Orthogonal Polynomials, Paraorthogonal Polynomials and Point Perturbation

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Abstract

This thesis consists of three parts.

Part 1 starts with an introduction to orthogonal polynomials, to be followed by some well-known theorems pertinent to the results we shall discuss. It also states the new results that are going to be proven in Parts 2 and 3.

In Part 2, we consider a sequence of paraorthogonal polynomials and investigate their zeros. Then we introduce paraorthogonal polynomials of the second kind and prove that zeros of first and second kind paraorthogonal polynomials interlace.

In Part 3, we consider the point mass problem. First, we give the point mass formula for the perturbed Verblunsky coefficients. Then we investigate the asymptotics of orthogonal polynomials on the unit circle and apply the results to the point mass formula to compute the perturbed Verblunsky coefficients. Finally, we present two examples, one on $\partial\mathbb{D}$ and one on $\mathbb{R}$, such that adding a point mass will generate non-exponential perturbations of the recursion coefficients.

Contents

Part I

Introduction

Chapter 1

Notations

In this thesis, we present results in two areas of orthogonal polynomials - paraorthogonal polynomials and point perturbation - in Chapter 2 and Chapter 3 respectively.

Before we get to the main results, we provide a very brief introduction into the area of orthogonal polynomials (abbreviated as OP in the remaining discussions).

Orthogonal Polynomials on $\partial\mathbb{D}$ Let μ be a probability measure on the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ with infinite support. We form the inner product $\langle\, ,\, \rangle$ and the norm in $L^2(d\mu)$ as follows:

$$\langle f, g\rangle = \int_{\partial\mathbb{D}} \overline{f(z)} g(z) d\mu(z) \tag{1.0.1}$$

$$\|f\| = \langle f, f\rangle^{1/2} \tag{1.0.2}$$

By the Gram–Schmidt process, we orthogonalize $1, z, z^2, \dots$ to obtain the sequence of monic orthogonal polynomials $(\Phi_n)_{n=1}^{\infty}$, with $\Phi_n(z)$ being the unique monic polynomial that is orthogonal to polynomials of order $\leq n-1$. For $m \geq 0$, we define $\Phi_m^*(z)$ to be the polynomial

$$\Phi_m^*(z) = z^m \overline{\Phi_m(1/\overline{z})} \tag{1.0.3}$$

Note that for $z \in \partial\mathbb{D}$, $\Phi_n^*(z) = z^n \overline{\Phi_n(z)}$. Therefore, for $j = 1, \dots, n$,

$$\langle z^j, \Phi_n^*(z) \rangle = \langle \Phi_n(z), z^{n-j} \rangle = 0 \tag{1.0.4}$$

which explains why $\Phi_n^*(z)$ is the unique polynomial of degree $\leq n$ which is orthogonal to $z, \dots, z^n$ (up to multiplication by a constant). Moreover, it is easy to deduce that

$$\|\Phi_n\|^2 = \|\Phi_n^*\|^2 = \int \Phi_n^*(e^{i\theta}) d\mu(\theta) \tag{1.0.5}$$

Since $\Phi_{n+1}(z) - z\Phi_n(z)$ is a polynomial of degree at most n and it is orthogonal to $z, z^2, \dots, z^n$, it is a multiple of $\Phi_n^*(z)$. In fact, orthogonal polynomials satisfy the well-known Szegő recursion relations:

$$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha_n}\Phi_n^*(z) \tag{1.0.6}$$

$$\Phi_{n+1}^*(z) = \Phi_n^*(z) - \alpha_n z \Phi_n(z) \tag{1.0.7}$$

α_n is called the n^{th} *Verblunsky coefficient* of $d\mu$. Notice that $\alpha_n = -\overline{\Phi_{n+1}(0)}$. By rearranging (1.0.11) above, we have

$$z\Phi_n(z) = \Phi_{n+1}(z) + \overline{\alpha_n}\Phi_n^*(z) \tag{1.0.8}$$

Consider the norm of each side of (1.0.8). Since $\Phi_n^*(z)$ is a polynomial of degree at most n, it is orthogonal to $\Phi_{n+1}(z)$. Moreover, we know that $\|z\Phi_n\|^2 = \|\Phi_n\|$. Therefore, the norm of the left hand side is $\|\Phi_n\|^2$ while the norm of the right hand side is $\|\Phi_{n+1}\|^2 + |\alpha_n|^2\|\Phi_n\|^2$. Upon regrouping, that becomes

$$\|\Phi_{n+1}\|^2 = \left(1 - |\alpha_n|^2\right)\|\Phi_n\|^2 \tag{1.0.9}$$

Inductively,

$$\|\Phi_{n+1}\|^2 = \prod_{j=0}^{n}\left(1 - |\alpha_j|^2\right) \tag{1.0.10}$$

Since $\|\Phi_n\|^2 > 0$, this implies $|\alpha_n| < 1$.

Now we see that for every probability measure μ there correspond a family of Verblunsky coefficients $(\alpha_n)_{n=0}^{\infty} \in \mathbb{C}^{\infty}$. In fact, the converse is also true:

Theorem 1.0.1 (Verblunsky's Theorem)**.** *Let $(\beta_j)_{j=0}^{\infty}$ be a sequence in $\mathbb{C}^{\infty}$. Then there is a unique measure μ with Verblunsky coefficients $(\beta_j)_{j=0}^{\infty}$.*

Hence, there is a bijection between any probability measure with infinite support on $\partial\mathbb{D}$ and its family of Verblunsky coefficients.

If we let $\varphi_n(z) = \Phi_n(z)/\|\Phi_n\|$ be the orthonormal polynomial of order n, then the Szegő recursion relations become:

$$\varphi_{n+1}(z) = (1 - |\alpha_n|^2)^{-1/2} \left(z\varphi_n(z) - \overline{\alpha_n}\varphi_n^*(z)\right) \tag{1.0.11}$$

$$\varphi_{n+1}^*(z) = (1 - |\alpha_n|^2)^{-1/2} \left(\varphi_n^*(z) - \alpha_n z\varphi_n(z)\right) \tag{1.0.12}$$

Such recursion relations can be expressed in matrix form:

$$\begin{pmatrix} \varphi_{n+1}(z) \\ \varphi_{n+1}^*(z) \end{pmatrix} = (1 - |\alpha_n|^2)^{-1/2} \begin{pmatrix} z & -\overline{\alpha_n} \\ -z\alpha_n & 1 \end{pmatrix} \begin{pmatrix} \varphi_n(z) \\ \varphi_n^*(z) \end{pmatrix} \tag{1.0.13}$$

Together with the fact that $\varphi_0(z) = \varphi_0^*(z) = 1$, we have

$$\begin{pmatrix} \varphi_{n+1}(z) \\ \varphi_{n+1}^*(z) \end{pmatrix} = A_n(z)A_{n-1}(z)\cdots A_0(z) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \equiv T_n(z) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.0.14}$$

where

$$A_n(z) = (1 - |\alpha_n|^2)^{-1/2} \begin{pmatrix} z & -\overline{\alpha_n} \\ -z\alpha_n & 1 \end{pmatrix} \tag{1.0.15}$$

and $T_n(z)$ is called the *Transfer Matrix*. These $A_n(z)$'s will play a major role in Part 3.

The Szegő function, which will be involved in Theorem 2.0.8, is defined as follows

Definition 1.0.1. *If* $d\mu = w(\theta)\frac{d\theta}{2\pi} + d\mu_s$ *and* $\sum_{j=0}^{\infty} |\alpha_j|^2 < \infty$*, the Szegő* *function is defined as*

$$D(z) = \exp\left(\frac{1}{4\pi}\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\log w(\theta)d\theta\right) \tag{1.0.16}$$

The well-known Szegö's Theorem asserts the following equality

$$\prod_{j=0}^{\infty}(1-|\alpha_j|^2) = \exp\left(\int_0^{2\pi}\log(w(\theta))\frac{d\theta}{2\pi}\right) \tag{1.0.17}$$

Hence, if (α_n) is ℓ^2, $\log w(\theta)$ is integrable and $D(z)$ defines an analytic function on $\mathbb{D}$. For a thorough discussion of the Szegő function, the reader may refer to Chapter 2 of [46].

For a more detailed introduction to orthogonal polynomials on the unit circle, the reader should refer to [20, 46, 47, 48, 50].

Orthogonal Polynomials on $\mathbb{R}$ Let $d\gamma$ be a probability measure on $\mathbb{R}$. We can define an inner product and norm on $L^2(\mathbb{R}, d\gamma)$ as in (1.0.1) and (1.0.2), except that in this case it does not involve any conjugation. By the Gram–Schmidt process, we can orthogonalize $1, x, x^2, \ldots$ and form the family of monic orthogonal polynomials, $(P_n(x))_{n=0}^{\infty}$. Upon normalization, we obtain the family of orthonormal polynomials, $(p_n(x))_{n=0}^{\infty}$.

Note that $xP_n(x) - P_{n+1}(x)$ is a polynomial of degree at most n, we can express it as

$$xP_n - P_{n+1} = \sum_{j=0}^{n}\beta_j^{(n)}P_j \tag{1.0.18}$$

where $\beta_j = \langle P_j, xP_n \rangle / \|P_j\|^2$. Moreover, observe that

$$\beta_j = \langle P_j, xP_n \rangle = \langle xP_j, P_n \rangle = 0 \tag{1.0.19}$$

for $0 \leq j \leq n-2$; and

$$\beta_{n-1} = \frac{\langle xP_{n-1}, P_n \rangle}{\|P_{n-1}\|^2} = \frac{\|P_n\|^2}{\|P_{n-1}\|^2} > 0 \tag{1.0.20}$$

If we let

$$a_n = \frac{\|P_n\|}{\|P_{n-1}\|} \tag{1.0.21}$$

$$b_n = \frac{\langle xP_n, P_n \rangle}{\|P_n\|^2} \tag{1.0.22}$$

then by (1.0.18) above, we deduce that the orthogonal polynomials satisfy the following three-term recursion relation:

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_{n+1}p_n(x) + a_n p_{n-1}(x) \tag{1.0.23}$$

$(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are called the recursion coefficients of the measure $d\gamma$.

Furthermore, by iterating (1.0.21), one gets:

$$\|P_n\| = a_n a_{n-1} \cdots a_1 \tag{1.0.24}$$

which implies that the recurrence relation for the orthonormal polynomials is

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_{n+1}p_n(x) + a_n p_{n-1}(x) \tag{1.0.25}$$

For more on orthogonal polynomials on the real line, the reader may refer to [14, 46].

Chapter 2

Summary of Results

Paraorthogonal polynomials were introduced at least as early as in [24]. An $(n+1)^{th}$ degree paraorthogonal polynomial is of the form (up to multiplication by a constant):

$$H_{n+1}(z, \beta_n, d\mu) = z\Phi_n(z) - \overline{\beta_n}\Phi_n^*(z) \tag{2.0.1}$$

with $\beta_n \in \partial\mathbb{D}$; $\Phi_n(z)$ being the n^{th} monic orthogonal polynomial associated with $d\mu$, and $\Phi_n^*(z) = z^n\overline{\Phi_n(1/\overline{z})}$.

Paraorthogonal polynomials have a lot in common with orthogonal polynomials on the real line $(p_n(x))_{n=0}^{\infty}$. For instance, a paraorthogonal polynomial has simple zeros on the unit circle while $p_n(x)$ has simple zeros on the real line.

We shall consider a specific family of paraorthogonal polynomials $(h_n(z,\lambda))_{n=0}^{\infty}$, defined for some fixed $\lambda \in \partial\mathbb{D}$ as:

$$h_n(z, \lambda) := (1 - \overline{\lambda} z) K_{n-1}(z, \lambda) \tag{2.0.2}$$

where

$$K_{n-1}(z, \lambda) = \sum_{j=0}^{n-1} \overline{\varphi_j(\lambda)} \varphi_j(z) \tag{2.0.3}$$

It will be clear in the proof that this definition of $h_n(z)$ is consistent with the definition given in (2.0.1).

It has been proven by Cantero–Moral–Velázquez [11] and Golinskii [22] that zeros of $h_n(z, \lambda)$ and $h_{n+1}(z, \lambda)$ strictly interlace. In fact, this interlacing property is shared by $p_n(x)$ and $p_{n+1}(x)$.

With the same λ that defines $h_n(z, \lambda)$, we introduce *paraorthogonal polynomials of the second kind*, $s_n(z, \lambda)$, and prove that zeros of $h_n(z, \lambda)$ and $s_n(z, \lambda)$ (with the exception of λ) strictly interlace. This resembles the fact that the zeros of $p_n(x)$ and $q_n(x)$ strictly interlace.

We prove four results concerning $h_n(z, \lambda)$, $h_{n+1}(z, \lambda)$, $s_n(z, \lambda)$ and $s_{n+1}(z, \lambda)$. The first result is as follows:

Theorem 2.0.2 ([53])**.** *Suppose $z_0 \in \partial\mathbb{D}$ distinct from λ and $\delta = \operatorname{dist}(z_0, \operatorname{supp}(d\mu))$ strictly positive. Then in the open disk around z_0 with radius*

$$\rho = \frac{\delta^3}{8 + \delta^2} \tag{2.0.4}$$

either h_n or h_{n+1} (or both) has no zero inside, with the possible exception of

λ. *Furthermore, if* $L = \text{dist}(\lambda, \text{supp}(d\mu) > 0$, *then the radius could be taken as:*

$$\rho' = \frac{\delta^2 L}{8 + \delta L} \tag{2.0.5}$$

Note that when $L > \delta$, $\rho' > \rho$, hence (2.0.5) improves (2.0.4).

There is a related conjecture concerning double limit points which was proposed in [22] and proven in [12]. The result says that the set of double limit points of h_n coincides with $\text{supp}(d\mu)$, except at most the point λ. In other words, if $\text{dist}(z_0, \text{supp}(d\mu)) > 0$, then for any sequence of integers I, there exists a subsequence $I' \subset I$ and $\epsilon_I > 0$ such that for $n \in I'$, either h_n or h_{n+1} (or both) has no zero in the open disk $B(z_0, \epsilon_I)$.

However, Theorem 2.0.2 is clearly stronger because we found an explicit radius ρ for which the double zero result holds (2.0.4) and the result does not depend on n.

The second result we prove is the following:

Theorem 2.0.3 ([53]). *The zeros of* h_n *and* s_n *strictly interlace, that is, between any two zeros of* h_n *(or* s_n*), there is one and only one zero of* s_n *(or* h_n *respectively) in between.*

Theorem 2.0.3 is an analogue of the following well-known fact that zeros of the first and second kind orthogonal polynomials on the real line strictly interlace.

At the same time when Theorem 2.0.3 was proven, Simon [46] demonstrated another way of proving the result using the theory of rank one perturbations of unitary operators. He made the observation that the CMV matrix associated to s_n is just the original one with the signs of α_j and β_{n-1} reversed, and it is unitarily equivalent to one where the signs are not reversed but the first column has opposite sign.

The main tools of the proof are the two real-valued functions σ_n and η_n which we will define in (4.2.3) and (4.2.4). They were used in [11] to prove that zeros of h_n and h_{n+1} interlace, but the method employed in our proof is different.

The remaining two results concerning paraorthogonal polynomials are:

Lemma 2.0.1 ([53]). *Suppose z_0 is an isolated point in $\operatorname{supp}(d\mu)$. Then*

$$\tilde{\delta} = \operatorname{dist}(z_0, \operatorname{supp}(d\nu)) > 0 \tag{2.0.6}$$

and in the ball around z_0 with radius

$$\tilde{\rho} = \frac{\tilde{\delta}^2 |z_0 - \lambda|}{8 + |z_0 - \lambda| \tilde{\delta}} \tag{2.0.7}$$

either s_n or s_{n+1} (or both) has no zeros inside.

Theorem 2.0.4 ([53]). *Suppose z_0 is an isolated point of $\operatorname{supp}(d\mu)$ and $\tilde{\delta}$ is as defined in (2.0.6). Then in the open disk around z_0 with radius*

$$\tilde{\rho} = \frac{\tilde{\delta}^2 |z_0 - \lambda|}{8 + |z_0 - \lambda| \tilde{\delta}} \tag{2.0.8}$$

either h_n or h_{n+1} (or both) has at most one zero inside.

Theorem 2.0.2 and Theorem 2.0.4 are analogues of the following results of Denisov–Simon [17]:

Theorem 2.0.5. *Let $\delta = \mathrm{dist}(x_0, \mathrm{supp}(d\mu)) > 0$. Suppose a_{n+1} is the recursion coefficient as given by $xp_n(x) = a_{n+1}p_{n+1}(x) + b_{n+1}p_n(x) + a_n p_{n-1}(x)$. Let $r_n = \delta^2/(\delta + \sqrt{2}a_{n+1})$. Then either p_n or p_{n+1} (or both) has no zeros in $(x_0 - r_n, x_0 + r_n)$.*

Theorem 2.0.6. *Let x_0 be an isolated point of $\mathrm{supp}(d\mu)$ on the real line. Then there exists $d_0 > 0$ so that if $\delta_n = d_0^2/(d_0 + \sqrt{2}a_{n+1}))$, then at least one of p_n and p_{n+1} has no zeros or one zero in $(x_0 - \delta_n, x_0 + \delta_n)$.*

This concludes Part 2 of this thesis.

Part 3 is dedicated to the point mass problem. Suppose $d\mu$ is a probability measure on the unit circle and $0 < \gamma < 1$. Let $d\nu$ be the probability measure formed by adding a point mass $\zeta = e^{i\omega} \in \partial\mathbb{D}$ to $d\mu$ in the following manner

$$d\nu = (1-\gamma)d\mu + \gamma\delta_\omega \tag{2.0.9}$$

Our goal is to investigate the Verblunsky coefficients of ν.

The history of the problem is as follows. The earliest work related to adding point masses was done by Wigner–von Neumann [52], where they constructed a potential with an embedded eigenvalue. Later, Gel'fand–Levitan

[18] constructed a potential V so that $-\frac{d^2}{dx^2}+V$ has a spectral measure with a pure point mass at a positive energy and was otherwise equal to the free measure. A more systematic approach to adding point masses to a potential was then taken by Jost–Kohn [25, 26].

Unaware of the Jost–Kohn work and of each other, Uvarov [51] and Nevai [38] discovered the formulae for adding point masses for orthogonal polynomials on the real line. They found the perturbed polynomials, and Nevai computed the perturbed recursion coefficients.

Jost–Kohn theory for orthogonal polynomials on the unit circle appeared previously in Cachafeiro–Marcellán [7, 8, 10], Marcellán–Maroni [32], and Peherstorfer–Steinbauer [33]. In particular, if $d\nu$ and $d\mu$ are as defined in (2.0.9), Peherstorfer–Steinbauer [33] proved that boundedness of the first and second kind orthonormal polynomials of $d\mu$ at the pure point ζ implies that $\lim_{n\to\infty}\alpha_n(d\nu)-\alpha_n(d\mu)=0$, but they did not establish any rate of convergence. We are going to prove more quantitative results concerning $\alpha(d\nu)$ than that.

The first result that we present is the following *point mass formula*:

Theorem 2.0.7. *[55] Suppose $d\mu$ is a probability measure on the unit circle and $0<\gamma<1$. Let $d\nu$ be the probability measure formed by adding a point mass $\zeta=e^{i\omega}\in\partial\mathbb{D}$ to $d\mu$ in the following manner*

$$d\nu=(1-\gamma)d\mu+\gamma\delta_\omega \tag{2.0.10}$$

Then the Verblunsky coefficients of $d\nu$ are given by

$$\alpha_n(d\nu) = \alpha_n + \Delta_n(\zeta) \tag{2.0.11}$$

where

$$\Delta_n(\zeta) = \frac{(1-|\alpha_n|^2)^{1/2}}{(1-\gamma)\gamma^{-1} + K_n(\zeta)} \overline{\varphi_{n+1}(\zeta)}\varphi_n^*(\zeta) \tag{2.0.12}$$

and

$$K_n(\zeta) = \sum_{j=0}^{n} |\varphi_j(\zeta)|^2 \tag{2.0.13}$$

and all objects without the label $(d\nu)$ are associated with the measure $d\mu$.

In fact, when we proved Theorem 2.0.7, we were totally unaware of the following formula found by Geronimus [20]:

$$\Phi_n(z, d\nu) = \Phi_n(z) - \frac{\Phi_n(\zeta)K_{n-1}(z,\zeta)}{(1-\gamma)\gamma^{-1} + K_{n-1}(\zeta,\zeta)} \tag{2.0.14}$$

Years after Geronimus proved (2.0.14), a similar formula for the real case was rediscovered by Nevai [38], and the same formula for the unit circle case was rediscovered by Cachafeiro-Marcellan [10]. Unaware of Geronimus' result and the fact that Nevai's result also applies to the unit circle, Simon reconsidered this problem independently using a totally different method (see Theorem 10.13.7 in [47]). However, a more useful form of his result (see formula (5.0.7) in Section 5) is disguised in his proof and it lays the foundation for Theorem 2.0.7.

Now that we have the point mass formula, we will demonstrate its first application to the point mass problem. Before we state the results, it is necessary that we introduce the notion of *p-generalized bounded variation* which is the class of sequences defined as follows:

Definition 2.0.2. *We say that a sequence* $(\alpha_n)_{n=0}^{\infty}$ *is of* p-generalized bounded variation *if each* α_n *can be decomposed into* p *components*

$$\alpha_n = \sum_{k=1}^{p} \beta_{n,k} \tag{2.0.15}$$

with $\beta_{n,k} \in \mathbb{C}$ *and there exist* $\zeta_1, \zeta_2, \ldots, \zeta_p \in \partial\mathbb{D}$ *such that for each* $1 \leq k \leq p$

$$\sum_{n=0}^{\infty} |\zeta_k \beta_{n+1,k} - \beta_{n,k}| < \infty \tag{2.0.16}$$

We denote by $W_p(\zeta_1, \zeta_2, \ldots, \zeta_p)$ *the class of sequences* $(\alpha_n)_{n=0}^{\infty}$ *that satisfy (2.0.15) and (2.0.16).*

In particular, when $p = 1$ and $\zeta_1 = 1$, then it becomes the conventional bounded variation. This is why we gave the name p-generalized bounded variation.

For the sake of simplicity, we shall write $d\mu \in W_p(\zeta_1, \zeta_2, \ldots, \zeta_p)$ if the family of Verblunsky coefficients of $d\mu$ is in the class $W_p(\zeta_1, \zeta_2, \ldots, \zeta_p)$.

Now we are ready to state the first two results:

Theorem 2.0.8. *[54] Let* $\zeta_j = e^{i\omega_j} \in \partial\mathbb{D}$, $1 \leq j \leq p$ *be distinct. Suppose we have a measure* $d\mu$ *with* $d\mu \in W_p(\zeta_1, \zeta_2, \ldots, \zeta_p)$ *such that for each* j,

$(\beta_{n,j})_{n=0}^{\infty} \in \ell^2$. *The following two results hold:*

(1) For any compact subset K of $\partial\mathbb{D}\backslash\{\zeta_1, \zeta_2, \ldots, \zeta_p\}$,

$$\sup_{n;z\in K} |\Phi_n^*(z)| < \infty \tag{2.0.17}$$

(2) The following limits are continuous at $z \neq \zeta_1, \zeta_2, \ldots, \zeta_p$

$$\Phi_\infty^*(z) = \lim_{n\to\infty} \Phi_n^*(z) = D(0)D(z)^{-1} \tag{2.0.18}$$

$$\varphi_\infty^*(z) = \lim_{n\to\infty} \varphi_n^*(z) = D(z)^{-1} \tag{2.0.19}$$

and the convergence is uniform on any compact subset $K \subset \partial\mathbb{D}\backslash\{\zeta_1, \zeta_2, \ldots, \zeta_p\}$. Moreoever, $d\mu_s$ is a pure point measure supported on a subset of $\{\zeta_1, \zeta_2, \ldots, \zeta_p\}$.

Theorem 2.0.9. *[54] Suppose $d\mu_0 \in W_1(1)$ and $(\alpha_n(d\mu_0))_{n=0}^{\infty} \in \ell^2$. We add m distinct pure points $z_j = e^{i\omega_j}$, $\omega_j \neq 0$, to $d\mu_0$ with weights γ_j to form the probability measure $d\mu_m$ as follows*

$$d\mu_m = \left(1 - \sum_{j=1}^{m} \gamma_j\right) d\mu_0 + \sum_{j=1}^{m} \gamma_j \delta_{\omega_j} \tag{2.0.20}$$

under the conditions that $0 < \gamma_j$ and $\sum_{j=1}^{m} \gamma_j < 1$. Then

$$d\mu_m \in W_{m+1}(1, z_1, z_2, \ldots, z_m) \tag{2.0.21}$$

and

$$\alpha_n(d\mu_m) = \alpha_n(d\mu_0) + \sum_{j=1}^{m} \frac{\overline{z_j}^n c_j}{n} + E_n \tag{2.0.22}$$

where $c_j = \overline{z_j}|D(z_j, d\mu_0)|^2 D(z_j, d\mu_0)^{-2}$ *are constants independent of the weights* $\gamma_1, \gamma_2, \ldots, \gamma_m$ *and of* n*; and*

$$E_n = E_n(z_1, z_2, \ldots, z_m, \gamma_1, \gamma_2, \ldots, \gamma_m) = o\left(\frac{1}{n}\right) \tag{2.0.23}$$

Furthermore, for $z \in \partial\mathbb{D}\backslash\{1, z_1, z_2, \ldots, z_m\}$*,* $\varphi^*_\infty(z, d\mu_m)$ *is continuous and is equal to* $(1 - \sum_{j=1}^m \gamma_j)^{-1/2} D(z, d\mu_0)^{-1}$*.*

Remark: Note that $d\mu_{m_{a.c.}}$ is just $(1-\sum_{j=1}^m \gamma_j) d\mu_{0_{a.c.}}$ and that $\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\frac{d\theta}{2\pi} = 1$. Hence, $D(z, d\mu_m) = (1 - \sum_{j=1}^m \gamma_j)^{1/2} D(z, d\mu_0)$.

Theorem 2.0.8 is a generalization of the following result:

Theorem 2.0.10 (Nevai [39], Nikishin [40])**.** *Suppose* $\sum_{j=0}^\infty |\alpha_j|^2 < \infty$ *and*

$$\sum_{j=0}^{\infty} |\alpha_{j+1} - \alpha_j| < \infty \tag{2.0.24}$$

Then, for any $\delta > 0$*,* $\sup_{n;\delta<\arg(z)<2\pi-\delta} |\Phi^*_n(z)| < \infty$ *and away from* $z = 1$*, we have that* $\lim_{n\to\infty} \Phi^*_n(z)$ *exists, is continuous and equal to* $D(0)D(z)^{-1}$*. Furthermore,* $d\mu_s = 0$ *or else a pure point at* $z = 1$*.*

The reader may refer to Theorem 10.12.5 of [47] for the proof of Theorem 2.0.10.

In addition to Nevai, Uvarov and Simon's result mentioned earlier, we use Prüfer variables as the main tool to prove that $\lim_{n\to\infty} \Phi_n^*(z)$ exists in Theorem 2.0.8. Prüfer variables are named after Prüfer [41]. Their initial introduction in the spectral theory of orthogonal polynomials on the unit circle was made by Nikishin [40] with a significant follow up by Nevai [39]. Both [39] and [40] had results related to Theorem 2.0.10 and they arrived at the result by essentially the same proof. Later, Prüfer variables were used as a serious tool in spectral theory by Kiselev–Last–Simon [28] and Last–Simon [31].

Most recently, in [46] (Example 1.6.3, p. 72) Simon considered the measure $d\nu$ with one pure point:

$$d\nu = (1-\gamma)\frac{d\theta}{2\pi} + \gamma\delta_0 \tag{2.0.25}$$

He proved that the n-th degree orthogonal polynomial of $d\nu$ is as follows

$$\Phi_n(z) = z^n - \frac{\gamma}{1+(n-1)\gamma}(z^{n-1} + z^{n-2} + \cdots + 1) \tag{2.0.26}$$

and since $\alpha_n = -\overline{\Phi_{n+1}(0)}$,

$$\alpha_n(d\nu) = \frac{\gamma}{1+\gamma n} \approx \frac{1}{n} + \frac{1}{\gamma n^2} + O\left(\frac{1}{n^3}\right) \tag{2.0.27}$$

Here is a sketch of Simon's proof: he considered L_n, the $(n+1)\times(n+1)$ matrix defined as $(L_n)_{jk} = c_{j-k}$, where $c_j = \int e^{-ij\theta} d\mu(\theta)$ is the j-th moment of the measure. It is well-known that if $\Phi_n(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$, $\delta_n = (0, 0, \ldots, 0, 1)$ and $\langle\, ,\, \rangle$ being the Euclidean norm,

$$(a_0, a_1, \ldots, a_n) = \left\langle \delta_n, L_n^{-1}\delta_n \right\rangle^{-1} L_n^{-1}\delta_n \tag{2.0.28}$$

Therefore, the aim is to compute L_n^{-1}. By (2.0.25), $c_n = (1-\gamma)\delta_{n0} + \gamma$. Let P_j be the $j \times j$ matrix which is j^{-1} times the matrix of all 1's, so it is a rank one projection. L_n could be decomposed as

$$L_n = (1-\gamma)\mathbf{1} + (n+1)\gamma P_{n+1} \tag{2.0.29}$$

From (2.0.29), one could deduce that the inverse of L_n is

$$L_n^{-1} = (1-\gamma)^{-1}(\mathbf{1} - P_{n+1}) + (1+n\gamma)^{-1} P_{n+1} \tag{2.0.30}$$

Unfortunately, the method used to prove the result above no longer gives such a nice result when there are two pure points. For instance, we won't have the decomposition as in (2.0.29), because L_n will be a rank m perturbation of $(1 - \sum_{j=1}^{m} \gamma_j)\mathbf{1}$ instead, so the computations will be much more complicated. Besides, this method only works for adding one point to $d\theta/2\pi$ but fails for more general measures.

After considering measures with ℓ^2 Verblunsky coefficients of bounded variation in Part 2, we turn our attention to probability measures on $\partial\mathbb{D}$ with asymptotically periodic Verblunsky coefficients of p-bounded variation in Part 3. The essential spectrum of these measures consists of a finite number of bands and gaps and our goal is to understand the effect of adding a point mass to a gap in the essential spectrum.

We start with asymptotically identical Verblunsky coefficients. We present a new method to compute the asymptotics of $\varphi_n(z)$ in the gap of the spectrum (see formulae (7.4.54) and (7.4.55)). Applying that to the point mass formula, we prove the following result:

Theorem 2.0.11. *[56] Let $(\alpha_n)_{n=0}^{\infty}$ be the Verblunsky coefficients of the probability measure $d\mu$ on $\partial\mathbb{D}$ such that*

$$\alpha_n \to L \in \mathbb{D}\backslash\{0\} \tag{2.0.31}$$

$$\sum_{j=0}^{\infty} |\alpha_{j+1} - \alpha_j| < \infty \tag{2.0.32}$$

Let G_L be the gap of the essential spectrum (not including the endpoints). We add a pure point $z = e^{i\theta} \in G_L$ to $d\mu$ to form $d\nu$ as in (2.0.9). Then either one of the following is true:

1. *If $\mu(z) > 0$, then $|\varphi_n(z)|$ decreases exponentially, $\Delta_n(z) \to 0$ exponentially fast, and $\alpha_n(d\nu) - \alpha_n(d\mu)$ is exponentially small.*

2. *If $\mu(z) = 0$, then*

(a) $\lim_{n\to\infty} \Delta_n(z)$ *exists, and*

$$\Delta_\infty(z) \equiv \lim_{n\to\infty} \Delta_n(z) = \overline{h(z)^{1/2}} \left[\frac{(z-1) - h(z)^{1/2}}{2\overline{L}} \right] \tag{2.0.33}$$

where

$$h(z) = (z-1)^2 + 4z|L|^2 \tag{2.0.34}$$

and we choose the branch of logarithm such that $(1)^{1/2} = 1$.

(b) Furthermore, $|\Delta_\infty(z) + L| = |L|$ *and*

$$\lim_{n\to\ \infty} \alpha_n(d\nu) = Le^{i\omega} \tag{2.0.35}$$

where

$$\cos\omega = \frac{2\sin^2\left(\frac{\theta}{2}\right) - |L|^2}{|L|^2} \tag{2.0.36}$$

$$\sin\omega = \frac{2\sin\left(\frac{\theta}{2}\right)\sqrt{|L|^2 - \sin^2\left(\frac{\theta}{2}\right)}}{|L|^2} \tag{2.0.37}$$

(c) $(\Delta_n)_n$ *is of bounded variation, i.e.,*

$$\sum_{n=0}^{\infty} |\Delta_{n+1}(z) - \Delta_n(z)| < \infty \tag{2.0.38}$$

A few remarks about Theorem 2.0.11:

(i) Since $\alpha_n \to L \neq 0$, this measure has the same essential spectrum as the measure $d\mu_0$ with Verblunsky coefficients $\alpha_n(d\mu_0) \equiv L$, which is supported on the arc $\Gamma_{|L|}$ as defined in (2.0.45).

(ii) Case (1) is a special case of Corollary 24.3 of [49], where Simon proved that varying the weight of an isolated pure point in the gap will result in exponentially small perturbation to $\alpha_n(d\mu)$.

(iii) By (2c), adding a pure point to the gap will preserve the bounded variation property of $(\alpha_n)_n$. Hence, we can add a finite number of points inductively and generalize the result to finitely many pure points in the gap.

Next, we will generalize the technique developed in the proof of Theorem 2.0.11 and prove the following result about measures with asymptotically periodic Verblunsky coefficients:

Theorem 2.0.12 ([56]). *Let $(\beta_n)_n$ be a periodic family of Verblunsky coefficients of period p, i.e., $\beta_n = \beta_{n+p}$ for all n, and let $d\mu_\beta$ be the measure associated to it. Let Γ_β be the union of open arcs which are the interiors of the bands that form* ess supp$(d\mu_\beta)$. *Suppose the measure $d\mu$ has Verblunsky coefficients $(\alpha_n)_n$ that are asymptotically p-periodic of bounded variation, i.e.,*

$$\lim_{n\to\infty} \alpha_n(d\mu) - \beta_n = 0 \tag{2.0.39}$$

$$\sum_{n=0}^{\infty} |\alpha_{n+p} - \alpha_n| < \infty \tag{2.0.40}$$

Now we add a pure point $\zeta \in \partial\mathbb{D} \setminus \overline{\Gamma}_\beta$ as in (2.0.9). Then one of the *following is true:*

1. $\mu(\zeta) > 0$, *then for each fixed* $0 \leq j < p$, $\lim_{k\to\infty} \Delta_{kp+j}(\zeta) = 0$ *exponentially fast.*

2. $\mu(\zeta) = 0$, *then for each fixed* $0 \leq j < p$, $\lim_{k\to\infty} \Delta_{kp+j}(\zeta)$ *exists and*

$$\sum_{k=0}^{\infty} |\Delta_{(k+1)p+j} - \Delta_{kp+j}| < \infty \tag{2.0.41}$$

Then we will prove the following result where $(\alpha_n)_n$ is not necessarily of bounded variation:

Theorem 2.0.13. *[56] Let* $\zeta \in \partial\mathbb{D}$ *and* $\mu(\zeta) = 0$. *Suppose* $\lim_{n\to\infty} \zeta^n \alpha_n = L$. *Then*

$$\lim_{n\to\infty} \zeta^n \Delta_n = -2L \tag{2.0.42}$$

As a result,

$$\lim_{n\to\infty} \zeta^n \alpha_n(d\nu) = -\lim_{n\to\infty} \zeta^n \alpha_n(d\mu) \tag{2.0.43}$$

Next, we use Theorem 2.0.13 to prove Corollary 2.0.1 below to illustrate the non-exponential rate of convergence of $\Delta_n(\zeta)$ towards its limit. One might have guessed that the convergence should be exponentially fast, but we will show that it is not the case!

Corollary 2.0.1. *Let* $\alpha_n = L + c_n$*, where* $L < 0$*,* $c_n \in \mathbb{R}$ *and* $c_n \to 0$*. Then*

$$\Delta_n(1) = -2L - 2c_n + o\left(c_n\right). \tag{2.0.44}$$

There are many papers about measures supported on an interval/arc, and about the perturbation of orthogonal polynomials with periodic recursion coefficients. For example, the reader may refer to [4, 16, 33, 37, 3, 2, 15].

Bello-López [4] extended the well-known work of Rakhmanov [42, 43, 44] and proved the following: let $0 < a < 1$ and $\theta_a = 2\arcsin(a)$. If $d\mu$ is supported on the arc

$$\Gamma_a = \{z \in \partial\mathbb{D} || \arg(z)| > \theta_a\} \tag{2.0.45}$$

such that $w(\theta) > 0$ on Γ_a, then $\lim_{n\to\infty} |\alpha_n| = a$. Bellos–López's result is restricted to measures that are absolutely continuous on the arc, and it was later extended to measures with infinitely many mass points outside the a.c. part of the support (see for example, [3] and Theorem 13.4.4 of [47]). However, unlike Theorem 2.0.11, these results do not tell us whether $\Delta_n(z)$ approaches a single point.

In [33], Peherstorfer–Steinbauer considered the situation where $d\mu$ is an absolutely continuous measure on $\operatorname{supp}(d\mu) = \Gamma_a$ with $w(\theta)$ satisfying the

Szegö condition on Γ_a, i.e.,

$$\int_{\Gamma_a} \log w(\theta) \frac{\sin(\frac{\theta}{2})}{\sqrt{\cos^2(\frac{\theta_{|\alpha|}}{2}) - \cos^2(\frac{\theta}{2})}} d\theta > -\infty \tag{2.0.46}$$

They proved that if we add a finite number of pure points to the gap to form the measure to $d\tau$, then $\lim_{n\to\infty} \alpha_n(d\tau)$ exists and the limit has norm $|a|$.

In Section 7.8, we are going to work out an example that demonstrates the existence of a large class of measures with Verblunsky coefficients $\alpha_n \to L$ of bounded variation that fail the Szegő condition (2.0.46).

Given such a result for orthogonal polynomials on the unit circle, one would expect a similar result for the real line. In [37], Peherstorfer–Yuditskii gave the following result: for any Jacobi matrix J whose spectrum is a finite gap set with the a.c. part of the spectral measure satisfying the Szegö condition, then there is a unique Jacobi matrix J_∞ in the isospectral torus such that the orthogonal polynomials of J and J_∞ have the same asymptotics away from the spectrum as $n \to \infty$. In particular, this implies that the Jacobi parameters of J converge to the parameters of J_∞ as $n \to \infty$.

We conclude the thesis by presenting the following result:

Theorem 2.0.14 ([57])**.** *There exists a purely absolutely continuous measure $d\gamma_0$ supported on $[-2,2]$ with no eigenvalues outside of $[-2,2]$, such that if we add a pure point $x_0 \in \mathbb{R}\backslash[-2,2]$ in the following manner*

$$d\tilde{\gamma}(x) = (1-\beta)d\gamma_0(x) + \beta\delta_{x_0} \qquad \beta > 0 \tag{2.0.47}$$

it will result in non-exponential perturbation of the recursion coefficients $a_n(d\gamma_0)$ *and* $b_n(d\gamma_0)$.

This example is of particular interest because of the following: in 1946, Borg [5] proved a well-known result concerning the Sturm–Liouville problem that, in general, a single spectrum is insufficient to determine the potential. Later, Gel'fand–Levitan [18] showed that in order to recover the potential one also needs the norming constants.

Norming constants correspond to the weights of pure points and it is known that in the short range case (in orthogonal polynomials language, $a_n - 1, b_n \to 0$ fast), varying the norming constants will result in exponential change in the potential.

Moreover, when considering the effect of varying the weight of discrete point masses on orthogonal polynomials (both on $\mathbb{R}$ and $\partial\mathbb{D}$), Simon proved that it will result in exponential perturbation of the recursion coefficients (see Corollary 24.4 and Corollary 24.3 of [49]).

All the results mentioned above lent to a few experts the intuition that if the recursion coefficients $a_n \to 1$ and $b_n \to 1$ fast, then adding a pure point will result in exponential change in the recursion coefficients. However, that was proven to be wrong by Theorem 2.0.14 above.

Part II

Paraorthogonal Polynomials

Chapter 3

Background

3.1 Properties

A major difference between orthogonal polynomials and paraorthogonal polynomials lies in the fact that $\alpha_n \in \mathbb{D}$ is determined uniquely by the measure, while $\beta_n \in \partial\mathbb{D}$ could be chosen arbitrarily on the unit circle. These differences give rise to the following properties of paraorthogonal polynomials which are not shared by $\Phi_n(z)$:

1. Zeros in $\partial\mathbb{D}$ Unlike orthogonal polynomials which have zeros strictly inside the unit disk, paraorthogonal polynomials have zeros in $\partial\mathbb{D}$. To see that it suffices, to note that

$$\left|\frac{z\Phi_n(z)}{\Phi_n^*(z)}\right| = 1 \Leftrightarrow z \in \partial\mathbb{D} \tag{3.1.1}$$

Recall that all the zeros of $\Phi_n(z)$ lie in $\mathbb{D}$. Moreover, since $\Phi_n^*(z) = z^n\overline{\Phi_n(1/\bar{z})}$ and $\Phi_n^*(0) = 1$, $\Phi_n^*(z)$ is non-vanishing on $\overline{\mathbb{D}}$. Therefore, the function $g(z) = z\Phi_n(z)/\Phi_n^*(z)$ is analytic in a neighborhood of $\mathbb{D}$. Furthermore, $|g(z)| = 1$ on $\partial\mathbb{D}$, so the maximal modulus principle implies that $|g(z)| < 1$ on $\mathbb{D}$. In other words,

$$|\Phi_n(z)| \leq |\Phi_n^*(z)| \quad z \in \overline{\mathbb{D}} \tag{3.1.2}$$

and equality is attained if and only if $z \in \partial\mathbb{D}$.

2. Orthogonality An n-th degree paraorthogonal polynomial is orthogonal to $\{z, z^2, \ldots, z^{n-1}\}$ because both $z\Phi_{n-1}(z)$ and $\Phi_{n-1}^*(z)$ are orthogonal to $z, \ldots, z^{n-1}$. However, we note that H_n is never orthogonal to 1 or z^n because

$$\langle 1, H_n \rangle = (\overline{\alpha_{n-1}} - \overline{\beta_{n-1}})\|\Phi_{n-1}\|^2 \neq 0 \tag{3.1.3}$$

$$\langle z^n, H_n \rangle = \left(1 - \overline{\beta_{n-1}}\alpha_{n-1}\right) \|\Phi_{n-1}\|^2 \neq 0 \tag{3.1.4}$$

3. Representation Suppose λ is a zero of $H_n(z, \beta_{n-1})$. We prove that H_n could be represented using the reproducing kernel $K_n(z, \lambda) = \sum_{j=0}^n \varphi_j(z)\overline{\varphi_j(\lambda)}$ and a constant C as follows:

$$H_n(z, \beta_{n-1}) = C(z - \lambda)\sum_{j=0}^{n-1} \varphi_j(z)\overline{\varphi_j(\lambda)} = C(z - \lambda)K_{n-1}(z, \lambda) \tag{3.1.5}$$

The argument is related to Szegő [50] when he proved the Christoffel–Darboux formula. It goes as follows: since λ is a zero of H_n, $H_n(z) = (z-\lambda)h(z)$ for some polynomial h of degree $n-1$. By the orthogonality of H_n against $\{z, \dots, z^{n-1}\}$, $\langle zh, z^m\rangle = \langle \lambda h, z^m\rangle$ for $1 \le m \le n-1$, which implies that $\overline{\lambda}\langle z^{m-1}, h\rangle = \langle z^m, h\rangle$. Applying this formula recursively, we conclude that

$$\langle z^m, h\rangle = \overline{\lambda^m}\langle 1, h\rangle, \text{ for } 0 < m \le n-1 \tag{3.1.6}$$

When $m = 0$ the argument is trivial. If $\varphi_s(z) = \sum_{j=0}^s a_j z^j$, then for $0 \le s \le n-1$,

$$\langle \varphi_s, h\rangle = \langle 1, h\rangle \sum_{j=0}^{s} \overline{a_j \lambda^j} = \langle 1, h\rangle \overline{\varphi_s(\lambda)} \tag{3.1.7}$$

If we express $h(z)$ using Fourier series,

$$h(z) = \sum_{j=0}^{n-1} \langle \varphi_j, h\rangle \varphi_j(z) = \sum_{j=0}^{n-1} \langle 1, h\rangle \overline{\varphi_j(\lambda)} \varphi_j(z) = \langle 1, h\rangle K_{n-1}(z, \lambda) \tag{3.1.8}$$

4. Simple Zeros Let λ and $h(z)$ be defined as above. By (3.1.8), $\langle h, 1\rangle = 0$ implies $h = 0$, hence $\langle h, 1\rangle \ne 0$. In addition, $\varphi_0 = 1$ implies $K_{n-1}(\lambda, \lambda) > 0$. Therefore $h(\lambda) = \langle h, 1\rangle K_{n-1}(\lambda, \lambda) \ne 0$. This shows that zeros of paraorthogonal polynomials are simple.

5. Linear Independence The argument for property (3) above also tells us that a paraorthogonal polynomial could vanish at one arbitrary point on the unit circle, and that particular zero fixes the remaining ones. Therefore, two paraorthogonal polynomials of the same degree are linearly independent if and only if all their zeros are distinct.

For a more comprehensive introduction to orthogonal polynomials and paraorthogonal polynomials, the reader should refer to [48, 46, 50].

3.2 Equivalent Definitions of $h_n(z)$

Fix $\lambda \in \partial\mathbb{D}$. We define the family of paraorthogonal polynomials $(h_n(z,\lambda))_n$ as follows:

$$h_n(z,\lambda) := (1-\overline{\lambda}z)K_{n-1}(z,\lambda) \tag{3.2.1}$$

We will soon see that there are three equivalent definitions of $h_n(z)$ by the Christoffel–Darboux formula. The formula says that for $\overline{y}z \neq 1$, the reproducing kernel $K_{n-1}(z,y)$ could be expressed in the following ways:

$$K_{n-1}(z,y) = \frac{\overline{\varphi_n^*(y)}\varphi_n^*(z) - \overline{\varphi_n(y)}\varphi_n(z)}{1-\overline{y}z} \tag{3.2.2}$$

$$= \frac{\overline{\varphi_{n-1}^*(y)}\varphi_{n-1}^*(z) - \overline{y}z\overline{\varphi_{n-1}(y)}\varphi_{n-1}(z)}{1-\overline{y}z} \tag{3.2.3}$$

Hence, we have the following three equivalent definitions of $h_n(z, \lambda)$:

$$\begin{aligned} h_n(z) &= (1 - \overline{\lambda} z) \sum_{j=0}^{n-1} \varphi_j(z) \overline{\varphi_j(\lambda)} && (3.2.4) \\ &= \overline{\varphi_n^*(\lambda)} \varphi_n^*(z) - \overline{\varphi_n(\lambda)} \varphi_n(z) && (3.2.5) \\ &= \overline{\varphi_{n-1}^*(\lambda)} \varphi_{n-1}^*(z) - z \overline{\lambda} \overline{\varphi_{n-1}(\lambda)} \varphi_{n-1}(z) && (3.2.6) \end{aligned}$$

By rewriting (3.2.6) in the form of (2.0.1),

$$h_n(z) = -\overline{\lambda \varphi_{n-1}(\lambda)} \left(z\varphi_{n-1}(z) - \lambda \frac{\overline{\varphi_{n-1}^*(\lambda)}}{\overline{\varphi_{n-1}(\lambda)}} \varphi_{n-1}^*(z) \right) \tag{3.2.7}$$

we see that the coefficients (β_{n-1}) of this particular family of paraorthogonal polynomials are

$$\beta_{n-1}(h_n) = \overline{\lambda} \frac{\varphi_{n-1}^*(\lambda)}{\varphi_{n-1}(\lambda)} \tag{3.2.8}$$

3.3 Paraorthogonal Polynomials of the Second Kind

Paraorthogonal polynomials of the second kind arise from orthogonal polynomials of the second kind, namely $\psi_k(z)$, which are orthogonal polynomials associated to the measure ν with Verblunsky coefficients

$$\alpha_n(d\nu) = -\alpha_n(d\mu) \tag{3.3.1}$$

The existence of the measure is guaranteed by Verblunsky's theorem which says that for any given sequence of complex numbers inside $\mathbb{D}$, there corresponds a measure on the unit circle with such as Verblunsky coefficients.

With the same λ as we used to define $h_n(z, \lambda)$, we define our *Paraorthogonal Polynomials of the Second Kind* s_n as follows:

$$s_n(z) = \overline{\varphi_{n-1}^*(\lambda)}\psi_{n-1}^*(z) + z\overline{\lambda\varphi_{n-1}(\lambda)}\psi_{n-1}(z) \tag{3.3.2}$$

If we rewrite (3.3.2) in the form of (3.2.7)

$$s_n(z) = \overline{\lambda\varphi_{n-1}(\lambda)}\left(z\psi_{n-1}(z) + \lambda\frac{\overline{\varphi_{n-1}^*(\lambda)}}{\overline{\varphi_{n-1}(\lambda)}}\psi_{n-1}^*(z)\right) \tag{3.3.3}$$

we see that the β_n coefficient of this family of paraorthogonal polynomials $(s_n)_n$ is given by:

$$\beta_n(s_n) = -\beta_n(h_n) \tag{3.3.4}$$

As in the case of h_n, we shall see that there are three equivalent definitions of s_n by means of the *Mixed Christoffel–Darboux Formulae*, which state that:

$$\overline{\varphi_{n-1}^*(y)}\psi_{n-1}^*(z) + z\overline{y}\,\overline{\varphi_{n-1}(y)}\psi_{n-1}(z) = \overline{\varphi_n^*(y)}\psi_n^*(z) + \overline{\varphi_n(y)}\psi_n(z) \tag{3.3.5}$$

$$\sum_{j=0}^{n-1} \overline{\varphi_j(y)}\psi_j(z) = \frac{2 - \overline{\varphi_n^*(y)}\psi_n^*(z) - \overline{\varphi_n(y)}\psi_n(z)}{1 - \overline{y}z} \quad \text{for } y \neq z \tag{3.3.6}$$

The reader should refer to Chapter 3.2 of [46] for the proof.

By (3.3.5) and (3.3.6), $s_n(z, \lambda)$ has the following three equivalent definitions:

$$s_n(z) = \overline{\varphi_{n-1}^*(\lambda)}\psi_{n-1}^*(z) + z\overline{\lambda}\,\overline{\varphi_{n-1}(\lambda)}\psi_{n-1}(z) \tag{3.3.7}$$

$$= \overline{\varphi_n^*(\lambda)}\psi_n^*(z) + \overline{\varphi_n(\lambda)}\psi_n(z) \tag{3.3.8}$$

$$= -(1 - \overline{\lambda}z) \sum_{j=0}^{n-1} \overline{\varphi_j(\lambda)}\psi_j(z) + 2 \tag{3.3.9}$$

Chapter 4

Proofs

4.1 Proof of Theorem 2.0.2

Before we start the proof, we refer to a theorem about zeros of h_n in a gap of the measure:

Theorem 4.1.1 (Corollary 2 of [11], Theorem 2 of [22], Theorem 2.3 of [46])**.** *Let an arc $\Gamma = (\alpha, \beta)$ on $\partial\mathbb{D}$ be a gap in $\operatorname{supp}(d\mu)$, that is, $\operatorname{supp}(d\mu) \cap \Gamma = \emptyset$ and α goes to β counterclockwise. Then for each n, the paraorthogonal polynomial h_n has at most one zero in $\overline{\Gamma} = [\alpha, \beta]$.*

If λ is in a gap Γ, since λ is zero of all h_n, by Theorem 4.1.1 above, there are no other zeros of h_n or h_{n+1} in Γ. In other words, if z_0 and λ are in the same gap, in a radius $\delta = \operatorname{dist}(z_0, \operatorname{supp}(d\mu))$ around z_0 there could be no zeros other than λ. Since $\delta > \rho$, Theorem 2.0.2 holds. Hence if λ is in a gap, it suffices to look at the case when z_0 sits in gaps other than Γ. In such a situation, $|z_0 - \lambda| \geq \operatorname{dist}(z_0, \operatorname{supp}(d\mu))$.

However, if λ is not in a gap, that is, λ is in the support of a measure, then clearly $|z_0 - \lambda| \geq \operatorname{dist}(z_0, \operatorname{supp}(d\mu))$.

Without loss of generality, we may assume that $|z_0 - \lambda| \geq \delta$ in this section.

We shall divide the proof into two lemmas:

Lemma 4.1.1.

$$\left| \frac{h_i(z_0)}{K_{n-1}(z_0, z_0)^{1/2}} \right| \geq \frac{1}{4} |\varphi_n(\lambda)| \delta^2 \tag{4.1.1}$$

where $i = \begin{cases} n & \textit{if } |h_{n+1}(z_0)| \leq |h_n(z_0)| \\ n+1 & \textit{if } |h_n(z_0)| \leq |h_{n+1}(z_0)| \end{cases}$

Proof. Suppose $|h_{n+1}(z_0)| \leq |h_n(z_0)|$.

First, we give a bound for the $L^2(\mu)$ norm of $\|(z_0 - \cdot) K_{n-1}(z_0, \cdot)\|$.

By the parallelogram equality and the fact that $|\varphi_n^*(z_0)| = |\varphi_n(z_0)|$,

$$
\begin{aligned}
\|(z_0 - \cdot) \quad & K_{n-1}(z_0, \cdot)\|^2 \\
&= \|\overline{\varphi_n^*(\cdot)}\varphi_n^*(z_0) - \overline{\varphi_n(\cdot)}\varphi_n(z_0)\|^2 \\
&\leq 2|\varphi_n^*(z_0)|^2 + 2|\varphi_n(z_0)|^2 \\
&= 4\left|\frac{h_{n+1}(z_0) - h_n(z_0)}{(z_0 - \lambda)\overline{\varphi_n(\lambda)}}\right|^2 \qquad (4.1.2) \\
&\leq \frac{4|h_{n+1}(z_0)|^2 + 4|h_n(z_0)|^2 + 8|h_{n+1}(z_0)h_n(z_0)|}{|\varphi_n(\lambda)|^2|z_0 - \lambda|^2} \\
&\leq \frac{16|h_n(z_0)|^2}{|\varphi_n(\lambda)|^2|z_0 - \lambda|^2}
\end{aligned}
$$

Remark: Note that $h_{n+1}(z_0) - h_n(z_0) = (1 - \overline{\lambda}z_0)\overline{\varphi_n(\lambda)}\varphi_n(z_0)$, so it is impossible that both $h_{n+1}(z_0)$ and $h_n(z_0)$ are zero because φ has zeros inside the unit circle.

On the other hand, we observe that

$$
\|K_{n-1}(z_0, \cdot)\| = \left(\int_{\partial\mathbb{D}} K_{n-1}(z_0, y)\overline{K_{n-1}(z_0, y)}d\mu(y)\right)^{1/2} = K_{n-1}(z_0, z_0)^{1/2} \qquad (4.1.3)
$$

Hence,

$$
\|(z_0 - \cdot)K_{n-1}(z_0, \cdot)\|^2 \geq \mathrm{dist}(z_0, \mathrm{supp}(d\mu))^2 K_{n-1}(z_0, z_0) \qquad (4.1.4)
$$

As a result,

$$\text{dist}(z_0, \text{supp}(d\mu))^2 K_{n-1}(z_0, z_0) \leq \frac{16|h_n(z_0)|^2}{|\varphi_n(\lambda)|^2 |z_0 - \lambda|^2} \tag{4.1.5}$$

This proves the case when $|h_{n+1}(z_0)| \leq |h_n(z_0)|$.

Now suppose $|h_{n+1}(z_0)| \leq |h_n(z_0)|$. The proof could be carried out in a similar manner, only that after (4.1.2) all appearances of h_n will be replaced by h_{n+1}.

□

Lemma 4.1.2. *Suppose τ is a zero of h_n which is distinct from λ. Let $T = \text{dist}(\tau, \text{supp}(d\mu))$, then*

$$|z_0 - \tau| \geq \frac{|h_n(z_0)|}{K_{n-1}(z_0, z_0)^{1/2} \|h_n\|} \quad T \tag{4.1.6}$$

Proof. Since τ is a zero of h_n, $g(z) = \frac{h_n(z)}{(z-\tau)}$ is a polynomial of degree $n-1$, so we can express it as

$$\frac{h_n(z)}{(z-\tau)} = \int_{\partial\mathbb{D}} K_{n-1}(z, y) g(y) d\mu(y) \tag{4.1.7}$$

By the Schwarz inequality,

$$\left| \frac{h_n(z_0)}{(z_0 - \tau)} \right| \leq \|K_{n-1}(z_0, \cdot)\| \|g\| = K_{n-1}(z_0, z_0)^{1/2} \|g\| \tag{4.1.8}$$

Also note that $\|g\| = \left\| \frac{h_n(z)}{(z-\tau)} \right\| \leq \frac{\|h_n\|}{T}$. Therefore,

$$|z_0 - \tau| \geq \frac{|h_n(z_0)|}{K_{n-1}(z_0, z_0)^{1/2}\|h_n\|} \quad T \tag{4.1.9}$$

□

Proof of Theorem 2.0.2. Notice that either one of the following must be true:

$$|h_{n+1}(z_0)| \leq |h_n(z_0)| \tag{4.1.10}$$

$$|h_n(z_0)| \leq |h_{n+1}(z_0)| \tag{4.1.11}$$

We observe that

$$\|h_n\| = \|\overline{\varphi_n^*(\lambda)}\varphi_n^*(y) - \overline{\varphi_n(\lambda)}\varphi_n(y)\|_{L^2(d\mu(y))} \leq 2|\varphi_n(\lambda)| \tag{4.1.12}$$

If (4.1.10) is true, combining this with Lemma 4.1.1 and Lemma 4.1.2, we obtain that:

$$|z_0 - \tau| \geq \left(\frac{\delta^2 |\varphi_n(\lambda)|}{4} \frac{1}{2|\varphi_n(\lambda)|} \right) T = \frac{\delta^2 T}{8} \tag{4.1.13}$$

Finally, by the triangle inequality,

$$T = \mathrm{dist}(\tau, \mathrm{supp}(d\mu)) \geq \mathrm{dist}(z_0, \mathrm{supp}(d\mu)) - |z_0 - \tau| = \delta - |z_0 - \tau| \tag{4.1.14}$$

This gives

$$|z_0 - \tau| \geq \frac{\delta^2(\delta - |z_0 - \tau|)}{8} \tag{4.1.15}$$

and the result follows.

On the other hand, if (4.1.11) is true, then instead of (4.1.12) we use the definition of h_{n+1} in (3.2.6) which will give the same bound of $\|h_{n+1}\|$ as in (4.1.12). Hence the same argument applies to h_{n+1}.

Now consider the special case where $L = \text{dist}(\lambda, \text{supp}(d\mu)) > 0$. Without loss of generality, suppose (4.1.10) is true. Since τ and λ are distinct zeros of h_n, we could apply a similar argument as in Lemma 4.1.2 to $\frac{h_n(z)}{(z-\tau)(z-\lambda)}$ and obtain the following

$$|z_0 - \tau||z_0 - \lambda| \;\; \geq \;\; \frac{|h_n(z_0)|}{K_{n-2}(z_0, z_0)^{1/2}\|h_n\|} TL \tag{4.1.16}$$

Since $K_{n-2}(z_0, z_0)^{1/2} \leq K_{n-1}(z_0, z_0)^{1/2}$, the desired inequality follows. Now we combine (4.1.16) with Lemma 4.1.1. The $|z_0 - \lambda|$ term cancels on both sides and it gives us

$$|z_0 - \tau| \geq \frac{\delta L T}{8} \tag{4.1.17}$$

Again, we use the triangle inequality on T and the result follows. Clearly, if (4.1.11) is true, we could still apply the same argument to h_{n+1}. □

4.2 Proof of Theorem 2.0.3

Proof. According to the definitions of φ_n^* and ψ_n^*,

$$s_n(z) = \overline{\lambda^n} z^n \varphi_n(\lambda)\overline{\psi_n(z)} + \overline{\varphi_n(\lambda)}\psi_n(z) \tag{4.2.1}$$

$$h_n(z) = \overline{\lambda^n} z^n \varphi_n(\lambda)\overline{\varphi_n(z)} - \overline{\varphi_n(\lambda)}\varphi_n(z) \tag{4.2.2}$$

If we define for $z \in \partial\mathbb{D}$

$$\sigma_n(z) := \frac{s_n(z)}{(\overline{\lambda} z)^{n/2}} \tag{4.2.3}$$

$$\eta_n(z) := \frac{h_n(z)}{i(\overline{\lambda} z)^{n/2}} \tag{4.2.4}$$

with $\mathrm{Arg}((\overline{\lambda}\mathrm{z})^{1/2}) \in [0, \pi)$, then σ_n and η_n are real-valued C^∞ functions and they have the same zeros as s_n and h_n respectively.

To prove the interlacing condition of Theorem 2.0.3, it suffices to prove the following:

$$\frac{d\eta_n(e^{i\theta})}{d\theta}\sigma_n(e^{i\theta}) < 0 \text{ at every zero } e^{i\theta} \text{ of } \eta_n(z) \tag{4.2.5}$$

We shall prove condition (4.2.5) for $n+1$.

Suppose ζ is a zero of h_{n+1}. By (3.1.5), h_{n+1} could be expressed by the reproducing kernel. Hence η_{n+1} can be represented as

$$\eta_{n+1}(z) = \frac{1}{i(\overline{\lambda} z)^{(n+1)/2}} \frac{-\overline{\lambda\varphi_n(\lambda)}}{\overline{\varphi_n(\zeta)}}(z-\zeta)\sum_{j=0}^{n} \varphi_j(z)\overline{\varphi_j(\zeta)} \tag{4.2.6}$$

The constant $\frac{-\overline{\lambda\varphi_n(\lambda)}}{\overline{\varphi_n(\zeta)}}$ is obtained by comparing the leading coefficients of the right hand side of (4.2.6) and that of h_{n+1} when expressed in terms of (3.2.6).

As a result, the derivative of η_{n+1} at ζ is

$$\begin{aligned} \frac{d\eta_{n+1}}{dz}(\zeta) &= \lim_{z\to\zeta} \frac{\eta_{n+1}(z)-\eta_{n+1}(\zeta)}{z-\zeta} \\ &= \lim_{z\to\zeta} \frac{\eta_{n+1}(z)}{z-\zeta} \\ &= \frac{-\overline{\lambda\varphi_n(\lambda)}}{i\overline{\varphi_n(\zeta)}} \left(\frac{\lambda}{\zeta}\right)^{\frac{n+1}{2}} K_n(\zeta,\zeta) \end{aligned} \tag{4.2.7}$$

Let $\zeta = e^{i\theta}$ and $z = e^{i\omega}$. By the chain rule,

$$\begin{aligned} \frac{d\eta_{n+1}}{d\omega}(\theta) &= i\zeta \frac{d\eta_{n+1}}{dz}(\zeta) \\ &= -\frac{\overline{\varphi_n(\lambda)}}{\overline{\varphi_n(\zeta)}} \left(\frac{\lambda}{\zeta}\right)^{\frac{n-1}{2}} K_n(\zeta,\zeta) \end{aligned} \tag{4.2.8}$$

Now we go back to $\frac{d\eta_n(e^{i\theta})}{d\theta}\sigma_n(e^{i\theta})$ and compute:

$$\begin{aligned} &\frac{d\eta_{n+1}(e^{i\theta})}{d\theta}\sigma_{n+1}(e^{i\theta}) \\ = \quad & -\frac{\overline{\varphi_n(\lambda)}}{\overline{\varphi_n(\zeta)}} \left(\frac{\lambda}{\zeta}\right)^n K_n(\zeta,\zeta) \left(\overline{\varphi_n^*(\lambda)}\psi_n^*(\zeta) + \bar{\lambda}\zeta\overline{\varphi_n(\lambda)}\psi_n(\zeta)\right) \\ = \quad & -\left(\frac{\lambda}{\zeta}\right)^n K_n(\zeta,\zeta) \left(|\varphi_n(\lambda)|^2 \left(\frac{\zeta}{\lambda}\right)^n \frac{\overline{\psi_n(\zeta)}}{\overline{\varphi_n(\zeta)}} + \bar{\lambda}\zeta \frac{\overline{\varphi_n(\lambda)}}{\overline{\varphi_n(\zeta)}}\overline{\varphi_n(\lambda)}\psi_n(\zeta)\right) \end{aligned} \tag{4.2.9}$$

Recall that $\eta_{n+1}(\zeta) = 0$, which implies that

$$\frac{\overline{\varphi_n(\lambda)}}{\overline{\varphi_n(\zeta)}} = \frac{\varphi_n(\lambda)}{\varphi_n(\zeta)} \left(\frac{\zeta}{\lambda}\right)^{n-1} \tag{4.2.10}$$

We then apply this to the second part of the summand in (4.2.9):

$$
\begin{aligned}
(4.2.9) &= -\left(\frac{\lambda}{\zeta}\right)^n K_n(\zeta,\zeta)\left(|\varphi_n(\lambda)|^2\left(\frac{\zeta}{\lambda}\right)^n\frac{\overline{\psi_n(\zeta)}}{\overline{\varphi_n(\zeta)}}+\left(\frac{\zeta}{\lambda}\right)^n\frac{\varphi_n(\lambda)}{\varphi_n(\zeta)}\overline{\varphi_n(\lambda)}\psi_n(\zeta)\right)\\
&= -K_n(\zeta,\zeta)|\varphi_n(\lambda)|^2\left(\frac{\overline{\psi_n(\zeta)}}{\overline{\varphi_n(\zeta)}}+\frac{\psi_n(\zeta)}{\varphi_n(\zeta)}\right)\\
&= -K_n(\zeta,\zeta)\left|\frac{\varphi_n(\lambda)}{\varphi_n(\zeta)}\right|^2\left(\overline{\psi_n(\zeta)}\varphi_n(\zeta)+\overline{\varphi_n(\zeta)}\psi_n(\zeta)\right) \qquad (4.2.11)
\end{aligned}
$$

Now we use a formula that relates φ_n and ψ_n (see Chapter 3.2 in [46]):

$$\overline{\psi_n(z)}\varphi_n(z)+\overline{\varphi_n(z)}\psi_n(z)=2 \text{ in } \partial\mathbb{D} \qquad (4.2.12)$$

We apply (4.2.12) to (4.2.11). This gives us the result that at any zero ζ of η_{n+1}:

$$\frac{d\eta_{n+1}(e^{i\theta})}{d\theta}\sigma_{n+1}(e^{i\theta})=(4.2.9)=-2K_n(\zeta,\zeta)\left|\frac{\varphi_n(\lambda)}{\varphi_n(\zeta)}\right|^2<0 \qquad (4.2.13)$$

The interlacing theorem is proven. □

4.3 Proof of Lemma 2.0.1

We prove Lemma 2.0.1 by stating several lemmas which are similar to those in the proof of Theorem 2.0.2.

Lemma 4.3.1. *Suppose* $\tilde{\delta}=\operatorname{dist}(z_0,\operatorname{supp}(d\nu))>0$ *and* $\tilde{K}_n(x,y)=\sum_{j=0}^{n}\psi_j(x)\overline{\psi_j(y)}$

is the reproducing kernel with respect to the measure ν. Then

$$\left|\frac{s_i(z_0)}{\tilde{K}_{n-1}(z_0, z_0)^{1/2}}\right| \geq \frac{1}{4}|\varphi_n(\lambda)||z_0 - \lambda|\tilde{\delta} \tag{4.3.1}$$

where $i = \begin{cases} n & \text{if} \quad |s_{n+1}(z_0)| \leq |s_n(z_0)| \\ n+1 & \text{if} \quad |s_n(z_0)| \leq |s_{n+1}(z_0)| \end{cases}.$

Proof. The proof is essentially the same as the one of Lemma 4.1.1, except for a few differences. The L^2 norm here refers to the one taken with respect to ν and h_n is replaced by s_n.

It is also worth noting that by the definition of s_n in (3.3.9),

$$s_{n+1}(z) - s_n(z) = -(1 - \overline{\lambda} z)\overline{\varphi_n(\lambda)}\psi_n(z) \neq 0 \text{ on } \partial\mathbb{D} \tag{4.3.2}$$

As a result,

$$|\psi_n(z_0)| = \left|\frac{s_{n+1}(z_0) - s_n(z_0)}{(z_0 - \lambda)\varphi_n(\lambda)}\right| \tag{4.3.3}$$

which allows us to proceed in the same way as in the proof of Lemma 4.1.1.

□

Lemma 4.3.2. *Suppose $\tilde{\tau}$ is a zero of s_n. Let $\widetilde{T} = \operatorname{dist}(\tilde{\tau}, \operatorname{supp}(d\nu))$, then*

$$|z_0 - \tilde{\tau}| \geq \frac{|s_n(z_0)|}{\tilde{K}_{n-1}(z_0, z_0)^{1/2}\|s_n\|_{L^2(d\nu)}}\widetilde{T} \tag{4.3.4}$$

The proof of this lemma is omitted because it resembles that of Lemma 4.1.2.

Finally, we state the following lemma relating the support of μ and ν:

Lemma 4.3.3. *Suppose z_0 is an isolated point in the support of μ. Then*

$$\tilde{\delta} = \operatorname{dist}(z_0, \operatorname{supp}(d\nu)) > 0 \tag{4.3.5}$$

The reader may refer to Chapter 3.2, p. 225 of [46] for the proof.

Next, we are going to finish the proof of Lemma 2.0.1.

Proof. Suppose z_0 is an isolated point in the support of $d\mu$ which is distinct from λ. By Lemma 4.3.3, $\operatorname{dist}(z_0, \operatorname{supp}(d\nu)) > 0$.

Either $|s_n(z_0)| \geq |s_{n+1}(z_0)|$ or $|s_n(z_0)| \leq |s_{n+1}(z_0)|$ is true. Without loss of generality, we assume that $|s_n(z_0)| \geq |s_{n+1}(z_0)|$ and use Lemma 4.3.1.

Furthermore, we observe that

$$\|s_n\| \leq 2|\varphi_n(\lambda)|\|\psi_n\|_{L^2(d\nu)} = 2|\varphi_n(\lambda)| \tag{4.3.6}$$

Then we combine these results to get

$$|z_0 - \tilde{\tau}| \geq \frac{|z_0 - \lambda|\tilde{\delta}\widetilde{T}}{8} \tag{4.3.7}$$

Finally, we apply the triangle inequality to $\widetilde{T}$:

$$\widetilde{T} = \operatorname{dist}(\tilde{\tau}, \operatorname{supp}(d\nu)) \geq \operatorname{dist}(z_0, \operatorname{supp}(d\nu)) - |z_0 - \tilde{\tau}| = \tilde{\delta} - |z_0 - \tilde{\tau}| \tag{4.3.8}$$

This gives us the following inequality which finishes the proof:

$$|z_0 - \tilde{\tau}| \geq \frac{\tilde{\delta}^2 |z_0 - \lambda|}{8 + |z_0 - \lambda|\tilde{\delta}} \tag{4.3.9}$$

□

4.4 Proof of Theorem 2.0.4

Proof. By Lemma 2.0.1, inside the ball $B(z_0, \tilde{\rho})$ either s_n or s_{n+1} (or both) has no zero inside, with $\tilde{\rho}$ given by (4.3.9) above. Without loss of generality, we assume that s_n does not have zeros inside. By Theorem 2.0.3 the zeros of h_n and s_n interlace, therefore h_n cannot have more than two zeros inside $B(z_0, \tilde{\rho})$. □

Part III

Point Perturbation

Chapter 5

Proof of the Point Mass Formula

Most of the proof of Theorem 2.0.7 (Lemma 5.0.1, Lemma 5.0.2 and Theorem 5.0.1) follows the methods developed by Simon in the proof of Theorem 10.13.7 in [47]. The proof is concluded by a few observations of ours using the Christoffel–Darboux formula [55].

Lemma 5.0.1. *Let* $\beta_{jk} = \langle \Phi_j(d\mu), \Phi_k(d\mu) \rangle_{d\nu}$. *Then*

$$\Phi_n(d\nu)(z) = \frac{1}{D^{(n-1)}} \begin{vmatrix} \beta_{00} & \beta_{0\,1} & \dots & \beta_{0\,n} \\ \vdots & & & \vdots \\ \beta_{n-1\,0} & \beta_{n-1\,1} & \dots & \beta_{n-1\,n} \\ \Phi_0(d\mu) & \dots & \dots & \Phi_n(d\mu) \end{vmatrix} \tag{5.0.1}$$

where

$$D^{(n-1)} = \begin{vmatrix} \beta_{0\,0} & \beta_{0\,1} & \cdots & \beta_{0\,n-1} \\ \vdots & & & \vdots \\ \beta_{n-1\,0} & \beta_{n-1\,1} & \cdots & \beta_{n-1\,n-1} \end{vmatrix} \tag{5.0.2}$$

Proof. Let $\tilde{\Phi}_n(d\nu)$ be the right hand side of (5.0.1). We observe that the inner product $\langle \Phi_j(d\mu), \tilde{\Phi}_n(d\nu) \rangle_{d\nu}$ is zero for $j = 0, 1, \ldots, n-1$ as the last row and the j-th row of the determinant are the same. By expanding in minors, we see that the leading coefficient of $\tilde{\Phi}_n(d\nu)$ in (5.0.1) is one. In other words, $\tilde{\Phi}_n(d\nu)$ is an n-th degree monic polynomial which is orthogonal to $1, z, \ldots, z^{n-1}$ with respect to $\langle\,,\,\rangle_{d\nu}$, hence $\tilde{\Phi}_n(d\nu)$ equals $\Phi_n(d\nu)$. □

Lemma 5.0.2. *Let C be the following $(n+1) \times (n+1)$ matrix*

$$\begin{pmatrix} A & v \\ w & \beta \end{pmatrix} \tag{5.0.3}$$

where A is an $n \times n$ matrix, β is in $\mathbb{C}$, v is the column vector $(v_0, v_1, \ldots, v_{n-1})^T$ and w is the row vector $(w_0, w_1, \ldots, w_{n-1})$. If $\det(A) \neq 0$, we have

$$\det(C) = \det(A) \left(\beta - \sum_{0 \le j,k \le n-1} w_k v_j (A^{-1})_{jk} \right) \tag{5.0.4}$$

Proof. We expand in minors, starting from the bottom row to get

$$\det(C) = \beta \det(A) + \sum_{0 \le j,k \le n-1} w_k v_j (-1)^{j+k+1} \det(\tilde{A}_{jk}) \tag{5.0.5}$$

where $\tilde{A}_{jk}$ is the matrix A with the j-th row and k-th column removed.

By Cramer's rule, since $\det(A) \neq 0$,

$$\tilde{A}_{jk} = (-1)^{j+k} \det(A)(A^{-1})_{jk} \tag{5.0.6}$$

proving Lemma 5.0.2. □

Next, we are going to prove the following formula by Simon [47]:

Theorem 5.0.1. *The Verblunsky coefficient of* $d\nu$ (*as defined in* (2.0.10)) *is given by*

$$\alpha_n(d\nu) = \alpha_n - q_n^{-1} \gamma \overline{\varphi_{n+1}(\zeta)} \left(\sum_{j=0}^{n} \alpha_{j-1} \frac{\|\Phi_{n+1}\|}{\|\Phi_j\|} \varphi_j(\zeta) \right) \tag{5.0.7}$$

where

$$K_n(\zeta) = \sum_{j=0}^{n} |\varphi_j(\zeta)|^2 \tag{5.0.8}$$

$$q_n = (1-\gamma) + \gamma K_n(\zeta) \tag{5.0.9}$$

$$\alpha_{-1} = -1 \tag{5.0.10}$$

and all objects without the label $(d\nu)$ *are associated with the measure* $d\mu$.

Proof. Since $\alpha_{n-1}(d\nu) = -\overline{\Phi_n(0, d\nu)}$ and $\overline{\beta_{jk}} = \beta_{kj}$, by Lemma 5.0.1,

$$\alpha_{n-1}(d\nu) = \frac{1}{D^{(n-1)}} \begin{vmatrix} \beta_{0\,0} & \beta_{1\,0} & \dots & \beta_{n\,0} \\ \vdots & & & \vdots \\ \beta_{0\,n-1} & \beta_{1\,n-1} & \dots & \beta_{n\,n-1} \\ -1 & \alpha_0 & \dots & \alpha_{n-1} \end{vmatrix} \tag{5.0.11}$$

Let C be the matrix with entries as in the determinant in (5.0.11) above. It could be expressed as follows

$$C = \begin{pmatrix} A & v \\ w & \alpha_{n-1} \end{pmatrix} \tag{5.0.12}$$

where A is the $n \times n$ matrix with entries $A_{jk} = \beta_{kj}$, v is the column vector $(\beta_{n0}, \dots, \beta_{nn-1})^T$ and w is the row vector $(-1, \alpha_0, \dots, \alpha_{n-2})$. Note that $\det(A) = D^{(n-1)}$, and it is real as A is Hermitian.

Now we use Lemma 5.0.2 to compute $det(C)$. To do that, we need to find out what A^{-1} is.

By the definition of ν,

$$A_{jk} = (1-\gamma)\|\Phi_k\|^2\delta_{kj} + \gamma\overline{\Phi_k(\zeta)}\Phi_j(\zeta) = \|\Phi_k\|\|\Phi_j\|M_{jk} \tag{5.0.13}$$

where

$$M_{jk} = (1-\gamma)\delta_{kj} + \gamma\overline{\varphi_k(\zeta)}\varphi_j(\zeta) \tag{5.0.14}$$

Observe that for any column vector $x = (x_0, x_1, \ldots, x_{n-1})^T$,

$$Mx = (1-\gamma)x + \gamma \left(\sum_{j=0}^{n-1} \varphi_j(\zeta) x_j \right) (\varphi_0(\zeta), \varphi_1(\zeta), \ldots, \varphi_0(\zeta))^T \qquad (5.0.15)$$

Therefore, if P_φ denotes the orthogonal projection onto the space spanned by the vector $\varphi = (\varphi_0(\zeta), \varphi_1(\zeta), \ldots, \varphi_0(\zeta))$, we can write

$$M = (1-\gamma)\mathbf{1} + \gamma K_{n-1} P_\varphi \qquad (5.0.16)$$

Hence, the inverse of M is

$$M^{-1} = (1-\gamma)^{-1}(\mathbf{1} - P_\varphi) + ((1-\gamma) + \gamma K_{n-1})^{-1} P_\varphi \qquad (5.0.17)$$

and the inverse of A is

$$A^{-1} = D^{-1} M^{-1} D^{-1} \qquad (5.0.18)$$

where $D_{ij} = \|\Phi_i\| \delta_{ij}$.

Recall that $v = (\beta_{n0}, \beta_{n1}, \ldots, \beta_{nn-1})^T$, which is a multiple of φ. Therefore,

$$(A^{-1}v)_j = ((1-\gamma) + \gamma K_{n-1})^{-1} \gamma \, \overline{\Phi_n(\zeta)} \, \|\Phi_j\|^{-1} \varphi_j(\zeta) \qquad (5.0.19)$$

(5.0.19), (5.0.11) and Lemma 5.0.2 then imply

$$\alpha_{n-1}(d\nu) = \alpha_{n-1} - ((1-\gamma) + \gamma K_{n-1})^{-1} \gamma \, \overline{\varphi_n(\zeta)} \left(\sum_{j=0}^{n-1} \alpha_{j-1} \frac{\|\Phi_n\|}{\|\Phi_j\|} \varphi_j(z_0) \right) \qquad (5.0.20)$$

This concludes the proof of Theorem 5.0.1. $\square$

Now we are going to prove Theorem 2.0.7.

Proof. First, observe that $\alpha_{j-1} = -\overline{\Phi_j(0)}$. Therefore, $\alpha_{j-1}/\|\Phi_j\| = -\overline{\varphi_j(0)}$. Second, observe that $\|\Phi_{n+1}\|$ is independent of j so it could be taken out from the summation. As a result, (5.0.7) in Theorem 5.0.1 becomes

$$\alpha_n(d\nu) = \alpha_n(d\mu) + q_n^{-1}\gamma\, \overline{\varphi_{n+1}(\zeta)}\, \|\Phi_{n+1}\| \left(\sum_{j=0}^{n} \overline{\varphi_j(0)} \varphi_j(\zeta) \right) \tag{5.0.21}$$

Then we use the Christoffel–Darboux formula, which states that for $x, y \in \mathbb{C}$ with $x\bar{y} \neq 1$,

$$(1 - \overline{x}y) \left(\sum_{j=0}^{n} \overline{\varphi_j(x)} \varphi_j(y) \right) = \overline{\varphi_n^*(x)} \varphi_n^*(y) - \overline{x}y \overline{\varphi_n(x)} \varphi_n(y) \tag{5.0.22}$$

Moreover, note that $q_n^{-1}\gamma = ((1-\gamma)\gamma^{-1} + K_n(\zeta))^{-1}$. Therefore, (5.0.21) could be simplified as follows

$$\alpha_n(d\nu) = \alpha_n + \frac{\overline{\varphi_{n+1}(\zeta)} \varphi_n^*(0) \varphi_n^*(\zeta)}{(1-\gamma)\gamma^{-1} + K_n(\zeta)} \|\Phi_{n+1}\| \tag{5.0.23}$$

Finally, observe that $\varphi_n^*(0) = \|\Phi_n\|^{-1}$ and that by (1.0.10), $\|\Phi_{n+1}\|/\|\Phi_n\| = (1 - |\alpha_n|^2)^{1/2}$. This completes the proof of Theorem 2.0.7. $\square$

Chapter 6

ℓ^2 Verblunsky Coefficients

From the *point mass formula* (2.0.11) we could make a few observations concerning successive Verblunsky coefficients $\alpha_{n+1}(d\nu)$ and $\alpha_n(d\nu)$: first, we use the fact that $\overline{\varphi_{n+1}(\zeta)} = \overline{\zeta^{n+1}}\varphi^*_{n+1}(\zeta)$ and rewrite the point mass formula as:

$$\alpha_n(d\nu) = \alpha_n + \frac{(1-|\alpha_n|^2)^{1/2}}{(1-\gamma)\gamma^{-1} + K_n(\zeta)}\overline{\zeta^{n+1}}\varphi^*_{n+1}(\zeta)\varphi^*_n(\zeta) \tag{6.0.1}$$

Let t_n be the tail term in the right hand side of (6.0.1) above. Suppose we can prove that $\varphi^*_n(\zeta)$ tends to some non-zero limit L as n tends to infinity, then $1/K_n = O(1/n)$. Hence,

$$\frac{1}{(1-\gamma)\gamma^{-1} + K_n(\zeta)} = \frac{1}{K_n(\zeta)} + O\left(\frac{1}{n^2}\right) \tag{6.0.2}$$

Besides, $(\alpha_n)_{n=0}^{\infty}$ is ℓ^2, therefore $(1-|\alpha_n|^2)^{1/2} \to 1$. As a result,

$$\alpha_n(d\nu) = \alpha_n + t_n \approx \alpha_n + \frac{\overline{\zeta^{n+1}} L^2}{n|L|^2} + o\left(\frac{1}{n}\right) \tag{6.0.3}$$

Indeed, we shall prove that if $\zeta t_{n+1} - t_n$ is summable, by Theorem 2.0.10, $\lim_{n\to\infty} \varphi_n^*(z, d\mu_1)$ exists away from $z = 1$. As a result, if we add another pure point to $d\mu_1$, we can use a similar argument to the one above and the point mass formula (2.0.11) to prove that $\alpha_n(d\nu)$ is the sum of $\alpha_n(d\mu_0)$ plus two tail terms and an error term.

In general, if we have a measure $d\mu_m$ as defined in (2.0.20), then we add one pure point after the other and use the point mass formula (2.0.11) inductively. Therefore, we shall be able to express $\alpha_n(d\mu_m)$ as the sum of $\alpha_n(d\mu_0)$ plus m tail terms, and an error term

$$\alpha_n(d\mu_m) = \alpha_n(d\mu_0) + t_{1,n} + t_{2,n} + \cdots + t_{m,n} + \text{error} \tag{6.0.4}$$

By an argument similar to the one above, we observe that $t_{j,n}$ is $O(1/n)$ and $z_j t_{j,n} - t_{j,n-1}$ is small. Of course, the 'smallness' has to be determined by rigorous computations that we shall present in the proof. Nonetheless, these observations led us to introduce the notion of generalized bounded variation W_m, and from that we could deduce that $\lim_{n\to\infty} \varphi_n^*(z, d\mu_m)$ exists.

6.1 Proof of Theorem 2.0.8

The technique used in this proof is a generalization of the one used in proving Theorem 2.0.10. It involves Prüfer variables which are defined as follows

Definition 6.1.1. *Suppose* $z_0 = e^{i\eta} \in \partial\mathbb{D}$ *with* $\eta \in [0, 2\pi)$. *Define the Prüfer variables by*

$$\Phi_n(z_0) = R_n(z_0)\exp(i(n\eta + \theta_n(z_0))) \tag{6.1.1}$$

where θ_n *is determined by* $|\theta_{n+1} - \theta_n| < \pi$. *Here,* $R_n(z) = |\Phi_n(z)| > 0$, θ_n *is real. By the fact that* $\Phi_n^*(z) = z^n\overline{\Phi_n(z)}$ *on* $\partial\mathbb{D}$, *(6.1.1) is equivalent to*

$$\Phi_n^*(z) = R_n(z)\exp(-i\theta_n) \tag{6.1.2}$$

Under such a definition,

$$\log\left(\frac{\Phi_{n+1}^*}{\Phi_n^*}\right) = \log(1 - \alpha_n\exp(i[(n+1)\eta + 2\theta_n])) \tag{6.1.3}$$

For simplicity, we let

$$a_n = \alpha_n\exp(i[(n+1)\eta + 2\theta_n]) \tag{6.1.4}$$

Now write $\log\Phi_{n+1}^*$ as a telescoping sum

$$\log\Phi_{n+1}^*(z) = \sum_{j=0}^{n}\left(\log\Phi_{j+1}^*(z) - \log\Phi_j^*(z)\right) = \sum_{j=0}^{n}\log\left(\frac{\Phi_{j+1}^*(z)}{\Phi_j^*(z)}\right) \tag{6.1.5}$$

Note that for $|w| \leq Q < 1$, there is a constant K such that

$$|\log(1-w) - w| \leq K|w|^2 \tag{6.1.6}$$

Together with (6.1.3), we have

$$\log(\Phi_{n+1}^*(z)) = -\sum_{j=0}^{n} (a_j + L(a_j)) \tag{6.1.7}$$

where $|L(a_j)| \leq K|a_j|^2$.

Recall that by assumption, $(\alpha_n(d\mu_0))_{n=0}^\infty$ is ℓ^2. Therefore, by (6.1.4), $(a_n)_{n=0}^\infty$ is also ℓ^2, thus $\sum_{j=0}^\infty L(a_j) < \infty$. As a result, in order to prove that $\lim_{n\to\infty} \Phi_n^*(z)$ exists, it suffices to prove that $\sum_{j=0}^\infty a_j$ exists.

Let

$$h_n^{(k)} = \sum_{j=0}^{n-1} \overline{\zeta_k}^j e^{ij\eta} = \frac{\overline{\zeta_k}^n e^{in\eta} - 1}{\overline{\zeta_k} e^{i\eta} - 1} \tag{6.1.8}$$

Then

$$h_{n+1}^{(k)} - h_n^{(k)} = \overline{\zeta_k}^n e^{in\eta} \tag{6.1.9}$$

$$\text{and} \quad |h_n^{(k)}| \leq 2|\overline{\zeta_k} e^{i\eta} - 1|^{-1} \tag{6.1.10}$$

Let $g_j = \eta + 2\theta_j$ and recall that $\alpha_n = \sum_{k=1}^p \beta_{n,k}$. By rearranging the order of summation, we get

$$S_n = \sum_{j=0}^{n} \alpha_j e^{i(j\eta+g_j)} = \sum_{j=0}^{n} \left(\sum_{k=1}^{p} \beta_{j,k} \right) e^{i(j\eta+g_j)} = \sum_{k=1}^{p} B_n^{(k)} \tag{6.1.11}$$

where

$$B_n^{(k)} = \sum_{j=0}^{n} \beta_{j,k} e^{i(j\eta+g_j)} \tag{6.1.12}$$

We are going to sum by parts by Abel's formula. Suppose $(a_j)_{j=0}^{\infty}$ is a sequence, we define

$$(\delta^+ a)_j = a_{j+1} - a_j \tag{6.1.13}$$

$$(\delta^- a)_j = a_j - a_{j-1} \tag{6.1.14}$$

Abel's formula states that

$$\sum_{j=0}^{n} (\delta^+ a)_j b_j = a_{n+1} b_n - a_0 b_{-1} - \sum_{j=0}^{n} a_j (\delta^- b)_j \tag{6.1.15}$$

Now we apply Abel's formula to $B_n^{(k)}$

$$\begin{aligned} B_n^{(k)} &= \sum_{j=0}^{n} (\delta^+ h^{(k)})_j ({\zeta_k}^j \beta_{j,k} e^{ig_j}) \\ &= h_{n+1}^{(k)} {\zeta_k}^n \beta_{n,k} e^{ig_n} - h_0^{(k)} \zeta_k \beta_{-1,k} e^{ig_{-1}} - \sum_{j=0}^{n} h_j^{(k)} \delta^- ({\zeta_k}^j \beta_{j,k} e^{ig_j})_j \end{aligned} \tag{6.1.16}$$

Note that the term $h_0 \zeta_k^{-1} \beta_{-1,k} e^{ig_{-1}}$ will be canceled in (6.1.16), without loss of generality we may assume it to be 0.

We want to obtain a bound for $B_n^{(k)}$. Observe that

$$|\beta_{n,k}| \leq \sum_{q=1}^{n} |\beta_{q,k} - \overline{\zeta_k}\beta_{q-1,k}| + |\beta_0| \leq D_k \tag{6.1.17}$$

where

$$D_k = \sum_{q=0}^{\infty} |\beta_{q,k} - \overline{\zeta_k}\beta_{q-1,k}| \tag{6.1.18}$$

is finite because $d\mu \in W_p(\zeta_1, \zeta_2, \ldots, \zeta_p)$.

Next, we use the triangle inequality and $|e^{ix} - e^{iy}| \leq |x - y|$ to obtain

$$\begin{aligned} |\delta^-(\zeta_k^j \beta_{j,k} e^{ig_j})_j| &\leq |\beta_{j,k}(e^{ig_j} - e^{ig_{j-1}})| + |\zeta_k \beta_{j,k} - \beta_{j-1,k}| \\ &\leq |\beta_{j,k}(\theta_j - \theta_{j-1})| + |\zeta_k \beta_{j,k} - \beta_{j-1,k}| \end{aligned} \tag{6.1.19}$$

It has been proven for Prüfer variables (see Corollary 10.12.2 of [47]) that

$$|\theta_{n+1} - \theta_n| < \frac{\pi}{2}|\alpha_n|(1 - |\alpha_n|)^{-1} \tag{6.1.20}$$

Now recall our assumption that for $1 \leq k \leq p$, $(\beta_{n,k})_{n=0}^{\infty}$ is ℓ^2, therefore $\beta_{n,k} \to 0$, $\alpha_n \to 0$, which implies $Q = \sup_n |\alpha_n| < 1$ and $C = \sup_n (1 - |\alpha_n|)^{-1} = (1 - Q)^{-1}$. For any n we have

$$|B_{n,k}| \leq |\zeta_k e^{i\eta} - 1|^{-1} \left(2D_k + \frac{\pi}{2} \sum_{j=0}^{\infty} |\beta_{j+1,k}||\alpha_j|(1 - Q)^{-1} \right) < \infty \tag{6.1.21}$$

It follows that $\sup_n |S_n| < \infty$. This proves (2.0.17).

The computations above also show that the sum in the right hand side of (6.1.16) is absolutely convergent as $n \to \infty$ and the convergence is uniform on any compact subset of $\partial\mathbb{D}\backslash\{\zeta_1, \zeta_2, \dots, \zeta_p\}$. Therefore, $\lim_{j\to\infty} \beta_{j,k} = 0$ for all $1 \le k \le p$ implies that $\lim_{n\to\infty} B_{n,k}$ exists. Thus $\lim_{n\to\infty} S_n$ exists and is finite. This proves (2.0.19).

Moreover, for each fixed k, $(\beta_{n,k})_{n=0}^{\infty}$ is ℓ^2, $(\alpha_n)_{n=0}^{\infty}$ is also ℓ^2, hence the Szegő function $D(z)$ exists and it has boundary values a.e.. Now decompose $d\mu = w(\theta)\frac{d\theta}{2\pi} + d\mu_s$. It is well-known that $\Phi_n^* \to D(0)D^{-1}$ in $L^2(w(\theta)\frac{d\theta}{2\pi})$. Since $\Phi_n^* \to \Phi_\infty^*$ uniformly on any compact subset of $\partial\mathbb{D}\backslash\{\zeta_1, \zeta_2, \dots, \zeta_p\}$, the limit also converges in the L^2-sense. Besides, it is well known that $D(0) = \lim_{n\to\infty} \|\Phi_n\| = \prod_{n=0}^{\infty}(1 - |\alpha_n|^2)^{1/2}$. Hence,

$$\Phi_\infty^*(z) = D(0)D^{-1}(z) \tag{6.1.22}$$

$$\varphi_\infty^*(z) = D^{-1}(z) \tag{6.1.23}$$

on $\partial\mathbb{D}\backslash\{\zeta_1, \zeta_2, \dots, \zeta_p\}$.

6.2 Proof of Theorem 2.0.9

We proceed by induction.

Base Case Let any object without the label $(d\mu_1)$ be associated with the measure $d\mu_0$. First, we start by considering adding one pure point $z_1 = e^{i\omega_1} \in \partial\mathbb{D}$, $\omega_1 \neq 1$, to $d\mu_0 \in W_1(1)$ which has ℓ^2 Verblunsky coefficients.

Define $\tilde{\xi}_n(d\mu_1)$ as

$$\tilde{\xi}_n(d\mu_1) = \frac{(1-|\alpha_n|^2)^{1/2}}{(1-\gamma)\gamma^{-1} + K_n(z_1)} \overline{\varphi_{n+1}(z_1)} \varphi_n^*(z_1) \tag{6.2.1}$$

where $\alpha_j = \alpha_j(d\mu_0)$ and $(\Phi_n)_{n=0}^{\infty}$ is the family of orthogonal polynomials for $d\mu_0$. We want to simplify $\tilde{\xi}_n(d\mu_0)$.

Since $d\mu_0 \in W_1(1)$ and $\sum_{j=0}^{\infty} |\alpha_j|^2 < \infty$, by Theorem 2.0.8, $\lim_{n\to\infty} \varphi_n^*(z_1) = D(z_1)^{-1}$, which implies $1/K_n(z_1) = O(1/n)$. Hence,

$$\tilde{\xi}_n(d\mu_1) = \frac{(1-|\alpha_n|^2)^{1/2}}{K_n(z_1)} \overline{\varphi_{n+1}(z_1)} \varphi_n^*(z_1) + O\left(\frac{1}{n^2}\right) \tag{6.2.2}$$

Moreover, $\overline{\varphi_{n+1}(z_1)} = \overline{z_1^{n+1}} \varphi_{n+1}^*(z_1)$. We can further simplify and obtain

$$\alpha_n(d\mu_1) = \alpha_n + \overline{z_1}^{n+1} \frac{D(z_1)^{-2}}{|D(z_1)|^{-2}} \frac{1}{n} + o\left(\frac{1}{n}\right) \tag{6.2.3}$$

Let $c_1 = \overline{z_1} D(z_1)^2 / |D(z_1)|^2$. This proves (2.0.22) for $m = 1$.

Remark: Note that the error term in the right hand side of (6.2.3) is dependent on γ_1. This is because as $\gamma_0 \to 0$, $d\mu_1 \to d\mu_0$ weakly, which implies that for each n, $\alpha_n(d\mu_1) \to \alpha_n(d\mu_0)$. Since the tail term $\overline{z_1}^{n+1} \frac{D(z_1)^{-2}}{|D(z_1)|^{-2}} \frac{1}{n}$ in (6.2.3) is independent of γ_1, if the error term is also independent of γ_1, then $\alpha_n(d\mu_1) \not\to \alpha_n(d\mu_0)$.

It remains to show the claimed properties of $\Phi_n(d\mu_1)$. To do that, it suffices to show that $(\alpha_n(d\mu_1))_{n=0}^{\infty}$ is ℓ^2 and it is in the class $W_2(1, z_1)$, then we can conclude by Theorem 2.0.8.

First of all, it is clear that $(\alpha_n(d\mu_1))_{n=0}^{\infty}$ is ℓ^2 because $(\alpha_n)_{n=0}^{\infty}$ is ℓ^2 and $\tilde{\xi}_n(d\mu_1)$ is $O(1/n)$.

Next, we want to show that

$$\sum_{n=0}^{\infty} |z_1 \tilde{\xi}_{n+1} - \tilde{\xi}_n| < \infty \tag{6.2.4}$$

By (6.2.2), the error term is in the order of $O(1/n^2)$. Therefore, this is the same as showing the following is ℓ^1-summable

$$\left| \frac{\varphi_{n+2}^*(z_1)\varphi_{n+1}^*(z_1)(1-|\alpha_{n+1}|^2)^{1/2}}{K_{n+1}} - \frac{\varphi_{n+1}^*(z_1)\varphi_n^*(z_1)(1-|\alpha_n|^2)^{1/2}}{K_n} \right| \tag{6.2.5}$$

We are going to estimate term by term.

- Let $\rho_n = (1-|\alpha_n|^2)^{1/2}$. We estimate the following using the recurrence relation for orthogonal polynomials:

$$\begin{aligned} \varphi_{n+1}^*(z_1) - \varphi_n^*(z_1) &= (\rho_n \varphi_n^*(z_1) - \alpha_n \varphi_{n+1}(z_1)) - \varphi_n^*(z_1) \\ &= (\rho_n - 1)\varphi_n^*(z_1) - \alpha_n \varphi_{n+1}(z_1) \end{aligned} \tag{6.2.6}$$

Since $\rho_n - 1 = O(|\alpha_n|^2)$, $\varphi_n^*(z_1) = D(z_1)^{-1} + o(1)$ and $1/K_n = O(1/n)$,

$$\left|\varphi_{n+1}^*(z_1) - \varphi_n^*(z_1)\right| = (O(|\alpha_n|^2) + |\alpha_n|)|D(z_1)^{-1} + o(1)| = O(|\alpha_n|) \tag{6.2.7}$$

Hence,

$$\left|\frac{\left(\varphi_{n+1}^*(z_1) - \varphi_n^*(z_1)\right)\varphi_{n+1}^*(z_1)(1-|\alpha_n|^2)^{1/2}}{K_n}\right| = O\left(\frac{|\alpha_n|}{n}\right) \tag{6.2.8}$$

- If we change n to $n+1$, the same argument still holds. Therefore,

$$\left|\frac{\left(\varphi_{n+2}^*(z_1) - \varphi_{n+1}^*(z_1)\right)\varphi_n^*(z_1)(1-|\alpha_n|^2)^{1/2}}{K_n}\right| = O\left(\frac{|\alpha_{n+1}|}{n}\right) \tag{6.2.9}$$

- Observe that

$$|(1-|\alpha_{n+1}|)^{1/2} - (1-|\alpha_n|)^{1/2}| = O(|\alpha_n| + |\alpha_{n+1}|) \tag{6.2.10}$$

Hence,

$$\left|\frac{\left[(1-|\alpha_{n+1}|)^{1/2} - (1-|\alpha_n|)^{1/2}\right]\varphi_{n+1}^*(z_1)\varphi_n^*(z_1)}{K_n}\right| = O\left(\frac{|\alpha_{n+1}| + |\alpha_n|}{n}\right) \tag{6.2.11}$$

- Finally, note that

$$\left(\frac{1}{K_{n+1}} - \frac{1}{K_n}\right)\varphi_{n+1}^*(z_1)\varphi_n^*(z_1)(1-|\alpha_n|^2)^{1/2} = O\left(\frac{1}{n^2}\right) \tag{6.2.12}$$

Combining all the estimates above, we have

$$|z_1\tilde{\xi}_{n+1} - \tilde{\xi}_n| = O\left(\frac{|\alpha_n| + |\alpha_{n+1}|}{n}\right) + O\left(\frac{1}{n^2}\right) \tag{6.2.13}$$

As a result,

$$\sum_{n=0}^{\infty} |z_1\tilde{\xi}_{n+1} - \tilde{\xi}_n| < \infty \tag{6.2.14}$$

and by Theorem 2.0.8, the proof of the case $m = 1$ is complete.

Induction Step We consider $d\mu_m$ as defined in (2.0.20) as a measure formed by adding a pure point to $d\mu_{m-1}$ in the following manner

Let

$$\tilde{\gamma}_j = (1 - \gamma_m)^{-1}\gamma_j \tag{6.2.15}$$

and

$$d\mu_{m-1} = \left(1 - \sum_{l=1}^{m-1} \tilde{\gamma}_l\right) d\mu_0 + \sum_{l=0}^{m-1} \tilde{\gamma}_l \delta_{\omega_l} \tag{6.2.16}$$

Then we could write

$$d\mu_m = (1 - \gamma_m) d\mu_{m-1} + \gamma_m \delta_{\omega_m} \tag{6.2.17}$$

Recall that $0 < \sum_{l=1}^{m} \gamma_l < 1$, or equivalently, $\sum_{l=1}^{m-1} \gamma_l < 1 - \gamma_m$. Hence,

$$0 < \sum_{j=1}^{m-1} \tilde{\gamma}_j = (1 - \gamma_m)^{-1} \left(\sum_{j=1}^{m-1} \gamma_j\right) < 1 \tag{6.2.18}$$

Therefore, $d\mu_{m-1}$ satisfies the induction hypothesis, so its family of Verblunsky coefficients is ℓ^2 and $d\mu_{m-1} \in W_m(1, z_1, z_2, \ldots, z_{m-1})$. Hence,

$$\lim_{n\to\infty} \varphi_n^*(z_m, d\mu_{m-1}) \tag{6.2.19}$$

exists and is equal to $(1-\sum_{j=1}^{m-1}\gamma_j)^{1/2}D(z_m, d\mu_0)^{-1}$ (see the remark following Theorem 2.0.9). As a result, we can use a similar argument as in the base case and deduce that

$$\begin{aligned} \alpha_n(d\mu_m) &= \alpha_n(d\mu_{m-1}) + \overline{z_m}^{n+1}\frac{|D(z_m, d\mu_0)|^2}{D(z_m, d\mu_0)^2}\frac{1}{n} + E_n \\ &= \alpha_n(d\mu_0) + \sum_{j=1}^{m}\frac{\overline{z_j}^n c_j}{n} + E_n \end{aligned} \tag{6.2.20}$$

where $c_j = \overline{z_j}D(z_j, d\mu_0)^2/|D(z_j, d\mu_0)|^2$, $1 \leq j \leq m$, are constants independent of the weights $\gamma_1, \gamma_2, \ldots, \gamma_m$ and of n; and

$$E_n = E_n(z_1, z_2, \ldots, z_m, \gamma_1, \gamma_2, \ldots, \gamma_m) \tag{6.2.21}$$

is in the order of $o(1/n)$. This proves (2.0.22).

By estimating consecutive Verblunsky coefficients in the same way we did in the base case, we prove that $d\mu_m \in W_{m+1}(1, z_1, z_2, \ldots, z_m)$. Thus, we can apply Theorem 2.0.8 to prove that $\varphi_n^*(z_m)$ tends to $D(z_m, d\mu_m)^{-1}$. This completes the proof of Theorem 2.0.9.

Remark: Note that if $d\mu_0 \in W_p(\zeta_1, \zeta_2, \ldots, \zeta_p)$ and $z_j \neq \zeta_k$ for all j, k, we can use the same arguments as in the proof of Theorem 2.0.9 to prove similar results, i.e., $\alpha_n(d\mu_m)$ is in the form (2.0.22), $d\mu_m$ is in $W_{m+p}(\zeta_1, \zeta_2, \ldots, \zeta_p, z_1, z_2, \ldots, z_m)$ and that $\lim_{n\to\infty} \varphi_n(z, d\mu_m) = D(z, d\mu_m)^{-1}$ for $z \neq \zeta_1, \zeta_2, \ldots, \zeta_p, z_1, z_2, \ldots, z_m$.

Chapter 7

Asymptotically Periodic Verblunsky Coefficients

We present another application of the point mass formula.

In the previous section, we considered the probability measure $d\mu_0$ with ℓ^2 Verblunsky coefficients of bounded variation, i.e.,

$$\sum_{n=0}^{\infty} |\alpha_n|^2 < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty \tag{7.0.1}$$

In this section, we consider measures with asymptotically periodic Verblunsky coefficients of p-type bounded variation (this term was first introduced in [23]), i.e., given a periodic sequence β_n of period p,

$$\lim_{n\to\infty} \alpha_n - \beta_n = 0 \text{ and } \sum_{n=0}^{\infty} |\alpha_{n+p} - \alpha_n| < \infty \tag{7.0.2}$$

First, we handle the special case $p = 1$, then we generalize the method to any integer $p > 1$. It is well-known that any measure satisfying (7.0.2) has the same essential spectrum as $d\mu_\beta$ (the measure with periodic Verblunsky coefficients $(\beta_n)_n$) which is supported on a finite number of bands.

7.1 Gaps and Periodicity

Before we move on to stating the results, it would be helpful to have a brief discussion about gaps and periodicity.

By an application of Weyl's Theorem to the CMV matrix (see Theorem 4.3.5 of [46]), $\alpha_n \to L$ implies that $d\mu$ has the same essential spectrum as the measure $d\mu_0$ with Verblunsky coefficients $\alpha_n(d\mu_0) \equiv L$ (the measure $d\mu_0$ is known to be associated with the Geronimus polynomials). Besides, it is known that $d\mu_0$ is supported on the arc

$$\Gamma_L = [\theta_{|L|}, 2\pi - \theta_{|L|}] \tag{7.1.1}$$

where $\theta_{|L|} = 2\arcsin(|L|)$, and $d\mu_0$ admits at most one single pure point in $[-\theta_{|L|}, \theta_{|L|}]$. In other words, there is a gap in the spectrum, with at most one pure point inside.

Moreover, it is known that for $e^{i\beta/2} = (1+\overline{L})/|1+L|$,

$$w(\theta) = \begin{cases} \dfrac{1}{|1+L|} \dfrac{\sqrt{\cos^2(\theta_{|L|}/2) - \cos^2(\theta/2)}}{\sin((\theta-\beta)/2)} & \theta \in (\theta_{|L|}, 2\pi - \theta_{|L|}) \\ 0 & \theta \in [-\theta_{|L|}, \theta_{|L|}] \end{cases} \tag{7.1.2}$$

and

$$d\mu_s = \begin{cases} 0 & \text{if } \left|L + \dfrac{1}{2}\right| \leq \dfrac{1}{2} \\ \dfrac{1}{|1+L|^2} \left(\left|L + \dfrac{1}{2}\right|^2 - \dfrac{1}{4} \right) \delta_{\theta,\beta} & \text{if } \left|L + \dfrac{1}{2}\right| > \dfrac{1}{2} \end{cases} \tag{7.1.3}$$

The reader may refer to Example 1.6.12 in [46] for a detailed discussion.

Note that $\alpha_n \equiv L$ can be seen as a periodic sequence of period one, in fact, there is a more general result concerning gaps in the spectrum for measures with periodic Verblunsky coefficients. The precise statement reads as follows (see Theorem 11.1.2 of [47]): let $(\beta_n)_n$ be a periodic family of Verblunsky coefficients of period p, i.e., $\beta_n = \beta_{n+p}$ for all n. Let $d\mu_\beta$ be the associated measure. Then $\{e^{i\theta} || \mathrm{Tr}(T_p(e^{i\theta})| \leq 2\}$ is a closed set which is the union of p closed intervals $B_1, \ldots, B_p$ (which can only overlap at the endpoints). Let

$$B = \cup_{j=1}^{p} B_j \tag{7.1.4}$$

Moreover, $d\mu_s[B] = \emptyset$ and B is the essential support of the a.c. spectrum. In each disjoint open interval on $\partial\mathbb{D}\backslash B$, $d\mu$ has either no support or a single pure point.

As a result, in both cases that we consider, there are gaps in the spectrum and when $z \in \partial\mathbb{D}$ is in one of those open gaps, we have $|TrT_p(z)| > 2$.

The reader may refer to Chapter 11 of [47] for a detailed discussion of periodic Verblunsky coefficients.

7.2 Tools

For the convenience of the reader, a brief discussion of two major tools used in the proofs will be presented here.

7.2.1 The Stolz–Cesàro Theorem

One of the very important tools for the computation of the limit $\lim_{n\to\infty} \Delta_n(\zeta)$ is the Stolz–Cesàro Theorem, which reads as follows:

Theorem 7.2.1 (Stolz–Cesàro)**.** *Let* $(\Gamma_n)_{n\in\mathbb{N}}, (\Theta_n)_{n\in\mathbb{N}}$ *be two sequences of numbers such that* Θ_n *is strictly increasing and tends to infinity. If the following limit exists*

$$\lim_{n\to\infty} \frac{\Gamma_n - \Gamma_{n-1}}{\Theta_n - \Theta_{n-1}} \tag{7.2.1}$$

then it is equal to $\lim_{n\to\infty} \Gamma_n/\Theta_n$.

Proof. First, assume that $\Gamma_n \in \mathbb{R}$. From the definition of convergence, for every $\epsilon > 0$ there is an $N(\epsilon) \in \mathbb{N}$ such that $\forall n \geq N(\epsilon)$:

$$l - \epsilon < \frac{\Gamma_{n+1} - \Gamma_n}{\Theta_{n+1} - \Theta_n} < l + \epsilon \tag{7.2.2}$$

Since Θ_n is strictly increasing, we can multiply the last equation with $\Theta_{n+1} - \Theta_n > 0$ to get:

$$(l-\epsilon)(\Theta_{n+1} - \Theta_n) < \Gamma_{n+1} - \Gamma_n < (l+\epsilon)(\Theta_{n+1} - \Theta_n) \tag{7.2.3}$$

Let $k > N(\epsilon)$ be a natural number. By (7.2.3) above,

$$(l-\epsilon)\sum_{i=N(\epsilon)}^{k}(\Theta_{i+1}-\Theta_i) < \sum_{i=N(\epsilon)}^{k}(\Gamma_{n+1}-\Gamma_n) < (l+\epsilon)\sum_{i=N(\epsilon)}^{k}(\Theta_{i+1}-\Theta_i) \tag{7.2.4}$$

which is equivalent to

$$(l-\epsilon)(\Theta_{k+1} - \Theta_{N(\epsilon)}) < \Gamma_{k+1} - \Gamma_{N(\epsilon)} < (l+\epsilon)(\Theta_{k+1} - \Theta_{N(\epsilon)}) \tag{7.2.5}$$

Without loss of generality, we may assume that $\Theta_k > 0$ for all large k. Hence, we can divide the last relation by $\Theta_{k+1} > 0$ to get :

$$(l-\epsilon)(1 - \frac{\Theta_{N(\epsilon)}}{\Theta_{k+1}}) < \frac{\Gamma_{k+1}}{\Theta_{k+1}} - \frac{\Gamma_{N(\epsilon)}}{\Theta_{k+1}} < (l+\epsilon)(1 - \frac{\Theta_{N(\epsilon)}}{\Theta_{k+1}}) \tag{7.2.6}$$

Upon rearranging,

$$(l-\epsilon)(1 - \frac{\Theta_{N(\epsilon)}}{\Theta_{k+1}}) + \frac{\Gamma_{N(\epsilon)}}{\Theta_{k+1}} < \frac{\Gamma_{k+1}}{\Theta_{k+1}} < (l+\epsilon)(1 - \frac{\Theta_{N(\epsilon)}}{\Theta_{k+1}}) + \frac{\Gamma_{N(\epsilon)}}{\Theta_{k+1}} \tag{7.2.7}$$

Since $\Theta_k \to \infty$, $\Gamma_{N(\epsilon)}/\Theta_{k+1} \to 0$. Hence, tfor all large k we have

$$(l-\epsilon) < \frac{\Gamma_{k+1}}{\Theta_{k+1}} < (l+\epsilon) \tag{7.2.8}$$

which proves that $\lim_{n\to\infty} \Gamma_n/\Theta_n = l$.

It is easy to see that if $\Gamma_k = \Gamma_{1,k} + i\,\Gamma_{2,k}$, we can separate the real and imaginary cases and apply the same argument to $(\Gamma_{1,k}, \Theta_k)$ and $(\Gamma_{2,k}, \Theta_k)$.

□

7.2.2 Kooman's Theorem

Another very useful tool is an application of Kooman's Theorem to the family of $A_n(z)$'s as defined in (1.0.15). Kooman's Theorem, adopted for our proof, reads as follows:

Theorem 7.2.2 (Kooman [30, 29]). *Let A be an $\ell \times \ell$ matrix with distinct eigenvalues. Then there exists $\epsilon > 0$ and analytic functions $U(B)$ and $D(B)$ defined on $S_\epsilon = \{B : \|B - A\| < \epsilon\}$ such that*
(1) $B = U_B D_B U_B^{-1}$, D_B commutes with A.
(2) U_B is invertible for all $B \in S_\epsilon$.
(3) $U_A = 1$, $D_A = A$.
(4) By picking a basis such that A is diagonal, we can have all D_B diagonal with entries being the eigenvalues of B.

Remark: Theorem 7 basically follows the formulation of Theorem 12.1.7 of

[47], except that in [47] the statement was intended for quasi-unitary matrices. However, the same proof also holds when A has distinct eigenvalues.

The original Kooman's Theorem appeared in Theorem 1.3 of [30]. An application of Kooman's theorem to orthogonal polynomials was first made by Golinskii–Nevai [23]. They applied Kooman's result to the case when $\alpha_n \to 0$ and $\sum_n \|A_{n+1} - A_n\| < \infty$ to prove that $w(\theta) > 0$ a.e. on $\partial\mathbb{D}$, where $w(\theta)$ is the a.c. part of the measure.

7.3 Outline of the Proof

Recall the Szegő recursion relations for orthogonal polynomials on the unit circle in matrix form:

$$\begin{pmatrix} \varphi_{n+1}(z) \\ \varphi_{n+1}^*(z) \end{pmatrix} = (1 - |\alpha_n|^2)^{-1/2} \begin{pmatrix} z & -\overline{\alpha_n} \\ -z\alpha_n & 1 \end{pmatrix} \begin{pmatrix} \varphi_n(z) \\ \varphi_n^*(z) \end{pmatrix} \tag{7.3.1}$$

Let

$$A_n(z) = (1 - |\alpha_n|^2)^{-1/2} \begin{pmatrix} z & -\overline{\alpha_n} \\ -z\alpha_n & 1 \end{pmatrix} \tag{7.3.2}$$

$$A_\infty(z) = (1 - |L|^2)^{-1/2} \begin{pmatrix} z & -\overline{L} \\ -zL & 1 \end{pmatrix} \tag{7.3.3}$$

If we iterate (7.4.1) and use the fact that $\varphi_0(z) = \varphi_0^*(z) = 1$, the Szegö recursion relations can be expressed as a product of matrices:

$$\begin{pmatrix} \varphi_{n+1}(z) \\ \varphi^*_{n+1}(z) \end{pmatrix} = A_n(z)A_{n-1}(z)\cdots A_0(z)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \equiv T_n(z)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{7.3.4}$$

where

$$T_n(z) = A_n(z)A_{n-1}(z)\cdots A_0(z) \tag{7.3.5}$$

is known as the Transfer Matrix associated with the measure $d\mu$. In order to understand the asymptotics of $\varphi_n(z)$ and $\varphi^*_n(z)$, we will study the behavior of the product in (7.3.4).

First, note that $A_n(z)$ is hyperbolic for large enough n when $z \in G_L$, so $A_n(z)$ has distinct eigenvalues. This allows us to apply Kooman's result (see Theorem 7.2.2) to the product $A_n \cdots A_0$ to prove that there exists an integer N (which only depends on how fast $A_n(z) \to A_\infty(z)$) such that there exists $w \equiv w(N) \in \mathbb{C}^2$, matrices $G_j(z)$ and diagonal matrices $D_j(z)$ such that:

$$T_n(z)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = G_nD_nG_n^{-1}G_{n-1}D_{n-1}G_{n-1}^{-1}\cdots D_{N+1}G_{N+1}^{-1}G_N(z)w \tag{7.3.6}$$

The matrices G_n and D_n have the properties that

(1) $G_j(z) \to G(z)$ on $\mathbb{C}$, where $G(z)$ is the matrix that diagonalizes $A_\infty(z)$.

(2) Each $D_n(z)$ is a diagonal matrix with entries $\lambda_{1,n}$, $\lambda_{2,n}$ which are the two eigenvalues of $A_n(z)$.

(3) $G_{j+1}^{-1}G_j(z) = 1 + O(\|A_{j+1} - A_j\|)$.

Then in Lemma 7.4.2, we express the right hand side of (7.3.6) as:

$$G_n P_n \begin{pmatrix} f_{1,n} & 0 \\ 0 & f_{2,n} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \quad ; \quad P_n = \prod_{j=N+1}^{n} \lambda_{1,j} \tag{7.3.7}$$

We will consider two possible situations ((2a) and (2b) of Lemma 7.4.2), which will correspond to (1) and (2) of Theorem 2.0.11 respectively.

In the former case, we will prove that varying the weight of an existing point mass will result in a perturbation that is exponentially small.

In the latter case, we will prove that $f_{1,n}$ converges to some $f_1 \neq 0$ and $f_{2,n}/f_{1,n} \to 0$. Moreover, there are constants g_1, g_2 such that:

$$\varphi_n(z) = P_n \left(f_1 g_1 w_1 + o(1) \right) \tag{7.3.8}$$

$$\varphi_n(z)^* = P_n \left(f_1 g_2 w_1 + o(1) \right) \tag{7.3.9}$$

Then we are going to use the Stolz–Cesàro theorem as follows. We let

$$\Gamma_n(\zeta) = \overline{\varphi_{n+1}(\zeta)} \varphi_n^*(\zeta) \tag{7.3.10}$$

$$\Theta_n(\zeta) = (1-\gamma)\gamma^{-1} + K_n(\zeta, \zeta) \tag{7.3.11}$$

We observe that

$$(1-|\alpha_n|^2)^{1/2}\frac{\Gamma_n}{\Theta_n} = \Delta_n \tag{7.3.12}$$

$$\Theta_n - \Theta_{n-1} = |\varphi_n(\zeta)|^2 \tag{7.3.13}$$

$$\frac{\Gamma_n - \Gamma_{n-1}}{\Theta_n - \Theta_{n-1}} = \frac{\overline{\varphi_{n+1}(\zeta)}\varphi_n^*(\zeta) - \overline{\varphi_n(\zeta)}\varphi_{n-1}^*(\zeta)}{|\varphi_n(\zeta)|^2} \tag{7.3.14}$$

Furthermore, if $\mu(\zeta) = 0$,

$$K_n(\zeta, \zeta) \to \mu(\zeta)^{-1} = \infty \tag{7.3.15}$$

Therefore, by the Stolz–Cesàro theorem, if the limit in (7.3.14) exists it will be the same as $(1-|L|^2)^{-1/2}\lim_{n\to\infty}\Delta_n(\zeta)$.

We will prove that the limit actually exists by plugging (7.3.8) and (7.3.9) into (7.3.14).

Remark 7.3.1. *The original Kooman's Theorem appeared in Theorem 1.3 of [30]. An application of Kooman's Theorem to orthogonal polynomials was first made by Golinskii–Nevai [23]. They applied Kooman's result to the case when* $\alpha_n \to 0$ *and* $\sum_n \|A_{n+1} - A_n\| < \infty$ *and proved that* $w(\theta) > 0$ *a.e. on* $\partial\mathbb{D}$*, where* $w(\theta)$ *is the a.c. part of the measure.*

7.4 Proof of Theorem 2.0.11

The proof of Theorem 2.0.11 will be divided into many steps. First, we introduce a few objects and prove a lemma about them (see Lemma 7.4.2). Using Lemma 7.4.2, we will prove that $\lim_{n\to\infty} \Delta_n(\zeta)$ exists. Then we compute that limit explicitly and prove that the sequence $(\Delta_n(\zeta))_{n\in\mathbb{N}}$ is of bounded variation.

The matrix $A_n(\zeta)$ and its eigenvalues

Recall the Szegő recursion relations (1.0.11) and (1.0.12). Observe that they can be expressed in matrix form as follows:

$$\begin{pmatrix} \varphi_{n+1}(z) \\ \varphi_{n+1}^*(z) \end{pmatrix} = (1-|\alpha_n|^2)^{-1/2} \begin{pmatrix} z & -\overline{\alpha_n} \\ -z\alpha_n & 1 \end{pmatrix} \begin{pmatrix} \varphi_n(z) \\ \varphi_n^*(z) \end{pmatrix} \tag{7.4.1}$$

Let

$$A_n(z) = (1-|\alpha_n|^2)^{-1/2} \begin{pmatrix} z & -\overline{\alpha_n} \\ -z\alpha_n & 1 \end{pmatrix} \tag{7.4.2}$$

$$A_\infty(z) = (1-|L|^2)^{-1/2} \begin{pmatrix} z & -\overline{L} \\ -zL & 1 \end{pmatrix} \tag{7.4.3}$$

It is known (see Theorem 11.1.2 of [47]) that $e^{i\theta} \in \mathbb{G}_L$ if and only if

$$|TrA_\infty(e^{i\theta})| = (1 - |L|^2)^{-1/2} 2 \left|\cos\left(\frac{\theta}{2}\right)\right| > 2 \tag{7.4.4}$$

Since ζ is in the gap, $A_\infty \equiv A_\infty(\zeta)$ is hyperbolic, which implies A_∞ has two distinct eigenvalues $\lambda_1 \equiv \lambda_1(\zeta)$ and $\lambda_2 \equiv \lambda_2(\zeta)$ such that $|\lambda_1| > 1 > |\lambda_2|$ and $\lambda_2 = (\overline{\lambda_1})^{-1}$ (see Chapter 10.4 of [47] for an introduction to the group $\mathbb{U}(1,1)$, to which $A_\infty(\zeta)$ belongs).

Let $A_n \equiv A_n(\zeta)$. Since $A_n \to A_\infty$ and $|\mathrm{Tr}A_\infty| > 2$, for some large N_1,

$$|\mathrm{Tr}A_n| > 2 \quad \forall n \geq N_1 \tag{7.4.5}$$

Hence, for all $n > N_1$, A_n is hyperbolic and has distinct eigenvalues $\lambda_{1,n}$ and $\lambda_{2,n}$ such that $|\lambda_{1,n}| > 1 > |\lambda_{2,n}|$ and $\lambda_{2,n} = (\overline{\lambda_{1,n}})^{-1}$.

$A_n(\zeta)$ and Kooman's Theorem

As seen in Section 7.4 above, A_∞ is hyperbolic. Hence, it has distinct eigenvalues and we can apply Kooman's Theorem (Theorem 7.2.2). By Kooman's Theorem, there is an open neighborhood S_ϵ around A_∞ and an integer N_2 such that

$$A_n \in S_\epsilon \quad \forall n \geq N_2 \tag{7.4.6}$$

and there exist matrices U_{A_n} and D_{A_n} such that

$$A_n = U_{A_n}\, D_{A_n}\, U_{A_n}^{-1} \tag{7.4.7}$$

Perform a change of basis to make A_∞ diagonal, i.e., express

$$A_\infty = G\, D_\infty\, G^{-1} \tag{7.4.8}$$

where

$$D_\infty = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{7.4.9}$$

By the construction of the function D, D_{A_n} is diagonal under this new basis, so there exists a diagonal matrix

$$D_n = \begin{pmatrix} \lambda_{1,n} & 0 \\ 0 & \lambda_{2,n} \end{pmatrix} \tag{7.4.10}$$

such that

$$D_{A_n} = G\, D_n\, G^{-1} \tag{7.4.11}$$

Now we define

$$G_n = U_{A_n}\, G, \tag{7.4.12}$$

and by (7.4.7), we have the following representation of A_n:

$$A_n = G_n\, D_n\, G_n^{-1} \tag{7.4.13}$$

The vector w

Let N be an integer such that

$$N > \max\{N_1, N_2\}, \tag{7.4.14}$$

where N_1 and N_2 are defined in (7.4.5) and (7.4.6) respectively. Let w be the vector such that

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = D_N G_N^{-1} A_{N-1} A_{N-2} \cdots A_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{7.4.15}$$

We prove the following result about w_1 and w_2:

Lemma 7.4.1. *Both w_1 and w_2 are non-zero.*

Proof. First of all, observe that either w_1 or w_2 must be non-zero, because both $\varphi_N(\zeta)$ and $\varphi_N^*(\zeta)$ are non-vanishing on $\partial\mathbb{D}$, and both D_N and G_N^{-1} are invertible.

Now we prove $w_2 \neq 0$ by contradiction. Suppose $w_2 = 0$. Observe that $G_N w = (\varphi_{N+1}(\zeta), \varphi_{N+1}^*(\zeta))^T$ and $|\varphi_n(\zeta)| = |\varphi_n^*(\zeta)|$ on $\partial\mathbb{D}$. Hence, $w_2 = 0$ implies that the matrix elements $(G_N)_{11}$ and $(G_N)_{21}$ satisfy

$$|(G_N)_{11}| = |(G_N)_{21}| \tag{7.4.16}$$

It will be shown later (see the discussion after (7.4.63)) that $|G_{21}/G_{11}| = |L| < 1$. Since $G_N \to G$, (7.4.16) cannot be true if N is sufficiently large. By a similar argument, we can also prove that $w_1 \neq 0$.

□

Definitions and Asymptotics of $f_{1,n}$ and $f_{2,n}$

For $n > N$ (N as defined in (7.4.14)), we let:

$$P_n = \prod_{k=N+1}^{n} \lambda_{1,k} \,. \tag{7.4.17}$$

Furthermore, let $f_{1,n}$ and $f_{2,n}$ be defined implicitly by the equation below:

$$D_n G_n^{-1} G_{n-1} D_{n-1} \cdots D_{N+1} G_{N+1}^{-1} G_N w = P_n \begin{pmatrix} f_{1,n} w_1 \\ f_{2,n} w_2 \end{pmatrix} . \tag{7.4.18}$$

We are going to prove the following lemma about the asymptotics of $f_{1,n}$ and $f_{2,n}$:

Lemma 7.4.2. *Let $f_{1,n}$ and $f_{2,n}$ be defined as in* (7.4.18)*. The following statements hold:*

(1) $f_{2,n} \to 0$;

(2) Either one of the following is true:

- *(2a) There exists a constant C such that $|f_{1,n}| \leq C|f_{2,n}|$. Moreover, given any $\epsilon > 0$, there exist an integer N_ϵ and a constant C_ϵ such that*

$$|f_{2,n}| \leq C_\epsilon \left(\left| \frac{\lambda_2}{\lambda_1} \right| + \epsilon \right)^n , \quad \forall n \geq N_\epsilon . \tag{7.4.19}$$

- *(2b)* $|f_{2,n}/f_{1,n}| \to 0$. *Furthermore,* $f_1 = \lim_{n\to\infty} f_{1,n}$ *exists and it is non-zero.*

Proof. **We prove statement (1) of Lemma 7.4.2**. For $n \geq N$, let the left-hand side of (7.4.18) be

$$\begin{pmatrix} w_{1,n} \\ w_{2,n} \end{pmatrix} \equiv w(n) = D_n G_n^{-1} G_{n-1} D_{n-1} \cdots D_{N+1} G_{N+1}^{-1} G_N w \,. \qquad (7.4.20)$$

First, we want to show that

$$\|w(n+1) - D_{n+1} w(n)\| \leq C \|A_{n+1} - A_n\| |P_n| \left(|f_{1,n}| + |f_{2,n}|\right). \qquad (7.4.21)$$

Note that

$$w(n+1) - D_{n+1} w(n) = D_{n+1} \left(G_{n+1}^{-1} G_n - 1\right) w(n). \qquad (7.4.22)$$

We aim to bound each of the components on the right hand side of (7.4.22). Since U is analytic on S_ϵ, on some compact subset of S_ϵ there exist constants $\eta_1, \eta_2 > 0$ such that

$$\|G_n - G_{n-1}\| \leq \|G\| \|U_{A_n} - U_{A_{n-1}}\| \leq \eta_1 \|A_n - A_{n-1}\| \qquad (7.4.23)$$

and

$$\|G_n^{-1}\| \leq \|G^{-1}\| \|U_{A_n}^{-1}\| \leq \eta_2. \qquad (7.4.24)$$

Therefore, for $\eta = \eta_1\eta_2$,

$$\|G_{n+1}^{-1}G_n - 1\| = \|G_{n+1}^{-1}\left(G_n - G_{n+1}\right)\| \leq \eta\|A_{n+1} - A_n\|. \tag{7.4.25}$$

Moreover, for $C_1 = max\{|w_1|, |w_2|\}$, we have the following bounds

$$\sup_{n\geq N}\|D_n\| = \sup_{n\geq N}|\lambda_{1,n}| < 2|\lambda_1|, \tag{7.4.26}$$

$$\|w(n)\| = \left\| \begin{pmatrix} f_{1,n}P_nw_1 \\ f_{2,n}P_nw_2 \end{pmatrix} \right\| < C_1|P_n|\left(|f_{1,n}| + |f_{2,n}|\right). \tag{7.4.27}$$

Combining all the inequalities above and applying them to (7.4.22), we have

$$\|w(n+1) - D_{n+1}w(n)\| \leq C_2\|A_{n+1} - A_n\||P_n|\left(|f_{1,n}| + |f_{2,n}|\right) \tag{7.4.28}$$

where C_2 is a constant. This proves (7.4.21). We shall see why (7.4.21) is useful as we prove (7.4.30) and (7.4.32) below.

Since $P_{n+1} = \lambda_{1,n+1}P_n$ and $w_{1,n} = P_nf_{1,n}w_1$, there is a constant C_3 such that

$$\begin{aligned} |f_{1,n+1} - f_{1,n}| &= \frac{1}{|w_1|}\left|\frac{w_{1,n+1} - \lambda_{1,n+1}w_{1,n}}{P_{n+1}}\right| \\ &\leq \frac{1}{|w_1P_{n+1}|}\|w(n+1) - D_{n+1}w(n)\|. \end{aligned} \tag{7.4.29}$$

By (7.4.28), this implies

$$|f_{1,n+1} - f_{1,n}| \le C_3 \|A_{n+1} - A_n\| \left(|f_{1,n}| + |f_{2,n}|\right). \tag{7.4.30}$$

Thus, by the triangle inequality,

$$\begin{aligned} |f_{1,n+1}| &\le |f_{1,n+1} - f_{1,n}| + |f_{1,n}| \\ &\le \left(1 + C_3\|A_{n+1} - A_n\|\right)|f_{1,n}| + C_3\|A_{n+1} - A_n\||f_{2,n}|. \end{aligned} \tag{7.4.31}$$

By a similar argument, one can prove that there is a constant C_4 such that

$$\left| f_{2,n+1} - \frac{\lambda_{2,n}}{\lambda_{1,n}} f_{2,n} \right| \le C_4 \|A_{n+1} - A_n\| \left(|f_{1,n}| + |f_{2,n}|\right). \tag{7.4.32}$$

Similarly, by (7.4.32) and the fact that $|\lambda_{2,n}/\lambda_{1,n}| < 1$,

$$|f_{2,n+1}| \le \left(1 + C_4\|A_{n+1} - A_n\|\right)|f_{2,n}| + C_4\|A_{n+1} - A_n\||f_{1,n}| \tag{7.4.33}$$

We add (7.4.31) to (7.4.33) to obtain

$$|f_{1,n+1}| + |f_{2,n+1}| \le \left(1 + 2C_5\|A_{n+1} - A_n\|\right)\left(|f_{1,n}| + |f_{2,n}|\right), \tag{7.4.34}$$

where $C_5 = \max\{C_3, C_4\}$.

By applying (7.4.34) recursively, we conclude that

$$\sup_n \left(|f_{1,n}| + |f_{2,n}|\right) < \infty. \tag{7.4.35}$$

Therefore, (7.4.30) and (7.4.32) imply that $|f_{1,n+1}-f_{1,n}|$ and $|f_{2,n+1}-\lambda_{2,n}f_{2,n}/\lambda_{1,n}|$ are bounded. Furthermore, by the triangle inequality, there is a constant C_6 such that

$$|f_{1,n+1}| \leq |f_{1,n}| + C_6\|A_{n+1} - A_n\|; \tag{7.4.36}$$

$$|f_{2,n+1}| \leq \left|\frac{\lambda_{2,n}}{\lambda_{1,n}} f_{2,n}\right| + C_6\|A_{n+1} - A_n\|. \tag{7.4.37}$$

By applying (7.4.36) and (7.4.37) recursively, we conclude that for any fixed M such that $N \leq M \leq n$,

$$|f_{1,n+1}| \leq |f_{1,M}| + C_6 \sum_{j=M}^{n} \|A_{j+1} - A_j\|; \tag{7.4.38}$$

$$|f_{2,n+1}| \leq \prod_{j=M}^{n} \left|\frac{\lambda_{2,j}}{\lambda_{1,j}}\right| |f_{2,M}| + C_6 \sum_{j=M}^{n} \|A_{j+1} - A_j\|. \tag{7.4.39}$$

Without loss of generality, consider $n = 2M$. Since $|\lambda_{2,n}/\lambda_{1,n}| \to |\lambda_2/\lambda_1| < 1$, $\prod_{j=M}^{n} \left|\frac{\lambda_{2,j}}{\lambda_{1,j}}\right| \to 0$ as $n \to \infty$. Moreover, $\sum_j \|A_{j+1} - A_j\| < \infty$ implies that $\sum_{j=M}^{n} \|A_{j+1} - A_j\| \to 0$ as $n \to \infty$.

Therefore, $|f_{2,n}| \to 0$ as $n \to \infty$. This proves (1) of Lemma 7.4.2.

We proceed to prove statement (2) of Lemma 7.4.2.

There are two possible cases concerning $f_{1,n}$ and $f_{2,n}$:

Case (1): There exist a fixed integer K and a constant C, $|f_{1,n}| \leq C|f_{2,n}|$ for all $n \geq K$.

Case (2): For any integer K and any constant M, there exists an integer $n_{K,M} \geq K$ such that $|f_{1,n_{K,M}}| > M|f_{2,n_{K,M}}|$.

Case (1): (7.4.32) implies that for $n \geq \max\{N, K\}$, there is a constant C_7 such that

$$|f_{2,n+1}| \leq \left(\left| \frac{\lambda_{2,n}}{\lambda_{1,n}} \right| + C_7 \|A_{n+1} - A_n\| \right) |f_{2,n}|. \tag{7.4.40}$$

Therefore, given any $\epsilon > 0$, there exist N_ϵ and a constant C_ϵ such that

$$|f_{2,n}| \leq C_\epsilon \left(\left| \frac{\lambda_2}{\lambda_1} \right| + \epsilon \right)^n \quad \forall n \geq N_\epsilon. \tag{7.4.41}$$

In other words, $f_{2,n}$ decays exponentially fast, hence, so does $f_{1,n}$. This proves (2a) of Lemma 7.4.2.

Case (2): Let $r_n = f_{2,n}/f_{1,n}$. **First, we want to show that given any $\epsilon > 0$ there exists an integer J_ϵ such that $|r_j| < \epsilon$ for all $j \geq J_\epsilon$.**

First, we show that both $f_{1,n}$ and $f_{1,n+1}$ are non-zero, as (7.4.43) below will involve $f_{1,n}$ and $f_{1,n+1}$ in the denominator.

By assumption, we are free to choose any M, so we choose an integer M such that $1/M < \epsilon$. Consider any fixed pair (K, M) (we will choose K later in the proof). We are guaranteed the existence of an integer $n = n_{K,M} > K$ such that $|r_n| < 1/M = \epsilon$, which also implies that $f_{1,n} \neq 0$. Furthermore, by the triangle inequality and (7.4.30),

$$\begin{aligned} \left|\frac{f_{1,n+1}}{f_{1,n}}\right| &\geq 1 - \left|\frac{f_{1,n+1} - f_{1,n}}{f_{1,n}}\right| \\ &\geq 1 - C_3\|A_{n+1} - A_n\|(1 + |r_n|) > 0. \end{aligned} \tag{7.4.42}$$

Thus, $f_{1,n+1}$ is also non-zero.

By the triangle inequality,

$$\begin{aligned} &\left|r_{n+1} - \frac{\lambda_{2,n}}{\lambda_{1,n}} r_n\right| \\ \leq &\left|\frac{f_{2,n+1}}{f_{1,n+1}} - \frac{\lambda_{2,n}}{\lambda_{1,n}} \frac{f_{2,n}}{f_{1,n+1}}\right| + \left|\frac{\lambda_{2,n}}{\lambda_{1,n}}\right| \left|\frac{f_{2,n}}{f_{1,n+1}} - \frac{f_{2,n}}{f_{1,n}}\right| \\ = &\left|\frac{f_{2,n+1} - (\lambda_{2,n}/\lambda_{1,n}) f_{2,n}}{f_{1,n+1}}\right| + \left|\frac{\lambda_{2,n}}{\lambda_{1,n}} r_n\right| \left|\frac{f_{1,n} - f_{1,n+1}}{f_{1,n+1}}\right|. \end{aligned} \tag{7.4.43}$$

By (7.4.30) and (7.4.32), there exists a constant C_8 such that

$$\begin{aligned} &\left|r_{n+1} - \frac{\lambda_{2,n}}{\lambda_{1,n}} r_n\right| \\ \leq &\frac{1 + |r_n||\lambda_{2,n}/\lambda_{1,n}|}{|f_{1,n+1}|} C_8 \|A_{n+1} - A_n\|(|f_{1,n}| + |f_{2,n}|) \\ = &C_8(1 + |r_n||\lambda_{2,n}/\lambda_{1,n}|)\|A_{n+1} - A_n\| \frac{|f_{1,n}|}{|f_{1,n+1}|}(1 + |r_n|). \end{aligned} \tag{7.4.44}$$

Furthermore, by inverting (7.4.42) one gets

$$\left|\frac{f_{1,n}}{f_{1,n+1}}\right| \le \frac{1}{1 - C_3\|A_{n+1} - A_n\|(1 + |r_n|)}\,. \tag{7.4.45}$$

Then we plug this into (7.4.44) to obtain

$$|r_{n+1}| \le \left|\frac{\lambda_{2,n}}{\lambda_{1,n}} r_n\right| + \frac{C_8(1 + |r_n||\lambda_{2,n}/\lambda_{1,n}|)(1 + |r_n|)}{1 - C_3\|A_{n+1} - A_n\|(1 + |r_n|)}\|A_{n+1} - A_n\|. \tag{7.4.46}$$

Let R_n be the second term on the right hand side of (7.4.46). Note that the quotient in front of $\|A_{n+1} - A_n\|$ is bounded. Hence, for any sufficiently large K, there exists $n \equiv n_{n,k} > K$ such that $|r_{n+1}| < |r_n| < \epsilon$.

Applying the same argument to r_{n+1}, we can prove that $|r_{n+2}| < \epsilon$. Inductively, $|r_j| < \epsilon$ for all large j. This proves $|f_{2,n}/f_{1,n}| \to 0$, the first claim of (2b) of Lemma 7.4.2.

It remains to show that $\lim_{n\to\infty} f_n$ **exists.** We divide both sides of (7.4.30) by $|f_{1,n}|$. Since $|r_n| \to 0$,

$$\left|\frac{f_{1,n+1}}{f_{1,n}} - 1\right| \le C\|A_{n+1} - A_n\|\,(1 + |r_n|) \to 0. \tag{7.4.47}$$

Moreover, log is analytic near 1, so in an ϵ-neighborhood of 1 there is a constant E such that

$$|\log z| = |\log \zeta - \log 1| \le E|z - 1|. \tag{7.4.48}$$

By (7.4.47),

$$\left|\log\left(\frac{f_{1,n+1}}{f_{1,n}}\right)\right| \leq C\|A_{n+1} - A_n\|. \tag{7.4.49}$$

Therefore, the series $\sum_{j=N}^{\infty} \log\left(f_{1,j+1}/f_{1,j}\right)$ is absolutely convergent. Furthermore, as we have seen in (7.4.42), $f_{1,j} \neq 0$ for all large j. Thus, $\log f_{1,j}$ is finite and the following limit

$$\lim_{n\to\infty} \log f_{1,n+1} = \lim_{n\to\infty} \sum_{j=p}^{n} \left(\log f_{1,j+1} - \log f_{1,j}\right) + \log f_{1,p} \tag{7.4.50}$$

exists and is finite. We call the limit $\lim_{n\to\infty} f_{1,n} = f_1$. This proves the second part of (2b) and concludes the proof of Lemma 7.4.2.

□

Proof of Theorem 2.0.11. By statement (2) of Lemma 7.4.2, there are two possible cases:

First Case. This corresponds to (2a) of Lemma 7.4.2. Recall that for $n > N$,

$$T_n \begin{pmatrix} 1 \\ 1 \end{pmatrix} = G_n P_n \begin{pmatrix} f_{1,n} w_1 \\ f_{2,n} w_2 \end{pmatrix} \tag{7.4.51}$$

and $G_n = U_{A_n} G \to G$ as $n \to \infty$. Hence, given any $\epsilon > 0$, there exists a constant K_ϵ such that

$$\left\| T_n \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| \leq \|G_n\| \prod_{j=N}^{n} |\lambda_{1,j}| \left\| \begin{pmatrix} f_{1,n} w_1 \\ f_{2,n} w_2 \end{pmatrix} \right\| \leq K_\epsilon \left(\left| \frac{\lambda_2}{\lambda_1} \right| + \epsilon \right)^n \left(|\lambda_1| + \epsilon \right)^n . \tag{7.4.52}$$

This means $|\varphi_n(\zeta)|$ is exponentially decaying. As a result, $K_n(\zeta,\zeta)$ converges, $\mu(\zeta) = \lim_{n\to\infty} K_n(\zeta,\zeta)^{-1} > 0$ and $\Delta_n(\zeta) \to 0$ exponentially fast. This proves claim (1) of Theorem 2.0.11.

Second Case. This corresponds to (2b) of Lemma 7.4.2.

First, we compute $\lim_{n\to\infty} \Delta_n(\zeta)$ **using the asymptotic expressions of** $\varphi_n(\zeta)$ **and** $\varphi_n^*(\zeta)$**.** By definition, $G_n \to G$. Suppose

$$G_n = \begin{pmatrix} g_{1,n} & g'_{1,n} \\ g_{2,n} & g'_{2,n} \end{pmatrix} \to G = \begin{pmatrix} g_1 & g'_1 \\ g_2 & g'_2 \end{pmatrix}. \tag{7.4.53}$$

Since $\varphi_n(\zeta)$ is the first component of the vector $G_n P_n (f_{1,n} w_1, f_{2,n} w_2)^T$,

$$\begin{aligned} \varphi_n(\zeta) &= P_n \left(g_{1,n} f_{1,n} w_1 + g'_{1,n} f_{2,n} w_2 \right) \\ &= P_n f_{1,n} \left(g_{1,n} w_1 + g'_{1,n} r_n w_2 \right) \\ &= P_n \left(f_1 g_1 w_1 + o(1) \right). \end{aligned} \tag{7.4.54}$$

Similarly,

$$\varphi_n^*(\zeta) = P_n \left(f_1 g_2 w_1 + o(1) \right). \tag{7.4.55}$$

Since $P_n \to \infty$, both $\varphi_n(\zeta)$ and $\varphi_n^*(\zeta) \to \infty$. As a result, $(K_n(\zeta,\zeta))_{n\in\mathbb{N}}$ is a positive sequence that tends to infinity. Hence, we can use the Stolz–Cesàro

Theorem. Let

$$\Gamma_n(\zeta) = \overline{\varphi_{n+1}(\zeta)}\varphi_n^*(\zeta) \tag{7.4.56}$$

$$\Theta_n(\zeta) = (1-\gamma)\gamma^{-1} + K_n(\zeta,\zeta). \tag{7.4.57}$$

By (7.4.54) and (7.4.55),

$$\Gamma_n(\zeta) = \overline{P_{n+1}}P_n\left(|f_1|^2|w_1|^2\overline{g_1}g_2 + o(1)\right); \tag{7.4.58}$$

$$\Theta_n(\zeta) - \Theta_{n-1}(\zeta) = |P_n|^2\left(|f_1|^2|w_1|^2|g_1|^2 + o(1)\right). \tag{7.4.59}$$

Using (7.4.58), (7.4.59) above and the fact that $\lambda_2 = (\overline{\lambda_1})^{-1}$, we compute

$$\begin{aligned}\frac{\Gamma_n(\zeta)-\Gamma_{n-1}(\zeta)}{\Theta_n(\zeta)-\Theta_{n-1}(\zeta)} &= \frac{\overline{P_{n+1}}P_n - \overline{P_n}P_{n-1}}{|P_n|^2}\left(\frac{\overline{g_1}g_2}{|g_1|^2}+o(1)\right)\\ &= \left(\overline{\lambda_{1,n+1}} - \frac{1}{\lambda_{1,n}}\right)\left(\frac{g_2}{g_1}+o(1)\right) \qquad (7.4.60)\\ &\to \left(\overline{\lambda_1}-\overline{\lambda_2}\right)\left(\frac{g_2}{g_1}\right).\end{aligned}$$

Since the limit in (7.4.60) exists, $\lim_{n\to\infty}\Gamma_n(\zeta)/\Theta_n(\zeta)$ exists and is equal to the limit in (7.4.60). It remains to compute g_2/g_1. Note that

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = G\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{7.4.61}$$

By definition, G is the change of basis matrix for A_∞. Therefore, $g = (g_1, g_2)$ is the eigenvector of A_∞ corresponding to the eigenvalue λ_1. It suffices to solve $(A_\infty - \lambda_1)g = 0$, which is equivalent to

$$\begin{pmatrix} \zeta - \tau_1 & -\overline{L} \\ -\zeta L & 1 - \tau_1 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \quad \tau_1 = (1 - |L|^2)^{1/2} \lambda_1 . \tag{7.4.62}$$

Since the matrix on the left hand side of (7.4.62) has a non-zero vector in its kernel, it must have rank 1, so the two rows are equivalent. For that reason we only have to look at the first row. Furthermore, note that we are only concerned about the ratio g_2/g_1, which is constant upon multiplication of G by any non-zero constant; therefore, by putting $g_1 = 1$ and we deduce that

$$\frac{g_2}{g_1} = \frac{\zeta - \tau_1}{\overline{L}}. \tag{7.4.63}$$

Then by (7.4.60),

$$\Delta_\infty(\zeta) \quad = (1 - |L|^2)^{1/2} \left(\overline{\lambda_1} - \overline{\lambda_2}\right) \frac{\zeta - \lambda_1 (1 - |L|^2)^{1/2}}{\overline{L}} \quad . \tag{7.4.64}$$

We will simplify (7.4.64) further. Let $\tau_2 = (1 - |L|^2)^{1/2} \lambda_2$. Observe that τ_1, τ_2 are eigenvalues of the matrix

$$M(\zeta) = (1 - |L|^2)^{1/2} A_\infty(\zeta) = \begin{pmatrix} \zeta & -\overline{L} \\ -\zeta L & 1 \end{pmatrix}. \tag{7.4.65}$$

The characteristic polynomial of $M(\zeta)$ is

$$f_M(y) = (\zeta - y)(1-y) - \zeta|L|^2 = y^2 - (\zeta+1)y + \zeta(1-|L|^2) \qquad (7.4.66)$$

and the eigenvalues of $M(\zeta)$ are

$$y_\pm(\zeta) = \frac{(\zeta+1) \pm \sqrt{(\zeta+1)^2 - 4\zeta(1-|L|^2)}}{2}. \qquad (7.4.67)$$

We do not know whether $y_+(\zeta)$ is τ_1 or τ_2. We decide in the following manner: observe that $y_\pm(\zeta)$ is continuous with respect to ζ, hence if $|\lambda_1(\zeta_0)| > 1$ for some ζ_0 in the gap, we must have $|\lambda_1(\zeta)| > 1$ for all ζ in the gap. Otherwise, there must be some ζ_1 in the gap such that $|\lambda_1(\zeta_1)| = 1$, contradicting the hyperbolicity of $A_\infty(\zeta)$ in the gap.

Since $\zeta = 1$ is in the gap, we plug it into (7.4.67) to obtain

$$y_\pm(1) = 1 \pm |L|\,. \qquad (7.4.68)$$

If we choose the branch of square root such that $\sqrt{|L|^2} = |L|$, we have $y_+(\zeta) = \tau_1(\zeta)$ and $y_-(\zeta) = \tau_2(\zeta)$, and

$$\tau_1 - \tau_2 = \sqrt{(z-1)^2 + 4z|L|^2}. \qquad (7.4.69)$$

Therefore,

$$\Delta_\infty(\zeta) = \overline{h(\zeta)^{1/2}} \left(\frac{(\zeta-1) - h(\zeta)^{1/2}}{2\overline{L}} \right), \qquad (7.4.70)$$

where

$$h(\zeta) = (\zeta - 1)^2 + 4\zeta |L|^2. \tag{7.4.71}$$

This proves statement (2a) of Theorem 2.0.11.

Next, we prove statement (2b) of Theorem 2.0.11. Recall the result of Bello–López mentioned in the Introduction. Because of it, we expect $\lim_{n\to\infty} |\alpha_n(d\nu)| = |\Delta_\infty(\zeta) + L| = |L|$.

First, observe that for $\zeta = e^{i\theta}$,

$$\zeta - 1 = \zeta^{1/2}\left(\zeta^{1/2} - \zeta^{-1/2}\right) = \zeta^{1/2}\, 2i \sin\left(\frac{\theta}{2}\right). \tag{7.4.72}$$

That implies

$$h(\zeta) = 4\zeta\left(|L|^2 - \sin^2\left(\frac{\theta}{2}\right)\right), \tag{7.4.73}$$

$$\overline{h(\zeta)^{1/2}}(\zeta - 1) = 4i\sin\left(\frac{\theta}{2}\right)\sqrt{|L|^2 - \sin^2\left(\frac{\theta}{2}\right)}. \tag{7.4.74}$$

Now we consider $\Delta_\infty(\zeta) + L$. Combining (7.4.70), (7.4.73) and (7.4.74), we have

$$\Delta_\infty(\zeta) + L = \frac{i\, 2\sin\left(\frac{\theta}{2}\right)\sqrt{|L|^2 - \sin^2\left(\frac{\theta}{2}\right)} + \left[2\sin^2\left(\frac{\theta}{2}\right) - |L|^2\right]}{\overline{L}}. \tag{7.4.75}$$

Since ζ is in the gap $\mathbb{G}_L$ if and only if $|L|^2 > \sin^2(\frac{\theta}{2})$, $\sqrt{|L|^2 - \sin^2(\theta/2)}$

is real (see Section 7.4 above). Therefore, (7.4.75) implies that

$$Re\,\overline{L}\,(\Delta_\infty(\zeta)+L) = 2\sin^2\left(\frac{\theta}{2}\right) - |L|^2 \tag{7.4.76}$$

$$Im\,\overline{L}\,(\Delta_\infty(\zeta)+L) = 2\sin\left(\frac{\theta}{2}\right)\sqrt{|L|^2-\sin^2\left(\frac{\theta}{2}\right)}. \tag{7.4.77}$$

Now that we have successfully separated the real and imaginary part of $\overline{L}(\Delta_\infty(\zeta)+L)$, with a direct computation we can show that

$$\left|\overline{L}(\Delta_\infty(\zeta)+L)\right| = |L|^2. \tag{7.4.78}$$

It remains to compute the phase. Suppose $\overline{L}\,(\Delta_\infty(\zeta)+L) = |L|^2e^{i\omega}$. $|L|^2\cos\omega$ and $|L|^2\sin\omega$, being the real and imaginary part of $\overline{L}(\Delta_\infty(\zeta)+L)$ respectively, will be given by (7.4.76) and (7.4.77). This proves statement (2b) of Theorem 2.0.11.

Now we are going to prove that $(\Delta_n(\zeta))_{n\in\mathbb{N}}$ is of bounded variation.

First, we note the following estimates:

(1) By the definition of $A_n(\zeta)$, $\|A_n(\zeta)-A_{n-1}(\zeta)\| = O\left(|\alpha_n-\alpha_{n-1}|\right)$.

(2) By (7.4.30), $|f_{1,n+1}-f_{1,n}| = O(\|A_{n+1}(\zeta)-A_n(\zeta)\|)$.

(3) By the definition of G_n in (7.4.12), both $|g_{1,n+1}-g_{1,n}|$ and $|g'_{1,n+1}-g'_{1,n}|$ are $O(\|A_{n+1}(\zeta)-A_n(\zeta)\|)$.

(4) Since $\lambda_{1,n}$, $\lambda_{2,n}$ are the eigenvalues $A_n(\zeta)$, $|\lambda_{1,n+1}-\lambda_{1,n}|$ and $|\lambda_{2,n+1}-\lambda_{2,n}|$ are $O(|\alpha_{n+1}-\alpha_n|)$.

(5) By (7.4.44), $|r_{n+1} - c_n r_n| = O(\|A_{n+1}(\zeta) - A_n(\zeta)\|)$ where

$$c_n = \frac{\lambda_{2,n}}{\lambda_{1,n}} \to c = \frac{\lambda_2}{\lambda_1} \tag{7.4.79}$$

has norm strictly less than 1. From now on, we will denote all error terms in the order of $O(|\alpha_n - \alpha_{n-1}|)$ as e_n.

Recall that $\Delta_n(\zeta) = (1-|\alpha_n|^2)^{1/2}\Gamma_n(\zeta)/\Theta_n(\zeta)$. To prove that $(\Delta_n(\zeta))_{n\in\mathbb{N}}$ is of bounded variation, we will consider $(1-|\alpha_n|^2)^{1/2}$ and $\Gamma_n(\zeta)/\Theta_n(\zeta)$ separately.

First, note that

$$(1 - |\alpha_{n+1}|^2)^{1/2} - (1 - |\alpha_n|^2)^{1/2} = e_{n+1}. \tag{7.4.80}$$

Recall that $f_{2,n}/f_{1,n} = r_n$. Hence, by (7.4.54) and (7.4.55),

$$\begin{aligned}
&\frac{\Gamma_n(\zeta)}{(1-\gamma)\gamma^{-1} + K_n(\zeta,\zeta)} \\
=\ &\frac{\overline{P_{n+1}}P_n}{(1-\gamma)\gamma^{-1} + K_n(\zeta,\zeta)}\overline{f_{1,n+1}}f_{1,n}\overline{\left(g_{1,n+1}w_1 + g'_{1,n+1}r_{n+1}w_2\right)}\left(g_{2,n}w_1 + g'_{2,n}r_n w_2\right) \\
=\ &\underbrace{\frac{\overline{\lambda_{n+1}}|P_n|^2}{(1-\gamma)\gamma^{-1} + K_n(\zeta,\zeta)}}_{(I)}\underbrace{\overline{f_{1,n+1}}f_{1,n}}_{(II)}\underbrace{\overline{\left(g_{1,n+1}w_1 + g'_{1,n+1}r_{n+1}w_2\right)}}_{(III)}\underbrace{\left(g_{2,n}w_1 + g'_{2,n}r_n w_2\right)}_{(IV)}.
\end{aligned} \tag{7.4.81}$$

Now we will show that (I), (II), (III) and (IV) of (7.4.81) are of bounded variation.

We start with the easiest. For (II), note that by estimate (2) above,

$$\overline{f_{1,n+1}}f_{1,n} - \overline{f_{1,n}}f_{1,n-1} = e_n + e_{n-1}. \tag{7.4.82}$$

The next term we will estimate is (III). We start by showing that $(r_n)_{n\in\mathbb{N}}$ is of bounded variation. Observe that

$$\begin{aligned} |r_{n+1} - r_n| &\leq |c_n r_n + e_{n+1} - c_{n-1} r_{n-1} + e_n| \\ &\leq |c_n||r_n - r_{n-1}| + e_n + e_{n+1} \\ &\vdots \\ &\leq |c_n \dots c_1||r_1 - r_0| + E_n + E_{n+1}, \end{aligned} \tag{7.4.83}$$

where

$$E_n = O(e_n + |c_n|e_{n-1} + |c_n c_{n-1}||e_{n-2}| + \dots + |c_n \dots c_2|e_1). \tag{7.4.84}$$

Hence,

$$\sum_{n=0}^{\infty} |r_{n+1} - r_n| \leq |r_1 - r_0| \sum_{n=1}^{\infty} |c_n \dots c_1| + 2 \sum_{n=0}^{\infty} E_n \ . \tag{7.4.85}$$

The first sum on the right hand side of (7.4.85) is finite because $|c_n| \to |c| < 1$. Now we turn to the second sum. Upon rearranging,

$$2\sum_{n=0}^{\infty} E_n = O\left(\sum_{n=0}^{\infty} e_n[1 + |c_{n+1}| + |c_{n+1}c_{n+2}| + \dots]\right) < \infty. \tag{7.4.86}$$

Then we observe that

$$\overline{\left(g_{1,n+1}w_1 + g'_{1,n+1}r_{n+1}w_2\right)} - \overline{\left(g_{1,n}w_1 + g'_{1,n}r_nw_2\right)} = e_{n+1} + O(|r_{n+1} - r_n|). \tag{7.4.87}$$

Therefore, (III) is of bounded variation. With a similar argument we can prove that the same goes for (IV).

It remains to prove that (I) is of bounded variation. We will make use of the simple equality

$$\frac{1}{a_{n+1}} - \frac{1}{a_n} = \frac{a_{n+1} - a_n}{a_{n+1}a_n}. \tag{7.4.88}$$

As a result, if $\lim_{n\to\infty} a_n = a \neq 0$ and $(a_n)_{n\in\mathbb{N}}$ is of bounded variation, then $(1/a_n)_{n\in\mathbb{N}}$ is also of bounded variation. **Thus, it suffices to prove that** $([(1-\gamma)\gamma^{-1} + K_n(\zeta,\zeta)]/|P_n|^2)_{n\in\mathbb{N}}$ **is of bounded variation and** $\lim_{n\to\infty}[(1-\gamma)\gamma^{-1} + K_n(\zeta,\zeta)]/|P_n|^2 = \mathcal{L} > 0$**.**

For the convenience of computation we will define a few more objects below. First, we let

$$\Lambda_n = \begin{cases} \lambda_{1,n} & \text{if } n \geq N+1 \\ 1 & \text{if } 0 \leq n \leq N \end{cases}. \tag{7.4.89}$$

Then by (7.4.17), $P_n = \prod_{j=0}^{n} \Lambda_j$. Moreover, recall the definition of $f_{1,n}$ in (7.4.18), which was only defined for $n \geq N$. For $0 \leq n \leq N$, let $f_{1,n}$ $f_{2,n}$ be defined implicitly by (7.4.54) and (7.4.55). We will see later that the introduction of these objects will not affect the result of our computation.

Note that $K_n(\zeta, \zeta)$ is the summation of $n+1$ terms, so we can express

$$\frac{(1-\gamma)\gamma^{-1} + K_n(\zeta,\zeta)}{|P_n|^2} = \frac{\gamma^{-1}}{|P_n|^2} + \mathcal{T}_n \,, \tag{7.4.90}$$

where

$$\mathcal{T}_n = \sum_{j=1}^{n} \frac{|\varphi_j(\zeta)|^2}{|P_n|^2} = \sum_{j=1}^{n} \frac{|f_{1,j}|^2 |g_{1,j} w_1 + g'_{1,j} r_j w_2|^2}{|\Lambda_{j+1} \cdots \Lambda_n|^2}. \tag{7.4.91}$$

with the convention that $\Lambda_{j+1} \cdots \Lambda_n = 1$ when $j = n$.

Next, we let

$$\mathcal{S}_n = \frac{K_{n-1}(\zeta,\zeta)}{|P_{n-1}|^2} = \sum_{j=0}^{n-1} \frac{|f_{1,j}|^2 |g_{1,j} w_1 + g'_{1,j} r_j w_2|^2}{|\Lambda_{j+1} \cdots \Lambda_{n-1}|^2}. \tag{7.4.92}$$

Then

$$\left| \frac{(1-\gamma)\gamma^{-1} + K_n(\zeta,\zeta)}{|P_n|^2} - \frac{(1-\gamma)\gamma^{-1} + K_{n-1}(\zeta,\zeta)}{|P_{n-1}|^2} \right| \le \frac{2(1+\gamma^{-1})}{|P_{n-1}|^2} + |\mathcal{T}_n - \mathcal{S}_n| \,. \tag{7.4.93}$$

We will show that each of the two terms on the right hand side of (7.4.93) is summable.

Since $|\Lambda_n|^{-1} \to |\lambda_1|^{-1} < 1$,

$$\sum_{n=0}^{\infty} \frac{2(1+\gamma^{-1})}{|P_n|^2} = O\left(\sum_{j=0}^{\infty} \frac{1}{|\lambda_1|^{2j}} \right) < \infty. \tag{7.4.94}$$

Now we will go on to prove that $\mathcal{T}_n - \mathcal{S}_n$ is summable. Upon relabeling the indices of $\mathcal{S}_n$ in (7.4.92), we have

$$\begin{aligned}
&\mathcal{T}_n - \mathcal{S}_n \\
= &\sum_{j=1}^{n} \left[\frac{|f_{1,j}|^2 |g_{1,j} w_1 + g'_{1,j} r_j w_2|^2}{|\Lambda_{j+1} \ldots \Lambda_n|^2} - \frac{|f_{1,j-1}|^2 |g_{1,j-1} w_1 + g'_{1,j-1} r_{j-1} w_2|^2}{|\Lambda_j \ldots \Lambda_{n-1}|^2} \right]
\end{aligned} \tag{7.4.95}$$

and we will compute term by term.

Let

$$\epsilon_j = |g_{1,j} w_1 + g'_{1,j} r_j w_2|^2. \tag{7.4.96}$$

Then by (7.4.95) above,

$$\begin{aligned}
|\mathcal{T}_n - \mathcal{S}_n| \le & \underbrace{\sum_{j=1}^{n} \frac{|f_{1,j}|^2 |\epsilon_j - \epsilon_{j-1}|}{|\Lambda_{j+1} \cdots \Lambda_n|^2}}_{(I)} + \underbrace{\sum_{j=1}^{n} \frac{||f_{1,j}|^2 - |f_{1,j-1}|^2| \epsilon_{j-1}}{|\Lambda_{j+1} \cdots \Lambda_n|^2}}_{(II)} \\
& + \underbrace{\sum_{j=1}^{n} |f_{1,j-1}|^2 \epsilon_{j-1} \left| \frac{1}{|\Lambda_{j+1} \cdots \Lambda_n|^2} - \frac{1}{|\Lambda_j \cdots \Lambda_{n-1}|^2} \right|}_{(III)}.
\end{aligned} \tag{7.4.97}$$

Now we will prove that each of the sums on the right hand side of (7.4.97) is summable. We will start with (II).

Recall that $|f_{1,j} - f_{1,j-1}| = O(\|A_j - A_{j-1}\|)$ and that $f_{1,j} \to f_1$. Therefore, for some constant C,

$$\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{||f_{1,j}|^2-|f_{1,j-1}|^2|\epsilon_{j-1}}{|\Lambda_{j+1}\cdots\Lambda_n|^2} < C\left(\sum_{n=1}^{\infty}|f_{1,n}-f_{1,n-1}|\right)\left(\sum_{j=1}^{\infty}\frac{1}{\lambda_1^{2j}}\right)<\infty. \quad (7.4.98)$$

Since $g_{1,j}$, $g'_{1,j}$ and r_j are all of bounded variation and their limits exist when j goes to infinity, ϵ_j is of bounded variation. Hence, there exists a constant C such that

$$\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{|f_{1,j}|^2|\epsilon_j-\epsilon_{j-1}|}{|\Lambda_{j+1}\cdots\Lambda_n|^2} < C\left(\sum_{j=1}^{\infty}|\epsilon_j-\epsilon_{j-1}|\right)\left(\sum_{j=1}^{\infty}\frac{1}{\lambda_1^{2j}}\right)<\infty. \quad (7.4.99)$$

Finally, we will consider (III). Observe that

$$\left|\frac{1}{|\Lambda_{j+1}\cdots\Lambda_n|^2}-\frac{1}{|\Lambda_j\cdots\Lambda_{n-1}|^2}\right| = \frac{|\Lambda_j|^2-|\Lambda_n|^2}{|\Lambda_j\cdots\Lambda_n|^2} \quad (7.4.100)$$

and that there exists a constant C independent of j,n such that

$$|\Lambda_j|^2-|\Lambda_n|^2 = \sum_{k=j}^{n-1}\left(|\Lambda_k|^2-|\Lambda_{k+1}|^2\right) < C\sum_{k=j}^{n-1}|\Lambda_k-\Lambda_{k+1}|. \quad (7.4.101)$$

Hence,

$$\sum_{n=1}^{\infty}\sum_{j=1}^{n}\left|\frac{1}{|\Lambda_{j+1}\cdots\Lambda_n|^2}-\frac{1}{|\Lambda_j\cdots\Lambda_{n-1}|^2}\right| < C\sum_{n=1}^{\infty}\sum_{j=1}^{n}\sum_{k=j}^{n-1}\frac{|\Lambda_{k+1}-\Lambda_k|}{|\Lambda_j\cdots\Lambda_n|^2}. \quad (7.4.102)$$

Next, we count the coefficient of $|\Lambda_{k+1}-\Lambda_k|$ in the sum above. From the expression, we know that $j \le k < n$. Therefore, the coefficient is

$$\sum_{n=k+1}^{\infty}\sum_{j=1}^{k}\frac{1}{|\Lambda_{j+1}\cdots\Lambda_n|^2} = \sum_{j=1}^{k}\sum_{n=k+1}^{\infty}\left(\frac{1}{|\Lambda_{j+1}\cdots\Lambda_n|^2}\right)$$
$$= \left(\sum_{j=1}^{k}\frac{1}{|\Lambda_2\cdots\Lambda_j|^2}\right)\left(\sum_{n=k+1}^{\infty}\frac{1}{|\Lambda_{k+1}\cdots\Lambda_n|^2}\right), \quad (7.4.103)$$

which is bounded above by a constant B independent of k. This implies that (III) is summable in n.

As a result, $((1-\gamma)\gamma^{-1}+K_n(\zeta,\zeta)/|P_n|^2)_{n\in\mathbb{N}}$ is of bounded variation and that implies $\lim_{n\to\infty}[(1-\gamma)\gamma^{-1}+K_n(\zeta,\zeta)]/|P_n|^2$ exists. Moreover,

$$\mathcal{L} = \lim_{n\to\infty}\frac{K_n(\zeta,\zeta)}{|P_n|^2} > \lim_{n\to\infty}\frac{|\varphi_n(\zeta)|^2}{|P_n|^2} > 0. \quad (7.4.104)$$

This concludes the proof of Theorem 2.0.11. □

7.5 Proof of Theorem 2.0.12

We will generalize the method developed in Theorem 2.0.11. First, we define

$$B_k(\zeta) = A(\alpha_{(k+1)p-1}, z)\cdots A(\alpha_{kp}, z); \quad (7.5.1)$$
$$B_\infty(\zeta) = A(\beta_{p-1}, z)\cdots A(\beta_0, z). \quad (7.5.2)$$

We need to check a few conditions concerning the $B_k(\zeta)$'s. First, note that there exists a constant C such that

$$\|B_{k+1}(\zeta) - B_k(\zeta)\| \leq C \sum_{j=0}^{p-1} |\alpha_{(k+1)p+j} - \alpha_{kp+j}| \tag{7.5.3}$$

Hence,

$$\begin{aligned} \sum_{k=0}^{\infty} \|B_{k+1}(\zeta) - B_k(\zeta)\| &\leq C \sum_{k=0}^{\infty} \sum_{j=0}^{p-1} |\alpha_{(k+1)p+j} - \alpha_{kp+j}| \\ &= C \sum_{m=0}^{\infty} |\alpha_{m+p} - \alpha_m| < \infty \end{aligned} \tag{7.5.4}$$

Furthermore, since ζ is in the gap, $|\mathrm{Tr} B_\infty(\zeta)| > 2$. Since $B_k(\zeta) \to B_\infty(\zeta)$, for all large k, $|\mathrm{Tr} B_k(\zeta)| > 2$. As a result, $B_k(\zeta)$ has distinct eigenvalues $\tau_{1,k}$ and $\tau_{2,k}$ such that $|\tau_{1,k}| > 1 > |\tau_{2,k}|$ and $|\tau_{1,k}\tau_{2,k}| = 1$. Moreover, $\tau_{i,k} \to \tau_i$, where τ_1, τ_2 are the eigenvalues of $B_\infty(\zeta)$.

Next, observe that for any fixed $0 \leq j \leq p-1$,

$$\begin{aligned} T_{kp+j}(\zeta) &= \left(A_{kp+j}(\zeta) \cdots A_{kp}(\zeta)\right) A_{kp-1} \cdots A_0(\zeta) \\ &= \left(A_{kp+j}(\zeta) \cdots A_{kp}(\zeta)\right) B_{k-1}(\zeta) B_{k-2}(\zeta) \cdots B_0(\zeta) \end{aligned} \tag{7.5.5}$$

and $A_{kp+j}(\zeta) \to A_{\infty,j}(\zeta)$, where

$$A_{\infty,j}(\zeta) = (1 - |\beta_j|^2)^{-1/2} \begin{pmatrix} \zeta & -\overline{\beta_j} \\ -\zeta\beta_j & 1 \end{pmatrix} ; 0 \leq j \leq p-1 \tag{7.5.6}$$

By Kooman's Theorem and a change of basis, we can express

$$B_n(\zeta) = G_n D_n G_n^{-1} \tag{7.5.7}$$

as in (7.4.13), where D_n is a diagonal matrix with entries being the eigenvalues of $B_n(\zeta)$, and $G_n \to G_\infty$, where G_∞ is the matrix that diagonalizes $B_\infty(\zeta)$.

By applying a similar argument as in Section 7.4 to the family of $B_n(\zeta)$'s, we can show that there exists a non-zero vector w and an integer N such that

$$B_n(\zeta)\cdots B_0(\zeta)\begin{pmatrix}1\\1\end{pmatrix} = G_n(\zeta)P_n\begin{pmatrix}f_{1,n} & 0\\0 & f_{2,n}\end{pmatrix}\begin{pmatrix}w_1\\w_2\end{pmatrix}, \tag{7.5.8}$$

where $P_n = \prod_{j=N+1}^{n}\tau_{1,j}$. Moreover, we can show that

$$f_{1,n} \to f_1; \quad f_{2,n} \to f_2; \quad \frac{f_{1,n}}{f_{2,n}} \to 0. \tag{7.5.9}$$

Furthermore, by (7.5.5), for each fixed j, we can express $T_{kp+j}(\zeta)$ as

$$T_{kp+j}(\zeta)v = \left(A_{kp+j}(\zeta)\cdots A_{kp}(\zeta)\right)G_{k-1}P_{k-1}\begin{pmatrix}f_{1,k-1} & 0\\0 & f_{2,k-1}\end{pmatrix}\begin{pmatrix}w_1\\w_2\end{pmatrix} \tag{7.5.10}$$

with the property that

$$A_{kp+j}(\zeta)\cdots A_{kp}(\zeta)G_{k-1} \to A_{\infty,j}(\zeta)\cdots A_{\infty,0}(\zeta)G_{\infty} \equiv M_j. \qquad (7.5.11)$$

Let

$$M_j = \begin{pmatrix} m_{1,j} & m_{1,j'} \\ m_{2,j} & m_{2',j} \end{pmatrix}. \qquad (7.5.12)$$

Note that for each n, there are two possible expressions for $T_n(\zeta)v$. We could either write it as in (7.5.10) or as follows

$$T_{kp+j}(\zeta)v = A_{kp+j}(\zeta)\cdots A_{(k-1)p}(\zeta)G_{k-2}P_{k-2}F_{k-2}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \qquad (7.5.13)$$

The reason will be apparent later in the proof.

Consider $n = kp+j$ where $0 \leq j \leq p$. The asymptotic formulae for $\varphi_n(\zeta)$ and $\varphi_n^*(\zeta)$ are of the form

$$\varphi_n(\zeta) = P_{k-1}(f_1 m_{1,j} w_1 + o(1)); \qquad (7.5.14)$$

$$\varphi_n^*(\zeta) = P_{k-1}(f_1 m_{2,j+p} w_1 + o(1)). \qquad (7.5.15)$$

The alternate formulae for $\varphi_n(\zeta)$ and $\varphi_n^*(\zeta)$ are:

$$\varphi_n(\zeta) = P_{k-2}(f_1 m_{1,p+j} w_1 + o(1)); \qquad (7.5.16)$$

$$\varphi_n^*(\zeta) = P_{k-1}(f_1 m_{2,j} w_1 + o(1)). \qquad (7.5.17)$$

We define $\Gamma_n(\zeta)$ and $\Theta_n(\zeta)$ as in (7.4.56) and (7.4.57) respectively. Then

$$\begin{aligned} \Gamma_n(\zeta) &= |P_{k-1}|^2 \left(|f_1|^2|w_1|^2\overline{m_{1,j+1}}m_{2,j} + o(1)\right), && (7.5.18) \\ \Theta_n(\zeta) &= |P_{k-1}|^2 \left(|f_1|^2|w_1|^2|m_{1,j}|^2 + o(1)\right). && (7.5.19) \end{aligned}$$

Moreover, observe that

$$\Gamma_{n+p}(\zeta) = |P_k|^2 \left(|f_1|^2|w_1|^2\overline{m_{1,j+1}}m_{2,j} + o(1)\right). \tag{7.5.20}$$

Instead of $(\Gamma_n - \Gamma_{n-1})/(\Theta_n - \Theta_{n-1})$ in the proof of Theorem 2.0.11, we compute

$$\begin{aligned} &\lim_{k\to\infty} \frac{\Gamma_{(k+1)p+j}(\zeta) - \Gamma_{kp+j}(\zeta)}{\Theta_{(k+1)p+j}(\zeta) - \Theta_{kp+j}(\zeta)} \\ = \;& \lim_{k\to\infty} \frac{(|P_k|^2 - |P_{k-1}|^2)\left(|f_1|^2|w_1|^2\overline{m_{1,j+1}}m_{2,j} + o(1)\right)}{|P_{k-1}|^2|f_1|^2|w_1|^2\left(|m_{1,j+p}|^2 + \cdots + |m_{1,j}|^2 + o(1)\right)} && (7.5.21) \\ = \;& (|\tau_1|^2 - 1)\frac{\overline{m_{1,j+1}}m_{2,j}}{|m_{1,j+p}|^2 + \cdots + |m_{1,j}|^2}. \end{aligned}$$

Combining with the fact that $\lim_{k\to\infty}(1 - |\alpha_{kp+j}|^2)^{1/2} = (1 - |\beta_j|^2)^{1/2}$, we conclude that for each fixed $0 \le j < p$, $\lim_{k\to\infty} \Delta_{kp+j}(\zeta)$ exists.

Finally, by a similar argument as in the proof of Theorem 2.0.11, one could prove that for each fixed j, $(\Delta_{kp+j}(\zeta))_k$ is of bounded variation.

7.6 Proof of Theorem 2.0.13

In this section, $\zeta^n \alpha_n \to L$ and $\mu(\zeta) = 0$ are the only assumptions we need. No bounded variation of the Verblunsky coefficients is required.

Let

$$P_n(\zeta) = (1 - |\alpha_n|^2)^{1/2} \overline{\varphi_{n+1}(\zeta)} \varphi_n^*(\zeta) \tag{7.6.1}$$

and $\Theta_n(z)$ be defined as in (7.4.57).

Note that $P_n(\zeta)/\Theta_n(\zeta) = \Delta_n(\zeta)$. Moreover, since $\mu(\zeta) = 0$, $K_n(\zeta, \zeta) \to \infty$, which allows us to use the Stolz–Cesàro Theorem.

Let $\rho_n = (1 - |\alpha_n|^2)^{1/2}$. Since $\zeta \in \partial\mathbb{D}$, we can rewrite $P_n(\zeta), P_{n-1}(\zeta)$ as follows:

$$P_n(\zeta) = \rho_n \zeta^{-1} \varphi_{n+1}^*(\zeta) \overline{\varphi_n(\zeta)}, \tag{7.6.2}$$

$$P_{n-1}(\zeta) = \rho_{n-1} \overline{\varphi_n(\zeta)} \varphi_{n-1}^*(\zeta). \tag{7.6.3}$$

Moreover,

$$\Theta_n(\zeta) - \Theta_{n-1}(\zeta) = |\varphi_n(\zeta)|^2 \tag{7.6.4}$$

and $\varphi_n \neq 0$ on $\partial\mathbb{D}$, therefore we could cancel $\overline{\varphi_n(\zeta)}$ and obtain

$$\frac{\zeta^n P_n(\zeta) - \zeta^{n-1} P_{n-1}(\zeta)}{\Theta_n(\zeta) - \Theta_{n-1}(\zeta)} = \frac{\zeta^{n-1}(\rho_n \varphi_{n+1}^*(\zeta) - \rho_{n-1} \varphi_{n-1}^*(\zeta))}{\varphi_n(\zeta)}. \tag{7.6.5}$$

By (1.5.24) and (1.5.43) in [46] respectively,

$$\rho_n \varphi_{n+1}^*(\zeta) = \varphi_n^*(\zeta) - \alpha_n \zeta \varphi_n(\zeta), \tag{7.6.6}$$

$$\rho_{n-1} \varphi_{n-1}^*(\zeta) = \varphi_n^*(\zeta) + \alpha_{n-1} \varphi_n(\zeta). \tag{7.6.7}$$

Therefore, (7.6.5) becomes

$$\begin{aligned} \frac{\zeta^n P_n(\zeta) - \zeta^{n-1} P_{n-1}(\zeta)}{\Theta_n(\zeta) - \Theta_{n-1}(\zeta)} &= \frac{\zeta^{n-1} \left(\varphi_n^*(\zeta) - \zeta \alpha_n \varphi_n(\zeta) - \varphi_n^*(\zeta) - \alpha_{n-1} \varphi_n(\zeta)\right)}{\varphi_n(\zeta)} \\ &= -(\zeta^n \alpha_n + \zeta^{n-1} \alpha_{n-1}). \end{aligned} \tag{7.6.8}$$

Since $\zeta^n \alpha_n \to L$, the limit of (7.6.8) as $n \to \infty$ exists and is equal to $-2L$. Moreover, since ζ is not a pure point of $d\mu$, $\Theta_n(\zeta)$ is a strictly increasing sequence that tends to $+\infty$, so we can apply the Stolz–Cesàro theorem and conclude that $\zeta^n \Delta_n(\zeta) = \zeta^n P_n(\zeta)/\Theta_n(\zeta) \to -2L$. This implies that

$$\zeta^n \alpha_n(d\nu) = \zeta^n \alpha_n + \zeta^n \Delta_n(\zeta) \to -L. \tag{7.6.9}$$

7.7 Proof of Corollary 2.0.1

First, note that α_n is real for all n, so by induction on (1.0.11) we have a closed form for $\varphi_n(1)$:

$$\varphi_n(1) = \prod_{j=0}^{n-1} \sqrt{\frac{1 - \alpha_j}{1 + \alpha_j}} \in \mathbb{R}. \tag{7.7.1}$$

Moreover, since $\alpha_n \to L < 0$, $\sqrt{\frac{1-\alpha_j}{1+\alpha_j}} > 1$ for large j, $\varphi_n(1)$ is exponentially increasing towards $+\infty$. Thus, $\lim_{n\to\infty} K_n(1,1) = \infty$ and $\mu(1) = 0$. By Theorem 2.0.11, we have $\Delta_n(1) \to -2L$.

To prove Corollary 2.0.1, we are going to show that

$$\lim_{n\to\infty} \frac{(\Delta_n(1) + 2L)}{c_n} = -2. \tag{7.7.2}$$

Observe that by (7.7.1),

$$(1 - |\alpha_n|^2)^{1/2} \varphi_{n+1}(1) = (1 - \alpha_n)\varphi_n(1). \tag{7.7.3}$$

Moreover, $K_n(1,1)$ is exponentially increasing. Therefore,

$$\Delta_n(1) + 2L = \frac{(1-\alpha_n)\varphi_n(1)^2 + 2L\, K_n(1,1)}{K_n(1,1)} + E_n \tag{7.7.4}$$

where E_n is exponentially small.

We shall use the Stolz–Cesàro theorem again to prove that the limit in (7.7.2) exists and is finite. Let

$$A_n = c_n^{-1} \left[(1-\alpha_n)\varphi_n(1)^2 + 2L\, K_n(1,1) \right]; \tag{7.7.5}$$

$$B_n = K_n(1,1). \tag{7.7.6}$$

First, note that $B_n - B_{n-1} = \varphi_n(1)^2$. Second, note that by (7.7.1),

$$(1-\alpha_{n-1})\varphi_{n-1}(1)^2 = (1+\alpha_{n-1})\varphi_n(1)^2. \tag{7.7.7}$$

Therefore,

$$\begin{aligned} A_n - A_{n-1} = &\left[c_n^{-1}(1-\alpha_n)\varphi_n(1)^2 - c_{n-1}^{-1}(1+\alpha_{n-1})\varphi_n(1)^2\right] \\ &+ c_n^{-1}(2L)K_n(1,1) - c_{n-1}^{-1}(2L)K_{n-1}(1,1). \end{aligned} \tag{7.7.8}$$

The first sum on the right hand side of (7.7.8) is

$$\left[c_n^{-1}(1-L) - c_{n-1}^{-1}(1+L) - 2\right]\varphi_n(1)^2, \tag{7.7.9}$$

while the second sum is

$$2L\left[c_n^{-1}\varphi_n(1)^2 + (c_n^{-1} - c_{n-1}^{-1})K_{n-1}(1,1)\right]. \tag{7.7.10}$$

Combining (7.7.9) and (7.7.10), we have

$$\frac{A_n - A_{n-1}}{B_n - B_{n-1}} = \left[(1+L)(c_n^{-1} - c_{n-1}^{-1}) - 2\right] + 2L(c_n^{-1} - c_{n-1}^{-1})\frac{K_{n-1}(1,1)}{\varphi_n(1)^2}. \tag{7.7.11}$$

Next, we are going to show that $\frac{K_{n-1}(1,1)}{\varphi_n(1)^2}$ exists. To do that, we use the

Stolz–Cesàro Theorem again. Let

$$C_n = K_{n-1}(1,1), \tag{7.7.12}$$

$$D_n = \varphi_n(1)^2. \tag{7.7.13}$$

Recall that by (7.7.1), $\varphi_n(1)^2 = \frac{1-\alpha_n}{1+\alpha_n}\varphi_{n-1}(1)^2$. Hence,

$$D_n - D_{n-1} = \left(\frac{1-\alpha_n}{1+\alpha_n} - 1\right)\varphi_{n-1}(1)^2. \tag{7.7.14}$$

Since $C_n - C_{n-1} = \varphi_{n-1}(1)^2$, we have

$$\lim_{n\to\infty} \frac{C_n - C_{n-1}}{D_n - D_{n-1}} = \lim_{n\to\infty} \left(\frac{1-\alpha_n}{1+\alpha_n} - 1\right)^{-1} = \frac{1+L}{-2L}. \tag{7.7.15}$$

Therefore, $K_{n-1}(1,1)/\varphi_n(1)^2 = -(1+L)/2L$. By (7.7.11) and the Stolz–Cesàro Theorem,

$$\lim_{n\to\infty} \frac{A_n - A_{n-1}}{B_n - B_{n-1}} = -2 = \lim_{n\to\infty} \frac{A_n}{B_n}. \tag{7.7.16}$$

As a result,

$$\Delta_n(1) = -2L - 2c_n + o\left(c_n\right). \tag{7.7.17}$$

This proves Corollary 2.0.1. In particular, if $L = -1/2$ and $c_n = 1/n$, we have the rate of convergence of $\Delta_n(1)$ being $O(1/n)$, which is clearly not exponential.

7.8 Szegő condition and bounded variation

Both the Szegő condition and bounded variation of recursion coefficients come up in the study of orthogonal polynomials very often. In this section, we will show that there is a very large class of measures with Verblunsky coefficients of bounded variation satisfying $\alpha_n \to L \neq 0$ yet failing the Szegő condition (2.0.46).

Let $d\gamma$ be a non-trivial measure on $\mathbb{R}$ such that for all n, $\int |x|^n d\gamma < \infty$.

It is well-known that the family of orthonormal polynomials $(p_n(x))_{n\in\mathbb{N}}$ obey the following recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_{n+1}p_n(x) + a_n p_{n-1}(x) \tag{7.8.1}$$

for $n \geq 0$. The reader should refer to [45, 46] for details.

Remark: The reader should be reminded that the a_n's and b_n's in [46] are different from those in [45]! In fact, $a_{n+1}([46]) = a_n([45])$ and $b_{n+1}([46])=$ $b_n([45])$. In this paper, we are following the notations of [46].

Now we consider the measure $d\gamma$ on $\mathbb{R}$ which has recursion coefficients satisfying:

$$b_n \equiv 0, \qquad a_n \nearrow 1, \tag{7.8.2}$$

$$\sum_{n=1}^{\infty} |a_n - 1|^2 = \infty. \tag{7.8.3}$$

This measure, supported on $[-2, 2]$, is purely a.c., and has no eigenvalues outside $[-2,2]$. Moreover, if we write $d\gamma(x) = f(x)dx$, $f(x)$ is symmetric. By the Killip–Simon Theorem [27], condition (7.8.3) implies that such a measure fails the quasi-Szegő condition, i.e.

$$\int_{[-2,2]} (4 - x^2)^{1/2} \log f(x)dx = -\infty, \tag{7.8.4}$$

which is weaker than the Szegő condition

$$\int_{[-2,2]} (4 - x^2)^{-1/2} \log f(x)dx = -\infty. \tag{7.8.5}$$

Now we consider $d\gamma_y$ supported on $[-y, y] \subset [-2, 2]$, which is defined by scaling $d\gamma$

$$d\gamma_y(x) = d\gamma\left(2xy^{-1}\right), \quad 0 < y < 2. \tag{7.8.6}$$

Then the a.c. part of $d\gamma_y(x)$, supported on $[-y, y]$, is

$$f_y(x) = f(2xy^{-1})\chi_{[-y,y]}. \tag{7.8.7}$$

It is well-known that

$$a_n(d\gamma_y) = \left(\frac{y}{2}\right) a_n(d\gamma), \quad b_n(d\gamma_y) = \left(\frac{y}{2}\right) b_n(d\gamma). \tag{7.8.8}$$

Now we apply the inverse Szegő map (see Chapter 13 of [47]) to $d\gamma_y$ to form the probability measure μ_y on $\partial\mathbb{D}$. Under this map, we have $d\mu_y(\theta) =$

$w_y(\theta)\frac{d\theta}{2\pi}$ with

$$w_y(\theta) = 2\pi|\sin(\theta)|f_y(2\cos\theta)\chi_{[\theta_y,\pi-\theta_y]}(\theta), \tag{7.8.9}$$

where

$$\theta_y = \cos^{-1}\left(\frac{y}{2}\right) \in \left(0, \frac{\pi}{2}\right). \tag{7.8.10}$$

For for any g measurable on $[-2, 2]$,

$$\int g(x)d\gamma_y(x) = \int g(2\cos\theta)d\mu_y(\theta). \tag{7.8.11}$$

By Corollary 13.1.8 of [47], $b_n(\gamma_y) \equiv 0$ if and only if $\alpha_{2n}(d\mu_y) \equiv 0$. Moreover, by Theorem 13.1.7 of [47], we know that

$$\begin{aligned} a_{n+1}^2(d\gamma_y) &= (1-\alpha_{2n-1}(d\mu_y))(1-\alpha_{2n}(d\mu_y)^2)(1+\alpha_{2n+1}(d\mu_y)) \\ &= (1-\alpha_{2n-1}(d\mu_y))(1+\alpha_{2n+1}(d\mu_y)). \end{aligned} \tag{7.8.12}$$

Note that $w_y(\theta)$ is supported on two arcs, $[\theta_y, \pi-\theta_y]$ and $[\pi+\theta_y, 2\pi-\theta_y]$, and we can decompose $w_y(\theta)$ into

$$w_y(\theta) = w_y(\theta)|_{[\theta_y,\pi-\theta_y]} + w_y(\theta)|_{[\pi+\theta_y,2\pi-\theta_y]}. \tag{7.8.13}$$

Moreover, because $\gamma_y(x)$ is symmetric, each of the two components on the right hand side of (7.8.13) is symmetric along the imaginary axis. Hence, we can view $d\mu_y$ as a two-fold copy of the probability measure

$$d\nu_y(\theta) = m_y(\theta)\frac{d\theta}{2\pi} \tag{7.8.14}$$

defined on $\partial\mathbb{D}$ with

$$m_y(\theta) = 2w_y\left(\frac{\theta}{2}\right)\chi_{[2\theta_y, 2\pi-2\theta_y]} \tag{7.8.15}$$

(this is also called the sieved orthogonal polynomials, see Example 1.6.14 of [46]). Hence,

$$\alpha_{2k-1}(d\mu_y) = \alpha_{k-1}(d\nu_y). \tag{7.8.16}$$

In other words, the Verblunsky coefficients of $d\mu_y$ are

$$0, \alpha_0(d\nu_y), 0, \alpha_1(d\nu_y), 0, \alpha_2(d\nu_y)\dots \tag{7.8.17}$$

Therefore, (7.8.12) becomes

$$\left(\frac{y}{2}\right)^2 a_{n+1}^2(d\gamma) = (1-\alpha_{n-1}(d\nu_y))(1+\alpha_n(d\nu_y)) \tag{7.8.18}$$

for $n = 0, 1, \dots,$ with the convention that $\alpha_{-1} = -1$.

Now note that $d\nu_y$ is supported on the arc $[2\theta_y, 2\pi - 2\theta_y]$, so by the

Bello-López result [4] (see also Theorem 9.9.1 of [47]), for $a_y = \sin(\theta_y)$,

$$\begin{aligned} \lim_{n\to\infty} |\alpha_n(d\nu_y)| &= a_y\,, && (7.8.19)\\ \lim_{n\to\infty} \overline{\alpha_{n+1}(d\nu_y)}\alpha_n(d\nu_y) &= a_y^2\,. && (7.8.20) \end{aligned}$$

Since $\alpha_n \in \mathbb{R}$, $\alpha_n(d\nu_y)$ actually converges. Moreover, recall that $\theta_y \in (0, \frac{\pi}{2})$ was defined such that $\cos(\theta_y) = \frac{y}{2}$. Hence,

$$a_y = \sqrt{1-\cos^2(\theta_y)} = \sqrt{1-\left(\frac{y}{2}\right)^2}. \tag{7.8.21}$$

We rewrite (7.8.18) as follows

$$\left(\frac{y}{2}\right)^2 \frac{a_{n+1}^2(d\gamma)}{1-\alpha_{n-1}(d\nu_y)} - 1 = \alpha_n(d\nu_y). \tag{7.8.22}$$

When $n = 0$, we have $\alpha_0 = (\frac{y}{2})\frac{a_1^2}{2} - 1 < 0$. Hence, by an inductive argument on (7.8.22) we can show that $\alpha_n < 0$ for all $n \geq 0$.

Next, we want to prove that $(\alpha_n(d\nu_y))_{n\in\mathbb{N}}$ is of bounded variation if $(a_n(d\gamma))_{n\in\mathbb{N}}$ is. From now on, we let $\alpha_n = \alpha_n(d\nu_y)$, $a_n = a_n(d\gamma)$ and $c = (y/2)^2 < 1$.

By (7.8.22) above,

$$\alpha_n - \alpha_{n-1} = \frac{c(a_{n+1}^2 - a_n^2)}{1-\alpha_{n-1}} + \frac{ca_n^2(\alpha_{n-1}-\alpha_{n-2})}{(1-\alpha_{n-1})(1-\alpha_{n-2})}. \tag{7.8.23}$$

Therefore, by an inductive argument we conclude that $\sum_n (\alpha_n(d\nu_y) -$

$\alpha_{n-1}(d\nu_y)) < \infty$ for any $0 < y < 2$. Hence for any monotonic sequence of $a_n \to 1$ and any $0 < y < 2$, there corresponds a family of $\alpha_n(d\nu_y)$'s of bounded variation that converge to $-a_y < 0$.

Finally, we have to show that $m_y(\theta)$ fails the Szegő condition (2.0.46). Since $f(x)$ fails the quasi-Szegő condition (7.8.4), it also fails the Szegő condition (7.8.5). Upon scaling, (7.8.5) becomes

$$\int_y^{-y} (\log f_y(x)) \frac{1}{\sqrt{y^2 - x^2}} dx = -\infty. \tag{7.8.24}$$

Finally, by the Szegő map and a change of variables, (7.8.24) is equivalent to (2.0.46).

Chapter 8

Non-exponential Perturbation

8.1 The Szegő Mapping

It turns out that one can relate measures supported on $[-2,2]$ with a certain class of measures on $\partial\mathbb{D}$.

Note that the map $\theta \mapsto 2\cos\theta$ is a two-one map from $\partial\mathbb{D}$ to $[-2,2]$. Therefore, given a non-trivial probability measure $d\xi$ on $\partial\mathbb{D}$ that is invariant under $\theta \to -\theta$, we can define a measure

$$\gamma = \mathrm{Sz}(d\xi) \tag{8.1.1}$$

using what is known as the Szegő map, such that for g measurable on $[-2,2]$,

$$\int g(2\cos\theta)d\xi(\theta) = \int g(x)d\gamma(x) \tag{8.1.2}$$

Conversely, if we have a probability measure β supported on $[-2, 2]$, we can obtain a probability measure

$$d\nu = \mathrm{Sz}^{-1}(d\gamma) \tag{8.1.3}$$

on $\partial\mathbb{D}$ by what is known as the Inverse Szegő Mapping, such that for $h(z)$ measurable on $\partial\mathbb{D}$,

$$\int h(\theta) d\nu(\theta) = \int h\left(\cos^{-1}\frac{x}{2}\right) d\gamma(x) \tag{8.1.4}$$

There are many interesting results about the Szegő mapping (see Chapter 13 of [47]), but the only relevant one for this thesis is the following by Geronimus [19] (see also Theorem 13.1.7 of [47]):

Theorem 8.1.1 (Geronimus [19])**.** *Let $d\xi$ be a probability measure on $\partial\mathbb{D}$ which is invariant under $\theta \to -\theta$ and let $d\gamma = \mathrm{Sz}(\xi)$. Let $\alpha_n \equiv \alpha_n(d\xi)$, $a_n \equiv a_n(d\gamma)$ and $b_n \equiv b_n(d\gamma)$. Then for $n = 0, 1, 2, \ldots,$*

$$a_{n+1}^2 = (1 - \alpha_{2n-1})(1 - \alpha_{2n})^2(1 + \alpha_{2n+1}) \tag{8.1.5}$$

$$b_{n+1} = (1 - \alpha_{2n-1})\alpha_{2n} - (1 + \alpha_{2n-1})\alpha_{2n-2} \tag{8.1.6}$$

with the convention that $\alpha_{-1} = -1$.

We will consider two cases: $x_0 > 2$ and $x_0 < 2$.

8.1.1 Case 1: $x_0 > 2$

Suppose $d\gamma_0$ is a measure on $\mathbb{R}$ with recursion coefficients (a_n) and (b_n) satisfying

$$a_n \nearrow 1 \qquad b_n \equiv 0 \tag{8.1.7}$$

$$\sum_n |a_n - 1|^2 = \infty \tag{8.1.8}$$

The measure $d\gamma_0$ is purely absolutely continuous and symmetrically supported on $[-2, 2]$, with no pure points outside $[-2, 2]$. We scale it by a factor $0 < y < 2$ to form the measure $d\gamma_y$ supported on $[-y, y] \subset [-2, 2]$ (we will show the connection between y and x_0 a bit later; see (8.1.9)).

Then we use the Inverse Szegő map on $d\gamma_y$ to obtain $d\mu_y$, a probability measure on $\partial\mathbb{D}$. By looking at the Direct Geronimus Relations (8.1.5) and (8.1.6), we will find necessary conditions for $\alpha_n(d\mu_y)$ so that both (8.1.7) and (8.1.8) hold.

Since $d\gamma_y$ is supported on $[-y, y] \subset [-2, 2]$, we know that $d\mu_y$ is supported on two identical bands. Besides, $d\mu_y$ is symmetric along both the x- and y-axes because of the symmetry of $d\gamma_y$ and the Szegő map.

Next, we add a pure point at $z = 1$ to $d\mu_y$ to form the measure $d\tilde{\mu}_y$ and compute the perturbed Verblunsky coefficients $\alpha_n(d\tilde{\mu}_y)$.

Then we use the Szegő map on $d\tilde{\mu}_y$ to obtain the probability measure $d\tilde{\gamma}_y$ on $\mathbb{R}$. Finally, we scale $d\tilde{\gamma}_y$ to form the measure $d\tilde{\gamma}$.

Note that if we have chosen y such that

$$\frac{y}{2} = \frac{2}{|x_0|} \tag{8.1.9}$$

then we have $d\tilde{\gamma} = (1-\beta)d\gamma_0 + \beta\delta_{x_0}$.

As the final step, we show that for some constants C_{x_0}, D_{x_0} (both dependent on x_0) such that

$$a_n(d\tilde{\gamma}) = a_n(d\gamma_0) + \frac{C_{x_0}}{n^{3/2}} + o\left(\frac{1}{n^{3/2}}\right) \tag{8.1.10}$$

$$b_n(d\tilde{\gamma}) = b_n(d\gamma_0) + \frac{D_{x_0}}{n^{3/2}} + o\left(\frac{1}{n^{3/2}}\right) \tag{8.1.11}$$

8.1.2 Case 2: $x_0 < -2$

Everything in Case 1 will follow except that we add a point $z = -1$ to $d\mu_y$ instead. As we shall see later in the proof, $d\mu_y$ is symmetric both along the $x-$ and $y-$ axes. Therefore, adding a pure point at $z = -1$ is the same as adding a pure point at $z = 1$ and then rotating the measure by an angle of π.

For the convenience of the reader, here is a diagram showing all the measures involved. We will start from the measure $d\mu_y$, and move along two directions:

$$d\gamma_0 \overset{\text{scaling}}{\longleftarrow} d\gamma_y \overset{\text{Sz}^{-1}}{\longleftarrow} d\mu_y \overset{\text{add } z=1}{\longrightarrow} d\tilde{\mu}_y \overset{\text{Sz}^{-1}}{\longrightarrow} d\tilde{\gamma}_y \overset{\text{scaling}}{\longrightarrow} d\tilde{\gamma} \tag{8.1.12}$$

8.2 The Proof

Let $d\gamma_0$ be a probability measure on $\mathbb{R}$ with recursion coefficients satisfying (8.1.7) and (8.1.8).

This measure, supported on $[-2, 2]$, is purely absolutely continuous, and has no eigenvalues outside $[-2, 2]$. Moreover, if we write $d\gamma_0(x) = f(x)dx$, $f(x)$ is symmetric.

Now we scale $d\gamma_0$ to form the measure $d\gamma_y$ defined by

$$d\gamma_y(x) = d\gamma\left(2xy^{-1}\right) \quad 0 < y < 2 \tag{8.2.1}$$

The measure $d\gamma_y$, supported on $[-y, y] \subset [-2, 2]$, is purely absolutely continuous and the a.c. part of $d\gamma_y(x)$ is

$$f_y(x) = f(2xy^{-1})\chi_{[-y,y]} \tag{8.2.2}$$

which is also symmetric.

It is well known that scaling has the following effects on the recursion coefficients

$$a_n(d\gamma_y) = \left(\frac{y}{2}\right) a_n(d\gamma_0) \quad b_n(d\gamma_y) = \left(\frac{y}{2}\right) b_n(d\gamma_0) \tag{8.2.3}$$

Now we apply the inverse Szegő map to $d\gamma_y$ to form the probability measure $d\mu_y$ on $\partial\mathbb{D}$, see figure below:

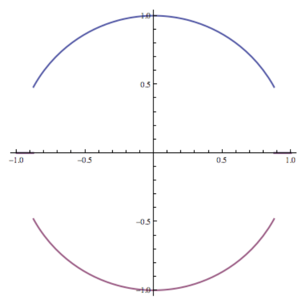

Figure 8.1: Graph of supp$(d\mu)$

The measure $d\mu_y$ is supported on two arcs, $[\theta_y, \pi - \theta_y]$ and $[\pi + \theta_y, 2\pi - \theta_y]$, with a.c. part

$$w_y(\theta) = w_y(\theta)|_{[\theta_y, \pi-\theta_y]} + w_y(\theta)|_{[\pi+\theta_y, 2\pi-\theta_y]} \tag{8.2.4}$$

where

$$w_y(\theta) = 2\pi|\sin(\theta)|f_y(2\cos\theta)\chi_{[\theta_y,\pi-\theta_y]}(\theta) labellasteqn110 \tag{8.2.5}$$

$$\theta_y = \cos^{-1}\left(\frac{y}{2}\right) \in \left(0, \frac{\pi}{2}\right) \tag{8.2.6}$$

By Theorem 8.1.1, $b_n(d\gamma_y) \equiv 0$ if and only if $\alpha_{2n}(d\mu_y) \equiv 0$. Therefore, we can express the Verblunsky coefficients of $d\mu_y$ as

$$0, \tau_0, 0, \tau_1, 0, \tau_2, \ldots \tag{8.2.7}$$

with $\tau_j = \alpha_{2j+1}$. Moreover, by Theorem 13.1.7 of [47], we know that

$$\begin{aligned} a_{n+1}^2(d\gamma_y) &= (1-\alpha_{2n-1}(d\mu_y))(1-\alpha_{2n}(d\mu_y)^2)(1+\alpha_{2n+1}(d\mu_y)) \\ &= (1-\tau_{n-1})(1+\tau_n) \end{aligned} \tag{8.2.8}$$

Now we will choose a suitable family of $\tau_n \in \mathbb{R}$ such that the corresponding $a_n(d\gamma_y)$ satisfy both (8.1.7) and (8.1.8).

Observe that by (8.2.8) above,

$$a_{n+1}(d\gamma_y)^2 - a_n(d\gamma_y)^2 = (1-\tau_{n-1})(\tau_n - \tau_{n-1}) + (1+\tau_{n-1})(\tau_{n-1} - \tau_{n-2}) \tag{8.2.9}$$

Therefore, if we have an increasing family of $\tau_n < 0$ such that

$$\tau_n \nearrow \tau_\infty = -\sqrt{1-\left(\frac{y}{2}\right)^2} < 0 \tag{8.2.10}$$

then $a_n(d\gamma_y) \nearrow y/2$ and the corresponding measure $d\mu_y$ has the desired properties.

In particular, if we let for $k \geq 1$

$$\tau_k = \tau_\infty - \frac{1}{\sqrt{k}} \tag{8.2.11}$$

then the goal is achieved.

Next, we prove the following lemma:

Lemma 8.2.1. *Let $d\mu_y$ be the measure on $\partial\mathbb{D}$ with Verblunsky coefficients* *as in* (8.2.7)*, where for all large* n*,*

$$\tau_n = \tau_\infty - \frac{1}{\sqrt{n}} \qquad -1 < \tau_\infty < 0 \tag{8.2.12}$$

Then for $n = 2m$ *or* $2m+1$*,* $\Delta_n(1)$ *(*Δ_n *as defined in* (2.0.7)*) has the following expansion*

$$\Delta_n(1) = -\tau_\infty + \frac{1}{\sqrt{m}} + 0 + \left(1 + \frac{1}{2\tau_\infty}\right)\frac{1}{m^{3/2}} + o\left(\frac{1}{m^{3/2}}\right) \tag{8.2.13}$$

Therefore, if we add a pure point at $z = 1$ *to* $d\mu_y$ *as in (2.0.9) to form* $d\tilde{\mu}_y$*, then the perturbed Verblunsky coefficients are given by*

$$\alpha_n(d\tilde{\mu}_y) = \begin{cases} -\tau_m + \left(1 + \frac{1}{2\tau_\infty}\right)\frac{1}{m^{3/2}} + e_m & n = 2m \\ \left(1 + \frac{1}{2\tau_\infty}\right)\frac{1}{m^{3/2}} + e_m & n = 2m+1 \end{cases} \tag{8.2.14}$$

where $e_m = o\left(m^{-3/2}\right)$*.*

Proof. Since all the Verblunsky coefficients of $d\mu_y$ are real, by induction on the recursion relation (1.0.11),

$$\varphi_n(1) = \prod_{j=0}^{n-1}\sqrt{\frac{1-\alpha_j}{1+\alpha_j}} \tag{8.2.15}$$

By (8.2.7), when $n = 2m$ or $2m+1$,

$$\varphi_n^*(1) = \varphi_n(1) = \prod_{j=0}^{m-1} \sqrt{\frac{1-\tau_j}{1+\tau_j}} \tag{8.2.16}$$

This formula will play a crucial role in the computation below.

8.2.1 n is even

First, we compute $\Delta_n(1)$ when $n = 2m$ using the point mass formula (2.0.11). Let

$$\begin{aligned} A_n &= \overline{\varphi_{n+1}(1)}\varphi_n^*(1) && (8.2.17) \\ B_n &= (1-\beta)\beta^{-1} + K_n(1,1) && (8.2.18) \end{aligned}$$

Then

$$\lim_{m\to\infty} \Delta_{2m}(1) = \lim_{m\to\infty} (1-|\alpha_{2m}|^2)^{1/2}\frac{A_{2m}}{B_{2m}} = \lim_{m\to\infty}\frac{A_{2m}}{B_{2m}} \tag{8.2.19}$$

because $\alpha_{2m} = 0$ for all m. However, instead of computing this directly, we use the Stolz–Cesàro theorem.

First, note that $\tau_k \to \tau_\infty < 0$. Thus,

$$B_m > K_n(1,1) > |\varphi_n(1)|^2 \to \infty \tag{8.2.20}$$

by (8.2.16). Hence, it is legitimate for us to use the Stolz–Cesàro Theorem.

Let $K_n \equiv K_n(1,1)$ and $\varphi_n \equiv \varphi_n(1)$. Observe that $\varphi_{2m+1} = \varphi_{2m}$ and $\varphi_{2m+2}^2 = \varphi_{2m}^2(1-\tau_m)/(1+\tau_m)$. Therefore, by (8.2.16),

$$B_{2(m+1)} - B_{2m} = \varphi_{2(m+1)}^2 + \varphi_{2m}^2 = \frac{2\varphi_{2m}^2}{1+\tau_m} \tag{8.2.21}$$

and

$$A_{2(m+1)} - A_{2m} = \varphi_{2(m+1)}^2 - \varphi_{2m}^2 = \left(\frac{-2\tau_m}{1+\tau_m}\right)\varphi_{2m}^2 \tag{8.2.22}$$

As a result,

$$\lim_{m\to\infty} \Delta_{2m}(1) = \lim_{m\to\infty} \frac{A_{2(m+1)} - A_{2m}}{B_{2(m+1)} - B_{2m}} = -\tau_\infty \tag{8.2.23}$$

Next, we will prove that the rate of convergence is

$$\Delta_{2m}(1) = -\tau_\infty + \frac{1}{\sqrt{m}} + o\left(\frac{1}{\sqrt{m}}\right) \tag{8.2.24}$$

by computing the following limit

$$\lim_{m\to\infty} \sqrt{m}\,(\Delta_{2m}(1) + \tau_\infty) = 1 \tag{8.2.25}$$

Recall the definition of $\Delta_n(1)$ and the facts that $\alpha_{2m} \equiv 0$ and $\varphi_{2m+1}\varphi_{2m} = \varphi_{2m}^2$. Thus, the left hand side of (8.2.25) can be expressed as X_n/Y_n, where

$$X_m = \sqrt{m}\left[\varphi_{2m}^2 + \tau_\infty(1-\beta)\beta^{-1} + \tau_\infty K_{2m}\right] \tag{8.2.26}$$

$$Y_m = (1-\beta)\beta^{-1} + K_{2m} \to \infty \tag{8.2.27}$$

We use the Stolz–Cesàro Theorem again. First, observe that

$$Y_{m+1} - Y_m = \frac{1-\tau_m}{1+\tau_m} + 1 = \frac{2}{1+\tau_m}\varphi_{2m}^2 \tag{8.2.28}$$

Then we compute

$$\begin{aligned} &X_{m+1} - X_m \\ &\quad = \underbrace{\left[\sqrt{m+1}\,\frac{1-\tau_m}{1+\tau_m} - \sqrt{m}\right]\varphi_{2m}^2}_{\text{(I)}} + \underbrace{\tau_\infty\left[\sqrt{m+1}K_{2(m+1)} - \sqrt{m}K_{2m}\right]}_{\text{(II)}} \\ &\qquad\qquad + \underbrace{\tau_\infty(1-\beta)\beta^{-1}(\sqrt{m+1} - \sqrt{m})}_{\text{(III)}} \end{aligned} \tag{8.2.29}$$

Note that (III) $\to 0$ as $m \to \infty$. Thus, it suffices to consider terms (I) and (II) in (8.2.29) above. Observe that

$$\begin{aligned} \frac{\text{(I)}}{Y_{2(m+1)} - Y_{2m}} &= \frac{\sqrt{m+1}\left(\frac{1-\tau_m}{1+\tau_m}\right) - \sqrt{m}}{\frac{2}{1+\tau_m}} \\ &= \frac{\sqrt{m+1}(1-\tau_m) - \sqrt{m}(1+\tau_m)}{2} \end{aligned} \tag{8.2.30}$$

Furthermore,

$$\text{(II)} = \tau_\infty\left[\sqrt{m+1}(K_{2(m+1)} - K_{2m}) + (\sqrt{m+1} - \sqrt{m})K_{2m}\right] \tag{8.2.31}$$

which implies that

$$\frac{(\mathrm{II})}{Y_{m+1}-Y_m} = \tau_\infty \left[\sqrt{m+1} + (\sqrt{m+1}-\sqrt{m})\frac{1+\tau_m}{2}\frac{K_{2m}}{\varphi_{2m}^2}\right] \tag{8.2.32}$$

Next, we are going to show that $\lim_{m\to\infty} K_{2m}/\varphi_{2m}^2 = -1/\tau_\infty$ by the Stolz–Cesàro Theorem. Observe that

$$\begin{aligned} \lim_{m\to\infty}\frac{K_{2m}}{\varphi_{2m}^2} &= \left(\lim_{m\to\infty}\frac{\varphi_{2(m+1)}^2-\varphi_{2m}^2}{K_{2(m+1)}-K_{2m}}\right)^{-1} \\ &= \lim_{m\to\infty}\left(\frac{1-\tau_m}{1+\tau_m}-1\right)^{-1}\left(\frac{1-\tau_m}{1+\tau_m}+1\right) \\ &= -\frac{1}{\tau_\infty} \end{aligned} \tag{8.2.33}$$

Combining (8.2.30), (8.2.32) and (8.2.33), we obtain (8.2.25).

Next, we are going to show that the second order term is zero. In other words,

$$\Delta_{2m}(1) = -\tau_\infty + \frac{1}{\sqrt{m}} + o\left(\frac{1}{m}\right) \tag{8.2.34}$$

We do so by proving that

$$L_2 \equiv \lim_{m\to\infty} m\left(\Delta_{2m}(1) - (-\tau_\infty) - \frac{1}{\sqrt{m}}\right) = 0 \tag{8.2.35}$$

Let

$$P_m = m\varphi_{2m}^2 + m\tau_\infty\left((1-\beta)\beta^{-1}+K_{2m}\right) - \sqrt{m}\left((1-\beta)\beta^{-1}+K_{2m}\right) \tag{8.2.36}$$

Then

$$P_{m+1} - P_m = \left[(m+1)\frac{1-\tau_m}{1+\tau_m} - m\right]\varphi_{2m}^2 + [(m+1)-m]\tau_\infty(1-\beta)\beta^{-1}$$
$$+ (m+1)\tau_\infty\left[K_{2(m+1)} - K_{2m}\right] + [(m+1)-m]\,\tau_\infty K_{2m}$$
$$- (\sqrt{m+1} - \sqrt{m})(1-\beta)\beta^{-1}$$
$$- \sqrt{m+1}\left[K_{2(m+1)} - K_{2m}\right] - (\sqrt{m+1} - \sqrt{m})K_{2m} \quad (8.2.37)$$

Combining with previous results about $Y_{m+1} - Y_m$ and K_{2m}/φ_{2m}^2, we have

$$L_2 = \lim_{m\to\infty} \frac{P_{m+1} - P_m}{Y_{m+1} - Y_m} = 0 \quad (8.2.38)$$

which proves (8.2.34).

Next, we will obtain the third-order term by computing

$$L_3 = \lim_{m\to\infty} m^{3/2}\left(\Delta_{2m} - (-\tau_\infty) - \frac{1}{\sqrt{m}}\right) \quad (8.2.39)$$

Let

$$J_m = m^{3/2}\varphi_{2m}^2 + \left(m^{3/2}\tau_\infty - m\right)\left((1-\beta)\beta^{-1} + K_{2m}\right) \quad (8.2.40)$$

By a similar argument as in (8.2.33),

$$\begin{aligned}J_{m+1} - J_m = &\left[(m+1)^{3/2}\frac{1-\tau_m}{1+\tau_m} - m^{3/2}\right]\varphi_{2m}^2 \\ &+ (1-\beta)\beta^{-1}\left[\left((m+1)^{3/2} - m^{3/2}\right)\tau_\infty - (m+1-m)\right] \\ &+ (m+1)^{3/2}\tau_\infty\left[K_{2(m+1)} - K_{2m}\right] + \left[(m+1)^{3/2} - m^{3/2}\right]\tau_\infty K_{2m} \\ &- (m+1)\left[K_{2(m+1)} - K_{2m}\right] - (m+1-m)K_{2m} \quad (8.2.41)\end{aligned}$$

which implies that

$$L_3 = \lim_{m\to\infty}\frac{J_m}{Y_m} = 1 + \frac{1}{2\tau_\infty} \quad (8.2.42)$$

8.2.2 n is odd

We compute $\Delta_n(1)$ when $n = 2m+1$ using the point mass formula and (2.0.12). Let A_n and B_n be defined as in (8.2.17) and (8.2.18). Then

$$\lim_{m\to\infty}\Delta_{2m+1}(1) = (1-|\tau_\infty|^2)^{1/2}\lim_{m\to\infty}\frac{A_{2m+1}}{B_{2m+1}} \quad (8.2.43)$$

We will use the Stolz–Cesàro Theorem again. Note that

$$A_{2(m+1)+1} - A_{2m+1} = \left(\sqrt{\frac{1-\tau_{m+1}}{1+\tau_{m+1}}}\frac{1-\tau_m}{1+\tau_m} - \sqrt{\frac{1-\tau_m}{1+\tau_m}}\right)\varphi_{2m}^2 \quad (8.2.44)$$

and because $\varphi_{2m+3} = \varphi_{2m+2}$,

$$B_{2(m+1)+1} - B_{2m+1} = 2\varphi_{2m+2}^2 = 2\left(\frac{1-\tau_m}{1+\tau_m}\right)\varphi_{2m}^2 \tag{8.2.45}$$

Therefore,

$$\lim_{m\to\infty} \Delta_{2m+1}(1) = \frac{-\tau_\infty(1-|\tau_\infty|^2)^{1/2}}{\sqrt{(1+\tau_\infty)(1-\tau_\infty)}} = -\tau_\infty \tag{8.2.46}$$

Next, we prove the rate of convergence by computing

$$\lim_{m\to\infty} \sqrt{m}\,(\Delta_{2m+1}(1) + \tau_\infty) = 1 \tag{8.2.47}$$

Since $\alpha_n \in \mathbb{R}$, the recursion relation becomes

$$(1-|\alpha_n|^2)^{1/2}\varphi_{n+1} = \varphi_n - \overline{\alpha_n}\varphi_n^* = (1-\alpha_n)\varphi_n \tag{8.2.48}$$

Therefore,

$$\Delta_{2m+1}(1) = \frac{(1-\alpha_{2m+1})\varphi_{2m+1}^2}{(1-\beta)\beta^{-1} + K_{2m+1}} = (1-\tau_m)\frac{\varphi_{2m}^2}{(1-\beta)\beta^{-1} + K_{2m+1}} \tag{8.2.49}$$

Let

$$P_m = \sqrt{m}\left[(1-\tau_m)\varphi_{2m}^2 + \tau_\infty((1-\beta)\beta^{-1} + K_{2m+1})\right] \tag{8.2.50}$$

$$Q_m = (1-\beta)\beta^{-1} + K_{2m+1} \to \infty \tag{8.2.51}$$

Note that

$$Q_{m+1} - Q_m = K_{2m+3} - K_{2m+1} = 2\varphi^2_{2(m+1)} \tag{8.2.52}$$

and

$$\begin{aligned} P_{m+1} - P_m = &\underbrace{\left[\sqrt{m+1}(1-\tau_{m+1})\varphi^2_{2(m+1)} - \sqrt{m}(1-\tau_m)\varphi^2_{2m}\right]}_{\text{(I)}} \\ &+ (1-\beta)\beta^{-1}\tau_\infty(\sqrt{m+1} - \sqrt{m}) + \tau_\infty\sqrt{m+1}\left[K_{2m+3} - K_{2m+1}\right] \\ &+ \underbrace{(\sqrt{m+1} - \sqrt{m})\tau_\infty K_{2m+1}}_{\text{(II)}} \end{aligned} \tag{8.2.53}$$

Since $(1-\tau_m)\varphi^2_{2m} = (1+\tau_m)\varphi^2_{2(m+1)}$,

$$\frac{\text{(I)}}{Q_{m+1} - Q_m} = \frac{\sqrt{m+1}(1-\tau_{m+1}) - \sqrt{m}(1+\tau_m)}{2} \tag{8.2.54}$$

Next, consider (II). We compute

$$\lim_{m\to\infty} \frac{\varphi^2_{2(m+1)} - \varphi^2_{2m}}{K_{2m+1} - K_{2m-1}} = \frac{\left(\frac{1-\tau_m}{1+\tau_m} - 1\right)\varphi^2_{2m}}{2\varphi^2_{2m}} = \frac{-\tau_\infty}{1+\tau_\infty} \tag{8.2.55}$$

which implies

$$\frac{\text{(II)}}{Q_{m+1} - Q_m} = -(1+\tau_\infty)(\sqrt{m+1} - \sqrt{m}) \tag{8.2.56}$$

Therefore,

$$\lim_{m\to\infty} m\left(\Delta_{2m+1}(1) - (-\tau_\infty)\right) = \lim_{m\to\infty} \frac{P_m}{Q_m} = 1 \tag{8.2.57}$$

Next, we will prove that

$$\Delta_{2m+1} = -\tau_\infty + \frac{1}{\sqrt{m}} + o\left(\frac{1}{m}\right) \tag{8.2.58}$$

by showing that

$$L_2' \equiv \lim_{m\to\infty} m\left(\Delta_{2m+1} + \tau_\infty - \frac{1}{\sqrt{m}}\right) = 0 \tag{8.2.59}$$

Let

$$H_m = m(1-\tau_m)\varphi_{2m}^2 + \left(m\tau_\infty - \sqrt{m}\right)\left((1-\beta)\beta^{-1} + K_{2m+1}\right) \tag{8.2.60}$$

$$\begin{aligned} H_{m+1} - H_m = {} & \underbrace{(m+1)(1-\tau_{m+1})\varphi_{2(m+1)}^2 - m(1-\tau_m)\varphi_{2m}^2}_{\text{(I)}} \\ & \underbrace{+(m+1)\tau_\infty K_{2m+3} - m\tau_\infty K_{2m+1}}_{\text{(II)}} \underbrace{-\sqrt{m+1}K_{2m+3} + \sqrt{m}K_{2m+1}}_{\text{(III)}} \\ & + \underbrace{(1-\beta)\beta^{-1}(\tau_\infty(m+1-m) - (\sqrt{m+1} - \sqrt{m}))}_{\text{(IV)}} \end{aligned} \tag{8.2.61}$$

First, note that

$$\lim_{m\to\infty}\frac{(\mathrm{IV})}{Q_{m+1}-Q_m}=0 \tag{8.2.62}$$

Since $(1-\tau_m)\varphi_{2m}^2=(1+\tau_m)\varphi_{2(m+1)}^2$, we have

$$\frac{(\mathrm{I})}{Q_{m+1}-Q_m}=\frac{(m+1)(1-\tau_{m+1})-m(1+\tau_m)}{2} \tag{8.2.63}$$

$$\frac{(\mathrm{II})}{Q_{m+1}-Q_m}=\tau_\infty(m+1)+(m+1-m)\tau_\infty\frac{K_{2m+1}}{2\varphi_{2(m+1)}^2} \tag{8.2.64}$$

$$\frac{(\mathrm{III})}{Q_{m+1}-Q_m}=\frac{(-\sqrt{m+1}+\sqrt{m})K_{2m+3}}{2\varphi_{2(m+1)}^2}+(-\sqrt{m}) \tag{8.2.65}$$

Hence, $L'=0$ and this proves (8.2.58).

Next, we compute

$$L_3'=\lim_{m\to\infty}m^{3/2}\left(\Delta_{2m+1}+\tau_\infty-\frac{1}{\sqrt{m}}\right) \tag{8.2.66}$$

By similar arguments as in (8.2.63), (8.2.64) and (8.2.65), we obtain

$$L_3'=1+\frac{1}{2\tau_\infty} \tag{8.2.67}$$

This concludes the proof of Lemma 8.2.1. □

Finally, we apply the Szegő map to this perturbed measure $d\tilde{\mu}_y$ to form the perturbed measure $d\tilde{\gamma}_y$ on $[-2, 2]$, which is defined by

$$\tilde{\gamma}_y(x) = (1-\gamma)d\gamma_y(x) + \gamma\delta_{x=2} \tag{8.2.68}$$

For the sake of convenience, we temporarily denote $\alpha_n \equiv \alpha_n(d\tilde{\mu}_y)$. Since $b_n(d\gamma_y) \equiv 0$, it suffices to consider

$$b_{n+1}(d\gamma_y) = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n-1}} + o\left(\frac{1}{n^{3/2}}\right) = \frac{-1}{2n^{3/2}} + o\left(\frac{1}{n^{3/2}}\right) \tag{8.2.69}$$

The computation of $a_n(d\tilde{\gamma}_y)$ is more complicated. Recall that

$$a_{n+1}(d\tilde{\gamma}_y)^2 = (1-\alpha_{2n-1})(1-\alpha_{2n}^2)(1+\alpha_{2n+1}) \tag{8.2.70}$$

and we know that

$$a_{n+1}(d\gamma_y)^2 = (1-\tau_{n-1})(1+\tau_n) \tag{8.2.71}$$

Therefore, upon solving the algebra, we obtain

$$a_{n+1}(d\tilde{\gamma}_y)^2 - a_{n+1}(d\gamma_y)^2 = \frac{1}{2(1+\tau_\infty)m^{3/2}} + o\left(\frac{1}{m^{3/2}}\right) \tag{8.2.72}$$

After scaling, we have

$$a_{n+1}^2(d\gamma) - a_{n+1}^2(d\gamma_0) = \frac{2}{y^2(1+\tau_\infty)m^{3/2}} + o\left(\frac{1}{m^{3/2}}\right) \tag{8.2.73}$$

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