Appendix A

Dispersion Relations for Metal-Insulator-Metal Waveguides

In this appendix, we analyze the thin film dispersion relations for a metal-insulator-metal waveguide.[1] Both the transverse magnetic and transverse electric conditions will be considered. The geometry used for this derivation is shown in Figure A.1.

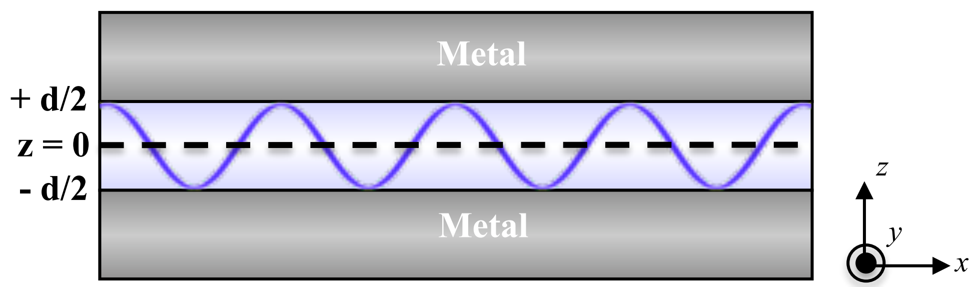

Figure A.1. The coordinate system used for the following three-layer derivation.

To begin, we first assume the form of the electric and magnetic fields given by:

$$\vec{E} = \vec{E}(z)\mathrm{e}^{i(k_x x - \omega t)} \tag{A.1a}$$

$$\vec{B} = \vec{B}(z)\mathrm{e}^{i(k_x x - \omega t)} \tag{A.1b}$$

where we assume there is no y-dependence in either field.

For this derivation, the curl can be written in it's full form for the any vector $\vec{U}$ as:

$$\nabla \times \vec{U} = \left[\frac{\partial U_z}{\partial y} - \frac{\partial U_y}{\partial z}\right]\hat{x} + \left[\frac{\partial U_x}{\partial z} - \frac{\partial U_z}{\partial x}\right]\hat{y} + \left[\frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y}\right]\hat{z} \tag{A.2}$$

[1] This appendix is based on texts by Professor Heinz Raether [97], Professor Stefan Maier [71], notes and discussions with Jennifer Dionne.

for for the curl components of $\vec{U}$ in the $\hat{x}$, $\hat{y}$, and $\hat{z}$ directions respectively.

We then plug the general form of the electric and magnetic fields into Maxwell's Equations. In the absence of space charge and currents, we have:

$$\nabla \cdot \vec{E} = 0 \tag{A.3a}$$

$$\nabla \cdot \vec{B} = 0 \tag{A.3b}$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{A.3c}$$

$$\nabla \times \vec{B} = \frac{1}{c}\varepsilon_i(\omega)\frac{\partial \vec{E}}{\partial t} \tag{A.3d}$$

We see that plugging (A.1a) and (A.1b) into (A.2) yields two sets of equations for either $\nabla \times \vec{E}$ or $\nabla \times \vec{B}$ which are given by:

$$\nabla \times \vec{E} :$$

$$\hat{x} : -\frac{\partial E_y}{\partial z} = \frac{i\omega}{c} B_x \tag{A.4a}$$

$$\hat{y} : -\frac{\partial E_x}{\partial z} + ik_x E_z = \frac{i\omega}{c} B_y \tag{A.4b}$$

$$\hat{z} : ik_x E_y = \frac{i\omega}{c} B_z \tag{A.4c}$$

$$\nabla \times \vec{B} :$$

$$\hat{x} : -\frac{\partial B_y}{\partial z} = -\frac{i\omega}{c}\epsilon_i E_x \tag{A.5a}$$

$$\hat{y} : \frac{\partial B_x}{\partial z} = -ik_x B_z = -\frac{i\omega}{c}\epsilon_i E_y \tag{A.5b}$$

$$\hat{z} : -ik_x B_y = -\frac{i\omega}{c}\epsilon_i E_z \tag{A.5c}$$

Initially we solve for generally for each component of $\vec{E}$ and $\vec{B}$. Also note that the sets of solutions either consist of (E_y, B_x, B_z) or (E_x, E_z, B_y). To simplify the general solution to solve for the set of transverse-magnetic (TM) modes, we can set $E_y = 0 \rightarrow B_x = B_z = 0$. To simplify the general solution to solve for the set of transverse-electric (TE) modes, we can set $B_y = 0 \rightarrow E_x = E_z = 0$.

A.1 The General Solution

We begin by solving for E_z. This is done by combining (A.4b) and (A.5c) :

$$\frac{\partial E_x}{\partial z} = -ik_x E_z = \frac{i\omega}{c}\left(-\frac{\omega}{ck_x}\right)\epsilon_i E_z \tag{A.6}$$

with (A.5a) and (A.5c):

$$-\frac{\partial\left(\frac{\omega}{ck_x}\cdot\epsilon_i E_z\right)}{\partial z} = \frac{i\omega}{c}\epsilon_i E_x \tag{A.7a}$$

$$\Rightarrow -\frac{\partial E_z}{\partial z} = ik_x E_x \tag{A.7b}$$

which yields :

$$E_x = \frac{1}{k_x}\frac{\partial E_z}{\partial z} \tag{A.8}$$

$$\frac{1}{k_x}\frac{\partial^2 E_z}{\partial z^2} - ik_x E_z = -\frac{i}{k_x}\left(\frac{\omega}{c}\right)^2\epsilon_i E_z \tag{A.9}$$

$$\frac{\partial^2 E_z}{\partial z^2} - k_x^2 E_z = -\left(\frac{\omega}{c}\right)^2\epsilon_i E_z \tag{A.10}$$

$$\frac{\partial^2 E_z}{\partial z^2} - \left(k_x^2 - \left(\frac{\omega}{c}\right)^2\epsilon_i\right)E_z = 0 \tag{A.11}$$

Here we use the definition: ${k_z}_i^2 \equiv k_x^2 - \left(\frac{\omega}{c}\right)^2\epsilon_i$ to yield:

$$\frac{\partial^2 E_z}{\partial z^2} - {k_z}_i^2 E_z = 0 \rightarrow \lambda^2 - {k_z}_i^2 = 0 \rightarrow \lambda = \pm k_{zi} \tag{A.12}$$

$$\boxed{E_z = \mathcal{A}\mathrm{e}^{k_{zi}\cdot z} \pm \mathcal{B}\mathrm{e}^{-k_{zi}\cdot z}} \tag{A.13}$$

Where the A = B condition yields the even (symmetric) solutions and the A = -B conditions yields the odd (anti-symmetric) solutions. To solve for E_x, we combine this result with that of (A.7b) to get:

$$\frac{\partial E_z}{\partial z} = -ik_x E_x = \mathcal{A}{k_z}_i\mathrm{e}^{k_{zi}\cdot z} \mp \mathcal{B}{k_z}_i\mathrm{e}^{-k_{zi}\cdot z} \tag{A.14}$$

$$\boxed{E_x = -\frac{k_{zi}}{ik_x}\left(\mathcal{A}\mathrm{e}^{k_{zi}\cdot z} \mp \mathcal{B}\mathrm{e}^{-k_{zi}\cdot z}\right)} \tag{A.15}$$

Finally, to solve for E_y, we combine (A.4a), (A.5b), and (A.4c) to obtain:

$$B_x = -\frac{c}{i\omega}\frac{\partial E_y}{\partial z} \tag{A.16}$$

$$\frac{\partial B_x}{\partial z} - ik_x B_z = -\frac{i\omega}{c}\epsilon_i E_y \tag{A.17}$$

$$\frac{\partial\left(-\frac{c}{i\omega}\cdot\frac{\partial E_y}{\partial z}\right)}{\partial z} - ik_x\left(\frac{ck_x}{\omega}\right)E_y = -\frac{i\omega}{c}\epsilon_i E_y \tag{A.18}$$

$$-\frac{c}{i\omega}\cdot\frac{\partial^2 E_y}{\partial z^2} + \frac{c}{i\omega}\cdot k_x^2 E_y + \frac{i\omega}{c}\epsilon_i E_y = 0 \tag{A.19}$$

$$\frac{\partial^2 E_y}{\partial z^2} - \left(k_x^2 + \left(\frac{i\omega}{c}\right)^2\epsilon_i\right)E_y = 0 \tag{A.20}$$

$$\frac{\partial^2 E_y}{\partial z^2} - \left(k_x^2 - \left(\frac{\omega}{c}\right)^2\epsilon_i\right)E_y = 0 \tag{A.21}$$

Here we use the definition $\beta_i^2 \equiv -k_x^2 + \left(\frac{\omega}{c}\right)^2\epsilon_i \rightarrow \beta_i \equiv ik_{zi}$ to obtain:

$$\frac{\partial^2 E_y}{\partial z^2} + \beta_i^2 E_y = 0 \tag{A.22}$$

$$\boxed{E_y = \mathcal{C}\mathrm{e}^{i\beta_i\cdot z} \pm \mathcal{D}\mathrm{e}^{-i\beta_i\cdot z} = \mathcal{C}\mathrm{e}^{-k_{zi}\cdot z} \pm \mathcal{D}\mathrm{e}^{k_{zi}\cdot z}} \tag{A.23}$$

To obtain the z-component of $\vec{B}$, we use the fact that $ik_xE_y = \frac{i\omega}{c}B_z$ from (A.4c). It then follows that:

$$\boxed{B_z = \frac{ck_x}{\omega}\left(\mathcal{C}\mathrm{e}^{i\beta_i\cdot z} \pm \mathcal{D}\mathrm{e}^{-i\beta_i\cdot z}\right) = \frac{ck_x}{\omega}\left(\mathcal{C}\mathrm{e}^{-k_{zi}\cdot z} \pm \mathcal{D}\mathrm{e}^{k_{zi}\cdot z}\right)} \tag{A.24}$$

and from (A.4a) it follows that:

$$-\frac{\partial E_y}{\partial z} = -\mathcal{C}\beta_i\mathrm{e}^{i\beta_i\cdot z} \pm \mathcal{D}\beta_i\mathrm{e}^{-i\beta_i\cdot z} = \frac{i\omega}{c}\cdot B_x \tag{A.25}$$

$$\boxed{B_x = \frac{c}{\omega}\beta_i\left(-\mathcal{C}\mathrm{e}^{i\beta_i\cdot z} \pm \mathcal{D}\mathrm{e}^{-i\beta_i\cdot z}\right)} \tag{A.26}$$

Finally, using the fact that $ik_xB_y = -\frac{i\omega}{c}\cdot\epsilon_i E_z$ from (A.5c) we obtain:

$$\boxed{B_y = -\frac{\omega}{c}\left(\frac{1}{k_x}\right)\epsilon_i\left(\mathcal{A}\mathrm{e}^{k_{zi}\cdot z} \mp \mathcal{B}\mathrm{e}^{-k_{zi}\cdot z}\right)} \tag{A.27}$$

Now that we have each component of $\vec{E}$ and $\vec{B}$, we introduce the boundary conditions necessary for the metal-insulator-metal waveguide structure. We know that outside the waveguide, both $\vec{E}$ and $\vec{B}$ must decay to 0 as z $\rightarrow \infty$; however, within the waveguide, no such restrictions exist. For the general solution, we have A = B for the even (symmetric) solutions and A = -B for the odd (anti-symmetric) solutions. Combining these assumptions we get two sets of equations for the general solution for waves either inside, or outside of the layered, waveguide structure.

Inside the waveguide:

$$E_x = -\frac{k_{z1}}{ik_x}\left(\mathrm{e}^{k_{z1}z} \mp \mathrm{e}^{-k_{z1}z}\right) \tag{A.28a}$$

$$E_y = \mathrm{e}^{-k_{z1}z} \pm \mathrm{e}^{k_{z1}z} \tag{A.28b}$$

$$E_z = \mathrm{e}^{k_{z1}z} \pm \mathrm{e}^{-k_{z1}z} \tag{A.28c}$$

$$B_x = \frac{ic}{\omega}k_{z1}\left(-\mathrm{e}^{-k_{z1}z} \pm \mathrm{e}^{k_{z1}z}\right) \tag{A.28d}$$

$$B_y = -\frac{\omega}{c}\left(\frac{1}{k_x}\right)\epsilon_l\left(\mathrm{e}^{k_{z1}z} \mp \mathrm{e}^{-k_{z1}z}\right) \tag{A.28e}$$

$$B_z = \frac{ck_x}{\omega}\left(\mathrm{e}^{-k_{z1}z} \pm \mathrm{e}^{k_{z1}z}\right) \tag{A.28f}$$

Outside the waveguide:

$$E_x = -\frac{k_{z2}}{ik_x}\left(\mp\mathcal{B}\mathrm{e}^{-k_{z2}z}\right) \tag{A.29a}$$

$$E_y = \mathcal{C}\mathrm{e}^{-k_{z2}z} \tag{A.29b}$$

$$E_z = \pm\mathcal{B}\mathrm{e}^{-k_{z2}z} \tag{A.29c}$$

$$B_x = -\frac{ic}{\omega}k_{z2}\mathcal{C}\mathrm{e}^{-k_{z2}z} \tag{A.29d}$$

$$B_y = -\frac{\omega}{c}\left(\frac{1}{k_x}\right)\epsilon_l\left(\mp\mathcal{B}\mathrm{e}^{-k_{z2}z}\right) \tag{A.29e}$$

$$B_z = \frac{ck_x}{\omega}\mathcal{C}\mathrm{e}^{-k_{z2}z} \tag{A.29f}$$

A.2 Boundary Conditions

For all solutions, we assume that:

- E_x and D_z are continuous at $z = \pm\frac{d}{2}$

- E_y is continuous at $z = \pm\frac{d}{2}$
- B_z is continuous at $z = \pm\frac{d}{2}$
- $\frac{1}{\mu}B_x$ is continuous at $z = \pm\frac{d}{2}$
- $\frac{1}{\mu}B_y$ is continuous at $z = \pm\frac{d}{2}$

A.2.1 E_x and D_z are continuous:

From (A.28a) and (A.29a) we have:

$$\frac{ik_{z1}}{k_x}\left(\mathrm{e}^{k_{z1}z} \mp \mathrm{e}^{-k_{z1}z}\right) = \pm\frac{k_{z2}}{ik_x}\left(\mathcal{B}\mathrm{e}^{-k_{z2}z}\right) \tag{A.30}$$

From (A.28c) and (A.29c) we have:

$$\epsilon_1\left(\mathrm{e}^{k_{z1}z} \pm \mathrm{e}^{-k_{z1}z}\right) = \pm\epsilon_2\left(\mathcal{B}\mathrm{e}^{-k_{z2}z}\right) \tag{A.31}$$

and combining the two we derive:

$$ik_{z1}\left(\mathrm{e}^{k_{z1}z} \mp \mathrm{e}^{-k_{z1}z}\right) = \frac{k_{z2}}{i}\left(\frac{\epsilon_1}{\epsilon_2}\right)\left(\mathrm{e}^{k_{z1}z} \pm \mathrm{e}^{-k_{z1}z}\right) \tag{A.32}$$

$$-\epsilon_2 k_{z1}\left(\mathrm{e}^{k_{z1}z} \mp \mathrm{e}^{-k_{z1}z}\right) = \epsilon_1 k_{z2}\left(\mathrm{e}^{k_{z1}z} \pm \mathrm{e}^{-k_{z1}z}\right) \tag{A.33}$$

$$-\epsilon_2 k_{z1} = \epsilon_1 k_{z2}\begin{Bmatrix}\coth(k_{z1}d/2)\\ \tanh(k_{z1}d/2)\end{Bmatrix} \tag{A.34}$$

which yields the transverse-magnetic dispersion relation:

$$\boxed{\epsilon_1 k_{z2} + \epsilon_2 k_{z1}\begin{Bmatrix}\coth(k_{z1}d/2)\\ \tanh(k_{z1}d/2)\end{Bmatrix} = 0} \tag{A.35}$$

Here, the "coth" function represents the symmetric plasmon modes and the "tanh" function represents the antisymmetric modes.

A.2.2 E_y is continuous:

From (A.28b) and (A.29b) we have:

$$\mathrm{e}^{-k_{z1}z} \pm \mathrm{e}^{k_{z1}z} = \mathcal{C}\mathrm{e}^{-k_{z2}z} \tag{A.36}$$

which yields:

$$\boxed{\mathcal{C} = \mathrm{e}^{k_{z2}d/2}\left(\mathrm{e}^{-k_{z1}d/2} \pm \mathrm{e}^{k_{z1}d/2}\right)} \tag{A.37}$$

A.2.3 H_z, H_y, and B_z are continuous:

From (A.28e) and (A.29e) we have:

$$-\frac{\omega}{c}\left(\frac{1}{k_x}\frac{\epsilon_1}{\mu_1}\right)\left(\mathrm{e}^{k_{z1}d/2} \mp \mathrm{e}^{-k_{z1}d/2}\right) = -\frac{\omega}{c}\left(\frac{1}{k_x}\frac{\epsilon_2}{\mu_2}\right)\left(\mp\mathcal{B}\mathrm{e}^{-k_{z2}d/2}\right) \tag{A.38}$$

$$\epsilon_1\mu_2\left(\mathrm{e}^{k_{z1}d/2} \mp \mathrm{e}^{-k_{z1}d/2}\right) = \epsilon_2\mu_1\left(\mp\mathcal{B}\mathrm{e}^{-k_{z2}d/2}\right) \tag{A.39}$$

which yields:

$$\boxed{\mathcal{B} = \frac{\epsilon_1\mu_2}{\epsilon_2\mu_1}\mathrm{e}^{k_{z2}d/2}\left(\mathrm{e}^{k_{z1}d/2} \mp \mathrm{e}^{-k_{z1}d/2}\right)} \tag{A.40}$$

From (A.28f) and (A.29f) we have:

$$\frac{ck_x}{\omega}\left(\mathrm{e}^{-k_{z1}d/2} \pm \mathrm{e}^{k_{z1}d/2}\right) = \frac{ck_x}{\omega}\mathcal{C}\mathrm{e}^{-k_{z2}d/2} \tag{A.41}$$

$$\mathcal{C} = \mathrm{e}^{k_{z2}d/2}\left(\mathrm{e}^{-k_{z1}d/2} \pm \mathrm{e}^{k_{z1}d/2}\right) \tag{A.42}$$

From (A.28d) and (A.29d) we have:

$$\frac{ick_{z1}}{\omega\mu_1}\left(-\mathrm{e}^{-k_{z1}d/2} \pm \mathrm{e}^{k_{z1}d/2}\right) = -\frac{ick_{z2}}{\omega\mu_2}\mathcal{C}\mathrm{e}^{-k_{z2}z} \tag{A.43}$$

$$\mathcal{C} = -\frac{k_{z1}}{k_{z2}}\frac{\mu_2}{\mu_1}\mathrm{e}^{k_{z2}d/2}\left(-\mathrm{e}^{-k_{z1}d/2} \pm \mathrm{e}^{k_{z1}d/2}\right) \tag{A.44}$$

From this we can infer that:

$$-\frac{k_{z1}}{k_{z2}}\left(-\mathrm{e}^{-k_{z1}d/2} \pm \mathrm{e}^{k_{z1}d/2}\right) = \mathrm{e}^{-k_{z1}d/2} \pm \mathrm{e}^{k_{z1}d/2} \tag{A.45}$$

$$\frac{k_{z1}\mu_2}{k_{z2}\mu_1}\left(\mathrm{e}^{-k_{z1}d/2} \mp \mathrm{e}^{k_{z1}d/2}\right) = \left(\mathrm{e}^{-k_{z1}d/2} \pm \mathrm{e}^{k_{z1}d/2}\right) \tag{A.46}$$

$$\mp k_{z1}\mu_2\left(\mathrm{e}^{k_{z1}d/2} \mp \mathrm{e}^{-k_{z1}d/2}\right) = \pm k_{z2}\mu_1\left(\mathrm{e}^{k_{z1}d/2} \pm \mathrm{e}^{-k_{z1}d/2}\right) \tag{A.47}$$

$$\mp k_{z1}\mu_2 = \pm k_{z2}\mu 1\begin{Bmatrix}\coth(k_{z1}d/2)\\ \tanh(k_{z1}d/2)\end{Bmatrix} \tag{A.48}$$

which yields the transverse-electric dispersion relation:

$$\boxed{\pm\mu_1 k_{z2} \pm \mu_2 k_{z1}\begin{Bmatrix}\tanh(k_{z1}d/2)\\ \coth(k_{z1}d/2)\end{Bmatrix} = 0} \tag{A.49}$$

Thus, by comparing (A.35) and (A.49) we see that the sets of solutions either consist of (E_y, B_x, B_z) or (E_x, E_z, B_y) as was stated at the beginning of the Appendix. From here it can be shown, [71], that there are no non-zero modal solutions to the transverse-electric modes operating within this structure. As a result, we can state that surface plasmon polaritons are strictly transverse-magnetic.