THE BOUNDARY LAYER AND SKIN FRICTION FOR A BODY OF REVOLUTION AT LARGE REYNOLDS' NUMBERS

Thesis by

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Summary:

Clark Millikan's work on the boundary layer and skin friction for a figure of revolution is extended, with particular reference to the completely turbulent boundary layer, in two ways:

a) His expressions for the completely turbulent regime are generalised so as to hold for the assumption of a one n-th power law for the velocity distribution in the boundary layer;

b) von Karman's logarithmic velocity distribution is introduced into the analysis.

b) leads to a practical method by which the drag of a full-scale dirigible can be predicted from wind tunnel tests.

Comparison of the theory as gotten from b) with experiments leads to the conclusion that the present theory can be safely used to predict drags at large Reynolds' numbers, whereas drags predicted on the basis of a one seventh power law may be from 20% to 30% low.

Introduction:

After experiments had shown that the power law theory of von Kármán[1] and Prandtl[2], for the skin friction on a pointed flat plate in a two-dimensional flow parallel to the surface and perpendicular to the leading edge, could be used satisfactorily in determining the skin friction at the curved surface of a streamline body in a two-dimensional flow when the radius of curvature is large compared to the boundary layer thickness, Jones[3] and Dryden and Kuethe[4] extended the theory so as to give an account of the skin friction acting on a dirigible model in an axial flow. In both these discussions the dirigible was replaced by an "equivalent flat plate".

Although Boltze[5] had given the equations for the laminar boundary layer for a body with axial symmetry about an axis in the direction of flow, and Levi-Civita[6] had attacked the general problem of the turbulent boundary layer for a body of any shape, it remained for Millikan[7] to give a satisfactory analysis applicable to dirigible models.

Millikan derived the approximate form of the boundary layer equations in the neighborhood of the surface of a figure of revolution from the Navier-Stokes and continuity equations for the steady motion of a viscous incompressible fluid, letting the kinematic viscosity go to zero and neglecting powers of the square root of the kinematic viscosity higher than the first. The equations are valid when the boundary layer thickness is small compared to the distance from the axis. For the turbulent boundary layer, Millikan assumed the one-seventh power law for the velocity profile. However, von Kármán has shown[8), 9)] that the power law is merely an interpolation of or approximation to the Logarithmic Law; and experiments, especially those of Nikuradse[10)], show that the 1/7-th power does indeed become a 1/8-th, then a 1/9-th, etc., power, as the Reynolds' Number increases.

Both Millikan[7)] and von Kármán[11)] have pointed out the desirability, where the analysis is to be used to predict drag coefficients at large Reynolds' Numbers, of extending Millikan's work by the introduction of the Logarithmic Law; this then is the object of this paper, as is indicated by the title.

Notation:

Our notation follows very closely that employed by Millikan[7] and by von Kármán in his Hamburg paper[12]:--

x = distance along the surface measured from stagnation point.

y = distance normal to the surface.

a = axial distance from nose of the airship.

L = length of airship.

r_o = radius of the airship surface.

C_r = longitudinal radius of curvature of airship surface.

δ = boundary layer thickness.

U = velocity just outside the boundary layer.

U_o = axial velocity of the flow far from the airship and relative to it.

ρ = mass density.

ν = kinematic viscosity.

τ_o = shearing stress at the surface.

u = velocity in the boundary layer.

$\beta = \sqrt{1 - r_o'^2} = \cos\alpha$, where $r_o' = dr_o/dx$ and α gives the inclination of the boundary to the axis.

$V_r = \sqrt{\frac{2\tau_o}{\rho}}$ = the "friction velocity".

K = von Kármán's universal constant of the turbulent exchange.

$R = U_o L/\nu$, $R_V = (U_o V^{1/3})/\nu$.

V = volume of the airship.

Preliminary:

Millikan used

$$u^* = 8.74\ y^{*1/7} \tag{1.0}$$

$(u^* = u/\sqrt{\tau_0/\rho}\ ,\ y^* = \dfrac{y}{\nu/\sqrt{\tau_0/\rho}}\ )$

which for

$$u/U = (y/\delta)^{1/7} \tag{1.1}$$

gives

$$\frac{\tau_0}{\rho} = 0.0225\ U^2\left(\frac{\nu}{U\delta}\right)^{\frac{1}{4}} . \tag{1.2}$$

von Kármán showed that (1.0) was only an approximation to or interpolation of

$$u^* = \text{constant} + (1/\kappa)\log_e y^* ,$$

which holds near the surface, except in the laminar sublayer, where

$$u^* = y^*.$$

As a first step we generalise Millikan's results for the completely turbulent boundary layer to hold for a one n-th power law. This was originally done to get an estimate of the drag coefficient of the N. P. L. Long Model at large Reynolds' Numbers, and thus determine whether or not it was worth while to go through the analysis using a logarithmic velocity distribution.

Suppose we have

$$u^* = a + b\log_{10} y^* \quad (b = 2.3/\kappa)$$

and wish to approximate to it by

$$u^* = B_n\ y^{*1/n},$$

then we can eliminate u* and y* and find that

$$B_n = (bn/2.3)\, e^{\left[(2.3\,a\,/\,bn) - 1\right]} .$$

Nikuradse[13) concludes from his experiments that

$$u^* = 5.5 + 5.75 \log_{10} y^* ,$$

i.e., $a = 5.5$ and $b = 5.75$. If we use these values, we find that $B_7 = 8.81$. But, for consistency in comparing with Millikan's results, we would like to have $B_7 = 8.74$.

If we impose, then, the further condition that $B_7 = 8.74$, it is easy to get a relation between a and b:

$$a = b\left[3.043 - 7 \log_{10}(b/2.872)\right] .$$

We now try to find a pair of values a, b near a value of b corresponding to $\kappa \doteq 0.40$, and such that Nikuradse's points are closely approximated. It is found that $a = 5.26$ and $b = 5.80$ ($\kappa = 0.397$) are satisfactory, and these indeed seem to fit Nikuradse's points even better than the values which he selected.

Thus, we use

$$\boxed{u^* = 5.26 + 5.80 \log_{10} y^*} \qquad (2)$$

as our "Base" curve. For the values of a and b which we use in (2),

$$B_n = 2.52\, n\, e^{\left[(2.086/n) - 1\right]} .$$

Taking

$$u/U = (y/\delta)^{1/n} \qquad (4.0)$$

and eliminating u from this and from

$$u^* = B_n \, y_*^{1/n} \, , \tag{4.1}$$

we get the well-known expression for the shearing stress - density ratio:

$$\frac{\tau_0}{\rho} = B_n^{-2n/(n+1)} \, U^2 \left(\frac{\nu}{U\delta}\right)^{2/(n+1)} \, . \tag{4.2}$$

The equations (4.0) and (4.2) will be used in the following section, in place of the equations (1.0) and (1.2) used by Millikan.

Generalisation to the one n-th Power Law:

We start from Millikan's equation (13) governing the boundary layer thickness, which equation is his extension of the two-dimensional von Kármán integral relation:

$$\left.\begin{aligned} &\frac{d}{dx}\left[U^2\delta\int_0^1\left(\frac{u}{U}\right)^2 d\left(\frac{y}{\delta}\right)\right] - U\frac{d}{dx}\left[U\delta\int_0^1\left(\frac{u}{U}\right) d\left(\frac{y}{\delta}\right)\right] \\ &\quad + \frac{r_0'}{r_0}U^2\delta\left[\int_0^1\left(\frac{u}{U}\right)^2 d\left(\frac{y}{\delta}\right) - \int_0^1\left(\frac{u}{U}\right) d\left(\frac{y}{\delta}\right)\right] = UU'\delta - \frac{\tau_0}{\rho}. \end{aligned}\right\} \quad (5)$$

Letting $\frac{u}{U} = \left(y/\delta\right)^{1/n}$, then $\int_0^1\left(\frac{u}{U}\right) d\left(\frac{y}{\delta}\right) = \frac{n}{n+1}$

and $\int_0^1\left(\frac{u}{U}\right)^2 d\left(\frac{y}{\delta}\right) = \frac{n}{n+2}$,

and the equation becomes:

$$\frac{n}{n+2}\frac{d}{dx}(U^2\delta) - \frac{n}{n+1}U\frac{d}{dx}(U\delta) + \frac{r_0'}{r_0}U^2\delta\left[\frac{n}{n+2} - \frac{n}{n+1}\right] = UU'\delta - \frac{\tau_0}{\rho}.$$

Carrying out the differentiations, rearranging, and substituting

$$\tau_0/\rho = B_n^{-\frac{2n}{n+1}}\,U^2\left(\frac{\nu}{U\delta}\right)^{\frac{2}{n+1}}$$

we get

$$n\,\delta^{\frac{2}{n+1}}\frac{d\delta}{dx} + \left[(3n+2)\frac{U'}{U}\delta + n\frac{r_0'}{r_0}\delta\right]\delta^{2/n+1} = (n+1)(n+2)\,B_n^{-\frac{2n}{n+1}}\left(\frac{\nu}{U}\right)^{\frac{2}{n+1}}$$

which can be put in the form

$$\frac{d}{dx}\left(\delta^{\frac{n+3}{n+1}}\right) + \left[\frac{(n+3)(3n+2)}{n(n+1)}\frac{U'}{U} + \frac{n+3}{n+1}\frac{r_0'}{r_0}\right]\delta^{\frac{n+3}{n+1}} = C_n\left(\frac{\nu}{U}\right)^{\frac{2}{n+1}}$$

where

$$C_n = \frac{(n+2)(n+3)}{n}B_n^{-\frac{2n}{n+1}}.$$

Introducing the obvious integrating factor,

$$U^{\frac{(n+3)(3n+2)}{n(n+1)}} \cdot r_0^{\frac{n+3}{n+1}},$$

the equation becomes

$$\frac{d}{dx}\left[U^{\frac{(n+3)(3n+2)}{n(n+1)}} \cdot r_0^{\frac{n+3}{n+1}} \cdot \delta^{\frac{n+3}{n+1}}\right] = C_n\, U^{\frac{(n+3)(3n+2)}{n(n+1)}} \cdot r_0^{\frac{n+3}{n+1}} \cdot \left(\frac{\nu}{U}\right)^{\frac{2}{n+1}},$$

which can now be integrated. We are interested only in the "completely turbulent regime," therefore the limits of x are taken as 0 and x. Thus:

$$\delta^{\frac{n+3}{n+1}} = \frac{C_n\, \nu^{\frac{2}{n+1}}}{U^{\frac{(n+3)(3n+2)}{n(n+1)}} \cdot r_0^{\frac{n+3}{n+1}}} \cdot \int_0^x U^{\left[\frac{(n+3)(3n+2)}{n(n+1)} - \frac{2}{n+1}\right]} \cdot r_0^{\frac{n+3}{n+1}}\, dx .$$

Now, we introduce the fact that $x = a/\beta$, and introduce also the Reynolds' Number, $R = U_0 L/\nu$, with the result that we find:

$$\delta/L = \frac{C_n^{\frac{n+1}{n+3}}}{R^{\frac{2}{n+3}}} \cdot N_n(a/L),$$

where

$$N_n(a/L) = \frac{1}{\left(\frac{U}{U_0}\right)^{\frac{3n+2}{n}} \cdot \frac{r_0}{L}} \left[\int_0^{a/L} \frac{\left(\frac{U}{U_0}\right)^{\frac{3(n+2)}{n}} \cdot \left(\frac{r_0}{L}\right)^{\frac{n+3}{n+1}}}{\beta}\, d\left(\frac{a}{L}\right)\right]^{\frac{n+1}{n+3}} \qquad (6)$$

This is Millikan's equation (19), generalised from $n = 7$ to $n = n$, but specialised to the completely turbulent regime. (Our δ here is, of course, Millikan's δ_t).

Let D_t (= $\text{drag}_{\text{turbulent}}$) be the skin friction due to the turbulent boundary layer, then

$$D_t = 2\pi\rho\int_0^L r_0 \frac{\tau_0}{\rho}\, da .$$

Substituting for τ_0/ρ, this becomes

$$D_t = 2\pi\rho\, B_n^{-\frac{2n}{n+1}} \int_0^L r_0 U^2 \left(\frac{\nu}{U\delta}\right)^{\frac{2}{n+1}} da .$$

Now, defining $\quad C_V = \dfrac{D_t}{\frac{1}{2}\rho U_0^2 V^{2/3}}$,

and making use of the fact that from (6),

$$\delta^{\frac{2}{n+1}} = \frac{L^{\frac{2}{n+1}} C_n^{\frac{2}{n+3}} \left[N_n(a/L)\right]^{\frac{2}{n+1}}}{R^{\frac{4}{(n+3)(n+1)}}}$$

we get

$$\boxed{C_V = \frac{H_n}{R^{\frac{2}{n+3}}} \cdot \frac{L^2}{V^{2/3}} \cdot \int_0^1 \frac{\left(\frac{U}{U_0}\right)^{\frac{2n}{n+1}} \cdot \frac{r_0}{L}}{\left[N_n(a/L)\right]^{\frac{2}{n+1}}} \cdot d(a/L)} \qquad (7)$$

where $H_n = 4\pi B_n^{-\frac{2n}{n+1}} C_n^{-\frac{2}{n+3}}$

and N_n is as given in (6).

This corresponds to Millikan's equation (20).

The coefficients appearing in the expressions (4.1), (4.2), (6), and (7) are tabulated below for the usual values of n:--

n	B_n	$B_n^{-\frac{2n}{n+1}}$	$C_n^{\frac{n+1}{n+3}}$	H_n
7	8.74	0.0225	0.3708	0.3623
8	9.63	.0178	.3163	.2889
9	10.52	.0145	.2752	.2359
10	11.42	.0119	.2406	.1937

The Logarithmic Analysis:

The drag coefficient for the N. P. L. Long Model was calculated by equation (7) for several values of n, and it was found that at Reynolds' Numbers corresponding to full scale, quite considerable differences in the drag coefficient were obtained for the various powers. A comparison of the C_v versus R_v curves for the N. P. L. Long Model may be seen in Figure 1, where, among others, the curves for $n = 7$ and $n = 10$ are plotted. It thus appeared that repitition of the analysis, using the logarithmic velocity distribution, would indeed be desirable.

We therefore assume that the logarithmic velocity distribution, as developed by von Kármán for the flat plate, holds also for the boundary layer on the axially symmetric body.

We remember that von Kármán showed that

$$u^* = B_n y^{*1/n}$$

was merely an interpolation formula holding in a limited range, and which approximated to

$$u^* = \text{constant} + (1/\kappa)\log_e y^*.$$

The corresponding velocity profile,

$$u = U\,(y/\delta)^{1/n}$$

is then replaced by

$$u = U - (1/\kappa\sqrt{2})\, V_r \left\{\log_e(\delta/y) + \log C_2 - h(y/\delta)\right.$$

where

$$h(1) = \log C_2.$$

In our analysis we will be integrating u and u(U - u) over y, and it can be shown that for our purpose it suffices to take

$$\boxed{u \doteq U - \frac{V_r}{\kappa\sqrt{2}} \log_e(\delta/y).} \tag{8}$$

If this velocity distribution be introduced directly into Millikan's equation (13) (our equation (5)), then after a rather lengthy process, equation (9) in the following paragraph is obtained.

However, if we go back to Millikan's basic equation (10),

$$\frac{d}{dx}\int_0^\delta u^2 dy - U\frac{d}{dx}\int_0^\delta u\,dy + \frac{r_0'}{r_0}\left[\int_0^\delta u^2 dy - U\int_0^\delta u\,dy\right] = UU'\delta - \frac{\tau_0}{\rho},$$

we find that it may be written in a more convenient form:

$$\boxed{\frac{\tau_0}{\rho} = UU'\delta + \frac{1}{r_0}\frac{d}{dx}\left\{r_0\left[\int_0^\delta u(U-u)\,dy\right]\right\} - U'\int_0^\delta u\,dy.}$$

Now, since

$$\int_0^\delta \left(\log\frac{\delta}{y}\right)^2 dy = 2\int_0^\delta \log\frac{\delta}{y}\,dy = 2\delta,$$

we have

$$\int_0^\delta u(U-u)\,dy = \delta\left[\frac{UV_r}{\kappa\sqrt{2}} - \frac{V_r^2}{\kappa^2}\right]$$

and

$$\int_0^\delta u\,dy = \delta\left[U - \frac{V_r}{\kappa\sqrt{2}}\right],$$

whence the following relation comes out immediately:

$$\boxed{\frac{\tau_0}{\rho} \equiv \frac{1}{2}V_r^2 = \frac{1}{r_0}\frac{d}{dx}\left\{r_0\delta\left[\frac{UV_r}{\kappa\sqrt{2}} - \frac{V_r^2}{\kappa^2}\right]\right\} + \frac{V_r\delta}{\kappa\sqrt{2}}\frac{dU}{dx}.} \tag{9}$$

We have, in addition to equation (8), the relation

$$U = \frac{V_r}{\kappa\sqrt{2}} \left\{ \log \frac{V_r \delta}{\nu} + \log C_2 \right\}.$$

(cf. von Kármán[12], equation (44)).

It is convenient to write this in the form

$$V_r = \frac{\nu}{C_2 \delta} e^{\kappa\sqrt{2}\frac{U}{V_r}}, \tag{10}$$

and to introduce the parameter

$$\boxed{z = \kappa\sqrt{2}\,\frac{U}{V_r}} \tag{11}$$

whence

$$V_r = \frac{\nu}{C_2 \delta} e^{z}$$

and

$$V_r = \kappa\sqrt{2}\,\frac{U}{z}.$$

Introducing the parameter z into equation (9),

we get:

$$\kappa^2 \frac{U^2}{z^2} = \frac{1}{r_0}\frac{d}{dx}\left\{ r_0 \frac{U\nu}{\kappa\sqrt{2}\,C_2} e^{z} - \frac{r_0}{\kappa}\frac{\sqrt{2}}{z} U\nu \frac{e^{z}}{C_2} \right\} + \frac{\nu e^{z}}{C_2 \kappa\sqrt{2}} \frac{dU}{dx}$$

or,

$$\frac{\sqrt{2}\,C_2 \kappa^3}{\nu} U^2 = \frac{z^2}{r_0}\frac{d}{dx}\left\{ r_0 U e^{z}\left(1 - \frac{2}{z}\right)\right\} + z^2 e^{z} \frac{dU}{dx}.$$

After carrying out the differentiation, this last can be put in the form:

$$\frac{\sqrt{2}\,C_2 \kappa^3}{\nu} U = (z^2 - 2z - 2) e^{z} \frac{dz}{dx} + z^2 e^{z}\left[\left(2\frac{U'}{U} + \frac{r_0'}{r_0}\right) - \frac{2}{z}\left(\frac{U'}{U} + \frac{r_0'}{r_0}\right)\right].$$

We now integrate and get:

$$\frac{\sqrt{2}\,C_2\,\kappa^3}{\nu}\int_0^x U\,dx = (z^2-4z+6)e^z - 6 + \int_0^x z e^z\left[z\left(2\frac{U'}{U}+\frac{r_0'}{r_0}\right)-2\left(\frac{U'}{U}+\frac{r_0'}{r_0}\right)\right]dx\,.$$

The term -6 comes from an assumption that as a first approximation, $z = 0$ at $x = 0$. This is true only for the special case of the flat plate. More will be said on this point later.

We now introduce the axial length L and the length Reynolds' Number $R = U_0 L/\nu$, and $a = x\beta$, with the result that the above equation can be put in the following dimensionless form:

$$\sqrt{2}\,C_2\,\kappa^3\cdot R\cdot\int_0^{a/L}\frac{U/U_0}{\beta}\,d\left(\frac{a}{L}\right) = (z^2-4z+6)e^z - 6$$

$$+\int_0^{a/L} z e^z\left\{z\,\frac{\frac{d}{d(a/L)}\left(\frac{U^2}{U_0^2}\frac{r_0}{L}\right)}{\frac{U^2}{U_0^2}\frac{r_0}{L}} - 2\,\frac{\frac{d}{d(a/L)}\left(\frac{U}{U_0}\frac{r_0}{L}\right)}{\frac{U}{U_0}\frac{r_0}{L}}\right\}d\left(\frac{a}{L}\right).$$

For simplicity, we write

$$\boxed{\gamma(z) = (z^2 - 4z + 6)e^z - 6} \qquad (12)$$

and

$$\boxed{\xi(\ ) = \frac{\frac{d}{d(a/L)}(\ )}{(\ )}} \qquad (13)$$

so that our final expression is:

$$\boxed{\gamma(z) = C_2\,\kappa^3\sqrt{2}\cdot R\cdot\int_0^{a/L}\frac{U/U_0}{\beta}\,d\left(\frac{a}{L}\right) + \int_0^{a/L} z e^z\left[2\xi\left(\frac{U}{U_0}\frac{r_0}{L}\right) - z\,\xi\left(\frac{U^2}{U_0^2}\frac{r_0}{L}\right)\right]d\left(\frac{a}{L}\right)} \qquad (14)$$

If we can solve (14) for z , then from (10) and (11) it is easily shown that the shearing stress at the surface, and the boundary layer thickness, are given by the following:

$$\frac{\tau_0}{\rho} = \kappa^2 \frac{U^2}{z^2} \tag{15}$$

and

$$\frac{\delta}{L} = \frac{1}{C_2 \kappa \sqrt{2}} \cdot \frac{z e^z}{\left(\frac{U}{U_0}\right) \cdot R} . \tag{16}$$

Similarly, the turbulent drag coefficient is found to be:

$$C_V = 4\pi \kappa^2 \cdot \frac{L^2}{V^{2/3}} \cdot \int_0^1 \frac{\left(\frac{U}{U_0}\right)^2 \cdot \frac{r_0}{L}}{z^2} d(x/L) . \tag{17}$$

As to the two constants, κ and C_2, appearing in the above expressions, we have already selected $\kappa = 0.397$, but have as yet said nothing about C_2.

For evaluation of C_2, we digress briefly at this point and consider the special case of the flat plate. von Kármán showed that, approximately,

$$\frac{\kappa\sqrt{2}}{\sqrt{c_f}} = \log_e (R_x c_f) - \log_e \left(\frac{\sqrt{2}}{\kappa C_2}\right) \tag{18.0}$$

where

$$c_f = \frac{\tau_0}{\frac{1}{2}\rho U^2} = \text{local friction coefficient,}$$

and $R_x = U_0 x/\nu$.

(Note: $U = U_0$ for flat plate).

Kempf's measurements[14] of the local friction on smooth flat plates were used to obtain a value for C_2, and for $\kappa = 0.397$, it was found that $\underline{C_2 = 10.84}$ gave good agreement with the experimental results. These values of κ and C_2 give

$$1/\sqrt{c_f} = 4.09 \log_{10}(R_x c_f) + 1.97 \tag{18.1}$$

which compares favorably with the expressions given by von Kármán and by Schoenherr.[15] The question of the flat plate will be discussed further in a later section thereon.

Returning to the axially symmetric body: with $\kappa = 0.397$ and $C_2 = 10.84$, our equations become:

$$\boxed{\begin{aligned}
&\gamma(z) = 0.959\, R \int_0^{a/L} \frac{U/U_0}{\beta}\, d(a/L) + \int_0^{a/L} z e^z \left[2\,\xi\!\left(\frac{U}{U_0}\frac{r_0}{L}\right) - z\,\xi\!\left(\frac{U^2}{U_0^2}\frac{r_0}{L}\right)\right] d(a/L) \\
&\left(\gamma(z) = (z^2 - 4z + 6)e^z - 6, \quad \xi(\;) = \frac{d/d(a/L)\,(\;)}{(\;)} \right) \\
&\frac{\tau_0}{\rho} = 0.158\, \frac{U^2}{z^2} \\
&\frac{\delta}{L} = 0.164\, \frac{z e^z}{\left(\frac{U}{U_0}\right) \cdot R} \\
&C_V = 1.98\, \frac{L^2}{V^{2/3}} \int_0^1 \frac{\left(\frac{U}{U_0}\right)^2 \cdot \frac{r_0}{L}}{z^2}\, d(a/L)
\end{aligned}} \tag{19}$$

Equation (14) is treated by simple iteration, i.e.,

$$\gamma(z_1) = 0.959 \cdot R \cdot \int_0^{a/L} \frac{U/U_0}{\beta}\, d(a/L),$$

$$\gamma(z_2) = \gamma(z_1) + \int_0^{a/L} z_1 e^{z_1} \left[2\xi\left(\frac{U r_0}{U_0 L}\right) - z_1 \xi\left(\frac{U^2 r_0}{U_0^2 L}\right) \right] d(a/L),$$

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.
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etc.

r_0/L and β as functions of a/L are known from the shape of the dirigible, and U/U_0 as a function of a/L is obtainable from wind tunnel tests. The integrals are most conveniently handled by means of a Coradi Integraph. It is found that as far as the drag coefficient (with which we are mainly concerned) it is unnecessary to go beyond $\gamma(z_2)$.

To get z from $\gamma(z)$, a table was made up, giving $\gamma(z)$ for values of z between 7.0 and 20.0, in steps of 0.1. This table allowed rapid solution of the transcendental equation, with a maximum variation in z of $\pm$ 0.05, which accuracy could be improved with judicious interpolation.

We saw that the term -6 in

$$\gamma(z) = (z^2 - 4z + 6)e^z - 6$$

was an approximation, exact only if $z = 0$ at $x = 0$. However, at large Reynolds' Numbers, this is practically the case. The term -6 is the constant of integration, and depends upon the thickness of the boundary layer at the stagnation point. But, to calculate the thickness of the boundary layer it is necessary to know the integration constant. An estimate of the stagnation δ, δ_s, can be gotten by considering the behavior of the power law differential equation for δ near the singular point $r_o = 0$, assuming that in this region the potential flow is that which would occur near the stagnation point of a sphere, of radius equal to the radius of curvature of the nose of the dirigible, in a uniform flow U_o, in a manner analogous to that used by Millikan in his treatment of the laminar δ_s. The analysis is not of sufficient interest to be reproduced here. We say only that our calculation of δ from z as gotten from equation (14) does not hold close to the stagnation point (nor, of course, near the tail), but that the larger the Reynolds' Number, the more confidence we can have in our calculated values of δ near the nose.

In the following two sections are given results of calculations for two models. It might be pointed out here that, from the standpoint of time and labor, the calculation of C_v from equations (19) for various Reynolds' Numbers is easier than that necessary if one were to get an approximation to the curve of C_v versus R_v by calculating for several different power laws from equation (7).

The N. P. L. Long Model:

Our first applications of the relations developed above are made on the so-called N. P. L. Long Model, a 1/225 size replica of the "H.M.A.-R 33." The results are presented in Figure 1. The curve of C_v versus R_v as obtained from z_2 is marked "present theory", and may be seen to lie well above Millikan's curve, marked "$n = 7$". In fact, if we consider $R_v = 10^8$ to be an upper limit for the R_v of this particular ship, and assume that the logarithmic analysis gives us the correct value of the drag coefficient, then the one-seventh power law gives a value 22.4% low at this R_v.

The curve for the one-tenth power law (marked "$n = 10$") and the curve calculated by Dryden and Kuethe (marked "D. & K.") are shown for comparison.

The circles in Figure 1 are experimental points obtained in the N. A. C. A. Variable Density Wind Tunnel[16), and it is interesting to note that these points cross over and above both Dryden and Kuethe's and Millikan's curves, and follow the curve for the present theory, with the exception of the one point at the highest Reynolds' Number. This last point is marked with a question mark in the figure, and was neglected by Higgins[16) himself in plotting up his results. It is inconceivable that this point may be explained by a roughness effect. In this connection, the crosses in the figure represent experimental points in the same wind tunnel on the N. P. L. Short Model, a similar model, and it is seen that these points follow those of the Long Model, and follow the curve for the present theory, but without an

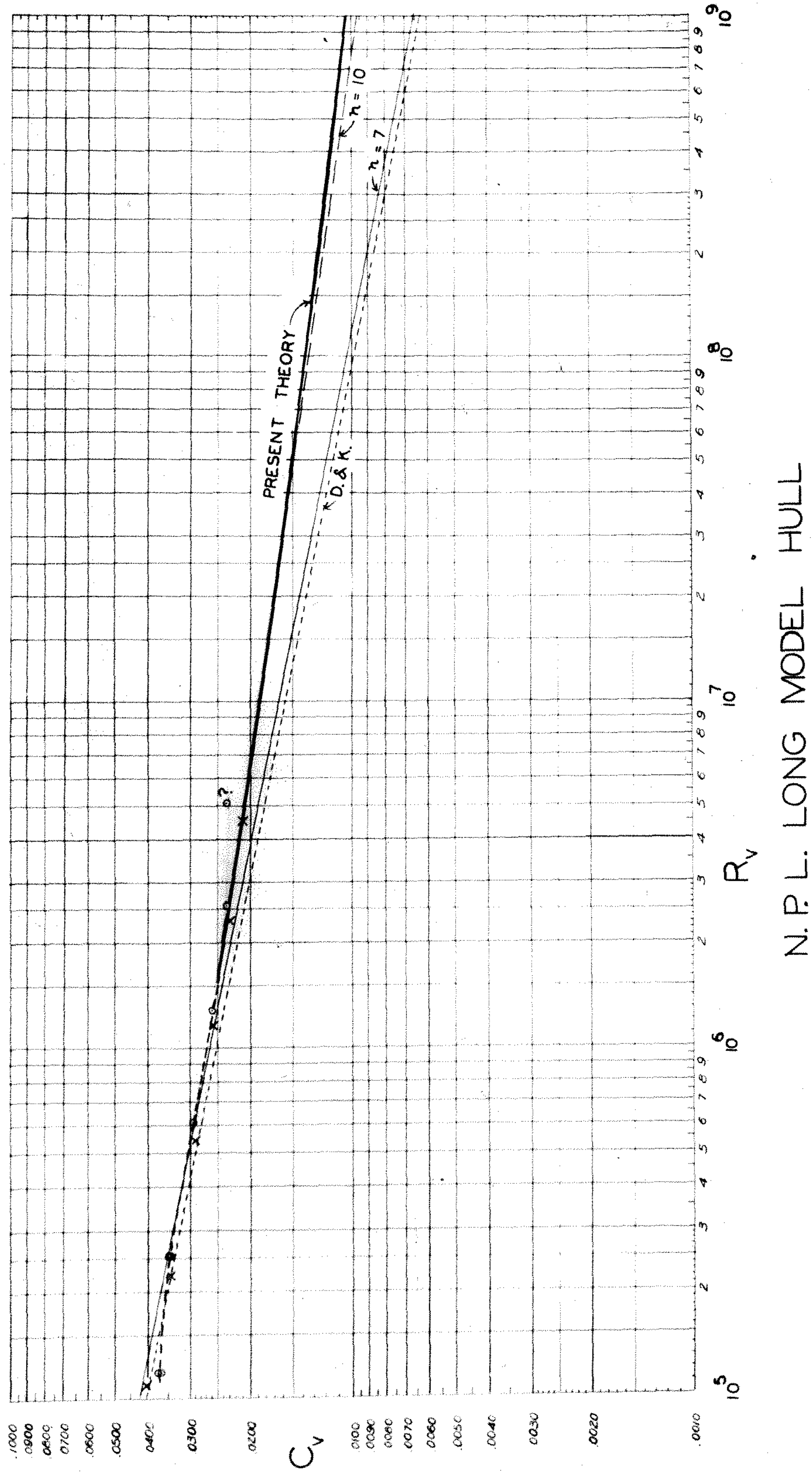

N. P. L. LONG MODEL HULL
PRESENT THEORY
D. & K.
n = 10
n = 7
R_V
C_V
.1000
.0900
.0800
.0700
.0600
.0500
.0400
.0300
.0200
.0100
.0090
.0080
.0070
.0060
.0050
.0040
.0030
.0020
.0010
10^5
10^6
10^7
10^8
10^9

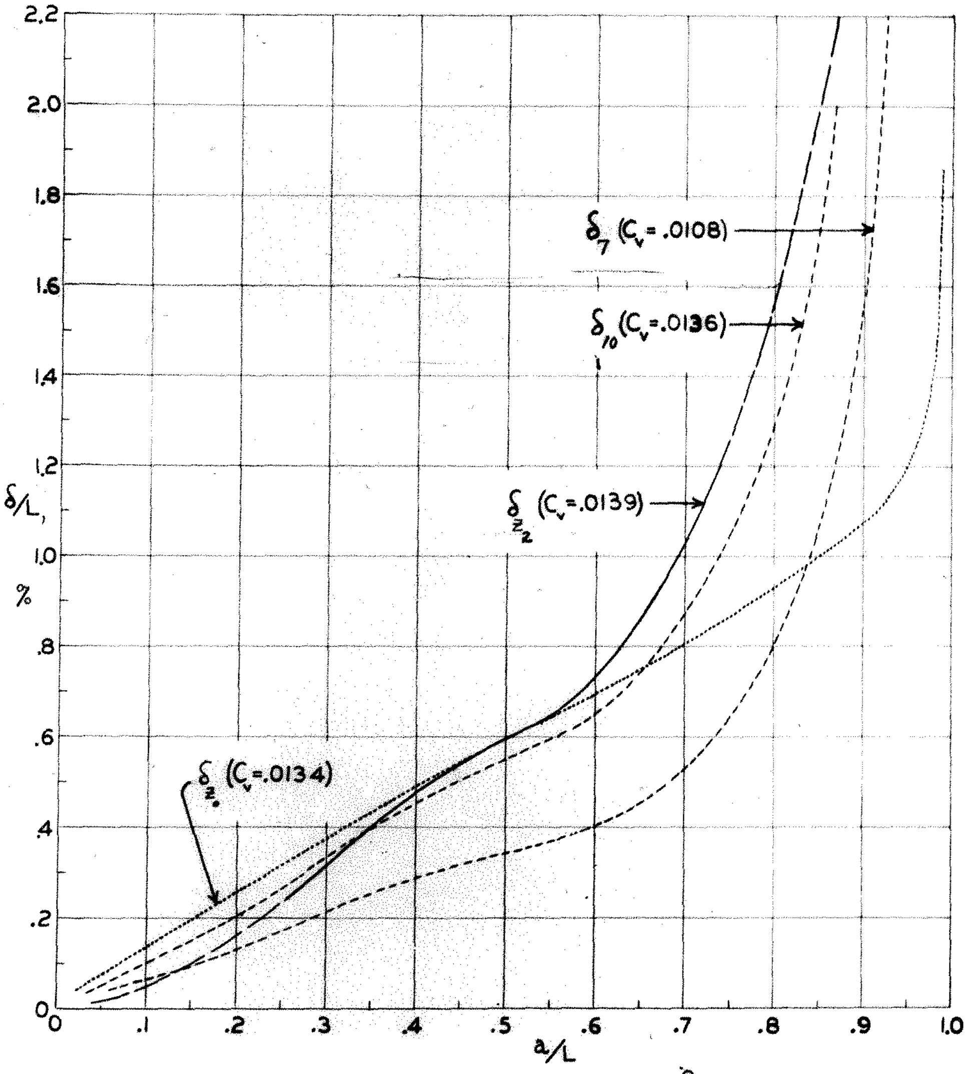
Fig. 1

upward slope at the last point. The general agreement of the present theory with the experimental points leads one to conclude that the logarithmic analysis gives a quite accurate method for predicting full-scale drag coefficients from wind tunnel tests.

In Figure 2 are plotted boundary layer thickness in percent axial length for the N. P. L. Long Model, calculated under various assumptions. An examination of equations (16) and (17) will show that the boundary layer thickness is much more susceptible to variations in z than is the drag coefficient; this fact is brought out in Figure 2. We have plotted here, for $R_v = 10^8$, the thicknesses based on the one-seventh and on the one-tenth power laws, and also that obtained by the second approximation in the logarithmic analysis, δ_{z_2}. The δ_{z_1} is not plotted, but is not much different from δ_{z_2} except near the nose and near the tail. Plotted also is what we will call δ_{z_0}, gotten from a very rough approximation to z, namely, from

$$z^2 e^z = 0.959 \int_0^{a/L} \frac{U/U_0}{\beta} \, d\left(\frac{a}{L}\right).$$

It is seen from the figure that, while this very rough approximation gives a drag coefficient not very different from the second approximation, it gives an entirely erroneous picture of the developement of the boundary layer along the body. Thus, if one is interested mainly in calculating the



N.P.L. LONG MODEL, R_v=10^8

Fig. 2

boundary layer thickness, one must be careful to get a good approximation to z . However, the drag coefficient is the main point of interest, and here even a rough approximation to z gives a good approximation to C_v. In our calculations, we have carried out the approximation to z_2 in getting points for the curves marked "present theory" in Figures 1 and 3. The corrections change the boundary layer thicknesses from the first approximations hardly at all in the middle half of the body, but decrease them somewhat near the nose and increase them at the tail.

It is interesting to note in Figure 2 the decided decrease in boundary layer thickness as obtained from z_2, as the nose is approached. In fact, the thickness approaches the order of thickness of the laminar thickness. From this we might infer that sudden jumps in boundary layer thickness in transition from laminar to turbulent, as postulated by Dryden and Kuethe and by Millikan, are not to be expected.

The Akron Model:

A very interesting application of the theory is afforded us by the Akron. The drag coefficient curve as obtained by the present theory is shown for this ship in Figure3, and is compared with Millikan's curve. Experimental points for several models are also plotted: those[17] from the propeller research tunnel (P.R.T.) and the variable density tunnel (V.D.T.), and from the Guggenheim Aeronautics Laboratory of the California Institute of Technology (G.A.L.C.I.T.) tunnel[18]. The general agreement of the trend of the points with the present theory is quite satisfactory, notably instanced in the case of the points for the wooden model in the variable density tunnel.

For the full-scale Akron with $U_o = 80$ miles per hour, $R = 5.7 \times 10^8$, $R_v = 1.4 \times 10^8$, and for this R_v, we see that the curve based on the one seventh-power law gives $C_v = 0.0090$, whereas the curve based on the logarithmic law gives $C_v = 0.0127$, i.e., 29.1% low.

There is sufficient data available for the Akron to allow us to make a rough estimate of the full-scale equivalent parasite area* and thus check the accuracy of our theory. For the full-scale ship, $V^{2/3} = 3530$ square meters, so that our curve gives, for the full-scale Akron at 80 miles per hour, an equivalent parasite area for the hull of 44.8 square meters, whereas Millikan's curve gives only 31.8 square

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* equivalent parasite area, $f = D/\frac{1}{2}\rho\, U_o^2 = C_D V^{2/3}$.

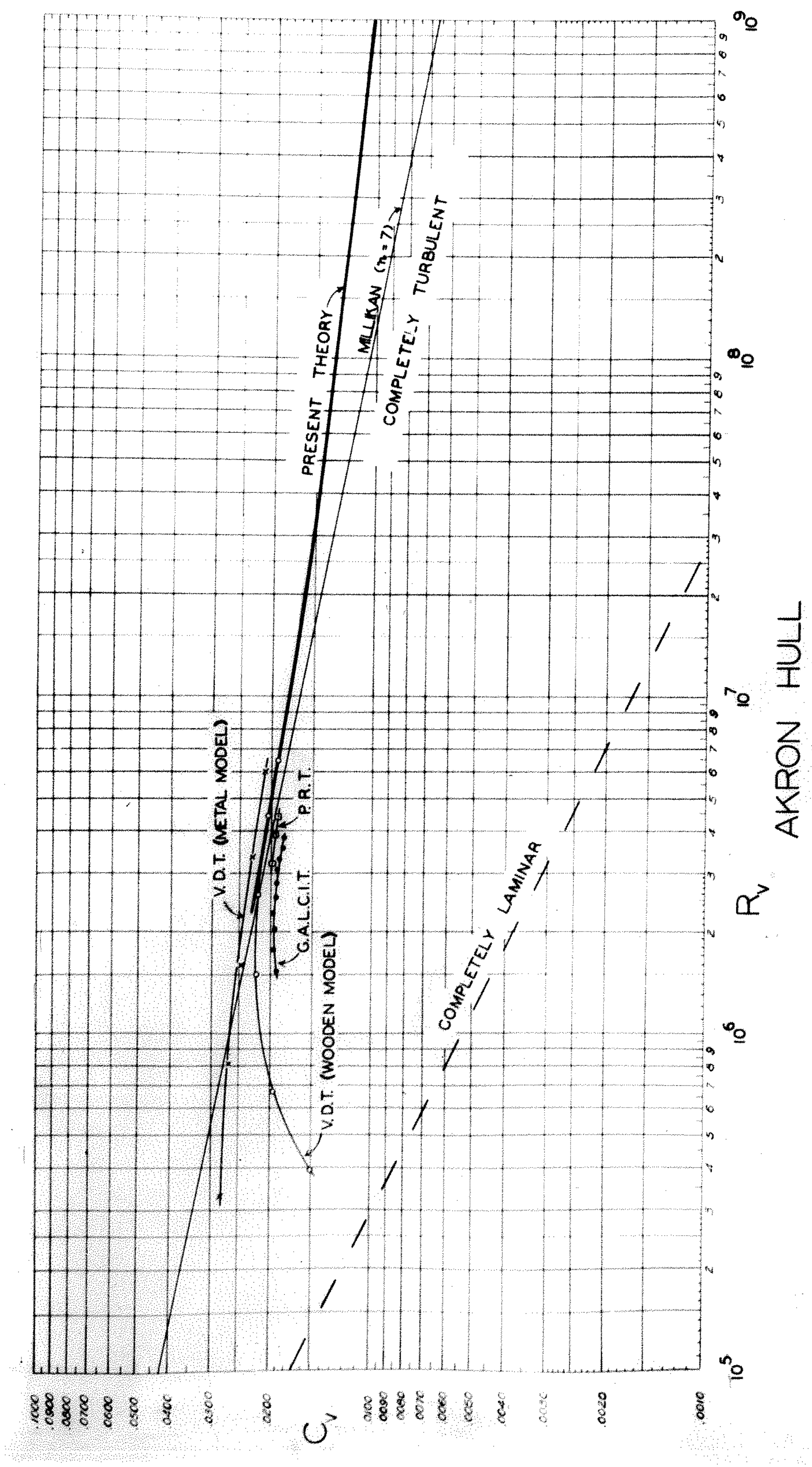
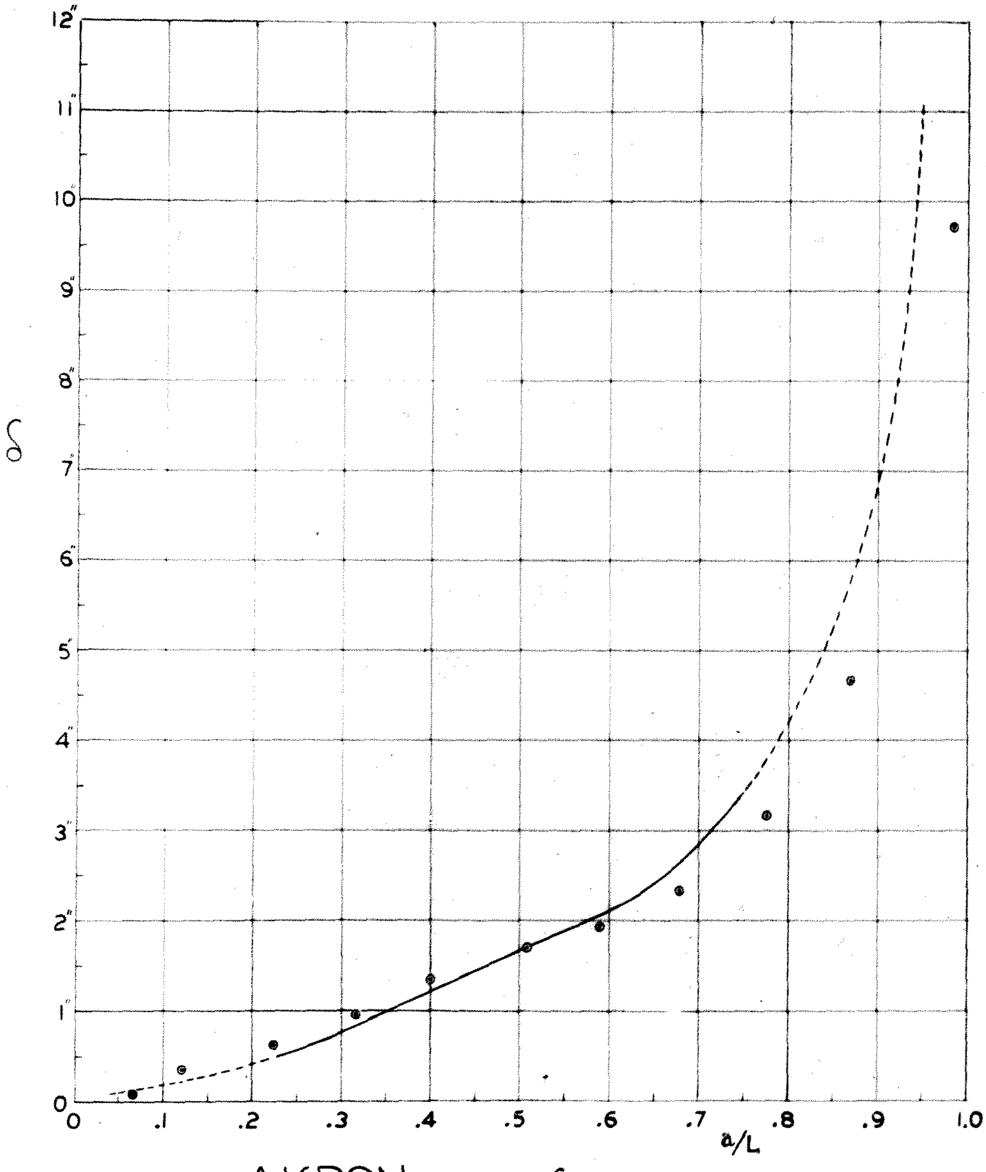

AKRON HULL
R_V
C_V
PRESENT THEORY
MILLIKAN (n=7)
COMPLETELY TURBULENT
COMPLETELY LAMINAR
V.D.T. (METAL MODEL)
V.D.T. (WOODEN MODEL)
G.A.L.C.I.T.
P.R.T.
10^5
10^6
10^7
10^8
10^9

Fig. 3

meters. From G.A.L.C.I.T. wind tunnel tests[18], the increment in drag coefficient added by ten hoods, control car with bumping ball, and bumping ball on the lower tail surface, is 0.0014, and is constant with Reynolds' Number. This gives a parasite of 4.9 square meters. From the same tests, the increment added by the tail surfaces, corrected to full scale (tail surfaces considered as flat plates), is about 0.0012, giving a parasite area of 4.2 square meters. Other GA.L.C.I.T. tests[19] give a full-scale parasite area of 20.3 square meters for the eight Akron propulsive units with no propellers, and 29.7 square meters for the same with stationary propellers.

From the above data, an estimate of the full-scale parasite area of the Akron gives between 75 and 84 square meters, based on our theory, whereas an estimate based on the seventh-power law gives only between 61 and 70 square meters. It is known that the actual equivalent parasite area for the Akron was about 80 square meters, as determined from deceleration tests. We thus have a very nice check on our theory. (The author is obtaining more accurate and further data from the Goodyear-Zeppelin Co., in order to make a closer check, but at the time of writing, this material is not available).

In Figure 4 is plotted the boundary layer thickness in inches for the Akron model tested by Freeman[17] at $R_v \doteq 3.8 \times 10^6$, as calculated by our theory from z_2, and compared with Freeman's measured thicknesses. The point shown nearest the nose is a laminar point (Freeman's tests were in the transition range) and does not concern us. The agreement with the other points is fairly good, but not as good as the



AKRON, $R_v = 3.8 \times 10^6$

Fig. 4

agreement found by Freeman between his points and thicknesses calculated from the one-seventh power law. This fact is, however, not at all disconcerting, since one can really only speak of the boundary layer thickness in terms of order of magnitude. The crux of the matter lies in the velocity distribution curve. This was calculated from equation (8) for Freeman's cases, and the agreement was found to be perfect. There is considerable latitude allowable in the choice of the boundary layer thickness from the measured velocity distribution curve, and in fact, Freeman's measurements could be made to fit our calculated boundary layer thicknesses just as well as he made them fit the one-seventh power calculations.

(Note: the curves in Figures 1 and 3 are run out to the exceedingly high R_V's of 10^9 because it is felt that future airships will have larger full-scale Reynolds' Numbers than at present, due to probable increase in both size and speed; the larger the R_V, the more pronounced the difference between the present logarithmic theory and the one seventh-power theory.)

The Flat Plate:

For the special case of the flat plate, $U = U_0$, $r_0 = \infty$, $r_0' = 0$, and our equation (14) reduces to

$$C_2\sqrt{2}\,\kappa^3 R_x = 0.959\,R_x = (z^2-4z+6)e^z - 6 = \gamma(z), \tag{20}$$

where $R_x = \dfrac{Ux}{\nu}$.

This was originally obtained by von Kármán. However, he then approximated to this by

$$C_2\sqrt{2}\,\kappa^3 R_x \doteq z^2 e^z$$

from which comes his equation,

$$\frac{\kappa\sqrt{2}}{\sqrt{c_f}} = \log_e(R_x c_f) - \log_e\left(\frac{\sqrt{2}}{\kappa C_2}\right).$$

(cf. equations (18.0), (18.1)).

We, however, having already constructed a table for $\gamma(z)$, can use this table to get more refined values of u, δ, and c_f for the flat plate. The velocity distribution is of course as in equation (8), and

$$c_f = \frac{0.316}{z^2}$$

and

$$\delta = 0.164\,\frac{x}{R_x}\,z e^z.$$

We find that the refinement does not affect the local friction coefficient to any extent, except slightly at low Reynolds' Numbers. The exact boundary layer thicknesses are all higher than those calculated by von Kármán's approximation, the difference of course being more pronounced at the lower Reynolds' Numbers. δ's calculated by the z-method for the experiments of Kempf[14)] were all rather considerably larger than the values announced by Kempf-- however, here again, the crux of the matter lies in the velocity profile. Logarithmic velocity profiles calculated by the z-method fitted Kempf's points beautifully. (Dr. Kempf kindly furnished Professor von Kármán with his original data). Here again the selection from the measured points of a definite distance as a δ is quite elastic.

Thus the z-method used in this paper is not so valuable for flat plates, as von Kármán's approximation is sufficient. The main value is in its application to the prediction of drag of full-scale airships from wind tunnel tests.

Acknowledgement:

The author takes this opportunity to express his indebtedness to Professor von Kármán, whose notable developements of the theory of skin friction inspired the present application of it.

References:

1) Th. von Kármán, "Über laminare und turbulente Reibung," Z.A.M.M., vol. 1 (1921), no. 4, p. 233. (Reprinted in Abhandlungen aus dem Aerodynamischen Institut an der Technischen Hochschule Aachen, Heft 1).

2) L. Prandtl, Ergebnisse der aerodynamischen Versuchsanstalt zu Göttingen, III Lieferung, p. 1; Oldenbourg (1927).

3) B. M. Jones, "Skin Friction and the Drag of Streamline Bodies," R. & M. No. 1199 (1928).

4) H. L. Dryden and A. M. Kuethe, "Effect of Turbulence in Wind Tunnel Measurements," N.A.C.A. Rep. No. 342 (1930).

5) E. Boltze, "Grenzschichten an Rotationskörpern in Flüssigkeiten mit kleiner Reibung," Dissertation, Göttingen, 1908.

6) T. Levi-Civita, "Allgemeine Folgerungen aus der Prandlschen Grenzschichttheorie," Vorträge aus dem Geb. der Aerod. und verw. Geb., Aachen (1929), Springer (1930).

7) Clark B. Millikan, "The Boundary Layer and Skin Friction for a Figure of Revolution," Trans. A.S.M.E., App. Mech. Section, (APM-54-3), pp. 29 - 43, Jan. 30, 1932.

8) Th. von Kármán, "Mechanische Ähnlichkeit und Turbulenz," Göttingen Nachrichten, Jan. 21, 1930.

9) Th. von Kármán, "Mechanische Ähnlichkeit und Turbulenz," Proc. III Intern. Cong. Appl. Mech., Stockholm, 1930.

10) J. Nikuradse, "Über turbulente Wasserströmungen in geraden Rohren bei sehr grossen Reynoldsschen Zahlen," Vorträge aus dem Geb. der Aerod. und verw. Geb., Aachen (1929), Springer (1930).

11) Th. von Kármán, "Turbulence and Skin Friction," Journal of the Aeronautical Sciences, Vol. 1, No. 1, pp. 1 - 20, (Jan., 1934).

12) Th. von Kármán, "Theorie des Reibungswiderstandes," Hydromechanische Probleme des Schiffsantriebs, pp. 50 - 73 (Hamburg, 1932).

13) J. Nikuradse, "Gesetzmässigkeiten der turbulenten Strömung in glatten Rohren," Forschungsheft 356 (1932).

14) G. Kempf, "Weitere Reibungsergebnisse an ebenen glatten und rauhen Flächen," Hydromechanische Probleme des Schiffsantriebs, pp. 74 - 82 (Hamburg, 1932).

15) K. E. Schoenherr, "Resistance of Flat Surfaces Moving Through a Fluid," paper read before the Soc. Nav. Arch. and Marine Eng., Nov., 1932.

16) G. J. Higgins, "Tests of the N. P. L. Airship Models in the Variable Density Wind Tunnel," N. A. C. A. Tech. Note No. 264 (1927).

17) H. B. Freeman, "Measurements of Flow in the Boundary Layer of a 1/40-scale Model of the U. S. Airship 'Akron'", N. A. C. A. Rep. No. 430 (1932).

18) R. Seiferth, "Wind Tunnel Tests on a 1/75 Scale Model of the Goodyear-Zeppelin Airship Akron Z.R.S.4 with Normal and Ring Tail Surfaces," G.A.L.C.I.T. Rep. No. 105. (Preprinted by the A.S.M.E. for presentation at the Pacific Coast Aeronautic Meeting, Berkeley, June 1932).

19) A. L. Klein and C. B. Millikan, "Wind Tunnel Tests at Comparatively Large Scale on Various Propulsive Units for the Macon Dirigible," (unpublished).