatrix} \quad x_t = \begin{bmatrix} m_t \\ z_t \\ x_t \\ \bar{z}_{1t} \\ \bar{z}_{2t} \\ \bar{x}_{1t} \\ \bar{x}_{2t} \end{bmatrix} \quad u_t = \begin{bmatrix} u_{1t} \\ u_{2t} \\ u_{3t} \\ u_{4t} \\ u_{5t} \\ u_{6t} \end{bmatrix} \tag{3.49}$$

and the dimensions of $m, z, x, \bar{z}_1, \bar{z}_2, \bar{x}_1$, and $\bar{x}_2$ are respectively $h_m, h_z, h_x, h_{\bar{z}_1}, h_{\bar{z}_2}, h_{\bar{x}_1}$, and $h_{\bar{x}_2}$.

Since the a priori restrictions are exclusion restrictions in our framework, the necessary condition for the identification of a specific equation is that the number of variables excluded from the equation be greater than or equal to the number of equations in the model minus one. By examining the parameter matrices Γ and $\boldsymbol{B}$, the following order conditions for the respective equations of the model are derived:

First equation: $1 + h_x + h_{\bar{z}_2} + h_{\bar{x}_1} + h_{\bar{x}_2} \geqq 6 - 1$

Second equation: $1 + h_x + h_{\bar{z}_1} + h_{\bar{x}_1} + h_{\bar{x}_2} \geqq 6 - 1$

Third equation: $1 + 1 + 1 + 1 + h_z + h_{\bar{z}_1} + h_{\bar{z}_2} + h_{\bar{x}_2} \geqq 6 - 1$

Fourth equation: $1 + 1 + 1 + 1 + h_z + h_{\bar{z}_1} + h_{\bar{z}_2} + h_{\bar{x}_1} \geqq 6 - 1$

Fifth equation: $1 + 1 + 1 + 1 + h_z + h_{\bar{z}_1} + h_{\bar{z}_2} + h_{\bar{x}_2} \geqq 6 - 1$

Sixth equation: $1 + 1 + 1 + 1 + h_z + h_{\bar{z}_1} + h_{\bar{z}_2} + h_{\bar{x}_1} \geqq 6 - 1.$

Rearranging, and regrouping conditions which are the same, we see that the necessary conditions for identification of the whole model are given by the inequalities marked by (3.42)-(3.45) in the proposition.

Q.E.D.

While propositions 7 through 11 above show that each of the six games can be represented by an econometric model which is nested in the general linear simultaneous-equation model, a natural question which arises is whether the representative models are nested in each other. The next proposition tells us that these models are nonnested.

Proposition 13. Given that the general linear simultaneous-equation model is identified, the models associated with the various games described in section 2 are nonnested (except for game 1 and game 2 which are represented by the same econometric model).

Proof. Note that, since they are nested in the general model which is identified, the models that represent the various games are also identified. Thus, we seek to show that any pair of these identified models are nonnested. In what follows, we shall demonstrate the proposition for the case of one pair of models. These models are presented in propositions 7 and 8 and will be designated, hereafter, by M1-2 and M3 respectively. The proof for any other pair of models makes use of an analogous argument and is omitted.

The models M1-2 and M3 are said to be strictly nonnested if their respective conditional distributions of the endogenous variables given the exogenous variables, for different parameter values, are different. The assumption of identification of M1-2 and M3 essentially means that different structural forms, corresponding to different parameter values, are not observationally equivalent, i.e., do not result in the same likelihood. Accordingly, there is a one-to-one correspondence between the family of conditional distributions under consideration, and the parameter spaces of the models. Thus, the models M1-2 and M3 will be strictly nonnested if their respective parameter spaces, Θ_1 and Θ_2 say, are different.

Let,

$$\theta \equiv \left[\alpha_{11},\alpha_{12},\alpha_{21},\alpha_{22},\gamma_{11},\gamma_{12},\gamma_{21},\gamma_{22},\Sigma,\lambda_1,\lambda_2,\delta_1,\delta_2,\delta_3,\delta_4,\mu_1,\mu_2\right]' \tag{3.50}$$

$$\omega \equiv \left[\alpha_{11},\alpha_{12},\alpha_{21},\alpha_{22},\gamma_{11},\gamma_{12},\gamma_{21},\gamma_{22},\Sigma,\eta_1,\eta_2\right]' . \tag{3.51}$$

The spaces Θ_1 and Θ_2 are then defined by $\Theta_1 = \{\theta\text{: } \theta = h_1(\omega)\}$ and $\Theta_2 = \{\theta\text{: } \theta = h_2(\omega)\}$ respectively, where the functions h_1 and h_2 are specified according to the restrictions (3.9)-(3.12) and (3.14)-(3.17) given above. These parameter spaces will be the same if ***for any*** ω such that $\theta \in \Theta_1$, we have also $\theta \in \Theta_2$ and vice versa, i.e., $\theta = h_1(\omega) \Leftrightarrow \theta = h_2(\omega)$. This latter equivalence implies that, for any ω, the two following equations hold simultaneously:

$$\alpha_{12}\frac{\alpha_{21}\gamma_{11}-2\alpha_{11}\gamma_{21}}{4\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}}=0 \tag{3.52}$$

$$\alpha_{21}\frac{\alpha_{12}\gamma_{22}-2\alpha_{22}\gamma_{12}}{4\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}}=0. \tag{3.53}$$

Clearly, this is the case only for specific ω's; namely, those for which $\alpha_{12}=\alpha_{21}=0$ or $\alpha_{21}\gamma_{11}-2\alpha_{11}\gamma_{21}=\alpha_{12}\gamma_{22}-2\alpha_{22}\gamma_{12}=0$.

Q.E.D.

NOTES

1. Though in recent decades much emphasis has been put on the examination of the validity of the Cournot assumption, the history of its criticism goes back to the end of the last century. See Fisher (1898).

2. In order to avoid problems associated with accounting methods and tax laws, as well as difficulties related to the sensitivity of profits to cyclical phenomena, authors have studied the effect of advertising on sales.

3. Without any loss of generality we assume that in the games where one firm has the ability to move first, firm 1 does so. Further, in the Stackelberg game, firm 1 is assumed to be the leader.

4. For an interesting economic analysis and comparison of equilibria for some of these games, the reader is referred to Economides (1986).

5. An alternative way of generating econometric models is to incorporate error terms into the demand and cost equations. While this lowers the number of parameters, the estimation procedure becomes more difficult as it is one under constraints on the covariance matrix of the error vector.

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CHAPTER 2. ECONOMETRIC ANALYSIS OF DUOPOLISTIC DISCRETE GAMES IN ADVERTISING

1. INTRODUCTION

In chapter 1, we have described a method of empirically analyzing various duopolistic games. We recall that, in those games, the strategies of the firms were levels of prices and advertising expenditures. Profit functions were the firms' payoffs from the games and those functions were maximized over continuous stategy sets. Also, recall, by assuming that the data have emerged as equilibrium outcomes of the different games, we were able to derive simultaneous-equation econometric models. In this chapter, we provide an analysis of the same nature for duopolistic games with discrete strategy spaces. More specifically, we shall be interested in generating econometric models from situations where two firms/players, having the possibility to take one of two available actions, are involved in games of competition.

The purpose of such an approach is twofold. First, from an empirical point of view, by taking the strategy sets to be, for example, levels of advertising such as "high" or "low" relative to a general trend or simply variations of these levels such as "increase" or "decrease", we are able to study the economic effect of the variable: advertising. Therefore, an econometric analysis of the type presented in the previous chapter can be conducted to measure the economic impact of advertising on which qualitative data are available. Second, from the point of view of econometric theory, it is seen that the econometric formulation of the 2×2 game described above can be related to standard systems of simultaneous-equation models with continuous and discrete variables, such as those analyzed by Heckman (1976, 1978) and Amemiya (1974).[1] Thus, our approach provides a structural economic framework for this type of econometric models.

While the econometric formulation of the discrete game developed in this chapter follows the lines of an approach presented in Bjorn and Vuong (1984, 1985), two features distinctive to the present analysis add to its above-mentioned import. First, we extend the strategy spaces to include random strategies, and by doing so we obtain a unique mixed strategy equilibrium in the cases where these authors obtain none. Second, whereas in the multiple equilibria cases these authors attribute an equal probability of occurrence to each of the equilibrium outcomes, we chose to let the data convey information on the magnitude of these probabilities. More specifically, in those multiple equilibria cases, we allocate endogenous weights to each of the various outcomes and allow, therefore, for the possibility of estimating these weights while estimating the parameters of the econometric models.

The remainder of this chapter is organized as follows. In section 2 we fully analyze a standard two-by-two noncooperative game. In order to characterize the set of Nash equilibria in pure and mixed strategies of this game, we study the possible orderings of the payoffs and define, for each player, four fundamental reaction functions based upon these orderings. The Nash equilibria are then the outcomes produced by each of the sixteen pairs of the above reaction functions. Section 3 derives from this structure of a two-player noncooperative discrete game, a system of simultaneous equations for two dichotomous qualitative variables. This is done essentially by incorporating random components into the payoffs, and expressing conditions of the occurrence of each of the equilibria as conditions on some specified random variables. We solve the technical difficulty of multiple equilibria cases by allocating a probability of occurrence to each of the possible outcomes in those situations. In section 4 we derive nine estimable models which can be used to analyze empirical discrete data.

2. A THEORETICAL DUOPOLISTIC DISCRETE GAME

Our point of departure is to analyze in full generality a two-person non-cooperative game in which each player can take one of two possible actions. More formally, we consider a game in normal form consisting of a set of players $N = \{1,2\}$, their strategy sets $S_1 = S_2 = \{1,0\}$ and payoff functions $\tilde{\pi}_i : S \longrightarrow \boldsymbol{R}$, $i = 1,2$, where S is the cartesian product of the two players' strategy sets and $\boldsymbol{R}$ the set of reals. An element of S may be referred to as an outcome of the game. Thus, we have four possible outcomes namely, all the possible pairs of actions taken by the two players. Table 1 below vizualizes such a game. Each entry of the table shows the pair of payoffs to the players, resulting from the corresponding outcome.

player 2 / player 1	1	0
1	$[\tilde{\pi}_1(1,1), \tilde{\pi}_2(1,1)]$	$[\tilde{\pi}_1(1,0), \tilde{\pi}_2(1,0)]$
0	$[\tilde{\pi}_1(0,1), \tilde{\pi}_2(0,1)]$	$[\tilde{\pi}_1(0,0), \tilde{\pi}_2(0,0)]$

***Table* 1.** A 2×2 game

As already mentioned, in this chapter we are principally interested in modelling duopolistic competition in advertising. It should be clear, however, that other types of multi-agent interactions can be formally described by the simple two-by-two game analyzed below. Along the lines of the empirical study conducted in this work, one can model duopolistic competition in more conventional variables such as output, price, capital or R & D investment.

Alternatively, such a game has been used by Bjorn and Vuong (1984, 1985) to study the simultaneous decision of members of a household to join the work force.

Turning to the game outlined above, we extend the players' strategy sets to include mixed strategies i.e. probability distributions on the strategy sets. Note, that since each player has only two possible actions or "pure strategies," mixed strategies for respectively player 1 and player 2, reduce to vectors $(p, 1-p)$ and $(q, 1-q)$ say, where $0 \leq p \leq 1$ and $0 \leq q \leq 1$. If player 1 uses the mixed strategy $(p, 1-p)$ and player 2 uses $(q, 1-q)$ the outcomes (1,1), (1,0), (0,1) and (1,1) occur with the respective probabilities pq, $p(1-q)$, $(1-p)q$ and $(1-p)(1-q)$. Thus, the resulting expected payoffs for players 1 and 2 are given by:

$$E\tilde{\pi}_i = pq\tilde{\pi}_i(1,1) + p(1-q)\tilde{\pi}_i(1,0) + (1-p)q\tilde{\pi}_i(0,1) + (1-p)(1-q)\tilde{\pi}_i(0,0)$$

for $i = 1,2$.

We seek now to characterize the Nash equilibria of the underlying game.[2] An important step in doing so is the derivation of the reaction functions (correspondences) of the two players. Any such function prescribes for a player, his best response to any strategy adopted by his opponent. More formally, this function associates to any mixed strategy of his opponent, the mixed strategy that maximizes his expected payoff.

In the next section, we will specify an econometric model where randomness is introduced via (unobserved) random variables. Consequently, in what follows we will focus our attention only on reaction functions which can occur with a *strictly* positive probability.[3] As we shall show in the next section, we need to identify only four basic types of reaction functions for each player. Denoting these reaction functions (correspondences) p_1, p_2, p_3, p_4 and q_1, q_2, q_3, q_4 respectively for player 1 and 2, these are:

p_1: $[0,1] \longrightarrow [0,1]$

$$q \longrightarrow p_1(q) = 1 \text{ for } \mathit{all}\ 0 \le q \le 1$$

p_2: $[0,1] \longrightarrow [0,1]$

$$q \longrightarrow p_2(q) = 0 \text{ for } \mathit{all}\ 0 \le q \le 1$$

p_3: $[0,1] \twoheadrightarrow [0,1]$

$$q \longrightarrow p_3(q) = \begin{cases} 1 \text{ for } 0 \le q < r_q \\ [0,1] \text{ for } q = r_p \\ 0 \text{ for } r_q < q \le 1 \end{cases}$$

p_4: $[0,1] \twoheadrightarrow [0,1]$

$$q \longrightarrow p_4(q) = \begin{cases} 0 \text{ for } 0 \le q < r_q \\ [0,1] \text{ for } q = r_q \\ 1 \text{ for } r_q < q \le 1 \end{cases}$$

q_1: $[0,1] \longrightarrow [0,1]$

$$p \longrightarrow q_1(p) = 1 \text{ for } \mathit{all}\ 0 \le p \le 1$$

q_2: $[0,1] \longrightarrow [0,1]$

$$p \longrightarrow q_2(p) = 0 \text{ for } \mathit{all}\ 0 \le p \le 1$$

q_3: $[0,1] \twoheadrightarrow [0,1]$

$$p \longrightarrow q_3(p) = \begin{cases} 1 \text{ for } 0 \le p < r_p \\ [0,1] \text{ for } p = r_p \\ 0 \text{ for } r_p < p \le 1 \end{cases}$$

q_4: $[0,1] \twoheadrightarrow [0,1]$

$$p \longrightarrow q_4(p) = \begin{cases} 0 \text{ for } 0 \le p < r_p \\ [0,1] \text{ for } p = r_q \\ 1 \text{ for } r_p < p \le 1 \end{cases}$$

$$\textit{where} \quad r_p = -\frac{\bar{\pi}_2(0,1) - \bar{\pi}_2(0,0)}{\bar{\pi}_2(1,1) - \bar{\pi}_2(1,0) + \bar{\pi}(0,0) - \bar{\pi}_2(0,1)}$$

$$r_q = -\frac{\bar{\pi}_1(1,0) - \bar{\pi}_1(0,0)}{\bar{\pi}_1(1,1) - \bar{\pi}_1(1,0) + \bar{\pi}_1(0,0) - \bar{\pi}_1(0,1)}$$

As shown in proposition 2 below, we point out that, (r_p, r_q) defines the Nash equilibrium in mixed strategies of the game.

Figure 1 below shows each of the reaction functions on the mixed strategy space i.e., the unit square, for arbitrary r_p and r_q.

The next proposition gives conditions on the payoffs which characterize each of the above reaction functions.

Proposition 1. The reaction functions $p_i, q_i, i = 1,2,3,4$ occur under the following conditions on the payoffs:

p_1: $0 < \bar{\pi}_1(1,0) - \bar{\pi}_1(0,0) < \bar{\pi}_1(1,1) - \bar{\pi}_1(0,1)$ *or*
$0 < \bar{\pi}_1(1,1) - \bar{\pi}_1(0,1) < \bar{\pi}_1(1,0) - \bar{\pi}_1(0,0)$

p_2: $\bar{\pi}_1(1,0) - \bar{\pi}_1(0,0) < \bar{\pi}_1(1,1) - \bar{\pi}_1(0,1) < 0$ *or*
$\bar{\pi}_1(1,1) - \bar{\pi}_1(0,1) < \bar{\pi}_1(1,0) - \bar{\pi}_1(0,0) < 0$

p_3: $\bar{\pi}_1(1,1) - \bar{\pi}_1(0,1) < 0 < \bar{\pi}_1(1,0) - \bar{\pi}_1(0,0)$

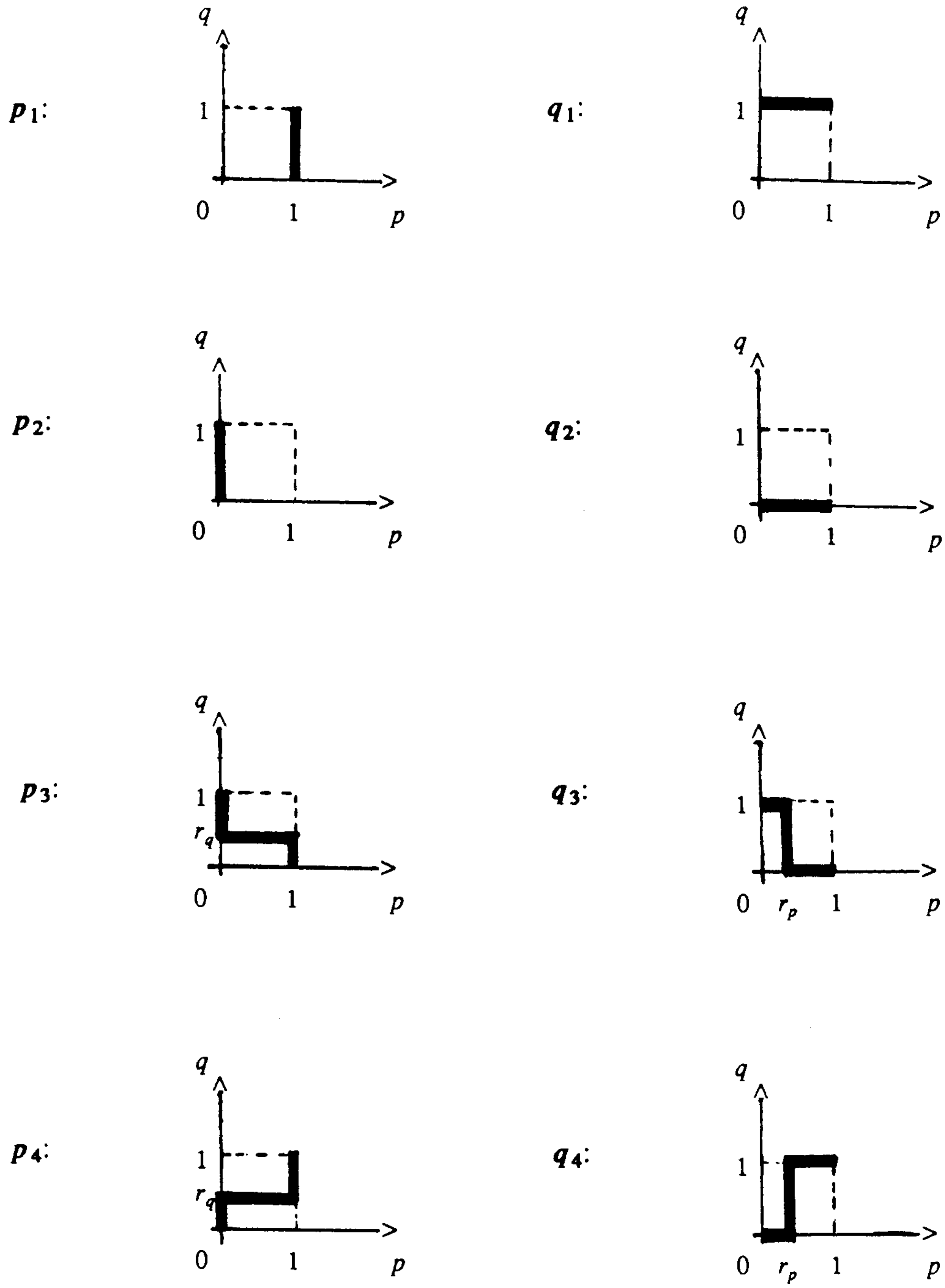

Figure 1. The Players' Reaction Functions on the Unit Square

p_4: $\tilde{\pi}_1(1,0) - \tilde{\pi}_1(0,0) < 0 < \tilde{\pi}_1(1,1) - \tilde{\pi}_1(0,1)$

q_1: $0 < \tilde{\pi}_2(0,1) - \tilde{\pi}_2(0,0) < \tilde{\pi}_2(1,1) - \tilde{\pi}_2(1,0)$ *or*
$0 < \tilde{\pi}_2(1,1) - \tilde{\pi}_2(1,0) < \tilde{\pi}_2(0,1) - \tilde{\pi}_2(0,0)$

q_2: $\tilde{\pi}_2(0,1) - \tilde{\pi}_2(0,0) < \tilde{\pi}_2(1,1) - \tilde{\pi}_2(1,0) < 0$ *or*
$\tilde{\pi}_2(1,1) - \tilde{\pi}_2(1,0) < \tilde{\pi}_2(0,1) - \tilde{\pi}_2(0,0) < 0$

q_3: $\tilde{\pi}_2(1,1) - \tilde{\pi}_2(1,0) < 0 < \tilde{\pi}_2(0,1) - \tilde{\pi}_2(0,0)$

q_4: $\tilde{\pi}_2(0,1) - \tilde{\pi}_2(0,0) < 0 < \tilde{\pi}_2(1,1) - \tilde{\pi}_2(1,0)$.

Proof. In what follows, we shall expose in detail the proof of the results established for player 1. The remainder of the proposition follows from an analogous and symmetric argument.

Rearranging the expression of the expected payoff (see section 1 above), when player 1 uses the mixed strategy $(p, 1-p)$ and player 2 uses $(q, 1-q)$, we obtain:

$$\begin{aligned} E\tilde{\pi}_1 &= q\tilde{\pi}_1(0,1) + (1-q)\tilde{\pi}_1(0,0) + p\,[\tilde{\pi}_1(1,0) - \tilde{\pi}_1(0,0) \\ &\quad + q\,(\tilde{\pi}_1(1,1) - \tilde{\pi}_1(1,0) + \tilde{\pi}_1(0,0) - \tilde{\pi}_1(0,1))] \\ &= q\tilde{\pi}_1(0,1) + (1-q)\tilde{\pi}_1(0,0) + p \cdot \psi_1(q) \end{aligned}$$

where,

$$\psi_1(q) \equiv \tilde{\pi}_1(1,0) - \tilde{\pi}_1(0,0) + q\,[\tilde{\pi}_1(1,1) - \tilde{\pi}_1(1,0) + \tilde{\pi}_1(0,0) - \tilde{\pi}_1(0,1)].$$

Define $r_q \equiv -\dfrac{\tilde{\pi}_1(1,0) - \tilde{\pi}_1(0,0)}{\tilde{\pi}_1(1,1) - \tilde{\pi}_1(1,0) + \tilde{\pi}_1(0,0) - \tilde{\pi}_1(0,1)}$, namely r_q is the value of q for which $\psi_1(q)$ vanishes.

We have the following six cases:

Case 1: $\tilde{\pi}_1(1,1) - \tilde{\pi}_1(0,1) > \tilde{\pi}_1(1,0) - \tilde{\pi}_1(0,0) > 0$

It follows that $r_q < 0$. Furthermore, the linear function $\psi_1(q)$ has a positive slope. Consequently, it exhibits the shape shown in figure 2 below. Thus, for $0 \le q \le 1$, $\psi_1(q) > 0$. By examining the expression of $E\tilde{\pi}_1$, we see that it is maximized for $p = 1$. Thus, the above conditions lead to the reaction function p_1 described in the proposition.

Case 2: $\tilde{\pi}_1(1,1) - \tilde{\pi}_1(0,1) < 0 < \tilde{\pi}_1(1,0) - \tilde{\pi}_1(0,0)$.

This implies that $0 < r_q < 1$. Thus, the function $\psi_1(q)$ has a negative slope and is displayed in figure 2.

$$\begin{aligned} \textit{Thus}, \text{ for } 0 \le q < r_q, \; & \psi_1(q) > 0 \\ r_p < q \le 1, \; & \psi_1(q) < 0 \\ q = r_q, \; & \psi_1(q) = 0. \end{aligned}$$

The reaction correspondence p_3 follows.

Case 3: $0 < \tilde{\pi}_1(1,1) - \tilde{\pi}_1(0,1) < \tilde{\pi}_1(1,0) - \tilde{\pi}_1(0,0)$

It follows that $r_q > 1$. The function $\psi_1(q)$ is as shown in figure 2. Thus for $0 \le q \le 1$, $\psi_1(q) > 0$. The reaction function p_1 follows.

Case 4: $\tilde{\pi}_1(1,1) - \tilde{\pi}_1(0,1) < \tilde{\pi}_1(1,0) - \tilde{\pi}_1(0,0) < 0$

This implies $r_q < 0$ and a function $\psi_1(q)$ as shown in figure 2. Hence, for $0 \le q \le 1$, $\psi_1(q) < 0$ and the reaction function p_2 follows.

Case 5: $\tilde{\pi}_1(1,0) - \tilde{\pi}_1(0,0) < 0 < \tilde{\pi}_1(1,1) - \tilde{\pi}_1(0,1)$

In this case we have $0 < r_q < 1$, and the linear function $\psi_1(q)$ exhibits the shape shown in figure 2.

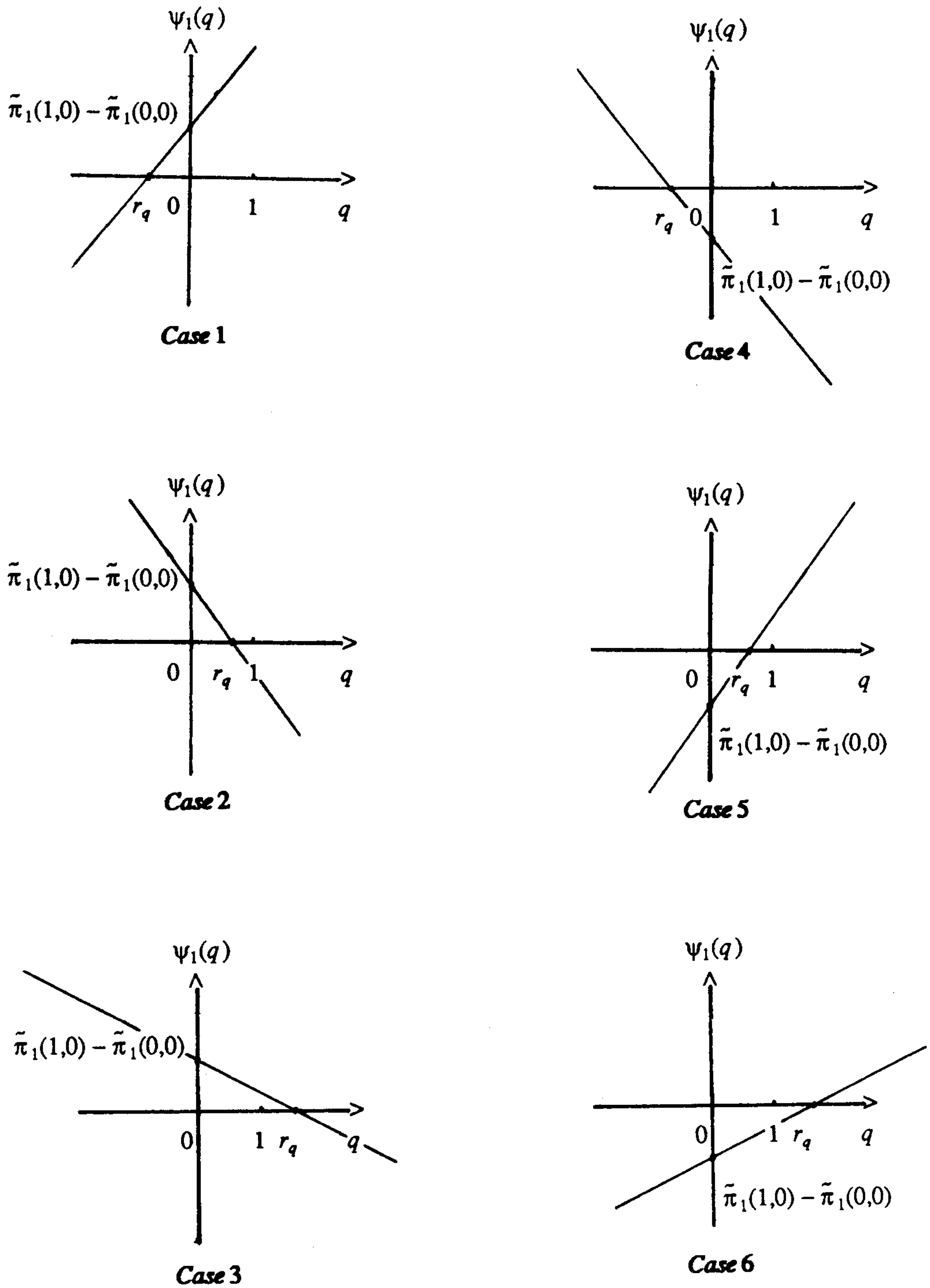

Figure 2. The Function $\psi_1(q)$ Used in the Proof of Proposition 1

Thus for $0 \le q < r_q$, $\psi_1(q) < 0$

$r_q < q \le 1$, $\psi_1(q) > 0$

$q = r_q$, $\psi_1(q) = 0$.

Consequently, the reaction correspondence p_4 described above follows.

Case 6: $\tilde{\pi}_1(1,0) - \tilde{\pi}_1(0,0) < \tilde{\pi}_1(1,1) - \tilde{\pi}_1(0,1) < 0$.

This implies $r_q > 1$ and a function $\psi_1(q)$ of the form given below. Thus for $0 \le q \le 1$, $\psi_1(q) < 0$. The reaction function p_2 described above follows.

Now, it suffices to collect the conditions leading to the same reaction function.

Q.E.D.

A Nash equilibrium results when no player has an incentive to deviate from his equilibrium strategy given that the other player does not alter his strategy. Thus, an equilibrium (if it exists) corresponds precisely to the intersection of a pair of reaction functions of the two players. Since each player has four possible reaction functions, we need to consider sixteen pairs of these. The next proposition presents the Nash equilibria corresponding to each of the sixteen pairs of reaction functions.

Proposition 2. For each of the sixteen pairs of reaction functions the following are the corresponding Nash equilibria:

	(q_1)	(q_2)	(q_3)	(q_4)
(p_1)	(1,1)	(1,0)	(1,0)	(1,1)
(p_2)	(0,1)	(0,0)	(0,1)	(0,0)
(p_3)	(0,1)	(1,0)	(0,1) (r_p, r_q) (1,0)	(r_p, r_q)
(p_4)	(1,1)	(0,0)	(r_p, r_q)	(0,0) (r_p, r_q) (1,1)

where (r_p, r_q) refers to the Nash equilibrium where player 1 uses the mixed strategy $(r_p, 1 - r_p)$ and player 2 uses $(r_q, 1 - r_q)$.[4]

Proof. It suffices to solve simultaneously for each pair of the reaction function (correspondences). Equivalently, one can examine the representations of these functions (correspondences) on the unit square (see figure 1 above), and perform the different intersections.

Q.E.D.

3. GENERAL ECONOMETRIC MODEL OF A DUOPOLISTIC DISCRETE GAME

In section 2, we have described a two-person noncooperative game where each player can take one of two possible actions available to him. In this game, and in most of the fundamental theory of games as developed by von Neumann and Morgenstern (1944), implicit is the assumption that each player is aware of the alternatives (or actions) available both to him and his opponent, as well as the relationship between these choices and the different outcomes. Furthermore, for any given configuration of payoffs, each player knows it and thus knows his opponent's preference pattern for the outcomes of the game.

In the spirit of econometric analysis we shall incorporate some uncertainty in the payoff functions. More specifically, we will regard the payoff functions as the sum of two components: a systematic or deterministic part which includes the important factors that influence the payoffs and, a nonsystematic or random part which incorporate other factors that are unobservable. The deterministic part will incorporate for example a linear combination of a set of observed exogenous variables which are believed to influence the payoffs. This approach stands along the line of the so-called random utility maximization theory first developed in psychometrics (see e.g. Tversky, 1972a and 1972b) and introduced in econometrics by McFadden (1974, 1981).

The following assumption translates the previous discussion:

Assumption 1: The payoffs are random functions. Namely, for any outcome $s = (s_1, s_2) \in S$

$$\tilde{\pi}_i(s) = \pi_i(s) + \eta_i(s) \quad \text{for } i = 1,2$$

where $\pi_i(s)$ and $\eta_i(s)$ are respectively the deterministic and random components.

In proposition 1, we have seen that each of the reaction functions (correspondences) of the players is characterized by inequality conditions involving differences of payoffs. Assumption 1 above implies that these conditions involve differences of π_i's and η_i's. As in Bjorn and Vuong (1984) the following assumptions are made for computational tractability. We shall motivate them below.

Assumption 2:

$$\eta_1(1,1) - \eta_1(0,1) = \eta_1(1,0) - \eta_1(0,0) = \varepsilon_1$$

$$\eta_2(1,1) - \eta_2(1,0) = \eta_2(0,1) - \eta_2(0,0) = \varepsilon_2$$

where the two-dimensional random vector $(\varepsilon_1,\varepsilon_2)$ follows a bivariate normal distribution with zero means, unit variances and correlation ρ.

Assumption 3:

$$\pi_1(1,1) - \pi_1(0,1) = \beta_1 + \pi_1(1,0) - \pi_1(0,0) = \beta_1 + \Delta_1$$

$$\pi_2(1,1) - \pi_2(1,0) = \beta_2 + \pi_2(0,1) - \pi_2(0,0) = \beta_2 + \Delta_2$$

with β_1, β_2 being constants,

$$\Delta_1 = x_1'\gamma_1, \ \Delta_2 = x_2'\gamma_2,$$

where x_1' and x_2' are vectors of exogenous variables and γ_1 and γ_2 are respectively $(K_1 \times 1)$ and $(K_2 \times 1)$ vectors of parameters.[5]

To relate the econometric formulation of the two-by-two game proposed previously to a more standard framework, we shall represent the actions taken by the two players by two dichotomous qualitative variables. We let:

$$y_1 = \begin{cases} 1 \text{ if } \textit{player } 1 \textit{ takes action } 1 \\ 0 \textit{ otherwise} \end{cases}$$

$$y_2 = \begin{cases} 1 \text{ if } \textit{player } 2 \textit{ takes action } 1 \\ 0 \textit{ otherwise} \end{cases}$$

Next, note that assumptions 1-3 together with the above definition of the qualitative variables y_1 and y_2, enable us to write the following for $y_1, y_2 = 0,1$:

$$\tilde{\pi}_1(1,y_2) - \tilde{\pi}_1(0,y_2) = \beta_1 y_2 + x_1'\gamma_1 + \varepsilon_1$$

$$\tilde{\pi}_2(y_1,1) - \tilde{\pi}_2(y_1,0) = \beta_2 y_1 + x_2'\gamma_2 + \varepsilon_2$$

Define,

$$y_1^* = \tilde{\pi}_1(1,y_2) - \tilde{\pi}_1(0,y_2)$$

$$y_2^* = \tilde{\pi}_2(y_1,1) - \tilde{\pi}_2(y_1,0).$$

Thus, the system of two equations above can be viewed as a system of simultaneous equations for the continuous latent random variables y_1^* and y_2^* as described by Heckman (1978), where the dummy variables y_1 and y_2 are assumed to be generated by the following dichotomization:

$$y_1 = \begin{cases} 1 \text{ if } y_1^* > 0 \\ 0 \textit{ otherwise}, \end{cases} \quad y_2 = \begin{cases} 1 \text{ if } y_2^* > 0 \\ 0 \textit{ otherwise} \end{cases}$$

The next proposition characterizes the occurrence of the two players' reaction functions (correspondences) in terms of conditions on the random variables ε_1 and ε_2.

Proposition 3. The following are conditions on the random variables ε_1 and ε_2 for the occurrence of the two players' reaction functions:

p_1 *occurs* if $\varepsilon_1 > -\Delta_1 - \min\{0,\beta_1\}$

p_2 *occurs* if $\varepsilon_1 < -\Delta_1 - \max\{0,\beta_1\}$

p_3 *occurs* if $\beta_1 < 0$ and $-\Delta_1 < \varepsilon_1 < -\Delta - \beta_1$

p_4 *occurs* if $\beta_1 > 0$ and $-\Delta_1 - \beta_1 < \varepsilon_1 < -\Delta_1$

q_1 *occurs* if $\varepsilon_2 > -\Delta_2 - \min\{0,\beta_2\}$

q_2 *occurs* if $\varepsilon_2 < -\Delta_2 - \max\{0,\beta_2\}$

q_3 *occurs* if $\beta_2 < 0$ and $-\Delta_2 < \varepsilon_2 < -\Delta_2 - \beta_2$

q_4 *occurs* if $\beta_2 > 0$ and $-\Delta_2 - \beta_2 < \varepsilon_2 < -\Delta_2$

Proof. Recall, from proposition 1 that p_1 occurs if,

$$0 < \tilde{\pi}_1(1,0) - \tilde{\pi}_1(0,0) < \tilde{\pi}_1(1,1) - \tilde{\pi}_1(0,1) \textit{ or } 0 < \tilde{\pi}_1(1,1) - \tilde{\pi}_1(0,1) < \tilde{\pi}_1(1,0) - \tilde{\pi}_1(0,0).$$

Substituting for the expressions of $\tilde{\pi}_1(1,0)$, $\tilde{\pi}_1(0,0)$, $\tilde{\pi}_1(1,1)$, and $\tilde{\pi}_1(0,1)$ these two conditions become:

$$\varepsilon_1 > \Delta_1 \text{ and } \beta_1 > 0 \textit{ or } \varepsilon_1 > -\Delta_1 - \beta_1 \text{ and } \beta_1 < 0$$

Finally, regrouping the two we obtain $\varepsilon_1 > -\Delta_1 - \min\{0,\beta_1\}$.

p_2 occurs if,

$$\tilde{\pi}_1(1,1) - \tilde{\pi}_1(0,1) < \tilde{\pi}_1(1,0) - \tilde{\pi}_1(0,0) < 0 \textit{ or}$$

$$\tilde{\pi}_1(1,0) - \tilde{\pi}_1(0,0) < \tilde{\pi}_1(1,1) - \tilde{\pi}_1(0,1) < 0.$$

Substituting, we have $\varepsilon_1 < -\Delta_1$ and $\beta_1 < 0$ or $\varepsilon_1 < -\Delta_1 - \beta_1$ and $\beta_1 > 0$ which can be written compactly as, $\varepsilon_1 < -\Delta_1 - \max\{0,\beta_1\}$.

p_3 occurs if,

$$\tilde{\pi}_1(1,1) - \tilde{\pi}_1(0,1) < 0 < \tilde{\pi}_1(1,0) - \tilde{\pi}_1(0,0).$$

This translates as: $-\Delta_1 < \varepsilon_1 < -\Delta_1 - \beta_1$ and $\beta_1 < 0$.

p_4 occurs if,

$$\tilde{\pi}_1(1,0) - \tilde{\pi}_1(0,0) < 0 < \tilde{\pi}_1(1,1) - \tilde{\pi}_1(0,1).$$

Again, substituting for the payoffs we obtain: $-\Delta_1 - \beta_1 < \varepsilon_1 < -\Delta_1$ and $\beta_1 > 0$. The conditions on the random variable ε_2 which characterize player 2's reaction functions (correspondences) are derived using a similar argument.

Q.E.D.

We seek now to determine the probabilities $Pr(i,j)$, $i,j = 0,1$. This task is however rendered somewhat difficult by the existence of multiple equilibria when the pairs of reaction correspondences (p_3,q_3) and (p_4,q_4) occur (see proposition 2). Since we maintain the assumption of simultaneity of the moves in this game, all of the equilibria are a priori reasonable outcomes in the above situations. How can one go about solving this difficulty?

When one of the pairs, (p_3,q_3) or (p_4,q_4) occurs, the outcome is not well defined. In this case, we propose to attach a probability of occurrence to each of the four possible outcomes. Further, we shall show that these probabilities, or weights, can be either a priori or endogenously determined. By doing so, we attribute to our econometric model a full level of generality. For instance, the two following solutions can be applied. First, a simple solution would be to allocate an a priori positive constant probability to each of the Nash equilibria which could arise in the multiple equilibria cases. These probabilities may be totally ad-hoc and entirely dependent upon the judgment of the econometrician. Indeed, they may reflect the econometrician's a priori information about the possible outcome in these situations. Alternatively, a natural solution would be to allow these probabilities to be endogenous. We think that this solution is somewhat superior to the previous one, as the econometrician has only to decide on a "general" functional form for the weights. We shall illustrate these solutions below. We shall also argue that our approach is quite general in the sense that it can handle any outcome that may arise in situations of multiple Nash equilibria.

Some definitions and notations are in order. Let $\theta = (\beta_1, \gamma_1, \beta_2, \gamma_2, \rho)'$ be the parameter vector of the model and $\omega = (\beta_1, \gamma_1, \beta_2, \gamma_2, \rho, \varepsilon_1, \varepsilon_2)'$. For any $i, j = 0,1$, let $h_{ij}(\omega)$ and $k_{ij}(\omega)$, $i, j = 0,1$ be the probabilities assigned to the outcome (i, j) when the pairs (p_3, q_3) and (p_4, q_4) respectively occur. Further, let $F(x, y; \rho)$ be the c.d.f. evaluated at (x, y) of the bivariate normal distribution with zero means, unit variances and correlation ρ, and $J(a, b, c, d; \rho)$ be the integral over the range $c \le \varepsilon_1 \le a$, $d \le \varepsilon_2 \le b$ of the bivariate normal density $f(\varepsilon_1, \varepsilon_2; \rho)$. Define,

$$H_{00}^{++}(\theta) = \frac{1}{J_{++}} \iint_{D_{++}} [k_{10}(\omega) + k_{01}(\omega) + k_{11}(\omega)] \cdot f(\varepsilon_1, \varepsilon_2; \rho) d\varepsilon_1 d\varepsilon_2$$

$$H_{10}^{++}(\theta) = \frac{1}{J_{++}} \iint_{D_{++}} k_{10}(\omega) \cdot f(\varepsilon_1, \varepsilon_2; \rho) d\varepsilon_1 d\varepsilon_2$$

$$H_{01}^{++}(\theta) = \frac{1}{J_{++}} \iint_{D_{++}} k_{01}(\omega) \cdot f(\varepsilon_1, \varepsilon_2; \rho) d\varepsilon_1 d\varepsilon_2$$

$$H_{11}^{++}(\theta) = \frac{1}{J_{++}} \iint_{D_{++}} [k_{00}(\omega) + k_{10}(\omega) + k_{01}(\omega)] \cdot f(\varepsilon_1, \varepsilon_2; \rho) d\varepsilon_1 d\varepsilon_2$$

$$H_{00}^{+-}(\theta) = \frac{1}{J_{+-}} \iint_{D_{+-}} \frac{(\Delta_1 + \beta_1 + \varepsilon_1)(\Delta_2 + \beta_2 + \varepsilon_2)}{\beta_1 \beta_2} \cdot f(\varepsilon_1, \varepsilon_2; \rho) d\varepsilon_1 d\varepsilon_2$$

$$H_{10}^{+-}(\theta) = -\frac{1}{J_{+-}} \iint_{D_{+-}} \frac{(\Delta_1 + \beta_1 + \varepsilon_1)(\Delta_2 + \varepsilon_2)}{\beta_1 \beta_2} \cdot f(\varepsilon_1, \varepsilon_2; \rho) d\varepsilon_1 d\varepsilon_2$$

$$H_{01}^{+-}(\theta) = -\frac{1}{J_{+-}} \iint_{D_{+-}} \frac{(\Delta_1 + \varepsilon_1)(\Delta_2 + \beta_2 + \varepsilon_2)}{\beta_1 \beta_2} \cdot f(\varepsilon_1, \varepsilon_2; \rho) d\varepsilon_1 d\varepsilon_2$$

$$H_{11}^{+-}(\theta) = \frac{1}{J_{+-}} \iint_{D_{+-}} \frac{(\Delta_1 + \varepsilon_1)(\Delta_2 + \varepsilon_2)}{\beta_1 \beta_2} \cdot f(\varepsilon_1, \varepsilon_2; \rho) d\varepsilon_1 d\varepsilon_2$$

$$H_{00}^{-+}(\theta)=\frac{1}{J_{-+}}\iint_{D_{-+}}\frac{(\Delta_1+\beta_1+\varepsilon_1)(\Delta_2+\beta_2+\varepsilon_2)}{\beta_1\beta_2}\cdot f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$H_{10}^{-+}(\theta)=-\frac{1}{J_{-+}}\iint_{D_{-+}}\frac{(\Delta_1+\beta_1+\varepsilon_1)(\Delta_2+\beta_2)}{\beta_1\beta_2}\cdot f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$H_{01}^{-+}(\theta)=-\frac{1}{J_{-+}}\iint_{D_{-+}}\frac{(\Delta_1+\varepsilon_1)(\Delta_2+\beta_2+\varepsilon_2)}{\beta_1\beta_2}\cdot f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$H_{11}^{-+}(\theta)=\frac{1}{J_{-+}}\iint_{D_{-+}}\frac{(\Delta_1+\varepsilon_1)(\Delta_2+\varepsilon_2)}{\beta_1\beta_2}\cdot f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$H_{00}^{--}(\theta)=\frac{1}{J_{--}}\iint_{D_{--}}h_{00}(\omega)\cdot f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$H_{10}^{--}(\theta)=\frac{1}{J_{--}}\iint_{D_{--}}[h_{00}(\omega)+h_{01}(\omega)+h_{11}(\omega)]\cdot f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$H_{01}^{--}(\theta)=\frac{1}{J_{--}}\iint_{D_{--}}[h_{00}(\omega)+h_{10}(\omega)+h_{11}(\omega)]\cdot f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$H_{11}^{--}(\theta)=\frac{1}{J_{--}}\iint_{D_{--}}h_{11}(\omega)\cdot f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

where,

$$J_{++}=J(-\Delta_1,-\Delta_2,-\Delta_1-\beta_1,-\Delta_2-\beta_2;\rho)$$

$$J_{+-}=J(-\Delta_1,-\Delta_2-\beta_2,-\Delta_1-\beta_1,-\Delta_2;\rho)$$

$$J_{-+}=J(-\Delta_1-\beta_1,-\Delta_2,-\Delta_1,-\Delta_2-\beta_2;\rho)$$

$$J_{--}=J(-\Delta_1-\beta_1,-\Delta_2-\beta_2,-\Delta_1,-\Delta_2;\rho)$$

and

$$D_{++}=\{(\varepsilon_1,\varepsilon_2):\ -\Delta-\beta_1<\varepsilon_1<-\Delta_1,-\Delta_2-\beta_2<\varepsilon_2<-\Delta_2\}$$

$$D_{+-}=\{(\varepsilon_1,\varepsilon_2):\ -\Delta_1-\beta_1<\varepsilon_1<-\Delta_1,-\Delta_2<\varepsilon_2<-\Delta_2-\beta_2\}$$

$$D_{-+}=\{(\varepsilon_1,\varepsilon_2):\ -\Delta_1<\varepsilon_1<-\Delta_1-\beta_1,-\Delta_2<\varepsilon_2<-\Delta_2-\beta_2\}$$

$$D_{--}=\{(\varepsilon_1,\varepsilon_2):\ -\Delta_1<\varepsilon_1<-\Delta_1-\beta_1,-\Delta_2<\varepsilon_2<-\Delta_2-\beta_2\}.$$

The following proposition gives a general formula for the probabilities $Pr(i,j),\ i,j=0,1$.

Proposition 4: The probabilities are given by:

$$Pr[0,0]=\begin{cases} F(-\Delta_1,-\Delta_2;\rho)-H_{00}^{++}(\theta)\cdot J_{++} & \text{if } \beta_1>0 \text{ and } \beta_2>0 \\ F(-\Delta_1,-\Delta_2;\rho)+H_{00}^{+-}(\theta)\cdot J_{+-} & \text{if } \beta_1>0 \text{ and } \beta_2<0 \\ F(-\Delta_1,-\Delta_2;\rho)+H_{00}^{-+}(\theta)\cdot J_{-+} & \text{if } \beta_1<0 \text{ and } \beta_2>0 \\ F(-\Delta_1,-\Delta_2;\rho)+H_{00}^{--}(\theta)\cdot J_{--} & \text{if } \beta_1<0 \text{ and } \beta_2<0 \end{cases}$$

$$Pr[1,0]=\begin{cases} F(\Delta_1,-\Delta_2-\beta_2;-\rho)+H_{10}^{++}(\theta)\cdot J_{++} & \text{if } \beta_1>0 \text{ and } \beta_2>0 \\ F(\Delta_1,-\Delta_2-\beta_2;-\rho)+H_{10}^{+-}(\theta)\cdot J_{+-} & \text{if } \beta_1>0 \text{ and } \beta_2<0 \\ F(\Delta_1,-\Delta_2-\beta_2;-\rho)+H_{10}^{-+}(\theta)\cdot J_{-+} & \text{if } \beta_1<0 \text{ and } \beta_2>0 \\ F(\Delta_1,-\Delta_2-\beta_2;-\rho)+H_{10}^{--}(\theta)\cdot J_{--} & \text{if } \beta_1<0 \text{ and } \beta_2<0 \end{cases}$$

$$Pr[0,1]=\begin{cases} F(-\Delta_1-\beta_1,\Delta_2;-\rho)+H_{01}^{++}(\theta)\cdot J_{++} & \text{if } \beta_1>0 \text{ and } \beta_2>0 \\ F(-\Delta_1-\beta_1,\Delta_2;-\rho)+H_{01}^{+-}(\theta)\cdot J_{+-} & \text{if } \beta_1>0 \text{ and } \beta_2<0 \\ F(-\Delta_1-\beta_1,\Delta_2;-\rho)+H_{01}^{-+}(\theta)\cdot J_{-+} & \text{if } \beta_1<0 \text{ and } \beta_2>0 \\ F(-\Delta_1-\beta_1,\Delta_2;-\rho)+H_{01}^{--}(\theta)\cdot J_{--} & \text{if } \beta_1<0 \text{ and } \beta_2<0 \end{cases}$$

$$Pr[1,1]=\begin{cases} F(\Delta_1+\beta_1,\Delta_2+\beta_2;\rho)-H_{11}^{++}(\theta)\cdot J_{++} & \text{if } \beta_1>0 \text{ and } \beta_2>0 \\ F(\Delta_1+\beta_1,\Delta_2+\beta_2;\rho)+H_{11}^{+-}(\theta)\cdot J_{+-} & \text{if } \beta_1>0 \text{ and } \beta_2<0 \\ F(\Delta_1+\beta_1,\Delta_2+\beta_2;\rho)+H_{11}^{-+}(\theta)\cdot J_{-+} & \text{if } \beta_1<0 \text{ and } \beta_2>0 \\ F(\Delta_1+\beta_1,\Delta_2+\beta_2;\rho)+H_{11}^{--}(\theta)\cdot J_{--} & \text{if } \beta_1<0 \text{ and } \beta_2<0 \end{cases}$$

where the $H_{ij}(\theta)$'s are nonnegative functions of the parameter vector θ and satisfy:

$$H_{00}^{++}(\theta) + H_{11}^{++}(\theta) - H_{10}^{++}(\theta) - H_{01}^{++}(\theta) = 1$$

$$H_{00}^{+-}(\theta) + H_{10}^{+-}(\theta) + H_{01}^{+-}(\theta) + H_{11}^{+-}(\theta) = 1$$

$$H_{00}^{-+}(\theta) + H_{10}^{-+}(\theta) + H_{01}^{-+}(\theta) + H_{11}^{-+}(\theta) = 1$$

$$H_{10}^{--}(\theta) + H_{01}^{--}(\theta) - H_{00}^{--}(\theta) - H_{11}^{--}(\theta) = 1.$$

Proof. We shall write (p_i, q_j) to designate the simultaneous occurrence of the reaction functions (correspondences) p_i and p_j, $i, j = 1,2,3,4$. From proposition 2, we have:

$$Pr[0,0] = Pr[(p_2,q_2)] + Pr[(p_2,q_4)] + Pr[(p_3,q_3)] \cdot Pr[(0,0) \mid (p_3,q_3)]$$

$$+ \iint_{\{(\varepsilon_1,\varepsilon_2):(p_3,q_4)\ occurs\}} (1-r_p)(1-r_q) f(\varepsilon_1,\varepsilon_2;\rho)\, d\varepsilon_1 d\varepsilon_2$$

$$+ Pr[(p_4,q_2)] + \iint_{\{(\varepsilon_1,\varepsilon_2):(p_4,q_3)\ occurs\}} (1-r_p)(1-r_q) f(\varepsilon_1,\varepsilon_2;\rho)\, d\varepsilon_1 d\varepsilon_2$$

$$+ Pr[(p_4,q_4)] \cdot Pr[(0,0) \mid (p_4,q_4)]$$

$$Pr[(0,0) \mid (p_3,q_3)] = \frac{Pr[(0,0)\ \&\ (p_3,q_3)]}{Pr[(p_3,q_3)]}$$

$$= \frac{1}{Pr[(p_3,q_3)]} \iint_{\{(\varepsilon_1,\varepsilon_2):(p_3,q_3)\ occurs\}} h_{00}(\omega) f(\varepsilon_1,\varepsilon_2;\rho)\, d\varepsilon_1 d\varepsilon_2.$$

Similarly,

$$Pr[(0,0) \mid (p_4,q_4)] = \frac{1}{Pr[(p_4,q_4)]} \iint_{\{(\varepsilon_1,\varepsilon_2):(p_4,q_4)\ occurs\}} k_{00}(\omega) f(\varepsilon_1,\varepsilon_2;\rho)\, d\varepsilon_1 d\varepsilon_2.$$ *Thus*,

$$Pr[0,0] = Pr[(p_2,q_2)] + Pr[(p_2,q_4)]$$

$$+\iint_{\{(\varepsilon_1,\varepsilon_2):(p_3,q_3)\ occurs\}} h_{00}(\omega) f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$+\iint_{\{(\varepsilon_1,\varepsilon_2):(p_3,q_4)\ occurs\}} (1-r_p)(1-r_q) f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2 + Pr[(p_4,q_2)]$$

$$+\iint_{\{(\varepsilon_1,\varepsilon_2):((p_4,q_3)\ occurs\}} (1-r_p)(1-r_q) f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$+\iint_{\{(\varepsilon_1,\varepsilon_2):(p_4 q_4)\ occurs\}} k_{00}(\omega) f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$Pr[1,0] = Pr[(p_1,q_2)] + Pr[(p_1,q_3)] + Pr[(p_3,q_2)]$$

$$+\iint_{\{(\varepsilon_1,\varepsilon_2):(p_3,q_3)\ occurs\}} h_{10}(\omega) f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$+\iint_{\{(\varepsilon_1,\varepsilon_2):(p_3,q_4)\ occurs\}} r_p(1-r_q) f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$+\iint_{\{(\varepsilon_1,\varepsilon_2):(p_4,q_3)\ occurs\}} r_p(1-r_q) f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$+\iint_{\{(\varepsilon_1,\varepsilon_2):(p_4,q_4)\ occurs\}} k_{10}(\omega) f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$Pr[0,1] = Pr[(p_2,q_1)] + Pr[(p_2,q_3)] + Pr[(p_3,q_1)]$$

$$+\iint_{\{(\varepsilon_1,\varepsilon_2):(p_3,q_3)\ occurs\}} h_{01}(\omega) f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$+\iint_{\{(\varepsilon_1,\varepsilon_2):(p_3,q_4)\ occurs\}} (1-r_p) r_q f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$+\iint_{\{(\varepsilon_1,\varepsilon_2):(p_4,q_3)\ occurs\}} (1-r_p) r_q f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$+\iint_{\{(\varepsilon_1,\varepsilon_2):(p_4,q_4)\ occurs\}} k_{01}(\omega) f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$Pr[1,1] = Pr[(p_1,q_1)] + Pr[(p_1,q_4)] + \iint_{\{(\varepsilon_1,\varepsilon_2):(p_3,q_3)\ occurs\}} h_{11}(\omega) f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$+\iint_{\{(\varepsilon_1,\varepsilon_2)(p_3,q_4)\ occurs\}} r_p r_q f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2 + Pr[(p_4,q_1)]$$

$$+\iint_{\{(\varepsilon_1,\varepsilon_2):(p_4,q_3)\ occurs\}} r_p r_q f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$+\iint_{\{(\varepsilon_1,\varepsilon_2):(p_4,q_4)\ occurs\}} k_{11}(\omega) f(\varepsilon_1\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

where for all ω,

$$h_{00}(\omega)+h_{10}(\omega)+h_{01}(\omega)+h_{11}(\omega)=1$$

$$k_{00}(\omega)+k_{10}(\omega)+k_{01}(\omega)+k_{11}(\omega)=1$$

We shall consider four cases according to the signs of β_1 and β_2.

Case 1: $\beta_1 > 0$ and $\beta_2 > 0$.

In this case, we see from proposition 3 that the reaction correspondences p_3 and q_3 cannot occur. Consequently, all the pairs involving these two correspondences occur with null probability. Using proposition 3, we obtain:

$$Pr[0,0] = Pr[\varepsilon_1 < -\Delta_1 - \beta_1, \varepsilon_2 < -\Delta_2 - \beta_2] + Pr[\varepsilon_1 < -\Delta_1 - \beta_1, -\Delta_2 - \beta_2 < \varepsilon_2 < -\Delta_2]$$

$$+ Pr[-\Delta_1 - \beta_1 < \varepsilon_1 < -\Delta_1, \varepsilon_2 < -\Delta_2 - \beta_2] + \iint_{D_{++}} k_{00}(\omega) f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$= F(-\Delta_1,-\Delta_2;\rho) - \iint_{D_{++}} [k_{10}(\omega)+k_{01}(\omega)+k_{11}(\omega)] f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$Pr[1,0] = Pr[\varepsilon_1 > -\Delta_1, \varepsilon_2 < -\Delta_2 - \beta_2] + \iint_{D_{++}} k_{10}(\omega) f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$= F(\Delta_1, -\Delta_2 - \beta_2; -\rho) + \iint_{D_{++}} k_{10}(\omega) f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$Pr[0,1] = Pr[\varepsilon_1 < -\Delta_1 - \beta_1, \varepsilon_2 > -\Delta_2] + \iint_{D_{++}} k_{01}(\omega) f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$= F(-\Delta_1 - \beta_1, \Delta_2; -\rho) + \iint_{D_{++}} k_{01}(\omega) f(\varepsilon_1, \varepsilon_2; \rho) d\varepsilon_1 d\varepsilon_2$$

$$Pr[1,1] = Pr[\varepsilon_1 > -\Delta_1, \varepsilon_2 > -\Delta_2] + Pr[\varepsilon_1 > -\Delta_1, -\Delta_2 - \beta_2 < \varepsilon_2 < -\Delta_2]$$

$$+ Pr[-\Delta_1 - \beta_1 < \varepsilon_1 < -\Delta_1, \varepsilon_2 > -\Delta_2] + \iint_{D_{++}} k_{11}(\omega) f(\varepsilon_1, \varepsilon_2; \rho) d\varepsilon_1 d\varepsilon_2$$

$$= F(\Delta_1 + \beta_1, \Delta_2 + \beta_2; \rho) - \iint_{D_{++}} [k_{00}(\omega) + k_{10}(\omega) + k_{01}(\omega)] f(\varepsilon_1, \varepsilon_2; \rho) d\varepsilon_1 d\varepsilon_2$$

Case 2: $\beta_1 > 0$ and $\beta_2 < 0$.

In this case we see by proposition 3 that, p_3 and q_4 cannot occur. We have then,

$$Pr[0,0] = Pr[\varepsilon_1 < -\Delta_1 - \beta_1, \varepsilon_2 < -\Delta_2] + Pr[-\Delta_1 - \beta_1 < \varepsilon_1 < -\Delta_1, \varepsilon_2 < -\Delta_2]$$

$$+ \iint_{D_{+-}} (1 - r_p)(1 - r_q) f(\varepsilon_1, \varepsilon_2; \rho) d\varepsilon_1 d\varepsilon_2$$

$$= F(-\Delta_1, -\Delta_2; \rho) + \iint_{D_{+-}} (1 - r_p)(1 - r_q) f(\varepsilon_1, \varepsilon_2; \rho) d\varepsilon_1 d\varepsilon_2$$

$$Pr[1,0] = Pr[\varepsilon_1 > -\Delta_1, \varepsilon_2 < -\Delta_2] + Pr[\varepsilon_1 > -\Delta_1, -\Delta_2 < \varepsilon_2 < -\Delta_2 - \beta_2]$$

$$+ \iint_{D_{+-}} r_p (1 - r_q) f(\varepsilon_1, \varepsilon_2; \rho) d\varepsilon_1 d\varepsilon_2$$

$$= F(\Delta_1, -\Delta_2 - \beta_2; -\rho) + \iint_{D_{+-}} r_p (1 - r_q) f(\varepsilon_1, \varepsilon_2;) d\varepsilon_1 d\varepsilon_2$$

$$Pr[0,1] = Pr[\varepsilon_1 < -\Delta_1 - \beta_1, \varepsilon_2 > -\Delta_2 - \beta_2] + Pr[\varepsilon_1 < -\Delta_1 - \beta_1, -\Delta_2 < \varepsilon_2 < -\Delta_2 - \beta_2]$$

$$+ \iint_{D_{+-}} (1 - r_p) r_q f(\varepsilon_1, \varepsilon_2; \rho) d\varepsilon_1 d\varepsilon_2$$

$$= F(-\Delta_1 - \beta_1, \Delta_2; -\rho) + \iint_{D_{+-}} (1 - r_p) r_q f(\varepsilon_1, \varepsilon_2; \rho) d\varepsilon_1 d\varepsilon_2$$

$$Pr[1,1] = Pr[\varepsilon_1 > -\Delta_1, \varepsilon_2 > -\Delta_2 - \beta_2] + Pr[-\Delta_1 - \beta_1 < \varepsilon_1 < -\Delta_1, \varepsilon_2 > -\Delta_2 - \beta_2]$$

$$+\iint_{D_{+-}} r_p r_q f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$=F(\Delta_1+\beta_1,\Delta_2+\beta_2;\rho)+\iint_{D_{+-}} r_p r_q f(\varepsilon_1,\varepsilon_2)d\varepsilon_1 d\varepsilon_2$$

Case 3: $\beta_1 < 0$ and $\beta_2 > 0$.

The reaction correspondences p_4 and q_3 cannot occur in this case. Thus, the probabilities become:

$$Pr[0,0]=Pr[\varepsilon_1<-\Delta_1,\varepsilon_2<-\Delta_2-\beta_2]+Pr[\varepsilon_1<-\Delta_1,-\Delta_2-\beta_2<\varepsilon_2<-\Delta_2]$$

$$+\iint_{D_{-+}}(1-r_p)(1-r_q)f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$=F(-\Delta_1,-\Delta_2;\rho)+\iint_{D_{-+}}(1-r_p)(1-r_q)f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$Pr[1,0]=Pr[\varepsilon_1>-\Delta_1-\beta_1,\varepsilon_2<-\Delta_2-\beta_2]+Pr[-\Delta_1<\varepsilon_1<-\Delta_1-\beta_1,\varepsilon_2<-\Delta_2-\beta_2]$$

$$+\iint_{D_{-+}} r_p(1-r_q)f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$=F(\Delta_1,-\Delta_2-\beta_2;-\rho)+\iint_{D_{-+}} r_p(1-r_q)f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$Pr[0,1]=Pr[\varepsilon_1<-\Delta_1,\varepsilon_2>-\Delta_2]+Pr[-\Delta_1<\varepsilon_1<-\Delta_1-\beta_1,\varepsilon_2>-\Delta_2]$$

$$+\iint_{D_{-+}}(1-r_p)r_q f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$=F(-\Delta_1-\beta_1,\Delta_2;-\rho)+\iint_{D_{-+}}(1-r_p)r_q f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$Pr[1,1]=Pr[\varepsilon_1>-\Delta_1-\beta_1,\varepsilon_2>-\Delta_2]+Pr[\varepsilon_1>-\Delta_1-\beta_1,-\Delta_2-\beta_2<\varepsilon_2<-\Delta_2]$$

$$+\iint_{D_{-+}} r_p r_q f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$= F(\Delta_1 + \beta_1, \Delta_2 + \beta_2; \rho) + \iint_{D_{-+}} r_p r_q f(\varepsilon_1, \varepsilon_2; \rho) d\varepsilon_1 d\varepsilon_2$$

Case 4: $\beta_1 < 0$ and $\beta_2 < 0$.

Again, examining proposition 3 we note that the reaction correspondences p_4 and q_4 cannot occur in this case. It follows that,

$$Pr[0,0] = Pr[\varepsilon_1 < -\Delta_1, \varepsilon_2 < -\Delta_2] + \iint_{D_{--}} h_{00}(\omega) f(\varepsilon_1, \varepsilon_2; \rho) d\varepsilon_1 d\varepsilon_2$$

$$= F(-\Delta_1, -\Delta_2; \rho) + \iint_{D_{--}} h_{00}(\omega) f(\varepsilon_1, \varepsilon_2; \rho) d\varepsilon_1 d\varepsilon_2$$

$$Pr[1,0] = Pr[\varepsilon_1 > -\Delta_1 - \beta_1, \varepsilon_2 < -\Delta_2] + Pr[\varepsilon_1 > -\Delta_1 - \beta_1, -\Delta_2 < \varepsilon_2 < -\Delta_2 - \beta_2]$$

$$+ Pr[-\Delta_1 < \varepsilon_1 < -\Delta_1 - \beta_1, \varepsilon_2 < -\Delta_2] + \iint_{D_{--}} h_{10}(\omega) f(\varepsilon_1, \varepsilon_2; \rho) d\varepsilon_1 d\varepsilon_2$$

$$= F(\Delta_1, -\Delta_2 - \beta_2; -\rho) - \iint_{D_{--}} [h_{00}(\omega) + h_{01}(\omega) + h_{11}(\omega)] f(\varepsilon_1, \varepsilon_2; \rho) d\varepsilon_1 d\varepsilon_2$$

$$Pr[0,1] = Pr[\varepsilon_1 < -\Delta_1, \varepsilon_2 > -\Delta_2 - \beta_2] + Pr[\varepsilon_1 < -\Delta_1, -\Delta_2 < \varepsilon_2 < -\Delta_2 - \beta_2]$$

$$+ Pr[-\Delta_1 < \varepsilon_1 < -\Delta_1 - \beta_1, \varepsilon_2 > -\Delta_2 - \beta_2] + \iint_{D_{--}} h_{01}(\omega) f(\varepsilon_1, \varepsilon_2; \rho) d\varepsilon_1 d\varepsilon_2$$

$$= F(-\Delta_1 - \beta_1, \Delta_2; -\rho) - \iint_{D_{--}} [h_{00}(\omega) + h_{10}(\omega) + h_{11}(\omega)] f(\varepsilon_1, \varepsilon_2; \rho) d\varepsilon_1 d\varepsilon_2$$

$$Pr[1,1] = Pr[\varepsilon_1 > -\Delta_1 - \beta_1, \varepsilon_2 > -\Delta_2 - \beta_2] + \iint_{D_{--}} h_{11}(\omega) f(\varepsilon_1, \varepsilon_2; \rho) d\varepsilon_1 d\varepsilon_2$$

$$= F(\Delta_1 + \beta_1, \Delta_2 + \beta_2; \rho) + \iint_{D_{--}} h_{11}(\omega) f(\varepsilon_1, \varepsilon_2; \rho) d\varepsilon_1 d\varepsilon_2$$

Recall that,

$$r_p = -\frac{\tilde{\pi}_2(0,1) - \tilde{\pi}_2(0,0)}{\tilde{\pi}_2(1,1) - \tilde{\pi}_2(1,0) + \tilde{\pi}_2(0,0) - \tilde{\pi}_2(0,1)} \text{ and}$$

$$r_q = -\frac{\tilde{\pi}_1(1,0) - \tilde{\pi}_1(0,0)}{\tilde{\pi}_1(1,1) - \tilde{\pi}_1(1,0) + \tilde{\pi}_1(0,0) - \tilde{\pi}_1(0,1)}$$

Using assumptions 2 and 3 we see that:

$$r_p = -\frac{\Delta_2 + \varepsilon_2}{\beta_2} \quad \text{and } r_q = -\frac{\Delta_1 + \varepsilon_1}{\beta_1}.$$

Thus,

$$(1 - r_p)(1 - r_q) = \frac{(\Delta_1 + \beta_1 + \varepsilon_1)(\Delta_2 + \beta_2 + \varepsilon_2)}{\beta_1\beta_2}$$

$$r_p(1 - r_q) = -\frac{(\Delta_1 + \beta_1 + \varepsilon_1)(\Delta_2 + \varepsilon_2)}{\beta_1\beta_2}$$

$$(1 - r_p)r_q = -\frac{(\Delta_1 + \varepsilon_1)(\Delta_2 + \beta_2 + \varepsilon_2)}{\beta_1\beta_2}$$

$$r_p r_q = \frac{(\Delta_1 + \varepsilon_1)(\Delta_2 + \varepsilon_2)}{\beta_1\beta_2}$$

Noting for example that,

$$\iint_{D_{++}} [k_{10}(\omega) + k_{01}(\omega) + k_{11}(\omega)] \cdot f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$= \left\{ \frac{1}{J_{++}} \cdot \iint_{D_{++}} [k_{10}(\omega) + k_{01}(\omega) + k_{11}(\omega)] \cdot f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2 \right\} \cdot J_{++}$$

where, $J_{++} = \iint_{D_{++}} f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$, we see that,

$$\iint_{D_{++}} [k_{10}(\omega) + k_{01}(\omega) + k_{11}(\omega)] \cdot f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

is the portion of J_{++} that must be substracted from $F(-\Delta_1,-\Delta_2,;\rho)$ to recover $Pr[0,0]$ in the case where $\beta_1 > 0$ and $\beta_2 > 0$. A similar argument holds for the other probabilities and cases.

Therefore, one can write each of the four probabilities in an attractive form consisting of a sum of two parts: a common part, irrespective of the signs of β_1 and β_2, and a specific or individual part. More precisely, the common part consists of the c.d.f. of the bivariate normal evaluated at an appropriate argument vector, and the specific part of a portion of

$$Pr[\boldsymbol{K}] = \iint_{\boldsymbol{K}} f(\varepsilon_1, \varepsilon_2; \rho) d\varepsilon_1 d\varepsilon_2 \text{ (where } \boldsymbol{K} \text{ is as defined below)}$$

assigned to each of the four probabilities, in each case.

Clearly, the weights involved in this allocation of the above area are functions of θ, since they depend upon the functions $h_{ij}(\omega)$ and $k_{ij}(\omega)$, $i,j, = 0,1$, and ε_1 and ε_2 are integrated out. Furthermore, one can readily check that these weights (the H_{ij}'s) satisfy the conditions formulated in the proposition.

Q.E.D.

One can easily verify that the different probabilities $Pr(i,j)$ sum up to one, irrespective of the sign of β_1 and β_2.[6] It is worthwhile to make two remarks. First, as pointed out earlier, though the econometric model described in proposition 4 assumes mainly that the outcomes emerge as Nash equilibria, it allows for any other type of solution concept when more than one equilibrium exists. In a subsequent section we will elaborate more on this issue. In particular, we will argue that since the concept of Nash equilibrium is used in all of the other situations, models derived from proposition 4 that use the Nash equilibrium concept in the multiple Nash equilibria cases, are relevant models to consider. We shall retain nine such models. Second, each of the different models which can be generated from the generic model displayed in proposition 4 corresponds to a given specification of the functional form of the

h_{ij}'s and k_{ij}'s, and therefore of the H_{ij}'s. Furthermore, the latter constitutes a distribution of the area $\boldsymbol{K}$ defined by:

$$\{(\varepsilon_1,\varepsilon_2):\ -\Delta_1 - \max\{0,\beta_1\} < \varepsilon_1 < -\Delta_1 - \min\{0,\beta_1\}, -\Delta_2 - \max\{0,\beta_2\} < \varepsilon_1 < -\Delta_2 - \min\{0,\beta_2\}\}$$

among the four outcomes. Hence, there is a one-to-one correspondence between the set of models described by proposition 4, and the set of all possible divisions of the above region.

To illustrate proposition 4, suppose that exogenous weights are assigned to each of the Nash equilibria which could arise when the pairs of reaction correspondences (p_3,q_3) and (p_4,q_4) occur. Further, suppose that equal weights are allocated to the pure strategy Nash equilibria whereas no weight is attributed to the mixed strategy equilibrium. It is easy to see that this is equivalent to setting:

$$h_{00}(\omega) = h_{11}(\omega) = k_{01}(\omega) = k_{10}(\omega) = 0$$

and,

$$h_{01}(\omega) = h_{10}(\omega) = k_{00}(\omega) = k_{11}(\omega) = 1/2$$

One can then easily derive the model generated by these assumptions from proposition 4. A particular case of this model has been considered by Bjorn and Vuong (1984). These authors assign, however, exogenous weights to each possible outcome in the case where the pairs of reaction functions (p_3,q_4) and (p_4,q_3) occur, i.e., in the case where there is no Nash equilibrium in pure stategies. This difficulty does not arise in our present framework since this case has a unique Nash equilibrium in mixed strategies (see proposition 2).

Another interesting illustration of proposition 4 is derived by assuming that in the multiple equilibria cases, the players choose their randomized strategies.[7] The weights become endogenous, and it is easy to see that we have:

$$h_{00}(\omega) = k_{00}(\omega) = (1 - r_p)(1 - r_q) = \frac{(\Delta_1 + \beta_1 + \varepsilon_1)(\Delta_2 + \beta_2 + \varepsilon_2)}{\beta_1\beta_2}$$

$$h_{10}(\omega) = k_{10}(\omega) = r_p(1 - r_q) = -\frac{(\Delta_1 + \beta_1 + \varepsilon_1)(\Delta_2 + \varepsilon_2)}{\beta_1\beta_2}$$

$$h_{01}(\omega) = k_{01}(\omega) = r_p(1 - r_q) = -\frac{(\Delta_1 + \varepsilon_1)(\Delta_2 + \beta_2 + \varepsilon_2)}{\beta_1\beta_2}$$

$$h_{11}(\omega) = k_{11}(\omega) = r_p r_q = \frac{(\Delta_1 + \varepsilon_1)(\Delta_2 + \varepsilon_2)}{\beta_1\beta_2}$$

It follows that, for example:

$$H_{00}^{++}(\theta) = \left[\frac{1}{J_{++}}\iint_{D_{++}}[1 - k_{00}(\omega)]\right] \cdot f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$= 1 - \frac{1}{J_{++}}\iint_{D_{++}} \frac{(\Delta_1 + \beta_1 + \varepsilon_1)(\Delta_2 + \beta_2 + \varepsilon_2)}{\beta_1\beta_2} \cdot f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$= 1 - \frac{(\Delta_1 + \beta_1)(\Delta_2 + \beta_2)}{\beta_1\beta_2} - \frac{1}{J_{++}}\frac{(\Delta_2 + \beta_2)}{\beta_1\beta_2}\iint_{D_{++}} \varepsilon_1 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$- \frac{1}{J_{++}}\frac{(\Delta_1 + \beta_1)}{\beta_1\beta_2}\iint_{D_{++}} \varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$- \frac{1}{J_{++}}\frac{1}{\beta_1\beta_2}\iint_{D_{++}} \varepsilon_1\varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

implying,

$$Pr[0,0] = F(-\Delta_1,-\Delta_2;\rho) + \left[\frac{(\Delta_1 + \beta_1)(\Delta_2 + \beta_2)}{\beta_1\beta_2} - 1\right] J_{++}$$

$$+ \frac{1}{\beta_1\beta_2}\Big[(\Delta_2 + \beta_2)\iint_{D_{++}} \varepsilon_1 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2 + (\Delta_1 + \beta_1)\iint_{D_{++}} \varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$+\iint_{D_{++}} \varepsilon_1\varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

in the case where, $\beta_1 > 0$ and $\beta_2 > 0$.

Finally, proposition 4 can be used to generate a model where, in the multiple equilibria cases, communication between the two players is permitted, thereby leading for instance to an agreement on a collusive solution, or simply to a solution resulting from bargaining between the two players. An important step is the derivation of the weights h_{ij} and k_{ij}'s, corresponding to the game theoretic solution concept in question.

4. EMPIRICAL MODELS OF A DUOPOLISTIC DISCRETE GAME

In the previous section, we have described a method of generating a general econometric model for two dichotomous variables, under the hypothesis that these variables represent actions of two players involved in a noncooperative game. While analyzing this discrete game by means of the Nash equilibrium solution concept, we have encountered the possibility of a configuration of the players' payoffs which yields multiple equilibria. In order to confer a full level of generality to our econometric model, we have attached, in those multiple equilibria cases, a probability of occurrence to each of the four possible outcomes. In the present section, we aim to derive various estimable models which can be used to analyze empirical data available in a discrete form.

From propositon 2, we see that when the payoffs are such that the reaction correspondence pairs (p_3,q_3) or (p_4,q_4) occur, noncooperative behavior leads to multiple equilibria. The outcomes (0,1), (r_p,r_q) and (1,0) can all emerge as a consequence of the pair (p_3,q_3) and the outcomes (0,0), (r_p,r_q) and (1,1) as a consequence of the pair (p_4,q_4). In what follows, we derive nine particular econometric probability models. These models are associated with the nine possible pairs of outcomes which can be formed from the above two sets of Nash equilibria. They are designated by M_1–M_9 and characterized by the following assumptions:

Assumption $\mathbf{M_1}$:

When the reaction correspondence pair (p_3,q_3) occurs, the equilibrium outcome is (0,1) and when the reaction correspondence (p_4,q_4) occurs, the equilibrium outcome is (0,0).

Assumption $\mathbf{M_2}$:

When the reaction correspondence pair (p_3,q_3) occurs, the equilibrium outcome is (0,1) and when the reaction correspondence (p_4,q_4) occurs, the equilibrium outcome is (r_p,r_q).

Assumption $\mathbf{M_3}$:

When the reaction correspondence pair (p_3,q_3) occurs, the equilibrium outcome is (0,1) and when the reaction correspondence (p_4,q_4) occurs, the equilibrium outcome is (1,1).

Assumption $\mathbf{M_4}$:

When the reaction correspondence pair (p_3,q_3) occurs, the equilibrium outcome is (r_p,r_q) and when the reaction correspondence (p_4,q_4) occurs, the equilibrium outcome is (0,0).

Assumption $\mathbf{M_5}$:

When the reaction correspondence pair (p_3,q_3) occurs, the equilibrium outcome is (r_p,r_q) and when the reaction correspondence (p_4,q_4) occurs, the equilibrium outcome is (r_p,r_q).

Assumption $\mathbf{M_6}$:

When the reaction correspondence pair (p_3,q_3) occurs, the equilibrium outcome is (r_p,r_q) and when the reaction correspondence (p_4,q_4) occurs, the equilibrium outcome is (1,1).

Assumption M_7:

When the reaction correspondence pair (p_3,q_3) occurs, the equilibrium outcome is (1,0) and when the reaction correspondence (p_4,q_4) occurs, the equilibrium outcome is (0,0).

Assumption M_8:

When the reaction correspondence pair (p_3,q_3) occurs, the equilibrium outcome is (1,0) and when the reaction correspondence (p_4,q_4) occurs, the equilibrium outcome is (r_p,r_q).

Assumption M_9:

When the reaction correspondence pair (p_3,q_3) occurs, the equilibrium outcome is (1,0) and when the reaction correspondence (p_4,q_4) occurs, the equilibrium outcome is (1,1).

The following notations and definitions are used in the derivation of the probability models M_1–M_9 associated with, respectively, asumptions M_1–M_9 above:

$\Phi(x)-$ *c.d.f. evaluated at x of the standard univariate normal distribution*

$\phi(x)-$ *p.d.f. evaluated at x of the standard univariate normal distribution*

$A(.,.)-$ *real valued function defined by*: $A(x,y)=\dfrac{x-\rho y^2}{(1-\rho^2)^{1/2}}$

$a_i \equiv \Delta_i + \beta_i,\ i=1,2$ *and*, $\bar{x} \equiv \dfrac{x}{\beta_1\beta_2}$ for *any real x*

$A_1 \equiv A(-\Delta_1,-\Delta_2)$; $A_5 \equiv A(-\Delta_2,-\Delta_1)$

$A_2 \equiv A(-\Delta_1,-a_2)$; $A_6 \equiv A(-\Delta_2,-a_1)$

$A_3 \equiv A(-a_1, -\Delta_2)$; $A_7 \equiv A(-a_2, -\Delta_1)$

$A_4 \equiv A(-a_1, -a_2)$; $A_8 \equiv A(-a_2, -a_1)$

$\mathbf{1}_R$ – *real valued function defined by*: $\mathbf{1}_R(x) = 1$ if $x > 0$ *and*, $\mathbf{1}_R(x) = 0$ if $x \le 0$

$$A(\theta) \equiv \frac{1}{F(a_1, a_2; \rho)}\left[\bar{a}_2\phi(a_1)\{1-\Phi(A_8)\} + \bar{a}_1\phi(a_2)\{1-\Phi(A_4)\} + (\overline{1-\rho^2})f(a_1, a_2; \rho)\right]$$

$$B(\theta) \equiv \frac{1}{F(\Delta_1, \Delta_2; \rho)}\Big[(\bar{a}_2 + \rho\bar{\beta}_1)\phi(\Delta_1)\{1-\Phi(A_5)\} + (\bar{a}_1 + \rho\bar{\beta}_2)\phi(\Delta_2)\{1-\Phi(A_1)\}$$

$$+ (\overline{1-\rho^2})f(\Delta_1, \Delta_2; \rho)$$

$$C(\theta) \equiv \frac{1}{F(\Delta_1, a_2; \rho)}\Big[(\bar{a}_2 + \rho\bar{\beta}_1)\phi(\Delta_1)\{1-\Phi(A_7)\} + \bar{a}_1\phi(a_2)\{1-\Phi(A_2)\}$$

$$+ (\overline{1-\rho^2})f(\Delta_1, a_2; \rho)$$

$$D(\theta) \equiv \frac{1}{F(a_1, \Delta_2; \rho)}\Big[\bar{a}_2\phi(a_1)\{1-\Phi(A_6)\} + (\bar{a}_1 + \rho\bar{\beta}_2)\phi(\Delta_2)\{1-\Phi(A_3)\}$$

$$+ (\overline{1-\rho^2})f(a_1, \Delta_2; \rho)$$

$$\Gamma(\theta) \equiv \frac{1}{F(\Delta_1, a_2; \rho)}\left[\phi(\Delta_1)\{1-\Phi(A_7)\} + \rho\phi(a_2)\{1-\Phi(A_2)\}\right]$$

$$+ \frac{1}{F(a_1, \Delta_2; \rho)}\left[\phi(a_1)\{1-\Phi(A_6)\} + \rho\phi(\Delta_2)\{1-\Phi(A_3)\}\right]$$

$$- \frac{1}{F(a_1, a_2; \rho)}\left[\phi(a_1)\{1-\Phi(A_8)\} + \rho\phi(a_2)\{1-\Phi(A_4)\}\right]$$

$$- \frac{1}{F(\Delta_1, \Delta_2; \rho)}\left[\phi(\Delta_1)\{1-\Phi(A_5)\} + \rho\phi(\Delta_2)\{1-\Phi(A_1)\}\right]$$

$$\Psi(\theta) \equiv \frac{1}{F(a_1,a_2;\rho)}\left[\rho\phi(a_1)\{1-\Phi(A_8)\} + \phi(a_2)\{1-\Phi(A_4)\}\right]$$

$$+ \frac{1}{F(\Delta_1,\Delta_2;\rho)}\left[\rho\phi(\Delta_1)\{1-\Phi(A_5)\} + \phi(\Delta_2)\{1-\Phi(A_1)\}\right]$$

$$- \frac{1}{F(\Delta_1,a_2;\rho)}\left[\rho\phi(\Delta_1)\{1-\Phi(A_7)\} + \phi(a_2)\{1-\Phi(A_2)\}\right]$$

$$- \frac{1}{F(a_1,\Delta_2;\rho)}\left[\rho\phi(a_1)\{1-\Phi(A_6)\} + \phi(\Delta_2)\{1-\Phi(A_3)\}\right]$$

$$\mu(\theta) \equiv F(a_1,a_2;\rho) + F(\Delta_1,\Delta_2;\rho) - F(\Delta_1,a_2;\rho) - F(a_1,\Delta_2;\rho)$$

$$\Sigma(\theta) \equiv A(\theta) + B(\theta) - C(\theta) - D(\theta).$$

We now present in a set of propositions (5-13) the probability models M_1–M_9.

Proposition 5: The probabilities of the model M_1 are given by:

$$Pr[0,0] = F(-\Delta_1,-\Delta_2;\rho) - \left[\bar{a}_1 a_2 \mu(\theta) + \Sigma(\theta)\right]\{1 - \mathbf{1}_{\mathbf{R}}(\beta_1\beta_2)\}$$

$$Pr[1,0] = F(\Delta_1,-a_2;-\rho) + \left[(\bar{a}_1\Delta_2 - 1)\mu(\theta) + \Sigma(\theta) + \bar{\beta}_2\Gamma(\theta)\right]\{1 - \mathbf{1}_{\mathbf{R}}(\beta_1\beta_2)\}$$
$$+ \mu(\theta)\{1 - \mathbf{1}_{\mathbf{R}}(\beta_1)\mathbf{1}_{\mathbf{R}}(\beta_2)\}$$

$$Pr[0,1] = F(-a_1,\Delta_2;-\rho) + \left[\bar{a}_2\Delta_1\mu(\theta) + \Sigma(\theta) - \bar{\beta}_1\Psi(\theta)\right]\{1 - \mathbf{1}_{\mathbf{R}}(\beta_1\beta_2)\}$$

$$Pr[1,1] = F(a_1,a_2;\rho) - \left[\bar{\Delta}_1\Delta_2\mu(\theta) + \Sigma(\theta) + \bar{\beta}_2\Gamma(\theta) - \bar{\beta}_1\Psi(\theta)\right]\{1 - \mathbf{1}_{\mathbf{R}}(\beta_1\beta_2)\}$$
$$- \mu(\theta)\mathbf{1}_{\mathbf{R}}(\beta_1)\mathbf{1}_{\mathbf{R}}(\beta_2)$$

Proposition 6: The probabilities of the model M_2 are given by:

$$Pr[0,0] = \begin{cases} F(-\Delta_1,-\Delta_2;\rho) + \left[(\bar{a}_1 a_2 - 1)\mu(\theta) + \Sigma(\theta) \right] \mathbf{1}_{\boldsymbol{R}}(\beta_1)\mathbf{1}_{\boldsymbol{R}}(\beta_2) & \text{if } \beta_1\beta_2 > 0 \\ F(-\Delta_1,-\Delta_2;\rho) - \left[\bar{a}_1 a_2 \mu(\theta) + \Sigma(\theta) \right] & \text{if } \beta_1\beta_2 < 0 \end{cases}$$

$$Pr[1,0] = \begin{cases} F(\Delta_1,-a_2;-\rho) - \left[(\bar{a}_1 \Delta_2 + 1)\mu(\theta) + \Sigma(\theta) + \bar{\beta}_2\Gamma(\theta) \right] \mathbf{1}_{\boldsymbol{R}}(\beta_1)\mathbf{1}_{\boldsymbol{R}}(\beta_2) & \\ +\mu(\theta) & \text{if } \beta_1\beta_2 > 0 \\ F(\Delta_1,-a_2;-\rho) + \left[\bar{a}_1 \Delta_2 \mu(\theta) + \Sigma(\theta) + \bar{\beta}_2\Gamma(\theta) \right] & \text{if } \beta_1\beta_2 < 0 \end{cases}$$

$$Pr[0,1] = \begin{cases} F(-a_1,\Delta_2;-\rho) - \left[\bar{a}_2 \Delta_1 \mu(\theta) + \Sigma(\theta) - \bar{\beta}_1\Psi(\theta) \right] \mathbf{1}_{\boldsymbol{R}}(\beta_1)\mathbf{1}_{\boldsymbol{R}}(\beta_2) & \text{if } \beta_1\beta_2 > 0 \\ F(-a_1,\Delta_2;-\rho) + \left[\bar{a}_2 \Delta_1 \mu(\theta) + \Sigma(\theta) - \bar{\beta}_1\Psi(\theta) \right] & \text{if } \beta_1\beta_2 < 0 \end{cases}$$

$$Pr[1,1] = \begin{cases} F(a_1,a_2;\rho) + \left[(\bar{\Delta}_1 \Delta_2 - 1)\mu(\theta) + \Sigma(\theta) + \bar{\beta}_2\Gamma(\theta) \right. & \\ \left. -\ \bar{\beta}_1\Psi(\theta) \right] \mathbf{1}_{\boldsymbol{R}}(\beta_1)\mathbf{1}_{\boldsymbol{R}}(\beta_2) & \text{if } \beta_1\beta_2 > 0 \\ F(a_1,a_2;\rho) - \left[\bar{\Delta}_1 \Delta_2 \mu(\theta) + \Sigma(\theta) + \bar{\beta}_2\Gamma(\theta) - \bar{\beta}_1\Psi(\theta) \right] & \text{if } \beta_1\beta_2 < 0 \end{cases}$$

Proposition 7: The probabilities of the model M_3 are given by:

$$Pr[0,0] = F(-\Delta_1,-\Delta_2;\rho) - \left[\bar{a}_1 a_2 \mu(\theta) + \Sigma(\theta) \right] \{1 - \mathbf{1}_{\boldsymbol{R}}(\beta_1\beta_2)\} - \mu(\theta)\mathbf{1}_{\boldsymbol{R}}(\beta_1)\mathbf{1}_{\boldsymbol{R}}(\beta_2)$$

$$Pr[1,0] = F(\Delta_1,-a_2;-\rho) + \left[(\bar{a}_1 \Delta_2 - 1)\mu(\theta) + \Sigma(\theta) + \bar{\beta}_2\Gamma(\theta) \right] \{1 - \mathbf{1}_{\boldsymbol{R}}(\beta_1\beta_2)\}$$
$$+\ \mu(\theta)\{1 - \mathbf{1}_{\boldsymbol{R}}(\beta_1)\mathbf{1}_{\boldsymbol{R}}(\beta_2)\}$$

$$Pr[0,1] = F(-a_1,\Delta_2;-\rho) + \left[\bar{a}_2 \Delta_1 \mu(\theta) + \Sigma(\theta) - \bar{\beta}_1\Psi(\theta) \right] \{1 - \mathbf{1}_{\boldsymbol{R}}(\beta_1\beta_2)\}$$

$$Pr[1,1] = F(a_1,a_2;\rho) - \left[\bar{\Delta}_1\Delta_2\mu(\theta) + \Sigma(\theta) + \bar{\beta}_2\Gamma(\theta) - \bar{\beta}_1\Psi(\theta)\right]\{1 - \mathbf{1}_R(\beta_1\beta_2)\}$$

Proposition 8: The probabilities of the model M_4 are given by:

$$Pr[0,0] = \begin{cases} F(-\Delta_1,-\Delta_2;\rho) + \left[\bar{a}_1 a_2\mu(\theta) + \Sigma(\theta)\right]\{1 - \mathbf{1}_R(\beta_1)\mathbf{1}_R(\beta_2)\} & \text{if } \beta_1\beta_2 > 0 \\ F(-\Delta_1,-\Delta_2;\rho) - \left[\bar{a}_1 a_2\mu(\theta) + \Sigma(\theta)\right] & \text{if } \beta_1\beta_2 < 0 \end{cases}$$

$$Pr[1,0] = \begin{cases} F(\Delta_1,-a_2;-\rho) - \Big[(\bar{a}_1\Delta_2 + 1)\mu(\theta) + \Sigma(\theta) & \\ \quad + \bar{\beta}_2\Gamma(\theta)\Big]\{1 - \mathbf{1}_R(\beta_1)\mathbf{1}_R(\beta_2)\} & \text{if } \beta_1\beta_2 > 0 \\ F(\Delta_1,-a_2;-\rho) + \left[\bar{a}_1\Delta_2\mu(\theta) + \Sigma(\theta) + \bar{\beta}_2\Gamma(\theta)\right] & \text{if } \beta_1\beta_2 < 0 \end{cases}$$

$$Pr[0,1] = \begin{cases} F(-a_1,\Delta_2;-\rho) - \Big[(\bar{a}_2\Delta_1 + 1)\mu(\theta) + \Sigma(\theta) & \\ \quad - \bar{\beta}_1\Psi(\theta)\Big]\{1 - \mathbf{1}_R(\beta_1)\mathbf{1}_R(\beta_2)\} & \text{if } \beta_1\beta_2 > 0 \\ F(-a_1,\Delta_2;-\rho) + \left[\bar{a}_2\Delta_1\mu(\theta) + \Sigma(\theta) - \bar{\beta}_1\Psi(\theta)\right] & \text{if } \beta_1\beta_2 < 0 \end{cases}$$

$$Pr[1,1] = \begin{cases} F(a_1,a_2;\rho) + \Big[\bar{\Delta}_1\Delta_2\mu(\theta) + \Sigma(\theta) + \bar{\beta}_2\Gamma(\theta) & \\ \quad - \bar{\beta}_1\Psi(\theta)\Big]\{1 - \mathbf{1}_R(\beta_1)\mathbf{1}_R(\beta_2)\} - \mu(\theta)\mathbf{1}_R(\beta_1)\mathbf{1}_R(\beta_2) & \text{if } \beta_1\beta_2 > 0 \\ F(a_1,a_2;\rho) - \left[\bar{\Delta}_1\Delta_2\mu(\theta) + \Sigma(\theta) + \bar{\beta}_2\Gamma(\theta) - \bar{\beta}_1\Psi(\theta)\right] & \text{if } \beta_1\beta_2 < 0 \end{cases}$$

Proposition 9: The probabilities of the model M_5 are given by:

$$Pr[0,0] = F(-\Delta_1,-\Delta_2;\rho) - \left[\bar{a}_1 a_2\mu(\theta) + \Sigma(\theta)\right]\{1 - 2 \,.\, \mathbf{1}_R(\beta_1\beta_2)\} - \mu(\theta)\mathbf{1}_R(\beta_1)\mathbf{1}_R(\beta_2)$$

$$Pr[1,0] = F(\Delta_1,-a_2;-\rho) + \left[(\mathbf{1}_R(\beta_1\beta_2) + \bar{a}_1\Delta_2)\mu(\theta) + \Sigma(\theta) + \bar{\beta}_2\Gamma(\theta)\right]\{1 - 2 \,.\, \mathbf{1}_R(\beta_1\beta_2)\}$$

$$+\mu(\beta_1\beta_2)\mathbf{1}_{\boldsymbol{R}}(\beta_1)\mathbf{1}_{\boldsymbol{R}}(\beta_2)$$

$$Pr[0,1] = F(-a_1,\Delta_2;-\rho) + \left[(\mathbf{1}_{\boldsymbol{R}}(\beta_1\beta_2) + \bar{a}_2\Delta_1)\mu(\theta) + \Sigma(\theta) - \bar{\beta}_1\Psi(\theta)\right]\{1 - 2\,.\,\mathbf{1}_{\boldsymbol{R}}(\beta_1\beta_2)\}$$
$$+\mu(\beta_1\beta_2)\mathbf{1}_{\boldsymbol{R}}(\beta_1)\mathbf{1}_{\boldsymbol{R}}(\beta_2)$$

$$Pr[1,1] = F(a_1,a_2;\rho) - \left[\bar{\Delta}_1\Delta_2\mu(\theta) + \Sigma(\theta) + \bar{\beta}_2\Gamma(\theta) - \bar{\beta}_1\Psi(\theta)\right]\{1 - 2\,.\,\mathbf{1}_{\boldsymbol{R}}(\beta_1\beta_2)\}$$
$$-\mu(\beta_1\beta_2)\mathbf{1}_{\boldsymbol{R}}(\beta_1)\mathbf{1}_{\boldsymbol{R}}(\beta_2)$$

Proposition 10: The probabilities of the model M_6 are given by:

$$Pr[0,0] = \begin{cases} F(-\Delta_1,-\Delta_2;\rho) + \left[\bar{a}_1 a_2\mu(\theta) + \Sigma(\theta)\right]\{1 - \mathbf{1}_{\boldsymbol{R}}(\beta_1)\mathbf{1}_{\boldsymbol{R}}(\beta_2)\} & \\ \quad -\mu(\theta)\mathbf{1}_{\boldsymbol{R}}(\beta_1)\mathbf{1}_{\boldsymbol{R}}(\beta_2) & \text{if } \beta_1\beta_2 > 0 \\ F(-\Delta_1,-\Delta_2;\rho) - \left[\bar{a}_1 a_2\mu(\theta) + \Sigma(\theta)\right] & \text{if } \beta_1\beta_2 < 0 \end{cases}$$

$$Pr[1,0] = \begin{cases} F(\Delta_1,-a_2;-\rho) - \left[(\bar{a}_1\Delta_2 + 1)\mu(\theta) + \Sigma(\theta)\right. & \\ \quad \left. + \bar{\beta}_2\Gamma(\theta)\right]\{1 - \mathbf{1}_{\boldsymbol{R}}(\beta_1)\mathbf{1}_{\boldsymbol{R}}(\beta_2)\} & \text{if } \beta_1\beta_2 > 0 \\ F(\Delta_1,-a_2;-\rho) + \left[\bar{a}_1\Delta_2\mu(\theta) + \Sigma(\theta) + \bar{\beta}_2\Gamma(\theta)\right] & \text{if } \beta_1\beta_2 < 0 \end{cases}$$

$$Pr[0,1] = \begin{cases} F(-a_1,\Delta_2;-\rho) - \left[(\bar{a}_2\Delta_1 + 1)\mu(\theta) + \Sigma(\theta)\right. & \\ \quad \left. - \bar{\beta}_1\Psi(\theta)\right]\{1 - \mathbf{1}_{\boldsymbol{R}}(\beta_1)\mathbf{1}_{\boldsymbol{R}}(\beta_2)\} & \text{if } \beta_1\beta_2 > 0 \\ F(-a_1,\Delta_2;-\rho) + \left[\bar{a}_2\Delta_1\mu(\theta) + \Sigma(\theta) - \bar{\beta}_1\Psi(\theta)\right] & \text{if } \beta_1\beta_2 < 0 \end{cases}$$

$$Pr[1,1] = \begin{cases} F(a_1,a_2;\rho) + \left[\bar{\Delta}_1\Delta_2\mu(\theta) + \Sigma(\theta) + \bar{\beta}_2\Gamma(\theta)\right. & \\ \quad \left. - \bar{\beta}_1\Psi(\theta)\right]\{1 - \mathbf{1}_{\boldsymbol{R}}(\beta_1)\mathbf{1}_{\boldsymbol{R}}(\beta_2)\} & \text{if } \beta_1\beta_2 > 0 \\ F(a_1,a_2;\rho) - \left[\bar{\Delta}_1\Delta_2\mu(\theta) + \Sigma(\theta) + \bar{\beta}_2\Gamma(\theta) - \bar{\beta}_1\Psi(\theta)\right] & \text{if } \beta_1\beta_2 < 0 \end{cases}$$

Proposition 11: The probabilities of the model M_7 are given by:

$$Pr[0,0] = F(-\Delta_1,-\Delta_2;\rho) - \left[\bar{a}_1 a_2 \mu(\theta) + \Sigma(\theta)\right]\{1 - \mathbf{1}_{\boldsymbol{R}}(\beta_1\beta_2)\}$$

$$Pr[1,0] = F(\Delta_1,-a_2;-\rho) + \left[\bar{a}_1 \Delta_2 \mu(\theta) + \Sigma(\theta) + \bar{\beta}_2\Gamma(\theta)\right]\{1 - \mathbf{1}_{\boldsymbol{R}}(\beta_1\beta_2)\}$$

$$Pr[0,1] = F(-a_1,-\Delta_2;-\rho) + \left[(\bar{a}_2\Delta_1 - 1)\mu(\theta) + \Sigma(\theta) - \bar{\beta}_1\Gamma(\theta)\right]\{1 - \mathbf{1}_{\boldsymbol{R}}(\beta_1\beta_2)\}$$
$$+ \mu(\theta)\{1 - \mathbf{1}_{\boldsymbol{R}}(\beta_1)\mathbf{1}_{\boldsymbol{R}}(\beta_2)\}$$

$$Pr[1,1] = F(a_1,a_2;\rho) - \left[\bar{\Delta}_1\Delta_2\mu(\theta) + \Sigma(\theta) + \bar{\beta}_2\Gamma(\theta) - \bar{\beta}_1\Psi(\theta)\right]\{1 - \mathbf{1}_{\boldsymbol{R}}(\beta_1\beta_2)\}$$
$$- \mu(\theta)\mathbf{1}_{\boldsymbol{R}}(\beta_1)\mathbf{1}_{\boldsymbol{R}}(\beta_2)$$

Proposition 12: The probabilities of the model M_8 are given by:

$$Pr[0,0] = \begin{cases} F(-\Delta_1,-\Delta_2;\rho) + \left[(\bar{a}_1 a_2 - 1)\mu(\theta) + \Sigma(\theta)\right]\mathbf{1}_{\boldsymbol{R}}(\beta_1)\mathbf{1}_{\boldsymbol{R}}(\beta_2) & \text{if } \beta_1\beta_2 > 0 \\ F(-\Delta_1,-\Delta_2;\rho) - \left[\bar{a}_1 a_2 \mu(\theta) + \Sigma(\theta)\right] & \text{if } \beta_1\beta_2 < 0 \end{cases}$$

$$Pr[1,0] = \begin{cases} F(\Delta_1,-a_2;-\rho) - \left[(\bar{a}_1\Delta_2 + 1)\mu(\theta) + \Sigma(\theta) + \bar{\beta}_2\Gamma(\theta)\right]\mathbf{1}_{\boldsymbol{R}}(\beta_1)\mathbf{1}_{\boldsymbol{R}}(\beta_2) & \\ + \mu(\theta) & \text{if } \beta_1\beta_2 > 0 \\ F(\Delta_1,-a_2;-\rho) + \left[\bar{a}_1\Delta_2\mu(\theta) + \Sigma(\theta) + \bar{\beta}_2\Gamma(\theta)\right] & \text{if } \beta_1\beta_2 < 0 \end{cases}$$

$$Pr[0,1] = \begin{cases} F(-a_1,\Delta_2;-\rho) - \left[(\bar{a}_2\Delta_1 + 1)\mu(\theta) + \Sigma(\theta) - \bar{\beta}_1\Psi(\theta)\right]\mathbf{1}_{\boldsymbol{R}}(\beta_1)\mathbf{1}_{\boldsymbol{R}}(\beta_2) & \text{if } \beta_1\beta_2 > 0 \\ F(-a_1,\Delta_2;-\rho) + \left[\bar{a}_2\Delta_1\mu(\theta) + \Sigma(\theta) - \bar{\beta}_1\Psi(\theta)\right] & \text{if } \beta_1\beta_2 < 0 \end{cases}$$

$$Pr[1,1] = \begin{cases} F(a_1,a_2;\rho) + \Big[(\bar{\Delta}_1\Delta_2 - 1)\mu(\theta) + \Sigma(\theta) + \bar{\beta}_2\Gamma(\theta) \\ \quad - \bar{\beta}_1\Psi(\theta)\Big]\, \mathbf{1}_{\boldsymbol{R}}(\beta_1)\mathbf{1}_{\boldsymbol{R}}(\beta_2) & \text{if } \beta_1\beta_2 > 0 \\ F(a_1,a_2;\rho) - \Big[\bar{\Delta}_1\Delta_2\mu(\theta) + \Sigma(\theta) + \bar{\beta}_2\Gamma(\theta) - \bar{\beta}_1\Psi(\theta)\Big] & \text{if } \beta_1\beta_2 < 0 \end{cases}$$

Proposition 13: The probabilities of the model M_9 are given by:

$$Pr[0,0] = F(-\Delta_1,-\Delta_2;\rho) - \Big[\bar{a}_1 a_2\mu(\theta) + \Sigma(\theta)\Big]\{1 - \mathbf{1}_{\boldsymbol{R}}(\beta_1\beta_2)\} - \mu(\theta)\mathbf{1}_{\boldsymbol{R}}(\beta_1)\mathbf{1}_{\boldsymbol{R}}(\beta_2)$$

$$Pr[1,0] = F(\Delta_1,-a_2;-\rho) + \Big[\bar{a}_1\Delta_2\mu(\theta) + \Sigma(\theta) + \bar{\beta}_2\Gamma(\theta)\Big]\{1 - \mathbf{1}_{\boldsymbol{R}}(\beta_1\beta_2)\}$$

$$Pr[0,1] = F(-a_1,-\Delta_2;-\rho) + \Big[(\bar{a}_2\Delta_1 - 1)\mu(\theta) + \Sigma(\theta) - \bar{\beta}_1\Gamma(\theta)\Big]\{1 - \mathbf{1}_{\boldsymbol{R}}(\beta_1\beta_2)\} \\ + \mu(\theta)\{1 - \mathbf{1}_{\boldsymbol{R}}(\beta_1)\mathbf{1}_{\boldsymbol{R}}(\beta_2)\}$$

$$Pr[1,1] = F(a_1,a_2;\rho) - \Big[\bar{\Delta}_1\Delta_2\mu(\theta) + \Sigma(\theta) + \bar{\beta}_2\Gamma(\theta) - \bar{\beta}_1\Psi(\theta)\Big]\{1 - \mathbf{1}_{\boldsymbol{R}}(\beta_1\beta_2)\}.$$

Proof. We shall present below the proof of proposition 9 which establishes the probabilities for model 5 where, recall, the players choose to randomize in both multiple equilibria cases. Indeed, because model 5 is the most complex, this exercise will enable us to go through various arguments which can be used to calculate the probabilities of the other models. We will, therefore, omit the proofs of propositions 5-8 and 10-13 as they are straightforward applications of elements of the proof presented here.

Define,

$$m_{rs}(x,y) \equiv \int_y^\infty \int_x^\infty \varepsilon_1^r \varepsilon_2^s f(\varepsilon_1,\varepsilon_2;\rho)\, d\varepsilon_1 d\varepsilon_2.$$

Then, for $c \le a$ and $d \le b$, we have:

$$\int_d^b\int_c^a \varepsilon_1 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2 = m_{10}(c,d)+m_{10}(a,b)-m_{10}(a,d)-m_{10}(c,b)$$

$$\int_d^b\int_c^a \varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2 = m_{01}(c,d)+m_{01}(a,b)-m_{01}(a,d)-m_{01}(c,b)$$

$$\int_d^b\int_c^a \varepsilon_1\,\varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2 = m_{11}(c,d)+m_{11}(a,b)-m_{11}(a,d)-m_{11}(c,b)$$

$$\int_d^b\int_c^a f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2 = m_{00}(c,d)+m_{00}(a,b)-m_{00}(a,d)-m_{00}(c,b).$$

As the $p.d.f.$ $f(\varepsilon_1,\varepsilon_2;\rho)$ is symmetric with respect to (0,0) note also that,

$$m_{00}(x,y)=F(-x,-y;\rho).$$

Thus,

$$\int_d^b\int_c^a f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2 = F(-c,-d;\rho)+F(-a,-b;\rho)-F(-a,-d;\rho)-F(-c,-b;\rho).$$

On the other hand (see, for example, Johnson and Kotz (1972)), we have:

$$m_{00}(x,y).m_{10}(x,y)=\phi(x)\{1-\Phi(A(y,x))\}+\rho\phi(y)\{1-\Phi(A(x,y))\}$$

$$m_{00}(x,y).m_{01}(x,y)=\phi(y)\{1-\Phi(A(x,y))\}+\rho\phi(x)\{1-\Phi(A(y,x))\}$$

$$m_{00}(x,y).m_{11}(x,y)=\rho\Big[x\phi(x)\{1-\Phi(A(y,x))\}+y\phi(y)\{1-\Phi(A(x,y))\}+m_{00}(x,y)\Big]$$

$$+(1-\rho^2)f(x,y;\rho)$$

where, $\phi(.)$, $\Phi(.)$ and $A(.,.)$ are as defined above.

The weights $h_{ij}(\omega)$ and $k_{ij}(\omega)$, $i,j=0,1$ of model 5 are given in section 3 above. After substituting these into the expressions of $H_{ij}^{++}(\theta)$, H_{ij}^{+-}, H_{ij}^{-+} and H_{ij}^{--}, $i,j=0,1$ given in that section, we obtain:

$$H_{00}^{++}(\theta)=1-\bar{a}_1 a_2-\frac{\bar{a}_2}{J_{++}}\iint_{D_{++}}\varepsilon_1 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$-\frac{\bar{a}_1}{J_{++}}\iint_{D_{++}}\varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2-\frac{\bar{1}}{J_{++}}\iint_{D_{++}}\varepsilon_1\varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$H_{10}^{++}(\theta)=-\bar{a}_1\Delta_2-\frac{\bar{\Delta}_2}{J_{++}}\iint_{D_{++}}\varepsilon_1 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$-\frac{\bar{a}_1}{J_{++}}\iint_{D_{++}}\varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2-\frac{\bar{1}}{J_{++}}\iint_{D_{++}}\varepsilon_1\varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$H_{01}^{++}(\theta)=-\bar{a}_2\Delta_1-\frac{\bar{a}_2}{J_{++}}\iint_{D_{++}}\varepsilon_1 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$-\frac{\bar{\Delta}_1}{J_{++}}\iint_{D_{++}}\varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2-\frac{\bar{1}}{J_{++}}\iint_{D_{++}}\varepsilon_1\varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$H_{11}^{++}(\theta)=1-\bar{\Delta}_1\Delta_2-\frac{\bar{\Delta}_2}{J_{++}}\iint_{D_{++}}\varepsilon_1 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$-\frac{\bar{\Delta}_1}{J_{++}}\iint_{D_{++}}\varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2-\frac{\bar{1}}{J_{++}}\iint_{D_{++}}\varepsilon_1\varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$H_{00}^{+-}(\theta)=\bar{a}_1 a_2+\frac{\bar{a}_2}{J_{+-}}\iint_{D_{+-}}\varepsilon_1 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$+\frac{\bar{a}_1}{J_{+-}}\iint_{D_{+-}}\varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2+\frac{\bar{1}}{J_{+-}}\iint_{D_{+-}}\varepsilon_1\varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$H_{10}^{+-}(\theta)=-\bar{a}_1\Delta_2-\frac{\bar{\Delta}_2}{J_{+-}}\iint_{D_{+-}}\varepsilon_1 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$-\frac{\bar{a}_1}{J_{+-}}\iint_{D_{+-}}\varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2-\frac{\bar{1}}{J_{+-}}\iint_{D_{+-}}\varepsilon_1\varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$H_{01}^{+-}(\theta) = -\bar{a}_2\Delta_1 - \frac{\bar{a}_2}{J_{+-}}\iint_{D_{+-}} \varepsilon_1 f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$-\frac{\bar{\Delta}_1}{J_{+-}}\iint_{D_{+-}} \varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2 - \frac{\bar{1}}{J_{+-}}\iint_{D_{+-}} \varepsilon_1\varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$H_{11}^{+-}(\theta) = \bar{\Delta}_1\Delta_2 + \frac{\bar{\Delta}_2}{J_{+-}}\iint_{D_{+-}} \varepsilon_1 f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$+\frac{\bar{\Delta}_1}{J_{+-}}\iint_{D_{+-}} \varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2 + \frac{\bar{1}}{J_{+-}}\iint_{D_{+-}} \varepsilon_1\varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$H_{00}^{-+}(\theta) = \bar{a}_1 a_2 + \frac{\bar{a}_2}{J_{-+}}\iint_{D_{-+}} \varepsilon_1 f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$+\frac{\bar{a}_1}{J_{-+}}\iint_{D_{-+}} \varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2 + \frac{\bar{1}}{J_{-+}}\iint_{D_{-+}} \varepsilon_1\varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$H_{10}^{-+}(\theta) = -\bar{a}_1\Delta_2 - \frac{\bar{\Delta}_2}{J^{0}_{-+}}\iint_{D_{-+}} \varepsilon_1 f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$-\frac{\bar{a}_1}{J_{-+}}\iint_{D_{-+}} \varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2 - \frac{\bar{1}}{J_{-+}}\iint_{D_{-+}} \varepsilon_1\varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$H_{01}^{-+}(\theta) = -\bar{a}_2\Delta_1 - \frac{\bar{a}_2}{J_{-+}}\iint_{D_{-+}} \varepsilon_1 f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$-\frac{\bar{\Delta}_1}{J_{-+}}\iint_{D_{-+}} \varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2 - \frac{\bar{1}}{J_{-+}}\iint_{D_{-+}} \varepsilon_1\varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$H_{11}^{-+}(\theta) = \bar{\Delta}_1\Delta_2 + \frac{\bar{\Delta}_2}{J_{-+}}\iint_{D_{-+}} \varepsilon_1 f(\varepsilon_1,\varepsilon_2;\rho)\,d\varepsilon_1 d\varepsilon_2$$

$$+\frac{\bar{\Delta}_1}{J_{-+}}\iint_{D_{-+}} \varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2+\frac{\bar{1}}{J_{-+}}\iint_{D_{-+}} \varepsilon_1\varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$H_{\bar{0}\bar{0}}^{-}(\theta)=\bar{a}_1 a_2+\frac{\bar{a}_2}{J_{--}}\iint_{D_{--}} \varepsilon_1 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$+\frac{\bar{a}_1}{J_{--}}\iint_{D_{--}} \varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2+\frac{\bar{1}}{J_{--}}\iint_{D_{--}} \varepsilon_1\varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$H_{\bar{1}\bar{0}}^{-}(\theta)=1+\bar{a}_1\Delta_2+\frac{\bar{\Delta}_2}{J_{--}}\iint_{D_{--}} \varepsilon_1 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$+\frac{\bar{a}_1}{J_{--}}\iint_{D_{--}} \varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2+\frac{\bar{1}}{J_{--}}\iint_{D_{--}} \varepsilon_1\varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$H_{\bar{0}\bar{1}}^{-}(\theta)=1+\bar{a}_2\Delta_1+\frac{\bar{a}_2}{J_{--}}\iint_{D_{--}} \varepsilon_1 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$+\frac{\bar{\Delta}_1}{J_{--}}\iint_{D_{--}} \varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2+\frac{\bar{1}}{J_{--}}\iint_{D_{--}} \varepsilon_1\varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$H_{\bar{1}\bar{1}}^{-}(\theta)=\bar{\Delta}_1\Delta_2+\frac{\bar{\Delta}_2}{J_{--}}\iint_{D_{--}} \varepsilon_1 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

$$+\frac{\bar{\Delta}_1}{J_{--}}\iint_{D_{--}} \varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2+\frac{\bar{1}}{J_{-}}\iint_{D_{--}} \varepsilon_1\varepsilon_2 f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$$

where, the J's and the D's are as defined in section 3. Next, we replace the quantities $\int_d^b\int_c^a \varepsilon_1^r\,\varepsilon_2^s\, f(\varepsilon_1,\varepsilon_2;\rho)d\varepsilon_1 d\varepsilon_2$ by their expressions given above. This yields for example,

$$H_{\bar{0}\bar{0}}^{+}(\theta)=1-\bar{a}_1 a_2-\frac{\bar{a}_2}{J_{++}}\left[m_{10}(-a_1,-a_2)+m_{10}(-\Delta_1,-\Delta_2)-m_{10}(-\Delta_1,-a_2)-m_{10}(-a_1,-\Delta_2)\right]$$

$$-\frac{\bar{a}_1}{J_{++}}\left[m_{01}(-a_1,-a_2)+m_{01}(-\Delta_1,-\Delta_2)-m_{01}(-\Delta_1,-a_2)-m_{01}(-a_1,-\Delta_2)\right]$$

$$-\frac{\bar{1}}{J_{++}}\left[m_{11}(-a_1,-a_2)+m_{11}(-\Delta_1,-\Delta_2)-m_{11}(-\Delta_1,-a_2)-m_{11}(-a_1,-\Delta_2)\right]$$

Substituting for the expressions of the $m_{rs}(.,.)$'s given above, we obtain,

$$H_{00}^{++}(\theta)=1-\bar{a}_1a_2-\frac{\bar{a}_2}{J_{++}}\left\{\frac{1}{F(a_1,a_2;\rho)}\left[\phi(a_1)\{1-\Phi(A_8)\}+\rho\phi(a_2)\{1-\Phi(A_4)\}\right]\right.$$

$$+\frac{1}{F(\Delta_1,\Delta_2;\rho)}\left[\phi(\Delta_1)\{1-\Phi(A_5)\}+\rho\phi(\Delta_2)\{1-\Phi(A_1)\}\right]$$

$$-\frac{1}{F(\Delta_1,a_2;\rho)}\left[\phi(\Delta_1)\{1-\Phi(A_7)\}+\rho\phi(a_2)\{1-\Phi(A_2)\}\right]$$

$$\left.-\frac{1}{F(a_1,\Delta_2;\rho)}\left[\phi(a_1)\{1-\Phi(A_6)\}+\rho\phi(\Delta_2)\{1-\Phi(A_3)\}\right]\right\}$$

$$-\frac{\bar{a}_1}{J_{++}}\left\{\frac{1}{F(a_1,a_2;\rho)}\left[\phi(a_2)\{1-\Phi(A_4)\}+\rho\phi(a_1)\{1-\Phi(A_8)\}\right]\right.$$

$$+\frac{1}{F(\Delta_1,\Delta_2;\rho)}\left[\phi(\Delta_2)\{1-\Phi(A_1)\}+\rho\phi(\Delta_1)\{1-\Phi(A_5)\}\right]$$

$$-\frac{1}{F(\Delta_1,a_2;\rho)}\left[\phi(a_2)\{1-\Phi(A_2)\}+\rho\phi(\Delta_1)\{1-\Phi(A_7)\}\right]$$

$$\left.-\frac{1}{F(a_1,\Delta_2;\rho)}\left[\phi(\Delta_2)\{1-\Phi(A_3)\}+\rho\phi(a_1)\{1-\Phi(A_6)\}\right]\right\}$$

$$-\frac{\bar{1}}{J_{++}}\left\{\frac{1}{F(a_1,a_2;\rho)}\left[\rho\left[-a_1\phi(a_1)\{1-\Phi(A_8)\}-a_2\phi(a_2)\{1-\Phi(A_4)\}+F(a_1,a_{2};\rho)\right]\right.\right.$$

$$\left. +\ (1-\rho^2)f(a_1,a_2;\rho)\right]$$

$$+\frac{1}{F(\Delta_1,\Delta_2;\rho)}\left[\rho\left[-\Delta_1\phi(\Delta_1)\{1-\Phi(A_5)\}-\Delta_2\phi(\Delta_2)\{1-\Phi(A_2)\}+F(\Delta_1,\Delta_{2};\rho)\right]\right.$$

$$\left. +\ (1-\rho^2)f(\Delta_1,\Delta_2;\rho)\right]$$

$$-\frac{1}{F(\Delta_1,a_2;\rho)}\left[\rho\left[-\Delta_1\phi(\Delta_1)\{1-\Phi(A_7)\}-a_2\phi(a_2)\{1-\Phi(A_2)\}+F(\Delta_1,a_{2};\rho)\right]\right.$$

$$\left. +\ (1-\rho^2)f(\Delta_1,a_2;\rho)\right]$$

$$-\frac{1}{F(a_1,\Delta_2;\rho)}\left[\rho\left[-a_1\phi(a_1)\{1-\Phi(A_6)\}-\Delta_2\phi(\Delta_2)\{1-\Phi(A_3)\}+F(a_1,\Delta_{2};\rho)\right]\right.$$

$$\left.\left. +\ (1-\rho^2)f(a_1,\Delta_2;\rho)\right]\right\}.$$

Similar substitutions can be done to calculate $H_{00}^{+-}(\theta)$, $H_{00}^{-+}(\theta)$, and $H_{00}^{--}(\theta)$ in terms of the *p.d.f.*'s and *c.d.f.*'s of the univariate and bivariate normal distributions. After regrouping terms and using proposition 4, we obtain for example,

$$Pr[0,0]=\begin{cases} F(-\Delta_1,-\Delta_2;\rho)+(\bar{a}_1a_2-1)\mu(\theta)+\Sigma(\theta) & \text{if } \beta_1>0 \text{ and } \beta_2>0 \\ F(-\Delta_1,-\Delta_2;\rho)-\bar{a}_1a_2\mu(\theta)-\Sigma(\theta) & \text{if } \beta_1\beta_2>0 \\ F(-\Delta_1,-\Delta_2;\rho)+\bar{a}_1a_2\mu(\theta)+\Sigma(\theta) & \text{if } \beta_1<0 \text{ and } \beta_2<0 \end{cases}$$

where, $\mu(\theta)$ and $\Sigma(\theta)$ are as defined above. Now, it suffices to use the function $\mathbf{1}_{\boldsymbol{R}}(.)$ defined above to see that this probability can be written more compactly as in the proposition.

Q.E.D.

NOTES

1. As is shown in Gasmi and Vuong (1987b) however, econometric models resulting from a proper account of the interactions between players are more general than the ones that have been considered up to now in the literature.
2. The main difference between the results of this section and those is Bjorn and Vuong (1984) comes from the fact that we extend the players' strategy sets to include mixed strategies.
3. Indeed, as it will be seen in proposition 3, each of the reaction functions (correspondences) will occur under some inequality conditions on the unobserved random variables introduced in the econometric model. These latter variables being continuous, only strict inequalities are relevant.
4. This table which displays the Nash equilibria of the game is similar to the one derived in Bjorn and Vuong (1984). However, when each of the pairs (p_3,q_4), (p_4,q_3) occurs these authors obtain a lack of existence of a Nash equilibrium.
5. For notational simplicity, we have omitted the subscript t indexing the observations. Hence, assumption 3 actually means that the parameters β_1, β_2, γ_1, γ_2 are constant across observations.
6. This shows that we were justified in considering only the basic reaction functions p_1, p_2, p_3, p_4, q_1, q_2, q_3, and q_4 (see note 2).
7. The model generated by such an assumption is analyzed in much detail in section 4 of the chapter.

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CHAPTER 3. COMPETITION IN THE SOFT DRINK INDUSTRY: An Econometric Study

1. INTRODUCTION

The empirical analysis presented in this chapter of the thesis is directed at evaluating the effect of advertising expenditures on market demands for two major cola drinks in the United States, Coca Cola and Pepsi Cola. The market for the regular carbonated cola drink is one which is largely dominated by these two brands, where large allocations of funds are made by the two main producing firms, The Coca Cola Company and Pepsico, Inc., for the promotion and marketing of their products. In 1986, cola drinks accounted for 69% of the total soft drink industry. The brands Coca Cola and Pepsi Cola represented 21.5% and 18.6% of this industry, thus jointly accounting for about 60% of the total cola drinks sub-market. Through the '80s, The Coca Cola Company and Pepsico, Inc. have allocated $281.1 million and $232 million respectively, in advertising the brands Coca Cola and Pepsi Cola.

In this chapter, we use the theoretical economic framework developed in chapter 1 and the econometric models generated from it, to analyze the market demands for the products Coca Cola and Pepsi Cola. This work is organized as follows. In the next section we examine the relevant literature on advertising, and discuss the methodological import of our study. Section 3 gives a complete description of the data used and their sources. As in any empirical study which involves industry or firm-specific data, we have encountered considerable difficulties to gather quantitative information on quantity, prices and costs. Even when some parts of the data were available, we still had to face the problem of incompatibility of the frequencies. A quarterly data set covering the period 1968-1986 is utilized to estimate the econometric models developed in chapter 1 and to perform various statistical tests.

Section 4 contains the estimation. Recall that our starting point was to specify economic theoretical models for a duopoly and then generate from these structures a set of econometric models. More specifically, each econometric model is derived from a specified game which is supposed to describe the competitive behavior in advertising and prices of the firms. Thus, the essential hypothesis is that, at any given period, observations on the variables of any of the models have arisen as equilibrium outcomes of a specific game of competition between firms. We specify six duopolistic games. The first game allows firms to set prices and advertising levels simultaneously. In the second game, firms choose prices in a first stage and advertising in a second stage. The fourth game gives the opportunity to one of the firms to set its price level in a first stage, while its advertising and the other firm's price and advertising decisions are made in a second stage. The third and fifth games are respectively identical to the second and fourth games with the role of advertising and price reversed. Finally, the sixth game allows one of the firms to act as a leader in making prices and advertising decisions. The standard noncooperative Nash equilibrium solution concept is used in the first five games. In addition, the notion of subgame perfection of this equilibrium is required in order to sustain credibility of commitments in the two-stage games. We use the usual Stackelberg equilibrium notion to solve the sixth game.

In order to empirically analyze the different games described above, we have formulated a linear simultaneous-equation model, the general model, from which each of the models corresponding to these games is derived provided that a specific set of nonlinear constraints are imposed on the parameters. We provide ordinary least squares, two-stage least squares and three-stage least squares parameter estimates of the general linear model. Due to the presence of more than one endogenous variable in each equation of the model, OLS estimates will be biased and inconsistent since regressors and residuals will be correlated. Estimating the model by 2SLS gives consistent parameter estimates. A gain in asymptotic efficiency is achieved by the 3SLS method which is an application of the generalized least squares technique to the complete system of equations. The next step is to compute full information maximum likelihood parameter estimates of the general linear model and the various nonlinear models. We also consider the estimation of the models associated with three additional games, the seventh, eighth, and ninth. These games are identical to the fourth, fifth, and sixth games respectively, with the role of the firms interchanged. FIML is asymptotically efficient as its asymptotic covariance matrix reaches the Cramer-Rao bound, and so is 3SLS for the linear model.

The estimation results obtained from section 4 are employed in section 5 to perform various specification and model selection tests. First, we derive some specification tests of the type proposed by Hausman (1978), for the presence of endogeneity in the linear model using OLS and 2SLS parameter estimates. Indeed, one would expect these tests to reveal endogeneity of price and advertising variables, since they are decision variables of the firms. Second, since each of the specific models is equivalent to the general linear model given a set of nonlinear restrictions on the parameters, following the work of Atkinson (1969, 1970) we perform some likelihood ratio tests of these restrictions under the assumption of correct specification of the general linear model. These tests are interpreted as tests of each of the specific nonlinear models versus the general linear model. But, since the general linear model appears to be a quite simplified representation of the competitive behavior of the firms and, indeed, is a simple version of somewhat more structural models (the specific models derived from the games), it is likely to be misspecified. We investigate this issue by means of a Hausman model specification test making use of 2SLS and 3SLS parameter estimates. The next task is to construct a model selection procedure, used to choose among the specific models, based on some tests which can be applied independently of the correct or incorrect specification of the models. For this purpose, we consider some recently developed tests of nonnested hypotheses based on likelihood ratio statistics (see Vuong (1988)).

Finally, the results of the estimation and the model selection procedures are discussed in section 6 of this chapter. In particular, the model selection conducted in the previous section will enable us to draw some conclusions on the types of competitive price and advertising behavior of the two companies under study, which are most compatible with the analyzed data.

2. SELECTED LITERATURE ON ADVERTISING AND INDUSTRIAL ECONOMICS

The purpose of this section is to give an overview of the literature on advertising. We do not examine here the numerous studies in the marketing and management literature on advertising, as these studies are concerned chiefly with the marketing effect of advertising expenditures on sales and do not emphasize economic behavioral issues related to advertising. See for example Rao (1986) and Weinberg and Weiss (1986) among others. Instead, we shall examine the works which, in our view, represent the main streams of the traditional literature on the economics of advertising.

By far the main question that has animated the debates in the empirical economic literature on advertising over the last three decades is that of its impact on competition. These debates have generated two principal streams of thoughts. We shall survey below the theoretical works corresponding to each of these school of thoughts, and the studies which have empirically investigated the relevant issues.

On one hand, authors like Simon (1970, 1980), Ferguson (1974) and Ornstein (1977) view advertising as an aid to consumers in their purchasing decisions making in that it provides information on the diverse goods and services available in the market. By conveying varied information, advertising facilitates rapid buyer exposure to new products. This process should then operate in the sense of enhancing competition and, therefore, promoting an efficient allocation of resources.

On the other hand, a view shared by authors like Kaldor (1950), Bain (1951, 1956, 1959), and more recently Comanor and Wilson (1974, 1979, 1980), suggests that advertising serves the purpose of increasing monopoly power by enhancing industrial concentration and creating barriers to entry. The argument here is that advertising is used by firms to artificially differentiate their products from other similar ones, thereby promoting brand loyalties. The high advertising costs of surmounting these brand loyalties constitute barriers to entry for new firms seeking to enter the industry. The barriers to entry phenomenon is accentuated by economies of scale in advertising and volume discounts for large-scale purchase of advertising which benefit large companies. In sum, this view holds that firms that are heavily advertising can charge higher prices and enjoy excess profits.

While from a theoretical point of view no definite answer has been provided, the question of the effects of advertising on competition has induced a large number of empirical studies. In addition to the studies discussed in detail below, the vast literature which addresses directly or indirectly this issue includes the works of Dorward and Steiner (1961), Meehan and Duchesneau (1973), Vernon and Nourse (1973), Wilder (1974), Phillips (1976), Hart and Morgan (1977), Jones et al. (1977), Pitts (1979), Bloch (1980), Simon (1970, 1980), Kelton and Kelton (1982), Schmalensee (1976, 1982), Mertens and Ginsburgh (1985), Blair et al. (1987), and Roberts and Samuelson (1987). However, in most of the cases, the findings of these studies conflict. Using cross-sectional data on twenty manufacturing industries, Bain (1959) concluded that as product differentiation became an increasing feature of the market, industry concentration rose. By contrast, Telser (1964) found that the correlation between advertising and concentration is not impressive. Using a sample of forty-two firms, Mann et al. (1967) ran regressions (for the years 1954, 1958, and 1963) of a measure of industrial concentration on average advertising to sales revenues ratio. The results of their regressions showed a marked correlation between these two variables, thus agreeing with the earlier findings of Bain's study and its underlying theoretical hypothesis. In a comment to the study by Mann et al., Ekelund and Maurice (1969) pointed out that these authors have used a sample which was biased towards industries with relatively high concentration. Articles by Mann et al. (1969a, 1969b) and Telser (1969) followed this question but, as Mann et al. put it, "...no one has said the last word on this issue".

In response to the argument that high advertising costs are associated with new products of incoming firms, it has been advanced that despite the fact that advertising effects persist over time, its cost has been treated as current expenses rather than capitalized over time. When the capital nature of advertising has been properly taken into account, it appears (see for example the conclusions of Bloch (1974) and Ayanian (1975)) that the gap in cost of advertising between new and old products would substantially decrease, thus lowering initial costs for the new entrant.[1] This conclusion is, however, contested by Comanor and Wilson (1979) who point out that the results of Bloch and Ayanian are strongly contingent upon the low depreciation rates that they used in computing advertising capital.

Traditional work on advertising has focused on its impact on market performance indicators such as demand, prices, and competition. It appears, however, that most of this work was based on pure (ad hoc) statistical investigations. In recent years, researchers have been more and more inclined toward understanding the process by which firms' strategic variables are determined. This has led to a set of studies that has examined the stucture of industries using information about the strategic behavior of firms. One such group of studies has investigated the forces that determine the demand for advertising. These studies bring out the implicit role of buyers as well as sellers in influencing market demand for advertising. Among others, see Shramm and Sherman (1976), Ehrilich and Fisher (1982), Baye (1981,1983), and Clarke and Else (1983). Another group of studies followed an approach that examines the optimal (price-) advertising decision of firms (see, e.g., Dorfman and Steiner (1961), Nerlove and Arrow (1962), Jacquemin (1973)). See also Dehez and Jacquemin (1975), Ireland and Law (1977), and more recently Brick and Jagpal (1981), and Luptacik (1982).

Over the last two decades, the field of industrial economics has experienced a great deal of theoretical development due in large part to the introduction of the theory of games. A large number of interesting theoretical questions on the structure of industries have been addressed, and approached succesfully using modern game-theoretic tools. As a result, in recent years, a large body of literature on empirical industrial economics has emerged seeking to confront the main hypotheses of the theoretical models to actual data. See the recent special issue of The Journal of Industrial Economics edited by Bresnahan and Schmalensee (1987) on the renaissance of empirical industrial economics, and the various articles contained in this issue. See also Telser (1972), Murphy and Ng (1974), Porter (1980), Caves and Williamson (1985), Neven and Phlips (1985), and the special issue of The Journal of Industrial Economics edited by Geroski et al. (1985).

We now discuss an article (Metwally (1976)) in which the author considers both the optimizing and interdependence of firms, and derives empirically testable optimal criteria for advertising. Seeking to investigate a question that has been originally raised by Kaldor (1950), that of excessiveness of advertising, Metwally develops two basic optimization models in advertising.

The first model is a short-term model which leads to a necessary condition for optimal behavior in advertising by firms of the form $\theta = -\varepsilon/\eta$, where θ is the short-run optimal advertising-sales ratio, ε is the advertising-elasticity of demand, and η is the price-elasticity of demand. A similar result is derived by Dorfman and Steiner (1961).

The long-run model takes into account the stock of goodwill of the firms which reflects the impact of past advertising expenditures. Using a framework described in Jacquemin (1973), Metwally uses optimal control techniques to derive a necessary condition for an optimum in advertising given by $\tilde{\theta} = -\varepsilon/\eta(1-\lambda)$, where $\tilde{\theta}$ is the long-run counterpart of θ described above, and λ is the Lagrange multiplier attached to the constraint that describes the process of depreciation, over time, of the stock of goodwill. Similar types of results have been obtained by Nerlove and Arrow (1962).

These two criteria are then used by the author to test whether advertising expenditures on the product, washing powder, are "too excessive" in Australia. To do so, Metwally specifies econometric models which enable him to estimate the different elasticities, and a reaction coefficient which appears in the advertising elasticity. This coefficient is introduced via a reaction function of the firm, to the advertising strategies of other firms, which captures the competitive interdependence among firms. A comparison of the actual ratios of advertising to sales, and those given by the optimization models above, allows the author to draw conclusions about whether advertising is excessive. The empirical part of this study showed that advertising was too excessive in the short-run, but in the long-run, firms were following a profit maximization policy in advertising.

While we think that the study by Metwally is quite innovative and thoughts provocative, we would like to comment on three points. First, the author introduces the oligopolistic interdependence into his model by formulating a reaction function. The latter is written as, $a_j = a_j(a_i)$ where a_i is advertising of firm i. While this clearly means that he is well aware of the competitive interdependence of firms in advertising, Metwally ignores such an interdependence in prices. Indeed, this fact is shown even further in the specification of a representative firm's demand, where only the firm's own price appears as an argument.

The second remark is that, in order to solve for the optimal criteria in advertising, the author indeed solves for prices and advertising, i.e., he considers the Nash equilibrium outcome in those variables. Therefore, though he does not explicitly formulate a reaction function in price, this variable is taken as endogenous in his model. Extending the frame of oligopolistic interdependence to include a price reaction function would not alter any of the theoretical results of the study and would enlarge its scope.

Finally, we turn to the discussion of the empirical application contained in Metwally's paper. In order to estimate the coefficients necessary to evaluate the optimal criteria, the author specifies three different models for a typical firm's demand. Only one of these models incorporates a coefficient which somewhat represents competitive interdependence in prices. However, in order to estimate the reaction function coefficient, the author uses the same advertising reaction equation for the three models. First, we do not think that such a reaction function is compatible with the three different models. Second, we do believe that the reaction function must be endogenous to the underlying competitive game, and thus be determined uniquely once the firm's demand model is specified.

We conclude our survey by drawing attention to some recent studies that have analyzed dynamic oligopolistic games in prices and advertising by means of optimal control theory and differential games. These studies include the works of Bourguignon and Sethi (1981), Feichtinger (1982), Jorgensen (1982), Thepot (1983), Bagchi (1984), Gaugusch (1984), Dockner and Feichtinger (1985), and Bagwell and Ramey (1988) among others.

3. DESCRIPTION OF THE DATA

As in any empirical research which involves analysis of industry or firm-specific data, we have encountered a great deal of difficulty in collecting quantitative information on volumes, prices, and costs. The starting point of our data search was to contact, directly, the companies of interest. While the data requested were available, we were told that the information could not be shared with people outside the companies. Consequently, we had to follow another route. We have contacted various governmental as well as private organizations which compile data on the soft drink industry. Some success has been achieved with some of these institutions (see the data description below), but the data needed some additional work. Even when data on some of the variables of interest were available, the frequencies were not always compatible. In particular, we had to devise a method of calculating quarterly figures of firm-specific variables starting from annual data and quarterly information on these variables at a more aggregated level. For example, we used information about the specific market under consideration in which the firms are involved, namely, the cola drink market, and in some instances information on the whole soft drink market.

While we think that the data used can be improved, two points are worth mentioning. First, we believe that though the data will be difficult to obtain directly from The Coca Cola Company and Pepsico, Inc. themselves, these companies are so large and important that some success in improving the quality of the data can be achieved by gathering information through private marketing and publishing corporations. We shall, in the near future, devote some effort towards contacting some of these corporations. Second, we think that the data used show some empirical results which lead to reasonable economic interpretations. In addition, we believe that the results can serve as a useful basis for future improvements of the data set.

The raw data was collected from the following sources: *Leading National Advertisers, Inc.* (LNA), *Beverage Industry* (BI), *National Soft Drink Association* (NSDA), *Bureau of Labor Statistics* (BLS), *Department of Commerce, Bureau of Economic Analysis* (BEA), and *Annual Reports of The Coca Cola Company and Pepsico, Inc., 1968-1986* (AR).

Annual consumption of Coca Cola and Pepsi Cola were obtained from BI and quarterly figures were constructed using an index of seasonality obtained from NSDA. The formula used is, $q_i^{st} = q_i^s \cdot (I^t / \sum_{l=1}^4 I^l)$, where q_i^{st} is consumption of brand i in quarter t of year s, q_i^s is consumption of this brand in year s and I^t is index of seasonality of quarter t. Average annual prices represented soft drink sales (obtained from AR) divided by annual consumption. Quarterly prices were calculated using weights generated from the consumer price index for nonalcoholic beverages, J, obtained from BLS. For this purpose, we used the formula, $p_i^{st} = 4p_i^s \cdot (J^{st} / \sum_{l=1}^4 J^{sl})$. Quarterly advertising expenditures were obtained from LNA. Quarterly disposable income was computed from monthly figures acquired from BEA. Quarterly figures on price index of sugar and unit cost of labor in the nondurable manufacturing sector were computed using monthly data obtained from BLS. Bond yields (Moody's AAA and BAA corporate) were obtained from BLS. All nominal variables were deflated using the general consumer price index obtained from BLS.

Tables 2 and 3 below display descriptive statistics of the data and correlation coefficients among the variables.

TABLE 2

SUMMARY STATISTICS OF THE VARIABLES

Variable	Mean	Minimum	Maximum	Standard Deviation
QCO	30.22	18.89	43.82	5.90
QPE	22.72	11.32	37.83	6.54
PCO	12.96	10.89	17.79	1.20
PPE	8.16	6.65	9.52	0.85
ADCO	5.89	2.39	8.48	1.18
ADPE	5.28	2.66	7.13	1.06
INC	20.63	15.31	26.26	3.07
SUG	1.15	0.78	4.22	0.54
LABOR	0.55	0.48	0.66	0.06
RAAA	3.01	-5.05	12.12	4.21
RBAA	4.27	-4.05	14.11	4.46

TABLE 3

CORRELATIONS AMONG VARIABLES

	QCO	*QPE*	*PCO*	*PPE*	*ADCO*	*ADPE*	*INC*	*SUG*	*LABOR*	*RAAA*	*RBAA*
QCO	1.00	0.93	-0.07	0.44	0.34	0.40	0.61	-0.04	-0.62	0.17	0.19
QPE		1.00	-0.09	0.59	0.24	0.34	0.84	-0.12	-0.82	0.33	0.36
PCO			1.00	0.30	0.10	0.19	-0.08	0.13	0.02	-0.43	-0.41
PPE				1.00	0.06	0.11	0.68	0.13	-0.72	0.05	0.10
ADCO					1.00	0.58	-0.0003	-0.06	-0.03	-0.27	-0.27
ADPE						1.00	0.12	-0.17	-0.08	-0.24	-0.24
INC							1.00	-0.13	-0.96	0.48	0.50
SUG								1.00	0.03	-0.35	-0.31
LABOR									1.00	-0.42	-0.46
RAAA										1.00	0.99
RBAA											1.00

4. ESTIMATION OF GAMES IN PRICES AND ADVERTISING

The main purpose in this section is to estimate, by full information maximum likelihood techniques, the nonlinear simultaneous-equation models generated from the different games in prices and advertising between firms described in section 2 of chapter 1. We will refer to The Coca Cola company and Pepsico, Inc. as firm 1 and firm 2 respectively. Models M4, M5, and M6 are associated with the games where firm 1 moves first and M7, M8, and M9 with those where firm 2 moves first. As seen in propositions 7-12, the nonlinear models are all nested in a linear simultaneous-equation model called the general model. Thus, in a first step, we estimate this model by standard linear regression and full information maximum likelihood methods.

Following assumption 1 in chapter 1, we specify the constant terms in the demand equations γ_{i0} and the real average costs c_i to be of the form:

$$\gamma_{i0} = \beta_{i0} + \beta_{i1}DUM + \beta_{i2}INC\,,\ i = 1{,}2 \tag{4.1a}$$

$$c_1 = \eta_{11}SUG + \eta_{12}LABOR + \eta_{13}RAAA \tag{4.1b}$$

$$c_2 = \eta_{21}SUG + \eta_{22}LABOR + \eta_{23}RBAA \tag{4.1c}$$

Thus, we obtain the following general linear simultaneous-equation model (see equations (3.3)-(3.8) in chapter 1) that nests all the particular models:

$$QCO_t - \alpha_{11}PCO_t - \alpha_{12}PPE_t - \gamma_{11}ADCO_t - \gamma_{12}ADPE_t - \beta_{10} - \beta_{11}DUM_t - \beta_{12}INC_t = u_{1t} \tag{4.2}$$

$$QPE_t - \alpha_{21}PCO_t - \alpha_{22}PPE_t - \gamma_{21}ADCO_t - \gamma_{22}ADPE_t - \beta_{20} - \beta_{21}DUM_t - \beta_{22}INC_t = u_{2t} \tag{4.3}$$

$$QCO_t + \lambda_1 PCO_t - \delta_{11}SUG_t - \delta_{12}LABOR_t - \delta_{13}RAAA_t = u_{3t} \tag{4.4}$$

$$QPE_t + \lambda_2 PPE_t - \delta_{21}SUG_t - \delta_{22}LABOR_t - \delta_{23}RBAA_t = u_{4t} \tag{4.5}$$

$$\mu_1 PCO_t + ADCO_t - \delta_{31}SUG_t - \delta_{32}LABOR_t - \delta_{33}RAAA_t = u_{5t} \tag{4.6}$$

$$\mu_2 PPE_t + ADPE_t - \delta_{41}SUG_t - \delta_{42}LABOR_t - \delta_{43}RBAA_t = u_{6t} \tag{4.6}$$

where the variables of the model are described below:

QCO, QPE Consumption volumes of Coca Cola and Pepsi Cola, respectively

PCO, *PPE*	Real prices of Coca Cola and Pepsi Cola
ADCO, *ADPE*	Square root of real advertising expenditures on Coca Cola and Pepsi Cola
DUM	Dummy variable coded 1 for quarters 2 and 3, and 0 for the others
INC	Real disposable income
SUG	Real price of sugar
LABOR	Real unit cost of labor in the non-durable manufacturing sector
RAAA	Real bond yield: Moody's AAA corporate
RBAA	Real bond yield: Moody's BAA corporate

The endogenous variables are $\{QCO, QPE, PCO, PPE, ADCO, ADPE\}$, the exogenous variables are $\{DUM, C, INC, SUG, LABOR, RAAA, RBAA\}$, and the error terms u_{it}, $i = 1,\ldots,6$ are normally distributed with covariance matrix Σ. We should note that the parameters of this model, δ_{ij}, $i = 1,\ldots,4$ and $j = 0,\ldots,3$, are not structural parameters. However, as seen in propositions 7-11 in chapter 1, they are related to the structural parameters which parameterize the nonlinear econometric models generated from our game theoretical structures.

The vectors of exogenous variables z_i and x_i, $i = 1,2$, given in assumption 2 of chapter 1 are now written as:

$$z_1' = \left[C \; DUM \; INC\right]$$

$$z_2' = \left[C \; DUM \; INC\right]$$

$$x_1' = \left[SUG \; LABOR \; RAAA\right]$$

$$x_2' = \left[SUG \; LABOR \; RBAA\right]$$

The dimensions of the different components of these vectors (see assumption 2 in chapter 1) are then given by:

$$h_m = 0,\ h_z = 3,\ h_x = 2,\ h_{\bar{z}_1} = h_{\bar{z}_2} = 0,\ h_{\bar{x}_1} = h_{\bar{x}_2} = 1.$$

Thus, the above specification of the general linear model satisfies the necessary conditions given in proposition 12 of chapter 1, for total identification of the system.

Next, we present in tables 4-6 successively, the results of OLS, 2SLS, and 3SLS estimations of the general linear simultaneous-equation model. As we will see in the next section, these estimates will be used to calculate specification tests. In particular, we will use OLS and 2SLS to check for endogeneity of price and advertising variables which, in the games analyzed, are the firms' decision variables. Further, the 2SLS and 3SLS estimates are used to perform a specification test on the demand equations contained in the general linear model.

TABLE 4

OLS PARAMETER ESTIMATES OF THE GENERAL MODEL

Variable	Estimated Coefficient	Standard Error	T-Statistic	Variable	Estimated Coefficient	Standard Error	T-Statistic
Equation 1				*Equation 2*			
PCO	-0.28	0.20	-1.41	*PCO*	-0.34	0.17	-2.04
PPE	0.53	0.38	1.40	*PPE*	0.60	0.31	1.93
ADCO	0.11	0.22	0.50	*ADCO*	0.005	0.18	0.26
ADPE	-0.22	0.26	-0.86	*ADPE*	0.001	0.22	0.005
C	3.85	2.85	1.35	*C*	-15.52	2.37	-6.56
DUM	8.74	0.49	17.88	*DUM*	6.41	0.41	15.79
INC	1.06	0.10	10.48	*INC*	1.66	0.008	19.71
Equation 3				*Equation 4*			
PCO	2.54	0.46	5.56	*PPE*	4.38	0.39	11.27
SUG	-0.003	1.51	-0.002	*SUG*	-1.26	1.04	-1.22
LABOR	-8.27	9.86	-0.84	*LABOR*	-23.19	5.13	-4.52
RAAA	0.54	0.18	3.04	*RBAA*	0.28	0.13	2.22
Equation 5				*Equation 6*			
PCO	0.36	0.009	4.09	*PPE*	0.51	0.009	5.78
SUG	-0.39	0.29	-1.34	*SUG*	-0.57	0.24	-2.41
LABOR	3.12	1.90	1.65	*LABOR*	3.69	1.17	3.16
RAAA	-0.002	0.003	-0.76	*RBAA*	-0.006	0.003	-2.24

TABLE 5
2SLS PARAMETER ESTIMATES OF THE GENERAL MODEL

Variable	Estimated Coefficient	Standard Error	T-Statistic	Variable	Estimated Coefficient	Standard Error	T-Statistic
Equation 1				*Equation 2*			
PCO	-0.36	0.92	-0.39	*PCO*	-1.15	0.76	-1.52
PPE	1.79	1.38	1.29	*PPE*	2.35	1.14	2.06
ADCO	0.90	1.07	0.85	*ADCO*	0.19	0.88	0.22
ADPE	-1.75	0.83	-2.10	*ADPE*	-0.50	0.69	-0.73
C	1.23	7.33	0.17	*C*	-10.76	6.04	-1.78
DUM	9.45	1.17	8.08	*DUM*	6.77	0.96	7.03
INC	0.89	0.29	3.03	*INC*	1.32	0.24	5.50
Equation 3				*Equation 4*			
PCO	4.60	0.67	6.83	*PPE*	5.06	0.43	11.85
SUG	-2.73	1.80	-1.52	*SUG*	-1.90	1.07	-1.78
LABOR	-49.82	14.16	-3.52	*LABOR*	-31.16	5.56	-5.60
RAAA	0.40	0.20	1.99	*RBAA*	0.18	0.13	1.40
Equation 5				*Equation 6*			
PCO	0.60	0.12	4.95	*PPE*	0.61	0.1	6.33
SUG	-0.70	0.32	-2.17	*SUG*	-0.66	0.24	-2.74
LABOR	-1.63	2.52	-0.65	*LABOR*	2.58	1.25	2.06
RAAA	-0.04	0.04	-1.14	*RBAA*	-0.08	0.03	-2.70

TABLE 6
3SLS PARAMETER ESTIMATES OF THE GENERAL MODEL

Variable	Estimated Coefficient	Standard Error	T-Statistic	Variable	Estimated Coefficient	Standard Error	T-Statistic
Equation 1				*Equation 2*			
PCO	-0.93	0.77	-1.21	*PCO*	-2.00	0.62	-3.24
PPE	2.65	1.01	2.62	*PPE*	4.04	0.77	5.23
ADCO	2.34	0.82	2.84	*ADCO*	0.71	0.64	1.10
ADPE	-2.56	0.65	-3.92	*ADPE*	-0.81	0.51	-1.57
C	2.18	6.60	0.33	*C*	-7.10	5.38	-1.32
DUM	7.38	1.01	7.33	*DUM*	5.73	0.81	7.06
INC	0.71	0.21	3.31	*INC*	0.97	0.16	5.94
Equation 3				*Equation 4*			
PCO	-4.57	0.65	-7.02	*PPE*	-5.00	0.41	-12.14
SUG	-2.74	1.75	-1.57	*SUG*	-1.87	1.04	-1.80
LABOR	-48.77	13.66	-3.57	*LABOR*	-30.30	5.37	-5.64
RAAA	0.37	0.20	1.90	*RBAA*	0.18	0.13	1.39
Equation 5				*Equation 6*			
PCO	-0.60	0.12	-5.16	*PPE*	-0.61	0.09	-6.57
SUG	-0.71	0.31	-2.28	*SUG*	-0.66	0.23	-2.84
LABOR	-1.75	2.45	-0.71	*LABOR*	1.21	0.718	2.08
RAAA	-0.04	0.03	-1.26	*RBAA*	0.03	0.29	-2.76

The next table gives the full information maximum likelihood parameter estimates of the general linear model assuming normally distributed error terms with 3SLS estimates as initial values for the parameters. The results of this estimation will be utilized in the next section to carry out some classical tests of the various nonlinear models versus the linear model under the assumption of correct specification of the latter.

TABLE 7
FIML PARAMETER ESTIMATES OF THE GENERAL MODEL

Variable	Estimated Coefficient	Standard Error	T-Statistic	Variable	Estimated Coefficient	Standard Error	T-Statistic
Equation 1				*Equation 2*			
PCO	-14.59	2.08	-7.01	*PCO*	-23.33	2.57	-9.09
PPE	0.81	2.38	0.34	*PPE*	8.98	2.94	3.06
ADCO	45.71	0.61	75.22	*ADCO*	58.99	0.57	102.83
ADPE	-25.40	0.80	-31.69	*ADPE*	-30.28	0.78	-38.83
C	-22.54	27.83	-0.81	*C*	-39.32	27.74	-1.42
DUM	14.81	1.30	11.42	*DUM*	13.35	1.26	10.58
INC	4.49	0.52	8.65	*INC*	4.67	0.53	8.88
Equation 3				*Equation 4*			
PCO	-4.77	0.32	-15.06	*PPE*	-6.04	0.57	-10.57
SUG	-2.29	3.00	-0.76	*SUG*	-2.51	2.64	-0.95
LABOR	-54.53	9.04	-6.03	*LABOR*	-43.59	8.70	-5.01
RAAA	0.45	0.21	2.12	*RBAA*	0.13	0.13	1.01
Equation 5				*Equation 6*			
PCO	-0.42	0.04	-10.40	*PPE*	-0.72	0.11	-6.34
SUG	-0.32	0.30	-1.09	*SUG*	-0.73	0.66	-1.10
LABOR	1.45	0.73	1.99	*LABOR*	1.10	1.33	0.82
RAAA	-0.04	0.01	-3.28	*RBAA*	-0.08	0.03	-2.98
Observations	76						
Likelihood	-626.553						

We now turn to the estimation of the nonlinear simultaneous-equation models derived from the various games described in chapter 1. Recall that in order to evaluate the impact of price and advertising decisions by firms in the context of a duopoly, we have postulated six games in those decision variables. On the basis of these six economic structural models describing a duopolistic industry, we then designed a set of nonlinear simultaneous-equation models.

Each of the econometric models formulated has the feature that, at any given period, observations on its variables have arisen as the values of these variables at the equilibrium of the corresponding game. We described the competitive behavior in advertising and prices of the firms by the following six games which will be designated by G1-G6. Game G1 allows firms to set prices and advertising levels simultaneously. In game G2, firms choose prices in a first stage and advertising in a second stage. The fourth game, G4, gives the opportunity to one of the firms to set its price level in a first stage, while its advertising and the other firm's price and advertising decisions are made in a second stage. The third and fifth games, G3 and G5, are respectively identical to the second and fourth games with the role of advertising and price reversed. Finally, game G6 allows one of the firm to act as a leader in making price and advertising decisions. The standard noncooperative Nash equilibrium solution concept is used in the first five games, G1-G5. In addition, the notion of subgame perfection of this equilibrium is required in order to sustain credibility of commitments in the games taking place in two stages. We use the usual Stackelberg equilibrium notion to solve the sixth game G6. We have noted in proposition 2 of chapter 1, that games G1 and G2 share the same Nash equilibrium solution. Thus, the same econometric model, model 1-2 below, represents these two games.

As established in propositions 7-11, the econometric models associated with the different games, model 1-2 through model 6, are obtained by imposing some nonlinear restrictions on the parameters of the general linear model. These restrictions are also given in propositions 7-11. A simple way to deal with the nonlinearity of these restrictions is to estimate each of these models by nonlinear methods. Because of parameter efficiency and the necessity of the results for the model selection procedure performed in the next section, we chose to estimate these models by nonlinear full information maximum likelihood.

The results of the estimations of model 1-2 through model 9 (models 7-9 are equivalent to models 4-6 with the role of the firms reversed) by nonlinear FIML are displayed in tables 8-15. We provided initial values of the parameters for the Gauss maximization algorithm used by the statistical package TSP. These starting values were computed from the FIML parameter estimates of the general linear model using the expressions of the nonlinear restrictions on these parameters imposed by each specific model.

TABLE 8
FIML PARAMETER ESTIMATES OF MODEL 1-2

Variable	Estimated Coefficient	T-Statistic	Variable	Estimated Coefficient	T-Statistic
***Demand Firm* 1**			***Demand Firm* 2**		
PCO	17.66	18.24	*PCO*	-5.65	-12.14
PPE	-4.73	-4.46	*PPE*	28.22	15.32
ADCO	-6.78	-17.69	*ADCO*	-0.68	-2.13
ADPE	-22.84	-18.43	*ADPE*	-12.86	-16.84
C	-134.29	-7.17	*C*	-178.23	-9.63
DUM	53.81	11.57	*DUM*	26.11	9.61
INC	5.16	6.81	*INC*	4.90	10.34
***AC Firm* 1**					
SUG	1.01	3.46	*SUG*	0.50	3.21
LABOR	24.08	29.35	*LABOR*	14.96	29.38
RAAA	0.07	1.37	*RBAA*	0.04	1.65
Observations	76				
Likelihood	-734.756				

TABLE 9
FIML PARAMETER ESTIMATES OF MODEL 3

Variable	Estimated Coefficient	T-Statistic	Variable	Estimated Coefficient	T-Statistic
Demand Firm 1			***Demand Firm 2***		
PCO	-2.50	-14.11	*PCO*	-2.73	-12.01
PPE	-4.48	-7.94	*PPE*	-3.94	-8.91
ADCO	-2.80	-9.52	*ADCO*	-6.10	-21.17
ADPE	-3.28	-8.75	*ADPE*	-0.54	-1.49
C	38.69	4.72	*C*	31.85	3.95
DUM	18.82	12.81	*DUM*	18.31	12.78
INC	4.12	14.49	*INC*	4.27	15.19
AC Firm 1					
SUG	0.13	0.10	*SUG*	-0.29	-0.44
LABOR	-3.05	-0.94	*LABOR*	1.40	0.74
RAAA	0.90	9.18	*RBAA*	0.51	8.39
Observations	76				
Likelihood	-749.732				

TABLE 10
FIML PARAMETER ESTIMATES OF MODEL 4

Variable	Estimated Coefficient	T-Statistic	Variable	Estimated Coefficient	T-Statistic
Demand Firm 1			***Demand Firm 2***		
PCO	13.43	8.64	*PCO*	15.45	8.31
PPE	7.75	8.88	*PPE*	8.91	9.36
ADCO	-2.07	-11.75	*ADCO*	-1.98	-8.25
ADPE	-2.16	-5.79	*ADPE*	-4.14	-10.50
C	-388.30	-12.05	*C*	-473.12	-12.74
DUM	33.68	13.35	*DUM*	37.43	13.56
INC	9.15	11.86	*INC*	11.57	13.66
AC Firm 1					
SUG	0.42	1.34	*SUG*	0.61	3.35
LABOR	32.57	30.32	*LABOR*	17.49	25.38
RAAA	0.04	1.07	*RBAA*	0.07	2.22
Observations	76				
Likelihood	-761.567				

TABLE 11
FIML PARAMETER ESTIMATES OF MODEL 5

Variable	Estimated Coefficient	T-Statistic	Variable	Estimated Coefficient	T-Statistic
***Demand Firm* 1**			***Demand Firm* 2**		
PCO	8.85	13.60	*PCO*	12.09	12.81
PPE	19.44	12.19	*PPE*	16.22	12.61
ADCO	-9.13	-16.74	*ADCO*	-10.49	-18.12
ADPE	-8.08	-14.78	*ADPE*	-7.83	-14.63
C	-377.99	-15.83	*C*	-416.32	-16.19
DUM	44.28	13.76	*DUM*	45.52	13.75
INC	10.47	15.39	*INC*	11.50	15.99
***AC Firm* 1**					
SUG	0.07	0.25	*SUG*	0.10	0.59
LABOR	28.83	35.08	*LABOR*	16.54	30.13
RAAA	0.005	0.15	*RBAA*	-0.001	-0.06
Observations	76				
Likelihood	-739.130				

TABLE 12
FIML PARAMETER ESTIMATES OF MODEL 6

Variable	Estimated Coefficient	T-Statistic	Variable	Estimated Coefficient	T-Statistic
***Demand Firm* 1**			***Demand Firm* 2**		
PCO	-5.86	-23.81	*PCO*	1.55	6.25
PPE	-0.40	-0.81	*PPE*	-6.59	-16.91
ADCO	2.21	20.06	*ADCO*	-0.63	-3.75
ADPE	0.07	0.26	*ADPE*	2.91	16.27
C	83.50	14.25	*C*	10.43	1.68
DUM	11.62	18.43	*DUM*	6.50	13.92
INC	0.33	2.09	*INC*	1.54	11.98
***AC Firm* 1**					
SUG	-0.02	-0.17	*SUG*	0.08	0.80
LABOR	13.95	22.63	*LABOR*	8.23	15.60
RAAA	0.02	1.37	*RBAA*	0.01	0.95
Observations	76				
Likelihood	-672.648				

TABLE 13

FIML PARAMETER ESTIMATES OF MODEL 7

Variable	Estimated Coefficient	T-Statistic	Variable	Estimated Coefficient	T-Statistic
***Demand Firm* 1**			***Demand Firm* 2**		
PCO	-5.67	-24.63	*PCO*	1.67	5.50
PPE	0.06	0.15	*PPE*	-6.78	-17.00
ADCO	2.14	22.57	*ADCO*	-0.40	-2.14
ADPE	1.59	6.39	*ADPE*	2.85	15.82
C	81.56	15.09	*C*	8.87	1.34
DUM	9.41	17.53	*DUM*	6.21	13.28
INC	-0.18	-1.22	*INC*	1.58	11.57
***AC Firm* 1**					
SUG	-0.05	-0.44	*SUG*	0.07	0.70
LABOR	13.60	22.48	*LABOR*	8.11	15.24
RAAA	0.02	1.51	*RBAA*	0.01	1.11
Observations	76				
Likelihood	-672.651				

TABLE 14

FIML PARAMETER ESTIMATES OF MODEL 8

Variable	Estimated Coefficient	T-Statistic	Variable	Estimated Coefficient	T-Statistic
***Demand Firm* 1**			***Demand Firm* 2**		
PCO	-4.18	-22.30	*PCO*	-10.28	-22.21
PPE	-11.35	-15.71	*PPE*	-6.01	-15.59
ADCO	1.58	17.46	*ADCO*	2.39	23.67
ADPE	2.67	10.62	*ADPE*	2.87	9.97
C	112.92	8.64	*C*	131.40	8.94
DUM	19.46	17.80	*DUM*	22.04	17.89
INC	1.51	5.16	*INC*	1.60	4.86
***AC Firm* 1**					
SUG	-0.47	-0.51	*SUG*	-0.03	-0.39
LABOR	10.21	12.48	*LABOR*	8.00	14.67
RAAA	-0.01	-1.67	*RBAA*	-0.009	-1.76
Observations	76				
Likelihood	-672.685				

TABLE 15
FIML PARAMETER ESTIMATES OF MODEL 9

Variable	Estimated Coefficient	T-Statistic	Variable	Estimated Coefficient	T-Statistic
***Demand Firm* 1**			***Demand Firm* 2**		
PCO	-2.40	-14.68	*PCO*	-6.60	-12.32
PPE	6.35	10.49	*PPE*	18.82	11.85
ADCO	0.95	12.69	*ADCO*	3.46	11.33
ADPE	-3.60	-8.22	*ADPE*	-9.65	-10.76
C	-31.32	-4.95	*C*	-174.78	-8.59
DUM	13.83	13.01	*DUM*	27.24	11.99
INC	2.34	11.93	*INC*	7.27	14.69
***AC Firm* 1**					
SUG	1.64	3.65	*SUG*	0.50	2.73
LABOR	-2.91	-1.23	*LABOR*	16.44	28.56
RAAA	0.07	0.99	*RBAA*	0.05	1.64
Observations	76				
Likelihood	-751.798				

Though it would be interesting to discuss the salient features of these tables as each one of them is associated with a particular economic game, we shall discuss, in a subsequent section, only the tables associated with the models that are most supported by the data according to our model selection procedure presented below.

5. SPECIFICATION AND MODEL SELECTION TESTS

In this section, we use various results from the estimations obtained in the previous section to calculate specification and model selection tests. As our models are generated from games between firms, the outcome of the model selection tests will give us some indications about the price and advertising competitive behavior of the firms.[2]

First, we derive some specification tests of the type proposed by Hausman (1978), for the presence of endogeneity in the general linear model, using OLS and 2SLS parameter estimates. Indeed, one would expect these tests to reveal endogeneity of price and advertising variables since they are decision variables of the firms. Second, as each of the specific models is obtained from the general linear model by imposing a set of nonlinear restrictions on the parameters, following Atkinson (1969, 1970), we perform some likelihood ratio tests of these restrictions, under the assumption that the general linear model is correctly specified. These tests are, in fact, tests of each of the specific nonlinear models versus the general linear model. However, since the general linear model appears to be a quite simplified representation of the competitive behavior of the firms and, indeed, is a simple or rather less restricted version of somewhat more structural models (the specific models derived from the games), it is likely to be misspecified. We attack this question by means of a Hausman model specification test making use of 2SLS and 3SLS parameter estimates.

The next step is to elaborate on a more direct model selection procedure, used to discriminate among the specific models, based on some tests which are applicable whether or not the models are correctly specified. We consider some recently developed tests. We have seen in proposition 13 of chapter 1, that the specific models are nonnested in each other. Thus, we consider model selection tests developed for nonnested models. We subsequently derive tests based on likelihood ratio statistics developed by Vuong (1988).

In order to check for the presence of endogeneity in the demand equations of the general linear model and show, therefore, that OLS is inappropriate, we calculate a Hausman specification test, using the ordinary least squares and the two-stage least squares parameter estimates of this model. This test, based on the chi-square distribution, is performed using:

$$H \equiv (\hat{\theta}^{2SLS} - \hat{\theta}^{OLS})' \cdot \left[Var(\hat{\theta}^{2SLS}) - Var(\hat{\theta}^{OLS}) \right]^{-} \cdot (\hat{\theta}^{2SLS} - \hat{\theta}^{OLS}) \rightarrow \chi_r^2$$

where, $\hat{\theta}^{2SLS}$ and $\hat{\theta}^{OLS}$ represent the 2SLS and OLS estimate of the parameter vector attached to the "endogenous" variables, $Var(\cdot)$ is the covariance matrix, r its rank, and where the notation $\left[Var(\cdot) - Var(\cdot) \right]^-$ designates a generalized inverse of the matrix in the brackets. See Rao and Mitra (1971) and Hausman and Taylor (1981). Thus, in the case where each individual component θ_i (of the coefficient vector) is tested separately, one can use:

$$H_i = \frac{\left[\hat{\theta}_i^{2SLS} - \hat{\theta}_i^{OLS} \right]^2}{Var(\hat{\theta}_i^{2SLS}) - Var(\hat{\theta}_i^{OLS})} \rightarrow \chi_1^2.$$

The results of such a test are displayed in table 16 below.

TABLE 16
ENDOGENEITY TEST USING OLS AND 2SLS ESTIMATES OF THE GENERAL MODEL

Parameter	Test Statistic*	Degrees of Freedom
α_{11}	0.007	1
α_{12}	0.901	1
γ_{11}	0.60	1
γ_{12}	3.73	1
α_{21}	1.21	1
α_{22}	2.55	1
γ_{21}	0.05	1
γ_{22}	0.60	1

* Chi-square statistic.

The results shown in table 16 above, indicate the presence of endogeneity in the demand equations of the linear system. More specifically, at a 10% significance level, we see that the null hypothesis is rejected for the parameters γ_{12} (the coefficient of ADPE in the demand of firm 1), and α_{22} (the coefficient of PPE in the demand of firm 2). Thus, the results of this test justify our considering price and advertising levels of firms as endogenous in defining games in those variables.

Under the assumption of correct specification of the general linear model, we now turn to the testing of each of the specific models. More specifically, we consider tests of the nonlinear restrictions on the parameters of the general model leading to each of the specific models. Table 17 below, displays the results of a likelihood ratio test of these restrictions.

TABLE 17
LIKELIHOOD RATIO TEST OF THE NONLINEAR MODELS

Model Designation	Corresponding Game	Test Statistic*	Degrees of Freedom
Model 1-2	*G1 & G2*	216.41	10
Model 3	*G3*	246.36	10
Model 4	*G4*	270.03	10
Model 5	*G5*	225.15	10
Model 6	*G6*	92.19	10
Model 7	*G7*	92.20	10
Model 8	*G8*	92.26	10
Model 9	*G9*	250.49	10

* Chi-square statistic.

From table 17 above, it appears that all of the nonlinear models are rejected. The results of the tests performed above are, however, contingent upon the assumption of correct specification of the general linear model. We now briefly describe a general model selection procedure which starts by questioning such an assumption. First, we could test the specification of each model separately. For, such a model specification analysis may distinguish those models that adequately explain the data from those that do not. The former will therefore be considered as the best models. However, since each of the structural models is a simplified representation of the competitive behavior of the firms, it is likely to be incorrectly specified. To examine and confirm this rather unfortunate conjecture, we use the property that each of the specific models is nested in the general linear model and test instead the specification of this larger model: Misspecification of the larger model will necessarly imply misspecification of every particular model. Next, given the misspecification of all the particular models, we might seek to determine which behavioral model most closely explains the data.

Instead of using Cox type tests (Cox (1961, 1962), Pesaran and Deaton (1978)), we consider a more direct model selection procedure based on likelihood ratio statistics.

To test the specification of the general linear model, we derive some Hausman-type statistics based on the difference between 2SLS and 3SLS parameter estimates of this model. See Hausman (1978). The next table displays a Hausman statistic for the two demand equations of this model. For each demand equation, the chi-square test based on such a statistic will assess the validity of the overidentifying restrictions which appear in the other equations of the general linear model. The number of degrees of freedom of such a test is not greater than the minimum of the dimension of the parameter vector in the equation and the total number of overidentifying restrictions on the other equations of the system.[3] Note that equations 1 and 2 are both just identified. Thus, each test will actually assess the validity of the overidentifying restrictions imposed on the last four equations which result from the first-order conditions of each of the games under consideration.

TABLE 18
SPECIFICATION TESTS OF THE GENERAL LINEAR MODEL

Equation	Test Statistic*	Maximum Degrees of Freedom
Equation 1	16.88	7
Equation 2	9.70	7

* Chi-square statistic.

The Hausman statistic for the first equation shows that the overidentifying constraints on the last four equations of the system can be globally rejected at least at the 2.5% significance level. Thus, the general linear model, and accordingly each of the behavioral game theoretic models, are misspecified. More pecisely, the preceding test shows that the first-order conditions associated with everyone of our games are at most an approximation to the first-order conditions of the game that actually takes place between the two firms.

The next two tables display the values of the log-likelihood for the various models and likelihood ratio statistics for pairwise model selection based on the Normal (0,1) distribution. See Vuong (1988).

TABLE 19
LOG-LIKELIHOOD OF THE NON-LINEAR MODELS

Model	Log-likelihood
M1–2	-734.756
M3	-749.732
M4	-761.567
M5	-739.130
M6	-672.648
M7	-672.651
M8	-672.685
M9	-751.798

TABLE 20
LIKELIHOOD RATIO STATISTICS FOR MODEL SELECTION

	M1–2	M3	M4	M5	M6	M7	M8	M9
M1–2		106.82	234.61	31.62	-405.44	-427.43	-460.54	105.80
M3	-106.82		75.51	-60.40	-460.38	-573.90	-518.93	12.89
M4	-234.61	-75.51		-151.19	-474.29	-534.79	-625.92	-72.08
M5	-31.62	60.40	151.19		-490.89	-511.96	-497.82	67.30
M6	405.44	460.38	474.29	490.89		0.02	0.28	460.52
M7	427.43	573.90	534.79	511.96	-0.02		0.26	586.82
M8	460.54	518.93	625.92	497.82	-0.28	-0.26		579.60
M9	-105.80	-12.89	72.08	-67.30	-460.52	-586.82	-579.60	

For each pair of models $\{M_i, M_j\}$ we perform a directional test based on the standard normal distribution. More specifically, we calculate a statistic as the (appropriately normalized) difference in the maximum log-likelihood values for the two models. Then, given a critical value c from the standard normal distribution for some significance level, we reject the null hypothesis that both models are equivalent and conclude that M_i is better than M_j if the value of the statistic is greater than c. If the value of the statistic is smaller than -c, we conclude that M_j is better. Finally, in the case where the calculated value of the statistic is between -c and c, the data analyzed do not enable us to discriminate between the two models.[4]

By examining table 21 above, which gives the value of the likelihood ratio statistic for discriminating between any two models, we now perform our model selection procedure. We let $M_i \equiv M_j$ designate the instance where one cannot discriminate between the two models, M_i and M_j, and $M_i > M_j$ ($M_i < M_j$) the case where M_i is better (worse) than M_j. At a 5% significance level, we arrive at the following ordering of the models:

$$M6 \equiv M7 \equiv M8 > M1\text{–}2 > M5 > M3 > M9 > M4$$

Note that we obtain a transitive ordering of our competing models eventhough our analysis is based upon pairwise comparisons.

6. CONCLUSION

The purpose of this chapter was twofold. First, we sought to determine the economic games which were most adequate to describe the structure of a particular market. Second, by taking explicitly into account the nature of the firms' behavior and interaction in this market, our purpose was to obtain more efficient estimates of the effects of the control variables (prices and advertising) on the firms' market demands. Starting from a specification of individual firms' demand and cost functions, we considered six theoretical noncooperative games and derived and estimated (by full information maximum likelihood methods) nine econometric models (three of these models are obtained by merely reversing the role of the firms). Our statistical analysis shows that models $M6$, $M7$, and $M8$ best support the data. These models correspond, respectively, to a Stackelberg game in which firm 1 is the leader, a game in which firm 2 precommits in price, and a game in which firm 2 precommits in advertising.

Instead of relying on a model selection based on a purely statistical analysis, we could have invoked theoretical economic reasoning. Indeed, economic theory strongly tells us that own price must have a negative effect while prices of close substitutes must have a positive effect on the firm's demand. According to the first criterion, models $M\,1$–2, $M\,4$, $M\,5$, and $M\,9$ would be rejected (see tables 8, 10, 11, and 15). According to the second economic criterion, models $M\,1$–2, $M\,3$, $M\,8$, and $M\,9$ would be rejected (see tables 8, 9, 14, and 15). Therefore, models $M\,1$–2, $M\,3$, $M\,4$, $M\,5$, $M\,8$, and $M\,9$ would be rejected as non-satisfactory on the basis of either of the two above economic criteria. It is quite satisfying to observe that model selection procedures based on statistical analysis and on economic reasoning lead to similar conclusions, namely, models $M\,6$ and $M\,7$ are undistinguishable and are the most appropriate models.

Models $M\,6$ and $M\,7$ postulate quite different economic structures. Model $M\,6$ states that firm 1 (The Coca Cola Company) is the Stackelberg leader in both prices and advertising. On the other hand, model $M\,7$ says that firm 2 (Pepsico, Inc.) is the leader in prices only. Except for the significance of the (positive) effect of advertising of Pepsico, Inc., on the demand of The Coca Cola Company that is found in model $M\,7$, both models produce similar results in terms of magnitude and signs of coefficient estimates (see tables 12 and 13). All the significant coefficients in the demand and cost specifications have the theoretical expected signs. Namely, the effects of own price and own advertising on each firm's demand are significant and negative and positive, respectively.[5] Furthermore, in both models the effects of the price charged by The Coca Cola Company and its advertising expenditure have a significant positive and negative effect, respectively, on the demand of Pepsico, Inc..

Our analysis also reveals some asymmetry between the two firms. In particular, both retained models show an insignificant effect of the price charged by Pepsico, Inc., on the demand of The Coca Cola Company whereas, as noted above, the reverse effect is significantly positive. In addition, while the effect of advertising of The Coca Cola Company on the demand for the product of Pepsico, Inc., is significantly negative, the reverse effect, namely, that of advertising of Pepsico, Inc., on the demand for the product of The Coca Cola Company, is insignificant in model $M6$ and positively significant in model $M7$. This apparently strong asymmetry in the effects of advertising and prices suggests the presence of a substantial brand loyalty for the product Coca Cola which has been extensively discussed in the traditional literature on the soft drink industry (see, e.g., Mongoven (1975), Katz (1976, 1979), and Uri (1986)).

This study has contributed to the applied industrial organization literature in many ways. In particular, it has proposed a new methodology that combines game theoretic considerations and recently developed econometric tools for the analysis of micro data. Our empirical results have indeed demonstrated how fruitful such a combination is. Even though our findings are satisfactory, they should, nevertheless, be regarded with care. First, as our specification analysis has demonstrated, even our best models are only approximations to the actual behavior of the firms in the particular market studied. Second, our analysis assumes that the same structural game is used over the whole analyzed period. Clearly, a promising avenue for further research would be to allow for different games across time. Moreover, one can generalize the framework developed in this paper to include collusive behavior along the lines of Porter (1983), Green and Porter (1984), and Lee and Porter (1984).

NOTES

1. A study by Clarke (1976) showed that the cumulative effect of advertising on sales lasts for 3 to 9 months.

2. From a theoretical point of view, the issue of choosing among different forms of oligopolistic competition has been examined to some extent in Dowrick (1986). Empirical work on the subject is found in Koutsoyiannis (1984), and Roberts (1984).

3. Hausman tests frequently involve generalized inverses. See Hausman and Taylor (1981). Constructions of quadratic forms based on g-inverses are known to present some theoretical difficulties. See Andrews (1987) and Vuong (1987).

4. When comparing models M_f and M_g, the variance statistic used to normalize the likelihood ratio is, in our case, given by:

$$\hat{\omega}_n^2 = (1/4n)\sum_{i=1}^{n}(\hat{u}_{fi}'\hat{\Sigma}_f^{-1}\hat{u}_{fi} - \hat{u}_{gi}'\hat{\Sigma}_g^{-1}\hat{u}_{gi})^2$$

where, $\hat{u}$ are the estimated residuals, and $\hat{\Sigma}$ is the estimated covariance matrix. See Vuong (1988).

5. Note that own-price effect in the case of Pepsico, Inc. would have been incorrect had we used a limited information estimation method (2SLS) or even a full information estimation method (3SLS and FIML) of the unconstrained general linear model (see tables 5-7 in the text). Thus, our analysis illustrates the usefulness of determining the appropriate game and imposing its structure on the econometric modelling so as to obtain the correct signs and parameter estimates.

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