Boundaries of Smooth Sets and Singular Sets of Blaschke Products in the Little Bloch Class

Thesis by

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Abstract

A subset of R is called smooth if the integral of its characteristic function is smooth in the sense defined by Zygmund. It is shown that such a set is either trivial or its boundary has Hausdorff dimension 1. Sets are constructed here which are as close to smooth as one likes but whose boundaries do not have dimension 1.

It was conjectured by T. Wolff that if B is Blaschke product in the Little Bloch class, its zeroes accumulate to a set of dimension 1. This conjecture is proven here.

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I. *Statement of Two Theorems.*

In this paper I prove two theorems, the first about sets which I will call "smooth," and the second about the singularities of Blaschke products in the Little Bloch class.

First I need a few definitions.

Zygmund [8, pg. 43] defined a function F as uniformly almost smooth on an interval I (or in class Λ_*) if F is continuous and satisfies

$$(1) \qquad \lim_{\epsilon \to 0} \sup\left\{\frac{F(x+h)+F(x-h)-2F(x)}{h} : 0 < h < \epsilon \text{ and } x+h \in I \text{ and } x-h \in I\right\} = M < \infty.$$

I will denote the limit on the left hand side of (1) by S(F). If S(F) = 0, Zygmund called such a function uniformly smooth, or in class λ_*. In this paper the words "smooth" and "almost smooth" should be understood to mean "uniformly smooth" and "uniformly almost smooth."

I will call a measure μ on I smooth or almost smooth if its integral,

$$F(x) = \mu[0,x]$$

is smooth or almost smooth. Let $s(\mu) = S(F)$. If $J \subseteq I$, bisect J into two intervals, denoted by J^+, J^-, as shown in Figure 1.

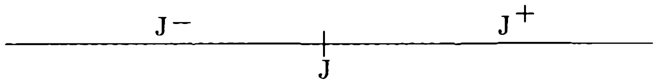


Figure 1.

Then if μ has a continuous integral,

$$(2) \qquad s(\mu) = \lim_{\epsilon \to 0} \sup_{\substack{J \subseteq I \\ 0 < |J| < \epsilon}} 2 \frac{|\mu(J^+) - \mu(J^-)|}{|J|} .$$

For any measurable set A, define the measure μ_A by

$$\mu_A(E) = |A \cap E|,$$

where $|\cdot|$ denotes Lebesgue measure.

Trivially, from (2), $s(\mu_A) \leq 1$.

Define A to be a smooth set if μ_A is a smooth measure.

I am going to prove in section II the following theorem, conjectured by T. Wolff.

__Theorem__ __1__. If $A \subseteq [0, 1]$ and A is a smooth set, then either $|A| = 1$ or $|A| = 0$ or the boundary of A has Hausdorff dimension equal to 1.

The following well-known fact is suggestive of Theorem 1.

Fact 1: If μ is a positive almost smooth measure on [0, 1], and E is a set such that $\mu(E) > 0$, then E has Hausdorff dimension equal to 1.

I will prove the fact from the following lemma, which will also prove useful later.

__Lemma__ __1__. Let μ be a positive measure on [0, 1] and assume there exist constants C,

α such that for all intervals $I \subseteq [0, 1]$, $\mu(I) \leq C|I|^{\alpha}$. Then if $E \subseteq [0, 1]$ and $\mu(E) > 0$, then E has Hausdorff dimension at least α.

Proof. Let $\{I_j\}$ be a covering of E by intervals. Then $0 < \frac{\mu(E)}{C} \leq \frac{\Sigma\, \mu(I_j)}{C} \leq \Sigma|I_j|^{\alpha}$. Therefore E has Hausdorff dimension $\geq \alpha$. □

From a theorem on the modulus of continuity of smooth functions (see [8, pg. 44]), one can deduce for any almost smooth μ that

$$\mu(I) = 0\left(|I| \log \frac{1}{|I|}\right) \text{ as } |I| \to 0 \quad . \tag{3}$$

From Lemma 1 and (3) the above fact follows. I will now show how to deduce (3) directly.

Let $M_1 = \mu\,[0,1]$ and let

$$M_2 = \sup_{J \subseteq [0,1]} \frac{2}{|J|}\, |\mu(J^+) - \mu(J^-)|.$$

Let $M_3 = \max\,(2M_1, M_2)$.

For $1 \leq i \leq 2^n$ let $K_i^n = [(i-1)\, 2^{-n}, i\, 2^{-n}]$.

It's easy to see by induction that

$$\frac{\mu(K_i^n)}{|K_i^n|} \leq \frac{(n+1)}{2} M_3 \quad . \tag{4}$$

Let I be any interval contained in [0,1] and let n be the smallest integer such that I contains some K_i^n. Then there are four intervals,

$$K_j, K_{j+1}^n, K_{j+2}^n, K_{j+3}^n, \text{ which cover I.}$$

$$\frac{\mu(I)}{|I|} \le \frac{1}{|K_j^n|} \sum_{k=0}^{3} \mu(K_{j+k}^n) \le 4 \frac{(n+1)M_3}{2} .$$

Noting $n + 1 = 0 (\log \frac{1}{|I|})$, (3) is proven. □

Fact 1 and Theorem 1 suggest the following question:

Does there exist $\in \ge 0$ such that if $s(\mu_A) \le \in$, then $|A| = 1$ or $|A| = 0$, or the boundary of A has Hausdorff dimension 1?

I will provide specific examples in section III which will show that the answer is "No." I will also show in section III that smooth sets exist, using a construction due to Kahane [4]. His construction was originally used to give an example of a smooth positive singular measure.

The second theorem I prove in this paper, as I mentioned, regards Blaschke products in the Little Bloch class. A function f, analytic on the unit disk $D = \{|z| < 1\}$, is said to be in the Little Bloch class (denoted by B_0) if it satisfies

$$\lim_{r \to 1} \sup_{0 \le \theta < 2\pi} (1 - r) |f'(re^{i\theta})| = 0.$$

The Little Bloch class is a subset of the Bloch class, B, of analytic functions satisfying

$$\sup_{z \in D} (1 - |z|)\,|f'(z)| < \infty .$$

The norm usually associated with the Bloch class is

$$||f||_* = |f(0)| + \sup_{z \in D} (1 - |z|)\,|f'(z)| \;.$$

Under this norm B_0 is the closure of the space of all polynomials.

Functions in the Bloch class and almost smooth measures arise in the context of extending the theory of H^p spaces to domains other than the unit disk. If Ω is a domain and $\varphi : D \to \Omega$ is conformal, we can define $E^p(\Omega)$ as the space of analytic functions f such that

$$\sup_{r<1} \int_{\Gamma_r} |f(z)|^p \; | \, d\, z \, | < \infty,$$

where Γ_r is the image under φ of the circle of radius r.

If Ω is bounded by a rectifiable Jordan curve C and $f \in E^p(\Omega)$, f has nontangential limits at almost every point of C, and the boundary function lies in $L^p(C)$. $\varphi'(z)$ is the form

$$\varphi'(z) = S(z)\; F(z),$$

where S is a singular inner function and F is an outer function in H^1. If $S(z) \equiv 1$, Ω is said to be of Smirnov type.

It is known (see [2, pg. 173]) that a domain is of Smirnov type if and only if for $1 \leq p < \infty$, every $f \in E^p(D)$ has boundary values lying in the $L^p(C)$ closure of the polynomials.

The question arose as to whether all domains bounded by rectifiable Jordan curves are Smirnov. Keldysh and Lavrentiev [5] were able to show, using a complicated geometric construction, that domains not of Smirnov type exist. Duren, Shapiro, and Shields [3] showed that finding a domain not of Smirnov type is equivalent to finding an almost smooth positive singular measure μ, and that the singular factor S(z) appearing above is of the form

$$S(z) = \exp\left(-a \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \, d\mu(t)\right)$$

where $a < 0$. Using techniques like those used earlier, one can show that such an S must lie in the Bloch class.

In section V, I will prove the following theorem about Blaschke products in B_0:

Theorem 2. Let B be an infinite Blaschke product in B_0, with zeroes $\{z_n\}$. Let S be the singularity set of B on the boundary of D. (That is, the set of accumulation points of $\{z_n\}$). Then S has Hausdorff dimension 1.

Remark: Using Frostman's Lemma, this result can be extended to inner functions which are not constants or finite Blaschke products.

At first glance it would seem that Theorems 1 and 2 are unrelated. I wish to demonstrate that the boundary behavior of a Blaschke product in B_0 shares some similarities with the behavior of the characteristic function of a smooth set.

A theorem due to Zygmund (see [2, pg. 75]) states:

Theorem A.

Let F(z) be analytic on $D = \{|z| < 1\}$. Then $F''(z) = 0\left(\frac{1}{1-|z|}\right)$ if and only if F can be extended continuously to $\{|z| \leq 1\}$ and $g(\theta) = F(e^{i\theta})$ lies in Λ_*. $F''(z) = o\left(\frac{1}{1-|z|}\right)$ if and only if F can be extended continuously to $|z| \leq 1$ and $g(\theta)$ is smooth. Thus, if f is in B_0, any primitive of f, say F, satisfies the hypothesis of Theorem A. Suppose, in addition, that f is bounded on the unit disk. Then it is well known that $\lim_{r \to 1} f(re^{i\theta})$ exists for almost all θ, and that f is the Poission integral of its boundary values. Furthermore,

$$\lim_{r \to 1} \int_0^{2\pi} |f(re^{i\theta}) - f(e^{i\theta})| \, d\theta = 0 \ . \tag{5}$$

I now claim that $d\lambda(\theta) = f(e^{i\theta})\, d\theta$ is a smooth measure under these conditions. As above, let F be a primitive of f, $g(\theta) = F(e^{i\theta})$ lies in λ_*. $g(\theta) = g(0) + \int_\gamma f(z)\, dz$, where γ is the path shown in Figure 2, consisting of two radial segments of length δ and one circular arc concentric with the unit circle.

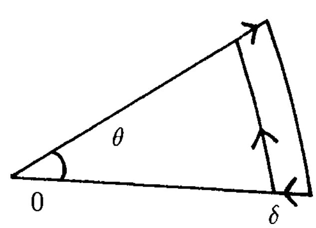


Figure 2.

As $\delta \to 0$, then integral over the radial segments goes to zero (f is bounded) and the integral over the circular arc tends to $g(\theta) - g(0) = \int_0^\theta f(e^{it})\, ie^{it}\, dt$ because $f(re^{it})$ converges in $L^1(d\theta)$ norm to $f(e^{it})$. $g(\theta)$ is smooth (being the boundary values of F) so

$$d\mu(\theta) = f(e^{i\theta})\, e^{i\theta}\, d\theta$$

is a smooth measure. We want to show $e^{-i\theta}d\mu(\theta)$ is smooth.

$$\frac{1}{h}\left|\int_{x_0-h}^{x_0} e^{-i\theta}d\mu(\theta) - \int_{x_0}^{x_0+h} e^{-i\theta}d\mu(\theta)\right| \le$$

$$\frac{1}{h}\int_{x_0-h}^{x_0} |e^{-i\theta}-e^{-ix_0}|d\mu(\theta) + \frac{1}{h}\int_{x_0}^{x_0+h} |e^{-i\theta}-e^{-ix_0}|d\mu(\theta) +$$

$$\frac{1}{h}\left|\int_{x_0-h}^{x_0} e^{-ix_0}d\mu(\theta) - \int_{x_0}^{x_0+h} e^{-ix_0}d\mu(t)\right| .$$

As $h \to 0$ the first two terms vanish uniformly in x_0 because $e^{i\theta}$ is Lipschitz and $d\mu$ is an absolutely continuous measure. The third term vanishes uniformly by the uniform smoothness of $d\mu(\theta)$. Hence the integral of the boundary values of f is a smooth function, as is the integral of the characteristic function of a smooth set.

II. *Proof of Theorem 1*.

Lemma 2. Let C, $\in$ be fixed constants, $0 < \in < C < 1$. Let $\{\mathcal{G}_j\}_{j=0}^{\infty}$ be collections of pairwise disjoint intervals with the following properties:

a) $\mathcal{G}_0$ contains at least one interval of non-zero length.

b) If $I \in \mathcal{G}_{j+1}$ then $\exists\ J \in \mathcal{G}_j$ such that $I \subseteq J$, and $|I| \leq \in |J|$.

c) If $J \in \mathcal{G}_j$, then

$$\sum_{\substack{I \in \mathcal{G}_{j+1} \\ I \subseteq J}} |I| \geq C\,|J| \ .$$

Let $E_j = \cup \{I \mid I \in \mathcal{G}_j\}$ and let $E = \cap E_j$. Then E has Hausdorff dimension of at least

$$1 - \frac{\log C}{\log \in} \ .$$

Remark: Let $\alpha\,(C, \in) = 1 - \dfrac{\log C}{\log \in}$.

Then $\alpha(C, \in)$ increases to 1 as C increases to 1 with $\in$ fixed, and

$\alpha(C, \in)$ increases to 1 as $\in$ decreases to 0 with C fixed.

Remark 2: When I prove Theorem 1, I will only need the special case $\in = 1/2$. However, to prove Theorem 2, I will need the general power of Lemma 2.

The proof proceeds by creating a sequence of measures, μ_j, on (0, 1), and then finding the limit, in some sense, of the μ_j's, which I will call μ. Next it is shown that $\mu(E) = 1$ and that for all intervals $J \subseteq [0, 1]$, $\mu\,(J) \leq A\,|J|^{\alpha}$, where A is a constant depending on C and $\in$, and $\alpha = \alpha(C, \in)$. Once this can be done, Lemma 2 follows immediately from Lemma 1.

We can assume without loss of generality that $\mathcal{G}_0 = \{[0, 1]\}$.

Let $E_j = \bigcup_{I \in \mathcal{G}_j} I$.

Let μ_0 = Lebesgue measure on [0, 1].

The μ_j's will be defined inductively. Let $D_j(I)$ denote, for any interval I, $\frac{\mu_j(I)}{|I|}$.

Suppose μ_{j-1} has already been defined. Define μ_j as that positive measure such that

(i) If $I \in \mathcal{G}_j$, $D_j(I) = \frac{D_{j-1}(J)}{\sum_{\substack{K \in \mathcal{G}_j \\ K \subseteq I}} |K|}$ and

(ii) For all measurable subsets $W \subseteq [0,1]$,

$$\mu_j(W) = \sum_{I \in \mathcal{G}_j} D_j(I) \, |W \cap I| \, .$$

In other words, μ_j is that measure whose restriction to any $I \in \mathcal{G}_j$ is simply a constant multiple of Lebesgue measure, the constant being determined by (i). Furthermore $\mu_j(E_j^c) = 0$. One can see that for all j and all $I \in \mathcal{G}_j$, $D_j(I) \leq C^{-j}$. This follows from (i), (c), and induction.

Observe from (b) that $|I| \leq \in^j$. Also, by construction, for all m,

(6) $$\mu_j(I) = \mu_{j+m}(I).$$

I next take the "limit" of the μ_j's, which I will call μ.

Let $F_k : [0,1] \to R$ by $F_k(x) = \mu_k([0, x])$.

If $x \in E_k^c$, $|F_{k+1}(x) - F_k(x)| = 0$. This follows from (6) applied to each of the intervals of $\mathcal{G}_k$ lying to the left of x.

Suppose $I \in \mathcal{G}_k$ has endpoints a and b, and $x \in I$. Both $F_{k+1}(x)$ and $F_k(x)$ lie between $F_k(a)$ and $F_k(b)$. Therefore,

$$|F_{k+1}(x) - F_k(x)| \le \mu_k(I) = D_k(I)\ |I| \le \left(\frac{\epsilon}{C}\right)^k.$$

Since $\sum_{k=0}^{\infty}(\epsilon/C)^k$ converges, it follows easily that the F_k's converge uniformly to some continuous monotonic function, F, with $F(0) = 0$ and $F(1) = 1$. Let μ be the Borel measure such that $\mu[0, x] = F(x)$. It is clear that for any interval I, $\mu(I) = \lim_{j\to\infty} \mu_j(I)$, and if $I \in \mathcal{G}_j$, $\mu(I) = \mu_j(I)$. Therefore $\mu(E_j) = \mu_j(E_j) = 1$. Since $E_{j+1} \subseteq E_j$ for all j, $1 = \lim_{j\to\infty} \mu(E_j) = \mu(\bigcap_j E_j) = \mu(E)$ (here using a basic property of finite measures). Our proof will be complete if we can show that for all intervals $K \subseteq [0, 1]$,

(7) $$\mu(K) \le 3\ C\ |K|^\alpha,\ \alpha = 1 - \frac{\log C}{\log \epsilon}\ .$$

First, choose j such that $\epsilon^j \ge |K| > \epsilon^{j+1}$.

Let $\mathcal{F}$ be the collection of all $I \in \mathcal{G}_{j+1}$ such that $I \cap K \neq \emptyset$.

Each $I \in \mathcal{F}$ satisfies $|I| \leq \epsilon^{j+1} < |K|$.

A little thought shows, then, that

$$\sum_{I \in \mathcal{F}} |I| \leq 3\,|K|. \text{ Therefore,}$$

$$\mu(K) \leq \sum_{I \in \mathcal{F}} \mu|I| \leq 3\,|K|\,C^{-(j+1)} = 3\,C|K|\,C^{-j}$$

(7) will be proven if we show that

$$C^{-j} \leq |K|^{\frac{-\log C}{\log \epsilon}} \tag{8}$$

$$\epsilon^{j} \geq |K| \Rightarrow j \log \epsilon \geq \log |K| \Rightarrow j \leq \frac{\log |K|}{\log \epsilon} \Rightarrow -j \log C \leq -\frac{\log C}{\log \epsilon} \log |K| \Rightarrow$$

$C^{-j} \leq |K|^{-\log C/\log \epsilon}$, which completes the proof of Lemma 2. □

I will use Lemma 2 to prove a stronger version of Theorem 1.

Theorem $1'$. If $0 < |A| < 1$, $A \subseteq [0, 1]$, and $s(\mu_A) < \epsilon$, then the Hausdorff dimension of ∂A is at least

$$\frac{\log 2\,(1-\epsilon)}{\log 2}.$$

Proof:

Notice that if τ is a linear function and $s(\mu_A) < \epsilon$, then $s(\mu_{\tau(A)}) < \epsilon$. This follows since $\frac{\mu_A(I)}{|I|} = \frac{\mu_{\tau(A)}(\tau(I))}{|\tau(I)|}$. Assume $|\partial A| = 0$, as otherwise ∂A has dimension 1.

Also, since

$$|A| = |\bar{A}| = |\mathrm{int}(A)|,$$

we can assume A is open. $0 < |A| < 1$ then implies that there are intervals I and J such that $\mu_A(I) = 0$ and $\mu_A(J) = 1$. If $h = \min(|I|, |J|)$, $g(x) = \frac{1}{h}\int_x^{x+h} d\mu_A(t)$ is continuous, and takes on all values from 0 to 1. Letting K be an interval, therefore, on which $\frac{\mu_A(K)}{|K|} = \frac{1}{2}$, and letting τ be the linear map taking K onto the interval [0, 1], we obtain $B = \tau(A) \cap [0, 1]$ with $|B| = 1/2$. Without loss, assume $|A| = 1/2$.

In fact, the same argument can be carried further to assure that

$$\sup_{I \subseteq [0,1]} \frac{2}{|I|} \left|\mu_A(I^+) - \mu_A(I^-)\right| < \epsilon .$$

Let $D(E) = \frac{\mu_A(E)}{|E|}$ for any measurable set E such that $|E| > 0$.

__Claim__ __1__. Suppose $I \subseteq [0, 1]$ is an interval such that $D(I) = 1/2$. Then there exists a collection of disjoint intervals $J_i \subseteq I$ such that $D(J_i) = 1/2$, $\Sigma|J_i| \geq (1 - \epsilon)\,|I|$, and such that $|J_i| \leq \frac{1}{2}\,|I|$ for all i.

Proof of Claim 1. Bisect I into I^+, I^-. If $D(I^+) = D(I^-) = 1/2$, the claim is true, so assume otherwise. Without loss of generality, assume $D(I^-) < 1/2$. We know from (2) that

$$D(I^-) > \frac{1}{2} - \epsilon/2 . \tag{9}$$

__Claim__ __2__. There is an interval $J_0^1 \subseteq I^-$ of maximal length with the property $D(J_0^1) = 1/2$ (I don't claim J_0^1 is unique).

First, there must be, by (9), some interval $B \subseteq I^-$ with $D(B) = 1$. I will only assume that $B \subseteq I^-$ is some interval with $D(B) > 1/2$. Let $B = (b_1, b_2)$. The map $f(x, y) = D((x, y))$, restricted to the domain U

$$U = \{(x, y) \mid x \in I^-, y \in I^-, x - y \geq b_2 - b_1\} \subseteq R^2$$

is a continuous function. Also, U is a connected domain, so f(U) is connected. $D(B) > 1/2$. $D(I^-) < 1/2$, so $1/2 \in f(U)$. $V = \{(x, y) \in : f(x, y) = 1/2\}$ is a compact subset of R^2, so $g(x, y) = y - x$ attains a maximum somewhere on V, say (x_1, y_1). Letting $J_1 = (x_1, y_1)$, Claim 2 is proven.

Notice that if, instead of U, we restrict f to the domain

$$U' = \{(x, y) \mid x \in I^-, y \in I^-, x \leq b_1, y \geq b_2\},$$

the same argument shows that if $D(B) > 1/2$ and $B \subseteq I^-$, then there is some interval K, $B \subset K \subset I^-$, such that $D(K) = 1/2$.

Neither endpoint of J_0^1 lies in the interior of $I^- \cap A$. Otherwise we could extend J_0^1 to an interval B on which $D(B) > 1/2$, and then by the above note, to a K on which $D(K) = 1/2$, contradicting the maximality of J_0^1. Let L_1^i be the components of $I \backslash J_0^1$ which contain more than one point, and which intersect A, so that $D(L_1^i) > 0$. $D(L_1^i) < 1/2$, as otherwise $D(J_0^1 \cup L_1^i) \geq 1/2$, and again we contradict the maximality of J_0^1.

We can find another interval inside L_1^i, say J_1^i, maximal with respect to the property

$$D(J_1^i) = 1/2 \ .$$

As before, neither endpoint of J_1^i lies in $A \cap I^-$.

We continue the process. Let L_2^i be a component of $I^- \backslash J_0^1 \cup (\bigcup_i J_1^i)$ which has more than one point and satisfies $D(L_2^i) > 0$. We create J_2^i's inside each L_2^i maximal again with respect to the property $D(L_2^i) = 1/2$. This process may or may not stop in a finite number of iterations. That is, there may exist a stage k, at which for all L_k^i, $|L_k^i \cap A| = 0$.

If the process stops, we have a finite collection of J_j^i's, and they must cover $A \cap I^-$. Otherwise, there would be a component L of $I \backslash (\bigcup_{ij} J_j^i)$ such that $D(L) > 0$, so we can continue the process. If the process never stops, I claim that the J_j^i's still cover $A \cap I^-$. For let B be any connected component of $I^- \cap A$ which is not covered. B must be completely uncovered, since the endpoints of J_j^i must always be in $A^c \cap I$. There can only be finitely many J_j^i's with length at least $|B|$. At some stage, therefore, there is an L_k^i containing B which also contains a J_k^i with $|J_k^i| < |B|$. But this is a contradiction, since there must be some interval, say K, with $B \subseteq K \subseteq L_k^i$ and $D(K) = 1/2$, contradicting the maximality of J_k^i. The J_k^i's are disjoint, $D(J_k^i) = 1/2$, and they cover $A \cap I^-$. Therefore

$$|\cup J_k^i| = 2\mu_A(\cup J_k^i) = 2\,|A \cap I^-| = 2\,|I^-|\, D(I^-).$$

(10) Therefore $$|\bigcup_{i,j} J_k^i| \geq (1 - \epsilon)|I^-| \ .$$

A similar procedure can be performed in I^+, and the combined collection of intervals which results satisfies the conclusion of Claim 1.

The proof of Theorem $1'$ now follows readily from Claim 1 and Lemma 2. Let $\mathcal{G}_0 = \{[0, 1]\}$ and let $\mathcal{G}_1 = \{H_i^1\}$ where H_i^1 are the intervals shown to exist in Claim 1. We now apply Claim 1 to each H_i^1 to obtain $\mathcal{G}_2 = \{H_i^2\}$, and so forth indefinitely.

$E = \bigcap_k (\bigcup_i H_i^k)$ has Hausdorff dimension at least $\frac{\log 2\,(1-\epsilon)}{\log(2)}$ from Lemma 2. Theorem $1'$ is proven when we show that $E \subseteq \partial A$. If $x \notin \partial A$, then there is an n such that $2^{-n} < \text{dist}(x, \partial A)$. Since each $|H_i^n| \leq 2^{-n}$, $x \notin H_i^n$ for all i, and thus $x \notin E$. Theorem $1'$ is therefore proven. □

Here is an extension of Theorem $1'$.

Theorem $1''$. Let $f \in L^1[0, 1]$, and suppose $f: [0, 1] \to R \backslash I$, where $I = (a, b)$ is some interval. Suppose $S(F) < (b - a)\,\epsilon$, where $F(x) \int_0^x f(t)\,dt$.
Let $A = \{t | f(t) \geq b\}$. Then either $|A| = 0$ or $|A| = 1$ or ∂A has Hausdorff dimension of at least

$$\frac{\log 2(1-\epsilon)}{\log 2} .$$

Corollary: Under the condition of Theorem $1''$, if F is smooth, then ∂A has Hausdorff dimension 1.

The proof is nearly the same as the proof of Theorem $1'$. We can first reduce Theorem $1''$ to the case where $(a, b) = (0, 1)$ by replacing f with

$$g(x) = \frac{f(x) - a}{b - a} .$$

Replace μ_A with the measure $\mu_f(E) = \int_E f(x)dx$, and let

$$D(E) = \mu_f(E)/|E|.$$

The rest of the proof holds until we try to prove (10). Instead, let $E = \bigcup_{i,j} J^i_j$ and note

$$\mu_f(I^-) \geq \frac{1}{2}(1 - \epsilon)\,|I^-|\ .$$

$$|E|\ D(E) + |E^c|\ D(E^c) \geq \frac{1}{2}(1 - \epsilon)\,(|E| + |E^c|)\ .$$

$$D(E) = 1/2, \text{ so } \frac{\epsilon|E|}{2} \geq \frac{(1 - \epsilon - D(E^c))\ |E^c|}{2}\ .$$

$$\text{Noting } D(E^c) \leq 0,\ \ |E|\ \frac{\epsilon}{1-\epsilon} \geq |E^c|\ .$$

$$|E|\left(\frac{\epsilon}{1-\epsilon} + 1\right) \geq |E^c| + |E| = |I^-|\ .$$

$$|\cup J^i_k| = |E| \geq (1 - \epsilon)\,|I^-|\ .$$

Thus, inequality (10) still holds, and the rest of the proof proceeds as before.

III. *Some Very Nearly Smooth Sets with Besicovitch Dimension Less Than One.*

For $E \subseteq [0, 1]$, the Besicovitch dimension of E is defined by

$$\text{Dim}_B(E) = \sup \{\alpha \mid \inf_{\epsilon} \{\Sigma |J_i|^\alpha \mid E \subseteq \cup J_i \text{ and } |J_i| = \epsilon \ \forall i\} > 0\} .$$

The Besicovitch dimension is always greater than the Hausdorff dimension of a set. It is well known that if E is closed, $E \subseteq [0, 1]$, and $|E| = 0$, then $\text{Dim}_B(E) = \inf \{\alpha | \ \Sigma |J_i|^\alpha < \infty\}$ where J_i are the connected components of E^c. Given $N \in \mathbb{Z}$, $N > 10$, I am going to construct a nontrivial set $A_N \subseteq [0, 1]$ such that

$$S(\mu_{A_N}) \leq \frac{10}{N} .$$

I will calculate $\text{Dim}_B(\partial A_N)$, which will turn out to be less than 1. This will answer the question asked in section I. There is no $\epsilon > 0$ such that $S(\mu_A) < \epsilon$ implies $|A| = 0$ or $|A| = 1$ or ∂A has Hausdorff dimension 1.

The construction I use is a minor modification of one due to Kahane [4]. It is necessary to describe the construction in detail here in order to be able to explain later the calculation of Besicovitch dimension.

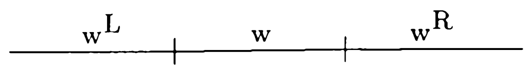
Modification of Kahane's Construction

I will construct a sequence of measures, $\mu_0, \mu_1, \mu_2 \cdots$, each of which is the integral of a step function, constant on all intervals of the form

$$w = [4^{-k}r, 4^{-k}(r+1)] .$$

That is, if $I \subseteq w$ and $|I| > 0$, then, the density,

$D_k(I) = \mu_k(I)/|I|$, will depend on w only , not in I. Given w as above, I will refer to $w_j = [4^{-(k+1)}(4\,r+j), 4^{-(k+1)}(4\,r+j+1)]$, $j = 0, 1, 2, 3$ as the "children" of w in the next generation, and w will be called the "parent" of w_0, w_1, w_2, w_3. Given w, let w^L and w^R denote the intervals adjacent to and of equal length to w, lying to the left and right of w.



Choose $N \in \mathbb{Z}$, $N > 10$.

Let $\mu_0 = 1/2$ Lebesgue measure.

Suppose μ_k has been constructed. Let μ_{k+1} be the measure with constant density on all intervals of the form $[4^{-(k+1)}s, 4^{-(k+1)}(s + 1)]$, the density on each such interval being determined as follows: Let $w = [4^{-k}r, 4^{-k}(r+1)]$, let w_0, w_1, w_2, w_3 be its children. If $D_k(w) = 0$ or 1, then let each w_j have the same density as its parent. Otherwise,

let
$$n_L = \begin{cases} 0 \text{ if } D_k(w^L) > D_k(w) \\ \quad 1 \quad \text{otherwise} \end{cases} .$$

Define n_R similarly, using w^R instead of w^L.

$$\text{Let } D_{k+1}(w_3) = D_k(w) + (-1)^{n_L} \frac{1}{4N}$$

$$D_{k+1}(w_2) = D_k(w) - (-1)^{n_L} \frac{1}{4N}$$

(11)

$$D_{k+1}(w_1) = D_k(w) - (-1)^{n_R} \frac{1}{4N}$$

$$D_{k+1}(w_0) = D_k(w) + (-1)^{n_R} \frac{1}{4N} \; .$$

So the density on w_3 moves toward $D_k(w^L)$ if $D_k(w^L) \neq D_k(w)$, and similarly for w_0. Each μ_k is of the form $\mu_k(A) = \int_A S_k(x)\,dx$, where S_k is a step function. As $k \to \infty$, S_k converges almost everywhere to the characteristic function of some set E, and the resulting measure is of the form μ_E. E is the union of all parent intervals which at some stage k have density 1. The argument by Kahane shows μ_E is almost smooth and

$$S(\mu_E) \leq \frac{10}{N} \; .$$

We wish next to calculate the Besicovitch dimension of ∂E.

A Note About Besicovitch Dimension

Let A be an open set such that $\overline{A} = [0, 1]$ and let $\{I^\tau\}$ be the connected components of A. Each I^τ can be written as the union of closed intervals of the form

$[r4^{-\ell}, (r+1)4^{-\ell}]$. In fact, for each I^τ, a collection of intervals of the above form can be found such that for any ℓ, no more than 6 of the subintervals have the same length. Call the collection $\{I^{\tau}_{\beta}\}$. For any $0 < \alpha < 1$,

$$|I^{\tau}|^{\alpha} \leq \Sigma |I^{\tau}_{\beta}|^{\alpha} \leq \sum_{\ell=0}^{\infty} 6\left(|I^{\tau}| 4^{-\ell}\right)^{\alpha} \leq M(\alpha)\ |I^{\tau}|^{\alpha}$$

where $M(\alpha) = 6 \sum_{\ell=0}^{\infty} 4^{-\ell\alpha}$. The point is, then, $\Sigma(I^{\tau})^{\alpha} = \infty \Leftrightarrow \Sigma|I^{\tau}_{\beta}|^{\alpha} = \infty$, and the calculation of Besicovitch dimension of ∂A is unaffected by the splitting of the connected components of A in the above manner. Such a splitting naturally occurs in calculating the Besicovitch dimension of the set E constructed above.

Calculation of the Besicovitch Dimension of ∂E.

Let E_k be the union of all closed intervals in the kth generation of intervals having density 1 whose parents have densities less than 1. Let F_k be the union of closed kth generation intervals with density zero whose parents have nonzero density. Let $E = \bigcup_k E_k$, and $F = \bigcup_k F_k$.

$$\partial(E) = [0, 1] \backslash (F \cup E).$$

Let $G_k = F_k \cup E_k$, so G_k is the union of intervals of length 4^{-k}. This provides the kind of splitting of the connected components of $E \cup F$ into smaller segments described above.

Let $N_k = 4^{-k}|G_k|$, so N_k is the number of intervals of length 4^{-k} composing G_k. Then

(12) $$\mathrm{Dim}_B(\partial E) = \sup \{\alpha | \Sigma N_k (4^{-k})^\alpha = \infty\} .$$

Estimation of $|G_k|$.

Let $A_{kj} = |\{x| S_k(x) = j/4N\}|$

$\mu_0 = 1/2$ Lebesgue measure, so $A_{0,2N} = 1$.

Note that if $k + j$ is odd, $A_{kj} = 0$.

For $2 \leq j \leq 4N-2$, $A_{k+1,j} = \frac{1}{2}(A_{k,j-1} + A_{k,j+1})$.

For $3 \leq j \leq 4N-3$, $A_{k+2,j} = \frac{1}{4}(A_{k,j-2} + A_{k,j+2} + 2 A_{k,j})$.

$$\text{Let } B_{kj} = \begin{cases} A_{2k,2j} & \text{if } 1 \leq j \leq 2N-1 \\ 0 & \text{if } j \leq 0 \text{ or } j \geq 2N \end{cases} .$$

For $1 \leq j \leq 2N-1$, $B_{k+1,j} = \frac{1}{4}(B_{k,j+1} + B_{k,j-1} + 2 B_{k,j})$. So

(13)

a) $B_{k,0} = B_{k,2N} = 0$

b) $B_{k+1,j} - B_{k,j} = \frac{1}{4}(B_{k,j+1} + B_{k,j-1} - 2 B_{k,j})$

c) $B_{0,j} = 1$ if $j = N$, 0 otherwise.

This is a discrete equation analogous to the Heat Equation

$$\frac{\partial f}{\partial t} = - C\frac{\partial^2 f}{\partial x^2},$$

where k is playing the role of time and j is playing the role of x. Clearly, for the system (13) a), (13) b), there is a unique solution corresponding every set of initial data of the form (13) c′) $B_{0,j} = b_j$, $j = 1, 2, 3, \cdots, 2N-1$.

We can solve this in a way analogous to the way the Heat Equation is solved. Let $\lambda_r = 2\cos\frac{\pi r}{2N} - 2$, $r = 1, 2, \cdots 2N-1$. Then for all r, $(1 + \frac{\lambda_r}{4})^k \sin\frac{\pi rj}{2N}$ is a solution to (13) a), (13), b).

For $r = 1, 2, \cdots 2N-1$, the functions $F_r(j) = \sin\frac{r\pi j}{2N}$ are orthogonal in the sense that

$$\sum_{j=1}^{2N-1} F_r(j)\, F_s(j) = 0 \text{ for } s \neq r,\ 1 \leq s \leq 2N-1.$$

Therefore, the Fr(j)'s are an orthogonal set spanning every set of initial values, (13) c′). $F_{j,k} = \sum_{r=1}^{2N-1} a_r \sin\frac{\pi rj}{2N}(1 + \frac{\lambda_r}{4})^k$ for some choice of $\{a_r\}$, is the general solution to (13) a) and (13) b). The solution corresponding to (13) c) clearly has $a_1 \neq 0$, since $\sum_{j=1}^{2N-1} \sin\left(\frac{\pi j}{2N}\right) B_{0,j} = 1$.

λ_r decreases monotonically in r for $1 \leq r \leq 2N-1$, so given $\in > 0$, for all k large enough,

$$(14) \quad (1 - \in)\, a_1 \sin\frac{\pi j}{2N}\left(1 + \frac{\lambda_1}{4}\right)^k \leq B_{k,j} \leq (1 + \in)\, a_1 \sin\frac{\pi j}{2N}\left(1 + \frac{\lambda_1}{4}\right)^k .$$

Going back to (12), we are in a position to estimate $|G_k|$ and hence N_k.

$$|G_{2k+1}| = 0$$

$$|G_{2k}| = \sum_{j=1}^{2N-1} B_{k-1,j} - \sum_{j=1}^{2N-1} B_{k,j} .$$

From (14), $\exists M$ such that if $k \geq M$,

$$\frac{1}{2}|G_{2k}| \leq \sum_{j=1}^{2N-1} a_1 \sin\left(\frac{\pi j}{2N}\right)\left((1+\frac{\lambda_1}{4})^{k-1} - (1+\frac{\lambda_1}{4})^{k}\right) \leq 2|G_{2k}|.$$

Thus there exists C_1, C_2, $M > 0$ such that if $k \geq M$,

$$C_1(1+\frac{\lambda_1}{4})^k \leq |G_{2k}| \leq C_2\,(1+\frac{\lambda_1}{4})^k$$

and

$$C_1(1+\frac{\lambda_1}{4})^k\, 4^{2k} \leq N_{2k} \leq C_2\,(1+\frac{\lambda_1}{4})^k\, 4^{2k}.$$

Therefore, $\mathrm{Dim}_B\, \partial E = \inf\{\alpha \mid \Sigma(1+\frac{\lambda_1}{4})^k\, 4^{2k}(4^{-2k})^\alpha = \infty\}$, or that α for which

$(1+\frac{\lambda_1}{4})\, 16^{1-\alpha} = 1.$

$$\alpha = \frac{\log 16 + \log(1+\frac{\lambda_1}{4})}{\log 16.}$$

Note $\lambda_1 < 0$, so $\alpha < 1$.

IV. *Remarks on the Previous Construction.*

In the preceding construction, we derived each measure from its predecessor. The rule used there specified that the construction stops on the interval ω whenever $D_k(\omega)$ equals 0 or 1, that is, $D_j(\omega) = D_k(\omega)$ for all $j \geq k$. Kahane's construction, from which my construction is derived, only comes to a stop when $D_k(\omega) = 0$. In other words, his construction has a "floor" but no "ceiling." The result is an almost smooth singular measure.

Kahane was able to make his measure smooth by modifying the construction. His initial measure, μ_0, is Lebesgue measure on [0, 1]. Instead of using a stepsize of 1/4N (see equation (11)), he uses a variable stepsize, c_j, satisfying:

(15) a) $c_0 = 1$

b) $c_{j+1} = c_j$ or $c_{j+1} = \frac{1}{2} c_j$

c) $\sum_{j=1}^{\infty} c_j^2 = \infty$.

He should have also added a condition such as

$$c_{2j+1} = c_{2j} \quad . \tag{16}$$

To explain why some condition such as (16) is needed would take me a little far from the point I wish to make. Let it suffice to say that without this constraint his construction leads to a smooth singular measure, but the proof is more complicated than the one he gives. Also, some of his later remarks assume that there exists $K_1 \in R$ such that

$$(17) \qquad K_1 c_j \geq \sup_{|I| \leq 4^{-j}} \left|D(I^+) - D(I^-)\right| \geq c_j .$$

This may not be true unless some constraint is placed on the c_j's beyond those in (15).

For a smooth measure μ, let

$$a_n(\mu) = \sup_{|I| \leq 4^{-n}} 2\frac{|\mu(I^+) - \mu(I^-)|}{|I|} .$$

Given any sequence $\{b_n\}$ decreasing monotonically to zero such that $b_{n+2} \geq b_n/2$ and $\Sigma\, b_n^2 = \infty$ we can choose c_j's meeting the constraints (15) and (16) with $c_j = 0\,(b_j)$. The measure derived from these c_j's, say μ, will satisfy

$$a_n(\mu) = 0(b_n).$$

Kahane pointed this out (in different terminology). He also pointed out that the condition $\Sigma\, b_n^2 = \infty$, or equivalently, $\Sigma\, a_n^2(\mu) = \infty$, is necessary as a result of a theorem of Stein and Zygmund [7, appendix].

The same techniques can be used to create a smooth set. We start with $\mu_0 = 1/2$ Lebesgue measure and a sequence of c_j's satisfying (15) and (16), except that we require $c_0 = 1/4$. We proceed as in section III, but substituting c_j for $1/4N$ in equations (11). The resulting sequence of measures converges to a measure of the form μ_A, $A \subseteq [0, 1]$, which is smooth for the same reasons as Kahane's measures. Again, given any sequence {bn} meeting the above criteria, we can create a smooth set A such that $a_n(\mu_A) = 0(b_n)$.

The question which arises is whether it is necessary, as in the case of smooth singular measures, that $\Sigma\, a_n^2(\mu_A) = \infty$. The condition is in fact necessary, as I shall now prove. Arguing by contradiction, assume

$$\sum_{n=1}^{\infty} a_n^2(\mu_A) = M < \infty.$$

There exists N such that $\sum_{n=N}^{\infty} a_n^2(\mu_A) < \frac{1}{16}$.

We can find an interval I such that $\log_4 \frac{1}{|I|} \in \mathbb{Z}$, $\log_4 \frac{1}{|I|} \geq N$, and $\frac{\mu_A(I)}{|I|} = 1/2$. Let τ be the linear function such that $\tau(I) = [0, 1]$, and let $B = \tau(I \cap A)$. $|B| = 1/2$ and $\sum_{n=1}^{\infty} a_n^2(\mu_B) < \frac{1}{16}$.

Next, define a sequence of step functions, $\{f_n(x)\}$ as follows:

Let $w_n(x)$ be the interval of the form $(r4^{-n}, (r + 1)\, 4^{-n}]$, $r \in \mathbb{Z}$, containing x.

Let

$$f_n(x) = 4^n\, \mu_B(w_n(x)) = \frac{1}{|w_n(x)|} \int_{w_n(x)} \chi_B(t)\, dt.$$

It is known that $f_n(x)$ converges almost everywhere to $\chi_B(x)$, and, by the Dominated Convergence Theorem, f_n converges in L^2 norm to χ_B. Therefore

$$\lim_{n\to\infty} \|f_n\|_2^2 = \|\chi_B\|_2^2 = 1/2. \tag{18}$$

However, note that $\|f_0\|_2^2 = 1/4$, and that $f_{n+1} - f_n$ is orthogonal to f_n (with respect to the usual inner product). Therefore

$$\|f_k\|_2^2 = 1/4 + \sum_{j=1}^{k} \|f_j - f_{j-1}\|^2$$

$$\leq 1/4 + \sum_{j=1}^{k} a_k^2(\mu_B) < 1/2.$$

(18) cannot hold and my claim is proven.

Finally, I wish to note how the existence of a positive smooth singular measure implies the existence of a Blaschke product in B_0. Consider first the Poisson integral of a positive smooth singular measure μ,

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} P(r, \theta - t)\, d\mu(t) \quad \text{where}$$

$$P(r, \theta) = \frac{1 - r^2}{1 - 2\, r\cos\theta + r^2} \; .$$

I claim first that $\lim_{r \to 1} (1 - r) \frac{\partial u}{\partial \theta} = 0$ and the limit is uniform in θ. Let

$$f(x) = \int_{-\pi}^{x} d\mu(t) \; .$$

If we extend f periodically to the real line, f is uniformly smooth on any closed interval not containing an odd multiple of π.

Let $F(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, (\theta - t)) f(t)\, dt$. It is known (see [8, pg. 109]) that for any $\delta > 0$,

$(1 - r) F_{\theta\theta}(r, \theta)$ tends to 0 uniformly on $(-\pi + \delta, \pi - \delta)$ as $r \to 1$.

$$
\begin{aligned}
F_\theta &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial}{\partial \theta} P(r, \theta - t) f(t)\, dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial}{\partial \theta} P(r, \theta - t) \int_{-\pi}^{t} d\mu(\lambda)\, dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{\lambda}^{\pi} \frac{\partial}{\partial \theta} P(r, \theta - t)\, dt\, d\mu(\lambda) \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{\lambda}^{\pi} \frac{\partial}{\partial t} P(r, \theta - t)\, dt\, d\mu(\lambda) \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} [P(r, \theta - \lambda)\, dt - P(r, \theta - \pi)]\, d\mu(\lambda)
\end{aligned}
$$

$$F_\theta = u(r, \theta) - \frac{1}{2\pi} P(r, \theta - \pi) \int_{-\pi}^{\pi} d\mu(\lambda)$$

$$F_{\theta\theta} = u_\theta(r, \theta) - \frac{1}{2\pi} P_\theta(r, \theta - \pi) f(\pi)$$

$\lim_{r \to 1} P_\theta(r, \theta - \pi) = 0$ for $\theta \in (-\pi + \delta, \pi - \delta)$ and the limit is uniform. Thus $(1 - r) u_\theta(r, \theta)$ tends to 0 uniformly for $\theta \in (-\pi + \zeta, \pi - \zeta)$.

The same argument can be repeated to obtain uniform convergence to 0 for $\theta \in (\delta, 2\pi - \delta)$, so my claim is proven.

$u(r, \theta)$ is known to be harmonic and since μ is singular, $\lim_{r \to 1} u(r, \theta) = 0$ almost everywhere.

Let $v(r, \theta)$ be the harmonic conjugate of u satisfying $v(0, 0) = 0$. The Cauchy-Riemann equations yield

$$u_\theta = - r v_r$$

$$v_\theta = r v_r \ .$$

Differentiating with respect to θ,

$$u_{\theta\theta} = - r v_{\theta r}$$

$$v_{\theta\theta} = v u_{\theta r} \ .$$

u_θ and v_θ then satisfy the Cauchy-Riemann equations, and are therefore harmonic conjugates. There is a slight problem when $r = 0$, since one cannot recover the Cauchy-Riemann equations in rectangular coordinates from the polar equations. However, $\int_0^{2\pi} u_\theta(r, \theta) d\theta = 0$ and $u_\theta = 0$ at $r = 0$. Thus from the mean value property we can deduce that u_θ is also harmonic in a neighborhood containing zero. Similarly for v_θ. Thus v_θ is the harmonic conjugate of u_θ. Therefore $\lim_{r \to 1} v_\theta = 0$, the limit being uniform (see [8, pg. 258]). Thus $f(r, \theta) = u + i v$ is analytic and in B_0. Let

$$g(z) = e^{-f}. \ \mathrm{Re}\, f \geq 0$$

so $|g'(z)| = |f'| e^{-\text{Re} f} \le |f'|$, so g is in B_0.

Furthermore,

$\lim_{r \to 1} |g(r, \theta)| = 1$ almost everywhere, so g is an inner function.

By Frostman's Lemma, for almost all complex α such that $|\alpha| < 1$,

$$h(z) = \frac{g(z) - \alpha}{1 - \bar{\alpha} g(z)}$$

is a Blaschke product. It is then easy to check that $h(z) \in B_0$ so Blaschke products in B_0 exist. This method of producing Blaschke products in B_0 was pointed out by T. Wolff (see [6].)

V. *Proof of Theorem 2*.

The proof of Theorem 2 will follow loosely the proof of Theorem 1. It will start with a "rectangle" near the boundary of the unit disk, on the top half of which our Blaschke product, B(z), is near zero. Inside this rectangle is found a collection of rectangles, on the top half of which B(z) is also near zero. The sum of the lengths of the rectangles will be at least a fraction, C, of the length of the original rectangle, but each will have length no more that $\in$ times the length of the original. We will be able to obtain a lower bound on the Hausdorff dimension of the singularity set of $1 - \log C/\log \in$ and will be able to force $\in$ to approach zero. It was T. Wolff who suggested a way of constructing these rectangles.

Recall that in proving Theorem 1 we started with an interval I with average density $D(I) = 1/2$, and generated intervals inside I also with average density equal to 1/2. The subintervals were all shorter than $\frac{1}{2}|I|$ and covered a subset of measure at least C. We were able to force C to tend to 1.

Note that $B(z_0)$ being near zero implies that the Poisson integral of its boundary values is near zero. The Poisson integral is a weighted average, so the condition that $B(z_0)$ is small is analogous to our condition in proving Theorem 1 that $D(I) = 1/2$.

We now begin our proof.

Given an arc $I = \{e^{i\theta}: a < \theta < b\}$, let $|I| = (b - a)$, and let

$R_I = \{re^{i\theta}: a < \theta < b, 1 > r > 1 - (b - a)\}$ and let

$T_I = \{re^{i\theta}: a < \theta < b,\ 1 - \frac{(b-a)}{2} > r > 1 - (b - a)\}$.

In proving a lemma, C. Bishop [1] proved the following:

Lemma 3.

Let $B \in B_0$ be an infinite Blaschke product. There is a $\delta > 0$ so that if $|I|$ is sufficiently small, and if B_1 is the Blaschke product whose zeroes are the zeroes of B belonging to $R_{\frac{1}{4}I}$, $z \in T_I$, and $|B(z)| < \delta$, then $|B_1(z)| < 1/2$.

I now prove:

Lemma 4.

Assume the singularity set of B has measure 0 with respect to arc length measure on the unit circle.

For any $\delta_1 > 0$, there is a $C > 0$ such that if I is an arc, $|I| < 1/2$, and B is a Blaschke product with zeroes in $R_{\frac{1}{4}I}$, then $|B(z_0)| < 1/2$ for some $z_0 \in T_I$ implies $\Sigma\ |I_j| \geq C\ |I|$, where I_j are the intervals in the dyadic decomposition of I maximal with respect to the property "$\exists\ z \in T_{I_j}$ such that $|B(z)| < \delta_1$."

Proof. Let $\Omega = D \backslash \cup R_I$. $|B(z)| \geq \delta_1$ on $\partial\ \Omega \backslash\ \partial D$, which follows from the maximality of the I_j's. If z_n is a sequence of points in D converging to $\omega \in \partial\ D \backslash \frac{1}{2}I$, then $\lim\ |B(z_n)| = 1$, since B can be extended analytically across the boundary of $\partial\ D \backslash \frac{1}{2}I$.

Furthermore, if z_n is a sequence contained in $\Omega \cap R_I$, then $|B(z_n)| > \delta_1$. Then clearly if $z_n \to \partial\ \Omega$, $\liminf\ |B(z_n)| \geq \delta_1$, so we can apply the maximum modulus theorem to $\frac{1}{|B(z_n)|}$ to conclude $|B(z_n)| \geq \delta_1$ on Ω. $-\text{Log}\ |B(z)|$ is therefore a positive, bounded harmonic function on Ω. Also,

$-\log\ |B(z)| \to 0$ as $z \to \partial\ D \backslash \overline{\cup I_j}$, so if $z_n \to \alpha \in \partial\ \Omega$

$$\limsup\ -\log\ |B(z_n)| \begin{cases} =0 \text{ if } z_n \to \partial D \backslash \overline{\cup I_j} \\ \leq -\log\ \delta_1 \text{ otherwise} \end{cases}.$$

Let ω (A, B, C) denote the harmonic measure, at the point C, of the set B (on the boundary of A) relative to the domain A.

By the maximum principle,

$-\log |B(z)| \leq \omega(\Omega, \partial \Omega \cap \overline{\cup R_{I_j}}, z)(-\log \delta_1)$. Therefore, if $|B(z_0)| < 1/2$,

$\omega(\Omega, \partial \Omega \cap \overline{\cup R_{I_j}}, z) \geq \frac{\log 2}{-\log \delta_1} = C_1(\delta_1)$.

I claim that

$$\omega(\Omega, \partial \Omega \cap \overline{\cup R_{I_j}}, z) = \omega(\Omega, \partial \Omega \backslash \partial D, z). \tag{19}$$

To prove this, it is enough to show that

$$\omega(\Omega, \overline{\cup I_j} \backslash \cup I_j, z) = 0. \tag{20}$$

If $\Omega \subseteq D$ and $A \subseteq \partial \Omega \cap \partial D$, it is known that

$$\omega(\Omega, A, z) \leq \omega(D, A, z).$$

We are assuming the singularity set of B has measure 0, so $\omega(D, \overline{\cup I_j} \backslash \cup I_j, z) = 0$. Hence (20) follows.

$$\omega(\Omega, \partial \Omega \backslash \partial D, z_0) \geq C_1(\delta_1) \text{ if } |B(z_0)| \leq 1/2 \tag{21}$$

I now wish to show $\exists$ C_2, C_3 such that

$$\omega(D, \cup I_j, z_0) \geq C_2\, \omega(\Omega, \partial \Omega \backslash \partial D, z_0) \quad \text{and} \tag{22}$$

(23) $$\Sigma|I_j| \geq C_3\, \omega(D, \cup I_j, z_0)\, |I| \ .$$

Consider first any $z \in \partial\, \Omega \backslash \partial D$, so $\exists\ I_j$ such that $z \in \partial\, R_{I_j}$. I claim there is a constant K independent of z and I_j such that

(24) $$\omega(D, I_j, z) \geq K.$$

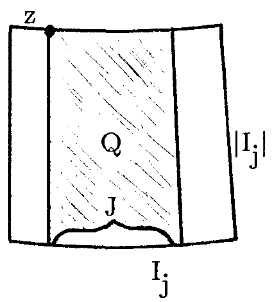


Figure 3.

Any z on ∂R_{I_j} is also on the corner of some rectangle Q contained in R_{I_j} whose height is 2 times its base, J (see figure 3).

(25) $$\omega(D, I_j, z) \geq \omega(D, J, z) = \frac{1}{2\pi}\int_0^{|J|} \frac{(1 - r^2)\, d\theta}{1 - 2r\cos\theta + r^2} \quad \text{(where } r = |z|)$$

$$= \frac{1}{2\pi}\int_0^{(1-r)/2} \frac{1 - r^2\ d\theta}{(1 - r)^2 + 2r\,(1 - \cos\theta)} \geq K' \int_0^{(1-r)/2} \frac{(1 - r)\, d\theta}{(1 - r)^2}$$

$$= K > 0$$

and hence (24).

Therefore $\omega(D, \cup I_j, z) \geq K$ for all $z \in \partial\,\Omega\backslash\partial D$, and (22) follows from this and the maximum principle. To show (23) it is enough to show for each I_j,

$$|I_j| \geq C_3\ \omega(D, I_j, z_0)\ |I|. \tag{26}$$

Let $I_j = \{e^{i\theta} : a_j < \theta < b_j\}$, I_j is contained in $\frac{1}{4}$ I as in figure 4. Let $z_0 = re^{i\emptyset}$.

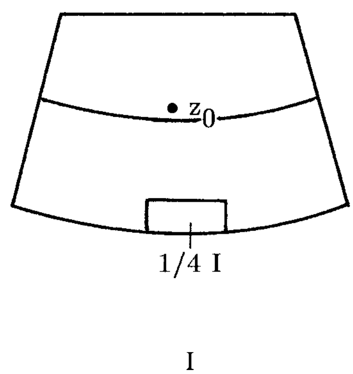


Figure 4.

$$\omega(D, I_j, z_0) = \int_{a_j}^{b_j} \frac{1 - r^2\ d\theta}{1 - 2r\cos(\theta - \emptyset) + r^2}$$

$$\leq (1 + r) \int_{a_j}^{b_j} \frac{(1 - r)}{(1 - r)^2}\, d\theta \leq \frac{2|I_j|}{1 - r} \leq 4\,\frac{|I_j|}{|I|}\ .$$

Therefore (26) and (23) hold.

Combining (21), (22) and (23) we obtain a $C > 0$ such that $\Sigma\ |I_j| > C|I|$ as desired.

Combining Lemmas 3 and 4, we obtain

Lemma 5. Let $B \in B_0$ be an infinite Blaschke product. There is a $\delta > 0$ and a $C > 0$ so that if I is a sufficiently small arc on ∂D and if there is a $z \in T_I$ with $|B(z)| \leq \delta$, then there are arcs $I_j \leq I$ such that $\Sigma|I_j| \geq C|I|$, and inside each T_{I_j}, $|B(\omega)| \leq \delta/8$.

We're now in a position to polish off the proof of Theorem 1. Let $B \in B_0$, let δ, C be as shown to exist by Lemma 5. Choose I_0 such that $\exists\ z \in T_{I_0}$ with $|B(z)| = \delta$ and sufficiently small to satisfy the hypothesis of Lemma 5. (Since there are an infinite collection of zeroes of B, many such I_0 exist.) Let $\{I_j\}$ be the intervals shown to exist by Lemma 5, such that T_{I_j} contains ω_0 such that $|B(\omega_0)| < \delta/8$. By choosing a smaller I_0 if necessary, we can guarantee that for all I_j, $\omega \in T_{I_j}$ implies

$$|B(\omega)| < \delta/4. \tag{27}$$

This is possible since $(1 - |z|)\ |B'(z)|$ tends to 0 as $|z|$ tends to 1.

Given any I_j, choose I_{jk}'s from the dyadic decomposition which are maximal with respect to the property

"$T_{I_{jk}}$ contains a point z where $|B(z)| = \delta$."

Since for almost all θ, $\lim_{r \to 1} |B(re^{i\theta})| = 1$, we can conclude that

$$\sum_k |I_{jk}| = |I_j|.$$

Inside each $T_{I_{jk}}$ we can guarantee that $|B(z)| > \delta/2$ for all z in $T_{I_{jk}}$. (The choice I_0 small enough to guarantee (27) also guarantees this.)

If $\omega_1 \in T_{I_j}$, and $w_2 \in T_{I_{jk}}$, therefore, $|B(\omega_1) - B(\omega_2)| > \delta/4$.

So if $(1 - |z|)\,|B'(z)| < a$,

$$\left| \int_{1-|I_{jk}|}^{1-|I_j|} \frac{a}{1-r}\, dr \right| > \delta/4$$

$$\log \frac{|I_j|}{|I_{jk}|} > \delta/4a \ .$$

Given any $\in > 0$, we can choose I_0 even smaller, if necessary, then, to assure that

$$|I_{jk}| < \in |I_j| \leq \in |I_0|.$$

So the I_{jk}'s are a set of intervals such that

a) $\Sigma\, |I_{jk}| > C\, |I_0|$

b) $|I_{jk}| < \in |I_0|$

c) $\exists\ z_{jk} \in T_{I_{jk}}$ such that $|B(z_{jk})| = \delta$.

Let $\mathcal{G}_0 = \{I_0\}$, let $\mathcal{G}_1 = \{I_{jk}\}$. c) implies that inside each I_{jk} we can repeat the process to obtain a collection $\mathcal{G}_2 = \{J_\alpha\}$ of intervals such that each J_α satisfies c) and such that for each I_{jk}

$$\sum_{J_\alpha \subseteq I_{jk}} |J_\alpha| > C\,|I_{jk}|\,,$$

$$J_\alpha \subseteq I_{jk} \Rightarrow |J_\alpha| < \epsilon\,|I_{jk}|\ .$$

We can then repeat the process indefinitely, creating a sequence of collections $\mathcal{G}_j$ satisfying the conditions of Lemma 2, and $E = \bigcap_j \Big(\bigcup_{I \in \mathcal{G}_j} I \Big)$ has Hausdorff dimension at least

$$1 - \frac{\log C}{\log \epsilon}.$$

E contains only points in the singularity set of B, as every point in E is the limit point of points β_j satisfying $|B(\beta_j)| < \delta$.

Thus, the singularity set has dimension at least

$$1 - \frac{\log C}{\log \epsilon},\ \text{and } \epsilon \text{ can be chosen}$$

arbitrarily close to zero. Hence Theorem 1 is proven. □

VI. *References*

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