Appendix D

Derivation of the Finite Time Lyapunov Exponent

Consider an arbitrary fluid particle $x(t)$ in a given domain $\in X$ at time t. Let Φ_t^{t+T} define a function which maps a particle at time $x(t)$ to time $x(t+T)$. Given that fluid is a continuum and behaves according to conservation of mass and conservation of momentum particles near $x(t)$ will behave similarly in time.

Consider the location $y(t)$ on the opposite side of a divergent manifold where:

$$y(t) - x(t) = \delta x(t) \tag{D.1}$$

Where $\delta x(t)$ is an infinitismal distance in an arbitrary orientation. After some time period T the particle has moved a distance of

$$\delta x(t+T) = y(t+T) - x(t+T) = \Phi_t^{t+T}(y) - \Phi_t^{t+T}(x) \tag{D.2}$$

Performing a Taylor expansion about x:

$$\delta x(t+T) = \Phi_x^{t+T} + \frac{d\Phi_x^{t+T}}{dx}(y(t) - x(t)) + ...O\left\|\delta x(t+T)\right\|^2 - \Phi_t^{t+T}(x) \tag{D.3}$$

$$\delta x(t+T) = \frac{d\Phi_x^{t+T}}{dx}\delta x(t) \tag{D.4}$$

$$\|\delta x(t+T)\| = \sqrt{\left\langle \frac{d\Phi_x^{t+T}}{dx}\delta x(t), \frac{d\Phi_x^{t+T}}{dx}\delta x(t) \right\rangle} \tag{D.5}$$

$$\|\delta x(t+T)\| = \sqrt{\left\langle \delta x(t), \frac{d\Phi_x^{t+T}}{dx}^{*} \frac{d\Phi_x^{t+T}}{dx}\delta x(t) \right\rangle} \tag{D.6}$$

Now we define the symmetric matrix ϵ given by:

$$\epsilon = \frac{d\Phi_x^{t+T}}{dx}^* \frac{d\Phi_x^{t+T}}{dx} \tag{D.7}$$

Since we are investigating stretching, we notice $\delta x(t)$ is maximum when aligned with the eigenvector associated with the maximum eigenvalue of ϵ.

If $\lambda_{max}(\epsilon)$ is the maximum eigenvalue of ϵ then:

$$\max_{\delta x(t)} \|\delta x(t+T)\| = \sqrt{\left\langle \overline{\delta x(t)}, \lambda_{max}(\epsilon)\overline{\delta x(t)} \right\rangle} \tag{D.8}$$

$$\max_{\delta x(t)} \|\delta x(t+T)\| = \sqrt{\overline{\delta x(t)}^2 \lambda_{max}(\epsilon)} \tag{D.9}$$

$$\max_{\delta x(t)} \|\delta x(t+T)\| = \left\|\overline{\delta x(t)}\right\| \sqrt{\lambda_{max}(\epsilon)} \tag{D.10}$$

Now define the finite time Lyapunov exponent (FTLE) at location x at time t with integration time T as:

$$\sigma_t^T(x) \equiv \frac{1}{|T|} \ln \sqrt{\lambda_{max}(\epsilon)} \tag{D.11}$$

Therefore the factor by which the flow is maximally stretched is given by:

$$e^{\sigma_t^T(x)|T|} = \sqrt{\lambda_{max}(\epsilon)} \tag{D.12}$$

Substituting equation (D.12) into (D.11) yields:

$$\max_{\delta x(t)} \|\delta x(t+T)\| = \left\|\overline{\delta x(t)}\right\| e^{\sigma_t^T(x)|T|} \tag{D.13}$$