NAVIER-STOKES SOLUTIONS AT LARGE DISTANCES FROM A FINITE BODY

Thesis by

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In Partial Fulfillment of the Requirement

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

June, 1959

ACKNOWLEDGMENTS

The author is deeply grateful to his advisor Professor P. A. Lagerstrom who suggested the research topic and whose close guidance and kind help throughout the course of the investigation have been a constant source of inspiration.

He is also greatly indebted to Dr. S. Kaplun who, with his unusual competence in this field, gave many suggestions which formed the basis of this study.

Lastly, thanks are due Mrs. A. Tingley for her capable help in the preparation of the manuscript.

In the course of this study, the author was supported by the Office of Naval Research, Contract Nonr-220(29), Project NR 041-156, and, for the final stage of the writing, by an Air Force Office of Scientific Research Contract AF 49 (638)-521.

ABSTRACT

The asymptotic expansion of the Navier-Stokes solutions at fixed Reynolds numbers and large distances from a finite object for an incompressible, stationary and two-dimensional flow is studied. The expansion is a coordinate-type expansion and differs in many mathematical aspects from the more familiar parameter-type expansions for large and small Reynolds number flows. These differences are noted and discussed in some detail. The technique chosen for dealing with the problem is that of the use of an artificial parameter. This is one possible method for using some of the techniques of parameter-type expansions. In particular, at large distances from the object one may distinguish a viscous wake region and a potential ("outer") flow region. The relation between these regions is very similar to the relation between the viscous boundary layer and the potential flow region for flow at large Reynolds numbers.

Several terms of the expansion are computed. However, the main emphasis is placed on discussing the methods for deriving these terms. The special features of expansions in artificial parameters are discussed in detail. The role of various properties of Navier-Stokes solutions, such as validity of integral theorems and rapid decay of vorticity is also brought out.

The original motivation of the study was an attempt to understand the Filon's paradox which historically was an error in evaluating the momentum integral of the asymptotic flow field. The present study, however, deals with the general problem of the flow at large distances from a finite object, and, more generally, with expansion techniques for similar problems. The author's explanation of Filon's paradox is only an incidental result.

TABLE OF CONTENTS

CHAPTER PAGE

I. INTRODUCTION

This paper is concerned with a theoretical investigation of the flow field at large distances from an object moving through a viscous fluid. The discussion will be restricted to the case of two-dimensional stationary incompressible flow. The object will be assumed to be of finite size. The domain of the fluid is infinite and it is assumed that there are no other boundaries for the fluid except that of the given object. The Reynolds number will be assumed to have a fixed value; thus we shall not consider the limiting cases of the Reynolds number tending to zero or to infinity.

The mathematical formulation of our problem is then the following: We consider a certain infinite domain consisting of the region of the plane outside of a fixed closed curve which represents the boundary of the object. The flow field in this domain is then represented by a time-independent solution of the Navier-Stokes equations which satisfies certain boundary conditions at the curve and at infinity (cf. Section 4). The existence of this solution is of course assumed. The results to be derived will actually be valid whether the solution is unique. We consider only time-independent solutions even though these solutions may not be stable. The complete solution of this problem will in principle give a vector function for the velocity $\vec{q}^*(r^*, \theta; Re)$ and a function for the pressure $p^*(r^*, \theta; Re)$. Here non-dimensional variables are used; the notation is explained on page 74. These functions cannot be obtained explicitly; however, their form at large distances may be obtained analytically. More precisely we shall study expansions of these functions which are asymptotically valid for large r^*, uniformly in θ and for a fixed value of the Reynolds number Re. Since the flow near the object is not considered the asymptotic expansions will involve certain undetermined constants. Some of these constants may be identified with the (unknown!) lift and drag of the body and with the torque on the body. The leading terms of the expansions will be found and methods given for successively computing the higher-order terms.

Actually, due to the fact that the flow near the body is not considered, the asymptotic expansions obtained give the solution of a somewhat more general problem: It is only assumed that there exists a closed finite curve C (which may be thought of as a curve enclosing the body, not necessarily the body contour itself) such that the flow is regular on C and that there are no boundaries for the fluid outside C. The exact boundary conditions at infinity are used, but the boundary condition at the body is replaced by the more general condition that the flow is regular on C. By regularity is meant that the velocity and the pressure as well as sufficiently many of their derivatives are finite on C. It will be assumed that the net mass flow through C is zero. This condition, which is a consequence of the no-slip condition at the body, could easily have been relaxed.

In the present paper we are concerned with the problem of expanding $\vec{q}^*$ for large values of the coordinate r^* (coordinate-type expansion). This type of problem differs mathematically from the problem of expanding $\vec{q}^*$ for large values of Reynolds number (parameter-type expansion). However, there is one important feature that is common to both problems. Even at a fixed Reynolds number there are two distinct flow regions at large distances from the object: A viscous wake region and a region of potential ("outer") flow. The relation between these regions is very similar to the relation between the viscous boundary layer and the potential flow region for flow at large Reynolds numbers. The latter problem may be studied with the aid of techniques applicable to certain parameter-type singular perturbation problems. One method of carrying over this technique to our present problem is to introduce a certain artificial parameter and then to treat our problem as a parameter-type expansion. The general nature of coordinate-type expansions and parameter-type expansions as well as the use of artificial parameters are discussed in Section 2. The specific artificial parameter used here is explained in Section 4. It must be emphasized that the author uses the technique of artificial parameters only for convenience. In principle, the material of Section 4 may be rephrased in a language which avoids the use of artificial parameters.

Section 3 contains a review of certain properties of exact solutions of the Navier-Stokes equations. The two main items are the conservation laws for mass, momentum, etc. and the principle of rapid decay of vorticity. The fundamental solution of the Oseen equations and the splitting theorem for Oseen solutions are also reviewed.

After the preparatory discussion in preceding sections a number of the leading terms of the expansions for velocity and pressure at large distances are derived in Section 4. Emphasis is placed on discussing the methods for determining these terms. The same methods may in principle be used for finding terms of arbitrarily high order. In these expansions certain intermediate terms of logarithmic nature appear. The mechanism which forces the introduction of these terms is called "switchback." This phenomenon is connected with certain deeper questions in the theory of asymptotic expansions.

The flow field at large distances from an object was originally studied by Filon (reference 1). The procedure was based on the Oseen equations. Filon, however, was misled by the discovery that his "second approximation" leads to an angular momentum integral which depends on contour of integration and which tends to infinity as the contour increases (Filon's "paradox"). Goldstein (reference 2) showed that the boundary-layer type approximation is sufficient to obtain Filon's term. He gave a simple explanation for the existence of the term responsible for paradox, and showed that Filon's approximation to the flow field is correct. The paradox was resolved by Imai (reference 3) by showing that a certain term of higher order gives a contribution to the angular momentum integral which is comparable to that of Filon's second approximation. This makes the angular momentum integral finite and contour independent. Imai used an expansion procedure based on the Oseen equations. He showed that in principle no essential difficulty is involved in this procedure, and obtained explicitly a large number of terms. However, his procedure is unduly complicated.

The present report uses a simple and systematic expansion procedure. The resulting simplifications are similar to those of Goldstein.

II. COORDINATE- AND PARAMETER-TYPE EXPANSIONS

2.1. Coordinates vs. Parameters

Consider the function $\vec{q^*}(r^*, \theta; Re)$ defined in the introduction. The velocity $\vec{q^*}$ depends on three independent variables r^*, θ, and Re. One calls r^* and θ coordinates and Re a parameter. The reason for this nomenclature is obvious from the physical meaning of the variables; however, for our purpose it is necessary to clarify the mathematical nature of the distinction between parameters and coordinates. The main problem of this paper is to find an asymptotic expansion of $\vec{q^*}$ for a fixed value of Re which is uniformly valid in θ for large values of r^*. We shall call such an expansion a coordinate-type expansion. Another type of problem (not studied here) is to find an expansion of $\vec{q^*}$ which is uniformly valid in r^* and θ for large values of Re. Such an expansion will be called a parameter-type expansion. The mathematical problems involved in constructing these two expansions from the equations and boundary conditions exhibit certain important differences. A discussion of these differences will form the main content of the present Section.

We shall begin with a discussion of the different nature of coordinates and parameters. General exact definitions will not be given; it is sufficient for our purpose to illustrate the difference with the aid of examples. To fix the ideas, consider a function $f(x, \epsilon)$ of two arguments x and ϵ, defined for $0 \leqslant x < \infty$, $0 \leqslant \epsilon < \infty$. If the function is given explicitly there is no mathematical reason for calling one of its arguments a coordinate and the other one a parameter (although from the point of view of applications x might be a position coordinate and ϵ a parameter in the physicist's sense). However, assume now that the function $f(x, \epsilon)$ is defined implicitly as follows: The function f satisfies, say, a second-order ordinary differential equation. The argument ϵ may appear in equation but all derivatives are taken with respect to x. In addition f satisfies two boundary conditions, say $f(0)$ and $f(\infty)$ are prescribed, possibly as functions of ϵ. (More generally one could have assumed that the value of f is prescribed for $x = x_1$ and $x = x_2$ where x_1 and x_2 may depend on ϵ.) With respect to this implicit definition of f we may call x a coordinate of f and ϵ a parameter of f. Note that in principle it is possible to give ϵ a definite value, say $\epsilon = 1$, and find $f(x, 1)$ for all x without considering values of $f(x, \epsilon)$ for $\epsilon \neq 1$; on the other hand, we cannot determine $f(1, \epsilon)$ without considering $f(x, \epsilon)$ for $x \neq 1$. In general, the region of influence of the point $x = 0$, $\epsilon = \epsilon_0$ = fixed extends over the whole line $0 \leq x \leq \infty$, $\epsilon = \epsilon_0$. If the value of $f(0, \epsilon_0)$ is changed, then, in principle, all the values $f(x, \epsilon_0)$ may be changed. A similar statement cannot be made for the dependence on ϵ. Changing the value of $f(0, \epsilon_0)$ does not imply that the values of $f(x_0, \epsilon_1)$ or even $f(0, \epsilon_0)$ have to be changed for $\epsilon_1 \neq \epsilon_0$.

If now $\vec{q}^*(r^*, \theta; Re)$ is implicitly defined by the Navier-Stokes equations and boundary conditions at infinity and, say, at the circle $r^* = 1$, it is clear that it is consistent with the terminology used above to call r^* and θ coordinates and Re a parameter. No derivatives with respect to Re occur in the Navier-Stokes equations. One may say that one set of boundary conditions is given on the cylinder $r^* = 1$, $0 \leq \theta \leq 2\pi$, $0 < Re < \infty$ in (r^*, θ, Re)-space. The region of influence of a point $(1, \theta_0, Re_0)$ on this cylinder is the planar region $r^* \geqslant 1$, $0 \leqslant \theta \leqslant 2\pi$, $Re = Re_0$ but does not extend to points in (r^*, θ, Re)-space for which $Re \neq Re_0$. Thus, with respect to the implicit definition of $\vec{q^*}$, r^* and Re play different roles. This accounts for the fact that when the expansions for large r^* or for large Re respectively are determined from the implicit definition, the mathematical problems in the two cases differ in certain important respects.

2.2. Parameter-Type Expansion. Singular Perturbation Problems and Matching.

Let the function $f(x, \epsilon)$ be defined by a second-order differential equation and by boundary conditions as in Section 2.1. One may then pose the problem of finding an asymptotic expansion of f which is uniformly valid in x for ϵ near zero. In the simplest case (regular perturbation problems) this expansion has the form of a convergent series

$$f(x) \simeq \sum_{n=0}^{\infty} \delta_n(\epsilon) f_n(x) \tag{2-1a}$$

where the $\delta_n(\epsilon)$ are functions of ϵ such that

$$\delta_0(\epsilon) = 1, \quad \lim_{\epsilon \downarrow 0} \frac{\delta_{n+1}}{\delta_n} = 0 \tag{2-1b}$$

By inserting this expansion into the equation and boundary conditions for f one obtains in typical cases a second-order equation and two boundary conditions for each f_n; these conditions determine each f_n uniquely.

In other problems (singular perturbation problems) it may turn out that f has an expansion of the form 2-1 for regions not including $x = 0$ but that a different expansion is needed near the origin. This expansion may have the form

$$f(x) \simeq \sum_{n=0}^{\infty} \delta_n(\epsilon) g_n(\overline{x}) \tag{2-2a}$$

where

$$a(\epsilon)\overline{x} = x, \quad \lim_{\epsilon \downarrow 0} a(\epsilon) = 0 \tag{2-2b}$$

The expansions 2-1 and 2-2 are then called the outer and inner expansions respectively. Introduction of each expansion into the differential equation leads to the differential equations for f_n and g_n. Inserting 2-1 in the boundary condition at infinity gives one boundary condition for f_n; inserting 2-2 into the boundary condition at $x = 0$ gives one boundary condition for g_n. Since some of the f_n and g_n will satisfy second order equations these boundary conditions are insufficient for the determination of f_n and g_n. Additional conditions are obtained from matching of the two expansions. It can be shown for many singular perturbation problems that there exist relations between a finite partial sum of the outer expansion and a finite partial sum of the inner expansion. Such conditions are referred to as matching conditions. They furnish the missing boundary conditions necessary for the complete determination of the f_n and g_n. A classical example of a matching condition occurs in boundary-layer theory. The first term in the outer expansion (for flow without separation) is the potential flow past a body; the first term in the inner expansion is the boundary-layer solution. These are related by the requirement that the tangential velocity component of the outer flow at the body surface equal the tangential velocity component of the boundary layer at infinity. This matching principle completes the boundary conditions for the boundary-layer equations.

It should be emphasized that a matching condition, say between f_0 and g_0, is an exact relation, not a numerical approximation. When ϵ is small one may think of g_0 as being approximately valid for x between zero and some rather arbitrarily chosen small value of x, say $x = x_0$, and one may consider f_0 to be approximately valid for $x \geqslant x_0$. One may then choose as a boundary condition that $f(x_0) = g(x_0)$. This, however, is only an approximate relation and will be referred to here as numerical patching. Such a numerical patching in a parameter-type expansion is not only logically less satisfactory than an exact matching condition but is in addition generally less practical. The nature of and the justification of exact matching conditions are discussed at length in references 4, 5 and 6.

A final remark regarding uniformly valid expansions will be made. Suppose that $f(x, \epsilon)$ has an outer and an inner expansion with matching conditions as described above. Once these have been found it is generally easy to construct a simple uniformly valid expansion

$$f \simeq \sum_{n=0}^{\infty} h_n(x, \epsilon) \qquad (2\text{-}3a)$$

which is asymptotic in the sense that

$$\lim \frac{f - \sum_{i=0}^{n} h_i(x, \epsilon)}{\zeta_n(\epsilon)} = 0 \quad \text{uniformly in } x \qquad (2\text{-}3b)$$

where the ζ_n are functions used to gauge the validity of the approximation, satisfying relations

$$\zeta_0(\epsilon) = 1, \quad \lim_{\epsilon \downarrow 0} \frac{\zeta_{n+1}}{\zeta_n} = 0 \qquad (2\text{-}3c)$$

This is discussed in references 4, 5 and 6.

2.3. Coordinate Type Expansions

As a special example consider the Blasius function occurring in the theory of viscous flow past a semi-infinite plate. It may be considered as defined by the equation

$$\frac{d^3 f}{dx^3} + f \frac{d^2 f}{dx^2} = 0 \qquad (2\text{-}4a)$$

$$f(0) = f'(0) = 0 \qquad (2\text{-}4b)$$

$$f'(\infty) = 1 \qquad (2\text{-}4c)$$

The argument is the coordinate x which actually is a combination of various physical quantities.

The equation and the boundary condition 2-4b give the following expansion of $f(x)$ near $x = 0$

$$f = \frac{\alpha}{2!} x^2 - \frac{\alpha^2}{5!} x^5 + \frac{11\alpha^3}{8!} x^8 + \ldots \qquad (2\text{-}5a)$$

In 2-5a α is a constant remaining undetermined.

Near $x = \infty$ one obtains, using 2-4a and 2-4c

$$f \sim x - \beta + \gamma \int_\infty^x dx \int_\infty^x e^{-\frac{1}{2}(x-\beta)^2} dx + \ldots \qquad (2\text{-}5b)$$

This asymptotic solution of $f(x)$ contains two unknown constants β and

γ, since only one of the three boundary conditions for f has been satisfied. Expansions 2-5a and 2-5b are examples of the coordinate-type expansion. Note that no exact relation exists between any finite number of terms of expansions 2-5a and 2-5b respectively. Thus α, β, γ cannot be determined by matching of a finite number of terms of 2-5a to a finite number of terms of 2-5b. For an exact determination of α, β, and γ, one needs the complete infinite series 2-5a. Let us denote by $f_\alpha(x)$ the function defined by this series. The leading term x in 2-5b may be said to match to the complete function $f_\alpha(x)$. The problem is therefore a matter of analytic continuation. In other words, if 2-5a were complete and continued analytically to infinity one could determine α by requiring that the leading term of the expansion of $f_\alpha(x)$ near infinity is x.

For numerical purposes it may be possible to use a finite number of terms of 2-5a and 2-5b and adjust the constants α, β, γ so that 2-5a and 2-5b agree approximately at some intermediate points. This numerical patching may be useful but is only an approximation and should be distinguished from the exact relations valid in matching for parameter-type expansions.

If a function $f(x, \epsilon)$ is defined as in Section 2.1, so that x appears as coordinate and ϵ as parameter, one may still try to obtain expansions of f for small and large x at fixed ϵ. These are coordinate-type expansions of the type described for Blasius function and no matching is possible between them in the sense valid for parameter-type expansions (cf. Section 2.2). It is a problem of this type that will be studied in the present article, namely the expansion of $\vec{q}^*(r^*, \theta; Re)$ for fixed Re and large r^*. It is then clear that expansion at large r^* cannot be matched to an expansion near the body unless the exact solution near the body is obtained. Thus the leading term in the expansion for large r^* may be said to match to the exact solution of the Navier-Stokes equation near the body, or, more precisely the analytic continuation thereof. Thus in principle this "matching" involves getting the exact solution of $\vec{q^*}$ and is consequently a very impractical procedure indeed. However, later in Section 4 it will be seen that matching of certain integrated values of the solution near the body and the solution near infinity is very useful in the study of coordinate-type expansions.

2.4. Artificial Parameters

Consider a function $f(x, \epsilon)$ defined by a differential equation as in Section 2.1 with x appearing as a coordinate and ϵ as a parameter. For certain special types of equation it may happen that the parameter ϵ, while appearing in the formulation, can be eliminated from the equation and the boundary conditions by coordinate transformation. As an example, consider the equation

$$\epsilon^2 \frac{d^2 f}{dx^2} + x \frac{df}{dx} = 0 \qquad (2\text{-}6a)$$

with the boundary conditions

$$f(0) = 0 \qquad f(\infty) = 1 \qquad (2\text{-}6b)$$

The parameter ϵ in 2-6 is then eliminable, because if one puts

$$\bar{x} = \frac{x}{\epsilon} \qquad (2\text{-}7a)$$

and re-formulates the problem in $\bar{x}$ coordinate, the parameter ϵ will disappear completely from the problem. A parameter which can be eliminated in this way is called an artificial parameter.

There are many physical problems in which artificial parameters appear. A typical example is the problem of incompressible viscous flow past a semi-infinite flat plate. In the original physical formulation two parameters occur, namely, the viscosity ν and the free-stream velocity U. These parameters may be completely eliminated from the mathematical formulation of the problem by replacing the coordinates x and y by $\frac{xU}{\nu}$ and $\frac{yU}{\nu}$. Thus in spite of the physical reality of these parameters they are, for the specific boundary-value problem studied, artificial from a mathematical point of view.

Another example for which a physically real parameter is from a mathematical point of view an artificial parameter is that of incompressible non-viscous flow past a paraboloid of revolution. Here we do have a true length, namely, the nose radius of curvature L of the paraboloid. Note, however, that no other length parameter occurs. The radius of curvature, L, being the only length, disappears after the equation and boundary conditions are expressed in non-dimensional coordinates. Thus L is in this sense an artificial parameter. Note that in the two examples given, each case has only one length: the semi-infinite plate problem has a viscous length and the paraboloid problem has a true length. It follows from dimensional considerations that if only one length parameter occurs in a physical problem, this parameter must be artificial.

Consider now a function $f(x, \epsilon)$ which has a parameter-type expansion as described by 2-3 except that uniformity in x is not required. If ϵ is an artificial parameter then the expansion is either not uniform, or the first term contains an exact solution. A more precise formulation is given by the lemma below (cf. reference 4)

<u>Lemma:</u> Let $f(x, \epsilon)$ be defined for $0 \leqslant x < \infty$ and for $0 < \epsilon < \epsilon_0$ where ϵ_0 is some arbitrary number greater than zero. Let $f^*(x, \epsilon)$ be an approximation to f uniformly valid to order unity in some interval $0 \leqslant x \leqslant x_0$, i. e.

$$\lim_{\epsilon \downarrow 0} |f(x,\epsilon) - f^*(x,\epsilon)| = 0, \quad \text{uniformly in } 0 \leqslant x \leqslant x_0 \tag{2-8a}$$

Furthermore, assume that f has the similarity form

$$f(x, \epsilon) = g(\overline{x}) \quad \text{where} \quad \overline{x} = \frac{x}{\epsilon} \tag{2-8b}$$

Then it follows that

$$\lim f^*, \ \epsilon \downarrow 0, \ \overline{x} \ \text{fixed, exists and equals} \ f(x, \epsilon) \tag{2-8c}$$

<u>Proof:</u> We write $f^*(x, \epsilon) = g^*(\overline{x}, \epsilon)$. We want to show that for a fixed value of $\overline{x}$ and for an arbitrary $\delta > 0$ one may find an $\epsilon_1(\delta)$ such that $|g(\overline{x}) - g^*(\overline{x},\epsilon)| < \delta$ for ϵ in the range $0 < \epsilon < \epsilon_1$. One first chooses an ϵ_2 such that $|f - f^*| < \delta$ in the rectangle $0 \leqslant x \leqslant x_0$, $0 < \epsilon < \epsilon_2$, and then one chooses an ϵ_3 such that for $0 < \epsilon < \epsilon_3$, the point $(\epsilon\overline{x}, \epsilon)$ is in the rectangle just described. For the required $\epsilon_1(\delta)$ we may then choose any positive number smaller than both ϵ_2 and ϵ_3.

<u>Corollary 1.</u> It follows that if f^* has the same similarity as f,

ie. if g^* depends on $\bar{x}$ only and not on ϵ, then $f^* = f$ under the assumptions of the lemma. On the other hand, if for example $f^* = f + \epsilon$ then the assumptions of the lemma are satisfied, but f^* becomes equal to f only in the limit 2-8c .

Corollary 2. If f^* is an approximation to f such that

$$\lim_{\epsilon \downarrow 0} |f^* - f| = 0 \text{ uniformly in } x_1 \leq x \leq x_2 \qquad (2\text{-}9)$$

and if f^* does not contain f in the sense of 2-8c then $x_1 > 0$, i. e. the approximation is not uniformly valid near the origin. For example, let

$$f = \left(1 + \frac{\epsilon}{x}\right)^{-1} \quad \text{and} \quad f^* = 1$$

Then f^* is valid to order unity uniformly in any region $0 < x_0 \leq x \leq \infty$ but not in a region including the origin. Clearly a parameter-type expansion of f for fixed x and ϵ small is equivalent to a coordinate-type expansion of $\left(1 + \frac{1}{\bar{x}}\right)^{-1}$ for $\bar{x}$ large.

2. 5. Expansion in an Artificial Parameter

One may be tempted to conclude that artificial parameters should be avoided altogether in expansion procedures. It is, however, interesting to note the following historical fact: A very important technique for finding parameter-type expansions for singular perturbation problems, the boundary-layer technique, was initiated by Prandtl in 1904 in his fundamental paper dealing with viscous flow for small values of the viscosity ν (reference 7). The main example chosen by Prandtl in this article was that of flow past a semi-infinite flat plate, in other words a case for which ν is an artificial parameter. Objectively speaking, it is logically possible to expand in an artificial parameter, although the expansion will not be uniformly valid unless the exact solution has been obtained. From a subjective point of view it may be convenient to work with artificial parameters in certain cases, as long as the difficulties resulting from non-uniformity are clearly realized. To illustrate this we shall briefly discuss an example which has certain features in common with the problem of viscous flow past a semi-infinite flat plate.

We consider stationary inviscid incompressible flow past a paraboloid of revolution with the x-axis as center-axis. The radius r_0 of a cross section at any positive x is given by

$$r_0^2 = 2Lx, \quad L = \text{radius of curvature at the nose} \tag{2-10}$$

The equation for the velocity potential is then in cylindrical coordinates

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{\partial^2\Phi}{\partial r^2} = 0 \tag{2-11a}$$

and the boundary conditions are

$$\frac{\partial\Phi}{\partial r}\bigg/\frac{\partial\Phi}{\partial x} = \frac{dr_0}{dx} \text{ at } r = r_0 \tag{2-11b}$$

$$\frac{\partial\Phi}{\partial x} \to U, \quad \frac{\partial\Phi}{\partial r} \to 0 \text{ as } r \to \infty \tag{2-11c}$$

If one uses the non-dimensional variables $\frac{\Phi}{UL}$, $\frac{x}{L}$, $\frac{r}{L}$ then the parameters U and L are completely eliminated from the equation. These parameters are hence artificial. Nevertheless we shall attempt an expansion for small values of L. It follows from the discussion in

Section 2.4 that, unless an exact solution is obtained, this expansion cannot be valid near the origin. Thus the expansion obtained should be equivalent to a coordinate-type expansion for large values of $\frac{x}{L}$ and $\frac{r}{L}$.

If L tends to zero the paraboloid becomes more slender at any given station x = constant. This suggests the use of a slender-body technique. We define an (exact) perturbation potential ϕ by

$$\phi = \frac{\Phi - Ux}{UL} \tag{2-12}$$

and assume that ϕ has an outer expansion

$$\phi \simeq \phi_0(x, r) + L\phi_1(x, r) + \ldots \tag{2-13a}$$

and an inner expansion

$$\phi \simeq \phi_0^*(x^*, r^*) + L\phi_1^*(x^*, r^*) + \ldots \tag{2-13b}$$

where the inner variables are defined by

$$x^* = x, \quad r^* = \frac{r}{\sqrt{L}} \tag{2-13c}$$

Note that in accordance with slender-body theory the radial distance from the body is measured in the body-scale, which for a fixed value of x is proportional to $\sqrt{L}$.

We now proceed as if we had a parameter-type expansion for a singular perturbation problem. By inserting the expansions into the original equation for Φ one obtains equations for ϕ_n and ϕ_n^*. The inner expansion should satisfy the inner boundary condition and the outer expansion should satisfy the outer boundary condition. Additional boundary conditions are furnished by matching of inner and outer expansions. However, the fact that L is an artificial parameter adds certain features to the problem which are discussed below.

First of all we have the <u>principle of eliminability.</u> It must be possible to eliminate L in the sense that ϕ depends on x/L and r/L only. Hence it follows that in the term $L^n\phi_n(x, r)$ the function ϕ_n must have the form

$$\phi_n = x^{-n}[\,\mathrm{fn}(\frac{r}{x})]; \text{ "fn" = "function of"} \tag{2-14a}$$

Similarly for the term $L^n\phi^*_n(x^*, r^*)$ one must have*

$$\phi^*_n = (x^*)^{-n}[\,\mathrm{fn}(\frac{r^{*2}}{x^*})] \tag{2-14b}$$

As an example consider the first two terms of the expansions. The equations and the inner boundary conditions give

$$\phi^*_0 = \log r^* + B(x^*) \tag{2-15a}$$

Eliminability shows that the function $B(x^*)$ must have the form

$$B(x^*) = b_0 \log x^* \tag{2-15b}$$

Finally matching with the outer solution gives

$$b_0 = -\frac{1}{2} \tag{2-15c}$$

The corresponding outer solution is

$$\phi_0 = \frac{1}{2}\log[\sqrt{x^2 + r^2} - x] \tag{2-16}$$

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*If the expansions proceed in powers of L each term must satisfy the principle of eliminability separately. In terms of order $L^n \log L$ occur eliminability is satisfied by combining various terms. Examples of this will be given in Section 4.

For the next approximation one obtains

$$\phi_1^* = \frac{b_1}{x^*} - \frac{r^{*2}}{\delta x^{*2}} \tag{2-17a}$$

$$\phi_1 = \frac{b_1}{\sqrt{x^2 + r^2}} \tag{2-17b}$$

The equations, boundary conditions and matching conditions are satisfied for an arbitrary choice of the constant b_1. Thus, the ordinary technique of parameter-type expansions leads to an indeterminacy. This is due to the fact that the origin $x = 0$, $r = 0$ is excluded, since the expansions are not uniform there. In fact the term ϕ_1 represents a source at the origin and b_1 is proportional to the source strength. The function ϕ_1 and the berm b_1/x^* of ϕ_1^* represent eigensolutions which may be added without violating the boundary conditions.

The indeterminacy just described is typical of expansions in artificial parameters. In the present case it can actually be removed by using an integral theorem. In principle b_1 should be determined by matching with the exact solution at the origin. However, in the special case considered it is sufficient to use one property of the exact solution namely the conservation of mass. Consider the cylindrical surface $-\infty < x < x_0$, $r = r_0 = 2Lx_0$. Since this surface extends to the paraboloid it follows that the outflow of mass through this surface must be equal to the inflow through the disc $x = -\infty$, $r \leqslant r_0$. If one requires this to be valid for all orders of L one finds that $b_1 = \frac{1}{4}$.

Similarly the solution for ϕ_2 and ϕ_2^* gives at first an undetermined constant b_2 corresponding to the strength of a dipole at the origin. This constant may again be determined by the integral theorem about the conservation of mass. Actually all indeterminacy of the higher order terms may be removed with the aid of this integral principle. While the present example illustrates some important principles of expansions in artificial parameters it must be emphasized that it is quite exceptional that all indeterminacy can be removed.

Instead of considering expansions in L one could have introduced an arbitrary length R and a non-dimensional parameter $\epsilon = \frac{L}{R}$. An expansion in powers of ϵ would, except for trivial changes, be obtained in the same way as the expansion in powers of L. The principle of eliminability would of course state that the artificial length R must actually disappear in all terms of the expansion.

Equivalent results for the paraboloid could of course have been obtained by considering a coordinate-type expansion directly. The technique using artificial parameters has no fundamental significance. The important thing is to realize clearly when a parameter is artificial and when it is not.

For a further example of an expansion in an artificial parameter see the discussion of the paraboloid of revolution in compressible flow, given in reference 8.

III. SOME PROPERTIES OF THE NAVIER-STOKES EQUATIONS AND THE OSEEN EQUATIONS

3.1. Navier-Stokes Equations. Vorticity Equation.

The Navier-Stokes equations for two-dimensional steady, incompressible viscous flow are

Momentum equation:

$$(\text{grad}\,\vec{q})\vec{q} + \frac{1}{\rho}\,\text{grad}\,p = \nu\nabla^2\vec{q} \tag{3-1a}$$

Continuity equation:

$$\text{div}\,\vec{q} = 0 \tag{3-1b}$$

where

ρ = density of fluid = constant

ν = kinematic viscosity = μ/ρ = constant

$\vec{q}$ = velocity; $\vec{q} = u\vec{i} + v\vec{j}$ in Cartesian coordinates

p = pressure

μ = viscosity = constant

For flow past a finite body we shall impose the following boundary conditions at infinity

$$\vec{q} = U\vec{i}, \quad p \to p_\infty = \text{constant as } r \to \infty, \text{ uniformly in } \theta \tag{3-2a}$$

where r and θ are polar coordinates. As boundary condition at the surface of the body one normally assumes the no-slip condition

$$\vec{q} = 0 \text{ at the surface of the body} \tag{3-2b}$$

Since we shall be dealing with solutions valid for large values of r the boundary condition 3-2a will be of crucial importance. In Section 4 it will be seen that the solutions obtained are valid even if 3-2b is replaced by a weaker condition.

In two-dimensional flows, one may introduce a stream function ψ:

$$u = \frac{\partial\psi}{\partial y} \qquad v = -\frac{\partial\psi}{\partial x} \tag{3-3a}$$

and write the vorticity ω in the form

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\nabla^2\psi \tag{3-3b}$$

Equation 3-1 may then be written

$$u\frac{\partial\omega}{\partial x} + v\frac{\partial\omega}{\partial y} = \nu\nabla^2\omega \tag{3-4a}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3-4b}$$

or, equivalently,

$$\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\nabla^2\psi = \nu\nabla^4\psi \tag{3-5}$$

It should be pointed out that equations 3-3 and 3-4 (or 3-5) are not fully equivalent to equation 3-1. For full equivalence one must require that the solution of equation 3-4 (or equation 3-5) gives a single-valued pressure field.*

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*As an example, the function $\psi = A\theta + Br^2$, A = constant, B = constant, solves equation 3-5. This stream function is multivalued which in itself is allowed since the stream function is not a direct physical quantity. The corresponding velocity field is single-valued. However, the pressure field is single-valued only if $A = 0$ or $B = 0$. It seems that failure to impose the condition of a single-valued pressure is responsible for the difficulties encountered in the discussion of the Stokes paradox in references 9 and 10.

3.2. Parameters. Non-dimensional Equations.

The differential equations contain the parameters ρ and ν. The boundary conditions at infinity contain the parameters U and p_∞. In addition we assume that a characteristic length L, say a body-dimension, is given. Using these parameters one may obtain various non-dimensional formulations of the Navier-Stokes equations and the boundary conditions. If one, for instance, introduces the variables

$$x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}, \quad p^* = \frac{p-p_\infty}{\rho U^2}, \quad \vec{q^*} = \frac{\vec{q}}{U} \qquad (3\text{-}6a)$$

and the parameter Reynolds number

$$Re = \frac{UL}{\nu} \qquad (3\text{-}6b)$$

then equation 3-1 may be written

$$(\text{grad } \vec{q^*})\,\vec{q^*} + \text{grad } p^* = \frac{1}{Re} \nabla^2 \vec{q^*} \qquad (3\text{-}7a)$$

$$\text{div } \vec{q^*} = 0 \qquad (3\text{-}7b)$$

Here the operators grad, ∇^2 and div are formed with respect to x^* and y^*: grad $= (\frac{\partial}{\partial x^*}, \frac{\partial}{\partial y^*})$, etc.

3.3. Divergence Relations. Integral Theorems.

The Navier-Stokes equations 3-1 are divergence relations: They state that the divergence of a certain vector or tensor is zero. Additional divergence relations may be derived from these equations. By integrating over a region and using Gauss' theorem one can then conclude that the flow of certain vectors or tensors through the boundary of the region is zero. We shall, mostly, consider regions of an annular shape as shown in figure 3.1. The boundary will then consist of an outer closed curve C* and an inner closed curve C. The body is located inside the curve C which may or may not coincide with the body surface. If the divergence of a quantity is zero one then obtains the result that the flow of this quantity through C* equals the flow of the same quantity through C. This gives an integral theorem which may be regarded as limited information about analytic continuation: One obtains exact relations between integrated value of flow quantitites at a curve C near or at the body and at a curve C* arbitrarily far away from the body. If C is taken to be the body surface the integral over C often has a simple physical interpretation such as the force on the body.

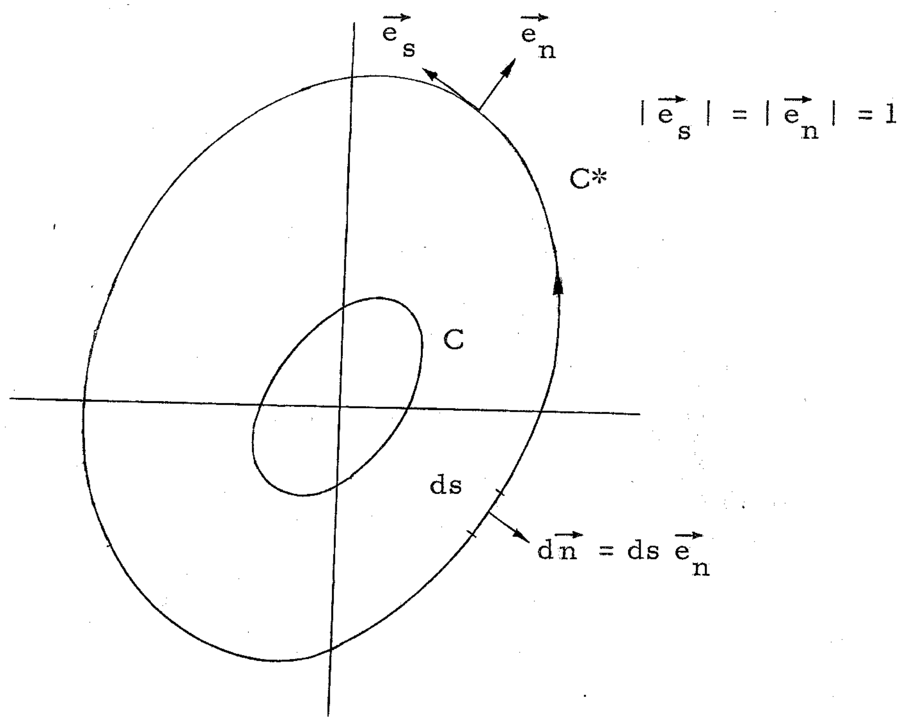

Figure 3.1

In writing the integrals we shall use the following notation (see figure 3.1). The distance along each contour measured in the counterclockwise direction will be denoted by s. The tangential unit vector in the counterclockwise direction will be denoted by $\vec{e}_s$. The distance normal to the surface, in the direction shown in the figure, will be denoted by n, the corresponding unit normal is $\vec{e}_n$. The components of a vectorial quantity, e.g. $\vec{q}$, will be subscripted in the same way, thus q_n is the normal velocity at a point on the contour and q_s is the tangential velocity along the contour. Unless otherwise stated, all formulas will be given in terms of physical variables, i.e. in dimensional forms.

Mass flow. The continuity equation 3-1b gives

$$\oint_{C^*} \vec{q} \cdot d\vec{n} = \oint_{C^*} q_n ds = \oint_C q_n ds \qquad (3\text{-}8a)$$

If we assume that $\vec{q}$ actually represents flow past a body and if the no-slip condition 3-2b holds, then one may write

$$\oint_{C^*} \vec{q} \cdot d\vec{n} = 0 \qquad (3\text{-}8b)$$

The same is of course true if one replaces 3-2b by the weaker condition that the flow be tangential at the surface of the body. More generally the integrals in equation 3-8a give the total source strength in the fluid if the contour encloses all sources.

In the following we shall assume the no-slip condition for simplicity. In Section 4 it will be pointed out the formulas obtained are easily generalized to cases for which the no-slip condition is not valid and even to cases for which the net source strength is not zero.

Flow of Vorticity. Using the continuity equation one may rewrite equation 3-4a as a divergence relation:

$$\operatorname{div}(\omega\vec{q} - \nu \operatorname{grad}\omega) = 0 \tag{3-9}$$

Letting C be the body contour B one then finds (cf. equations 3-22 and 3-23 below)

$$\oint_{C^*}(q_n\omega - \nu\frac{\partial\omega}{\partial n})ds = -\frac{1}{\rho}\oint_B \frac{\partial p}{\partial s}\,ds \tag{3-10}$$

If there is no pressure jump occurring at the boundary B, then

$$\oint_{C^*}(q_n\omega - \nu\frac{\partial\omega}{\partial n})ds = 0 \tag{3-11}$$

Flow of Momentum. Let

$$\underline{\underline{A}} = \rho\,\vec{q}\circ\vec{q} - \underline{\underline{\sigma}} \tag{3-12a}$$

$$\underline{\underline{\sigma}} = -pI + \underline{\underline{\tau}}, \quad I = \text{identity tensor} \tag{3-12b}$$

$$\underline{\underline{\tau}} = \mu \operatorname{def}\vec{q} \tag{3-12c}$$

$$\vec{F} = \oint_B \underline{\underline{\sigma}}\,d\vec{n} = \text{Total fluid force on body} \tag{3-12d}$$

Here B is actually the body contour. The dyadic product $\vec{a}\circ\vec{b}$ of two vectors is the tensor whose value for an arbitrary vector $\vec{c}$ is $(\vec{a}\circ\vec{b})\vec{c} = \vec{a}(\vec{b}\cdot\vec{c})$. If the Cartesian components of $\vec{q}$ are (u_i) then the (i, j)-component of def $\vec{q}$ is $\partial u_i/\partial x_j + \partial u_j/\partial x_i$. The momentum equation 3-1a is then equivalent to the divergence relation

$$\operatorname{div}\underline{\underline{A}} = 0 \tag{3-13}$$

This equation implies the following integral theorem:

$$\vec{F} = - \oint_{C^*} p\, d\vec{n} + \mu \oint_{C^*} \text{def}\, \vec{q}\, d\vec{n} - \rho \oint_{C^*} (\vec{q} \circ \vec{q}) d\vec{n} \quad (3\text{-}14)$$

Flow of Moment of Momentum. Since the tensor $\underline{\underline{A}}$ is symmetric the following identity is valid

$$\vec{r} \times \text{div}\, \underline{\underline{A}} = \text{div}\, (\vec{r} \times \underline{\underline{A}}) \quad (3\text{-}15)$$

and hence by equation 3-13

$$\text{div}\, (\vec{r} \times \underline{\underline{A}}) = 0 \quad (3\text{-}16)$$

The integrated form of equation 3-16 is

$$\vec{M} = \oint_{C^*} \vec{r} \times (-p + \underline{\underline{\tau}}) d\vec{n} - \rho \oint_{C^*} \vec{r} \times (\vec{q} \circ \vec{q}) d\vec{n} \quad (3\text{-}17a)$$

where

$$\vec{M} = M\vec{k} = \oint_{B} (\vec{r} \times \underline{\underline{\sigma}}) d\vec{n} \quad (3\text{-}17b)$$

3. 4. Integrals of Divergence-Free Tensors

In Section 3. 3 we considered certain tensors of zero divergence and their integrals over closed curves. In Section 4 we shall have occasion to use similar integrals which are taken over a path which is not closed. Some general properties of functions defined by such integrals will be considered here.

We shall assume that the flow field is regular outside a given curve C. For convenience the origin of the system of coordinates will be located inside C. Let $A^{(m)}$ be a tensor of order m which is defined in terms of velocity, pressure and their derivatives. The tensor $A^{(m)}$ is hence single-valued. Assume that

$$\operatorname{div} A^{(m)} = 0 \tag{3-18}$$

Examples of such tensors, for $m = 1$ and $m = 2$, were given in Section 3.3. Now let P be a fixed point in the plane and Q a variable point, both points being located outside the curve C. One may then define a tensor of order $(m-1)$, denoted by $A^{(m-1)}$, by the integral over a path from P to Q, the path lying outside the curve C:

$$A^{(m-1)}(Q) = \oint_P^Q A^{(m)} d\vec{n} \tag{3-19a}$$

The direction of $d\vec{n}$ is chosen so that a counterclockwise rotation of $d\vec{n}$ gives a tangent vector $d\vec{s}$ directed towards Q. Thus,

$$\text{if } d\vec{s} = dx\,\vec{i} + dy\,\vec{j}, \quad \text{then } d\vec{n} = dy\,\vec{i} - dx\,\vec{j} \tag{3-19b}$$

Equation 3-18 shows that the value of $A^{(m-1)}$ is the same for two different paths from P to Q provided that the combination of these two paths does not enclose C. In general $A^{(m-1)}$ is multiple-valued. It is single-valued if the plane is slit along the positive x-axis. The discontinuity across this slit is independent of x, as seen from the following relation, which is easily proved:

$$\begin{aligned} \Delta A^{(m-1)} &= A^{(m-1)}(x, 0+) - A^{(m-1)}(x, 0-) \\ &= -\oint_C A^{(m)} d\vec{n} = \text{constant} \end{aligned} \tag{3-20}$$

Here x is positive and the point $(x, 0)$ is located outside C.

Special Cases: If $m = 1$, then $A^{(m)}$ is a vector $\vec{a} = a_1\vec{i} + a_2\vec{j}$ and $A^{(m-1)}$ is a generalized stream function, i.e. a scalar function G such that (cf. equation 3-20)

$$a_1 = \frac{\partial G}{\partial y}, \quad a_2 = -\frac{\partial G}{\partial x} \tag{3-21}$$

In particular, if $\vec{a}$ is the velocity $\vec{q}$ then G is the ordinary stream-function ψ. The assumption 3-8b implies that ψ is single-valued even in the unslit plane. If instead

$$\vec{a} = \nu \operatorname{grad} \omega - \omega \vec{q} \tag{3-22a}$$

then a generalized streamfunction G may be defined to be the total pressure divided by the density

$$G = \frac{q^2}{2} + \frac{p}{\rho} \; ; \quad q^2 = u^2 + v^2 \tag{3-22b}$$

This statement follows from the fact that the momentum equation 3-1a may be written as

$$\frac{\partial}{\partial x}\left(\frac{u^2+v^2}{2} + \frac{p}{\rho}\right) - v\omega = -\nu\frac{\partial \omega}{\partial y} \tag{3-23a}$$

$$\frac{\partial}{\partial y}\left(\frac{u^2+v^2}{2} + \frac{p}{\rho}\right) + u\omega = \nu\frac{\partial \omega}{\partial x} \tag{3-23b}$$

Since in this case the generalized streamfunction is expressed in terms of pressure and velocity it is automatically single-valued in the unslit plane outside the curve C. It follows that the integral of $\vec{a}$ around C, i.e. the constant in equation 3-20, is zero. This fact was expressed by equation 3-11 above.

If $m = 2$, say $A^{(2)} = \underline{\underline{A}}$ where $\underline{\underline{A}}$ has Cartesian components A_{ij}

then $A^{(1)}$ is a vector streamfunction $\vec{G} = G_1\vec{i} + G_2\vec{j}$ such that

$$A_{i1} = \frac{\partial G_i}{\partial y}, \quad A_{i2} = -\frac{\partial G_i}{\partial x} \tag{3-24a}$$

In particular if $\underline{\underline{A}}$ is the tensor defined by equation 3-12 and if the flow field is due to a solid inside the curve C then (cf. equations 3-12d, 3-14, and 3-20) the force on the body is

$$\vec{F} = \Delta\vec{G} = \vec{G}(x, 0+) - \vec{G}(x, 0-) \tag{3-24b}$$

Finally if $\underline{\underline{A}}$ is a second-order tensor such that $\vec{r} \times \underline{\underline{A}}$ has zero divergence then one may write

$$\int_P^Q \vec{r} \times \underline{\underline{A}}\, d\vec{n} = H(Q)\vec{k} \tag{3-25a}$$

where H is a scalar function such that

$$xA_{21} - yA_{11} = \frac{\partial H}{\partial y}\ ; \quad xA_{22} - yA_{12} = -\frac{\partial H}{\partial x} \tag{3-25b}$$

In particular, if $\underline{\underline{A}}$ is the tensor defined by equation 3-12, then (cf. 3-17)

$$\text{Torque on body} = M = -\Delta H = H(x, 0+) - H(x, 0-) \tag{3-25c}$$

Harmonic Flow Fields. If the velocity field is harmonic, i.e. if

$$\text{curl}\,\vec{q} = 0, \quad \text{div}\,\vec{q} = 0 \tag{3-26}$$

then one may introduce a complex streamfunction χ such that

$$\chi(z) = \phi + i\psi, \qquad z = x + iy \tag{3-27a}$$

$$\frac{d\chi}{dz} = w(z) = u - iv \tag{3-27b}$$

Furthermore the pressure is given by Bernoulli's law

$$p + \frac{\rho q^2}{2} = \text{constant} \tag{3-28}$$

We shall give expressions for the flow of momentum and moment of momentum through a curve. Let $\vec{q}_1$ and $\vec{q}_2$ be two harmonic vector fields and let u_1, u_2, etc. be defined as in equation 3-27. A tensor $\underline{\underline{B}}$ is defined by

$$\underline{\underline{B}} = (\vec{q}_1 \circ \vec{q}_2) + (\vec{q}_2 \circ \vec{q}_1) - (\vec{q}_1 \cdot \vec{q}_2)\, I \tag{3-29a}$$

and an (infinitesimal) vector $\vec{b}$ by

$$\vec{b} = b_1\vec{i} + b_2\vec{j} = \underline{\underline{B}}\, d\vec{n} \tag{3-29b}$$

A direct computation then shows that

$$b_1 - ib_2 = -iw_1w_2dz \tag{3-29c}$$

and that

$$\vec{r} \cdot \vec{b} + i(\vec{r} \times \vec{b}) \cdot \vec{k} = z(b_1 - ib_2) \tag{3-29d}$$

Furthermore if an (infinitesimal) vector $\vec{c}$ is defined by

$$\vec{c} = c_1\vec{i} - c_2\vec{j} = (\text{def}\, \vec{q})\, d\vec{n} \tag{3-30a}$$

then

$$c_1 - ic_2 = -2i\frac{dw}{dz}dz \tag{3-30b}$$

We assume now that $\vec{q}$ is harmonic and the reference level for the pressure is chosen such that the Bernoulli constant (right-hand side of equation 3-28) is zero. Let the tensor $\underline{\underline{A}}$ be defined by equation

3-12 and the vector streamfunction $\vec{G}$ by

$$\vec{G} = G_1\vec{i} + G_2\vec{j} = \int_P^Q \underline{\underline{A}}\, d\vec{n} \tag{3-31a}$$

The formulas just proved then imply that

$$G_1 - iG_2 = -i\,\frac{\rho}{2}\int_P^Q w^2 dz + 2i\mu[\,w(Q) - w(P)] \tag{3-31b}$$

Furthermore, if H is defined by equation 3-26a then

$$H = \text{Real Part}\,[\int_P^Q (\frac{\rho}{2}\, zw^2 - 2\mu z\,\frac{dw}{dz})dz] \tag{3-32}$$

These formulas are derived by putting $\vec{q}_1 = \vec{q}_2 = \vec{q}$ in equations 3-29 ff. The more general formulas with $\vec{q}_1 \neq \vec{q}_2$ will be needed in Section 4 to handle certain "interference terms" when one considers a sum of different harmonic flow fields.

Note that even in a harmonic flow field the viscous surface stresses do not vanish, i.e. μ def $\vec{q}$ is in general not zero.

Generalization to Flow Fields with Discontinuities. We shall only consider flow fields which are regular outside the curve C. However, as will be seen in Section 4, the asymptotic expansion for such a field may contain terms which are discontinuous across the positive x-axis. The formulas derived above are easily generalized to this case. One may still define $A^{(m-1)}$ by equation 3-19. However, if $A^{(m)}$ has zero divergence in the slit plane but is not regular on the positive x-axis, then the path of integration should not cross the positive x-axis. Furthermore, the discontinuity of $A^{(m-1)}$ across the positive x-axis may no longer be independent of x, i.e. equation 3-20 is no longer valid.

3.5. The Rapid (Non-Algebraic) Decay of Vorticity

The vorticity equation 3-4 states that vorticity is transported with the streamlines of the flow and at the same time diffuses like heat. Furthermore, we note that the flow coming from upstream infinity is originally free of vorticity and there are no vorticity sources inside the fluid. The vorticity is generated at the surface of the body and then transported and diffused according to equation 3-4. It is therefore expected that the vorticity is small except in the following types of regions: Regions near the body, regions of closed streamlines (where the effect of diffusion may be strong) and regions downstream of the body composed of streamlines that have passed close to the body. The regions of closed streamlines are expected to occur adjacent to the rear part of the body. Together with the third type of region described above they constitute the wake. If one replaces the transporting streamlines by the free stream $U\vec{i}$ and assumes that all vorticity is generated at the origin by the vorticity dipole one may verify that vorticity decays as $1/\sqrt{x}$ downstream of the origin in a parabolic region. On the other hand, for a fixed value of x vorticity decays exponentially in y. (cf. the discussion of the Oseen fundamental solution and the discussion on p. 49 in Section 4). It is then expected that this result is still qualitatively correct for an actual solution of the Navier-Stokes equation for which the vorticity sources are distributed and for which the streamlines at infinity approach the free-stream flow. We shall therefore in the following (cf. Section 4) use the following <u>principle of rapid decay of vorticity</u>:

If one approaches infinity along a line whose angle with the streamlines never becomes small then vorticity decays along this line faster than any power of the distance. (3-33)

An exact proof of this principle cannot be given, although further plausibility argument will be given in Section 4. The use of this principle is the following: In applying certain expansion methods for finding Navier-Stokes solutions it may occur at a certain stage that the equations and boundary conditions allow solutions for which vorticity decays only algebraically. Such solutions must then be rejected by the principle of rapid decay of vorticity. Examples of this will be given in Section 4. Another example will be indicated here: We consider viscous flow past a semiinfinite flat plate and study expansions for small values of ν (as pointed out earlier this expansion cannot be valid near the origin; this is, however, irrelevant here). Leading terms are given by the Prandtl-Blasius boundary-layer solution and by the solution for flow due to displacement thickness. It was for a long time assumed that the next term is of order ν and various solutions were proposed in the literature. However, it was found in 1956 by S. Kaplun, Imai and others (cf. references 11 and 12) that all possible solutions of order ν have too slow decay of vorticity. This forced reconsideration of the problem and led to the correct solution, of the order $\nu \ln \nu$.

3.6. The Oseen Equations

The Oseen equations may be obtained by linearizing equation 3-1 about the free stream velocity $\vec{q} = U\vec{i}$, i.e. by neglecting terms which are quadratic in $\vec{q}' = \vec{q} - U\vec{i}$. The resulting equations are

$$U \frac{\partial \vec{q}}{\partial x} + \frac{1}{\rho} \nabla p = \nu \nabla^2 \vec{q} \tag{3-34a}$$

$$\operatorname{div} \vec{q} = 0 \tag{3-34b}$$

The same equations hold for the perturbation $\vec{q'}$. The boundary conditions are

$$\vec{q} = 0 \qquad \vec{q'} = -Ui \qquad \text{at body} \tag{3-35a}$$

$$\vec{q} \rightarrow U\vec{i} \qquad \vec{q'} \rightarrow 0 \qquad \text{at infinity} \tag{3-35b}$$

The corresponding equations for ω and ψ are

$$U \frac{\partial \omega}{\partial x} = \nu \nabla^2 \omega \tag{3-36}$$

and

$$U \frac{\partial}{\partial x} \nabla^2 \psi = \nu \nabla^4 \psi \tag{3-37}$$

The validity of the Oseen equations may be discussed based on two different points of view. For any fixed Reynolds number, it is to be expected that the linearized equations are approximately valid near infinity because at large distances from the body the perturbational velocity $\vec{q'}$ will be small compared to the free stream velocity $U\vec{i}$. Near the body, the linearization is not justified for any arbitrary Reynolds number. However, as the Reynolds number tends to zero the Oseen equations have uniform validity, at least for flow past finite bodies. An explanation of this was found by Kaplun who pointed out that as Re tends to zero the whole flow field may be regarded as a perturbation of uniform flow (details are given in references 5 and 6). In this article we are concerned with flow at a fixed Reynolds number.

Hence in the present context only the first viewpoint (validity of the Oseen equations at large distances) is relevant.

Finally, it may be pointed out, that one may sometimes use the Oseen equations as a qualitative model for the Navier-Stokes equations. The principle of rapid decay of vorticity is actually based on such a qualitative comparison.

3.7. Splitting of Oseen Solutions

It may be shown that any solution $\vec{q}$ of the Oseen equations may be expressed as the sum of two components

$$\vec{q} = \vec{q}_L + \vec{q}_T \tag{3-38}$$

where $\vec{q}_L$ and $\vec{q}_T$ satisfy the Oseen equations. The component, $\vec{q}_L$, called the longitudinal component satisfies the additional condition

$$\text{curl } \vec{q}_L = 0 \tag{3-39a}$$

The component $\vec{q}_T$, called the transversal component, satisfies the additional requirement that its associated pressure field is zero. It follows that the equations for $\vec{q}_L$ may be written

$$\vec{q}_L = \text{grad } \phi, \quad \nabla^2\phi = 0, \quad p - p_\infty = -\rho U u_L \tag{3-39b}$$

where u_L is the x-component of $\vec{q}_L$ (normalized in such a way that $u_L = 0$ at infinity). The equations for $\vec{q}_T$ may be written

$$U \frac{\partial \vec{q}_T}{\partial x} = \nu \nabla^2 \vec{q}_T, \quad \text{div } \vec{q}_T = 0 \tag{3-40}$$

Note that the longitudinal field contains the pressure disturbances and that the transversal field contains all the vorticity of the given

Oseen solution.

The decomposition 3-38 is often referred to as a splitting of an Oseen solution. It may be proved that this decomposition is unique provided both $\vec{q}_L$ and $\vec{q}_T$ are normalized to vanish at infinity. Examples of a splitting will be given for the fundamental solution below. Further discussion of splitting is given in reference 13.

The proof of the splitting theorem uses the fact that the Oseen equations form a system of linear equations with constant coefficients. Splitting theorems may be proved for many systems of equations of this type. Naturally one cannot expect a splitting theorem to hold for the non-linear Navier-Stokes equations. However, we shall see in the following section that the asymptotic expansion of a Navier-Stokes solution near infinity may be decomposed in a manner somewhat reminiscent of the splitting of the Oseen equations.

3.8. Fundamental Solutions of the Oseen Equations

In Section 4 we shall study the asymptotic flow field of the Navier-Stokes equation. For the sake of comparison the corresponding flow field given by the Oseen equation is of interest. At large distances the detailed effects of the body is not important, and the flow field may be studied by assuming a concentrated force at its origin. This resulting solution is called the fundamental solution of the equation. The x-component of the force is minus the drag, the y-component is minus the lift. Due to linearity one may study the effect of these two forces separately. Only certain results will be given here. For proofs the reader is referred to references 13 and 14.

<u>Drag Case</u>. We assume that the force is located at the origin and has the value $-\vec{i}$. This corresponds to a body of unit drag. The resulting flow field is (here $\vec{q}$ actually stands for the velocity perturbation)

$$\vec{q} = \vec{q}_L + \vec{q}_T \tag{3-41a}$$

where the longitudinal component is

$$\vec{q}_L = \frac{1}{2\pi\rho U} \text{ grad log } r \tag{3-41b}$$

and the transversal component is

$$\vec{q}_T = \frac{1}{2\pi\mu} \{\frac{1}{2\lambda} \text{ grad} [e^{\lambda x} K_0(\lambda r)] - e^{\lambda x} K_0(\lambda r) \vec{i} \} \tag{3-41c}$$

$$\lambda = \frac{U}{2\nu}$$

The pressure associated with the longitudinal component is

$$p' = p - p_\infty = -\rho U u_L = -\frac{x}{2\pi r^2} \tag{3-41d}$$

For large values of r the transversal wave is exponentially small except in a parabolic region for which $r - x$ remains finite as r tends to infinity. In this region, which is actually the wake region in the sense discussed in Section 3.5, the leading terms of the asymptotic expansion of $\vec{q}_T$ are

$$u_T(x, y) \simeq -\frac{1}{2\rho} \sqrt{\frac{1}{\pi \nu U x}} \; e^{-\frac{\lambda y^2}{2x}} \tag{3-42a}$$

$$v_T(x, y) \simeq -\frac{1}{4\rho} \sqrt{\frac{1}{\pi \nu U}} \; \frac{y}{x^{3/2}} \; e^{-\frac{\lambda y^2}{2x}} \tag{3-42b}$$

Note that $\vec{q}_L$ represents a potential source of strength $\frac{1}{\rho U}$. It follows, and may be verified explicitly from equation 3-41 or equation 3-42 that the transversal flow represents a sink of strength $\frac{1}{\rho U}$. Thus source strength of the potential flow equals sink strength of the wake equals the drag divided by ρU. Intuitively speaking the retarding action of the body on the fluid causes a velocity deficiency, i.e. a net sink strength in the wake. This displacement effect corresponds to an apparent semi-infinite body whose thickness downstream approaches a finite value. The longitudinal component corresponds to potential flow past this apparent body.

Lifting case. If the singular force is located at the origin and has the value $-\vec{j}$, corresponding to unit lift, the resulting velocity and pressure field is

$$\vec{q} = \vec{q}_L + \vec{q}_T \tag{3-43a}$$

where

$$\vec{q}_L = - \frac{1}{2\pi\rho U} \operatorname{grad} (\tan^{-1} \frac{y}{x}) \tag{3-43b}$$

$$\vec{q}_T = \operatorname{curl}[\vec{k} \; \frac{1}{2\pi\rho U} e^{\lambda x} K_0(\lambda r)] \tag{3-43c}$$

and

$$p' = - \rho U u_L = - \frac{y}{2\pi r^2} \tag{3-43d}$$

For large values of r the transversal flow field behaves as

$$u_T \simeq - \frac{1}{4\rho}\sqrt{\frac{1}{\pi U \nu}} \; \frac{y}{x^{3/2}} \, e^{-\frac{\lambda y^2}{2x}} \tag{3-44a}$$

$$v_T \simeq \frac{1}{4\rho}\sqrt{\frac{\nu}{\pi U^3}} \; \frac{1}{x^{3/2}} \, e^{-\frac{\lambda y^2}{2x}} \tag{3-44b}$$

Note that the Γ, the clockwise circulation around a large contour is given by $\vec{q}_L$ alone. It is equal to $\frac{1}{\rho U}$. In fact $\vec{q}_L$ represents a potential vortex.

For future reference we summarize the results regarding lift and drag. If the singular force on the fluid is $-F_1\vec{i} - F_2\vec{j}$ where F_1 = drag and F_2 = lift then

$$F_1 = \rho U m \; ; \qquad F_2 = \rho U \Gamma \tag{3-45}$$

where m = source strength of the longitudinal component = sink strength of the transversal component and Γ = clockwise circulation around a large contour of the longitudinal component. The circulation of the drag solution is zero by symmetry and the circulation of the transversal component of the lift solution vanishes as the contour tends to infinity. The source (or sink) strengths of the longitudinal and transversal components of the lift solution are zero individually.

IV. THE ASYMPTOTIC EXPANSIONS FOR VELOCITY AND PRESSURE

4.1. Artificial Parameter. General Form of the Expansion.

The assumptions about the flow field are the following. First we assume that the velocity $\vec{q^*}$ and the pressure p^* obey the Navier-Stokes equations for two-dimensional stationary incompressible flow (cf. Section 3.2)

$$(\text{grad } \vec{q^*})\vec{q^*} + \text{grad } p^* = \frac{1}{Re}\nabla^2\vec{q^*} \tag{4-1a}$$

$$\text{div } \vec{q^*} = 0 \tag{4-1b}$$

Secondly we assume the boundary conditions at infinity

$$\vec{q^*} \to \vec{i}, \quad p^* \to 0 \quad \text{as} \quad r^* \to \infty \tag{4-2}$$

Thirdly we assume that there exists a finite simple closed curve C such that

$\vec{q^*}$, p^* and their first derivatives are continuous on C (4-3a)

The net mass-flow through C is zero (4-3b)

Assumptions 4-3a, b are in particular valid if the flow field is caused by a finite body inside C on which the no-slip condition is satisfied.

The problem is to find expansions for $\vec{q^*}$ and p^* uniformly valid in θ for a fixed value of Re as r^* tends to infinity. We shall actually construct two expansions each for $\vec{q^*}$ and p^*, an outer expansion, valid outside the wake region and an inner expansion valid in the wake region.

The domain of validity of these two expansions will overlap; hence an expansion uniformly valid in θ may be constructed by a combination of outer and inner expansions. The form of these expansions will be given here, the justification for choosing this form will be discussed later.

We introduce an artificial length R and an artificial parameter

$$\epsilon = \frac{L}{R} \tag{4-4}$$

and propose the following expansions for ϵ small.

<u>Outer Expansion.</u> As independent variables (outer variables) we use the Cartesian coordinates

$$\tilde{x} = \frac{x}{R} = \epsilon x^*, \quad \tilde{y} = \frac{y}{R} = \epsilon y^* \tag{4-5a}$$

or, equivalently, the polar coordinates

$$\tilde{r} = \frac{r}{R} = \epsilon r^* \quad \text{and} \quad \theta \tag{4-5b}$$

The dependent variables are $\vec{q^*}$ and p^*. The outer expansions of velocity and pressure are assumed to have the form

$$\vec{q^*} = \vec{i} + \epsilon \vec{q_1} + \epsilon^{3/2} \vec{q}_{3/2} + \epsilon^2 \log \epsilon \, \vec{q}_{1a} + \epsilon^2 \vec{q_2} + \cdots \tag{4-6a}$$

$$p^* = \epsilon \tilde{p}_1 + \epsilon^{3/2} \tilde{p}_{3/2} + \epsilon^2 \log \epsilon \, \tilde{p}_{1a} + \epsilon^2 \tilde{p}_2 + \cdots \tag{4-6b}$$

Here $\vec{q_1}$, $\tilde{p}_1$, etc. are functions of the outer variables $\tilde{x}$ and $\tilde{y}$. Expressed in outer variables the Navier-Stokes equations take the form

$$(\text{grad}\, \vec{q^*})\, \vec{q^*} + \text{grad}\, p^* = \frac{\epsilon}{Re} \nabla^2 \vec{q^*} \tag{4-7a}$$

$$\text{div}\, \vec{q^*} = 0 \tag{4-7b}$$

where the differential operators are formed with respect to $\tilde{x}$ and $\tilde{y}$. The <u>outer limit</u> is defined as the limit as ϵ tends to zero for $\tilde{x}$ and $\tilde{y}$ fixed.

<u>Inner Expansion.</u> In the wake region we use the inner variables

$$\overline{x} = \tilde{x}, \quad \overline{y} = \frac{\tilde{y}}{\sqrt{\epsilon}} = \frac{y}{\sqrt{RL}} = \sqrt{\epsilon}\, y^* \tag{4-8a}$$

and the dependent variables

$$u^*, \quad \overline{v} = \frac{v^*}{\sqrt{\epsilon}} \quad \text{and} \quad p^* \tag{4-8b}$$

The inner expansions of velocity and pressure are assumed to have the form

$$u^* \sim 1 + \epsilon u_1 + \epsilon \log \epsilon\, u_{1a} + \epsilon u_2 + \epsilon^{3/2} \log \epsilon\, u_{2a} + \epsilon^{3/2} u_{2a} + \ldots \tag{4-9a}$$

$$\overline{v} \sim \epsilon v_1 + \epsilon \log \epsilon\, v_{1a} + \epsilon v_2 + \epsilon^{3/2} \log \epsilon\, v_{2a} + \epsilon^{3/2} v_2 + \ldots \tag{4-9b}$$

$$p^* \sim \epsilon p_1 + \epsilon^{3/2} p_2 + \epsilon^2 \log \epsilon\, p_{2a} + \epsilon^2 p_3 + \ldots \tag{4-9c}$$

Here u_1, v_1, etc. are functions of $\overline{x}$ and $\overline{y}$. Expressed in inner variables the Navier-Stokes equations take the form

$$u^* \frac{\partial u^*}{\partial \overline{x}} + \overline{v} \frac{\partial u^*}{\partial \overline{y}} + \frac{\partial p^*}{\partial \overline{x}} = \frac{1}{Re} \left[\epsilon \frac{\partial^2 u^*}{\partial \overline{x}^2} + \frac{\partial^2 u^*}{\partial \overline{y}^2} \right] \tag{4-10a}$$

$$u^* \frac{\partial \overline{v}}{\partial \overline{x}} + \overline{v} \frac{\partial \overline{v}}{\partial \overline{y}} + \frac{1}{\epsilon} \frac{\partial p^*}{\partial \overline{y}} = \frac{1}{Re} \left[\epsilon \frac{\partial^2 \overline{v}}{\partial \overline{x}^2} + \frac{\partial^2 \overline{v}}{\partial \overline{y}^2} \right] \tag{4-10b}$$

$$\frac{\partial u^*}{\partial \overline{x}} + \frac{\partial \overline{v}}{\partial \overline{y}} = 0 \tag{4-10c}$$

The <u>inner limit</u> is defined as the limit as ϵ tends to zero for $\overline{x}$ and $\overline{y}$

fixed.

Comments on Outer and Inner Limits. It is clear that the outer limit corresponds to r^* tending to infinity for θ fixed. In the inner limit x^* and y^* tend to infinity whereas the ratio $\frac{y^*}{x^*}$, and hence θ, tends to zero. Actually y^* and x^* move towards infinity along a parabola $y^* = (\text{constant}) \sqrt{x^*}$. The necessity for such an inner limit is strongly suggested by the form of the Oseen fundamental solution for r^* large (cf. Section 3.8). Note that in the inner limit $\theta \sqrt{r^*}$ tends to a finite value which is non-zero except when $y^* = 0$.

Vorticity. The non-dimensional vorticity is

$$\omega^* = \frac{L\omega}{U} = \frac{\partial v^*}{\partial x^*} - \frac{\partial u^*}{\partial y^*} \tag{4-11a}$$

It will be seen that the terms of the outer expansion are all potential, hence all the vorticity comes from the inner expansion. Using inner variables one may write

$$\omega^* = \epsilon^{3/2} \frac{\partial \overline{v}}{\partial \overline{x}} - \epsilon^{1/2} \frac{\partial u^*}{\partial \overline{y}} \tag{4-11b}$$

From equations 4-9 it then follows that the expansion of the vorticity is

$$\overline{\omega} = \frac{\omega^*}{\epsilon} = - \frac{\partial u_1}{\partial \overline{y}} - \sqrt{\epsilon} \log \epsilon \frac{\partial u_{1a}}{\partial \overline{y}} - \sqrt{\epsilon} \frac{\partial u_2}{\partial \overline{y}} + \dots \tag{4-12}$$

The first contribution of $\frac{\partial \overline{v}}{\partial \overline{x}}$ to $\overline{\omega}$ will occur in the term of order ϵ.

4.2. Comments on Mathematical Nature of Problem.

It follows from the discussion in Section 2 that one cannot expect the expansions to be completely determined. Since values for large r^* cannot be matched with an expansion for finite r^*, certain undetermined constants will occur.

The following principles will be used for finding the form of the expansion and for eliminating certain apparent indeterminacies (cf. Section 2.5).

1. Principle of Eliminability. (Similarity) The form of the expansion given indicates that one has to find certain functions of two independent variables. However, the principle of eliminability of ϵ requires that these variables occur in certain combinations so that effectively only functions of one variable have to be found. In the outer expansion a term $\epsilon^n f(\tilde{x}, \tilde{y})$ must have the form

$$\epsilon^n f(\tilde{x}, \tilde{y}) = \frac{\epsilon^n}{\tilde{r}^n} f\left(\frac{\tilde{x}}{\tilde{y}}\right) \qquad (4\text{-}13a)$$

and a term $\epsilon^n f(\bar{x}, \bar{y})$ of the inner expansion of u^*, v^* and p^* must have the form

$$\epsilon^n f(\bar{x}, \bar{y}) = \frac{\epsilon^n}{\bar{x}^n} f\left(\frac{\bar{y}^2}{\bar{x}}\right) \qquad (4\text{-}13b)$$

There are some modifications of the above rules due to the presence of terms involving $\log \epsilon$. These modifications will be obvious in each special case and the general rule need not be given here.

2. Matching of Inner and Outer Expansions. The principles for this matching are the same as for an expansion in a non-artificial parameter (cf. references 4, 5 and 6). Actually the relation of outer flow to wake flow in the present problem resembles the relation of outer flow to boundary-layer flow in the problem of expansions for large Reynolds number.

3. Integral Theorems. The conservation laws discussed in Section 3. 3 are exact theorems about Navier-Stokes solutions. With the aid of these theorems one may match certain integrals of the expansion for large r^* with the corresponding integrals at the given curve C. In this sense it is possible to match certain integrated quantities at finite r^* with the corresponding quantities at large r^*. This matching is quite different from the matching described under 2) above. It is not a consequence of some general theory about singular perturbations, but derives from the fact that certain properties of exact Navier-Stokes solutions are known.

4. Principle of Non-algebraic Decay of Vorticity. The rapid decay of vorticity is also regarded as a known property of exact Navier-Stokes solutions. This principle will be helpful in eliminating certain a priori possibilities.

4. 3. Determination of First-Order Terms

Determination of u_1, v_1 and p_1. By inserting the inner expansions into equations 4-10a, b, c one finds to order $\sqrt{\epsilon}$

$$\frac{\partial u_1}{\partial \bar{x}} - \frac{1}{Re} \frac{\partial^2 u_1}{\partial \bar{y}^2} = 0 \qquad (4\text{-}14a)$$

$$\frac{\partial p_1}{\partial \bar{y}} = 0 \qquad (4\text{-}14b)$$

$$\frac{\partial u_1}{\partial \bar{x}} + \frac{\partial v_1}{\partial \bar{y}} = 0 \qquad (4\text{-}14c)$$

The following solutions will be obtained

$$u_1 = - \frac{m\sqrt{Re}}{2\sqrt{\pi \bar{x}}} e^{-\eta^2} \; ; \; \eta^2 = \frac{Re\,\bar{y}^2}{4\bar{x}} \qquad (4\text{-}15a)$$

$$v_1 = v_1^{(1)} - \frac{\Gamma^*}{2\pi\bar{x}} \quad \text{where} \quad v_1^{(1)} = - \frac{m\sqrt{Re}\,\bar{y}}{4\sqrt{\pi}\,\bar{x}^{3/2}} e^{-\eta^2} \qquad (4\text{-}15b)$$

The corresponding streamfunction is

$$\psi_1 = \psi_1^{(1)} + \frac{\Gamma^*}{2\pi} \log \bar{x} \quad \text{where} \quad \psi_1^{(1)} = - \frac{m}{2} \operatorname{erf}\left(\frac{\sqrt{Re}\,\bar{y}}{2\sqrt{\bar{x}}}\right) \qquad (4\text{-}15c)$$

$$p_1 = - \frac{m}{2\pi\bar{x}} \qquad (4\text{-}15d)$$

Here m and Γ^* are undetermined constants. Later on (cf. equations 4-30 and 4-31) these constants will be identified with the drag coefficient and lift coefficient respectively.

The answers given may be justified as follows. Eliminability of ϵ implies that $\bar{x}$ is a function of $\bar{y}^2/\bar{x}$ only. This function must then satisfy the equation A-4a of Appendix A with $\alpha = 1$. Since u_1 alone determines the leading term of the vorticity (cf. equation 4-12) the principle of rapid decay of vorticity implies that of the two independent solutions of equation A-4a only the solution given by A-7a can be used. This proves equation 4-15a. The multiplicative constant has been written so as to exhibit the constant m which later on will be shown to be the drag coefficient (cf. p. 53). Incidentally, it follows from Appendix A and the principle of eliminability if one had assumed the inner expansion to start as $u^* \sim 1 + \epsilon^{\alpha/2} u_1$ with $0 < \alpha < 1$ then the decay of vorticity would have been algebraic. This justifies the choice of $\alpha = 1$.

By integrating the continuity equation one finds the first term of the right-hand side of equation 4-15b. The second term represents a "constant" of integration which, by eliminability, must be proportional to $\bar{x}^{-1}$. The undetermined constant Γ^* will later on be identified with the lift coefficient.

Similarly one finds from equation 4-14b that p_1 must be proportional to $\bar{x}^{-1}$. At the present stage one cannot justify the appearance of the same constant m in both equation 4-15a and equation 4-15d.

<u>Determination of $\vec{q_1}$</u>. Assume for convenience that the origin $x^* = 0$, $y^* = 0$ is located inside the given curve C (cf. equation 4-3a). Since r^* is finite on the curve C it follows that in the limit of ϵ tending to zero the curve C is represented by the origin in the $(\tilde{x}, \tilde{y})$-plane. The wake is represented by the positive $\tilde{x}$-axis. Thus $\vec{q_1}$ may be considered to be defined in the $(\tilde{x}, \tilde{y})$-plane slit along the positive $\tilde{x}$-axis.

It follows from equation 4-7 that q_1 obeys the linearized Euler equations

$$\frac{\partial \vec{q}_1}{\partial \tilde{x}} + \text{grad } p^* = 0 \tag{4-16a}$$

$$\text{div } \vec{q}_1 = 0 \tag{4-16b}$$

Hence vorticity is constant along the lines $\tilde{y}$ = constant (streamlines of the undisturbed flow). Since $\vec{q}_1$ is defined in the slit $(\tilde{x}, \tilde{y})$-plane and since vorticity is zero at upstream infinity it follows that the vorticity of $\vec{q}_1$ is zero.

By a similar reasoning, and using induction, one shows that all terms of the outer expansion are potential (cf. reference 4, p. 871). Incidentally, if one assumes that vorticity in the wake decays only algebraically then it would follow by matching that vorticity must exist in the outer flow. Hence the expansion scheme assumed necessitates the assumption of rapid decay of vorticity.

Since $\vec{q}_1$ is potential we may express it in terms of a complex velocity w_1 by the following definition. If

$$\vec{q}_1 = \tilde{u}_1(\tilde{x}, \tilde{y})\vec{i} + \tilde{v}_1(\tilde{x}, \tilde{y})\vec{j} \tag{4-17a}$$

then w_1 is defined by

$$w_1(z) = \tilde{u}_1 - i\tilde{v}_1, \quad z = \tilde{x} + i\tilde{y} \tag{4-17b}$$

The solution will be shown to be

$$w_1 = \frac{m}{2\pi z} + i\,\frac{\Gamma^*}{2\pi z} \tag{4-18a}$$

i. e.

$$\tilde{u}_1 = \frac{m\tilde{x} + \Gamma^*\tilde{y}}{2\pi\tilde{r}^2}\ , \quad \tilde{v}_1 = \frac{m\tilde{y} - \Gamma^*\tilde{x}}{2\pi\tilde{r}^2} \tag{4-18b}$$

The corresponding pressure is

$$\tilde{p}_1 = -\tilde{u}_1 \tag{4-19}$$

For future reference we note that the inner expansion of $\tilde{u}_1$ and $\tilde{v}_1$ is

$$\epsilon\tilde{u}_1 \sim \frac{\epsilon m}{2\pi\bar{x}} + \epsilon^{\frac{3}{2}} \frac{\Gamma^*\bar{y}}{2\pi\bar{x}^2} - \epsilon^2 \frac{m\bar{y}^2}{2\pi\bar{x}^3} + O(\epsilon^{\frac{5}{2}}) \tag{4-20a}$$

$$\epsilon\tilde{v}_1 \sim -\epsilon \frac{\Gamma^*}{2\pi\bar{x}} + \epsilon^{\frac{3}{2}} \frac{m\bar{y}}{2\pi\bar{x}^2} - \epsilon^2 \frac{\Gamma^*\bar{y}}{2\pi\bar{x}^3} + O(\epsilon^{\frac{5}{2}}) \tag{4-20b}$$

To justify equation 4-18 we first observe that eliminibility implies that w_1 must have the form

$$w_1 = \frac{C_1}{2\pi z} + i\frac{C_2}{2\pi z} \; ; \; C_1 \text{ and } C_2 \text{ real constants} \tag{4-21}$$

The first term of the right-hand side represents a potential source, the second term a potential vortex. By <u>matching</u> with the inner expansion we may identify C_2 with Γ^*. The correctness of this follows from a comparison of equations 4-15b and 4-20b. According to equation 4-15b the inner expansion of v^* for large values of $\bar{y}$ is

$$v^* = -\epsilon \frac{\Gamma^*}{2\pi\bar{x}} + o(\epsilon) \text{ as } \bar{y} \to \infty \tag{a}$$

This checks with the leading term of equation 4-20b.

On the other hand matching of inner and outer expansions will not determine the relation between C_1 and m. The inner expansion of u^* is

$$u^* = 1 + \sqrt{\epsilon}\, u_1 + o(\epsilon) \tag{b}$$

As $\bar{y}$ tends to infinity u_1 tends to zero as shown by equation 4-15a.

This checks with the fact that the outer expansion of u^* has no term of order $\sqrt{\epsilon}$ but it does not determine the relation between C_1 and m. This relation will now be determined by an integral theorem, namely the conservation of mass.

Conservation of mass. The following will be a typical example of the application of the reasoning of Section 3.4. Since the divergence of $\vec{q}$ is zero one may define a streamfunction ψ (cf. equations 3-19 and 3-21). If one defines a non-dimensional streamfunction ψ^* by

$$\psi^* = \frac{\psi}{UL} \tag{4-22a}$$

then

$$u^* = \frac{\partial \psi^*}{\partial y^*} \, , \quad v^* = -\frac{\partial \psi^*}{\partial x^*} \tag{4-22b}$$

Since by assumption 4-3b the mass flow through the given curve C is zero it follows that ψ^* is single-valued outside the given curve C (cf. equation 3-20).

If we put

$$\tilde{\psi} = \epsilon \psi^* \tag{4-23a}$$

then

$$u^* = \frac{\partial \tilde{\psi}}{\partial \tilde{y}} \, , \quad v^* = -\frac{\partial \tilde{\psi}}{\partial \tilde{x}} \tag{4-23b}$$

If we put

$$\overline{\psi} = \sqrt{\epsilon}\, \psi^* \tag{4-24a}$$

then

$$u^* = \frac{\partial \overline{\psi}}{\partial \overline{y}} \, , \quad \overline{v} = -\frac{\partial \overline{\psi}}{\partial \overline{x}} \tag{4-24b}$$

The outer expansion of the streamfunction is then

$$\tilde{\psi} = \tilde{y} + \epsilon\tilde{\psi}_1 + \ldots \tag{4-25b}$$

The inner expansion is

$$\overline{\psi} = \overline{y} + \sqrt{\epsilon}\,\overline{\psi}_1 + \ldots \tag{4-26a}$$

where

$$\frac{\partial\overline{\psi}_1}{\partial\overline{y}} = u_1, \quad \frac{\partial\overline{\psi}_1}{\partial\overline{x}} = -v_1 \tag{4-26b}$$

The matching conditions require that

$$\tilde{\psi}_1(\overline{x}, 0+) = \overline{\psi}_1(\overline{x}_1, \infty), \quad \tilde{\psi}_1(\overline{x}, 0-) = \overline{\psi}_1(\overline{x}_1, -\infty) \tag{4-27}$$

Since ψ^* is single-valued, i.e. there is no discontinuity at $y^* = 0$, one finds from equation 4-15a

$$\overline{\psi}_1(\overline{x}, \infty) - \overline{\psi}_1(\overline{x}, -\infty) = \int_{-\infty}^{\infty} u_1 d\overline{y} = -m \tag{4-28}$$

Hence, from equation 4-27

$$\tilde{\psi}_1(\overline{x}, 0-) - \tilde{\psi}_1(\overline{x}, 0+) = \oint \vec{q}_1 \, d\vec{n} = m \tag{4-29}$$

This equation requires that the constant C_1 in equation 4-21 be equal to m. Thus equation 4-18 has been justified.

The somewhat formal argument given above means intuitively the following: The inner term u_1 represents a sink of strength m. The outer term $\vec{q}_1$ represents a source whose strength must be m in order for the total mass flow to be zero.

Determination of Pressure. Since $\vec{q}_1$ represents a potential flow field equation 4-19 follows from Bernoulli's law. By matching of the inner and outer expansion for pressure one may now finally prove that the multiplicative constant for p_1 (equation 4-15d) is correctly chosen.

Lift and Drag. Momentum Integral. We define the non-dimensional force on the body by

$$\vec{F}^* = \frac{1}{\rho U^2 L} \vec{F} \tag{4-30}$$

We shall now prove that

$$m = \vec{F}^* \cdot \vec{i} = \frac{\text{Drag}}{\rho U^2 L} \tag{4-31a}$$

$$\Gamma^* = \vec{F}^* \cdot \vec{j} = \frac{\text{Lift}}{\rho U^2 L} \tag{4-31b}$$

Here m and Γ^* are the undetermined constants appearing in equations 4-15 and 4-18. Of course, equation 4-31 does not determine the value of these constants but it shows their physical meaning.

Equations 4-31a, b will be derived from the momentum integral as discussed in Sections 3.3 and 3.4. We define a non-dimensional tensor $\underline{\underline{A}}^*$ by (cf. equation 3-12)

$$\underline{\underline{A}}^* = \frac{1}{\rho U^2} [\underline{\underline{A}} - \rho(U\vec{i} \circ U\vec{i}) - p_\infty I]$$

$$= \vec{q}^* \circ \vec{q}^* - (\vec{i} \circ \vec{i}) + p^* I - \frac{1}{Re} \operatorname{def} \vec{q}^* \tag{4-32a}$$

and a vector streamfunction by

$$\vec{G}^* = \int_P^Q \underline{\underline{A}}^* \, d\vec{n}^* \tag{4-32b}$$

Here P is a given fixed point. The choice of P will be irrelevant for the following. In order to make $\vec{G}^*$ single-valued we slit the plane along the positive x^*-axis. Then (cf. equation 3-24b)

$$\vec{F}^* = \vec{G}^*(x^*, 0+) - \vec{G}^*(x^*, 0-) \equiv \Delta\vec{G}^* \tag{4-33}$$

In order to compute $\Delta\vec{G}^*$ we shall study the outer and inner expansions of $\underline{\underline{A}}^*$ and $\vec{G}^*$. The following notation will be useful. For two arbitrary vector fields $\vec{v}_1$ and $\vec{v}_2$ we define a tensor field $T(\vec{v}_1, \vec{v}_2)$ by

$$T(\vec{v}_1, \vec{v}_2) = (\vec{v}_1 \circ \vec{v}_2) + (\vec{v}_2 \circ \vec{v}_1) - \vec{v}_1 \cdot \vec{v}_2 I \tag{4-34}$$

It then follows from Bernoulli's law that, for the <u>outer expansion</u>,

$$\begin{aligned}(\vec{q}^* \circ \vec{q}^*) - (\vec{i} \circ \vec{i}) + p^* I \\ = \epsilon T(\vec{i} \circ \vec{q}_1) + \epsilon^{3/2} T(\vec{i} \circ \vec{q}_{3/2}) + \epsilon^2 \log \epsilon T(\vec{i} \circ \vec{q}_{1a}) \\ + \epsilon^2 [T(\vec{i} \circ \vec{q}_2) + \frac{1}{2} T(\vec{q}_1 \circ \vec{q}_1)] + o(\epsilon^2)\end{aligned} \tag{4-35a}$$

Furthermore

$$\text{def}^* \vec{q}^* = \epsilon^2 \,\widetilde{\text{def}}\, \vec{q}_1 + o(\epsilon^2) \tag{4-35b}$$

where def* is formed with respect to x^*, y^* and $\widetilde{\text{def}}$ with respect to $\tilde{x}, \tilde{y}$. The above formulas give the leading terms of the outer expansion of $\underline{\underline{A}}^*$.

By repeated application of the inner limit one obtains an inner expansion of $\underline{\underline{A}}^*$ which will be denoted by $\overline{\underline{\underline{A}}}$. The components of $\overline{\underline{\underline{A}}}$ are

$$\overline{A}_{11} = 2\sqrt{\epsilon}\, u_1 + 2\epsilon \log \epsilon\, u_{1a} + \epsilon(u_1^2 + 2u_2 - p_1) + o(\epsilon) \tag{4-36a}$$

$$\begin{aligned}\overline{A}_{12} = \overline{A}_{21} = \epsilon\left(v_1 - \frac{1}{Re}\frac{\partial u_1}{\partial \overline{y}}\right) + \epsilon^{3/2} \log \epsilon \left(v_{1a} - \frac{\partial u_{1a}}{\partial \overline{y}}\right) \\ + \epsilon^{3/2}\left(u_1 v_1 + v_2 - \frac{1}{Re}\frac{\partial u_2}{\partial \overline{y}}\right) + o(\epsilon^{3/2})\end{aligned} \tag{4-36b}$$

$$\overline{A}_{22} = \epsilon p_1 + o(\epsilon) \tag{4-36c}$$

The terms given by equations 4-35 and 4-36 are sufficient both for computing the integral of momentum and the integral of moment of momentum.

We shall also use the notation

$$\vec{\overline{G}} = \text{inner expansion of } \vec{G}^* \tag{4-37a}$$

$$\vec{\tilde{G}} = \text{outer expansion of } \vec{G}^* \tag{4-37b}$$

For the following, compare the computation of $\Delta\psi^*$ above. The $\tilde{x}$-argument will not be shown explicitly below. The argument shown is the value of $\overline{y}$ or $\tilde{y}$ for inner and outer terms respectively. Since the inner expansion is valid near the $\tilde{x}$-axis one finds

$$\Delta\vec{G}^* = \Delta\vec{\overline{G}} \equiv \vec{\overline{G}}(0+) - \vec{\overline{G}}(0-) \tag{4-38a}$$

The matching conditions are, in concentrated notation

$$\vec{\tilde{G}}(0+) = \vec{\overline{G}}(+\infty), \quad \vec{\tilde{G}}(0-) = \vec{\overline{G}}(-\infty) \tag{4-38b}$$

Furthermore (cf. equation 3-24)

$$\frac{\partial\vec{\overline{G}}}{\partial\overline{y}} = \frac{1}{\sqrt{\epsilon}}(\overline{A}_{11}\vec{i} + \overline{A}_{12}\vec{j}) \tag{4-38c}$$

Combining the above equations one finds (cf. equation 4-44 ff)

$$\Delta\vec{G}^* = \Delta\vec{\tilde{G}} - \int_{-\infty}^{\infty} \frac{1}{\sqrt{\epsilon}}(\overline{A}_{11}\vec{i} + \overline{A}_{12}\vec{j})\, d\overline{y} \tag{4-39}$$

The quantity represented by each side should be exactly of order unity, the terms of higher order should vanish identically. To order unity the first terms of equations 4-35a and 4-36a should be used, the term $\frac{1}{\sqrt{\epsilon}} \bar{A}_{12}$ gives no contribution. Thus

$$\Delta \vec{G}^* = -\oint T(\vec{i} \circ \vec{q}_1)\, d\tilde{\vec{n}} - 2\left(\int_{-\infty}^{\infty} u_1 d\bar{y} \right) \vec{i} \qquad (4\text{-}40)$$

With the aid of equations 3-29 and 4-18a one shows that the first term of the right-hand side is $-m\vec{i} + \Gamma^* \vec{j}$. By direct integration of equation 4-15a one finds that the second term is $2m\vec{i}$. This proves equation 4-31.

4.4. Some Higher-Order Terms

Results: The following formulas will be derived:

For the inner expansion

$$u_{1a} = -\frac{\Gamma^*}{2\pi} \frac{\partial u_1}{\partial \bar{y}} \qquad (4\text{-}41a)$$

$$v_{1a} = \frac{\Gamma^*}{2\pi} \frac{\partial u_1}{\partial \bar{x}} \qquad (4\text{-}41b)$$

There is no pressure term of order $\epsilon \log \epsilon$.

$$u_2 = u_2^{(1)} + u_2^{(2)}\,; \quad u_2^{(1)} \text{ even in } \bar{y}, \quad u_2^{(2)} \text{ odd in } \bar{y} \qquad (4\text{-}42a)$$

$$u_2^{(1)} = -\frac{u_1^2 + Rev^{(1)}\psi_1^{(1)}}{2} + \frac{m}{2\pi\bar{x}} \qquad (4\text{-}42b)$$

$$u_2^{(2)} = \frac{\Gamma^*}{2\pi} (\log \bar{x}) \frac{\partial u_1}{\partial \bar{y}} + C_1 \frac{\partial u_1}{\partial \bar{y}}\,; \quad C_1 = \text{constant} \qquad (4\text{-}42c)$$

$$v_2 = v_2^{(1)} + v_2^{(2)}; \quad v_2^{(1)} \text{ odd in } \bar{y}, \quad v_2^{(2)} \text{ even in } \bar{y} \qquad (4\text{-}42d)$$

$$v_2^{(1)} = -\frac{1}{Re}\frac{\partial u_2^{(1)}}{\partial \overline{y}} + u_1 v_1^{(1)} - \frac{\sqrt{Re}\, m^2}{4\sqrt{2\pi}\,\overline{x}^{3/2}}\left[\sqrt{\frac{2}{\pi}}\,\eta e^{-2\eta^2} + \frac{1}{2}\,\mathrm{erf}(\sqrt{2}\,\eta)\right]$$

$$+ \frac{m\overline{y}}{2\pi\overline{x}^2} \tag{4-42e}$$

$$v_2^{(2)} = -\frac{\partial \psi_2^{(2)}}{\partial \overline{x}} \tag{4-42f}$$

$$\psi_2^{(2)} = \frac{\Gamma^*}{2\pi}(\log \overline{x})u_1 + C_1 u_1; \qquad C_1 = \text{constant} \tag{4-42g}$$

Note that $u_2^{(1)}$ and $v_2^{(1)}$ represent a flow field which is symmetric with respect to the x-axis. The remaining terms of u_2 and v_2 represent an antisymmetric flow field. The corresponding pressure is

$$p_2 = -\frac{\Gamma^* \overline{y}}{2\pi\overline{x}^2} \tag{4-42h}$$

For the <u>outer expansion</u> we shall determine

$$\vec{q}_{3/2} = \tilde{u}_{3/2}\vec{i} + \tilde{v}_{3/2}\vec{j}; \quad w_{3/2} = \tilde{u}_{3/2} - i\tilde{v}_{3/2} = \frac{i\sqrt{Re}\, m^2}{8\sqrt{2\pi}\, z^{3/2}} \tag{4-43a}$$

$$\tilde{p}_{3/2} = -\tilde{u}_{3/2} \tag{4-43b}$$

$$\vec{q}_{1a} = \tilde{u}_{1a}\vec{i} + \tilde{v}_{1a}\vec{j}; \quad w_{1a} = \tilde{u}_{1a} - i\tilde{v}_{1a} = w_{1a}^{(1)} + w_{1a}^{(2)} \tag{4-44a}$$

The symmetrical part $w_{1a}^{(1)}$ will not be discussed here. The anti-symmetrical part is

$$w_{1a}^{(2)} = \frac{im\Gamma^*}{2\pi^2 z^2} \tag{4-44b}$$

$$\tilde{p}_{1a} = -\tilde{u}_{1a} \tag{4-44c}$$

For $\vec{q}_2$ we shall only compute the antisymmetric term involving a logarithmic function. This term is given by

$$w_2^{(2)} = - \frac{im\Gamma^*}{2\pi^2 z^2} \log z \qquad (4\text{-}45)$$

The formulas given above will now be derived.

Computations for u_2, v_2 and p_2. The equations for these terms are

$$H(u_2) \equiv \frac{\partial u_2}{\partial \bar{x}} - \frac{1}{Re} \frac{\partial^2 u_2}{\partial \bar{y}^2} = -\left(u_1 \frac{\partial u_1}{\partial \bar{x}} + v_1 \frac{\partial u_1}{\partial \bar{y}}\right) - \frac{\partial p_1}{\partial \bar{x}} \qquad (4\text{-}46a)$$

$$-\frac{\partial p_2}{\partial \bar{y}} = \frac{\partial v_1}{\partial \bar{x}} - \frac{1}{Re} \frac{\partial^2 v_1}{\partial \bar{y}^2} = \frac{\Gamma^*}{2\pi \bar{x}^2} \qquad (4\text{-}46b)$$

$$\frac{\partial u_2}{\partial \bar{x}} + \frac{\partial v_2}{\partial \bar{y}} = 0 \qquad (4\text{-}46c)$$

The even part of u_2, namely $u_2^{(1)}$ then satisfies the equation

$$H(u_2^{(1)}) = -\left(u_1 \frac{\partial u_1}{\partial \bar{x}} + v_1^{(1)} \frac{\partial u_1}{\partial \bar{y}}\right) - \frac{\partial p_1}{\partial \bar{x}} \qquad (4\text{-}47)$$

Note that the even part of the forcing function in equation 4-46a is obtained by replacing v_1 by its odd part $v_1^{(1)}$. If one assumes that $u_2^{(1)}$ is of the similarity form

$$v_2^{(1)} = \frac{1}{\bar{x}} f(\eta) \qquad (a)$$

then equation 4-47 becomes an ordinary differential equation for f. By straightforward computation one finds the particular solution given by equation 4-42b. This solution shows the correct exponential decay of vorticity. We have to consider the possibility of adding eigen-solutions, i. e. solutions of the homogeneous heat equation. The eigensolutions of the similarity form (a) are the function obtained by linear combination of the functions $f_2^{(1)}$ and $f_2^{(2)}$ defined by equation A-9. However, $f_2^{(1)}$ cannot be used since it is odd and $f_2^{(2)}$ cannot be used since it decays algebraically. Thus no eigenfunctions may be added and the solution obtained for $u_2^{(1)}$ is unique. Here and also below we are using the fact that $\partial u_2/\partial \overline{y}$ alone gives the vorticity to the corresponding order (cf. equation 4-12).

The odd part of u_2 obeys the equation

$$H(u_2^{(2)}) = \frac{\Gamma^*}{2\pi\overline{x}} \frac{\partial u_1}{\partial \overline{y}} \tag{4-48}$$

If one assumes that $u_2^{(2)}$ has the similarity form (a) then the equation for f is

$$L_2(f) = \frac{m\Gamma^* Re}{2\pi^{3/2}} \frac{d}{d\eta}(e^{-\eta^2}) \tag{b}$$

This is, within a multiplicative constant, equation A-19 for $n = 2$. Hence all solutions of (b) give algebraic decay of vorticity.

In other words $u_2^{(2)}$ cannot have the similarity form given by (a). However, one may verify directly that $u_2^{(2)}$ as defined by equation 4-42c is a solution of equation 4-48. The function $\partial u_1/\partial \overline{y}$ is an eigen-solution which is odd in $\overline{y}$ and which vanishes exponentially at $|\overline{y}| = \infty$. Thus an arbitrary multiple of this function may be added to $u_2^{(2)}$. This accounts for the undetermined constant C_1 in equation 4-42c.

We shall presently comment on the fact that $u_2^{(2)}$ does not have proper similarity form. However, first we shall discuss v_2 and p_2. Equations 4-46b, c show that these functions are obtained as integrals with respect to $\overline{y}$ of known functions. This integration leads to the expressions for v_2 and p_2 given by equation 4-42. To these expressions one may in principle add a function of $\overline{x}$. Any function of $\overline{x}$ may be regarded as eigensolution of the equations for v_2 and p_2. By similarity (eliminability) one finds that the eigen-solutions must be proportional to $\overline{x}^{-3/2}$. Thus

$$\text{Eigensolution for } v_2 = C_2 \overline{x}^{-3/2} \tag{4-49a}$$

$$\text{Eigensolution for } p_2 = C_3 \overline{x}^{-3/2} \tag{4-49b}$$

At the present stage one cannot determine C_2 and C_3. However, as will be shown later, one finds by matching with the outer expansion that both C_2 and C_3 must be zero.

<u>Switchback. Terms of order $\epsilon \log \epsilon$.</u> The particular solution of equation 4-48 used above was the function

$$u_{2p} = \frac{\Gamma^*}{2\pi\overline{x}} (\log \overline{x}) \frac{\partial u_1}{\partial \overline{y}} \tag{4-50}$$

This function does not have the correct similarity. Now, the reason for requiring similarity was the requirement of eliminability of ϵ. As is easily seen ϵ cannot be eliminated from ϵu_{2p} since

$$\epsilon u_{2p} = \frac{\Gamma^*}{2\pi} (\log x^* + \log \epsilon)(\epsilon \frac{\partial u_1}{\partial \overline{y}}) \tag{c}$$

where the factor $\epsilon(\partial u_1/\partial \overline{y})$ actually is independent of ϵ. The term $\log \epsilon$ can only be eliminated by a term of different order. Indeed one finds that $\epsilon u_{2p} + \epsilon \log \epsilon u_{1a}$ is independent of ϵ. Thus the function u_{1a} as given by equation 4-41a is determined by the requirement of eliminability. Note that it follows from equations 4-9a and 4-10a that u_{1a} must satisfy the homogeneous heat equation. This additional requirement is of course fulfilled for the u_{1a} as chosen.

In a similar way one finds v_{1a}, as given in equation 4-41b by requiring that ϵ must be eliminable from $\sqrt{\epsilon}\,(\epsilon \log \epsilon v_{1a} + \epsilon v_2)$. Note that u_{1a} and v_{1a}, as determined above, satisfy the correct continuity equation.

The form of the equations for u^*, v^* etc. seems to indicate that a term $\sqrt{\epsilon}\,u_1$ in the inner expansion is followed by a term ϵu_2. However, in trying to determine u_2 we were forced to introduce an additional term $\epsilon \log \epsilon u_{1a}$ whose order is intermediate between $\sqrt{\epsilon}$ and ϵ. This phenomenon will be referred to as switchback. Here we shall give a physical explanation for the special case of switchback just encountered. This explanation is similar to that given by Goldstein (reference 2). The terms u_{1a} and v_{1a} were found by Filon (reference 1). They will be referred to as Filon terms.

Physical Explanation of the Filon Terms. The appearance of the Filon terms may be explained from the fact that the wake is displaced by the antisymmetric part of the velocity field.

First we observe that if $u^* = f(\bar{x}, \bar{y})$ is a solution of a boundary layer equation then so is the function $f(\bar{x}, \bar{y} - y_o(\bar{x}))$ where y_o is any function of $\bar{x}$. The function $u_1(\bar{x}, \bar{y})$ is a solution of a boundary layer equation which is symmetric about the center line $\bar{y} = 0$. The solution for v^* contains the crossflow

$$-\frac{\epsilon\Gamma^*}{2\pi\overline{x}} = -\frac{\Gamma^*}{2\pi x^*} \tag{a}$$

This crossflow displaces the centerline of the wake. If we denote the displaced position of the centerline by

$$y^* = y_o(x^*), \quad \overline{y} = \sqrt{\epsilon}\, y_o(x^*) \tag{b}$$

then

$$\frac{dy_o}{dx^*} = -\frac{\Gamma^*}{2\pi x^*} \tag{c}$$

or

$$y_o = -\frac{\Gamma^*}{2\pi}\log x^*, \quad \epsilon\, y_o = -\frac{\epsilon\Gamma^*}{2\pi}(\log\overline{x} - \log\epsilon) \tag{d}$$

A solution of the equation for u_1 having a centerline corrected for crossflow is then

$$\epsilon\, u_1(\overline{x}, \overline{y} - \sqrt{\epsilon}\, y_o) = \epsilon\, u_1(\overline{x}, \overline{y}) - \epsilon\log\epsilon\,\frac{\Gamma^*}{2\pi}\frac{\partial u_1}{\partial\overline{y}} + \epsilon\,\frac{\Gamma^*}{2\pi}\log\overline{x}\,\frac{\partial u_1}{\partial\overline{y}} \tag{e}$$

The second term of the right-hand side is the Filon term $\epsilon\log\epsilon u_{1a}$, the third term is part of ϵu_2. Note that the present reasoning considers interaction of perturbation velocities and hence a nonlinear effect.

<u>Determination of $\vec{q}_{3/2}$</u>. The function v_2 of the inner expansion contains a term of the form $\overline{x}^{3/2} f(\eta)$. The value of this term at $|\eta| = \infty$ is

$$\mp \frac{\sqrt{Re}\ m^2}{8\sqrt{2\pi}\ \bar{x}^{3/2}} \quad \text{at} \quad \eta = \pm\infty \tag{f}$$

Thus there is inflow into the wake. The term ϵv_2 must then be matched with a term of the outer expansion which has a sink distribution along the x-axis. Since the term ϵv_2 occurs in the inner expansion of $\bar{v} = \frac{v^*}{\epsilon}$ it follows that the matching outer term must be of order $\epsilon^{3/2}$. By eliminability one concludes that the complex velocity function must be proportional to $z^{-3/2}$. The constant of proportionality is determined by matching with (f). This leads uniquely to the outer terms of order $\epsilon^{3/2}$ given by equation 4-43.

The constants C_2 and C_3 in equation 4-49 can now be determined. The eigensolution of v_2 must match with the outer flow of the order $\epsilon^{3/2}$, but from the above v_2 is already matched with $\vec{q}_{3/2}$, one should put $C_2 = 0$. By matching the constant C_3 should be zero also. This follows from the fact that $\vec{q}_{3/2}$ expanded in terms of inner variables $\bar{x}$, $\bar{y}$ does not contain terms of the order $\epsilon^{3/2}$, and consequently the pressure of the order $\epsilon^{3/2}$ is zero in the outer flow.

<u>Pressure Discontinuity Across the Wake.</u> We have thus far determined terms up to and including $O(\epsilon^{3/2})$ of the outer expansion and $O(\epsilon)$ of the inner expansion. The highest-order terms obtained gave a discontinuity in v^* across the wake but u^* and hence the pressure was continuous. Proceeding to higher-order terms one finds a pressure discontinuity across the wake.

We shall first consider the wake terms of order $\epsilon^{3/2}\log\epsilon$ and order $\epsilon^{3/2}$. The equations are

$O(\epsilon^{3/2}\log\epsilon)$:

$$\frac{\partial u_{2a}}{\partial \bar{x}} - \frac{1}{Re}\frac{\partial^2 u_{2a}}{\partial \bar{y}^2}$$

$$= -\left(u_1\frac{\partial u_{1a}}{\partial \bar{x}} + u_{1a}\frac{\partial u_1}{\partial \bar{x}} + v_1\frac{\partial u_{1a}}{\partial \bar{y}} + v_{1a}\frac{\partial u_1}{\partial \bar{y}}\right) \qquad (4\text{-}51a)$$

$$\frac{\partial p_{2a}}{\partial \bar{y}} = -\left(\frac{\partial v_{1a}}{\partial \bar{x}} - \frac{1}{Re}\frac{\partial^2 v_{1a}}{\partial \bar{y}^2}\right) \qquad (4\text{-}51b)$$

$$\frac{\partial u_{2a}}{\partial \bar{x}} + \frac{\partial v_{2a}}{\partial \bar{y}} = 0 \qquad (4\text{-}51c)$$

$O(\epsilon^{3/2})$:

$$\frac{\partial u_3}{\partial \bar{x}} - \frac{1}{Re}\frac{\partial^2 u_3}{\partial \bar{y}^2}$$

$$= -\left(u_1\frac{\partial u_2}{\partial \bar{x}} + u_2\frac{\partial u_1}{\partial \bar{y}} + v_1\frac{\partial u_2}{\partial \bar{y}} + v_2\frac{\partial u_1}{\partial \bar{y}}\right)$$

$$- \frac{\partial p_2}{\partial \bar{x}} + \frac{1}{Re}\frac{\partial^2 u_1}{\partial \bar{x}^2} \qquad (4\text{-}52a)$$

$$\frac{\partial p_3}{\partial \bar{y}} = -\left(\frac{\partial v_2}{\partial \bar{x}} - \frac{1}{Re}\frac{\partial^2 v_2}{\partial \bar{y}^2}\right) - \left(u_1\frac{\partial v_1}{\partial \bar{x}} + v_1\frac{\partial v_1}{\partial \bar{y}}\right) \qquad (4\text{-}52b)$$

$$\frac{\partial u_3}{\partial \bar{x}} + \frac{\partial v_3}{\partial \bar{y}} = 0 \qquad (4\text{-}52c)$$

By simple consideration one may come to the conclusion that solutions of equations 4-51 are simply the switchback terms of the solutions of equations 4-52. This comes from the fact that the principle of eliminability of the parameter ϵ may apply as well directly to the equations instead of the solutions. In fact, equations 4-51 may be regarded as a switchback equation of equations 4-52.

Since v_{1a} satisfies the homogeneous heat equation it follows that equation 4-51b reduces to

$$\frac{\partial p_{2a}}{\partial \bar{y}} = 0 \tag{4-53}$$

The pressure p_{2a} is therefore constant across the wake. By combination with p_1, p_2 one concludes that the pressure is continuous across the wake up to and including the order $O(\epsilon^{3/2} \log \epsilon)$. We shall show in the following that pressure jump occurs in p_3. This jump in pressure induces a switchback term in the outer flow.

To find the pressure jump one needs only consider the odd part in p_3. This part will be denoted by $p_3^{(2)}$. It obeys the equation

$$\frac{\partial p_3^{(2)}}{\partial \bar{y}} = \frac{\Gamma^*}{2\pi} \frac{\partial}{\partial \bar{x}} \left(\frac{u_1}{\bar{x}}\right) - u_1 \frac{\Gamma^*}{2\pi \bar{x}^2} + \frac{\Gamma^*}{2\pi \bar{x}} \frac{\partial v_1^{(1)}}{\partial \bar{y}} \tag{4-54}$$

Integrating, one finds

$$p_3^{(2)} = \frac{\Gamma^* m}{2\pi \bar{x}^2} \operatorname{erf}\left(\frac{\bar{y}}{2\sqrt{\bar{x}}}\right) \tag{4-55}$$

Hence

$$\Delta p_3^{(2)} = p_3^{(2)}(+\infty) - p_3^{(2)}(-\infty) = \frac{\Gamma^*}{\pi \bar{x}^2} \tag{4-56}$$

By matching it follows that the term $\tilde{p}_2$ of the outer expansion must have a discontinuity at $\tilde{y} = 0$ whose magnitude is given by equation 4-56. From Bernoulli's law one finds that

$$\tilde{p}_2 = -\tilde{u}_2 - \frac{1}{2}\left(\tilde{u}_1^2 + \tilde{v}_1^2\right) \tag{4-57}$$

Since $\tilde{u}_1$ and $\tilde{v}_1$ are continuous at $\tilde{y} = 0$ it follows that

$$\Delta\tilde{u}_2 = \tilde{u}_2(\tilde{x}, 0+) - \tilde{u}_2(\tilde{x}, 0-) = -\Delta\tilde{p}_2 = -\frac{\Gamma^* m}{\pi \tilde{x} 2} \qquad (4\text{-}58)$$

The corresponding complex velocity function of order ϵ^2 which gives this jump is

$$-\frac{\epsilon^2 i m \Gamma^*}{2\pi^2 z^2} \log z \qquad (4\text{-}59)$$

When the principle of eliminability is applied there arises the switchback term, which we shall refer to as the Imai term, (cf. reference 3),

$$\epsilon^2 \log \epsilon \frac{i m \Gamma^*}{2\pi^2 z^2} \qquad (4\text{-}60)$$

Physically, this term given by equation 4-60 represents a vertical dipole of strength $\frac{m\Gamma^*}{2\pi^2}$ in the outer flow. However, it must be pointed out here that due to the matching of velocity between the outer and the wake flows there arise other switchback terms which correspond to horizontal dipoles. These horizontal dipoles contribute nothing to the moment on the body and will be left out of the discussion.

4.5. Torque on Body. Filon's Paradox.

Let $\underline{\underline{A}}^*$ be the tensor defined by equation 4-32a. We define a scalar function $H^*(Q)$ by

$$H^* \vec{k} = \int_P^Q \vec{r}^* \times \underline{\underline{A}}^* \, d\vec{n} \qquad (4\text{-}61)$$

The position of the initial point P will be irrelevant for the following. The path of integration must lie outside the given curve C (cf. p. 41) and may not cross the x*-axis. It then follows (cf. equation 3-25) that

$$\frac{\text{Torque on Body}}{\rho^2 U^2 L^2} \equiv M^* = \Delta H^* \equiv H^*(x^*, 0+) - H^*(x^* 0-) \qquad (4\text{-}62)$$

where the point $(x^*, 0)$ is on the positive x^*-axis outside the curve C. The value of ΔH^* is then independent of x^*.

To evaluate M^* we first consider the outer expansions of H^*. Let $\tilde{H}$ and $\tilde{\underline{\underline{A}}}$ denote the formal outer expansions of H^* and $\underline{\underline{A}}^*$ respectively. Then (cf. equations 4-34 and 4-35)

$$\tilde{H}\vec{k} = \frac{1}{\epsilon^2} \int_P^Q \vec{\tilde{r}} \times \tilde{\underline{\underline{A}}}\, d\vec{\tilde{n}}$$

$$= \int_P^Q \vec{\tilde{r}} \times \{\frac{1}{\epsilon} T(\vec{i} \circ \vec{q}_1) + \frac{1}{\sqrt{\epsilon}} T(\vec{i} \circ \vec{q}_{3/2}) + \log \epsilon\, T(\vec{i} \circ \vec{q}_{1a})$$

$$+ [T(\vec{i} \circ \vec{q}_2) + \frac{1}{2}(\vec{q}_1 \circ \vec{q}_1) - \frac{1}{Re} \tilde{\text{def}}\ \vec{q}_1]\}\, d\vec{\tilde{n}} + o(1) \qquad (4\text{-}63)$$

We shall consider

$$\Delta\tilde{H} = \tilde{H}(\tilde{x}, 0+) - \tilde{H}(\tilde{x}, 0-) \qquad (4\text{-}64)$$

From the formulas on p. 31 it follows that if two harmonic velocity fields $\vec{v}^{(1)}$ and $\vec{v}^{(2)}$ have the complex velocity functions $w^{(1)}$ and $w^{(2)}$ then

$$\vec{k}\cdot \int_P^Q \tilde{r} \times T(\vec{v}^{(1)}, \vec{v}^{(2)})\, d\vec{\tilde{n}} = \text{Real Part}\left(\int_P^Q z w^{(1)} w^{(2)} dz\right) \qquad (4\text{-}65)$$

where now $z = \tilde{x} + i\tilde{y}$.

If we take $v^{(1)} = \vec{i}$, $v^{(2)} = \vec{q}_1$ then $w^{(1)} = 1$ and (cf. equations 4-17 and 4-18) $w^{(2)}$ = constant times z^{-1}. The integral in equation 4-65 is then single-valued. Hence the term of order $\frac{1}{\epsilon}$ in $\Delta\tilde{H}$ is zero. If we instead take $\vec{v}^{(2)} = \vec{q}_{3/2}$, then (cf. equation 4-43) $w^{(2)} = ibz^{-3/2}$, b = real constant. Then

$$\int_P^Q zw^{(1)}w^{(2)}dz = 2ib\sqrt{z} + \text{constant} \qquad \text{(a)}$$

The real part is

$$-2b\sqrt{r}\sin\frac{\theta}{2} \qquad \text{(b)}$$

which has the same value for $\theta = 0$ and $\theta = 2\pi$. Hence the term of order $\frac{1}{\sqrt{\epsilon}}$ is zero. If we take $\vec{v}^{(2)} = \vec{q}_{1a}$, then equation 4-43a shows that

$$w^{(2)} = \frac{ic}{z^2}, \quad c = \frac{m\Gamma^*}{2\pi^2} \qquad \text{(c)}$$

Hence

$$\int zw^{(1)}w^{(2)}dz = i \cdot c \log z + \text{constant} \qquad \text{(d)}$$

and

$$\text{Real part} = -c\theta + \text{constant} \qquad \text{(e)}$$

and

$$\Delta\tilde{H} = (\log\epsilon)\frac{m\Gamma^*}{\pi} + O(1) \qquad \text{(4-66)}$$

Thus the Imai term gives rise to torque of order $\log\epsilon$.

We shall now consider the inner expansion. Let $\overline{H}$ and $\overline{\underline{\underline{A}}}$ be the formal expansions of H^* and $\underline{\underline{A}}^*$ respectively. From equations

3-25b we find that

$$\frac{\partial H^*}{\partial y^*} = x^* A^*_{21} - y^* A^*_{11} \tag{4-68}$$

Hence from equation 4-36

$$\frac{\partial \overline{H}}{\partial \overline{y}} = \epsilon^{-3/2} \overline{x}\, \overline{A}_{21} - \epsilon^{-1} \overline{y}\, \overline{A}_{11}$$

$$= \frac{1}{\sqrt{\epsilon}} [\, \overline{x}(v_1 - \frac{1}{Re} \frac{\partial u_1}{\partial \overline{y}}) - 2\overline{y}\, u_1] + \log \epsilon [\, \overline{x}(v_{1a} - \frac{1}{Re} \frac{\partial u_{1a}}{\partial \overline{y}}) - 2\overline{y}\, u_{1a}]$$

$$+ [\, \overline{x}(u_1 v_1 + v_2 - \frac{1}{Re} \frac{\partial u_2}{\partial \overline{y}}) - \overline{y}(u_1^2 + 2u_2 - p_1)] + o(1) \tag{4-69}$$

The matching between inner and outer expansions of H^* has to be done carefully. We remind the reader of the following facts for the matching of the flow field. The outer source $\epsilon q_1^{(1)}$ where $q_1^{(1)}$ has the complex streamfunction $w_1^{(1)} = \frac{m}{2\pi z}$ (equation 4-18a) contains nothing of the inner sink with velocity components $\epsilon\, u_1$ and $\epsilon v_1^{(1)}$ (equation 4-15). The flow components of the inner sink are exponentially small at $\overline{y} = \infty$. On the other hand the outer vortex $\epsilon \vec{q}_1^{(2)}$ where $\vec{q}_1^{(2)}$ has the complex streamfunction $w_1^{(2)} = \frac{i\Gamma^*}{2\pi z}$ contains completely the inner "vortex" with vertical component $\epsilon v_1^{(2)} = - \frac{\epsilon \Gamma^*}{2\pi \overline{x}}$. In fact the inner expansion of $\tilde{v}_1^{(2)}$ gives the complete term $v_1^{(2)}$ (cf. p. 50). In computing the contribution of the inner expansion of H^* to ΔH^* to order $\frac{1}{\epsilon}$ we then neglect the component $v_1^{(1)}$ of v_1 since this flow component has already been considered in the outer expansion. In the first square bracket of equation 4-69 we hence replace v_1 by $v_1^{(1)}$. The remaining term is then odd in $\overline{y}$ and its integral from $\bar{y} = -\infty$ to $\bar{y} = +\infty$ is zero. Thus $\Delta H^* = 0$ to order $\frac{1}{\sqrt{\epsilon}}$.

Next we consider the term of order $\log \epsilon$ in equation 4-69. The functions v_{1a} and u_{1a} are exponentially small at $\bar{y} = \pm \infty$. Hence they contribute nothing to the outer expansion. The complete term of order $\log \epsilon$ should therefore be retained. One finds (cf. equation 4-41a)

$$-\int_{-\infty}^{\infty} [\bar{x} v_{1a} - \frac{1}{Re} \frac{\partial u_{1a}}{\partial \bar{y}} - 2\bar{y}\, u_{1a}]\, d\bar{y}$$

$$= -\frac{\Gamma^*}{\pi} \int_{-\infty}^{\infty} \bar{y} \frac{\partial u_{\Gamma}}{\partial \bar{y}}\, d\bar{y} = -\frac{m\Gamma^*}{\pi} \tag{4-70}$$

By analogy with equation 4-39 we then find that the Filon term contributes a torque of magnitude $-\log \epsilon \; \frac{m\Gamma^*}{\pi}$. This contribution is exactly cancelled by the contribution of the Imai term (equation 4-66). Since Filon had not computed the Imai term he was led to the paradoxical result that the torque is infinite (reference 1). This is Filon's paradox which was resolved by Imai in reference 2. Note that the Imai term is of higher order, i.e. smaller than the Filon term; however, their contributions to the torque are of the same order. Intuitively speaking this is due to the fact that the length of the path of integration for the outer terms is of larger order of magnitude than that for the inner terms. For the outer terms we integrate over a path of length $\sim r^*$, the inner terms are integrated over the parabolic wake over a length $\sim \sqrt{x^*} \sim \sqrt{r^*}$.

The torque to order unity can now be computed from the next terms in the expansions. This will relate one of the undetermined constants in the expansion to the value of M^*. These calculations will not be carried out here.

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LIST OF PRINCIPAL SYMBOLS

a. Dimensional Variables and Parameters

x, y	= Cartesian position coordinates
r, θ	= Polar coordinates
$\vec{i}, \vec{j}, \vec{k}$	= Cartesian unit vectors
$\vec{q}$	$= u\vec{i} + v\vec{j}$ = Flow velocity
p	= pressure
L	= characteristic length of body
$U\vec{i}$	= velocity at infinity
p_∞	= pressure at infinity
ρ	= density = constant
μ	= viscosity = constant
ν	$= \frac{\mu}{\rho}$ = kinematic viscosity
ω	$= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ = vorticity
ψ	= streamfunction, $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$
$\rho \vec{q} \circ \vec{q}$	= Dyadic product of $\rho \vec{q}$ and $\vec{q}$ = Flow-of-momentum tensor
$\underline{\underline{\tau}}$	$= \mu$ def $\vec{q}$ = viscous stress tensor
I	= identity tensor
$\underline{\underline{\sigma}}$	$= -pI + \tau$ = stress tensor
$\underline{\underline{A}}$	$= \rho \vec{q} \circ \vec{q} - \underline{\underline{\sigma}}$
C	= curve enclosing body. The origin is assumed to be inside C.
$\vec{F}$	= fluid force on body
$\vec{M}$	= moment exerted by fluid on body = $M\vec{k}$
M	= torque, positive if counter-clockwise.

b. Original non-dimensional variables and parameters

$$x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}, \quad r^* = \frac{r}{L}$$

$$\vec{q^*} = \frac{\vec{q}}{U}, \quad u^* = \frac{u}{U}, \quad v^* = \frac{v}{U}$$

$$p^* = \frac{p - p_\infty}{\rho U^2}$$

$$Re = \frac{UL}{\nu}$$

$$\psi^* = \frac{\psi}{UL}, \quad u^* = \frac{\partial \psi^*}{\partial y^*}, \quad v^* = -\frac{\partial \psi^*}{\partial x^*}$$

$$\underline{\underline{A}}^* = \frac{1}{\rho U^2}\left[\underline{\underline{A}} - (\vec{i} \circ \vec{i})\right]$$

$$\vec{F^*} = \frac{\vec{F}}{\rho U^2 L} = m\vec{i} + \Gamma^* \vec{j}$$

$$M^* = \frac{M}{\rho U^2 L^2}$$

c. Variables for outer expansion

R = an artificial length scale

$\epsilon = \frac{L}{R}$ = artificial parameter

$\tilde{x} = \frac{x}{R} = \epsilon x^*, \quad \tilde{y} = \frac{y}{R} = \epsilon y^*, \quad \tilde{r} = \frac{r}{R} = \epsilon r^*$

$z = \tilde{x} + i\tilde{y}$

$(\tilde{u}_n, \tilde{v}_n)$ = flow velocity terms in the outer expansions of u^* and v^*

$w_n = \tilde{u}_n - i\tilde{v}_n$ = complex velocity

$\tilde{p}_n$ = term in the outer expansion for pressure

d. Variables for inner expansion

$$\bar{x} = \tilde{x} = \epsilon x^*, \quad \bar{y} = \frac{\tilde{y}}{\sqrt{\epsilon}} = \sqrt{\epsilon}\, y^*, \quad \eta = \bar{y}\sqrt{\frac{Re}{4\bar{x}}}$$

$\vec{q^*} = (u^*, v^*) = (u^*, \sqrt{\epsilon}\,\bar{v})$ = flow velocity

(u_n, v_n) = flow velocity terms in the inner expansion

p_n = term in the inner expansion for pressure

e. Mathematical symbols

$$H(u) = \frac{\partial u}{\partial \bar{x}} - \frac{1}{Re}\frac{\partial^2 u}{\partial \bar{y}^2}$$

$$f_n^{(1)} = \frac{d^{n-1}}{d\eta^{n-1}}\, d^{-\eta^2}, \qquad f_n^{(2)} = \frac{d^{n-1}}{d\eta^{n-1}}\, d^{-\eta^2}\int_0^{\eta} e^{s^2}\, ds$$

$f_n^{(1)}$, $f_n^{(2)}$ are the two independent solutions of

$$L_n(f) \equiv f_n'' + 2\eta f_n' + 2nf = 0$$

K_o = modified Bessel function of the second kind of the zeroth order.

APPENDIX A: SIMILARITY SOLUTIONS OF THE HEAT EQUATION

Homogeneous Equation

We define a linear operator H by

$$H(u) = \frac{\partial u}{\partial \overline{x}} - \frac{1}{Re} \frac{\partial^2 u}{\partial \overline{y}^2} \tag{A-1}$$

If $u(\overline{x}, \overline{y})$ has the form

$$u(\overline{x}, \overline{y}) = \overline{x}^{-\alpha/2} f(\eta); \quad \eta = \frac{\sqrt{Re}\,\overline{y}}{2\sqrt{\overline{x}}} \tag{A-2}$$

then

$$H(u) = -\frac{x^{-(\alpha+2)/2}}{4} L_\alpha(f) \tag{A-3a}$$

where

$$L_\alpha(f) = f'' + 2\eta f' + 2\alpha f \tag{A-3b}$$

and prime denotes differentiation with respect to η.

Thus solutions of the ordinary differential equation

$$L_\alpha(f) = 0 \tag{A-4a}$$

give similarity solutions of the homogeneous heat equation

$$H(u) = 0 \tag{A-4b}$$

We note that if u is a solution of the homogeneous heat equation A-4b then $\partial u/\partial \overline{y}$ is another solution. For the special case of similarity solutions one finds

If u has the form A-2 then

$$\frac{\partial u}{\partial \overline{y}} = \frac{\sqrt{Re}}{2} x^{-(\alpha+1)/2} f' \tag{A-5}$$

This recursive formula for generating similarity solutions of the heat equation corresponds to the following recursive formula for the operator L_α,

$$[L_\alpha(f)]' = L_{\alpha+1}(f') \tag{A-6}$$

We shall now discuss the solutions of equation A-4a. For $\alpha = 1$ we find two linearly independent solutions

$$f_1^{(1)} = e^{-\eta^2} \tag{A-7a}$$

$$f_1^{(2)} = e^{-\eta^2}\int_0^\eta e^{s^2}\, ds \tag{A-7b}$$

For large values of η the asymptotic expansion of the second solution is

$$f_1^{(2)} \sim \frac{1}{2\eta} + \frac{1}{4\eta^3} + \dots \tag{A-8}$$

The recursive formula A-6 gives us the following pair of solutions of equation A-4a for $\alpha = n$ = positive integer

$$f_n^{(1)} = \frac{d^{n-1} f_1^{(1)}}{d\eta^{n-1}} \tag{A-9a}$$

$$f_n^{(2)} = \frac{d^{n-1} f_1^{(2)}}{d\eta^{n-1}} \tag{A-9b}$$

We note that $f_1^{(1)}$ has exponential decay at infinity and that $f_1^{(2)}$ has algebraic decay. From this we conclude that $f_n^{(1)}$ has exponential decay and $f_n^{(2)}$ has algebraic decay. In particular, equations A-9a, b define two linearly independent solutions.

When α is not a positive integer equation A-4a may be solved in terms of confluent hypergeometric functions. One may write equation A-4a as

$$z\frac{d^2f}{dz^2} + (\frac{1}{2} - z)\frac{df}{dz} - \frac{\alpha}{2}f = 0 \qquad \text{(A-10)}$$

It then follows (cf. reference 15 , p. 252) that there are two linearly independent solutions

$$f_\alpha^{(1)} = {}_1F_1(\frac{\alpha}{2}, \frac{1}{2}; -\eta^2) \qquad \text{(A-11a)}$$

$$f_\alpha^{(2)} = \eta\ {}_1F_1(\frac{\alpha+1}{2}, \frac{3}{2}; -\eta^2) \qquad \text{(A-11b)}$$

For large values of η, η real, one finds (reference 15, p. 277)

$$f_\alpha^{(1)} \sim \frac{\Gamma(1/2)}{\Gamma(\frac{1-\alpha}{2})} |\eta|^{-\alpha}\ {}_2F_0(\frac{\alpha}{2}, \frac{\alpha+1}{2}; \frac{1}{\eta^2}) \qquad \text{(A-12a)}$$

$$f_\alpha^{(2)} \sim \frac{\eta}{|\eta|} \frac{\Gamma(3/2)}{\Gamma(\frac{2-\alpha}{2})} |\eta|^{-\alpha}\ {}_2F_0(\frac{\alpha+1}{2}, \frac{\alpha}{2}; \frac{1}{\eta^2}) \qquad \text{(A-12b)}$$

where

$$ {}_2F_0(\alpha_1, \alpha_2; z) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1+n)\Gamma(\alpha_2+n)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \frac{z^n}{n!} \qquad \text{(A-12c)}$$

When α is an odd positive integer the expression for $f_\alpha^{(1)}$ gives zero and when α is an even positive integer the expression for $f_\alpha^{(2)}$ gives zero. However, for $\alpha \geqslant 0$ and not a positive integer the above formulas give the leading term of the asymptotic expansions of two linearly independent solutions of equation A-4a. Each of these solutions decays algebraically for $|\eta|$ large. Consider now a linear combination

$f_\alpha = C_1 f_\alpha^{(1)} + C_2 f_\alpha^{(2)}$. Since the expressions given by A-11a and A-11b differ only by a constant factor, one may choose C_1 and C_2 in such a way that the algebraic part of f_α vanishes for η near $+\infty$. Thus f_α will then decay exponentially near $\eta = +\infty$. However, near $\eta = -\infty$ $f_\alpha^{(2)}$ changes sign due to the factor $\eta/|\eta|$. Hence f_α will decay algebraically near $\eta = -\infty$. The role of $+\infty$ and $-\infty$ may be reversed, but it is impossible to find a f_α which decays exponentially both near $\eta = +\infty$ and $\eta = -\infty$. One may conclude directly that since $f_\alpha^{(1)}$ is even in η, $f_\alpha^{(2)}$ odd in η and since both have algebraic decay at $\eta = \pm\infty$ no linear combination of these two functions can have non-algebraic decay at both $\eta = +\infty$ and $\eta = -\infty$.

The results about the asymptotic behavior of f_α may be summarized as follows: Let f_α be any non-zero solution of the equation

$$L_\alpha f \equiv f'' + 2\eta f' + 2\alpha f = 0; \quad \alpha \geqslant 0 \qquad \text{(A-13)}$$

When α is not a positive integer there are no solutions which have a non-algebraic decay at both $\eta = +\infty$ and $\eta = -\infty$. When α is a positive integer the only solutions with non-algebraic decay are constant multiples of $f_n^{(1)}$ where

$$f_n^{(1)} = \frac{d^{n-1}}{d\eta^{n-1}}(e^{-\eta^2}), \quad n = 1, 2, \ldots \qquad \text{(A-14)}$$

<u>Non-homogeneous Equations</u>

We shall now study special cases of the equation

$$L_n(f) = g, \quad n = 1, 2, \ldots \qquad \text{(A-15)}$$

First we consider the case for which g is a solution of the corresponding homogeneous equation (case of resonance), in particular the case $g = f_n^{(1)}$ (cf. A-14). For $n = 1$ the equation then reads

$$L_1(f) \equiv f'' + 2\eta f' + 2f = f_1^{(1)} \tag{A-16}$$

A particular solution of this equation is

$$f_{1p} = \frac{\pi}{2} e^{-\eta^2} \int_0^{\eta} e^{\sigma^2} \operatorname{erf} \sigma \, d\sigma \tag{A-17a}$$

where

$$\operatorname{erf} s = \frac{2}{\pi} \int_0^{s} e^{-t^2} \, dt \tag{A-17b}$$

Since $\operatorname{erf} \infty = 1$, $\operatorname{erf}(-\infty) = -1$, a comparison of A-7b, A-9 and A-17 shows that f_{1p} decays as $1/|\eta|$ as $|\eta|$ tends to infinity. Since f_{1p} is even and $f_1^{(2)}$ is odd it follows that adding a multiple of $f_1^{(2)}$ to f_{1p} cannot remove the algebraic decay at $\pm\infty$ simultaneously. Adding a multiple of $f_1^{(1)}$ cannot remove the algebraic decay at either $\pm\infty$.

By the recursion formulas A-6 and A-9a one shows that the function

$$f_{np} = \frac{d^{n-1}}{d\eta^{n-1}} (f_{1p}) \tag{A-18}$$

is a particular solution of the equation

$$L_n(f) = f_n^{(1)} \tag{A-19}$$

By the same method as above one shows that an arbitrary solution of equation A-19 cannot decay faster than algebraically at both $\eta = +\infty$ and $\eta = -\infty$.