# Chapter 5

# LCS in a Model for Mediterranean Eddies

We now use the FTLE-LCS method, and in particular the utility of the method for identifying homoclinic tangles, to study kinematic models for geophysical flows. We begin with the elliptic Kida vortex, an exact solution to Euler's equations for incompressible flow, that provides a model for flow in a storm on Neptune. Next, we study a generalization of the Kida vortex to a three-dimensional quasi-geostrophic flow that provides an ellipsoidal model that faithfully captures motion of Mediterranean eddies. In both cases, computation of the LCS reveals a tangle structure and its consequent chaotic dynamics in the flow surrounding the vortex.

## 5.1 The Kida vortex

The Kirchoff ellipse is a particular solution to Euler's equations for an incompressible fluid in two-dimensions [Kirchhoff 1876]. Kirchhoff's solution is obtained by placing an elliptical patch of uniform vorticity in an otherwise quiescent fluid. The ellipse rotates about its center at a rate proportional to the strength of the vorticity.

In an otherwise quiescent flow, let the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{5.1}$$

define a region of constant vorticity, $\omega$, with $a$ and $b$ denoting the lengths of the principal axes of the ellipse. As shown in [Lamb 1945], the stream function for flow exterior to the ellipse in elliptical coordinates $(\xi, \eta)$ is

$$\psi_{\text{ext}} = \frac{1}{4}ab\omega e^{-2\xi}\cos 2\eta + \frac{1}{2}\omega ab\xi\,, \tag{5.2}$$

while interior to the ellipse, the stream function is

$$\psi_{\text{int}} = \frac{\omega}{2(a+b)}\left(bx^2 + ay^2\right)\,. \tag{5.3}$$

The ellipse rotates with constant frequency

$$n = \frac{ab}{(a+b)^2}\omega \tag{5.4}$$

in such a way that a no slip boundary condition is satisfied between the exterior and interior regions.

Morton investigated the character of individual Lagrangian trajectories in this flow for different aspect ratios of the ellipse [Morton 1913]. He uncovered complex Lagrangian trajectories - a notable feat since all the particle integrations were done by hand.

Much later, Kida introduced a more general solution for the case of linear background flow [Kida 1981]. In this case, an elliptical patch of vorticity remains an ellipse, but the aspect ratios of the ellipse evolve over time in such a way that the area of the ellipse is conserved. Moreover, the rotational rate of the ellipse is not constant, and for certain initial conditions the ellipse undergoes nutation.

Given an elliptical patch of constant vorticity, $\omega$, placed in a linear background flow with velocity components described by

$$u = ex - \gamma y \tag{5.5}$$

$$v = -ey + \gamma x\,, \tag{5.6}$$

Kida provided the evolution equations for the *orientation* of the ellipse, described by the angle, $\theta(t)$, arbitrarily chosen between the major axis and the horizontal; and the *aspect ratio* of the ellipse, denoted $r(t)$, and defined as

$$r(t) := \frac{a(t)}{b(t)} \,. \tag{5.7}$$

Written as a coupled system of ordinary differential equations, the evolution of $r(t)$ and $\theta(t)$ is

$$\dot{r} = 2er\cos 2\theta \tag{5.8}$$

$$\dot{\theta} = -e\frac{r^2+1}{r^2-1}\sin 2\theta + \frac{\omega r}{(r+1)^2} + \gamma. \tag{5.9}$$

The elliptical patch of fluid now has constant vorticity

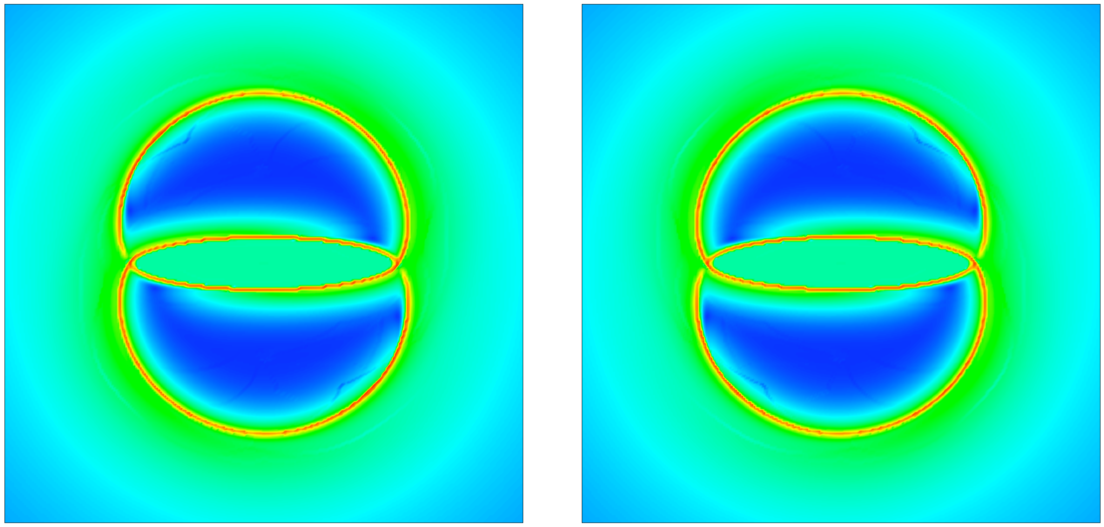
$$\omega' = \omega + 2\gamma \,, \tag{5.10}$$

and the stream function for determining the velocity of the fluid at any instant is determined using the stream functions as before for the Kirchhoff ellipse. Therefore, during simulation, we first integrate for the motion of the ellipse, and then integrate particle trajectories in the resulting velocity field. Fluid motion within the ellipse is integrable and the fluid particles follow circular trajectories. The flow exterior to the ellipse is not integrable, and trajectories must be computed numerically.

Polvani proposed that the Kida vortex serves as an excellent model for flow surrounding an elliptical storm in the atmosphere of Neptune – a storm known as Neptune's "Great Dark Spot", not unlike the "Great Red Spot" on Jupiter [Polvani 1990].[1] Using spacecraft imagery of the storm, they measured the evolution of the elliptical storm region, and assuming the Kida vortex model as a valid description of the flow, inferred the vorticity within the storm, as well as the background strain rates.

Here, we use the FTLE-LCS method to study the transport structures surrounding the Kida vortex. The system is akin to the perturbed pendulum, in that the Eulerian

[1]Neptune's Great Dark Spot has subsequently disappeared; however, during its lifetime, it exhibited the fastest winds ever recorded in the solar system.



(a) (b)

Figure 5.1: The repelling (a) and attracting (b) FTLE fields for Kirchoff's ellipse. This represents an unperturbed case in which the repelling and attracting LCS coincide, clearly demarcating an interior and exterior region.

velocity and vorticity fields yield little insight into Lagrangian transport structures.

The FTLE computed for the flow of a Kirchoff ellipse are shown in Figure 5.1 – recall that the ellipse remains unchanged in time and rotates with uniform angular velocity. The repelling and attracting LCS coincide to resemble a heteroclinic connection that separates the far-field quiescent flow from the region that rotates with the ellipse. The addition of a background linear flow introduces a perturbation that leads to the LCS in Figure 5.2. The intersection of the repelling and attracting LCS now indicate homoclinic-type points – the number of which goes to infinity as the integration time is increased. The LCS define lobe regions that are entrained and detrained in the flow surrounding the ellipse, indicating that the linear background flow enhances mixing through the mechanism of lobe dynamics.

## 5.2 The ellipsoidal vortex

We now proceed from the Kida vortex to a three-dimensional ellipsoidal model for a special class of oceanic eddies. Mediterranean eddies (or *meddies* as they are referred

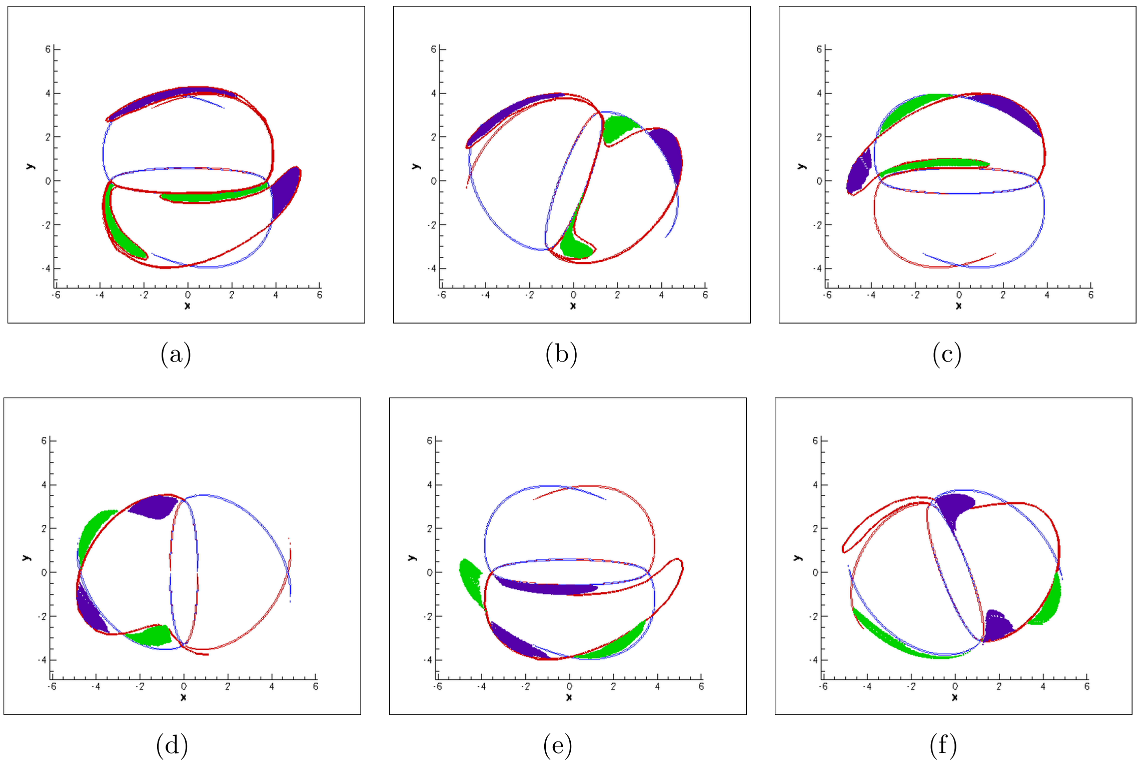



Figure 5.2: Snapshots of the repelling and attracting LCS are shown here in red and blue, respectively. The lobes defined by their intersections are filled with drifters: green drifters start inside the vortical region, while purple drifters start outside. Under the action of the time-dependent rotating homoclinic tangle, the green drifters are detrained while purple drifters are entrained.

to in the oceanographic comunity) are an interesting oceanic phenomenon in which dense salty water exits the Mediterranean Sea through the Straits of Gibraltar and descends into deeper water in the North Atlantic. Frictional and Coriolis effects induce rotation, resulting in a coherent eddy of high salinity with a typical radius of 100km at a depth of 1km [Armi 1989]. The three-dimensional eddy has the shape of an oblate ellipsoid, and hence is often referred to as a *salt lens*. The remarkable property of these eddies is that they retain their coherent shape and salinity, and have been tracked for several years as they migrate across the Atlantic toward the Carribbean.

A simple fluid model for the flow of these eddies has been proposed by [Meacham 1994], and an excellent treatment of the model can be found in [McKiver 2003]. The eddy model is a particular solution to a quasi-geostrophic model that approximates flow in rotating stratified flows. In the quasi-geostrophic model, the flow is entirely determined by the advection of a scalar quantity, the potential vorticity: $q(\mathbf{x}, t)$. Assuming constant values for the Coriolis and buoyancy frequencies yields the following simplified system:

$$\frac{dq(\mathbf{x},t)}{dt} = \frac{\partial q(\mathbf{x},t)}{\partial t} + \mathbf{u} \cdot \nabla q(\mathbf{x},t) = 0\,, \tag{5.11}$$

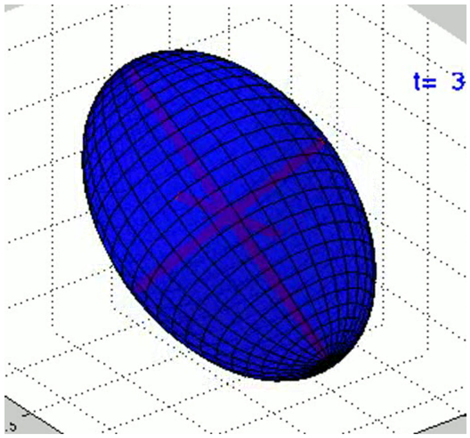
$$\nabla^2 \psi = q(\mathbf{x},t)\,, \tag{5.12}$$

$$\mathbf{u}(\mathbf{x},t) = (u,v,w) = \left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}, 0\right)\,. \tag{5.13}$$

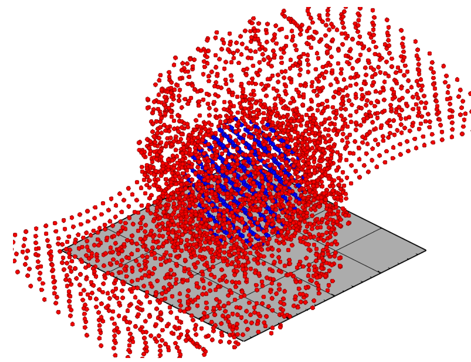
Meacham has shown that an ellipsoidal patch of constant vorticity placed in a background linear shear flow is an exact solution to this system. Under the action of the background flow, the aspect ratios of the ellipsoid change, but the ellipsoid nevertheless remains ellipsoidal while undergoing chaotic motions of rotation and nutation. The ellipsoidal solution for the three-dimensional quasi-geostrophic equations has an interesting counterpart in the gravitational potential theory of Laplace, and was also treated by Chandresekar [Chandrasekhar 1969].

Our purpose in studying this model with LCS is to understand the geometry of particle transport surrounding the ellipsoid. Figure 5.3(b) indicates the shape of drifter flow surrounding the ellipse – an animation of which does not immediately reveal insight in the transport mechanism at work.

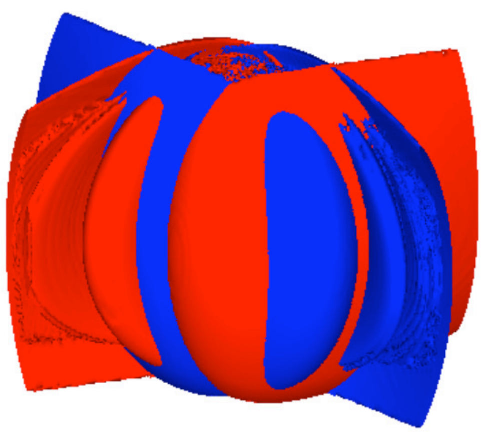
Figures 5.3(c) and 5.3(d) provide a side and top view of both the attracting and repelling LCS (now two-dimensional surfaces) surrounding the ellipsoid. Immediately, we perceive that the flow surrounding the ellipsoid resembles flow in the perturbed pendulum in that the perturbation provided by the background shear causes the formation of lobes, and that the mechanism by which fluid is transported into the vicinity of the ellipsoid is through the mechanism of lobe dynamics. Computing the volume of the lobes and the rate at which they form provides an indication of the rate at which mixing is occurring in the flow. Again, the background linear flow introduces a perturbation that induces and enhances transport via lobe dynamics.



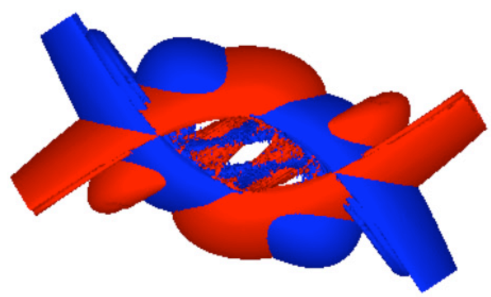

(a) The meddy is modeled as an ellipsoidal patch of constant vorticity in a linear shear flow. The ellipsoid undergoes rotation, nutation, and changes in aspect ratio.



(b) Drifters in the flow surrounding the ellipsoid. Blue particles are inside the ellipsoid and have integrable dynamics. Simply watching the movement of tracers does not reveal the transport mechanism outside the ellipsoid.



(c) Side view of the repelling (red) and attracting (blue) LCS in the the flow surrounding the ellipsoid. In time, the blue lobes are detrained out of the interior, while red lobes are entrained.



(d) Top view of the LCS in the flow surrounding the ellipsoid.

Figure 5.3: LCS in a quasi-geostrophic model for Mediterranean eddies.