Chapter 7

Cooling a single mode with hyperpolarized spins?

1 Hyperpolarized spins as a cold bath

The simulations presented in chapter 6 are based on the assumption that all modes of the mechanical oscillator are cooled to $\sim$ 10 mK by a dilution refrigerator. An alternative approach would be to extract energy from the single resonant mode, thereby cooling it to a temperature below that of the sample and the oscillator's remaining modes. A promising method for removing energy from a single mode is to use negative feedback to reduce the amplitude of the mode's thermal motion. Such "feedback cooling" of a single mechanical mode from a base temperature of 2.2 K down to 3 mK has been demonstrated experimentally [42].

We have considered the possibility of using cold spins to absorb the mode's energy. If a stream of hyperpolarized xenon nuclei passes by a warm mechanical oscillator whose frequency is resonant with the Larmor frequency of xenon, the spin-resonator interaction governed by the Hamiltonian (2.11) would cause the resonator to be cooled toward the spin temperature of the xenon. The scheme of using hyperpolarized spins to cool a resonator was particularly interesting to us because of the possibility of detecting entropy exchange between spins and resonator at sizes substantially larger than nanoscale. Numerical examples such as those presented in section 6 of chapter 5 suggest that a nanoscale resonator is needed to achieve measurable cooling of a spin system; we investigated the possibility that cooling of a larger resonator by many hyperpolarized spins might be detectable.

In studying the system consisting of a warm mechanical oscillator coupled to hyperpolarized spins, we begin with a heuristic example in which the spins are modelled as a cold bath which damps the oscillator. The oscillator is also coupled to a warm bath, and the master equation for the damped oscillator is

$$\frac{d}{dt}\rho = \frac{1}{i\hbar}\left[H_{\text{osc}}, \rho\right] + \Lambda_h\rho + \Lambda_c\rho, \tag{7.1}$$

where H_{osc} is the Hamiltonian for the undamped oscillator, and Λ_h, Λ_c are the superoperators associated with damping by the warm and cold baths, respectively. The formula for the relaxation superoperator associated with damping of a harmonic oscillator by a thermal bath [8] allows us to write Λ_h explicitly as

$$\begin{aligned}\Lambda_h\rho = &-\frac{n_h+1}{\tau_h}\left[a^\dagger a, \rho\right]_+ + 2\frac{n_h+1}{\tau_h}a\rho a^\dagger \\ &-\frac{n_h}{\tau_h}\left[aa^\dagger, \rho\right]_+ + 2\frac{n_h}{\tau_h}a^\dagger\rho a,\end{aligned} \tag{7.2}$$

where n_h is the number of quanta the resonator would have in equilibrium with the warm bath, and τ_h would be the ringdown time of the resonator if only the warm bath were present. By replacing n_h and τ_h by analogous quantities n_c and τ_c associated with the cold bath, we obtain an explicit expression for Λ_c. Letting Λ_∞ denote the sum $\Lambda_h + \Lambda_c$, we find that

$$\begin{aligned}\Lambda_\infty\rho = &-\frac{n_\infty+1}{\tau_\infty}\left[a^\dagger a, \rho\right]_+ + 2\frac{n_\infty+1}{\tau_\infty}a\rho a^\dagger \\ &-\frac{n_\infty}{\tau_\infty}\left[aa^\dagger, \rho\right]_+ + 2\frac{n_\infty}{\tau_\infty}a^\dagger\rho a,\end{aligned}$$

where

$$\frac{1}{\tau_\infty} = \frac{1}{\tau_h} + \frac{1}{\tau_c}, \tag{7.3}$$

$$n_\infty = \frac{\tau_h n_c + \tau_c n_h}{\tau_h + \tau_c}. \tag{7.4}$$

The system consisting of the resonator in contact with two baths is formally equivalent to a system in which only a single bath is present, with τ_∞ the ringdown time of the oscillator and

$$T_\infty = \frac{\hbar\omega_h}{k_B \ln(1 + 1/n_\infty)}$$

the temperature of the bath. The spectral density of the thermal torque can be obtained by substituting τ_∞ and n_∞ into equation (4.41):

$$\begin{aligned} S_{N'} &= \frac{4I_h\hbar\omega_h}{\tau_\infty}\left(\frac{1}{2} + n_\infty\right) \\ &= \frac{4I_h\hbar\omega_h}{\tau_h}\left(\frac{1}{2} + n_h\right) + \frac{4I_h\hbar\omega_h}{\tau_c}\left(\frac{1}{2} + n_c\right). \end{aligned} \tag{7.5}$$

We can interpret (7.5) to mean that the thermal torque responsible for introducing noise into the measurement is additive. Adding a cold bath will therefore not decrease the thermal torque exerted by the warm bath.

This argument highlights the possibility that the modification of a resonator's ringdown time by the cold spins could mitigate the advantages associated with cooling a single mode, but the model we used to obtain (7.5) is not in general correct for a system in which hyperpolarized spins flow past a mechanical resonator. Equation (7.1) describes relaxation associated with two baths, each of which acts independently of the other. However, we found in section 2 of chapter 2 that the rate constant for energy flow between spins and resonator depends on τ_h, that is, on the coupling between the resonator and the warm bath. Since energy exchange between spins and resonator depends on the collective properties of the spins, resonator, and warm bath, it is problematic to represent the cold spins as an independent cold bath.

Section 2 of this chapter presents a model based on the interaction Hamiltonian for the spin-resonator system, and sections 3 through 7 use this model to analyze the system. In summarizing here the results of this analysis, we use T_2 to denote a transverse spin decay time that can include contributions from ordinary transverse relaxation as well as from the flow of spins into and out of the region of space where the Larmor frequency is resonant with the mechanical oscillator. Our attention is focused on the regime where

$$\tau_h \ll T_2, \tag{7.6}$$

that is, the regime where the fluctuations of the resonator coordinate limit the magnitude of the spin-resonator correlations which can develop. In section 3, we find that the steady-state number of quanta n_∞ in the cooled resonator can in fact be calculated using equation (7.3), which we obtained above by treating the spins as a cold bath. In the regime defined by (7.6), the constant τ_c appearing in (7.3) is

$$\frac{1}{\tau_c} \equiv 2g^2\tau_h \left\langle I_z \right\rangle_\infty . \tag{7.7}$$

By way of contrast, the rate constant which governs cooling of spins by a single resonator at zero Kelvins is

$$R_0 = 2g^2\tau_h. \tag{7.8}$$

Since (7.7) is larger than (7.8) by a factor of $\left\langle I_z \right\rangle_\infty$, these equations suggest the possibility that entropy flow between spins and resonator might be most easily observed by performing an experiment in which hyperpolarized spins cool a resonator. Section 4 presents numerical examples to characterize the regime in which cooling may be possible, however, and for these examples, g^2 scales so strongly with size that cooling becomes negligible at size scales of order $10\,\mu$m or larger.

Since (7.3) can be used to calculate n_∞, it is tempting to conclude that the cold spins can be treated as a cold bath. Section 5 shows, however, that in the numerical examples where substantial cooling is possible, the spins and resonator are so strongly coupled that one cannot distinguish a mechanical mode or a spin mode. Instead, the two modes of the spin-resonator system include equal contributions from the spins and the mechanical resonator. It is only in the limit of weak spin-resonator coupling and short T_2 that (7.7) can be used to calculate the decay time of the mechanical mode. In general, the mechanical response to an external torque is also incorrectly predicted by the model which treats the hyperpolarized spins as a cold bath. Indeed, in the case where (7.6) holds, the mechanical resonator could be considered a device for transducing an external mechanical torque at frequency ω_h into precessing magnetization of hyperpolarized spins, since the energy donated by the external torque ends up as transverse spin excitation. For sufficiently strong spin-resonator coupling, however, we find that a resonant mechanical response can be obtained by driving the mechanical oscillator at one of the two eigenfrequencies $\omega_h \pm d$ of the spin-resonator system, where $d \approx 1/\sqrt{\tau_h \tau_c}$.

Section 7 uses the symmetric autocorrelation function for the oscillator' mechanical coordinate to characterize quantitatively the mechanical fluctuations. Equation (7.5) is obtained in the limit of weak spin-resonator coupling and short T_2. In the regime where (7.6) holds and substantial cooling is possible, the strong spin-resonator correlations which are responsible for cooling make a large contribution to mechanical fluctuations. As a result, the mechanical thermal noise is not decreased by the coupling to the hyperpolarized spins; indeed we find that when $n_\infty \gg 1/2$ and $n_\infty \gg n_c$, the thermal torque at the eigenfrequencies $\omega_h \pm d$ becomes larger than it would be at ω_h in the absence of the hyperpolarized spins. In this regime, as well as in the regime where the spins behave as a cold bath, the noisy thermal torque acting on the resonator is not decreased by the presence of the cold spins.

2 Model of the spin-resonator system

In analyzing a system consisting of a damped mechanical resonator and hyperpolarized spins which flow past it, we will use the master equation

$$\frac{d}{dt}\rho = \frac{1}{i\hbar}\left[H_0 + V, \rho\right] + \Lambda_h \rho + \Lambda_s \rho, \tag{7.9}$$

where H_0, V, and Λ_h are given respectively by (2.6), (2.11), and (2.25):

$$\begin{aligned} H_0 &= \omega_0 I_z + \omega_h \left(a^\dagger a + \frac{1}{2} \right), \\ V &= g \left(I_+ a^\dagger + I_- a \right), \\ \Lambda_h \rho &= -\frac{n_h + 1}{\tau_h} \left[a^\dagger a, \rho \right]_+ + 2 \frac{n_h + 1}{\tau_h} a \rho a^\dagger \\ &\quad - \frac{n_h}{\tau_h} \left[a a^\dagger, \rho \right]_+ + 2 \frac{n_h}{\tau_h} a^\dagger \rho a. \end{aligned} \tag{7.10}$$

(Note that for consistency with the notation used in section 1 of this chapter, we have used n_h rather than n_{th} to denote the thermal number of quanta in the warm bath which damps the resonator.) In order to reveal the fundamental properties of the spin-resonator system without complicating the analysis, we will assume that spins are perfectly resonant with the mechanical oscillator within a certain region of space but far off resonance outside of this region. The spin operators I_z, I_+, and I_- act only on the spins in the resonant region.

The superoperator Λ_s governs the decay of $\mathbf{I}$ due to spin-spin interactions, spin-lattice interactions, and the flow of spins into and out of the resonant region. For our purposes, it is sufficient to approximate the effects of Λ_s in an ad hoc way by assuming that it causes relaxation of $\langle I_z \rangle$ toward a hyperpolarized value $PN/2$ with a time constant denoted by T_1, and relaxation of transverse magnetization toward zero with a time constant denoted by T_2. In addition, we assume that Λ_s causes relaxation of $\left\langle I_+ a^\dagger - I_- a \right\rangle$ toward zero, with time constant T_2. These assumptions can be formally expressed as

$$\operatorname{Tr} \left\{ \left(\Lambda_s \rho \right) I_z \right\} = -\frac{1}{T_1} \left(\langle I_z \rangle - \frac{1}{2} PN \right), \tag{7.11}$$

$$\operatorname{Tr} \left\{ \left(\Lambda_s \rho \right) I_\pm \right\} = -\frac{1}{T_2} \langle I_\pm \rangle, \tag{7.12}$$

$$\operatorname{Tr} \left\{ \left(\Lambda_s \rho \right) \left(I_+ a^\dagger - I_- a \right) \right\} = -\frac{1}{T_2} \left\langle I_+ a^\dagger - I_- a \right\rangle. \tag{7.13}$$

The relaxation of $\langle I_z \rangle$ toward the hyperpolarized value $PN/2$ can be associated with the flow of spins through the resonator, and so T_1 is determined by the flow rate. In the case where the transverse decay described by equation (7.12) is due to the flow of spins, equation (7.13) can be motivated by the idea that the flow of spins during Δt causes a fraction of the spins in the cavity to be reset to the state having

$$\left\langle I_+ a^\dagger - I_- a \right\rangle = 0.$$

More generally, equation (7.13) can be motivated by first considering the way in which the oscillator's relaxation superoperator Λ_h contributes to the equation of motion of a product $G\left(a, a^\dagger\right) F\left(\mathbf{I}\right)$, where $g\left(a, a^\dagger\right)$ is a function of the oscillator's raising and lowering operators, and $F\left(\mathbf{I}\right)$ is an arbitrary spin operator. Using the cyclic property of the trace, we can express the term

$$\mathrm{Tr}\left\{\left(\Lambda_h \rho\right) G\left(a, a^\dagger\right) F\left(\mathbf{I}\right)\right\}$$

in the form

$$\mathrm{Tr}\left\{\rho \left[\Lambda_h' G\left(a, a^\dagger\right)\right] F\left(\mathbf{I}\right)\right\},$$

where

$$\begin{aligned}\Lambda_h' G = &-\frac{n_h+1}{\tau_h}\left[a^\dagger a, G\right]_+ + 2\frac{n_h+1}{\tau_h} a^\dagger G a \\ &-\frac{n_h}{\tau_h}\left[a a^\dagger, G\right]_+ + 2\frac{n_h}{\tau_h} a G a^\dagger.\end{aligned}$$

can be obtained from $\Lambda_h \rho$ by respectively replacing $\left[a^\dagger a, \rho\right]_+$, $\left[a a^\dagger, \rho\right]_+$, $a \rho a^\dagger$, and $a^\dagger \rho a$ by $\left[a^\dagger a, G\right]_+$, $\left[a a^\dagger, G\right]_+$, $a^\dagger G a$, and $a G a^\dagger$. Consider an example in which $G = a$. Since

$$\Lambda_h' a = -\frac{1}{\tau_h} a,$$

we find that

$$\mathrm{Tr}\left\{\left(\Lambda_h \rho\right) a F\left(\mathbf{I}\right)\right\} = -\frac{1}{\tau_h}\left\langle a F\left(\mathbf{I}\right)\right\rangle.$$

Similarly, since

$$\Lambda_h' a^\dagger a = -\frac{2}{\tau_h}\left(a^\dagger a - n_h\right),$$

we have

$$\mathrm{Tr}\left\{(\Lambda_h\rho)\, a^\dagger a F(\mathbf{I})\right\} = -\frac{2}{\tau_n}\left\langle\left(a^\dagger a - n_h\right) F(\mathbf{I})\right\rangle .$$

These observations regarding Λ_h support the use of equation (7.13) as a simple way to include the effects of spin relaxation in the model of the spin-resonator system. If we assume that the spin relaxation can be characterized by a superoperator of the form

$$\Lambda_s\rho = \sum_k f_k(\mathbf{I})\ \rho\, g_k(\mathbf{I}), \tag{7.14}$$

for some spin operators $f_k(\mathbf{I})$, $g_k(\mathbf{I})$, then the contribution of spin relaxation to the equation of motion for $\left\langle I_+a^\dagger - I_-a\right\rangle$ is given by

$$\mathrm{Tr}\left\{(\Lambda_s\rho)\left(I_+a^\dagger - I_-a\right)\right\} = \mathrm{Tr}\left\{\rho\left(\Lambda_s' I_+\right)a^\dagger - \rho\left(\Lambda_s' I_-\right)a\right\},$$

where

$$\Lambda_s' F(\mathbf{I}) = \sum_k g_k(\mathbf{I})\, F(\mathbf{I})\, f_k(\mathbf{I}) \ .$$

Equation (7.12) makes the assumption that the contribution of Λ_s to the equation of motion for $\langle I_\pm\rangle$ is

$$\begin{aligned}
-\frac{1}{T_2}\langle I_\pm\rangle &= \mathrm{Tr}\left\{(\Lambda_s\rho)\, I_\pm\right\}\\
&= \mathrm{Tr}\left\{\rho\Lambda_s' I_\pm\right\}\\
&= \langle\Lambda_s' I_\pm\rangle ,
\end{aligned}$$

which suggests the additional assumption

$$\Lambda_s' I_\pm = -\frac{1}{T_2} I_\pm. \tag{7.15}$$

Equation (7.13) follows from the assumptions (7.14) and (7.15).

3 Steady-state number of quanta in the resonator

Equations of motion for expectation values can be obtained by multiplying both sides of the master equation (7.9) by an operator and taking the trace. The following interaction-frame equations can be obtained in this way:

$$\frac{d}{dt}\left\langle a^{\dagger}a\right\rangle = K - \frac{2}{\tau_h}\left(\left\langle a^{\dagger}a\right\rangle - n_h\right), \tag{7.16}$$

$$K \equiv -ig\left\langle I_+a^{\dagger} - I_-a\right\rangle \tag{7.17}$$

$$\frac{d}{dt}\left\langle I_z\right\rangle = -\frac{1}{T_1}\left(\left\langle I_z\right\rangle - \frac{1}{2}PN\right) + K, \tag{7.18}$$

$$\frac{d}{dt}\left\langle I_+a^{\dagger} - I_-a\right\rangle = -\frac{1}{\tau_1}\left\langle I_+a^{\dagger} - I_-a\right\rangle - 4ig\left\langle I_za^{\dagger}a\right\rangle + 2ig\left\langle I_-I_+\right\rangle, \tag{7.19}$$

$$\frac{1}{\tau_1} \equiv \frac{1}{\tau_h} + \frac{1}{T_2}. \tag{7.20}$$

Note that K represents the rate at which $\langle I_z\rangle$ changes due to the spin-resonator interaction. Equation (A.10) of Appendix A gives a formula for this rate in the limiting case where the time constants T_1 and T_2 are long, with τ_h so short that the resonator is only weakly perturbed from thermal equilibrium:

$$\frac{\Delta\left\langle I_z\right\rangle}{\Delta t} = -4g^2\tau_h n_h\left\langle I_z\right\rangle + 2g^2\tau_h\left\langle I_-I_+\right\rangle. \tag{7.21}$$

By using (7.16) through (7.20) to do a steady-state calculation, we can lift this restriction on T_1 and T_2, and allow for the possibility that the resonator's state is strongly perturbed from equilibrium with the thermal reservoir. Setting the left side of (7.19) equal to zero and using (7.17) to eliminate $\left\langle I_+a^{\dagger} - I_-a\right\rangle$ gives

$$K_\infty = -4g^2\tau_1\left\langle I_za^{\dagger}a\right\rangle_\infty + 2g^2\tau_1\left\langle I_-I_+\right\rangle_\infty. \tag{7.22}$$

(Note that throughout this chapter, the subscript "∞" indicates a steady-state value.) The similarity between (7.21) and (7.22) is striking. The switch from τ_h in (7.21) to τ_1 in (7.22) is due to the fact that the superoperator Λ_s has been included in the model, and τ_1 is replaced by τ_h in the limit of long T_2. In both equations, the first term on the right-hand side of the equation characterizes stimulated emission and absorption by the spins, while the second term on the right-hand side characterizes spontaneous emission.

We assume that $\langle I_z \rangle$ is sufficiently large, and that the spins interact with the resonator for a short enough period that

$$\left\langle I_z a^\dagger a \right\rangle_\infty \approx \langle I_z \rangle_\infty \left\langle a^\dagger a \right\rangle_\infty .$$

For simplicity, we also assume that the flow of spins through the cavity is fast enough that the resonator-induced spin-spin correlations discussed in section 4 of chapter 3 remain weak:

$$\left\langle I_x^2 + I_y^2 \right\rangle \approx N/2, \tag{7.23}$$

where N is the number of spins interacting with the oscillator. These approximations allow us to express K_∞ as

$$K_\infty = -\frac{2}{\tau_c} \left(\left\langle a^\dagger a \right\rangle_\infty - n_c \right), \tag{7.24}$$

$$n_c \equiv \frac{1}{2} \left(\frac{N}{2 \langle I_z \rangle_\infty} - 1 \right), \tag{7.25}$$

$$\tau_c \equiv \left(2 g^2 \tau_1 \langle I_z \rangle_\infty \right)^{-1} . \tag{7.26}$$

Note that equation (7.25) defines n_c to be the number of quanta in the resonator when it is at the steady-state "spin temperature," that is, the temperature defined by the values of N and $\langle I_z \rangle_\infty$. In the steady state, equation (7.16) can be expressed as

$$0 = -\frac{2}{\tau_c} \left(\left\langle a^\dagger a \right\rangle_\infty - n_c \right) - \frac{2}{\tau_h} \left(\left\langle a^\dagger a \right\rangle_\infty - n_h \right), \tag{7.27}$$

where n_c and n_h are the equilibrium values of $\left\langle a^\dagger a \right\rangle$ at the respective temperatures associated with the spins and the warm bath. It is natural to interpret equation (7.27) as implying that $2/\tau_c$ is a rate constant for the relaxation of $\left\langle a^\dagger a \right\rangle$ toward the equilibrium value n_c determined by the "spin temperature," just as $2/\tau_h$ is the rate constant for relaxation of $\left\langle a^\dagger a \right\rangle$ toward the value n_h. As in the simpler analysis presented in section 1, the steady-state number of quanta in the resonator can then be expressed as

$$\left\langle a^{\dagger}a\right\rangle_{\infty} = \frac{\tau_h n_c + \tau_c n_h}{\tau_h + \tau_c}. \tag{7.28}$$

The rate constant $2/\tau_c$ characterizes the cooling of the resonator by many cold spins, while the rate constant R_0 characterizes the cooling of spins by a single resonator at zero Kelvins:

$$\frac{2}{\tau_c} = 2g^2\tau_1\left(2\left\langle I_z\right\rangle_{\infty}\right), \tag{7.29}$$

$$R_0 = 2g^2\tau_h. \tag{7.30}$$

In the case where $T_2 \gg \tau_h$, spin relaxation does not play a significant role in disrupting the development of spin-resonator correlations, and $\tau_1 \approx \tau_h$. Under these conditions, the rate constants given by (7.29) and (7.30) differ by the factor $2\left\langle I_z\right\rangle_{\infty}$, which can be considered the "effective number of spins at zero Kelvins" which are cooling the resonator. In considering numerical examples such as those presented in section 6 of chapter 5, we have found that R_0 achieved values of $\sim 1/\mathrm{s}$ when the dimensions of the resonator's magnets are of order 100 nm or less. The presence of the additional factor $2\left\langle I_z\right\rangle_{\infty}$ in equation (7.29) suggests the possibility of observing the exchange of entropy between spins and resonator at larger size scales, and in section 4 we present a numerical example to characterize the regime in which substantial cooling could be observed.

Additional support for the interpretation of $2/\tau_c$ as a rate constant for cooling may be obtained in the case where the spins pass by the resonator quickly enough that they are only weakly perturbed from the hyperpolarized state. In this case, the method of coarse-graining introduced in Appendix A can be used to derive a formula for

$$K \equiv -ig\left\langle I_+a^{\dagger} - I_-a\right\rangle$$

which is correct to second order in the coupling constant g. Equations (7.18) and (A.4), as well as equations of motion for I_-I_+ and $I_za^{\dagger}a$, are converted to integral equations. (In determining the contribution of Λ_s to these equations, we assume for simplicity that the flow of spins past the resonator causes $\langle I_z a^\dagger a\rangle$ and $_1\langle I_- I_+\rangle$ to relax with time constant $T_1 = T_2$.) We express K as an iterated integral and we evaluate the integral over a time step Δt which is long compared to τ_h and T_1. Since Δt is long compared to the period of time during which spin-resonator correlations survive, we can neglect initial spin-resonator correlations. Making the approximation (7.23) then yields the expression

$$K = -2g^2\tau_1 (PN) \left(\langle a^\dagger a\rangle - n_c\right), \tag{7.31}$$

which implies that

$$2g^2\tau_1 (PN)$$

is the rate constant for relaxation of $\langle a^\dagger a\rangle$ toward equilibrium with the spins. Note that this rate constant differs from that of (7.29) in replacing $2\langle I_z\rangle_\infty$ by the hyperpolarized value of $2\langle I_z\rangle$. This discrepancy is a result of the use of second-order perturbation theory in calculating the rate constant. Roughly speaking, we can say that changes in $\langle I_z\rangle$ due to interaction with the resonator are at least second-order in g. If we replace PN in (7.31) by an expression which includes effects which are second-order or higher in g, the resulting expression will include terms of 4th order or higher in g. Since the derivation of (7.31) only considers terms up to second order in g, it cannot incorporate the relaxation of $\langle I_z\rangle$ to $\langle I_z\rangle_\infty$.

At this point it may be tempting to conclude that the cold spins act as a bath characterized by ringdown time τ_c and temperature

$$T_c = \frac{\hbar\omega_h}{k_B \ln\left(1 + n_c^{-1}\right)}. \tag{7.32}$$

Although this approach yields the correct values for $\langle a^\dagger a\rangle_\infty$, the analysis in sections 5 through 7 shows that this model yields incorrect predictions for decay times of the spin-resonator modes, the resonator's response to a driving torque, and the mechanical fluctuations. Section 5 shows that in the regime where substantial cooling has occurred, with $\tau_h \ll T_2$, the spins and resonator are so strongly coupled that it is not possible to distinguish a mechanical mode or a spin mode.

4 Numerical example of cooling

This section presents a numerical example based on a simplified model in which polarized liquid xenon flows through a Halbach cylinder [43]. A Halbach cylinder is a circular tube of magnetic material for which the arrangement of magnetization produces a nominally uniform magnetic field within the tube and zero field outside of the tube. The Halbach cylinder is chosen to yield a simple, optimistic estimate of the size scale at which polarized spins could substantially cool a resonator, since the nominally uniform field inside the cylinder would allow a relatively large volume of cold spins to interact with the resonator. For this estimate, we set aside questions having to do with the technical feasibility of the experiment (e.g., questions about fabrication of the Halbach cylinder or its thermodynamic stability at small sizes). Our goal is simply to give a rough characterization of the regime in which polarized spins passing near a mechanical resonator could have a non-negligible effect on its temperature.

Consider a Halbach cylinder having inner radius R_i, outer radius R_o, and length $3R_i$, with magnetization $1.5\,\mathrm{T}/\mu_0$. The cylinder is mounted on a torsional beam which runs parallel to the cylinder's axis and has width and thickness equal to the cylinder's inner radius. The torsional beam length is adjusted to the value necessary for resonance with the Larmor frequency of xenon in the field generated by the Halbach cylinder. We suppose that xenon with a natural composition of isotopes fills half the volume of the cylinder and that the polarization of ^{129}Xe entering the cylinder is [44]

$$P = .70.$$

The triple point of xenon occurs at $0.81\,\mathrm{atm}$ and $161\,\mathrm{K}$, and the boiling point of xenon at $1\,\mathrm{atm}$ is $165\,\mathrm{K}$; within this temperature and pressure range the density of

Cylinder Dimensions	τ_h	T_h	T_∞	$\omega_h/2\pi$
$R_i = 500\,\text{nm},\ R_o = 1\,\mu\text{m},\ L = 1.5\,\mu\text{m}$	$290\,\mu\text{s}$	300 K	104 K	11 MHz
$R_i = 500\,\text{nm},\ R_o = 600\,\text{nm},\ L = 1.5\,\mu\text{m}$	1.3 ms	300 K	9 K	3 MHz

Table 7.1: Resonators cooled by hyperpolarized spins

liquid xenon is approximately 22.6 kmol / m^3 [45], and we assume this density for our estimate. If the resonator's quality factor is $Q = 10,000$, then we obtain the results shown in table 7.1, where

$$T_\infty = \frac{\hbar\omega_h}{k_B \ln\left(1 + n_\infty^{-1}\right)}, \qquad (7.33)$$
$$n_\infty \equiv \left\langle a^\dagger a \right\rangle_\infty .$$

(Decreasing the ratio R_o/R_i between the outer radius and the inner radius causes T_∞ to decrease continually toward $T_\infty \approx 8\,\text{K}$ as $R_o \to R_i$.)

The transverse decay time of liquid xenon has been measured at 1300 s [46], which allows us to consider T_2 to be determined by the rate at which spins flow through the cylinder. For the resonators of table 7.1, the rate at which quanta are donated to the spins is such that $\langle I_z \rangle$ changes by 0.6% or less during a time period of length τ_h, and so we consider the interaction time between a spin and the resonator to be substantially larger than τ_h without contradicting our assumptions that $\left\langle I_z a^\dagger a \right\rangle_\infty \approx \langle I_z \rangle_\infty n_\infty$ and $\left\langle I_x^2 + I_y^2 \right\rangle \approx N/2$. The disruption of spin-resonator correlations is thus primarily due to the thermal torque which acts on the resonator, and

$$\tau_1 \approx \tau_h .$$

The value of T_∞ scales sharply with resonator size. Scaling up the first cylinder in table 7.1 by a factor of 10 and the second by a factor of 100 while retaining the assumption that $Q = 10,000$, gives steady-state temperatures near 300 K, as shown in table 7.2. (Note that scaling the Halbach cylinder does not change the field at the spins, and so the frequency, quality factor, and ringdown time are all held constant as we scale up the resonators in table 7.1.)

Cylinder Dimensions	T_h	T_∞
$R_i = 5\,\mu$m, $R_o = 10\,\mu$m, $L = 15\,\mu$m	300 K	294 K
$R_i = 50\,\mu$m, $R_o = 60\,\mu$m, $L = 150\,\mu$m	300 K	297 K

Table 7.2: Scaled-up spin-resonator systems

It may be considered surprising that cooling becomes negligible at size scales of $\sim 10\,\mu$m, since one might have guessed that the presence of the term $2\langle I_z\rangle_\infty$ in (7.29) would permit cooling to be observed at larger size scales. The nature of the scaling can be clarified by noting that

$$\begin{aligned}\frac{2}{\tau_c} &= 2g^2\tau_h\left(2\langle I_z\rangle_\infty\right)\\ &= \frac{\hbar}{2}\left(\gamma\frac{dB_x}{d\theta}\right)^2\frac{\tau_h}{\omega_h}\left[\frac{\langle I_z\rangle_\infty}{I_h}\right].\end{aligned} \tag{7.34}$$

In these numerical examples, the two terms which vary as the resonators scale up are grouped in square brackets on the right side of (7.34). The torsional beams make a negligible contribution to the moment of inertia I_h in these examples, and we need only consider the cylinder's moment of inertia in estimating I_h. Since the shape of the cylinder does not change during the scaling, we have

$$\begin{aligned}I_h &\sim r^5,\\ \langle I_z\rangle_\infty &\sim r^3,\end{aligned}$$

where r is a characteristic dimension of the cylinder, such as the inner radius. It follows that

$$\frac{2}{\tau_c} \sim r^{-2}$$

in these examples. It is the strong scaling of g^2 with size which causes the cooling to become negligible as the resonator is scaled up to have dimensions of order $10\,\mu$m. (Note that although in our example, the size dependence of g^2 is determined solely by the moment of inertia, similar scaling is obtained for a translational resonator. In this case, the moment of inertia would be replaced by a mass, and the scale-invariant

Cylinder Dimensions	$2/\tau_h$	$2/\tau_c$
$R_i = 500\,\mathrm{nm}$, $R_o = 1\,\mu\mathrm{m}$	$7000\,\mathrm{s}^{-1}$	$13,000\,\mathrm{s}^{-1}$
$R_i = 5\,\mu\mathrm{m}$, $R_o = 10\,\mu\mathrm{m}$	$7000\,\mathrm{s}^{-1}$	$130\,\mathrm{s}^{-1}$

Table 7.3: Dependence of rate constants on size

term $dB_x/d\theta$ would be replaced by a gradient scaling as r^{-1}.) Table 7.3 shows how the rate constants $2/\tau_c$ and $2/\tau_h$ depend on size for the example resonator having $R_o/R_i = 2$.

5 Modes of the spin-resonator system

In estimating a "steady-state temperature" T_∞ based on the expectation value n_∞, we did not consider the question of whether the cooled oscillator "continues to look like a mechanical oscillator" in the regime where T_∞ differs substantially from T_h. In this section, we answer that question by studying the modes of the spin-resonator system. Although most of our results will be derived from the master equation (7.9), the nature of the system can initially be clarified using a model in which spins and resonator are coupled by the lab-frame Hamiltonian

$$\mathcal{H}_{sh} = -\gamma\hbar\frac{dB_x}{d\theta}I_x\theta,$$

rather than the interaction-frame Hamiltonian

$$V = g\left(I_+a^\dagger + I_-a\right)$$

obtained using the rotating-wave approximation. The lab-frame equations of motion are

$$\frac{d}{dt}\langle\theta\rangle = \frac{\langle p_\theta\rangle}{I_h} - \frac{\langle\theta\rangle}{\tau_h}, \tag{7.35}$$

$$\frac{d}{dt}\langle p_\theta\rangle = -I_h\omega_h^2\langle\theta\rangle - \frac{\langle p_\theta\rangle}{\tau_h} + \gamma\hbar\frac{dB_x}{d\theta}\langle I_x\rangle, \tag{7.36}$$

$$\frac{d}{dt}\langle I_x\rangle = -\omega_0\langle I_y\rangle - \frac{\langle I_x\rangle}{T_2}, \tag{7.37}$$

$$\frac{d}{dt}\langle I_y\rangle = \omega_0\langle I_x\rangle - \frac{\langle I_y\rangle}{T_2} + \gamma\frac{dB_x}{d\theta}\langle I_z\theta\rangle, \tag{7.38}$$

$$\frac{d}{dt}\langle I_z\rangle = -\frac{1}{T_1}\left(\langle I_z\rangle - \frac{PN}{2}\right) - \gamma\frac{dB_x}{d\theta}\langle I_y\theta\rangle\ . \tag{7.39}$$

Note the formal similarity between the equations for the oscillator variables $\langle\theta\rangle$, $\langle p_\theta\rangle$ and those of the transverse spin variables $\langle I_x\rangle$, $\langle I_y\rangle$. Indeed, we can write second-order differential equations for $\langle\theta\rangle$ and $\langle I_x\rangle$ which highlight the formal similarity:

$$\frac{d^2}{dt^2}\langle\theta\rangle + \frac{2}{\tau_h}\frac{d}{dt}\langle\theta\rangle + \left(\omega_h^2 + \frac{1}{\tau_h^2}\right)\langle\theta\rangle = \frac{\gamma\hbar}{I_h}\frac{dB_x}{d\theta}\langle I_x\rangle, \tag{7.40}$$

$$\frac{d^2}{dt^2}\langle I_x\rangle + \frac{2}{T_2}\frac{d}{dt}\langle I_x\rangle + \left(\omega_h^2 + \frac{1}{T_2^2}\right)\langle I_x\rangle = \omega_h\gamma\frac{dB_x}{d\theta}\langle I_z\theta\rangle. \tag{7.41}$$

For sufficiently large $\langle I_z\rangle$ and short interaction time between each spin and the resonator, we can approximate $\langle I_z\theta\rangle$ by $\langle I_z\rangle\langle\theta\rangle$ and consider $\langle I_z\rangle$ to be approximately constant. Under these conditions, the evolution of the variables $\langle\theta\rangle$, $\langle I_x\rangle$ is formally equivalent to that of two coupled oscillators, and interpreting the motion in this way can lead to an intuitive understanding of the system. We define the moment of inertia I_f of the formal oscillator associated with the variable $\langle I_x\rangle$ to be

$$I_f = \frac{\hbar}{\omega_h\langle I_z\rangle_\infty},$$

and we rewrite (7.40) and (7.41) as

$$I_h \frac{d^2}{dt^2} \langle \theta \rangle + \frac{2I_h}{\tau_h} \frac{d}{dt} \langle \theta \rangle + I_h \left(\omega_h^2 + \frac{1}{\tau_h^2} \right) \langle \theta \rangle = \gamma \hbar \frac{dB_x}{d\theta} \langle I_x \rangle , \tag{7.42}$$

$$I_f \frac{d^2}{dt^2} \langle I_x \rangle + \frac{2I_f}{T_2} \frac{d}{dt} \langle I_x \rangle + I_f \left(\omega_h^2 + \frac{1}{T_2^2} \right) \langle I_x \rangle = \gamma \hbar \frac{dB_x}{d\theta} \langle \theta \rangle . \tag{7.43}$$

The coupling between the two formal oscillators is associated with the potential function

$$V_1 = -\gamma \hbar \frac{dB_x}{d\theta} \langle I_x \rangle \langle \theta \rangle . \tag{7.44}$$

Note that a potential function of the same form is obtained when two linear harmonic oscillators are coupled by a spring. For instance, let x_1 and x_2 represent the coordinates of two linear oscillators, and suppose that they are coupled by a spring whose potential energy is

$$A (x_1 - x_2)^2 = Ax_1^2 + Ax_2^2 - 2Ax_1x_2. \tag{7.45}$$

In (7.45), the terms Ax_1^2 and Ax_2^2 can be considered to modify the potential wells of the individual oscillators, while the term $-2Ax_1x_2$ couples the two oscillators. We can therefore visualize the spin-resonator system as consisting of two oscillators coupled by a spring.

In order to obtain tractable solutions for the evolution of the system, we must replace (7.35) through (7.38) with equations obtained under the rotating-wave approximation. The master equation (7.9), in combination with the approximations

$$\begin{aligned} \langle I_z a \rangle &\approx \langle I_z \rangle \langle a \rangle , \\ \langle I_z \rangle &\approx \text{constant} \\ &\equiv \langle I_z \rangle_\infty , \end{aligned}$$

yields a linear equation in the two variables $\langle a\rangle$, $\langle I_+\rangle$:

$$\frac{d}{dt}\langle a\rangle = -\left(i\omega_h + \frac{1}{\tau_h}\right)\langle a\rangle - ig\langle I_+\rangle\,, \tag{7.46}$$

$$\frac{d}{dt}\langle I_+\rangle = -\left(i\omega_h + \frac{1}{T_2}\right)\langle I_+\rangle - 2ig\langle I_z\rangle_\infty\langle a\rangle\,. \tag{7.47}$$

We look for a steady-state solution to equations (7.46) and (7.47) of the form

$$\langle a\rangle(t) = e^{-(i\omega'+1/\tau')t}\langle a\rangle(0)\,, \tag{7.48}$$

$$\langle I_+\rangle(t) = e^{-(i\omega'+1/\tau')t}\langle I_+\rangle(0)\,, \tag{7.49}$$

$$\langle I_+\rangle(0) = \eta\langle a\rangle(0)\,. \tag{7.50}$$

A motivation for this ansatz is the fact that steady motion of the oscillator creates a sinusoidal transverse field; in the limit of weak spin-oscillator coupling, we expect the response of the spins to be similar to the linear response described by the steady-state solutions to the Bloch equations. Substituting (7.48) through (7.50) into (7.46) and (7.47) yields a solution for η, ω', τ'. We obtain

$$\eta = \frac{1 \pm \sqrt{1 - 8\langle I_z\rangle_\infty (gb)^2}}{2gb} i, \tag{7.51}$$

$$\frac{1}{b} \equiv \left(\frac{1}{\tau_h} - \frac{1}{T_2}\right), \tag{7.52}$$

and

$$\omega' = \omega_h + g\,\mathrm{Re}(\eta)\,, \tag{7.53}$$

$$1/\tau' = 1/\tau_h - g\,\mathrm{Im}(\eta)\,. \tag{7.54}$$

The physical content of these equations can be clarified by writing (7.53) and (7.54) in a more explicit form. Define

$$s \equiv 8\langle I_z\rangle_\infty (gb)^2$$

and consider two cases. For $s \leq 1$, we have

$$\omega' = \omega_h, \tag{7.55}$$

$$\frac{1}{\tau'} = \frac{1}{2}\left(\frac{1}{\tau_h} + \frac{1}{T_2}\right) \pm \frac{\sqrt{1-s}}{2}\left(\frac{1}{\tau_h} - \frac{1}{T_2}\right), \tag{7.56}$$

while for $s > 1$, we have

$$\omega' = \omega_h \pm \frac{\sqrt{s-1}}{2}\left(\frac{1}{\tau_h} - \frac{1}{T_2}\right), \tag{7.57}$$

$$\frac{1}{\tau'} = \frac{1}{2}\left(\frac{1}{\tau_h} + \frac{1}{T_2}\right). \tag{7.58}$$

Equations (7.55) through (7.58) can be understood as natural results for a system of two coupled oscillators. In the limit of strong coupling between the oscillators (that is, large g or large $\langle I_z \rangle_\infty$) or similar dissipation rates for the two oscillators (that is, large $|b|$), energy can be exchanged between the oscillators quickly enough that the net dissipation rate is just the average of $1/\tau_h$ and $1/T_2$. The ratio $|\langle a \rangle / \langle I_+ \rangle|$ that characterizes the relative excitation of the spins and the resonator is equal for the two modes, so neither mode can be specifically considered to be the mechanical mode. In the limit of weak coupling or dissimilar dissipation rates, equation (7.56) shows that the ringdown times for the two modes approach τ_h and T_2 as $g \to 0$. The solution with ringdown time $\sim \tau_h$ has larger excitation in the mechanical oscillator than the solution with ringdown time $\sim T_2$.

In section 4, we presented numerical examples in which resonators were cooled by hyperpolarized spins from the ambient temperature of 300 K to temperatures of 100 K or less. The results obtained in the current discussion imply that for these numerical examples, the spins and resonator are so strongly coupled that it is not possible to distinguish a mechanical mode or a spin mode. Indeed, we will now show that the value $s = 1$, which corresponds to the disappearance of distinct spin and mechanical modes, occurs when

$$T_\infty \approx \frac{4}{5} T_h.$$

Note first that when one of the decay times τ_h, T_2 is much longer than the other, the term s which determines the form of the modes can be written in a simpler way, since in this case

$$b \approx \min\{\tau_h, T_2\}.$$

We find that

$$s \approx \frac{4\tau_1}{\tau_c},$$

where the rate constant $2/\tau_c$ for cooling of the resonator by the spins is given by (7.29), and τ_1 is defined by (7.20). When $\tau_h \ll T_2$, as in the numerical examples of cooling that we considered, the transition to the strong-coupling regime occurs when

$$\frac{4\tau_h}{\tau_c} \approx 1. \tag{7.59}$$

Equation (7.59) implies that

$$\begin{aligned} n_\infty &= \frac{\tau_h n_c + \tau_c n_h}{\tau_h + \tau_c} \\ &\approx \frac{4}{5} n_h, \end{aligned}$$

where we have assumed that $4n_h \gg n_c$. If

$$n_\infty \gg 1,$$

if follows that

$$T_\infty \approx \frac{4}{5} T_h, \tag{7.60}$$

since

$$\begin{aligned} \frac{T_\infty}{T_h} &= \frac{\ln(1 + 1/n_h)}{\ln(1 + 1/n_\infty)} \\ &\approx \frac{n_\infty}{n_h}. \end{aligned}$$

Substantial cooling of the resonator by cold spins therefore requires that the coupling be strong enough to transform the mechanical mode into a mode which includes significant contributions from both mechanical motion and spin precession. Consistent with this observation is the fact that for the numerical example in which $T_h = 300\,\mathrm{K}$ to and $T_\infty =\sim 100\,\mathrm{K}$,

$$s \approx 8.$$

In section 1 we analyzed the spin-resonator system using a simple model in which the cold spins were represented by a cold bath whose properties were not affected by the warm bath. This model is incorrect in the general case, for two reasons. First, the theory supporting the use of a linear, time-independent superoperator to describe relaxation due to coupling with a reservoir [7] is valid only in the limit of weak coupling with the reservoir. As a result, it is only in the limit of weak-spin resonator coupling ($s \ll 1$) that the cold spins might be expected to behave as a cold reservoir. Second, the rate constant $2/\tau_c$, which characterizes the resonator's relaxation toward equilibrium with the spins, depends on τ_h, that is, on the coupling between the resonator and the warm bath. Except in the limiting case where $2/\tau_c$ is independent of the resonator's coupling to the warm bath, it is incorrect to represent the spins as a reservoir which acts independently of the warm bath. This limit corresponds to the condition $T_2 \ll \tau_h$, since this condition guarantees that it is spin relaxation rather than the ringdown time τ_h that limits the lifetime of spin-resonator correlations. In the regime defined by

$$s \ll 1, \tag{7.61}$$

$$T_2 \ll \tau_h, \tag{7.62}$$

however, we might expect to recover the results obtained from equation (7.1) to be valid. Indeed, replacing $\sqrt{1-s}$ by $1 - s/2$ in (7.56) and using condition (7.62) to simplify the resulting expression shows that the decay time of the mechanical mode is

$$\frac{1}{\tau_h} + 2g^2 T_2 \langle I_z \rangle_\infty = \frac{1}{\tau_h} + \frac{1}{\tau_c}$$
$$\equiv \frac{1}{\tau_\infty},$$

which agrees with equation (7.3), the result obtained by treating the cold spins as an independent thermal bath.

6 Response of the system to a torque acting on the resonator

In order to determine how the coupling to the hyperpolarized spins modifies the resonator's sensitivity as a detector of an external torque, we will calculate the system's response to a torque acting on the resonator. An external torque $f(t)$ corresponds to a term $-f(t)\theta$ added to the oscillator's Hamiltonian H_{osc} in (7.9), so that the equations governing $\langle\theta\rangle$ and $\langle I_+\rangle$ become

$$\frac{d}{dt}\langle a\rangle = -\left(i\omega_h + \frac{1}{\tau_h}\right)\langle a\rangle - ig\langle I_+\rangle + i\sqrt{\frac{1}{2I_h\hbar\omega_h}}f(t), \tag{7.63}$$

$$\frac{d}{dt}\langle I_+\rangle = -\left(i\omega_h + \frac{1}{T_2}\right)\langle I_+\rangle - 2ig\langle I_z\rangle_\infty\langle a\rangle. \tag{7.64}$$

We consider the case where

$$f(t) = Fe^{-i\omega t} \tag{7.65}$$

and we look for a steady-state solution of the form

$$\langle a\rangle(t) = A_\omega e^{-i\omega t}, \tag{7.66}$$

$$\langle I_+\rangle(t) = \eta_\omega\langle a\rangle(t) = \eta_\omega A_\omega e^{-i\omega t}. \tag{7.67}$$

The solution for A_ω and η_ω can be expressed as

$$\left\{ (\omega - \omega_h) f(\omega) + \frac{i}{\tau_d(\omega)} \right\} A_\omega = -\sqrt{\frac{1}{2 I_h \hbar \omega_h}} F, \tag{7.68}$$

$$\eta_\omega = \frac{2g \langle I_z \rangle_\infty}{(\omega - \omega_h) + i/T_2}, \tag{7.69}$$

where

$$f(\omega) \equiv 1 - \frac{1}{\tau_c \tau_1 \left\{ (\omega - \omega_h)^2 + 1/T_2^2 \right\}}, \tag{7.70}$$

$$1/\tau_d(\omega) \equiv \frac{1}{\tau_h} + \frac{1}{T_2 \tau_c \tau_1 \left\{ (\omega - \omega_h)^2 + 1/T_2^2 \right\}}. \tag{7.71}$$

The content of (7.68) becomes clearer if we compare it to the formula obtained in the case where the coupling constant g is zero:

$$\left\{ (\omega - \omega_h) + \frac{i}{\tau_h} \right\} A_\omega = -\sqrt{\frac{1}{2 I_h \hbar \omega_h}} F. \tag{7.72}$$

Although (7.72) is an unusual way to describe the steady-state response of the oscillator, it is straightforward to verify that in the limit of weak coupling to the reservoir, it yields the familiar steady-state expression for $\langle \theta \rangle (t)$. We may calculate the mechanical response to a torque at frequency ω_h as if the ringdown time were

$$\frac{1}{\tau_d} = \frac{1}{\tau_h} + 2g^2 T_2 \langle I_z \rangle_\infty . \tag{7.73}$$

At frequencies $\omega \neq \omega_h$, the resonator responds as if its ringdown time were $\tau_d(\omega)$ and the driving torque were off resonance by $(\omega - \omega_h) f(\omega)$.

In the case where (7.62) holds, we recover the expression τ_∞ obtained by treating the spins as a cold bath:

$$\begin{aligned} \frac{1}{\tau_d} &= \frac{1}{\tau_h} + 2g^2 \tau_1 \langle I_z \rangle_\infty \\ &= \frac{1}{\tau_\infty}. \end{aligned}$$

For the numerical examples we considered in section 4, however, the resonator's linear response will be considerably weaker at resonance than it would be for a mechanical oscillator with ringdown time τ_∞, since

$$\begin{aligned}\frac{1}{\tau_d} &= \frac{1}{\tau_h} + 2g^2 T_2 \langle I_z \rangle_\infty \\ &\gg \frac{1}{\tau_h} + 2g^2 \tau_h \langle I_z \rangle_\infty \\ &\approx \frac{1}{\tau_\infty}.\end{aligned}$$

A physical interpretation of this conclusion is that in the presence of the cold spins with long relaxation time, energy initially donating to the resonator by the driving torque is efficiently transferred onward to the cold spins, since the long relaxation time of the spins allows for a strong resonant response to the driving of the spins by the mechanical resonator. The transverse spin magnetization then exerts a torque on the resonator which counteracts the external torque and prevents a large mechanical response from developing.

The correctness of this interpretation can be demonstrated formally by considering the steady-state form of equation (7.63):

$$-i\omega A_\omega = -\left(i\omega_h + \frac{1}{\tau_h}\right) A_\omega - ig\eta_\omega A_\omega + i\sqrt{\frac{1}{2I_h\hbar\omega_h}}F. \tag{7.74}$$

The four terms in (7.74) represent distinct physical contributions which must cancel in the steady state. The last term on the right side of the equation represents the external torque, while the term $-\left(i\omega_h + 1/\tau_h\right) A_\omega$ represents the torques associated with the potential well and the damping by the warm bath. The term $-i\omega A_\omega$ can be interpreted as an "inertial torque." The remaining term, $-ig\eta_\omega A_\omega$, characterizes the torque exerted on the resonator by the spins. The torque associated with the imaginary part of $-ig\eta_\omega$ may be interpreted as modifying the resonator's potential energy, since it oscillates in phase with the torque exerted by the potential well, and it is responsible for replacing the term $(\omega - \omega_h)$ in (7.72) by $(\omega - \omega_h) f(\omega)$ in (7.68). The torque associated with the real part of $-ig\eta_\omega$ acts in phase with the damping torque exerted by the warm bath, and it can be considered to damp the mechanical motion, causing τ_h to be replace by $\tau_d(\omega)$. Equation (7.69) shows that the value of η_ω is peaked around $\omega = \omega_h$, and in the case where T_2 is long, the peak value of η_ω will be large. This peak value will be associated with a large transverse magnetization which exerts a damping torque on the resonator. Even in the case where the condition

$$\begin{aligned} \frac{1}{\tau_c} &= 2g^2\tau_h \langle I_z \rangle_\infty \\ &\ll \frac{1}{\tau_h} \end{aligned}$$

implies that cooling will be negligible, a sufficiently large value of T_2 will guarantee that during steady-state driving, most of the energy donated to the spin-resonator system by an external torque acting on the resonator will take the form of spin excitation. In the regime where $T_2 \gg \tau_h$, the mechanical resonator could be considered a device for transferring an external torque at frequency ω_h into excitation of hyperpolarized spins.

Although the mechanical response to a driving torque at ω_h becomes weak when $T_2 \gg \tau_h$, equations (7.70) and (7.71) allow for the possibility of a resonant mechanical response at frequencies which are out of resonance with the spins. This occurs at frequencies ω sufficiently far from ω_h that the damping torque exerted by the spins is negligible, but close enough to ω_h that the resonator's potential function is modified by the spins, yielding

$$(\omega - \omega_h) f(\omega) = 0.$$

Under these conditions, the mechanical response has the same amplitude that it would if the spins were absent and the driving torque were at frequency ω_h. To demonstrate this formally, note that when

$$(\omega - \omega_h)^2 \gg \frac{1}{T_2^2}, \tag{7.75}$$

the resonator responds as if its ringdown time were

$$\frac{1}{\tau_d} = \frac{1}{\tau_h}\left(1 + \frac{1}{T_2\tau_c(\omega - \omega_h)^2}\right), \tag{7.76}$$

and the driving torque were off resonance by

$$(\omega - \omega_h) f(\omega) = (\omega - \omega_h)\left(1 - \frac{1}{\tau_c \tau_h (\omega - \omega_h)^2}\right). \tag{7.77}$$

When ω satisfies

$$(\omega - \omega_h)^2 \gg \frac{1}{\tau_c T_2} \tag{7.78}$$

in addition to (7.75), then

$$\tau_d \approx \tau_h. \tag{7.79}$$

If

$$\tau_c \leq T_2, \tag{7.80}$$

then condition (7.75) automatically holds when (7.78) does, and we assume that this is the case, since $\tau_c \geq T_2 \gg \tau_h$ would otherwise imply that cooling is negligible. The condition $T_2 \gg \tau_h$ then allows us to chose ω such that (7.75) and

$$\frac{1}{\tau_c \tau_h (\omega - \omega_h)^2} \approx 1 \tag{7.81}$$

are both satisfied. Equation (7.77), (7.79), and (7.81) then imply that the resonator responds as if the spins were absent and ω were resonant with the mechanical frequency. Note that in the limit of strong-coupling, defined by $s \gg 1$, equation (7.81) is satisfied at the frequencies of the two modes of the system.

As an illustration, we consider the example resonator of table 7.1 which is cooled from $T_h = 300\,\mathrm{K}$ to $T_\infty = 104\,\mathrm{K}$. The rate at which quanta are donated to the spins is such that $\langle I_z \rangle$ changes by 0.1% during a time period of length τ_h, and so we may choose the flow rate such that the interaction time between spins and resonator is $T_2 = 50\tau_h$, without invalidating the assumption that the spins are only weakly perturbed from the hyperpolarized state during their interaction with the resonator. Since

$$\tau_h \approx 2\tau_c,$$

conditions (7.78) can be expressed as

$$(\omega - \omega_h)^2 \gg \frac{1}{25\tau_h^2},$$

and the zero of $f(\omega)$ occurs at

$$(\omega - \omega_h)^2 \approx \frac{2}{\tau_h^2}. \tag{7.82}$$

When ω satisfies (7.82), the mechanical response has the same magnitude that if would have if the spins were absent and ω were resonant with the mechanical frequency.

7 The cooled mode as a sensitive detector?

Equation (7.5), obtained by modelling the spins as a cold bath, predicts that the noisy thermal torque which acts on the resonator will not be diminished by the presence of the cold spins. To investigate the validity of this result, we use the symmetric correlation function

$$C(t_2 - t_1) = \frac{1}{2}\langle \theta(t_2)\,\theta(t_1) - \theta(t_1)\,\theta(t_2) \rangle$$

to evaluate the thermal fluctuations of a system. We assume that the mechanical fluctuations during driving by an external torque can be estimated using the steady-state correlation function during cooling in the absence of a external torque. As support for this approach, we note that our model of the spin-resonator system has yielded a linear system, and that the motion of such a system under the influence a driving force or torque is the sum of the steady-state driven motion plus motion identical with that of the undriven system.

The details of the derivation, as well as the general formula for $C(t)$, are presented in Appendix N. In the limit where (7.61) and (7.62) hold, we recover the results obtained by treating the cold spins as a thermal bath: the correlation function reduces to that of an oscillator which has ringdown time τ_∞ and is at temperature

T_∞. The examples we considered in which cooling was substantial had $T_2 \gg \tau_h$ and $s \gg 1$, and in this regime, $C(t)$ can be expressed as

$$C(t) \approx \exp(-t/2\tau_h)\cos(\omega_h t)\left\{\left\langle\theta^2\right\rangle_\infty \cos(dt) - c_2 \left\langle I_y\theta\right\rangle_\infty \sin(dt)\right\}, \quad t>0 \tag{7.83}$$

$$\approx \frac{\hbar}{I_h\omega_h} n_\infty \exp(-t/2\tau_h)\cos(\omega_h t)\left\{\cos(dt) - \sqrt{\tau_h/\tau_c}\sin(dt)\right\}, \quad t>0 \tag{7.84}$$

$$\approx \frac{\hbar}{I_h\omega_h} n_\infty \left(1+\frac{\tau_h}{\tau_c}\right)\exp(-t/2\tau_h)\times \tag{7.85}$$

$$\frac{\cos((\omega_h+d)t+\phi)+\cos((\omega_h-d)t-\phi)}{2}, \quad t>0,$$

where

$$d \approx 1/\sqrt{\tau_h\tau_c},$$

$$c_2 = -\sqrt{\frac{2\hbar}{I_h\omega_h}}\frac{2g\tau_h}{\sqrt{4\tau_h/\tau_c-1}},$$

with ϕ a phase constant that can be evaluated using equation (N.23). (In making this simplification, we have also assumed $n_\infty \gg 1/2$ and $n_\infty \gg n_c$.)

In the absence of the spin-resonator coupling, the correlation function would be

$$C_1(t) = \frac{\hbar}{I_h\omega_h} n_h \exp(-t/\tau_h)\cos(\omega_h t), \quad t>0. \tag{7.86}$$

By comparing (7.83) through (7.85) with (7.86), we can give a physical interpretation of the spins' effect on the mechanical fluctuations in this regime. From (7.84), we see that in cooling the resonator, the spins reduce the instantaneous correlation $C(0) = \langle\theta^2\rangle$ from $(\hbar/I_h\omega_h)\, n_h$ to $(\hbar/I_h\omega_h)\, n_\infty$, the same value it would have for an oscillator at temperature T_∞. The coupling to the spins also slows down the decay of the correlations, since the time constant in the exponential term increases from τ_h to $\tau' = 2\tau_h$, the decay constant for each of the spin-resonator modes. Since the modes of the system have frequencies $\omega_h \pm d$, the correlations oscillate at these two frequencies rather than at frequency ω_h, as shown by equation (7.85). Note that the resonant mechanical response will be observed at these frequencies, since they satisfy equation (7.81). Finally, we see from (7.83) and (7.84) that although the spin-resonator correlations characterized by $\langle I_y\theta\rangle_\infty$ do not contribute to $C(0)$, they are converted to correlations in θ within a time $t = \pi\sqrt{\tau_h\tau_c}/2$. The contribution made by the correlation $\langle I_y\theta\rangle_\infty$ increases the amplitude of $C(t)$ by a factor of $1+\tau_h/\tau_c$, as can be seen from equation (7.85).

The significance of the term $\langle I_y\theta\rangle_\infty$ can be understood by noting from (7.39) that in the absence of the rotating-wave approximation, the rate K at which quanta are transferred from spins to oscillator is given by

$$K = -\gamma\frac{dB_x}{d\theta}\langle I_y\theta\rangle .$$

The energy exchange characterized by K is mediated by fluctuating fields which induce correlations $I_y(t_1)\,\theta(t_1)$. The instantaneous value of these correlations can be viewed as a fluctuating random variable, and the conversion of this fluctuating variable into fluctuating values of $\theta(t_1)\,\theta(t_2)$ and $\theta(t_2)\,\theta(t_1)$ can make a significant contribution to the mechanical fluctuations. Since the term $n_\infty(1+\tau_h/\tau_c)$ appearing in (7.85) can be written as

$$n_\infty\left(1+\frac{\tau_h}{\tau_c}\right) = n_h + \frac{\tau_h}{\tau_c}n_c,$$

we can consider the effective number of quanta in the resonator to be greater than n_h for purposes of estimating the mechanical fluctuations. Although the instantaneous correlation $\langle\theta^2\rangle_\infty$ has the value characteristic of a cooled oscillator, the amplitude of $C_1(t)$ is not decreased by the cooling process.

The conditions $T_2 \gg \tau_h$ and $s \gg 1$ guarantee that the mechanical response to torques at the frequencies $\omega_h \pm d$ is that of a resonant mechanical oscillator with ringdown time τ_h. At these frequencies, the spectral density of the thermal torque can therefore be written as

$$S_N = \left(\frac{4I_h^2\omega_h^2}{\tau_h^2}\right)S_\theta,$$

where the spectral density S_θ for position is found by taking the Fourier transform of $C(t)$, and where the difference between ω_h^2 and $(\omega_h \pm d)^2$ has been neglected. We can obtain S_θ by noting that the spectral density obtained from a correlation function of the form

$$A \exp(-t/\tau) \cos(\omega_a t)$$

yields a spectral density which can be approximated as

$$A \frac{\tau}{1 + \tau^2 (\omega - \omega_a)^2}$$

provided that $\tau\omega_a \gg 1$ and $|\omega - \omega_a| \ll \omega_a$. Since the phase factors ϕ in (7.85) have negligible effect on the spectral density, S_θ can be written as the sum of two terms, each having the form

$$\frac{\hbar}{2I_h\omega_h} n_\infty \left(1 + \frac{\tau_h}{\tau_c}\right) \frac{2\tau_h}{1 + 4\tau_h^2 (\omega - \omega_i)^2},$$

where $\omega_i = \omega_h \pm d$ is a frequency of one of the modes of the spin-resonator system. We find that at each of these frequencies

$$\begin{aligned} S_\theta &\approx \frac{\tau_h \hbar}{I_h \omega_h} \left(n_h + \frac{\tau_h}{\tau_c} n_c\right), \\ S_N &\approx \frac{4 I_h \hbar \omega_h}{\tau_h} \left(n_h + \frac{\tau_h}{\tau_c} n_c\right). \end{aligned} \tag{7.87}$$

The spectral density of the mechanical fluctuations and the thermal torque at frequencies $\omega_h \pm d$ are thus larger than they would be at ω_h in the absence of the hyperpolarized spins.

In conclusion, the use of hyperpolarized spins to cool a mechanical resonator does not improve its sensitivity as a detector of an applied torque, since the mechanical thermal noise, characterized by equation (7.87), is not decreased by the coupling to the hyperpolarized spins; indeed, we have found that when $n_\infty \gg 1/2$ and $n_\infty \gg n_c$, the thermal torque at the eigenfrequencies $\omega_h \pm d$ becomes larger than it would be at ω_h in the absence of the hyperpolarized spins. In this regime, as well as in the regime where the spins behave as a cold bath, the noisy thermal torque acting on the resonator is not decreased by the presence of the cold spins.