Appendix A

Derivation of the spin-relaxation equations from the full master equation

In this appendix we illustrate the method we have used in deriving equations of motion for spin operators using the interaction-frame master equation for the spin-resonator system:

$$\frac{d}{dt}\rho = \frac{1}{i\hbar}\left[H_s + V, \rho\right] + \Lambda\rho. \tag{A.1}$$

We assume that the resonator's ringdown time τ_h is so short that the resonator functions as a reservoir, remaining near thermal equilibrium during its interaction with the spins, and we derive a coarse-grained derivative $\Delta \langle I_z \rangle / \Delta t$, where

$$\Delta t \gg \tau_h. \tag{A.2}$$

In addition to satisfying (A.2), the time step Δt must be short compared to the time required for relaxation of $\langle I_z \rangle$.

We use time-dependent perturbation theory to evaluate the coarse-grained derivative to lowest order in the coupling constant g. To motivate the approach, we first recall that a master equation of the form

$$\frac{d}{dt}\rho\left(t\right) = \mathcal{L}\rho\left(t\right)$$

can be transformed into an integral equation:

$$\rho(t) = \rho(0) + \int_0^t \mathcal{L}\rho(t_1)\, dt_1.$$

Replacing the density matrix $\rho(t_1)$ appearing in the integrand by an integral equation for $\rho(t_1)$ yields

$$\rho(t) = \rho(0) + \int_0^t \mathcal{L}\rho(0)\, dt_1 + \int_0^t \int_0^{t_1} \mathcal{L}(\mathcal{L}\rho(t_2))\, dt_2\, dt_1.$$

Repeating the process of substituting an integral equation for the time-dependent integrand yields a series expansion in which successive terms depend on higher powers of the superoperator $\mathcal{L}$.

An analogous process can be used to obtain a series expansion of $\Delta\langle I_z\rangle/\Delta t$. We use (A.1) to find the instantaneous derivative $d\langle I_z\rangle/dt$, and this derivative is transformed to an integral equation for $\langle I_z\rangle$. Time-dependent quantities appearing in the integrand are themselves replaced by integral equations, and the process is repeated to yield a series expansion for $\langle I_z\rangle$. Terms of high-order in the coupling constant g are discarded, and the remaining integrals are evaluated to yield an explicit formula for $\Delta\langle I_z\rangle/\Delta t$.

In carrying out this procedure, we will use the following equations:

$$\langle I_z\rangle(t) = \langle I_z\rangle(0) + \int_0^t (-ig)\left\langle I_+a^\dagger - I_-a\right\rangle(t_1)\, dt_1, \tag{A.3}$$

$$\begin{aligned}\left\langle I_+a^\dagger - I_-a\right\rangle(t) = {} & e^{-t/\tau_h}\left\langle I_+a^\dagger - I_-a\right\rangle(0) \\ & + e^{-t/\tau_h}\int_0^t e^{t_1/\tau_h}(-4ig)\left\langle I_z a^\dagger a\right\rangle(t_1)\, dt_1 \\ & + e^{-t/\tau_h}\int_0^t e^{t_1/\tau_h} 2ig\langle I_-I_+\rangle(t_1)\, dt_1,\end{aligned} \tag{A.4}$$

$$\begin{aligned}\left\langle I_z a^\dagger a\right\rangle(t) = {} & e^{-2t/\tau_h}\left\langle I_z a^\dagger a\right\rangle(0) \\ & + e^{-2t/\tau_h}\int_0^t e^{2t_1/\tau_h}\left(\frac{2n_{\text{th}}}{\tau_h}\right)\langle I_z\rangle(t_1)\, dt_1 + O(g),\end{aligned} \tag{A.5}$$

$$\langle I_-I_+\rangle(t) = \langle I_-I_+\rangle(0) + O(g). \tag{A.6}$$

These are derived by transforming derivatives obtained from (A.1) into integral equations. Replacing t in (A.3) by Δt gives

$$\begin{aligned}\frac{\Delta\left\langle I_z\right\rangle}{\Delta t} &= \frac{\left\langle I_z\right\rangle(t)-\left\langle I_z\right\rangle(0)}{\Delta t}\\ &= \frac{-ig}{\Delta t}\int_0^{\Delta t}\left\langle I_+a^\dagger - I_-a\right\rangle(t_1)\,dt_1. \end{aligned}\tag{A.7}$$

From (A.4), we obtain an integral equation for $\left\langle I_+a^\dagger - I_-a\right\rangle(t_1)$ which is substituted into the integrand of (A.7):

$$\begin{aligned}\frac{\Delta\left\langle I_z\right\rangle}{\Delta t} &= \frac{-ig}{\Delta t}\left\langle I_+a^\dagger - I_-a\right\rangle(0)\int_0^{\Delta t} e^{-t_1/\tau_h}\,dt_1\\ &+\frac{-ig}{\Delta t}(-4ig)\int_0^{\Delta t} e^{-t_1/\tau_h}\int_0^{t_1} e^{t_2/\tau_h}\left\langle I_z a^\dagger a\right\rangle(t_2)\,dt_2\,dt_1\\ &+\frac{-ig}{\Delta t}(2ig)\int_0^{\Delta t} e^{-t_1/\tau_h}\int_0^{t_1} e^{t_2/\tau_h}\left\langle I_-I_+\right\rangle(t_2)\,dt_2\,dt_1.\end{aligned}$$

Continuing in this way, we obtain

$$\begin{aligned}\frac{\Delta\left\langle I_z\right\rangle}{\Delta t} &= \frac{-ig}{\Delta t}\left\langle I_+a^\dagger - I_-a\right\rangle(0)\int_0^{\Delta t} e^{-t_1/\tau_h}\,dt_1\\ &+\frac{-ig}{\Delta t}(-4ig)\left\langle I_z a^\dagger a\right\rangle(0)\int_0^{\Delta t} e^{-t_1/\tau_h}\int_0^{t_1} e^{t_2/\tau_h}e^{-2t_2/\tau_h}\,dt_2\,dt_1\\ &+\frac{-ig}{\Delta t}(-4ig)\,n_{\mathrm{th}}\left\langle I_z\right\rangle(0)\int_0^{\Delta t} e^{-t_1/\tau_h}\int_0^{t_1} e^{t_2/\tau_h}\left(1-e^{-2t_2/\tau_h}\right)dt_2\,dt_1\\ &+\frac{-ig}{\Delta t}(2ig)\left\langle I_-I_+\right\rangle(0)\int_0^{\Delta t} e^{-t_1/\tau_h}\int_0^{t_1} e^{t_2/\tau_h}\,dt_2\,dt_1+O\left(g^3\right),\end{aligned}$$

which is evaluated as

$$\begin{aligned}\frac{\Delta\left\langle I_z\right\rangle}{\Delta t} &= \frac{\tau_h}{\Delta t}\left(1-e^{-\Delta t/\tau_h}\right)(-ig)\left\langle I_+a^\dagger - I_-a\right\rangle(0)\\ &+\frac{1}{\Delta t}\left[\tau_h^2\left(\frac{1}{2}-e^{-\Delta t/\tau_h}+\frac{1}{2}e^{-2\Delta t/\tau_h}\right)\right]\left(-4g^2\right)\left\langle I_z a^\dagger a\right\rangle(0)\\ &+\frac{1}{\Delta t}\left[\tau_h\Delta t+\tau_h^2\left(-\frac{3}{2}+2e^{-\Delta t/\tau_h}-\frac{1}{2}e^{-2\Delta t/\tau_h}\right)\right]\left(-4g^2\right)n_{\mathrm{th}}\left\langle I_z\right\rangle(0)\\ &+\frac{1}{\Delta t}\left[\tau_h\Delta t-\tau_h^2\left(1-e^{-\Delta t/\tau_h}\right)\right]\left(2g^2\right)\left\langle I_-I_+\right\rangle(0)+O\left(g^3\right).\end{aligned}\tag{A.8}$$

Equation (A.8) is correct to second order in the coupling constant, regardless of the relative sizes of τ_h and Δt. In the case where $\tau_h \ll \Delta t$, negligible error is introduced by considering the resonator to be uncorrelated with the spins and in a thermal state at the beginning of the time step. A similar approximation is made in the general derivation of the master equation given in reference [7], where it is shown that correlations present at the beginning of the time step make a contribution to the motion only during a time period of order τ_h. Since the initial spin-resonator correlations decay almost immediately on the scale of the time step Δt, these correlations do not make a significant contribution to the motion of $\langle I_z \rangle$ during Δt. Relaxation of $\langle I_z \rangle$ depends on the new correlations which develop continually during Δt, and the contribution of these correlations is not affected by the approximation of treating the resonator and spins as initially uncorrelated. This approximation yields

$$\begin{aligned}
\left\langle I_+ a^\dagger - I_- a \right\rangle (0) &= \langle I_+ \rangle (0) \left\langle a^\dagger \right\rangle_{\text{th}} - \langle I_- \rangle (0) \langle a \rangle_{\text{th}} \\
&= 0, \\
\left\langle I_z a^\dagger a \right\rangle (0) &= \langle I_z \rangle (0) \left\langle a^\dagger a \right\rangle_{\text{th}} \\
&= \langle I_z \rangle (0)\, n_{\text{th}},
\end{aligned}$$

from which it follows that

$$\begin{aligned}
\frac{\Delta \langle I_z \rangle}{\Delta t} &= \frac{1}{\Delta t} \left[\tau_h^2 \left(\frac{1}{2} - e^{-\Delta t/\tau_h} + \frac{1}{2} e^{-2\Delta t/\tau_h} \right) \right] \left(-4g^2 \right) n_{\text{th}} \langle I_z \rangle \\
&+ \frac{1}{\Delta t} \left[\tau_h \Delta t + \tau_h^2 \left(-\frac{3}{2} + 2e^{-\Delta t/\tau_h} - \frac{1}{2} e^{-2\Delta t/\tau_h} \right) \right] \left(-4g^2 \right) n_{\text{th}} \langle I_z \rangle \\
&+ \frac{1}{\Delta t} \left[\tau_h \Delta t - \tau_h^2 \left(1 - e^{-\Delta t/\tau_h} \right) \right] \left(2g^2 \right) \langle I_- I_+ \rangle .
\end{aligned} \tag{A.9}$$

Since τ_h^2 is negligible compared to $\tau_h \Delta t$, we can discard terms in (A.9) proportional to τ_h^2 and obtain an equation of motion for $\langle I_z \rangle$:

$$\begin{aligned}
\frac{\Delta \langle I_z \rangle}{\Delta t} &= -4g^2 \tau_h n_{\text{th}} \langle I_z \rangle + 2g^2 \tau_h \langle I_- I_+ \rangle \\
&= R_0 \left(n_{\text{th}} + 1 \right) \langle I_- I_+ \rangle - R_0 n_{\text{th}} \langle I_+ I_- \rangle .
\end{aligned} \tag{A.10}$$

Appendix B

Relative magnitudes of the rate constants for lifetime and secular broadening

Section 2 of chapter 2 presents the interaction-frame equations for resonator-induced transverse relaxation. In the case where the magnetic field $\mathbf{B}(\theta)$ is approximated to second order in θ, the equations can be expressed in the form

$$\begin{aligned}\frac{d}{dt}\langle I_x\rangle &= -\left(R_{\text{lifetime}}+R_{\text{secular}}\right)\langle I_x\rangle - R_0\left\langle\frac{1}{2}\left(I_xI_z+I_zI_x\right)\right\rangle,\\ \frac{d}{dt}\langle I_y\rangle &= -\left(R_{\text{lifetime}}+R_{\text{secular}}\right)\langle I_y\rangle - R_0\left\langle\frac{1}{2}\left(I_yI_z+I_zI_y\right)\right\rangle,\end{aligned}$$

where

$$\begin{aligned}R_{\text{lifetime}} &= g^2\tau_h\left(2n_{\text{th}}+1\right)\\ &= \left(\frac{\gamma}{2}\frac{dB_x}{d\theta}\right)^2\frac{\hbar}{2I_h\omega_h}\tau_h\left(2n_{\text{th}}+1\right)\end{aligned}$$

and

$$\begin{aligned}R_{\text{secular}} &= \frac{1}{2}f^2\tau_h n_{\text{th}}\left(n_{\text{th}}+1\right)\\ &= \frac{1}{2}\left(\gamma\frac{d^2B_z}{d\theta^2}\frac{\hbar}{2I_h\omega_h}\right)^2\tau_h n_{\text{th}}\left(n_{\text{th}}+1\right).\end{aligned}$$

We estimate the relative magnitude of R_{lifetime} and R_{secular} for the example resonator described by table 5.3. Section 5.1 of chapter 5 shows that the field can be expressed as

$$\mathbf{B}(\theta) = \mathbf{B}_a + B_h \left(\frac{3}{2}\theta, 0, 1 - 3\theta^2 \right), \tag{B.1}$$

where B_h is the magnitude of the resonator's field at the spins. Equation (B.1) implies that

$$\frac{dB_x}{d\theta} = \frac{3}{2} B_h,$$
$$\frac{d^2 B_z}{d\theta^2} = -3 B_h,$$

which yields

$$\frac{R_{\text{secular}}}{R_{\text{lifetime}}} = \frac{4 n_{\text{th}} (n_{\text{th}} + 1)}{(2 n_{\text{th}} + 1)} \frac{\hbar}{I_h \omega_h}.$$

For the example resonator, we have

$$I_h = 6.3 \times 10^{-33}\,\text{kg m}^2\,,$$
$$\omega_h = (2\pi)\,628\,\text{MHz}\,,$$
$$n_{\text{th}} = 0.052,$$

and

$$\frac{R_{\text{secular}}}{R_{\text{lifetime}}} \approx 10^{-14}.$$

Note that the rate constant R_{secular} becomes comparable to R_{lifetime} at temperatures high enough that

$$\frac{R_{\text{secular}}}{R_{\text{lifetime}}} \approx 2 n_{\text{th}} \frac{\hbar}{I_h \omega_h}$$

is of order unity or greater. Using the high-temperature approximation

$$n_{\text{th}} \approx \frac{k_B T}{\hbar \omega},$$

we find that this occurs when

$$T \gtrsim \frac{I_h \omega_h^2}{2k_B},$$

which is of order 10^9 K for the example resonator.

Appendix C

Longitudinal relaxation when the resonator's field is inhomogeneous

In this appendix, we remove the constraint that the resonator's field is uniform across the sample. The method used in Appendix A to obtain a series expansion for $\Delta \langle I_z \rangle / \Delta t$ can be extended to this more general problem in a natural way. We first consider a problem in which the spins all experience the same field but are not perfectly resonant with the mechanical oscillator. We define the frequency offset β by

$$\omega_0 = -\omega_h + \beta.$$

As in Appendix A, a series expansion for $\Delta \langle I_z \rangle / \Delta t$ is obtained by repeatedly replacing time-dependent integrands with integral equations. The expansion is in powers of the coupling constant g as well as the offset β. Term of order g^3 or higher are discarded, but the series in β is not truncated, since we wish to allow for the possibility that $\beta \gg g$.

Including the frequency offset in the spin Hamiltonian introduces an additional term into the integral equation (A.4), while leaving (A.3), (A.5), and (A.6) unchanged.

The full set of integral equations needed for the derivation is

$$\langle I_z\rangle(t) = \langle I_z\rangle(0) + \int_0^t (-ig)\left\langle I_+a^\dagger - I_-a\right\rangle(t_1)\,dt_1 \tag{C.1}$$

$$\begin{aligned}\left\langle I_+a^\dagger - I_-a\right\rangle(t) &= e^{-t/\tau_h}\int_0^t e^{t_1/\tau_h}(i\beta)\left\langle I_+a^\dagger + I_-a\right\rangle(t_1)\,dt_1 \\ &\quad + e^{-t/\tau_h}\int_0^t e^{t_1/\tau_h}(-4ig)\left\langle I_za^\dagger a\right\rangle(t_1)\,dt_1 \\ &\quad + e^{-t/\tau_h}\int_0^t e^{t_1/\tau_h}(2ig)\left\langle I_-I_+\right\rangle(t_1)\,dt_1\end{aligned} \tag{C.2}$$

$$\left\langle I_+a^+ + I_-a\right\rangle(t) = e^{-t/\tau_h}\int_0^t e^{t_1/\tau_h}(i\beta)\left\langle I_+a^\dagger - I_-a\right\rangle(t_1)\,dt_1 \tag{C.3}$$

$$\left\langle I_za^\dagger a\right\rangle(t) = n_{\text{th}}\langle I_z\rangle(0) + O(g) \tag{C.4}$$

$$\langle I_-I_+\rangle(t) = \langle I_-I_+\rangle(0) + O(g). \tag{C.5}$$

Note that the we have used the approximations

$$\left\langle I_za^\dagger a\right\rangle(0) = n_{\text{th}}\langle I_z\rangle(0)$$

and

$$\begin{aligned}\left\langle I_+a^\dagger - I_-a\right\rangle(0) &= \langle I_+\rangle(0)\left\langle a^\dagger\right\rangle_{\text{th}} - \langle I_-\rangle(0)\langle a\rangle_{\text{th}} \\ &= 0,\end{aligned}$$

which introduce negligible error provided $\tau_h \ll \Delta t$. After two iterations of replacing time-dependent integrands by integral equations, we obtain

$$\begin{aligned}\langle I_z\rangle(t) &= \langle I_z\rangle(0) + \left[-4g^2 n_{\text{th}}\langle I_z\rangle(0) + 2g^2\langle I_-I_+\rangle(0)\right]\times \\ &\quad \int_0^t e^{-t_1/\tau_h}\int_0^{t_1} e^{t_2/\tau_h}\,dt_2\,dt_1 \\ &\quad + (-ig)(i\beta)^2\int_0^t e^{-t_1/\tau_h}\int_0^{t_1}\int_0^{t_2} e^{t_3/\tau_h}\left\langle I_+a^\dagger - I_-a\right\rangle(t_3)\,dt_3\,dt_2\,dt_1,\end{aligned}$$

where terms of order g^3 or higher have been discarded. Repeating this process yields a series expansion for $\langle I_z \rangle (t)$:

$$\langle I_z \rangle (t) = \langle I_z \rangle (0) + \left[-4g^2 n_{\text{th}} \langle I_z \rangle (0) + 2g^2 \langle I_- I_+ \rangle (0) \right] \times C, \tag{C.6}$$

with

$$\begin{aligned} C = &\int_0^t e^{-t_1/\tau_h} \int_0^{t_1} e^{t_2/\tau_h}\, dt_2\, dt_1 + \\ &(i\beta)^2 \int_0^t e^{-t_1/\tau_h} \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} e^{t_4/\tau_h}\, dt_4 \ldots dt_1 + \\ &(i\beta)^4 \int_0^t e^{-t_1/\tau_h} \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} \int_0^{t_4} \int_0^{t_5} e^{t_6/\tau_h}\, dt_6 \ldots dt_1 + \cdots . \end{aligned} \tag{C.7}$$

Note that the expansion given by equations (C.6) and (C.7) includes arbitrarily high powers of the offset β. To estimate C, we replace t in (C.7) by $\Delta t \gg \tau_h$ and evaluate the integral:

$$\begin{aligned} C \approx & \left(\tau_h \Delta t - \tau_h^2\right) - (\beta\tau_h)^2 \left(\tau_h \Delta t - 3\tau_h^2\right) \\ &+ (\beta\tau_h)^4 \left(\tau_h \Delta t - 5\tau_h^2\right) + \cdots \end{aligned} \tag{C.8}$$

Although the factors $(\tau_h \Delta t - 3\tau_h^2)$ and $(\tau_h \Delta t - 5\tau_h^2)$ are each close to $\tau_h \Delta t$, the approximation of replacing $(\tau_h \Delta t - j\tau_h^2)$ by $\tau_h \Delta t$ will be invalid for high-order terms in the series (C.8). Provided that $|\beta\tau_h|$ is sufficiently small that C is well approximated by a sum of the initial terms for which

$$\tau_h \Delta t - j\tau_h^2 \approx \tau_h \Delta t,$$

we obtain

$$\begin{aligned} C &\approx \tau_h \Delta t \left[1 - (\beta\tau_h)^2 + (\beta\tau_h)^4 - \cdots \right] \\ &\approx \tau_h \Delta t \frac{1}{1 + (\beta\tau_h)^2}. \end{aligned} \tag{C.9}$$

Equations (C.6) and (C.9) yield

$$\frac{\Delta \langle I_z \rangle}{\Delta t} = \left[R_0 \left(n_{\mathrm{th}} + 1 \right) \langle I_- I_+ \rangle - R_0 n_{\mathrm{th}} \langle I_+ I_- \rangle \right] \frac{1}{1 + \left(\beta \tau_h \right)^2}. \tag{C.10}$$

In the case where the resonator's field varies across the sample, we define at spin k a local coordinate frame such that the static field is directed along the z-axis and the resonator's field is confined to the xz-plane. The spin operators $I_{z,k}$, $I_{+,k}$, and $I_{-,k}$ are defined relative to the local frame. Under the rotating wave approximation, the two terms of the Hamiltonian which act on the k^{th} spin are $\hbar \left(-\omega_h + \beta_k \right) I_{k,z}$ and $\hbar g_k \left(I_{+,k} a^\dagger + I_{-,k} a \right)$, where $\left(-\omega_h + \beta_k \right)$ and g_k are the respective Larmor frequency and coupling constant for the k^{th} spin. We define the lab-frame operator I'_z by

$$I'_z = \sum_k I_{z,k}.$$

The derivation of equation (C.10) can be adapted to the problem of finding an expression for $\Delta \langle I'_z \rangle / \Delta t$ in this more general case, and we find that

$$\frac{\Delta \langle I'_z \rangle}{\Delta t} = -\sum_k \frac{4 g_k^2 \tau_h n_{\mathrm{th}}}{1 + \left(\beta_k \tau_h \right)^2} \langle I_{z,k} \rangle + \sum_k \sum_j \frac{2 g_j g_k \tau_h}{1 + \left(\beta_k \tau_h \right)^2} \langle I_{-,j} I_{+,k} \rangle ,$$

which reduces to (C.10) if all spins experience the same frequency offset and the same coupling to the resonator.

Appendix D

Derivation of the semiclassical equation for longitudinal relaxation

In deriving a semiclassical equation for longitudinal relaxation, we note first that the steps used in Appendix A to obtain the equation (A.10) from the set of integral equations (A.3) through (A.6) are purely mathematical; given a similar set of integral equations for the semiclassical system, the same steps could be performed to yield an equation analogous to (A.10). We therefore proceed by defining semiclassical variables analogous to those appearing in the quantum mechanical integral equations of Appendix A, and we will use standard rules of calculus, in combination with some physical reasoning, to obtain integral equations for the semiclassical system. In order to simplify notation, we drop the superscript c used to distinguish semiclassical variables from analogous quantum operators.

The equation of motion of a semiclassical spin $\mathbf{I}$ is

$$\frac{d}{dt}\mathbf{I} = \gamma \mathbf{I} \times \mathbf{B}, \tag{D.1}$$

while the motion of a driven torsional resonator with coordinate θ and momentum p_θ is governed by the equations

$$\frac{d}{dt}\theta = \frac{1}{I_h}p_\theta - \frac{1}{\tau_h}\theta, \tag{D.2}$$

$$\frac{d}{dt}p_\theta = -I_h\omega_h^2\theta - \frac{1}{\tau_h}p_\theta + f\left(t\right), \tag{D.3}$$

where $f(t)$ is the driving torque. The interaction energy between spins and resonator is

$$W = -\mu \cdot \mathbf{B}(\theta),$$

where

$$\mu = \gamma\hbar\mathbf{I}$$

is the magnetic dipole associated with the spins. The driving torque exerted by the spins is

$$-\frac{\partial W}{\partial \theta} = \frac{dB_x}{d\theta}\mu_x.$$

and the total torque $f(t)$ acting on the resonator is

$$f(t) = \frac{dB_x}{d\theta}\gamma\hbar I_x(t) + N(t),$$

where $N(t)$ is the thermal torque.

Semiclassical analogs of the raising and lowering operators for the spin and the resonator are defined in the same way as the quantum operators:

$$\begin{aligned} a &= \frac{1}{\sqrt{2}}\left(\sqrt{\frac{I_h\omega_h}{\hbar}}\theta + i\sqrt{\frac{1}{I_h\omega_h\hbar}}p_\theta\right), \\ a^\dagger &= \frac{1}{\sqrt{2}}\left(\sqrt{\frac{I_h\omega_h}{\hbar}}\theta - i\sqrt{\frac{1}{I_h\omega_h\hbar}}p_\theta\right), \\ I_+ &= I_x + iI_y, \\ I_- &= I_x - iI_y, \end{aligned}$$

and we move to the "semiclassical interaction frame" by multiplying these variables by exponentials which cancel the time-dependence associated with the fast, unperturbed motion:

$$\begin{aligned}\tilde{a} &= e^{i\omega_h t} a, \\ \tilde{a}^\dagger &= e^{-i\omega_h t} a^\dagger, \\ \tilde{I}_+ &= e^{i\omega_h t} I_+, \\ \tilde{I}_- &= e^{-i\omega_h t} I_-.\end{aligned} \tag{D.4}$$

The right side of equation (D.1) is expressed in terms of these interaction frame variables, and the quickly oscillating terms are discarded, as in the rotating-wave approximation. Simplification of the resulting equations yields

$$\frac{d}{dt} I_z = -ig \left(\tilde{I}_+ \tilde{a}^\dagger - \tilde{I}_- \tilde{a} \right) \tag{D.5}$$

and

$$\frac{d}{dt} \tilde{I}_+ = -2igI_z \tilde{a}. \tag{D.6}$$

The first-order approximation to $\mathbf{B}(\theta)$ is used in calculating these derivatives:

$$\mathbf{B}(\theta) = \left(\frac{dB_x}{d\theta} \theta, 0, B_z \right).$$

The derivative of $\tilde{a}$ is found by differentiating (D.4), substituting (D.2) and (D.3) into the derivative, expressing the resulting equation in the interaction frame, and using the rotating-wave approximation:

$$\frac{d}{dt} \tilde{a} = -\frac{1}{\tau_h} \tilde{a} - ig\tilde{I}_+ + \frac{i}{\sqrt{2 I_h \omega_h \hbar}} e^{i\omega_h t} N. \tag{D.7}$$

The product rule of elementary calculus, in combination with equations (D.5), (D.6), and (D.7) is used to obtain integral equations similar to equations (A.3) through (A.6) of Appendix A. In order to obtain equations which do not include the thermal torque N, an average is taken over the statistical ensemble, and correlations between spin variables and the thermal torque acting on the resonator are neglected. The quickly fluctuating thermal torque N can be considered an impulse which acts on the resonator during the short correlation time of the torque, and the impulse response of the resonator appears as a weak correlation between the resonator motion and the torque. The thermal motion of the resonator is thus a sum of decaying responses to many uncorrelated impulses, with each impulse response contributing only weakly to the motion. Correlation between the instantaneous thermal torque $N(t)$ and the spin motion depends on the spins' response to the small fraction of the resonator motion which results from the impulse occurring at time t, and can thus be neglected.

We obtain in this way the differential equations

$$\frac{d}{dt}\langle I_z\rangle = -ig\left\langle \tilde{I}_+\tilde{a}^\dagger - \tilde{I}_-\tilde{a}\right\rangle, \tag{D.8}$$

$$\frac{d}{dt}\left\langle \tilde{I}_+\tilde{a}^\dagger - \tilde{I}_-\tilde{a}\right\rangle = \frac{1}{\tau_h}\left\langle \tilde{I}_+\tilde{a}^\dagger - \tilde{I}_-\tilde{a}\right\rangle - 4ig\left\langle I_z\tilde{a}^\dagger\tilde{a}\right\rangle + 2ig\left\langle \tilde{I}_-\tilde{I}_+\right\rangle, \tag{D.9}$$

$$\frac{d}{dt}\left\langle \tilde{I}_-\tilde{I}_+\right\rangle = O(g), \tag{D.10}$$

which yield integral equations analogous to (A.3), (A.4), and (A.6). Note that these equations are unchanged if they are transformed from the interaction frame to the lab frame, since

$$\begin{aligned} \tilde{I}_+\tilde{a}^\dagger &= I_+a^\dagger, \\ \tilde{I}_-\tilde{a} &= I_-a, \\ I_z\tilde{a}^\dagger\tilde{a} &= I_za^\dagger a, \\ \tilde{I}_-\tilde{I}_+ &= I_-I_+. \end{aligned}$$

Since the integral equations being derived are valid in both frames, we simplify notation by dropping tildes from the variables.

The semiclassical analog of (A.5) is

$$\frac{d}{dt}\left\langle I_za^\dagger a\right\rangle = -\frac{2}{\tau_h}\left\langle I_za^\dagger a\right\rangle + \frac{1}{I_h\omega_h\hbar}\left\langle I_zp_\theta N\right\rangle + O(\Omega). \tag{D.11}$$

In evaluating $\langle I_zp_\theta N\rangle$, we note that the resonator evolves under the influence of distinct torques which are associated with the spins and the reservoir fluctuations, so we may write p_θ as a sum of the two terms:

$$p_\theta = p_\theta^{(S)} + p_\theta^{(R)}.$$

Here $p_\theta^{(S)}$ and $p_\theta^{(R)}$ give the resonator's response to the respective torques associated with the spins and the reservoir. Since the spins and the damping torque N are considered uncorrelated, both I and $p_\theta^{(S)}$ are statistically independent of the thermal torque N, and we can write

$$\begin{aligned}\langle I_z p_\theta N\rangle &\approx \left\langle I_z p_\theta^{(S)}\right\rangle \langle N\rangle + \left\langle I_z p_\theta^{(R)} N\right\rangle \\ &= \left\langle I_z p_\theta^{(R)} N\right\rangle.\end{aligned}$$

Similarly, we may neglect correlations between I_z and the thermal function $p_\theta^{(R)}N$:

$$\langle I_z p_\theta N\rangle \approx \langle I_z\rangle \left\langle p_\theta^{(R)} N\right\rangle.$$

In order to obtain an explicit expression for the thermal average $\left\langle p_\theta^{(R)} N\right\rangle$, we consider a resonator which interacts only with a reservoir, simplifying notation by dropping the superscript 'R' from the resonator momentum. The correlation between the momentum and the thermal torque can be found by considering the derivative dE_h/dt, where

$$E_h = \frac{1}{2I_h} p_\theta^2 + \frac{I_h \omega_h^2}{2}\theta^2$$

is the resonator energy. Substituting the equations of motion (D.2) and (D.3) into the expression

$$\frac{d}{dt}E_h = \frac{1}{I_h} p_\theta \frac{d}{dt} p_\theta + I_h \omega_h^2 \theta \frac{d}{dt}\theta$$

gives

$$\frac{d}{dt}E_h = \frac{-2}{\tau_h} E_h + \frac{1}{I_h} p_\theta N,$$

where $N(t)$ is the thermal torque acting on the resonator. Taking the mean value of both sides yields

$$\frac{d}{dt}\left\langle E_h\right\rangle = \frac{-2}{\tau_h}\left\langle E_h\right\rangle + \frac{1}{I_h}\left\langle p_\theta N\right\rangle .$$

Since

$$\left\langle E_h\right\rangle = k_B T,$$

we have $d\left\langle E_h\right\rangle/dt = 0$ and

$$\left\langle p_\theta N\right\rangle = \frac{2I_h}{\tau_h}\left\langle E_h\right\rangle .$$

Defining

$$n_c = \frac{\left\langle E_h\right\rangle}{\hbar\omega_h},$$

we may express (D.11) as

$$\frac{d}{dt}\left\langle I_z a^\dagger a\right\rangle = -\frac{2}{\tau_h}\left\langle I_z a^\dagger a\right\rangle + \frac{2}{\tau_h} n_c \left\langle I_z\right\rangle + O\left(g\right) . \tag{D.12}$$

Converting (D.8), (D.9), (D.10), and (D.12) to integral equations and using these to derive a coarse-grained derivative equation yields the semiclassical relaxation equation

$$\frac{d\left\langle I_z\right\rangle}{dt} = -2R_0 n_c \left\langle I_z\right\rangle + R_0\left\langle I_- I_+\right\rangle .$$

Appendix E

Longitudinal relaxation due to coupling between product-state populations

Section 4 of chapter 3 presents a heuristic argument for the idea that if product states can be chosen as eigenstates, and if the spin-resonator interaction does not induce couplings between populations and coherences, then spin-spin correlations make no contribution to the relaxation of $\langle I_z \rangle$. To formalize this argument, we note first that the spin-resonator interaction Hamiltonian (2.11) couples product-state populations ρ_{aa} and ρ_{cc} only if eigenstates $|a\rangle$, $|c\rangle$ differ by exactly one spin flip. The rate constants $\Gamma_{c\to a}$, $\Gamma_{a\to c}$ for population transfer have the same value as they do for transfer between the two states of a single spin interacting with the resonator.

Consider the changes in populations which occur during a time step Δt. All such changes can be accounted for by summing the population transfers associated with the set of transition probabilities $\Gamma_{n\to m}$. These population transfers may be considered to occur in any order we choose, and each $\Gamma_{n\to m}$ couples two states which differ by exactly one spin flip. We initially focus attention on spin 1, and we take Z_1 to be the set of all $\Gamma_{n\to m}$ which couple eigenstates differing by a flip of this spin. We will show that if all population transfers associated with Z_1 occur, and if these are the only transfers that occur, then $\langle I_{z,1} \rangle$ relaxes exactly as if spin 1 were an isolated spin interacting with its own resonator, while $\langle I_{z,j} \rangle$ is unchanged, for $j \neq 1$. By defining Z_j to be the set of all $\Gamma_{n\to m}$ which couple eigenstates differing by a flip of spin j, and then sequentially applying all population transfers associated with Z_2, Z_3, and so on, we find that during Δt each spin has relaxed toward its thermal population as if it were interacting with its own resonator.

To establish this argument, we must show that if all population transfers associated with $\Gamma_{n\rightarrow m} \in Z_k$ occur, and if these are the only transfers that occur, then $\langle I_{z,k} \rangle$ relaxes as if spin k were an isolated spin interacting with a resonator, while $\langle I_{z,j} \rangle$ is unchanged, for $j \neq k$. Group the product eigenstates into pairs, with the eigenstates of each pair differing by a flip of spin k, and let $|+\beta\rangle$, $|-\beta\rangle$ denote the respective eigenstates of pair β for which spin k is oriented parallel and antiparallel to B_z. In addition, let $\rho_{(+\beta)}$, $\rho_{(-\beta)}$ denote the respective populations of states $|+\beta\rangle$, $|-\beta\rangle$, and define

$$\rho_+ = \sum_\beta \rho_{(+\beta)},$$
$$\rho_- = \sum_\beta \rho_{(-\beta)}.$$

Population transfers associated with Z_k cause $\rho_{(+\beta)}$ and $\rho_{(-\beta)}$ to evolve during the time step Δt exactly as if they were the populations of an isolated spin interacting with the resonator. Indeed, arguments similar to those used in deriving equation (2.29) show that population is transferred from $\rho_{(+\beta)}$ to $\rho_{(-\beta)}$ at rate $R_0 n_{\mathrm{th}} \rho_{(+\beta)}$ and from $\rho_{(-\beta)}$ to $\rho_{(+\beta)}$ at rate $R_0 \left(n_{\mathrm{th}} + 1\right) \rho_{(-\beta)}$. It follows that ρ_+ and ρ_- evolve under the same differential equations as the populations of an isolated spin relaxing due to its interactions with a resonator:

$$\frac{d}{dt}\rho_+ = -R_0 n_{\mathrm{th}} \rho_+ + R_0 \left(n_{\mathrm{th}} + 1\right) \rho_-,$$
$$\frac{d}{dt}\rho_- = -R_0 \left(n_{\mathrm{th}} + 1\right) \rho_- + R_0 n_{\mathrm{th}} \rho_+,$$

and since $\langle I_{z,k} \rangle$ can be expressed as

$$\langle I_{z,k} \rangle = \frac{1}{2}\rho_+ - \frac{1}{2}\rho_-,$$

these transitions cause $\langle I_{z,k} \rangle$ to relax as if it were an isolated spin.

Note that for each pair β, the sum

$$\rho_\beta \equiv \rho_{(+\beta)} + \rho_{(-\beta)}$$

does not change during these transitions, and that for $j \neq k$, we have

$$\langle I_{z,j} \rangle = \sum_\beta \lambda_{z,j}\, \rho_\beta,$$

where $\lambda_{z,j}$ is the eigenvalue of $I_{z,j}$ for the two states in pair β. Since ρ_β does not change during these transitions, $\langle I_{z,j} \rangle$ remains constant. This establishes our claim that direct coupling between populations, in the absence of any coupling between populations and coherences, causes $\langle I_z \rangle$ to relax exponentially with rate constant R_h to thermal equilibrium with the resonator.

Appendix F

Transverse relaxation due to coupling between product-state coherences

Although section 5 of chapter 3 shows that the damping constant for a coherence between product states increases with the size of the sample, transfer between product-state coherences can yield exponential transverse relaxation with rate constant $R_h/2$, regardless of the size of the sample. In particular, suppose that the single-quantum coherences are grouped into sets Z_k, where the coherences in set Z_k are between states which differ by a flip of spin k. Recall that in section 3 of chapter 3, we argued that the transfer between coherences ρ_{cd} and ρ_{ab} that is characterized by $\mathcal{R}_{abcd}$ will be suppressed if the frequency difference $|\omega_{ab} - \omega_{cd}|$ is perturbed to a value larger than $2\pi\mathcal{R}_{abcd}$. We show here that if the frequency differences between coherences within each set Z_k are small enough that transfers within Z_k are preserved, while transfers between coherences belonging to Z_k and other coherences are suppressed by frequency differences, then the transverse relaxation induced by the resonator is exponential with rate constant $R_h/2$.

In demonstrating this result, we first define s_k to be the sum of all coherences within Z_k:

$$s_k = \sum_{\rho_{ab} \in Z_k} \rho_{ab},$$

and we claim that

$$\langle I_{x,k} \rangle = \frac{1}{2} s_k. \tag{F.1}$$

Equation (F.1) can be established by expanding the density matrix as

$$\rho = \sum \rho_{ab} \left| a \right\rangle \left\langle b \right|,$$

and writing $I_{x,k}$ as

$$I_{x,k} = \frac{1}{2} \left(I_{+,k} + I_{-,k} \right).$$

For coherences ρ_{ab} belonging to Z_k, we have one of two possibilities:

$$\begin{aligned} \mathrm{Tr} \left\{ \rho_{ab} I_{+,k} \left| a \right\rangle \left\langle b \right| \right\} &= \rho_{ab}, \\ \mathrm{Tr} \left\{ \rho_{ab} I_{-,k} \left| a \right\rangle \left\langle b \right| \right\} &= 0, \end{aligned}$$

or

$$\begin{aligned} \mathrm{Tr} \left\{ \rho_{ab} I_{+,k} \left| a \right\rangle \left\langle b \right| \right\} &= 0, \\ \mathrm{Tr} \left\{ \rho_{ab} I_{-,k} \left| a \right\rangle \left\langle b \right| \right\} &= \rho_{ab}, \end{aligned}$$

while for coherences ρ_{cd} not belonging to Z_k, we have

$$\mathrm{Tr} \left\{ \rho_{cd} I_{+,k} \left| c \right\rangle \left\langle d \right| \right\} = \mathrm{Tr} \left\{ \rho_{cd} I_{-,k} \left| c \right\rangle \left\langle d \right| \right\} = 0.$$

Summing over the coherences belonging to Z_k, we obtain equation (F.1).

Since

$$\langle I_x \rangle = \sum_k \langle I_{x,k} \rangle,$$

it suffices to show that

$$\frac{d}{dt} s_k = -\frac{1}{2} R_h s_k \tag{F.2}$$

if transfers within Z_k are preserved while transfers between coherences belonging to Z_k and other coherences are suppressed. Two types of couplings contribute to $(d/dt)\, s_k$.

First, coupling constants $\mathcal{R}_{abab}$ contribute terms of the form

$$\mathcal{R}_{abab}\rho_{ab} = -\frac{1}{2}\left(\sum_{n\neq a}\Gamma_{a\to n} + \sum_{n\neq b}\Gamma_{b\to n}\right)\rho_{ab}. \tag{F.3}$$

Equation (F.3) can be written as

$$\mathcal{R}_{abab}\rho_{ab} = -\frac{1}{2}\left(\Gamma_{a\to b} + \Gamma_{b\to a}\right)\rho_{ab} - \frac{1}{2}\left(\sum_{n\neq a,b}\Gamma_{a\to n} + \sum_{n\neq b,a}\Gamma_{b\to n}\right)\rho_{ab} \tag{F.4}$$

$$= -\frac{1}{2}R_h\rho_{ab} - \frac{1}{2}\left(\sum_{n\neq a,b}\Gamma_{a\to n} + \sum_{n\neq b,a}\Gamma_{b\to n}\right)\rho_{ab}. \tag{F.5}$$

In going from (F.4) to (F.5), we used the fact that the rate constants $\Gamma_{m\to n}$ for transfer of population between product states which differ by a single spin flip are the same as for the two states of a single-spin system.

Assume that coherences in Z_k are coupled only to other coherences belonging to the same set. These couplings are associated with processes in which two transitions $|a\rangle \to |c\rangle$ and $|b\rangle \to |d\rangle$ occur, with both transitions involving a flip of spin $j \neq k$ in the same direction. Without loss of generality, we assume that both transitions involve a flip up of spin j:

$$I_{+,j}\,|a\rangle = |c\rangle\,,$$
$$I_{+,j}\,|b\rangle = |d\rangle\,.$$

The product of matrix elements which contributes to $\mathcal{R}_{cdab}$ is $\langle \mu, b\,|V|\,\nu, d\rangle\,\langle \nu, c\,|V|\,\mu, a\rangle$. We first demonstrate that the matrix elements $\langle \nu, d\,|V|\,\mu, b\rangle$ and $\langle \nu, c\,|V|\,\mu, a\rangle$ have the value $\left\langle \nu \left| a^\dagger \right| \mu \right\rangle$:

$$\begin{aligned}\langle \nu, d \left| V \right| \mu, b \rangle &= \left\langle \nu, d \left| I_+ a^\dagger \right| \mu, b \right\rangle \\ &= \langle d \left| I_+ \right| b \rangle \left\langle \nu \left| a^\dagger \right| \mu \right\rangle \\ &= \langle d \left| I_{+,j} \right| b \rangle \left\langle \nu \left| a^\dagger \right| \mu \right\rangle \\ &= \langle d | d \rangle \left\langle \nu \left| a^\dagger \right| \mu \right\rangle \\ &= \left\langle \nu \left| a^\dagger \right| \mu \right\rangle .\end{aligned}$$

Since similar steps can be used to obtain $\langle \nu, c \left| V \right| \mu, a \rangle = \left\langle \nu \left| a^\dagger \right| \mu \right\rangle$, we have

$$\langle \nu, c \left| V \right| \mu, a \rangle = \langle \nu, d \left| V \right| \mu, b \rangle$$

and

$$\begin{aligned}\langle \mu, b \left| V \right| \nu, d \rangle \langle \nu, c \left| V \right| \mu, a \rangle &= \left| \langle \nu, c \left| V \right| \mu, a \rangle \right|^2 \\ &= \left| \langle \nu, d \left| V \right| \mu, b \rangle \right|^2 .\end{aligned}$$

It follows from (3.6) and (3.9) that

$$\begin{aligned}\mathcal{R}_{cdab} &= \Gamma_{a \to c} = \Gamma_{b \to d} \\ &= \frac{1}{2} \left(\Gamma_{a \to c} + \Gamma_{b \to d} \right) .\end{aligned}$$

We can thus write (F.5) as

$$\mathcal{R}_{abab} \rho_{ab} = -\frac{1}{2} R_h \rho_{ab} + \sum_{Z_k} -\mathcal{R}_{cdab} \rho_{ab}, \tag{F.6}$$

where the sum is over all coherences in Z_k except ρ_{ab}. Equation (F.6) can be interpreted to mean that the evolution governed by the coefficient $\mathcal{R}_{abab}$ of the master equation includes a contribution associated with the "intrinsic decay "of ρ_{ab}, for which the rate constant is $R_h/2$, and a contribution associated with transfers from ρ_{ab} to other coherences belonging to the set Z_k. If we sum the derivatives of all coherences in Z_k, all contributions of the form $\pm\mathcal{R}_{cdab}\rho_{ab}$ and $\pm\mathcal{R}_{abcd}\rho_{cd}$ cancel, and the only remaining terms have the form $-\left(R_h/2\right)\rho_{ab}$ or $-\left(R_h/2\right)\rho_{cd}$. It follows that equation (F.2) holds, i.e., transverse relaxation induced by the resonator is exponential and has rate constant $R_h/2$.

Appendix G

Comparison between the use of an optimal filter and least-squares fitting

Unknown parameters in a measured signal are often estimated by least-squares fitting, rather than by applying an optimal filter. In this section, we compare the two methods of data analysis for signals of the form $Gm_0(t)$, with $m_0(t)$ a known function. The signal and the noise are assumed to be continuous functions of time, and errors arising from digitization of the signal are neglected. We show that if the noise is white, least-squares fitting yields the same value of G as that obtained from an optimal filter. If the noise is not white, however, the two methods in general yield different values of G, and the least-squares fit corresponds to an estimate made using a non-optimal filter.

Given a function $f(t) = m(t) + n(t)$, the least-squares fit is the function $Gm_0(t)$ that minimizes the integral

$$\|f - Gm_0\|^2 = \int_{-\infty}^{\infty} [f(t) - Gm_0(t)]^2 \, dt. \tag{G.1}$$

If the functions $f(t)$ and $m_0(t)$ belong to a Hilbert space such as L_2, the problem of finding G can be cast in geometric language. The set of scalar multiples of $m_0(t)$ constitutes a one-dimensional subspace, and the least-squares fit $Gm_0(t)$ is the projection of $f(t)$ onto this subspace. Indeed, $Gm_0(t)$ will be given by

$$Gm_0(t) = m_0(t)\frac{\langle m_0,\, f\rangle}{\|m_0\|^2},$$

where

$$\langle m_0,\, f\rangle = \int_{-\infty}^{\infty} m_0(t)\, f(t)\; dt$$

and

$$\|m_0\|^2 = \int_{-\infty}^{\infty} m_0^2(t)\; dt.$$

We see that least-squares fitting produces the amplitude estimate

$$G = \frac{\int_{-\infty}^{\infty} m_0(t)\, f(t)\; dt}{\int_{-\infty}^{\infty} m_0^2(t)\; dt}. \tag{G.2}$$

(Note that under the standard convention, which has $n(t) \not\to 0$ as $t \to \pm\infty$, it is not true that $f(t)$ belongs to $L_2(\mathbb{R})$. This is merely a matter of convention, however. If we limit the domain of integration for equation G.1 to a finite interval $[a, b]$ that includes all times t for which $m_0(t)$ is non-negligible, then $f(t)$ and $m_0(t)$ can be assumed to belong to $L_2[a, b]$.)

We compare equation G.2 to the estimate that would be obtained using an optimal filter. The signal $f(t)$ is passed through the filter $\mathcal{K}$ having transfer function

$$K(\omega) = c\frac{M_0^*(\omega)}{S_n(\omega)}.$$

The amplitude estimate X is given by

$$X = \frac{\phi}{\mu_0},$$

where

$$\begin{aligned}\phi &= \frac{1}{2\pi}\int_{-\infty}^{\infty} K(\omega)\, F(\omega)\ d\omega \\ &= c\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{M_0^*(\omega)}{S_n(\omega)} F(\omega)\ d\omega \end{aligned} \tag{G.3}$$

and

$$\mu_0 = c\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{M_0^*(\omega)}{S_n(\omega)} M_0(\omega)\ d\omega. \tag{G.4}$$

If the noise is white, then the constant term S_n can be taken outside the integrals of equations G.3 and G.4, and we find that

$$\phi = \int_{-\infty}^{\infty} m_0(t)\, f(t)\ dt$$

and

$$\mu_0 = \int_{-\infty}^{\infty} m_0^2(t)\ dt.$$

The amplitude estimate obtained in this way is

$$X = \frac{\int_{-\infty}^{\infty} m_0(t)\, f(t)\ dt}{\int_{-\infty}^{\infty} m_0^2(t)\ dt},$$

which is identical to the value obtained with a least-squares fit. We conclude that using a least-square fit is equivalent to using a filter which is optimal for extracting the signal from white noise. In the case where $S_n(\omega)$ varies over the spectral width of $m_0(t)$, the least-squares fit does not take account of the structure of $S_n(\omega)$, whereas the optimal filter does.

Appendix H

Spectral density in signal-to-noise ratio estimates

In this appendix, we introduce the spectral density as a tool to be used in SNR calculations. Recall that in analyzing the variance of a measured amplitude due to noise superimposed on the signal, we considered a noisy signal

$$f(t) = m(t) + n(t),$$

where $m(t)$ is the useful signal and $n(t)$ is the noise. When $f(t)$ is passed into filter $\mathcal{K}$, the output function is

$$\phi(t) = \mu(t) + \nu(t),$$

where $\mu(t)$ and $\nu(t)$ would be the respective outputs if $m(t)$ and $\nu(t)$ were passed through $\mathcal{K}$ individually. The amplitude estimate, denoted by X and given by equation (4.2), differs from the actual amplitude by a term proportional to $\nu(t_0)$, where t_0 is a time determined by the filter's transfer function. (If we do not care about whether the filter is causal, then t_0 can be chosen arbitrarily. In section 1 of chapter 4, for instance, we set $t_0 = 0$.) To calculate SNR, we divide the mean value $\langle X \rangle$ by the standard deviation of X, which can be calculated if the variance $\nu(t_0)$ is known. In the context of calculating SNR, the spectral density is used only as a tool for calculating this variance.

Since $\nu(t_0)$ is obtained by passing the noise $n(t)$ through a filter, one natural approach might be to calculate the frequency components of $n(t)$ and then use the filter's transfer function to determine the frequency components of $\nu(t)$. This approach presents a technical difficulty, however, since the function $n(t)$ is conventionally assumed not to approach zero as $t \to \pm\infty$. As a result, the Fourier transform of $n(t)$ is not defined. Fortunately, we merely need the variance of $\nu(t_0)$, not the frequency components of $\nu(t)$. An alternate approach to obtaining this variance can be used if we assume that $n(t)$ is stationary and has zero mean, with $\mathcal{K}$ linear and time-invariant. An outline of this approach introduces the spectral density in a simple way.

The assumptions on $n(t)$ and $\mathcal{K}$ guarantee that $\nu(t)$ is also stationary and has zero mean [19]. The variance of $\nu(t_0)$ is represented by the notation $\langle \nu^2 \rangle$, and it is given by $C_\nu(0)$, where

$$C_\nu(t) = \langle \nu(t)\, \nu(0) \rangle .$$

Our strategy will be to calculate $C_\nu(0)$ in terms of the correlation function $C_n(t)$, which is defined similarly to $C_\nu(t)$:

$$C_n(t) = \langle n(t)\, n(0) \rangle .$$

We define the spectral densities $S_n(\omega)$, $S_\nu(\omega)$ to be the respective Fourier transforms of $C_n(t)$ and $C_\nu(t)$:

$$\begin{aligned} S_n(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega t} C_n(t)\, dt, \\ S_\nu(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega t} C_\nu(t)\, dt. \end{aligned}$$

Reference [19] shows that

$$S_\nu(\omega) = |K(\omega)|^2 S_n(\omega), \tag{H.1}$$

where the transfer function of $\mathcal{K}$ is denoted by $K(\omega)$. If the correlation function

$C_n(t)$ has been previously derived, then equation H.1 can be used to calculate $\langle \nu^2 \rangle$:

$$\begin{aligned}
\left\langle \nu^2 \right\rangle &= C_\nu(0) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_\nu(\omega)\, d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} |K(\omega)|^2 S_n(\omega)\, d\omega.
\end{aligned}$$

This integral can in principle be evaluated, since the transfer function $|K(\omega)|$ is assumed to be known, while the spectral density $S_n(\omega)$ can be calculated from knowledge of $C_n(t)$.

This short introduction to the spectral density includes the ideas needed to understand its use as a tool in calculating SNR. The mean-square magnitude of the unfiltered noise is expressed as a sum over Fourier components of the noise's correlation function. Filtering the noise modifies these Fourier components, and the mean-square magnitude of the filtered noise is calculated as a sum over the modified Fourier components.

Appendix I

Statistics of a classical resonator

In this appendix we derive an expression for the spectral density of the thermal torque exerted on a classical torsional oscillator in equilibrium with a reservoir. The first section provides a shortened derivation of the correlation function of a classical oscillator, adapted from a derivation originally given by McCombie [47], as well as formal justification for an assumption made by McCombie. The second section derives the spectral density of the thermal torque.

1 Correlation function of the oscillator's coordinate

McCombie shows that the correlation function for a classical torsional resonator is [47]

$$\begin{aligned} C_\theta (t) &\equiv \langle \theta (t) \theta (0) \rangle \\ &= \left\langle \theta^2 \right\rangle e^{-|t|/\tau_h} \left(\cos \omega_d t + \frac{1}{\tau_h \omega_d} \sin \omega_d |t| \right), \end{aligned} \tag{I.1}$$

where

$$I_h \ddot{\theta} (t) + \frac{2 I_h}{\tau_h} \dot{\theta} (t) + k\theta = N' (t) \tag{I.2}$$

is the Langevin equation of motion that governs the resonator,

$$\langle \theta^2 \rangle = \frac{k_B T}{k}$$

is the mean-square thermal displacement, and

$$\omega_d = \sqrt{\frac{k}{I_h} - \frac{1}{\tau_h^2}}$$

is the frequency of the freely-running damped resonator. From (I.1) it follows that the spectral density $S_\theta(\omega)$ of the thermal fluctuations is

$$S_\theta(\omega) = \frac{4k_B T}{\tau_h I_h} \left(\frac{1}{(\omega^2 - \omega_\theta^2)^2 + 4\omega^2/\tau_h^2} \right). \tag{I.3}$$

A significant assumption behind McCombie's derivation of (I.1) is that "subsequent to any given instant the history of the random couple is quite independent of the fluctuation in the deflection at that instant" [47]. Stated in mathematical notation, McCombie's assumption is

$$\langle N'(t')\, \theta(t) \rangle = 0, \;\; t < t'.$$

In investigating this assumption, we note first that the resonator's equation of motion (I.2) can be integrated [48] to give a formal expression for $\theta(t)$:

$$\theta(t) = \frac{1}{I_h} \int_{-\infty}^{t} e^{-(t-t')/\tau_h} \frac{\sin \omega_d (t - t')}{\omega_d} N'(t')\, dt'. \tag{I.4}$$

Inspection of this equation shows that $\theta(t)$ retains a memory of the fluctuating torque $N'(t')$ for a period on the order of the ringdown time τ_h, since $N'(t')$ in general contributes to the integral when $(t - t')/\tau_h$ is on the order of unity. Torques exerted after time t do not directly contribute to $\theta(t)$, so we do not expect them to be correlated with $\theta(t)$. These statements can be demonstrated formally by calculating $\langle N'(t')\, \theta(t) \rangle$. If we consider the torque to vary so quickly that its correlation function is approximated as

$$\langle N'(t)\,N'(t')\rangle = \sigma_{N'}^2 \delta(t-t'),$$

with $\sigma_{N'}^2$ the variance of N', and δ the Dirac delta function, then we can use equation (I.4) to find $\langle N'(t')\,\theta(t)\rangle$. Changing the variable of integration in (I.4) to t'', multiplying by $N'(t')$, and taking the mean of each side gives

$$\langle N'(t')\,\theta(t)\rangle = \begin{cases} \frac{\sigma_{N'}^2}{I_h}\frac{\sin\omega_d(t-t')}{\omega_d}e^{-(t-t')/\tau_h}, & t'<t \\ 0 & t<t' \end{cases} \tag{I.5}$$

Equation (I.5) shows that the resonator coordinate $\theta(t)$ is correlated with $N'(t')$ when t' precedes t by a time on the order of τ_h. Intuitively, we can say that the resonator retains a memory of the torques exerted on it in the past during a period whose length is on the order of τ_h, but it has no knowledge of the torques that will be exerted on it in the future, since the reservoir itself has no memory.

It is now simple to derive the correlation function $C_\theta(t)$. We multiply the Langevin equation

$$I_h\frac{d^2}{dt}\theta(t) + \frac{2I_h}{\tau_h}\frac{d}{dt}\theta(t) + k\theta(t) = N'(t)$$

by $\theta(t')$, with $t'<t$, and take the mean value of each side. Since $\langle\theta(t')\,N'(t)\rangle = 0$, we find that

$$I_h\frac{d^2}{dt^2}\langle\theta(t')\,\theta(t)\rangle + \frac{2I_h}{\tau_h}\frac{d}{dt}\langle\theta(t')\,\theta(t)\rangle + k\langle\theta(t')\,\theta(t)\rangle = 0.$$

The solution to this differential equation in t is

$$\langle\theta(t')\,\theta(t)\rangle = \left\langle\theta^2(t')\right\rangle e^{-(t-t')/\tau_h}\left[\cos(\omega_d(t-t')) + \frac{1}{\tau_h\omega_d}\sin(\omega_d(t-t'))\right], \quad t>t'. \tag{I.6}$$

Since $\theta(t)$ is a stationary random process, equation (I.6) depends only on the difference between t' and t. We can thus consider $t' = 0$ and write

$$\langle \theta(0)\,\theta(t)\rangle = \langle \theta^2 \rangle e^{-t/\tau_h} \left(\cos\omega_d t + \frac{1}{\tau_h \omega_d} \sin\omega_d t \right), \quad t > 0. \tag{I.7}$$

Alternatively, we can choose $t = 0$ to obtain

$$\langle \theta(t')\,\theta(0)\rangle = \langle \theta^2 \rangle e^{-|t'|/\tau_h} \left(\cos\omega_d |t'| + \frac{1}{\tau_h \omega_d} \sin\omega_d |t'| \right), \quad 0 > t'. \tag{I.8}$$

Equations (I.7) and (I.8) can be combined in the form

$$C_\theta(t) = \langle \theta(t)\,\theta(0)\rangle = \langle \theta^2 \rangle e^{-|t'|/\tau_h} \left(\cos\omega_d t + \frac{1}{\tau_h \omega_d} \sin\omega_d t \right),$$

which is the desired result.

2 Spectral density of the thermal torque

The thermal torque N' and the angular displacement θ are considered to be ergodic, stationary random processes with zero mean. For each sample function $N'(t)$, there is an associated sample function $\theta(t)$ giving the displacement of the resonator driven by $N'(t)$. For a given pair $N'(t)$, $\theta(t)$, we define truncated functions $N'_T(t)$ and $\theta_T(t)$ which have as their domain some large interval $[-T, T]$ and which coincide respectively with $N'(t)$, $\theta(t)$ on this interval. The spectral density of $N'_T(t)$ and $\theta_T(t)$ will be denoted by $S_{N',T}(\omega)$ and $S_{\theta,T}(\omega)$, respectively. In addition, $C_{N'}(t)$, and $S_{N'}(\omega)$ denote the respective correlation function and the spectral density of N', while $C_\theta(t)$ and $S_\theta(\omega)$ are defined analogously for θ. Our goal in this section is to find an expression for $S_{N'}(\omega)$, the spectral density of the thermal torque.

If $N'_T(t)$ is given by

$$N'_T(t) = \sum_{n=1}^{\infty} N'_n \cos(\omega_n t + \phi_n),$$

with $\omega_n = \pi n/T$, then

$$\theta_T(t) = \theta_{\text{ini}}(t) + \sum_{n=1}^{\infty} \theta_n \cos(\omega_n t + \psi_n),$$

where

$$\theta_n = \frac{N'_n}{I_h} \frac{1}{\sqrt{\left(\omega_n^2 - \omega_\theta^2\right)^2 + 4\omega_n^2/\tau_h^2}}. \tag{I.9}$$

Here I_h, ω_θ, and τ_h are the resonator's moment of inertia, frequency, and ringdown time, respectively. The function $\theta_{\text{ini}}(t)$ is included because the resonator's response to the driving torque during the interval $[-T, T]$ depends on the initial state of the resonator at time $t = -T$; that is, $\theta_{\text{ini}}(t)$ corresponds to the ringing down of an undriven resonator. If the time interval is sufficiently large compared to the ringdown time τ_h, then $\theta_{\text{ini}}(t)$ will make a negligible contribution to the Fourier components of $\theta_T(t)$, which we may consider to be given by θ_n. From equation I.9, we conclude that

$$\begin{aligned}
S_{\theta,T}(\omega_n) &= \frac{|\theta_n|^2}{2T} \\
&= \frac{|N'_n|^2}{2T} \frac{1}{I_h^2\left(\left(\omega_n^2 - \omega_\theta^2\right)^2 + 4\omega_n^2/\tau_h^2\right)} \\
&= S_{N',T}(\omega_n) \frac{1}{I_h^2\left(\left(\omega_n^2 - \omega_\theta^2\right)^2 + 4\omega_n^2/\tau_h^2\right)}.
\end{aligned}$$

In general, the spectral density of a random process can be obtained by calculating the spectral density of truncated functions such as $N'_T(t)$ and $\theta_T(t)$, averaging over the ensemble, and then taking the limit as $T \to \infty$ [19]:

$$\begin{aligned}
S_\theta(\omega) &= \lim_{T\to\infty} \langle S_{\theta,T}(\omega) \rangle, \\
S_{N'}(\omega) &= \lim_{T\to\infty} \langle S_{N',T}(\omega) \rangle.
\end{aligned}$$

We find that

$$\begin{aligned} S_\theta(\omega) &= \frac{1}{I_h^2\left(\left(\omega^2-\omega_\theta^2\right)^2+4\omega^2/\tau_h^2\right)}\lim_{T\to\infty}\left\langle S_{N',T}(\omega)\right\rangle \\ &= \frac{S_{N'}(\omega)}{I_h^2\left(\left(\omega^2-\omega_\theta^2\right)^2+4\omega^2/\tau_h^2\right)}. \end{aligned} \tag{I.10}$$

It follows from (I.10) and (I.3) that

$$S_{N'}(\omega) = \frac{4I_hk_BT}{\tau_h}.$$

The single-sided spectral density $S_{N'}^{\mathrm{s}}$ of the fluctuating torque is

$$\begin{aligned} S_{N'}^{\mathrm{s}} &= 2S_{N'}(\omega) \\ &= \frac{8I_hk_BT}{\tau_h}. \end{aligned} \tag{I.11}$$

Appendix J

Contribution of the induced electric field to the resonator's kinetic energy

Movement of the magnetic mechanical resonator will cause the magnetic field in the space surrounding the resonator to vary with time, and the oscillating magnetic field will induce an electric field. The energy of the induced electric field is proportional to the square of the magnet's angular velocity, and may thus be considered to contribute to the resonator's kinetic energy. A simple argument suggests that this contribution is negligible for a radio-frequency nanoscale resonator. Note first that the resonator's magnetic field $\mathbf{B}_h(\mathbf{x}, t)$ can be estimated using the quasistatic approximation, since the wavelength of light at a typical resonator frequency of 500 MHz is many orders of magnitude larger than the dimensions of the resonator [49, 50]. A quasistatic estimate of $\mathbf{B}_h(\mathbf{x}, t)$ is obtained by dropping the displacement current from the Maxwell equation

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{J.1}$$

and calculating $\mathbf{B}$ as if it were generated by a static current distribution

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

Note, however, that if the displacement current is removed from equation (J.1), then the magnetic field cannot exchange energy with the electric field unless $\mathbf{J}$ is changed by the induced electric field. In the case where $\mathbf{B}$ is generated by the bound current of magnetic material, then there is no mechanism for energy exchange between the magnetic and electric fields. Conservation of energy therefore implies that if the quasistatic approximation is valid, the energy of the induced electric field is negligible compared to the energy of $\mathbf{B}_h(\mathbf{x}, t)$.

We used a simple example resonator model to make a numerical estimate of the ratio

$$r = \frac{T_{\text{elec}}}{T_{\text{mech}}},$$

where T_{elec} is the energy of the induced electric field, and T_{mech} is the mechanical kinetic energy. A Halbach cylinder [43] is a circular tube of magnetic material for which the arrangement of magnetization produces a nominally uniform magnetic field within the tube and zero field outside of the tube. The simplicity of this magnetic field facilitates an estimate of the electric field induced by the rotation of the cylinder around its axis. For this estimate, we used an equation similar in form to the Biot-Savart law:

$$\mathbf{E}(\mathbf{x}, t) = \frac{-1}{4\pi} \int \frac{\frac{\partial \mathbf{B}}{\partial t}(\mathbf{x}', t) \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d^3x', \tag{J.2}$$

where $\mathbf{B}_h(\mathbf{x}, t)$ is calculated using the quasistatic approximation. We assumed

$$\frac{R_o}{R_i} = 3,$$

where R_i, R_o are the respective inner radius and outer radius of the Halbach cylinder, as well as a remanent magnetization of

$$\mu_0 M = 1.5\,\text{T}\,,$$

and a magnetic density equal to that of iron. For the ratio r we obtained a scale-invariant expression with the approximate value

$$r \sim 10^{-15}.$$

The contribution of the energy of the induced electric field to the kinetic energy of a nanoscale magnetic mechanical oscillator is therefore negligible.

Appendix K

General formula for the magnetic spring constant

For a magnetic mechanical oscillator whose magnetization remains constant in a reference frame fixed in the oscillator, a simple formula for the magnetic spring constant can be obtained. A magnetic dipole μ in a static applied field B_a has energy

$$U = -B_a \mu \cos\theta, \tag{K.1}$$

where θ is the angle between μ and B_a. For small θ, (K.1) can be approximated as

$$U = -B_a \mu + \frac{1}{2} B_a \mu \theta^2,$$

which is the potential energy of a harmonic oscillator with magnetic spring constant

$$k_{\mathrm{mag}} = B_a \mu. \tag{K.2}$$

In the case where a soft magnetic oscillator moves in a large applied field, the magnetization can be considered to remain continuously aligned with the applied field, so that

$$\theta \approx 0$$

throughout the motion. Naive use of equation (K.1) would suggest that in this case, magnetic energy makes no contribution to the oscillator's spring constant. This conclusion is incorrect, however, since (K.1) only takes account of the interaction between the magnetization and the applied field, without including the magnetostatic energy associated with the interaction between dipoles at different points within the magnetic material.

In this appendix, we derive a formula for the magnetic spring constant in the general case where magnetization can change as the oscillator moves. We begin by considering the Hamiltonian for a nonrelativistic system of particles evolving in an external electromagnetic field [51]. The vector and scalar potentials for the electromagnetic field are each expressed as the sum of a dynamical variable and an externally-determined function associated with the applied field. The particles evolve under the action of the total electromagnetic fields $\mathbf{E}$, $\mathbf{B}$, and act as sources for the fields $\mathbf{E}'$, $\mathbf{B}'$ associated with the dynamical variables. In the case where the applied field is purely magnetic, the Hamiltonian for the system is [51]

$$H = \sum_{\alpha} \frac{1}{2m_a} \left(\dot{\mathbf{r}}_\alpha\right)^2 - \sum_{\alpha} \mu_a \cdot \mathbf{B}\left(\mathbf{r}_\alpha\right) + V_{\text{Coul}} + H_R. \tag{K.3}$$

Here $\mathbf{r}_\alpha$ and μ_a are the respective position and magnetic moment of particle α, and the Coulomb energy V_{Coul} is given by

$$V_{\text{Coul}} = \varepsilon_{\text{Coul}}^{\alpha} + \sum_{\alpha>\beta} \frac{q_\alpha q_\beta}{4\pi\varepsilon_0 \left|\mathbf{r}_\alpha - \mathbf{r}_\beta\right|},$$

with q_α the charge on particle α, and $\varepsilon_{\text{Coul}}^{\alpha}$ the "self-energy" of its Coulomb field. The Hamiltonian H_R governs the dynamical fields $\mathbf{E}'$ and $\mathbf{B}'$:

$$H_R = \frac{1}{2} \int \varepsilon_0 \left(E'_\perp\right)^2 + \frac{1}{\mu_0} \left(B'\right)^2 \, d^3x. \tag{K.4}$$

In equation (K.4), $E'_\perp$ denotes the transverse electric field. Excitation of the transverse electric field can be interpreted as quanta in electromagnetic modes.

If the system can be characterized with sufficient accuracy by a single pair of conjugate dynamical variables θ and p_θ, and if the terms in (K.3) can be separated into two distinct expressions U and T, with U depending only on θ, and T depending only on p_θ, then an "effective potential energy" U and an "effective kinetic energy" T can be defined. Note that if the system is considered to be semiclassical, with dynamical variables represented by functions rather than operators, then $H = U+T$ is a constant of the motion. Since a change ΔT will be accompanied by a corresponding change $-\Delta U$, we see that U and T conform to our expectations for kinetic and potential energy. In particular, if θ is an angular coordinate, and if U and T can be approximated as

$$U = \frac{1}{2}k_h\theta^2,$$
$$T = \frac{1}{2}I_h\dot{\theta}^2$$

for some constants k_h, I_h, then we can consider the system to be a torsional harmonic oscillator with spring constant k_h and moment of inertia I_h.

In the case where the system of particles governed by (K.3) is a magnetic mechanical oscillator, we define

$$U = -\sum_\alpha \mu_\alpha \cdot \mathbf{B}(\mathbf{r}_\alpha) + V_{\text{Coul}} + \frac{1}{2\mu_0}\int (B')^2 \, d^3x,$$
$$T = \sum_\alpha \frac{1}{2m_\alpha}(\dot{\mathbf{r}}_\alpha)^2 + \frac{1}{2}\int \varepsilon_0 (E'_\perp)^2 \, d^3x.$$

We argued in Appendix J that the contribution of the electric field to the kinetic energy of a nanoscale magnetic oscillator is negligible, and so we assume that T can be approximated as

$$\sum_\alpha \frac{1}{2m_a}(\dot{\mathbf{r}}_\alpha)^2 = \frac{1}{2}I_h\dot{\theta}^2.$$

We seek a formula for the contribution made to U by the magnetic energy U_{mag}. Note first that V_{Coul} is responsible for magneto-crystalline anisotropy, as well as exchange interactions, and can therefore affect U_{mag}. To simplify the discussion, we assume that these forms of energy do not contribute significantly to the oscillator's potential. The field $\mathbf{B}'$ generated by the oscillator's magnetization can be expressed as the sum of an averaged field $\mathbf{B}_h$ and an internal field $\mathbf{B}_i$:

$$\mathbf{B}' = \mathbf{B}_h + \mathbf{B}_i.$$

The field $\mathbf{B}_h(\mathbf{r})$ is calculated by treating the ferromagnetic material as a continuum described by the magnetization $\mathbf{M}$, while $\mathbf{B}_i(\mathbf{r})$ corrects $\mathbf{B}_h(\mathbf{r})$ by subtracting the contribution made by the continuum in the immediate vicinity of $\mathbf{r}$ and by adding the actual contribution of the sources in this region. These approximations allow us to write U_{mag} as

$$\begin{aligned} U_{\mathrm{mag}} &= -\sum_\alpha \mu_\alpha \cdot (\mathbf{B}_a + \mathbf{B}_h + \mathbf{B}_i) + \frac{1}{2\mu_0}\int (\mathbf{B}_h + \mathbf{B}_i)^2 \, d^3x \\ &= -\mu \cdot \mathbf{B}_a - \sum_\alpha \mu_\alpha \cdot \mathbf{B}_h + \frac{1}{2\mu_0}\int B_h^2 \, d^3x \\ &\quad - \sum_\alpha \mu_\alpha \cdot \mathbf{B}_i + \frac{1}{\mu_0}\int \mathbf{B}_i \cdot \mathbf{B}_h \, d^3x + \frac{1}{2\mu_0}\int B_i^2 \, d^3x, \end{aligned} \tag{K.5}$$

where μ is the oscillator's net dipole moment.

We next observe that the integral of B_h^2 can be simplified using the vector identity

$$\int_V \mathbf{P} \cdot (\nabla \times \mathbf{Q}) \, d^3x = \int_V \mathbf{Q} \cdot (\nabla \times \mathbf{P}) \, d^3x + \int_S (\mathbf{Q} \times \mathbf{P}) \cdot da, \tag{K.6}$$

where $\mathbf{P}$ and $\mathbf{Q}$ are vector functions, V is a volume of integration, and S is the surface of V. We let $\mathbf{A}_h$ denote the vector potential of the bound current density $\mathbf{J}_h = \nabla \times \mathbf{M}$, and we apply (K.6) repeatedly, noting that in each case the surface integral vanishes:

$$\begin{aligned}\frac{1}{2\mu_0}\int B_h^2\,d^3x &= \frac{1}{2\mu_0}\int \mathbf{B}_h\cdot(\nabla\times\mathbf{A}_h)\;d^3x\\ &= \frac{1}{2\mu_0}\int \mathbf{A}_h\cdot(\nabla\times\mathbf{B}_h)\;d^3x\\ &= \frac{1}{2\mu_0}\int \mathbf{A}_h\cdot\mu_0\mathbf{J}_h\,d^3x\\ &= \frac{1}{2}\int \mathbf{A}_h\cdot(\nabla\times\mathbf{M})\;d^3x\\ &= \frac{1}{2}\int \mathbf{M}\cdot(\nabla\times\mathbf{A}_h)\;d^3x\\ &= \frac{1}{2}\int \mathbf{M}\cdot\mathbf{B}_h\,d^3x. \end{aligned}\tag{K.7}$$

If we replace the sum

$$-\sum_\alpha \mu_\alpha\cdot\mathbf{B}_h$$

appearing in (K.5) by an integral over the volume of magnetic material, the sum of the second and third terms in (K.5) can be expressed as

$$\begin{aligned}-\sum_\alpha \mu_\alpha\cdot\mathbf{B}_h + \frac{1}{2\mu_0}\int B_h^2\,d^3x &= -\int \mathbf{M}\cdot\mathbf{B}_h\,d^3x + \frac{1}{2\mu_0}\int B_h^2\,d^3x\\ &= -\int \mathbf{M}\cdot\mathbf{B}_h\,d^3x + \frac{1}{2}\int \mathbf{M}\cdot\mathbf{B}_h\,d^3x\\ &= -\frac{1}{2}\int \mathbf{M}\cdot\mathbf{B}_h\,d^3x,\end{aligned}$$

and we obtain

$$\begin{aligned}U_{\text{mag}} = -\mu\cdot\mathbf{B}_a - \frac{1}{2}\int \mathbf{M}\cdot\mathbf{B}_h\,d^3x \qquad\qquad\qquad\qquad\qquad (\text{K.8})\\ -\sum_\alpha \mu_\alpha\cdot\mathbf{B}_i + \frac{1}{\mu_0}\int \mathbf{B}_i\cdot\mathbf{B}_h\,d^3x + \frac{1}{2\mu_0}\int B_i^2\,d^3x.\end{aligned}$$

An alternative form which may be more convenient for some purposes is found by using (K.7) to replace the second term of (K.8) by an integral over B_h^2:

$$\begin{aligned} U_{\text{mag}} = -\mu \cdot \mathbf{B}_a - \frac{1}{2\mu_0} \int B_h^2 \, d^3x \\ - \sum_\alpha \mu_\alpha \cdot \mathbf{B}_i (\mathbf{r}_\alpha) + \frac{1}{\mu_0} \int \mathbf{B}_i \cdot \mathbf{B}_h \, d^3x + \frac{1}{2\mu_0} \int B_i^2 \, d^3x. \end{aligned} \tag{K.9}$$

In using equations (K.8) or (K.9) to analyze a device, it is natural to make the simplification of assuming that terms which depend on the internal field $\mathbf{B}_i$ do not vary during the motion. We can show that this assumption is consistent with a simple classical model by expressing $\mathbf{B}_i (\mathbf{r})$ as

$$\mathbf{B}_i (\mathbf{r}) = \mathbf{B}_{\text{near}} (\mathbf{r}) - \mathbf{B}_{\text{avg}} (\mathbf{r}) \, ,$$

where $\mathbf{B}_{\text{near}} (\mathbf{r})$ is the contribution to the magnetic field at $\mathbf{r}$ made by particles in the immediate vicinity of $\mathbf{r}$, and $\mathbf{B}_{\text{avg}} (\mathbf{r})$ is the contribution made by treating these particles as a continuum. In order to estimate $\mathbf{B}_{\text{near}}$, we consider a model in which $\mathbf{B}_{\text{near}}$ is generated by a distribution of classical magnetic dipoles. Reference [49] points out that for most materials, the total electric field acting on a particle due to contributions from nearby electric dipoles distributed either randomly or at lattice sites throughout the material can be considered to be approximately zero. Since magnetic and electric dipole fields have the same functional form, we assume that a similar result holds for $\mathbf{B}_{\text{near}}$, so that the only nonzero contribution to $\mathbf{B}_{\text{near}}$ comes from magnetic fields within each magnetic dipole.

For a current distribution localized in a sphere of radius R centered at the origin, we have

$$\int_{r<R} \mathbf{B} \, d^3x = \frac{2\mu_0}{3} \mu, \tag{K.10}$$

where $\mathbf{B}$ is the field generated by the current distribution, and μ is its dipole moment [49]. Since this result holds for an arbitrarily small sphere surrounding a magnetic dipole, we can consider the field of the dipole to have a delta function contribution at the dipole, so that

$$\mathbf{B}_{\text{near}}(\mathbf{r}) \approx \frac{2\mu_0}{3}\sum_{\alpha}\mu_\alpha \delta(\mathbf{r}_\alpha). \tag{K.11}$$

Averaging the near field given by (K.11) yields

$$\mathbf{B}_{\text{avg}}(\mathbf{r}) = \frac{2\mu_0}{3}\mathbf{M}. \tag{K.12}$$

From (K.11) and (K.12), we obtain

$$\mathbf{B}_i(\mathbf{r}) \approx \frac{2\mu_0}{3}\sum_{\alpha}\mu_\alpha \delta(\mathbf{r}_\alpha) - \frac{2\mu_0}{3}\mathbf{M}(\mathbf{r}). \tag{K.13}$$

We can use (K.13) to simplify (K.8). It follows from (K.13) that

$$\frac{1}{\mu_0}\int \mathbf{B}_i \cdot \mathbf{B}_h \, d^3x = 0.$$

If we assume that for a ferromagnetic dipole μ_α located at $\mathbf{r}_\alpha$, both $\mu_\alpha \cdot \mathbf{M}(\mathbf{r}_\alpha)$ and $|\mathbf{M}(\mathbf{r}_\alpha)|$ remain constant as the oscillator moves, then the remaining terms

$$-\sum_{\alpha}\mu_\alpha \cdot \mathbf{B}_i(\mathbf{r}_\alpha)$$

and

$$\frac{1}{2\mu_0}\int B_i^2 \, d^3x$$

appearing in (K.8) are constant and can be discarded. The oscillator's potential energy can therefore be expressed as

$$U_{\text{mag}} = -\mu \cdot \mathbf{B}_a - \frac{1}{2}\int \mathbf{M} \cdot \mathbf{B}_h \, d^3x, \tag{K.14}$$

or, equivalently, as

$$U_{\text{mag}} = -\mu \cdot \mathbf{B}_a - \frac{1}{2\mu_0}\int B_h^2 \, d^3x.$$

The magnetic spring constant k_{mag} is given by

$$k_{\mathrm{mag}} = \frac{d^2 U_{\mathrm{mag}}}{d\theta^2}.$$

As a check on (K.14), we note that it is closely related to the expression

$$E = -\mathbf{M} \cdot \mathbf{H} \tag{K.15}$$

which is frequently used in the literature of magnetic materials for the energy of a magnetization $\mathbf{M}$ in a averaged field $\mathbf{H}$. In the case of ferromagnetic materials, $|\mathbf{M}|$ can be considered constant on a microscopic scale (since to a first approximation the direction but not the magnitude of $\mathbf{M}$ varies between domains and within domain walls), and (K.15) can be written as

$$\begin{aligned} E &= -\frac{1}{\mu_0}\mathbf{M} \cdot \mathbf{B} + \mathbf{M} \cdot \mathbf{M} \\ &= -\frac{1}{\mu_0}\mathbf{M} \cdot \mathbf{B} + \text{constant}, \end{aligned}$$

which differs only by an additive constant and a proportionality constant from the energy expression

$$E = -\mathbf{M} \cdot \mathbf{B}. \tag{K.16}$$

If we had naively used (K.16) to obtain a potential energy expression for the oscillator, taking care to avoid double counting of spin-spin interactions, we would have obtained the same expression derived more carefully in this discussion.

Appendix L

Spring constant and moment of inertia of a torsion beam

The spring constant and moment of inertia of a torsion beam can be most naturally obtained from the Lagrangian for the fundamental mode, which can be derived from the Lagrangian that governs arbitrary motions of the beam. For this analysis, we consider a simple rectangular beam, and we suppose that the z-axis lies along the central axis of the beam. Let $\phi(z, t')$ be the angular displacement of the beam at position z and time t'. (The time is denoted by t' to distinguish it from the thickness t of the beam.)

Note first that the elastic potential energy U of the beam is [52, 53]

$$U = \frac{1}{2} C \int_0^l \left(\frac{\partial \phi}{\partial z} \right)^2 dz. \tag{L.1}$$

The constant C is known as the torsional rigidity of the beam. The explicit definition of C depends on the assumption that the displacement $u_z(x, y, z)$ in the direction of the z-axis is proportional to $\partial\phi/\partial z$, that is, there is a function $\psi(x, y)$ such that $u_z = \psi\, \partial\phi/\partial z$. The integral

$$\int\int \left(x^2 + y^2 + x\frac{\partial \psi}{\partial y} - y\frac{\partial \psi}{\partial x} \right) dx\, dy,$$

taken over the cross section of the beam, is called the torsional constant of the beam. (Formulas for torsional constants are available in reference [54].) If we denote the torsional constant by J, then C is given by

$$C = GJ,$$

where G is the beam's modulus of rigidity.

The kinetic energy T of the beam is [52, 53]

$$T = \frac{1}{2}\rho_b I_p \int_0^l \left(\frac{\partial \phi}{\partial t'}\right)^2 dz. \tag{L.2}$$

Here I_p is the polar moment of inertia of the cross section:

$$I_p = \int\int \left(x^2 + y^2\right) dx\, dy.$$

Note that equation (L.2) neglects the kinetic energy due to motion along the beam's axis, and that both (L.1) and (L.2) depend on the condition $R\frac{\partial \phi}{\partial z} \ll 1$, where R is the maximum transverse dimension of the rod. This condition is necessary in order to achieve consistency between the assumption that strains are infinitesimal and the assumption that $u_x = -yz\frac{\partial \phi}{\partial z}$ and $u_y = zx\frac{\partial \phi}{\partial z}$, where u_x and u_y are the displacements along the x-axis and y-axis, respectively. These assumptions on u_x and u_y lead to strains which depend on the products $x\frac{\partial \phi}{\partial z}$ and $y\frac{\partial \phi}{\partial z}$.

The Lagrangian $\mathfrak{L} = T - U$ can be used to derive a wave equation:

$$\rho_b I_p \frac{\partial^2 \phi}{\partial \left(t'\right)^2} + \frac{1}{2}C\frac{\partial^2 \phi}{\partial z^2} = 0.$$

For a rod of length l fixed at both ends, the (non-normalized) modes are [55]

$$\sin\left(\frac{n\pi}{l}z\right)\cos\left(\omega_n t'\right), \tag{L.3}$$

where

$$\rho_b I_p \omega_n^2 = C\left(\frac{n\pi}{l}\right)^2,$$

or

$$\omega_n = \frac{n\pi}{l}\sqrt{\frac{C}{\rho_b I_p}}. \tag{L.4}$$

We write ϕ in the form

$$\phi\,(z,t') = \sum_n q_n(t')\sin(\frac{n\pi}{l}z),$$

and observe that the Lagrangian can be written as a sum of Lagrangians governing independent modes, with mode n characterized by discrete variables q_n and $\dot{q}_n$. Focusing our attention on the fundamental mode, we define $q \equiv q_n$. The Lagrangian $\mathfrak{L}$ governing this mode is

$$\mathfrak{L} = \frac{1}{2}\left[\rho_b I_p \int_0^l \sin^2(\frac{\pi}{l}z)\,dz\right](\dot{q})^2 - \frac{1}{2}\left[C\int_0^l \left(\frac{\pi}{l}\right)^2 \cos^2(\frac{\pi}{l}z)dz\right]q^2,$$

which simplifies to

$$\mathfrak{L} = \frac{1}{2}\left(\frac{\rho_b I_p l}{2}\right)(\dot{q})^2 - \frac{1}{2}\left(\frac{C\pi^2}{2l}\right)q^2. \tag{L.5}$$

Equation (L.5) describes a harmonic resonator with spring constant

$$K_{\text{beam}} = \frac{C\pi^2}{2l}$$

and moment of inertia

$$I_{\text{beam}} = \frac{\rho_b I_p l}{2}.$$

Note that if the only excited mode is the fundamental, then the variable q gives the angular displacement at the center of the beam. For a rectangular beam, the torsional rigidity C and polar moment of inertia I_p are [54]

$$C = Gwt^3\left[\frac{1}{3} - 0.21\frac{t}{w}\left(1 - \frac{t^4}{12w^4}\right)\right],$$
$$I_p = \frac{1}{12}tw\left(t^2 + w^2\right).$$

We can express the beam's spring constant and moment of inertia as

$$K_{\text{beam}} = \frac{\pi^2 G w t^3}{2l} \left[\frac{1}{3} - 0.21 \frac{t}{w} \left(1 - \frac{t^4}{12 w^4} \right) \right] \tag{L.6}$$

and

$$I_{\text{beam}} = \frac{\rho_b t w l}{24} \left(t^2 + w^2 \right) . \tag{L.7}$$

Equations (L.6) and (L.7) were derived under the assumption that the elastic material is isotropic. If the material is a cubic crystal and the beam axis is perpendicular to one of the faces of the cubic cell, similar equations can be derived, with the modulus of rigidity G replaced by the stiffness constant C_{44}. Reference [14] tabulates adiabatic stiffness constants for several cubic crystals. The magnitude of C_{44} typically decreases by 10% or less as the crystal is cooled from room temperature to $\sim 0\,\text{K}$. For numerical examples, we used the room temperature adiabatic stiffness constant for Si as a characteristic value of C_{44} [14]:

$$C_{44} = 7.96 \times 10^{10}\,\text{N} / \text{m}^2 .$$

Appendix M

Eddy-current heating of metallic cylinders

In this appendix, we present a rough estimate of the temperature increase ΔT_h caused by eddy-current heating. Since experimental information regarding the heat conductivity of nanoscale beams at mK temperatures is not available in the literature, and since the dimensions of the ferromagnetic cylinders are small enough that the low-temperature conductivity of the cylinders may be nonlocal, an accurate estimate cannot be made using results available in the literature. The simplified analysis we present here illustrates the way in which ΔT_h is determined by physical parameters which depend strongly on the dimensions and temperature of the resonator. Estimates of the order of magnitude of these parameters based on the limited information available in the literature leave open the possibility that detection sensitivity would be decreased by the temperature change ΔT_h if the magnetic cylinders are metallic. The possibility of eliminating eddy-current heating by the use of semiconducting ferromagnets is discussed in section 7 of chapter 5.

We first consider eddy currents in a nonmagnetic conducting particle placed in a uniform alternating magnetic field, and we assume that the mean free path of the conduction electrons is short enough that Ohm's law holds. If the dimensions of the particle are small compared to the skin depth of the conducting material, then we can expect the fields generated within the particle by the eddy currents themselves to be negligible compared to the alternating applied field and the Faraday electric field induced by it. If the particle is a sphere, the electric field driving eddy currents within it will consist of circular loops, and surface charges will in general develop on the surface of the particle. Altering the geometry of the particle does not perturb the Faraday electric field but leads to an altered configuration of surface charges, with the result that the circular electric field loops within the particle are perturbed. If a fixed magnetization is added to the particle and a static magnetic field is present, we may in general expect additional modifications to the configuration of surface charges (as in the Hall effect) and the configuration of eddy currents. Provided that the particle bears some resemblance to a sphere, however, a rough estimate of the power dissipation associated with the eddy currents can be made by treating the particle as a nonmagnetic spherical conductor.

If the mean free path of the electrons is at least as large as the dimensions of the particle, the conductivity becomes nonlocal, since most electron trajectories would sample a variety of different electric fields. An electron in a trajectory that crosses a loop of the electric field lines, for instance, would be accelerated in different directions during different portions of the trajectory, and the current at each point in the particle would depend on an integral over all points of all trajectories. We might guess that eddy currents within the particle would be weakened by the nonlocal nature of the conductivity, since many of the trajectories would have an electron experiencing accelerations during different portions of the trajectory which do not add constructively. Note, however, that when the conductivity becomes nonlocal, heating of the particle by the Faraday electric field may not be correlated in a simple way with the size of the eddy currents. It is the kinetic energy donated to individual electrons during their trajectories which contributes to heating, not the kinetic energy associated with the net current at a given point. Even if the net current is negligible because the contributions from different trajectories do not add constructively, the acceleration of electrons by the induced electric field as they move along trajectories might lead to a temperature increase in the metal.

For the example resonator presented in table 5.3, each of the magnetic cylinders has a length of 40 nm and a diameter of 55 nm. In a reasonably pure conductor at a temperature of a few Kelvin or below, the mean free path of an electron in the bulk metal would be significantly larger than the dimensions of the cylinder. The mean free path of an electron in room-temperature bulk iron is about 5 nm [56], for instance, and this can increase by orders of magnitude at low temperatures, when the scattering of electrons by phonons is "frozen out." At temperatures below a few Kelvin, the conductivity of a normal conductor does not depend on temperature; its value depends instead on the extent to which electrons are scattered by surfaces, lattice defects, impurities, and the like. A high concentration of scattering centers would be needed within the ferromagnetic material to yield a mean free path which is small on the scale of the curving electric field lines within the magnetic particles of the resonator.

If the resistivity of the cylinders is high enough that the mean free path of the electrons is smaller than the dimensions of the cylinders by a factor of ~ 10 or more, an analysis of eddy currents based on Ohm's law is relevant. In order to illustrate the eddy-current heating which could occur in this case, we assume a mean free path of 4 nm for the electrons. Note that if the mean free path is reduced below this value, both the conductivity and eddy-current power dissipation will decrease, and so we may consider this heating estimate to be a "worst-case" estimate for the regime in which Ohm's law holds. In converting the mean free path into a resistivity, we assume that conductivity is proportional to mean free path, and we note that the mean free path and resistivity for Fe at room-temperature are 4.75 nm and $9 \times 10^{-8}\,\Omega\,\mathrm{m}$, respectively [56]. The resistivity ρ corresponding to a mean free path of 4 nm in Fe is therefore

$$\rho = 9 \times 10^{-8}\,\Omega\,\mathrm{m}\,\frac{4.75}{4.0} = 10.7 \times 10^{-8}\,\Omega\,\mathrm{m}\,.$$

The skin depth [49] associated with this resistivity at frequency 630 MHz is

$$\delta = 6.6\,\mu\mathrm{m}\,,$$

which is much larger than the dimensions of the magnetic particle.

Smythe has derived a general expression for eddy currents induced in a conducting sphere by a uniform alternating magnetic field in the quasi-static regime [57]. For a nonmagnetic particle whose radius r is small compared to the skin depth, Smythe's expression for the power dissipation in the particle reduces to [58, 59]

$$P_{\text{diss}} = \frac{\pi r^5 \omega_0^2 B_1^2}{15\rho},$$

where B_1 and ω_0 are the magnitude and frequency of the alternating field, ρ is the resistivity of the particle, and r is its radius. We model each ferromagnetic particle as a sphere of radius $r = 25\,\text{nm}$, and we assume continuous irradiation by a resonant field (frequency $\omega_0/2\pi = 630\,\text{MHz}$) strong enough to give protons a Rabi frequency of 20 kHz:

$$\begin{aligned} B_1 &= 2\,(2\pi)\,20\,\text{kHz} \,/\, \left(\frac{267.5\times 10^6}{\text{s}\,\text{T}}\right) \\ &= 9.395\,4\times 10^{-4}\,\text{T}\,. \end{aligned}$$

We find that the power deposited in each particle is

$$P_{\text{diss}} = 2.627\,1\times 10^{-19}\,\text{W}\,. \tag{M.1}$$

For a long, thin beam with rough surfaces but no scattering centers within the crystal, the predicted thermal conductance is [60, 61]

$$K = \frac{2\pi^2 k_B^4}{15\hbar^3 v_s}\frac{lA}{L}T^3, \tag{M.2}$$

where L and A are the length cross-sectional area of the beam, $v_s = 4500\,\text{m}/\text{s}$ is the speed of sound in silicon, T is the temperature, and l is the phonon mean free path length. A recent experimental test of the low-temperature thermal conductance of nanoscale Si beams of cross section $130\,\text{nm}\times 200\,\text{nm}$ found that although the conductance varied as T^3 above $T = 1.4\,\text{K}$, the temperature dependence was less strong below this temperature and appeared to flatten out at the lowest temperature

($\sim 0.5\,\mathrm{K}$) at which a measurement was taken [60]. Temperature dependence weaker than T^3 was also observed at temperatures between 20 K and 60 K for beams of width 37 nm and 22 nm [62].

Although these departures from the predicted T^3 dependence are promising for our purposes, they are not well understood, and there is no experimental information on the temperature dependence of thermal conductivity below $\sim 0.5\,\mathrm{K}$. In estimating the thermal conductance of the resonator's beam, we therefore start from the expression which was found to be valid above 1.4 K [60]:

$$K_{\mathrm{cond}} = 2.6 \times 10^{-11} T^3 \frac{\mathrm{W}}{\mathrm{K}}. \tag{M.3}$$

From equation M.2, we see that if the mean free path did not depend on the dimensions of the beam or the temperature, then at 10 mK the value of thermal conductance for a section of the resonator beam stretching from the sample to the bulk substrate is

$$\begin{aligned} K_{\mathrm{cond}} &= 2.6 \times 10^{-11} T^3 \frac{\mathrm{W}}{\mathrm{K}} \left(\frac{2.5\,\mu\mathrm{m}}{3.5\,\mu\mathrm{m}/2} \right) \left(\frac{50\,\mathrm{nm} \times 50\,\mathrm{nm}}{130\,\mathrm{nm} \times 200\,\mathrm{nm}} \right) \\ &= 3.5714 \times 10^{-18} \frac{\mathrm{W}}{\mathrm{K}}. \end{aligned} \tag{M.4}$$

The mean free path obtained by comparing (M.3) to (M.2) was $\sim 600\,\mathrm{nm}$ [60]. In a simplified model which assumes that phonon scattering occurs only at the surface of the beam, and that a fraction p of the phonons incident upon a surface is reflected specularly, while the remainder are scattered diffusely, the mean free path can be written as [61]

$$l = \frac{1+p}{1-p} l_0,$$

where l_0 is the mean free path in the case where no specular reflection occurs. For a circular cross-section of diameter d or a square cross-section of side d, we have $l_0 = d$ and $l_0 = 1.12d$, respectively. We might therefore guess that if the cross-section corresponding to equation (M.3) were scaled down from 130 nm $\times$200 nm to 50 nm $\times$50 nm, the mean free path would decrease by a factor between 2 and 4. Due to lack of experimental information regarding either the size dependence or the temperature dependence of the phonon mean free path for nanowires ≤ 1.4 K, however, we will use equation (M.4) for our estimate of eddy-current heating rather than attempting to incorporate this guess into the estimate. Combining equations (M.1) and (M.4) yields a temperature difference of

$$\Delta T_{\rm beam} = \frac{P_{\rm diss}}{K_{\rm cond}} = 75 \text{ mK} \tag{M.5}$$

between the center of the resonator beam and each of its ends.

An estimate of the temperature gradient across the magnet-silicon interface can also be made. For interfaces between bulk solids, simplified theories of thermal boundary resistance have been shown to agree with experiment at low temperatures down to $\sim$ 100 mK [63]. The acoustic mismatch theory and the diffuse mismatch theory estimate the probability of phonon transmission across the boundary in the respective limits of specular reflection and diffuse scattering at the boundary. These theories are found to yield similar values for the thermal resistance R_{Bd}, and in the case of interfaces between Si and transition metals, $R_{Bd}T^3$ is typically found to lie in the range 10 to 20 $\text{K}^4 / (\text{W} / \text{cm}^2)$ [63]. Setting

$$R_{Bd} = \frac{15}{T^3}\frac{\text{K}^4}{\text{W} / \text{cm}^2}, \quad T = 10 \text{ mK}$$

we obtain

$$\Delta T_{\rm boundary} = P_{\rm diss} R_{Bd}/A \tag{M.6}$$

$$= 0.166\,\text{K}\,. \tag{M.7}$$

In (M.6), A represents the area of the flat surface of each ferromagnetic cylinder.

Equations (M.5) and (M.7) depend on the assumption that the temperature differences ΔT are small enough that a single value of $T \approx 10$ mK can be used to characterize the beam and the magnetic particle. Since the temperature differences we obtained are roughly an order of magnitude greater than 10 mK, this assumption is clearly invalid. A simple correction can be made by assuming that the $T \approx 25$ mK, which yields

$$\Delta T_{\text{beam}} = 4.7 \text{ mK},$$
$$\Delta T_{\text{boundary}} = 11 \text{ mK}.$$

Since increasing the temperature from 10 mK to 25 mK decreases the polarization from 0.91 to 0.54, and increases the thermal noise in the resonator, the estimates we have made of thermal and electric conductivity suggest that sensitivity could be decreased by the temperature change ΔT_h. The use of semiconducting ferromagnetic material such as EuO may therefore be preferred, since the resistivity of the cylinders would be orders of magnitude larger than the values we used for this estimate [38].

Appendix N

Correlation function of the mechanical coordinate during cooling by hyperpolarized spins

This appendix derives the symmetric autocorrelation function $C(t)$ of the resonator's mechanical coordinate during cooling by hyperpolarized spins, required for the analysis in section 7 of chapter 7. In deriving a formula for $C(t)$, we will need a general expression for $\langle \theta \rangle (t)$. We define

$$\eta_\alpha = \frac{1 + \sqrt{1 - 8 \langle I_z \rangle_\infty (gb)^2}}{2gb} i, \tag{N.1}$$

$$\eta_\beta = \frac{1 - \sqrt{1 - 8 \langle I_z \rangle_\infty (gb)^2}}{2gb} i, \tag{N.2}$$

and

$$\omega'_k = \omega_h + g \, \mathrm{Re}(\eta_k), \tag{N.3}$$

$$1/\tau'_k = 1/\tau_h - g \, \mathrm{Im}(\eta_k), \tag{N.4}$$

for $k = \alpha,\ \beta$. The general solution to the system of differential equations given by (7.46) and (7.47) is

$$\langle a\rangle(t) = p\exp\left[-\left(i\omega_a' + 1/\tau_a'\right)t\right] + q\exp\left[-\left(i\omega_\beta' + 1/\tau_\beta'\right)t\right], \tag{N.5}$$

$$\langle I_+\rangle(t) = p\eta_\alpha\exp\left[-\left(i\omega_a' + 1/\tau_a'\right)t\right] + q\eta_\beta\exp\left[-\left(i\omega_\beta' + 1/\tau_\beta'\right)t\right], \tag{N.6}$$

$$p = \frac{\eta_\beta\langle a\rangle(0) - \langle I_+\rangle(0)}{\eta_\beta - \eta_\alpha}, \tag{N.7}$$

$$q = \frac{-\eta_\alpha\langle a\rangle(0) + \langle I_+\rangle(0)}{\eta_\beta - \eta_\alpha}. \tag{N.8}$$

Given the general expression for $\langle a\rangle(t)$, we can write $\langle\theta\rangle(t)$ as

$$\langle\theta\rangle(t) = \sqrt{\frac{2\hbar}{I_h\omega_h}}\,\mathrm{Re}\left\{\langle a\rangle(t)\right\}. \tag{N.9}$$

The method presented in reference [8] can be used to express the correlation function $C(t)$ as

$$C(t) = \langle\theta(t)\rangle,$$

where the initial conditions which determine $\langle\theta(t)\rangle$ are calculated as if the density matrix at time $t = 0$ were

$$\rho(0) = \frac{1}{2}\left(\rho_\infty\theta + \theta\rho_\infty\right),$$

with ρ_∞ the steady state density matrix of the spin-resonator system. From equations (N.5) through (N.9), it follows that it is sufficient to find formulas for

$$p = \frac{1}{2\left(\eta_\beta - \eta_\alpha\right)}\left\{\eta_\beta\,\langle a\theta + \theta a\rangle_\infty - \langle I_+\theta + \theta I_+\rangle_\infty\right\},$$

$$q = \frac{1}{2\left(\eta_\beta - \eta_\alpha\right)}\left\{-\eta_\alpha\,\langle a\theta + \theta a\rangle_\infty + \langle I_+\theta + \theta I_+\rangle_\infty\right\}.$$

We show below that the steady-state expectation values $\langle\theta^2\rangle_\infty$, $\langle p_\theta\theta + \theta p_\theta\rangle_\infty$, $\langle I_x\theta\rangle_\infty$, and $\langle I_y\theta\rangle_\infty$ can be approximated as

$$\left\langle\theta^2\right\rangle_\infty = \frac{\hbar}{I_h\omega_h}\left(n_\infty + \frac{1}{2}\right), \tag{N.10}$$

$$\langle p_\theta\theta + \theta p_\theta\rangle_\infty = 0, \tag{N.11}$$

$$\langle I_x\theta\rangle_\infty = 0, \tag{N.12}$$

$$\langle I_y\theta\rangle_\infty = -K_\infty/\left(\gamma dB_x/d\theta\right), \tag{N.13}$$

where n_∞ is given by (7.28) and K_∞ by equation (7.24).

The formula for $\langle I_y\theta\rangle_\infty$ can be estimated by noting from (7.39) that in the absence of the rotating-wave approximation, the rate K at which quanta are transferred from spins to oscillator is given by

$$K = -\gamma\frac{dB_x}{d\theta}\langle I_y\theta\rangle . \tag{N.14}$$

Equations (N.10) through (N.12) can be obtained by deriving the equations of motion for selected operators using the master equation (7.9), setting derivatives to zero, and solving the resulting set of equations. The equations of motion which are needed are

$$\frac{d}{dt}\left\langle\theta^2\right\rangle = -\frac{2}{\tau_h}\left\{\left\langle\theta^2\right\rangle - \frac{\hbar}{I_h\omega_h}\left(n + \frac{1}{2}\right)\right\} + \frac{1}{I_h}\langle p_\theta\theta + \theta p_\theta\rangle - \frac{dB_x}{d\theta}\frac{\hbar\gamma}{I_h\omega_h}\langle I_y\theta\rangle , \tag{N.15}$$

$$\frac{d}{dt}\langle p_\theta\theta + \theta p_\theta\rangle = -\frac{2}{\tau_h}\langle p_\theta\theta + \theta p_\theta\rangle - 4I_h\omega_h^2\left\langle\theta^2\right\rangle + 4\hbar\omega_h\left(\left\langle a^\dagger a\right\rangle + \frac{1}{2}\right) + \hbar\gamma\frac{dB_x}{d\theta}\left(\langle I_x\theta\rangle - \frac{1}{I_h\omega_h}\langle I_y p_\theta\rangle\right), \tag{N.16}$$

$$\frac{d}{dt}\langle I_y\theta\rangle = -\frac{1}{\tau_1}\langle I_y\theta\rangle - \omega_h\langle I_x\theta\rangle + \frac{1}{I_h}\langle I_y p_\theta\rangle - \frac{dB_x}{d\theta}\frac{\hbar\gamma}{2I_h\omega_h}\left\langle I_y^2\right\rangle + \frac{dB_x}{d\theta}\frac{\gamma}{2}\left\langle\theta^2 I_z\right\rangle , \tag{N.17}$$

$$\frac{d}{dt}\left\langle I_+a^\dagger + I_-a\right\rangle = -\frac{1}{\tau_1}\left\langle I_+a^\dagger + I_-a\right\rangle . \tag{N.18}$$

From (N.18), we obtain

$$\begin{aligned} 0 &= \left\langle I_+ a^\dagger + I_- a \right\rangle_\infty \\ &= \sqrt{\frac{2 I_h \omega_h}{\hbar}} \left\langle I_x \theta \right\rangle_\infty + \sqrt{\frac{2}{\hbar I_h \omega_h}} \left\langle I_y p_\theta \right\rangle_\infty , \end{aligned}$$

from which it follows that

$$\left\langle I_y p_\theta \right\rangle_\infty = -I_h \omega_h \left\langle I_x \theta \right\rangle_\infty . \qquad \text{(N.19)}$$

Setting the left sides of (N.16) through (N.15) to zero and making the assumption that

$$\left\langle \theta^2 I_z \right\rangle_\infty \approx \left\langle \theta^2 \right\rangle_\infty \left\langle I_z \right\rangle_\infty$$

yields the following solution for $\left\langle \theta^2 \right\rangle_\infty$:

$$\begin{aligned} \frac{I_h \omega_h}{\hbar} \left(1 + \frac{1}{\omega_h^2 \tau_h^2} + \frac{1}{2 \omega_h^2 \tau_c \tau_1} \right) \left\langle \theta^2 \right\rangle_\infty = n_\infty & \left(1 + \frac{1}{\omega_h^2 \tau_h \tau_\infty} - \frac{1}{\omega_h^2 \tau_h \tau_c} + \frac{1}{2 \omega_h^2 \tau_c \tau_1} \right) \\ & + \frac{1}{2} \left(1 + \frac{1}{\omega_h^2 \tau_h^2} + \frac{1}{2 \omega_h^2 \tau_c \tau_1} \right) \\ & + \frac{1}{\left\langle I_z \right\rangle_\infty} \left(\left\langle I_y^2 \right\rangle_\infty - \frac{N}{4} \right) \frac{1}{2 \omega_h^2 \tau_c \tau_1} . \end{aligned}$$

Each of the decay times τ_h, τ_c, τ_1, and τ_∞ can be assumed to be much longer than the period of the resonator; if we assume in addition that

$$\left\langle I_y^2 \right\rangle_\infty \approx N/4,$$

then we obtain the solution (N.10) through (N.12).

In the regime where τ_h is short and the coupling is strong ($4\tau_h/\tau_c > 1$), the correlation function $C(t)$ can be written as

$$C(t) = \exp(-t/2\tau_h)\cos(\omega_h t) \times \tag{N.20}$$
$$\left\{\left\langle\theta^2\right\rangle_\infty \cos(dt) - \left(c_1\left\langle\theta^2\right\rangle_\infty + c_2\left\langle I_y\theta\right\rangle_\infty\right)\sin(dt)\right\},\ \ t > 0,$$
$$d = \left(\sqrt{4\tau_h/\tau_c - 1}\right)/2\tau_h,$$
$$c_1 = 1/\sqrt{4\tau_h/\tau_c - 1},$$
$$c_2 = -\sqrt{\frac{2\hbar}{I_h\omega_h}}\frac{2g\tau_h}{\sqrt{4\tau_h/\tau_c - 1}},$$

and in the limit of strong coupling ($4\tau_h/\tau_c \gg 1$), this reduces to

$$C(t) \approx \exp(-t/2\tau_h)\cos(\omega_h t)\left\{\left\langle\theta^2\right\rangle_\infty \cos(dt) - c_2\left\langle I_y\theta\right\rangle_\infty \sin(dt)\right\},\ \ t > 0 \tag{N.21}$$
$$\approx \frac{\hbar}{I_h\omega_h}n_\infty \exp(-t/2\tau_h)\cos(\omega_h t)\left\{\cos(dt) - \sqrt{\tau_h/\tau_c}\sin(dt)\right\},\ \ t > 0 \tag{N.22}$$
$$d \approx 1/\sqrt{\tau_h\tau_c}.$$

(In making this simplification we have also assumed $n_\infty \gg 1/2$ and $n_\infty \gg n_c$.) The expression in curly brackets is a sinusoidal function which can be written as

$$\cos(dt) - \sqrt{\tau_h/\tau_c}\sin(dt) = (1 + \tau_h/\tau_c)\cos(dt + \phi). \tag{N.23}$$

We obtain

$$C(t) = \frac{\hbar}{I_h\omega_h}n_\infty\left(1 + \frac{\tau_h}{\tau_c}\right)\exp(-t/2\tau_h)\cos(\omega_h t)\cos(dt + \phi),\ \ t > 0 \tag{N.24}$$
$$= \frac{\hbar}{I_h\omega_h}n_\infty\left(1 + \frac{\tau_h}{\tau_c}\right)\exp(-t/2\tau_h) \times \tag{N.25}$$
$$\frac{\cos((\omega_h + d)t + \phi) + \cos((\omega_h - d)t - \phi)}{2},\ \ t > 0.$$