Chapter 2

Introduction to Phase Noise

A brief introduction into the subject of phase noise is given here. We first describe the conversion of the phase fluctuations into the noise sideband of the carrier. We then define phase noise, frequency noise, and Allan deviation with emphasis on their relationship with each other. Leeson's model is described and used to analyze the thermal noise of an ideal, linear LC oscillator. Finally, we give the general expression of the minimum measurable frequency shift in a noisy system.

2.1 Introduction

In general, circuit and device noise can perturb both the amplitude and phase of an oscillator's output.[1,2] Of necessity, all practical oscillators inherently possess an amplitude limiting mechanism of some kind. Because the amplitude fluctuations are attenuated, phase noise generally dominates. We will primarily focus on phase noise in our theoretical exposition and divide the theoretical investigation into two parts. The first part is the general conceptual foundation on how the frequency stability of an oscillator should be characterized, more commonly known as the subject of phase noise. The second part is the exposition on the *physical* phase noise mechanisms affecting NEMS devices. In this chapter, we will deal with the first part and defer the second part to chapter 3. We will also describe Leeson's model to analyze the thermal noise of an ideal, linear LC oscillator. Finally, we will give expressions translating the frequency noise into the minimum measurable frequency shift in a noisy system.

2.2 General Remark

The output of an oscillator of angular frequency ω_C is generally given by

$$X(t) = X_0(1 + A(t)) f[\omega_C t + \phi(t)]. \tag{2.1}$$

Here $\phi(t)$ and $A(t)$ are functions of time and f is a periodic function. Here X can be the output voltage from an electrical oscillator or the displacement of a mechanical oscillator. The output spectrum contains higher harmonics of ω_C if the waveform is not sinusoidal. For our purpose, we assume no higher harmonics from any nonlinearity of the devices or the circuits, and thus the output $X(t)$ is purely sinusoidal. For a sinusoidal oscillation, the output is given by

$$X(t) = X_0(1 + A(t))\sin[\omega_C t + \phi(t)]\,. \tag{2.2}$$

2.3 Phase Noise

The physical fluctuations in the oscillator can perturb the phase of the oscillation and produce phase fluctuations. We now describe how then phase fluctuations are converted into noise sidebands around the carrier. Considering a small phase variation $\phi(t) = \phi_0 \sin \omega t$, equation (2.2) can be expanded as

$$\begin{aligned} X(t) &= X_0(1 + A(t))\sin(\omega_C t + \phi_0 \sin \omega t + \theta) \\ &= X_0 \sin(\omega_C t + \theta) + X_0 \frac{\phi_0}{2}\sin[(\omega_C + \omega)t] - X_0 \frac{\phi_0}{2}\sin[(\omega_C - \omega)t]. \end{aligned} \tag{2.3}$$

The phase variation generates two sidebands spaced $\pm\omega$ from the carrier with amplitude $X_0\phi_0 / 2$. The upper sideband is phase-coherent with the lower sideband with the opposite sign. The generated sideband is characterized in the following definition: it is conventionally given by the ratio of noise power to carrier power for 1 Hz bandwidth with offset frequency from the carrier. In notation, the definition is given by

$$L_{total}(\omega) = 10\log\left(\frac{P_{sideband}(\omega_C + \omega, 1Hz)}{P_C}\right). \tag{2.4}$$

P_C is the carrier power and $P_{sidebank}(\omega_C + \omega, 1Hz)$ is the single sideband power at a frequency offset ω from the carrier frequency ω_C with the measurement bandwidth of 1 Hz as shown in figure 2.1. $L_{total}(\omega)$ is thus in units of decibel referred to the carrier power per hertz (dBc/Hz).

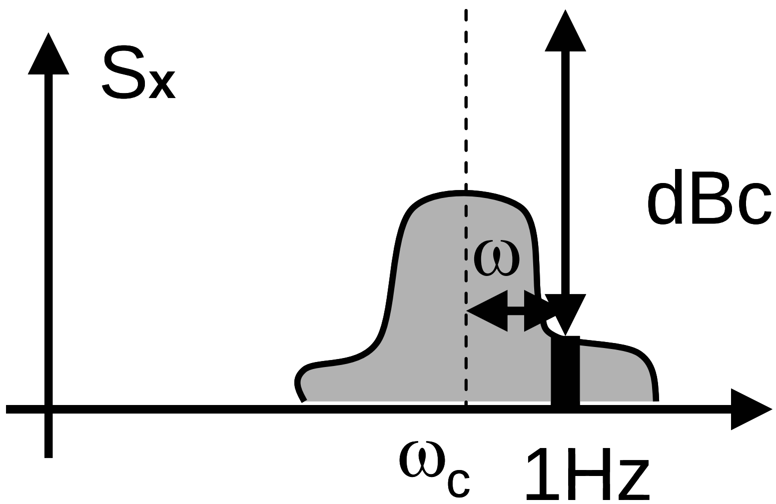

Figure 2.1. Definition of phase noise. The phase noise is conventionally expressed as the ratio of sideband noise power for 1 Hz bandwidth to the carrier power in units of dBc/Hz.

2.4 Frequency Noise

Phase is the integration of frequency over time, i.e.,

$$\phi(t) = \int_{-\infty}^{t} \omega(\tau) d\tau \,. \tag{2.5}$$

Conversely, frequency is the derivative of phase with respect to time, i.e.,

$$\omega(t) = \frac{d\phi}{dt} \,. \tag{2.6}$$

The spectral density of the phase noise is thus related to the spectral density of the frequency noise by

$$S_\phi(\omega) = \frac{1}{\omega^2} S_\omega(\omega) \,. \tag{2.7}$$

In addition to angular frequency, we introduce another commonly used quantity, fractional frequency, defined as ratio of frequency to carrier frequency.

$$y = \frac{\delta\omega}{\omega_C} \,. \tag{2.8}$$

The spectral density of fractional frequency is related to the spectral density of frequency by

$$S_y(\omega) = \frac{1}{\omega_C^2} S_\omega(\omega) \,. \tag{2.9}$$

The resonance frequency depends on many physical parameters of the resonator. The fluctuations of these parameters can translate into fractional frequency noise. The fractional noise is related to the fluctuation of the corresponding parameter by

$$S_y(\omega) = \left(\frac{\partial y}{\partial \chi} \right)^2 S_\chi \,. \tag{2.10}$$

χ is the physical parameter which the resonant frequency is dependent on. For example, if χ is the temperature T of the device, $\partial y/\partial T$ is simply the temperature coefficient of the resonant frequency.

2.5 Allan Variance and Allan Deviation

Allan variance is a quantity commonly used by the frequency standard community to compare the frequency stabilities of different oscillators. The phase and frequency noise are defined in the frequency domain; the Allan deviation is defined in the time domain. Allan deviation, $\sigma_A(\tau_A)$, is simply the square root of Allan variance, $\sigma_A^2(\tau_A)$. The defining expression of the Allan deviation is given by[1,3]

$$\sigma_A^2(\tau_A) = \frac{1}{2f_C^2}\frac{1}{N-1}\sum_{m=2}^{N_S}(\bar{f}_m - \bar{f}_{m-1})^2 . \tag{2.11}$$

$\bar{f}_m$ is the average frequency measured over the mth interval with zero dead time and N_S is the sample number. From this definition, the Allan deviation is related to the phase noise density by

$$\sigma_A^2(\tau_A) = 2\left(\frac{2}{\omega\tau_A}\right)^2 \int_0^\infty S_\phi(\omega)\sin^4(\omega\tau_A/2)d\omega . \tag{2.12}$$

In the experimental data, Allan deviation is usually presented with the error bar given by one standard deviation confidence interval (or 68% confidence interval), i.e., $\sigma_A/\sqrt{N_S-1}$. For example, for sample number N_S=101, the one standard deviation confidence interval is 10% of the Allan deviation.

The noise spectra with different power laws are commonly used so we give the formulas of the corresponding Allan deviations. For phase noise having $1/f^4$ component, i.e., $S_\phi(\omega) = C_4(\omega_C/\omega)^4$, the Allan deviation is given by

$$\sigma_A(\tau_A) = \sqrt{\frac{\pi}{3} C_4 \omega_C^2 \tau_A} \ . \tag{2.13}$$

For phase noise having $1/f^3$ component, i.e., $S_\phi(\omega) = C_3(\omega_C/\omega)^3$, the Allan deviation is given by

$$\sigma_A(\tau_A) = \sqrt{2\log_e 2 C_3 \omega_C} \ . \tag{2.14}$$

For phase noise having $1/f^2$ component, i.e., $S_\phi(\omega) = C_2(\omega_C/\omega)^2$, the Allan deviation is given by

$$\sigma_A(\tau_A) = \sqrt{\frac{\pi C_2}{\tau_A}} \ . \tag{2.15}$$

For the fractional frequency noise having the Lorentizian function form, i.e., $S_y(\omega) = A/(1+(\omega\tau_r)^2)$, the spectral density of phase noise is given by $S_\phi(\omega) = A(\omega_C/\omega)^2/(1+(\omega\tau_r)^2)$. Upon integration, the Allan deviation is given by

$$\sigma_A(\tau_A) = \sqrt{\frac{A}{2\pi} F(\frac{\tau_r}{\tau_A})} \ . \tag{2.16}$$

$F(x)$ is an analytic function defined by

$$F(x) = \frac{1}{x^2}\int_0^\infty \frac{\sin^4(\xi x/2)d\xi}{\xi^2(1+\xi^2)} = \frac{1}{2x} - \frac{1}{x^2}[(1-e^{-x}) - \frac{1}{4}(1-e^{-2x})]. \tag{2.17}$$

As shown in figure 2.2, $F(x)$ reaches a maximum at x=1.89 with the value 0.095. The asymptotic expressions of $F(x)$ are $F(x) = \frac{1}{2x}$ for x>>1 and $F(x) = \frac{1}{6}x$ for x <<1.

These behaviors can also be clearly seen in figure 2.2. In the limit $\tau_r << \tau_A$, equation (2.16) becomes

$$\sigma_A(\tau_A) = \sqrt{\frac{A\tau_r}{4\pi\tau_A}} \ . \tag{2.18}$$

In the other limit $\tau_A << \tau_r$, equation (2.16) becomes

$$\sigma_A(\tau_A) = \sqrt{\frac{A\tau_A}{12\pi\tau_r}} \ . \tag{2.19}$$

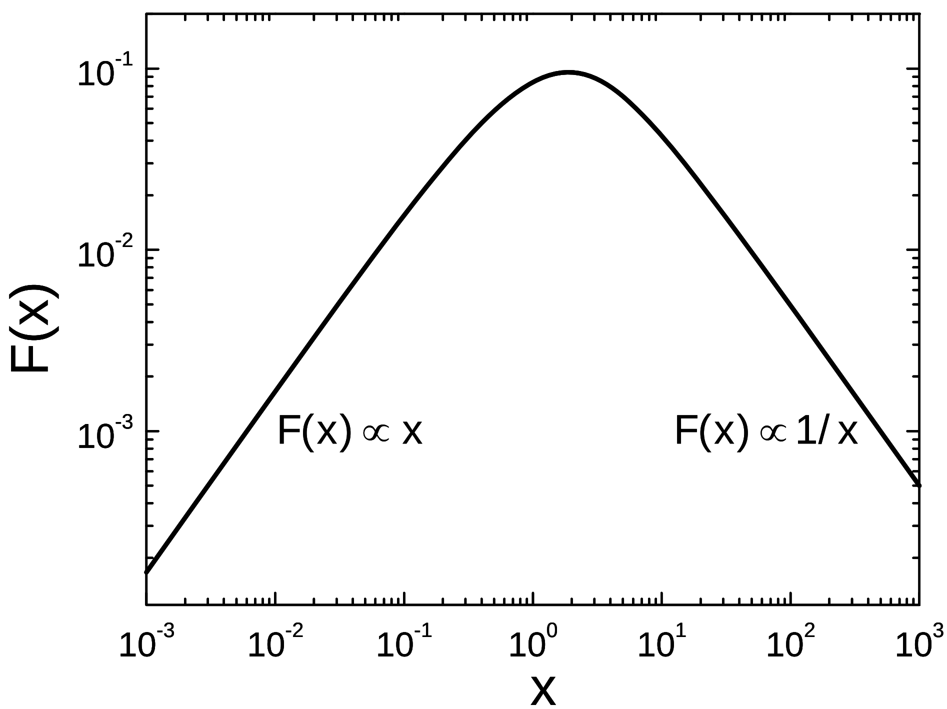

Figure 2.2. Plot of the function *F(x)*. $F(x)$ shows the dependence of Allan deviation, having frequency noise density of Lorentzian form, on the ratio of the correlation time τ_r to the averaging time τ_A. $F(x)$ reaches a maximum at x=1.85 with the value 0.095. Its asymptotic behaviors for x <<1 and for x>>1 are also shown.

2.6 Thermal Noise of an Ideal Linear LC Oscillator

The phase noise of an ideal linear LC oscillator due to the Nyquist-Johnson noise is analyzed by Leeson.[4] Figure 2.3 shows that the Nyquist-Johnson noise source associated with the resistor injects noise current into a LC tank circuit. The impedance of the LC tank with a quality factor Q and the resonant frequency ω_0 at offset frequency ω ($\omega << \omega_0$) is given by

$$Z(\omega_0+\omega)=\frac{1}{G}\frac{1}{1+j2Q\dfrac{\omega}{\omega_0}}. \tag{2.20}$$

To sustain oscillation, the active device must compensate the energy dissipation by positive feedback. Therefore, the active device behaves as a negative conductance $-G$. For steady state oscillation, the impedance of the oscillator model is given by

$$Z(\omega)=\frac{v_{out}(\omega_0+\omega)}{i_{in}(\omega_0+\omega)}=-j\frac{1}{G}\frac{\omega_0}{2Q\omega}. \tag{2.21}$$

The total equivalent parallel resistance of the tank has an equivalent mean square noise current density of $i_{in}^2/\Delta f=4k_BTG$. Using this effective current power, the phase noise can be calculated as

$$S_\phi(\omega)=\frac{v_{noise}^2}{v_{signal}^2}=\frac{\frac{1}{2}\left|Z(\omega)\right|^2 i_{in}^2/\Delta f}{\frac{1}{2}V_o^2}=\frac{k_BT}{2P_CQ^2}\left(\frac{\omega_0}{\omega}\right)^2. \tag{2.22}$$

P_C is the carrier power usually limited by saturation or nonlinearity of the active device. The Leeson model demonstrates explicitly the conversion of the current noise into sideband and explains the $1/\omega^2$ dependence of the phase noise density. Upon integration of the spectral density, we obtain the expression for the Allan deviation.

$$\sigma_A(\tau_A) = \sqrt{\frac{k_B T}{P_C}\frac{1}{Q^2 \tau_A}} \; . \tag{2.23}$$

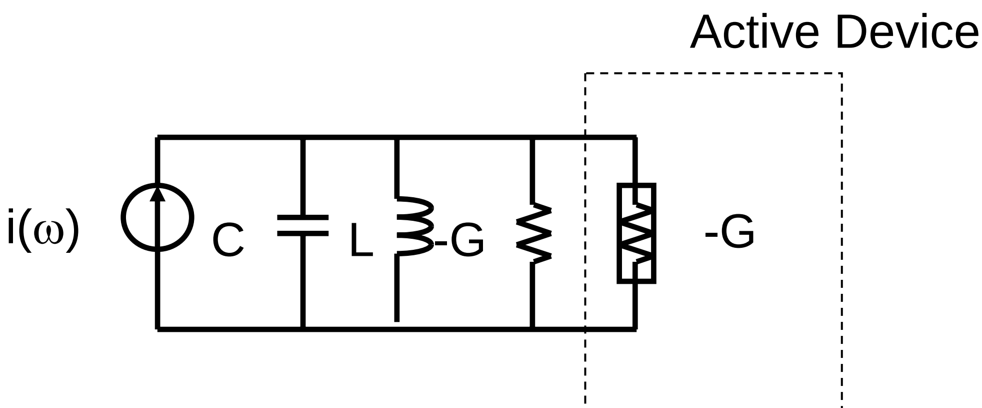

Figure 2.3. Leeson's model of phase noise for an ideal linear LC oscillator. Equivalent one-port circuit for phase noise calculation for an ideal linear LC oscillator is used in the model. The Nyquist-Johnson noise source associated with the resistor injects noise current in LC tank, producing the noise sideband around the carrier. Note that the active device, compensating the energy dissipation from the resistor, is modeled as a negative conductance.

2.7 Minimum Measurable Frequency Shift

Experimentally we measure the change in physical properties of the resonator by detecting the corresponding frequency shift and thus an important question needs to be addressed:what is the minimum measurable frequency shift, $\delta\omega_0$, that can be resolved in a (realistic) noisy system? In principle, a shift comparable to the mean square noise (the spread) in an ensemble average of a series of frequency measurements should be resolvable, i.e., $\delta\omega_0 \approx \frac{1}{N}\sqrt{\sum_{i=1}^{N}(\omega_i - \omega_0)^2}$ for signal-to-noise ratio equal to unity. An estimate for $\delta\omega_0$ can be obtained by integrating the weighted effective spectral density of the frequency fluctuations, $S_\omega(\omega)$, by the normalized transfer function of the measurement loop, $H(\omega)$:

$$\delta\omega_0 \approx [\int_0^\infty S_\omega(\omega)H(\omega)d\omega]^{1/2}. \tag{2.24}$$

Here, $S_\omega(\omega)$ is in units of $(rad/s^2)/(rad/s)$. We can further simplify equation (2.24) by replacing $H(\omega)$ with the square transfer function $H^{'}(\omega)$, which has the same integrated spectral weight, but is non-zero only within the passband delineated by $2\pi\Delta f$. Here, $\Delta f \approx 2\pi/\tau$ and is dependent upon the measurement averaging time, τ. Given this assumption, equation (2.24) takes the simpler, more familiar form.

$$\delta\omega_0 \approx [\int_0^{2\pi\Delta f} S_\omega(\omega)d\omega]^{1/2}. \tag{2.25}$$

This, of course, is an approximation to a real system — albeit a good one. If necessary, one can resort to the more accurate expression, equation (2.24).

2.8 Conclusion

We describe the conversion of phase fluctuations into the noise sideband of the carrier and present the definitions of phase noise, frequency noise, and Allan deviation, all commonly used to characterize the frequency stability of an oscillator. Figure 2.4 summarizes the relation between these quantities. We illustrate these definitions by analyzing the phase noise of an ideal, linear LC oscillator in the context of Leeson's model. In particular, Leeson's model explicitly demonstrates how the Nyquist-Johnson current noise produces noise sideband of carrier and explains the $1/\omega^2$ dependence of the phase noise density on the offset frequency. Finally, we give the expressions for the minimum measurable frequency shift in a noisy system for sensing applications involving oscillators.

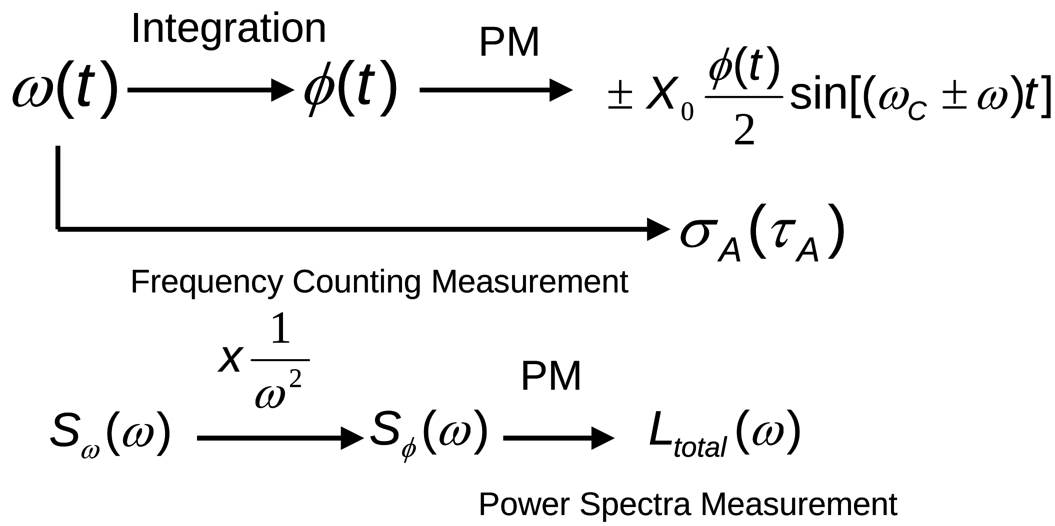

Figure 2.4. Summary of the relation between different quantities. In time domain, the phase variation $\phi(t)$, which is the integration of angular frequency variation $\omega(t)$, generates the sidebands $\pm x_0(\phi(t)/2)\sin[(\omega_c \pm \omega)t]$ through phase modulation (PM). The Allan deviation can be calculated with the frequency data from the frequency counting measurements. In the frequency domain, the frequency noise density $S_\omega(\omega)$ is related to the phase noise density $S_\phi(\omega)$ by $S_\phi(\omega) = 1/\omega^2 S_\omega(\omega)$. The noise sideband of the carrier is characterized by $L_{total}(\omega)$, which can be obtained from the power spectrum measurement.

References

1. A. N. Cleland and M. L. Roukes Noise processes in nanomechanical resonators. *J. Appl. Phys.* **92**, 2758 (2002).

2. A. Hajimiri and T. H. Lee *The design of low noise oscillator* (Norwell, Kluwer Academic Publisher, 1999).

3. D. W. Allan Statistics of atomic frequency standard. *Proc. IEEE* **54**, 221 (1966).

4. D. B Leeson A simple model of feedback oscillator noise spectrum. *Proc. IEEE* **54**, 329 (1996).