THE STRESS DISTRIBUTION IN REENFORCED PLATES UNDER CONCENTRATED EDGE LOADS

Thesis by

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SUMMARY

As part of the investigation of the stress distribution in metal covered wings leading to the most efficient distribution of materials, experiments have been made with flat plates reenforced with stiffeners under concentrated edge loads. Two theoretical solutions have been proposed: one by means of differential equations, considering the sheet between the stiffeners as the medium of transfer and the amount of load transferred to be proportional to the differences in the deflections; the other method, using the principle of the minimum of energy, approximates the accurate solution by considering the loaded system to be composed of columns (stiffeners acting with an effective width of sheet) under compression, the sheet being under shear.

While not a quantitative analysis, this investigation could be considered as a qualitative analysis, stating that:

1) The application of concentrated end loads results in an even stress distribution at reasonable distances from the point of application of load.

2) The shear modulus G_t used in the calculation for the transfer of load is found to be considerably smaller due to the buckling of the sheet between the stiffeners.

3) The area of sheet assumed to act with the stiffener seems to be slightly larger than that calculated by Sechler's method for panels under pure compression.

Extensive experimental investigations have to be made to verify these statements, taking into account variations in length and width of the panel, thickness of sheet, area and distance of stiffeners for different materials.

Experimental investigations of the stress distribution in reenforced plates under distributed side loads and end loads in the stiffeners are being carried out in the Laboratory at the present time.

INTRODUCTION

The strength characteristics of individual panels and stiffeners and their combination have been investigated and in time will be put on a rational design basis. The behavior of the individual panel in a compound structure presents a more complex problem. For efficient design it is important to know the amount of load which the individual stiffener or panel in a compound structure may carry in order to design to close strength limits.

The present investigation was suggested by Dr. Klein who was interested in determining how the material in a metal wing may best be distributed, especially as to the number and location of webs and distribution of flange material. This required a study of the stress distribution, particularly the distribution of stress in the flange due to the bending loads. Dr. von Kármán (reference 1) has shown that for beams having wide flanges, the effectiveness of the material at a distance from the webs may be considerably decreased. His work is based upon the Airy stress function and is applicable only to beams where the deformations occur in the plane of the flange such as concrete floor slabs.

The type of structures with which we are concerned consist of sheet stiffener combinations or corrugated sheet. Kármán's analysis may possibly be applied satisfactorily to flanges of corrugated sheet. The present interest, however, is in the sheet stiffener combination. This problem is also complicated by the fact that the flange material generally follows the contour of the wing section, hence sheet and stiffeners at a distance from the web may be more effective due to a possible greater distance from the neutral axis of the wing. The attempt here is to find how effective the sheet is in transferring loads among the various stiffeners with the hopes of finding a rational method of calculating the loads in the longitudinal stiffeners. In practice the spacing and size of the stiffeners may vary, hence an actual analysis based upon the formal elasticity equation would become too complicated.

The present investigation represents only the first step in a series of tests which have to be made with panels of various elastic and geometric properties under various loadings, in order to obtain any rational basis for design.

GENERAL SOLUTION BY DIFFERENTIAL EQUATIONS

In the present development, the formal elasticity equations for the stress distribution in the sheet have been avoided by considering the sheet as merely a medium for transferring the load from one stiffener to another. A portion of the sheet area is included with that of the stiffener to account for the load that the sheet may be expected to resist; hence our panel is replaced by a series of columns having axial distributed loads applied by the sheet. It is further assumed that the sheet has sufficient lateral restraint so that only axial deformation need be considered.

Before introducing assumptions as to how the load is distributed among the system of columns, it is convenient to develop the differential equation expressing the deformation of the individual column of constant cross-sectional area A, subjected to an arbitrary distributed axial loading w(x). Referring to Fig. 1 the stress σ_x at a distance x from the base of the column is

$$\sigma_x = \frac{1}{A}\int_x^l w(x)\,dx \qquad (1)$$

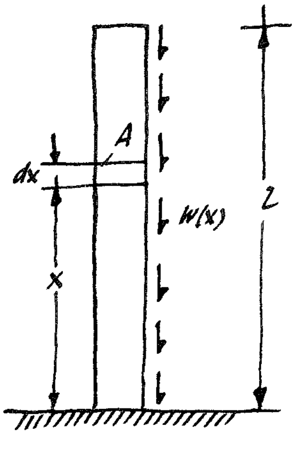


Fig. 1.

From the Hooke law relation

$$\mathcal{E} = \frac{\sigma_x}{\frac{dy}{dx}} \quad or \quad \frac{dy}{dx} = \frac{\sigma_x}{\mathcal{E}} \tag{2}$$

where y is the axial deflection at any distance x along the column. Combining equations (1) and (2)

$$y = \frac{1}{A\mathcal{E}} \int_0^x \int_x^L W(x)\, dx\, dx \tag{3}$$

For our purpose the differential form will be more convenient, so by differentiating twice both sides of equation (3), the final equation becomes, after introducing the subscript n to refer to the nth column:

$$A\mathcal{E} \frac{d^2 y_n}{dx^2} - W(x)_n = 0 \tag{4}$$

The next step is to connect the individual columns so as to obtain a structure that will carry the externally applied loads. This is done by assuming that an element of load on any column is transferred to the adjacent columns by an amount proportional to the difference of the deflections between the columns. Referring to Fig. 2 the load on an element dx of the nth stiffener becomes:

$$W(x)_n = \mu(-y_{n-1} + 2y_n - y_{n+1}) \tag{5}$$

where μ is termed the "transfer constant" having the dimensions $\left|\frac{F}{L^2}\right|$ or in the English

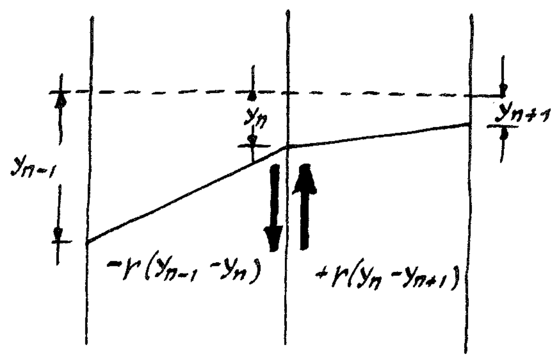


Fig. 2.

system of units lbs./in run/unit difference of deflection. This transfer constant will depend upon the thickness and width of sheet (between stiffeners) and probably on whether the sheet is in a buckled or unbuckled state. Combining (4) and (5) the differential equation for the deformation of the nth stiffener becomes:

$$\frac{A_n \mathcal{E}}{r} \cdot \frac{d^2 y_n}{dx^2} - 2y_n + y_{n-1} + y_{n+1} = 0$$

or where $D^2 = \frac{d^2 y_n}{dx^2}$ and $a_n = \sqrt{\frac{A_n \mathcal{E}}{r}}$

$$y_{n-1} + (a_n^2 D^2 - 2) y_n + y_{n+1} = 0 \qquad (6)$$

For a general case (Fig. 3), choose a panel having $(K + 1)$ stiffeners with external distributed loads $T_{0,K}(x)$ applied at the edge stiffeners and end loads P_n applied at the free end of the panel. Equation (6) combined with the distributed loads gives the following system of differential equations:

(7)

$$\begin{aligned}
&[a_0^2 D^2 - 1] y_0 + y_1 &= -\frac{T_0(x)}{r}\\
&y_0 + [a_1^2 D^2 - 2] y_1 &= 0\\
&- - - - - - - - - - - - &\\
&y_{n-1} + [a_n^2 D^2 - 2] y_n + y_{n+1} &= 0\\
&- - - - - - - - - - - - &\\
&[a_{K-1}^2 \cdot D^2 - 2] y_{K-1} + y_K &= 0\\
&y_{K-1} + [a_K^2 \cdot D^2 - 1] y_K &= -\frac{T_K(x)}{r}
\end{aligned}$$

Fig. 3

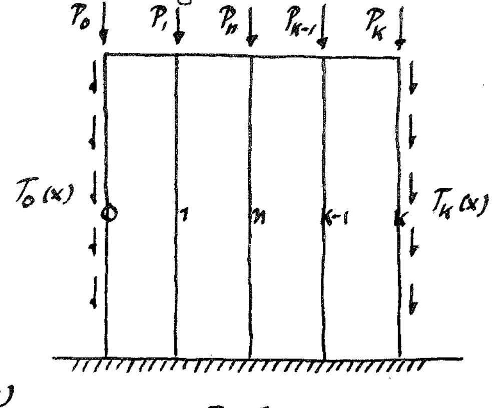


Fig. 3.

This system of linear second order differential equations is similar to that obtained for the classical problem of the loaded string (references 2 & 3), the difference being that $\sqrt{p}$ in the present case is real instead of imaginary, hence the solution is expressed in hyperbolic functions instead of trigonometric.

The complete solution will be considered composed of a general solution involving the boundary conditions and end loads P_n plus a particular solution which includes the effects of the distributed loads $T_n(x)$. To determine the general solution, the so-called "Eigenwerte" must first be found which involves finding the roots of a $(K + 1)$ degree polynomial. These values, along with the boundary conditions, give a set of $2(K + 1)$ linear equations to solve for the $2(K + 1)$ arbitrary constants. For symmetrical or asymmetrical distribution of the P_n loads about the center stiffener, the number of equations may be reduced to one-half if the values of A_n are in like symmetry. The cumbersomeness of the general solution for a number of stiffeners may readily be realized. So far, only the solution in the case of three stiffeners has been obtained. A solution for five or six stiffeners, even if obtained, would be too cumbersome to be of any practical value. The three stiffener solution may be applied to find the experimental value of ρ which in turn may be applied in developing a more approximate solution applicable to any number of stiffeners.

No attempt has been made to find the particular solution which involves the distributed load.

SOLUTION FOR A THREE STIFFENER PANEL

In this special case each of the edge stiffeners will be subjected to an end load P, and the effective area of each of the edge stiffeners will be r times that of the center stiffener. The differential equations (7) then reduce to

$$\begin{aligned} (ra^2D^2-1)y_0 + y_1 &= 0 \\ y_0 + (a^2D^2-2)y_1 + y_2 &= 0 \\ y_1 + (ra^2D^2-1)y_2 &= 0 \end{aligned} \qquad (8)$$

where: $r = \frac{A_0}{A_1} = \frac{A_2}{A_1}$; $a = \sqrt{\frac{A_1 E}{\rho}}$

Assuming a solution of the form $y_n = A_n \cdot e^{\lambda x}$

and substituting in (8), we obtain:

$$\begin{aligned} (ra^2\lambda^2-1)A_0 + A_1 &= 0 \\ A_0 + (a^2\lambda^2-2)A_1 + A_2 &= 0 \\ A_1 + (ra^2\lambda^2-1)A_2 &= 0 \end{aligned} \qquad (9)$$

This homogeneous system of equations will give values of A_n other than zero only when the determinate of their coefficients is zero, hence it will be possible to find the particular values of λ which represent the required "Eigenwerte".

$$\begin{vmatrix} (ra^2\lambda^2-1) & 1 & 0 \\ 1 & (a^2\lambda^2-2) & 1 \\ 0 & 1 & (ra^2\lambda^2-1) \end{vmatrix} = 0$$

Expanding the determinate

$$\left[r(a\lambda)^2-1\right]\left[(a\lambda)^2-(2+\tfrac{1}{r})\right]\left[(a\lambda)^2\right] = 0$$

which give the desired values of λ as

$$\lambda_0 = \pm\frac{\sqrt{\frac{1}{r}}}{a} \quad ; \quad \lambda_1 = \pm\frac{\sqrt{2+\frac{1}{r}}}{a} \quad ; \quad \lambda_2 = \pm 0$$

All roots are real and single except λ_2 which is real but double.

The general solution may now be written in the form:

$$\begin{aligned} y_n &= A_{0,n}\sinh\lambda_0 x + A_{1,n}\sinh\lambda_2 x + A_2 x \\ &+ B_{0,n}\cosh\lambda_0 x + B_{1,n}\cosh\lambda_2 x + B_2 \end{aligned} \qquad (10)$$

For each value of λ the coefficients A_n bear a definite ratio which may be determined from equation (9), hence for

$$\lambda_0 \longrightarrow \frac{A_{0,1}}{A_{0,0}} = 0 \quad ; \quad \frac{A_{0,2}}{A_{0,0}} = -1$$

$$\lambda_1 \longrightarrow \frac{A_{1,1}}{A_{1,0}} = -4 \quad ; \quad \frac{A_{1,2}}{A_{1,0}} = 1$$

$$\lambda_2 \longrightarrow \frac{A_{2,1}}{A_{2,0}} = 1 \quad ; \quad \frac{A_{2,2}}{A_{2,0}} = 1$$

The boundary conditions of the problem are

$$\text{at } x=0 \qquad \frac{dy_0}{dx} = \frac{P}{A_0 E} \;;\quad \frac{dy_1}{dx} = 0 \;;\quad \frac{dy_2}{dx} = \frac{P}{A_2 E}$$

and at $x = l$ $\qquad y_0 = y_1 = y_2 = 0$

With these conditions and equation (10), the values of $A_{0,0}$, $A_{1,0}$ and $A_{2,0}$ may be determined, giving the final solution which becomes, for an assumed value of r = 2:

$$y_0 = \left(\frac{P}{A_0 E}\right)\left(\frac{1}{5}\right)\left[\frac{1}{\alpha_1}(\sinh \alpha_1 x - \tanh \alpha_1 l \cdot \cosh \alpha_1 x) + 4(x-l)\right]$$

$$y_1 = \left(\frac{P}{A_0 E}\right)\left(\frac{4}{5}\right)\left[\frac{1}{\alpha_1}(\sinh \alpha_1 x - \tanh \alpha_1 l \cdot \cosh \alpha_1 x) - (x-l)\right] \qquad (11)$$

$$y_2 = y_0$$

Differentiating and multiplying by E, the expressions for the stress become

$$\sigma_0 = \left(\frac{P}{A_0}\right)\left(\frac{1}{5}\right)\left[\cosh \alpha_1 x - \tanh \alpha_1 l \cdot \sinh \alpha_1 x + 4\right]$$

$$\sigma_1 = \left(\frac{P}{A_0}\right)\left(\frac{4}{5}\right)\left[-\cosh \alpha_1 x + \tanh \alpha_1 l \cdot \sinh \alpha_1 x + 1\right] \qquad (12)$$

$$\sigma_2 = \sigma_0$$

APPLICATION OF DIFFERENTIAL EQUATIONS TO THE THREE STIFFENER PANEL

Equations (11) and (12) have been developed for $\frac{A_o}{A_1} = 2$. These equations will be applied to the experimental panel tested although an estimate of r is in the order of 1.8.

From Fig. 9 the stress at the point of application of the load is $\sigma_o = .860 \times 10^4$ from which the value of A_o for a load of 2P 4,000# becomes

$$A_o = \frac{P}{\sigma_o} = \frac{2000}{(.86 \times 10^4)} = .232 \text{ sq. in.}$$

$$A_1 = \frac{A_o}{r} = \frac{.232}{2} = .116 \text{ sq. in.}$$

Referring again to Fig. 9 at $x = l = 48''$, $\sigma_o = .70 \times 10^4$ and $\sigma_1 = .630 \times 10^4$. From equations (12) it can be shown that at $x = l = 48$ in.

$$\cosh \lambda_1 l = \frac{(1 + 4\frac{\varepsilon_o}{\varepsilon_1})}{4(\frac{\varepsilon_o}{\varepsilon_1} - 1)} \qquad \frac{\varepsilon_o}{\varepsilon_1} = \frac{\sigma_o}{\sigma_1}$$

for $\frac{\sigma_o}{\sigma_1} = \frac{.70 \times 10^4}{.63 \times 10^4} = 1.11$ and $l = 48''$, λ_1 will be .065

All of the necessary constants have been determined, hence equations (12) become

$$\sigma_0 = (.860 \times 10^4)\cdot\frac{1}{5}\cdot\left[\cosh .065x - \tanh(.065\times 48)\cdot\sinh(.065\cdot x) + 4\right]$$

$$\sigma_1 = (.860 \times 10^4)\cdot\frac{4}{5}\cdot\left[-\cosh .065x + \tanh(.065\times 48)\sinh(.065\cdot x) + 1\right]$$

$$\sigma_2 = \sigma_0$$

The stress curves have been calculated and are shown superimposed on the experimental results in Fig. 9 . The agreement with the experimental data seems quite reasonable. The value of the "transfer constant" p may be determined from λ_1 as

$$\lambda_1^2 = \frac{5}{2}\,\frac{p}{A_1 E}$$

or

$$p = \frac{2}{5} A_1 E \cdot \lambda_1^2 \qquad = \frac{2}{5}(.116)\,10^7\,(.065)^2 = 1960$$

As p is the shear force per unit length along the stiffener necessary to produce a unit shear deformation in the direction of the stiffener, it is possible to show the relation between and the shear modulus G by comparison with the calculated shear deformation of a rectangular plate. It can be shown that

$$p = K\frac{t}{w}\cdot G$$

where if $G = 4 \times 10^6$, $t = .025$, $w = 6"$ and $p = 1960$, k will have a value of .1175 or approximately 1/8th the shear modulus.

APPROXIMATE SOLUTION BY "ENERGY METHOD"

As can be seen from the preceding Chapter and also from the numerical solution ~~on pages~~ the "Accurate Solution" by differential equations becomes rather tedious, especially in the case of very wide panels with many reenforcements. Therefore, a solution has been proposed using the "Energy Method", the essential meaning of which is, that of all possible states of deformation that an elastic system under load might assume, it will assume that one which corresponds to a minimum of the entire energy of the system.

There are two modifications:

1) Let the energy of the system be expressed as a function of forces:

$$E = E(F_1, F_2, \ldots F_k \ldots F_n) = min$$

The solution of this equation will give us either the forces or the stresses for which the energy becomes a minimum, i.e.

$$\frac{\partial E}{\partial F} \quad or \quad \frac{\partial E}{\partial \sigma} = 0$$

2) Let the energy of the system be expressed as a function of the deflections. Then

Internal Energy $= E = \sum_n (F_n \cdot \delta_n)$

External Work =

Work of external forces $= W = F \cdot \delta$

Then the minimum condition requires that

$$E - W = min$$

and the solution of this equation will give us the unknown deflections, i.e.

$$\frac{\partial (E-W)}{\partial d} = 0$$

In applying this method, the usual procedure is to select functions for the unknowns which have to fit given boundary conditions. The partial differentiation of the entire energy expressed in terms of stresses or deflections then gives us the unknown parameters for which the energy of the system assumes a minimum value.

APPLICATION OF ENERGY METHOD TO THE PRESENT PROBLEM

Assuming that the load will be transmitted by shear stresses in the sheet connecting the stiffeners, we have to consider two cases: that of the panel of finite length and that of the panel of infinite length. For a panel of infinite length, the stress at the fixed end will be equally distributed over the entire width, and in expressing the energy of the system, it will be permissible to neglect the energy due to shear stresses in the region close to the point of application of the load, thus expressing the entire energy in terms of the compression stress in stiffeners and sheet. For the panel of finite length, this will not be allowable. Here the energy due to the shear stresses in the region of the applied load represents an important factor of the entire energy and cannot be neglected. Since in this particular case, the compressive energy in the sheet is very difficult to take into account, we make the simplified assumption that an equivalent amount of sheet is acting as effective width with the stiffeners.

Summation

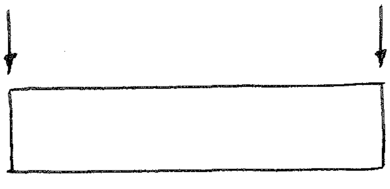
1) Panel of infinite length: Energy expressed as a function of the compressive stress in stiffeners and portion of sheet assumed to act as effective width with the stiffeners. (1) is a simplification of 2).

2) Panel of finite length: Energy expressed as a function of the compressive stress in stiffeners and portion of sheet assumed to act with the stiffeners and as function of the shear stresses in the sheet.

Obviously the function selected for the distribution of the compressive stress in the columns has to satisfy both cases, the infinite panel as well as the finite panel.

Consider the two extreme cases:

a).



b).

GENERAL SOLUTION

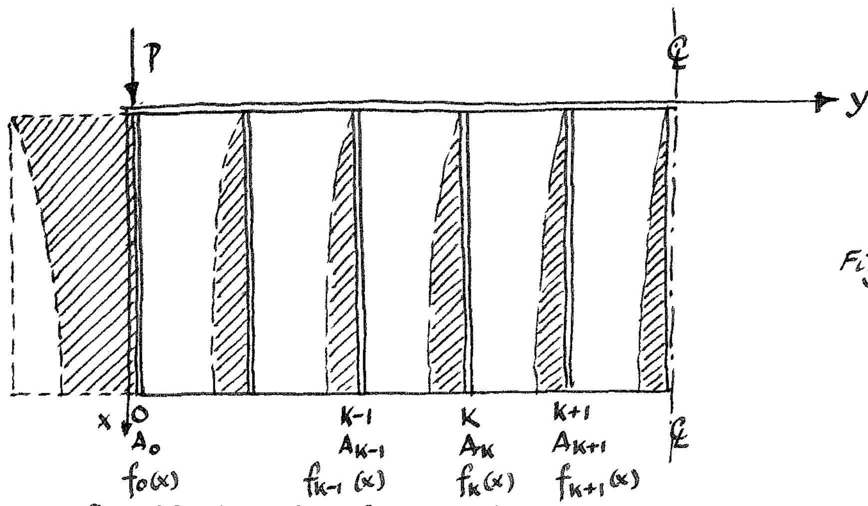


Fig. 4.

Considering the edge stiffener separately from the rest of the panel, the stress due to the external load P would be

$$\sigma_o' = \frac{P}{A_o}$$

However, due to the presence of the reenforced plate, this stress will be decreased and the distribution of the compressive stress in the edge stiffener will then follow a function

$$\sigma_o = \frac{P}{A_o} - a_o \cdot f_o(x)$$

where a_o is some unknown parameter with the dimensions of stress. Corresponding to the decrease in stress in the edge stiffener (#0), the stress in the center stiffeners will increase, following similar functions:

$$\sigma_1 = a_1 \cdot f_1(x)$$

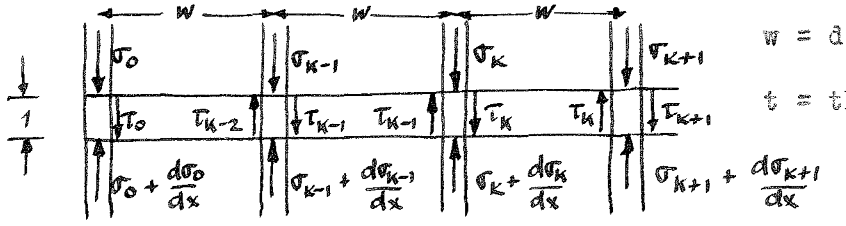
$$- - - - - - - - -$$

$$\sigma_{K-1} = a_{K-1} \cdot f_{K-1}(x)$$

$$\sigma_K = a_K \cdot f_K(x)$$

$$\sigma_{K+1} = a_{K+1} \cdot f_{K+1}(x)$$

The shear stresses in the sheet can be calculated from the following equilibrium condition:



w = distance between stiffeners

t = thickness of sheet

Fig. 5.

For the end stiffener, assuming the area to be constant

$$\left(\sigma_0 + \frac{d\sigma_0}{dx} \cdot 1\right) \cdot A_0 - \sigma_0 \cdot A_0 = \tau_0 \cdot 1 \cdot t$$

$$\tau_0 = \frac{d\sigma_0 \cdot A_0}{dx \cdot t} \qquad (\text{lbs}/\text{in}^2)$$

For the center stiffeners:

$$\left(\sigma_K + \frac{d\sigma_K}{dx}\right) \cdot A_K - \sigma_K \cdot A_K = (\tau_{K-1} - \tau_K) \cdot t$$

$$(\tau_{K-1} - \tau_K) = \frac{d\sigma_K \cdot A_K}{dx \cdot 1}$$

From this condition we can easily determine our values for τ and the energy expression then assumes the form:

$$E = \sum_{n} \frac{1}{2} \frac{A_n}{E} \cdot \int_0^l \sigma_n^2 \, dx + \sum_{n-1} \frac{1}{2} \frac{t \cdot w}{G} \int_0^l \tau_{n-1}^2 \cdot dx$$

Boundary Conditions

a) For the determination of the functions:

$$
\begin{array}{lll}
\text{at} & x=0\ ;\ y=0 & \sigma_0 = \frac{P}{A_0} \\
& x=0\ ;\ y=w,2w\ldots & \sigma_n = 0 \\
& x=L\gg 0\ ;\ y=0 & \sigma_0 = \frac{P}{A_0} - a_0 \\
& x=L\gg 0\ ;\ y=w,2w\ldots & \sigma_n = a_n
\end{array}
$$

b) At any cross-section through the panel, taking into account only the compressive energy in the columns:

$$\sum_n \sigma_n \cdot A_n = \sum P$$

i.e. $\sum_n a_n \cdot A_n = 0$

c) For L $\gg 0$, i.e. for constant distribution of stress:

$$\sigma_0 = \sigma_k = \cdots = \sigma_n = constant$$

For practical cases the stress distribution becomes constant for L $< \infty$.

Type of Stress Function

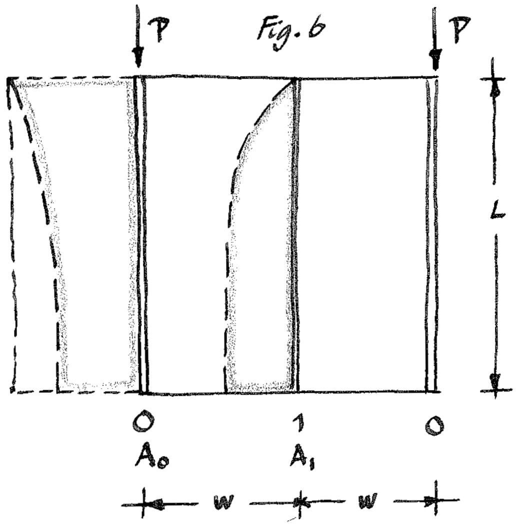
As has been pointed out, the stress in the columns assumes a constant and finite value at infinity; however, for practical purposes, it assumes a fairly constant value for panels of reasonable length. We therefore have to select a function which approaches the value at infinity also for panels of finite length, and it is suggested to select an exponential function or, as in the general case of a plate with more than one center stiffener, the sum of two exponential functions.

Note: This solution for the reenforced plate under concentrated edge loads can be extended for the more practical case of distributed side loads and end loads for each stiffener.

Numerical Example

As an example for the application of the Energy Method, the problem of the plate with three stiffeners has been calculated.

Fig. 6



$P = 2000$ #

$\mathcal{E} = 10^7$ lbs./in.^{2}

$L = 48''$

$2w = 12''$

Area of edge stiffeners alone = .180 in.^{2}

Area of center stiffener alone = .088 in.^{2}

$t = .025''$

Considering the result of the experimental investigation (see Fig. 9), the following functions for the stress distribution have been selected:

For the edge stiffener: $\sigma_0 = \frac{P}{A_0} - a_0(1 - e^{-\alpha x})$

for the center stiffener $\sigma_1 = a_1(1 - e^{-\alpha x})$

so that $\tau = \frac{A_0}{t}\frac{d\sigma_0}{dx} = -a_0 \cdot \alpha \cdot e^{-\alpha x} \cdot \frac{A_0}{t}$

Certain assumptions have to be made about the amount of sheet acting with the stiffeners as effective width.

a) From the experimental results we obtain:

$\frac{P}{A_0} = 0.86 \times 10^4$; for $P = 2000$ #

$$A_0 = \frac{2000}{0.86 \times 10^4} = 0.232 \text{ in.}^2$$

From the equilibrium of forces:

$$2\sigma_0 A_0 + \sigma_1 A_1 = 2P$$

$$2\left[\frac{P}{A_0} - a_0(1-e^{-\alpha x})\right]A_0 + a_1(1-e^{-\alpha x})A_1 = 2P$$

Assuming L to be large:

$$2P - 2a_0 \cdot A_0 + a_1 \cdot A_1 = 2P$$

$$-2a_0 \cdot A_0 + a_1 \cdot A_1 = 0$$

According to the experimental result (Fig. 9) a_0 approaches the value of 0.19×10^4 lbs./in.^{2} so that

$$A_1 = \frac{2a_0 \cdot A_0}{a_1} = \frac{2 \cdot 0.19 \cdot 0.232}{0.67} = 0.131 \text{ in.}^2$$

The following table shows the approximate values of sheet acting with the stiffeners based on above assumption:

	Stiffener A	Stiffener B
Area of stiffener and sheet	0.232 in.^{2}	0.131 in.^{2}
Area of stiffener alone	0.180 in.^{2}	0.088 in.^{2}
Sheet between stiffeners	0.006 in.^{2}	----
Area of sheet (A_s)	0.46 in.^{2}	0.43 in.^{2}
Effective width = $\frac{A_s}{t}$ =	$\frac{0.046}{0.025}$ = 1.84 in.	$\frac{0.043}{0.025}$ = 1.72 in. or $\frac{1.72}{2}$ = 0.86 in. on either side.

This comparison shows that the effective width of the sheet acting with the edge stiffener would be twice the effective width of the sheet acting with the center stiffener, which is a rather dubious result. Since the effective width is a function of $\frac{1}{\sqrt{\sigma}}$ according to Sechler's theory for sheet - stiffener combinations under compressive load, we should expect the effective width to be of the same magnitude for both the edge and the center column, since at least at the base of the panel the stresses are fairly equally distributed. A more accurate calculation of the effective width would have to take into account the change in stress, and the effective width, therefore, will change with the length of the panel until an even stress distribution has been obtained. This problem has not as yet been investigated.

b) Due to this uncertainty in the determination of the effective width, another assumption has been made and the numerical analysis has been carried out using both assumptions.

From Sechler's theory:

$$w = \frac{C_f}{2}\sqrt{\frac{E}{\sigma}} \cdot t$$

Using $\frac{C_f}{2} = 0.8$, $E = 10$, $t = 0.025$ and $\sigma \cong 7000$ lbs./in.^{2}, the effective width w will be about 0.8 inches; since the loading conditions are different, we will assume an effective width of 1.0 inches. The following Table shows the areas as calculated from this assumption:

	Edge stiffener	Center stiffener
Area of stiffener	0.180	0.088
Area of sheet between stiffener	0.006	—
Area of sheet acting with stiffener due to w = 1"	0.025	0.050
Total Area	A_0 = 0.211 in.^{2}	0.138 in.^{2}

The numerical analysis has been carried out using both assumptions, and the results have been plotted in Fig. 9 .

The total energy expression then assumes the form:

$$E = 2 \cdot \frac{1}{2} \frac{A_0}{E} \int_0^L \left[\frac{P}{A_0} - a_0 (1 - e^{-\alpha x}) \right]^2 dx + \frac{1}{2} \frac{A_1}{E} \int_0^L \left[a_1 (1 - e^{-\alpha x}) \right]^2 dx$$

$$+ \frac{1}{2} \frac{t \cdot w}{G_t} \cdot \frac{\alpha^2 \cdot A_0^2}{t^2} \int_0^L (-a_0 \cdot e^{-\alpha x})^2 dx$$

Integrating this equation we obtain:

$$E = \frac{A_0}{\mathcal{E}} \left[\left(\frac{P}{A_0} - a_0\right)^2 \cdot L + 2a_0 \left(\frac{P}{A_0} - a_0\right) \frac{1-e^{-\lambda L}}{\lambda} + a_0^2 \frac{1-e^{-2\lambda L}}{2\lambda} \right]$$

$$+ \frac{A_1}{2\mathcal{E}} \cdot a_1^2 \left[L - 2 \frac{1-e^{-\lambda L}}{\lambda} + \frac{1-e^{-2\lambda L}}{2\lambda} \right]$$

$$+ \frac{t \cdot w}{G_t} \cdot \frac{\lambda^2 \cdot A_0^2}{t^2} \cdot a_0^2 \left[\frac{1-e^{-2\lambda L}}{2\lambda} \right] = min$$

From the boundary condition (see page 19)

$$-2a_0 \cdot A_0 + a_1 \cdot A_1 = 0$$

$$a_1 = \frac{2a_0 \cdot A_0}{A_1}$$

Substituting this value, the minimum equation assumes the form:

$$\left[\left(\frac{P}{A_0} - a_0\right)^2 \cdot L + 2a_0 \left(\frac{P}{A_0} - a_0\right) \frac{1-e^{-\lambda L}}{\lambda} + a_0^2 \frac{1-e^{-2\lambda L}}{2\lambda} \right]$$

$$+ 2 \frac{A_0}{A_1} \cdot a_0^2 \left[L - 2 \frac{1-e^{-\lambda L}}{\lambda} + \frac{1-e^{-2\lambda L}}{2\lambda} \right]$$

$$+ \frac{\mathcal{E} \cdot w \cdot \lambda^2 \cdot A_0}{G_t \cdot t} \cdot a_0^2 \left[\frac{1-e^{-2\lambda L}}{2\lambda} \right] = min$$

This equation can be differentiated either with respect to a_0 or with respect to λ .

Differentiating with respect to a_o we obtain:

$$a_o\left\{L - \frac{2}{\lambda}(1-e^{-\lambda L}) + \frac{1}{2\lambda}(1-e^{-2\lambda L}) + 2\frac{A_o}{A_1}\left[L - \frac{2}{\lambda}(1-e^{-\lambda L}) + \frac{1}{2\lambda}(1-e^{-2\lambda L})\right]\right.$$

$$\left. + \frac{\mathcal{E}\cdot A_o\cdot w\cdot \lambda^2}{G_t\cdot t}\left(\frac{1-e^{-2\lambda L}}{2\lambda}\right)\right\} = \frac{P}{A_o}\left[1 - \frac{1}{\lambda}(1-e^{-\lambda L})\right]$$

In this case an assumption has to be made about the magnitude of λ. Judging from the experimental result, λ can be assumed to be $\frac{1}{12}$. Substituting all values we obtain:

<u>Using assumption a)</u>

$$a_o\left[138{,}5 + \frac{2{,}32\cdot 10^7}{G_t}\right] = 3.12\times 10^5$$

<u>Using assumption b)</u>

$$a_o\left[123{,}7 + \frac{2{,}11\times 10^7}{G_t}\right] = 3{,}5\times 10^5$$

Substituting the ordinary value of $G = \frac{E}{2(1+0.25)} = 0.4\times 10^7$

Assumption a):

$a_o = 0.218\times 10^4$

$a_1 = \frac{2a_oA_o}{A_1} = 0.765\times 10^4$

at ∞ :

1) $-2a_o\cdot A_o + a_1\cdot A_1 = 0$

from

2) $\frac{P}{A_o} - a_o = a_1$

$a_o = 0.19\times 10^4$

$a_1 = 0.672\times 10^4$

$\frac{P}{A_o} = a_o + a_1 = 0.862\times 10^4$

$\frac{P}{A_o}$ from exper. $= 0.862\times 10^4$

Assumption b):

$a_o = 0.271\times 10^4$

$a_1 = \frac{2a_oA_o}{A_1} = 0.83\times 10^4$

at ∞ :

$a_o = 0.231\times 10^4$

$a_1 = 0.715\times 10^4$

$\frac{P}{A_o} = a_o + a_1 = 0.946\times 10^4$

$\frac{P}{A_o}$ (using $A_o = 0.211$) $= 0.946\times 10^4$

Comparing in both cases the stresses at finite and at infinite length, we obtained the rather strange result that the stress at finite length should be greater than that at infinite length. Since this result cannot be affected by the areas, as can be seen from comparison of both calculations, there can be only one reason: that the modulus of shear transfer G_t cannot be of the same order as the shear modulus.

Assuming the following values of a_o for:

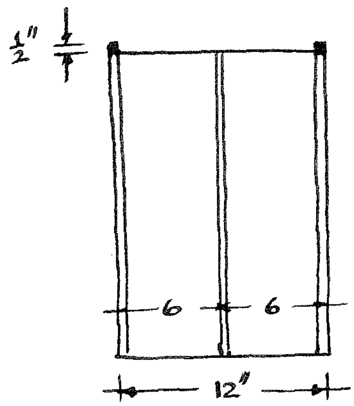
Assumption a)	Assumption b)
$a_o = 0.17 \times 10^4$ (from experiment)	$a_o = 0.21$ (as can be predicted from a_o = 0.251)
$G_t = 0.51 \times 10^6$	$G_t = 0.49 \times 10^6$
against	
$G = 4.0 \times 10^6$	$G = 4.0 \times 10^6$

The value for G was obtained from Appendix B of Reference 5. The effective shear modulus as determined here is in the order of 1/8 of the actual shear modulus, and is in agreement with that obtained from the solution by the differential equations. This value seems absurdly low and perhaps the G_t as determined here is not strictly comparable to G. This will need a more careful analysis. The rigidity of the buckled plate is expected to be less, due to the fact that the rigidity of the fibers in compression is lost. Kuhn has shown (reference 5) that the thickness of the plate should be reduced to about 5/8 of its actual thickness in computing the shear deformation of the buckled sheet which is the same as stating that an effective shear modulus should be used having a value of $\frac{5}{8}$ G. Our values are only about 25% of this value. The other 75% may possibly be due to the assumptions of our theory. As most of the shear occurs near the free end of the panel, it is possible that the free end of the sheet may cause a sufficient reduction of the shear rigidity to affect our results. Another possibility is that our stiffener spacing is decreased under load, due to insufficient transverse restraint. Further experiments where the shear transfer is distributed more uniformly over the length of the panel may give more representative data from which the value of G_t may be determined.

EXPERIMENTAL INVESTIGATION

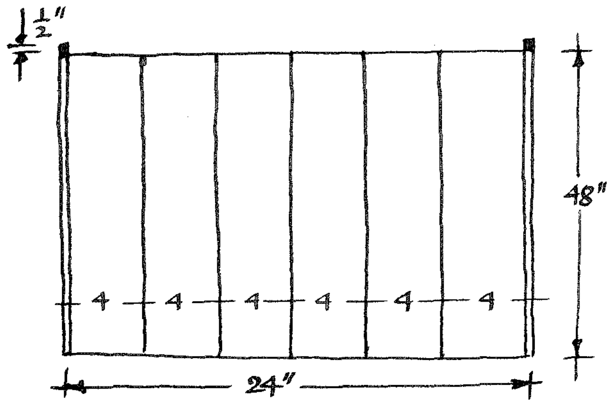
As mentioned before, the problem has been represented for both theoretical and experimental investigation in a very simplified manner in order to obtain some idea of the essential laws that are followed in the stress distribution due to shear transfer.

Two cases have been investigated: that of a narrow panel (Fig. 7a),and a wide panel (Fig. 7b),



Specimen No. I

Fig. 7a



Specimen No. II

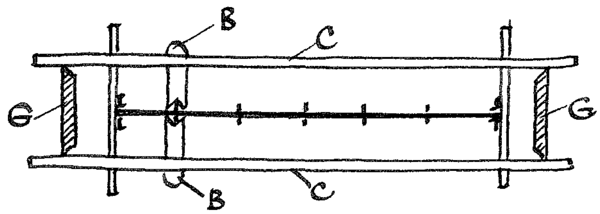
Fig. 7b

the dimensions of which are contained in the following table:

Spec.	Length	Width	No. of center stiff.	Dist. between stiff.	Area of Edge stiff.	Area of center stiff.	Thick. of sheet	Area of sheet between stiff.	Total area	Material
#1	48"	12"	1	6"	2 x .090	2 x .044	.025	.150	.748	24ST
#2	48"	24"	5	4"	2 x .090	2 x .044	.025	.100	1.400	24ST

Special consideration was given to the point of application of load. There the edge sections were reenforced over a short length so as to avoid local failure due to stress concentration. In order to obtain concentrated loading, the edge stiffeners were extended $\frac{1}{2}$" beyond the intermediate stiffeners.

To decrease the Euler length and avoid lateral buckling of the panel, angle reactions C were fastened to the edge stiffeners at intervals of 12 inches (see Figs. 8).



These were able to slide along guides G (cross section ▬) which were rigidly connected to the frame of the testing machine. This offered a lateral restraint without introducing any appreciable longitudinal stresses. Small plates B, with "V" grooves were bolted to the cross bars, thus giving the stiffeners the necessary lateral support. Fig. 14 shows the specimen ready to be set into the testing machine.

Testing Machine

The machine used for this experiment was the 15000# compression machine of the structural laboratory of the Guggenheim Aeronautics Laboratory (see Fig. 16). The load was applied by means of an hydraulic pressure gauge.

Measurement of Stresses

The stresses have been measured as unit extensions by means of Huggenberger extensometers.[x)] To obtain appreciable readings, extension bars were used increasing the gauge length to four times its original length.

With no accurate method available to measure stresses in the sheet due to large deformations, all measurements have been taken at the stiffeners. The total deflection of the panel was measured by means of Ames dial gauges.

Since the purpose of this experiment was not to find the ultimate strength of the panels, loads were applied only so far as to obtain well defined deformations in the sheet. No buckling or twisting deformation could be noticed in the stiffeners, while shear waves in the sheet appearing at the top of the panel merged into compression waves near the base.

x) Fig. 17!

DISCUSSION OF EXPERIMENTAL RESULTS

Table I contains the results of the stress measurements on the three stiffener panel. Values of the stresses are plotted in Fig. 9 for comparison with calculated theoretical curves. These same data are replotted in Fig. 10 in ratio $\frac{\sigma}{\sigma_m}$ form, all stress values having been divided by a mean stress computed by dividing the total applied load by the total cross-sectional area of the panel. Values of $\frac{\sigma}{\sigma_m}$ are greater than 1.0 at the base of the panel ($x = 0$), indicating that the stress distribution is not uniform over this cross-section. Had the panel been larger, this value would be expected to approach more closely to 1.0 but would not reach 1.0 as the material between the stiffeners is not as effective in carrying load as is that of the stiffener. The experimental data of stiffener A has more of a scatter than that of stiffener B. This is probably due to an inadequate extensometer mounting. A certain amount of error is also introduced in the reduction of the data.

The stress values measured on the seven stiffener panel are tabulated in ratio form in Table II, the mean stress being chosen similar to that for the three stiffener panel.

These data are plotted in Figs. 11_a and 11_b. Values of stresses for loads of 2P = 2000#, 4000# and 6000# were computed to determine if the stress distribution was affected by the magnitude of the load. Any effect of this nature was within the experimental accuracy. Here again the stress measurements on the first stiffener had the most scatter, which is thought to be due to the type of extensometer mounting. The values of $\frac{\sigma}{\sigma_m}$ at the base of the panel are more nearly equal to 1.0 as the sheet is stiffer in shear; hence the load is distributed more evenly for the same length of panel. Fig. 12 contains cross plots of these data, the values being taken from the curve drawn through the experimental points in Fig. $11_{a,b}$. These curves are designed to indicate the general nature of the stress distribution across the panel for different distances from the point of application of the load. In all cases, the stresses were computed from an assumed value of $E = 10 \times 10^6$.

CONCLUSIONS

1. The results of these experiments are not sufficient to determine whether or not satisfactory results may be obtained by assuming that the load is transferred between stiffeners according to the simple assumption made here.
2. Loads applied to the end of panels as tested here will be distributed such as to give equal stress distribution in all stiffeners at a sufficient distance from the point of application.
3. The values of the effective shear modulus G_t as determined from these tests, are believed to be too low due to reasons stated.
4. The application of the differential equations give too cumbersome expressions to be of any value for panels having more than three or four stiffeners. The energy method may be expected to give simpler results.
5. Future tests should be made on long panels having distributed side loads, in order to determine reasonable values of G_t.

REFERENCES

(1) Th. von Kármán "Beitrage zur technischen Mechanik und Technischen Physik", Springer 1924.

(2) W. V. Houston "Principles of Mathematical Physics", McGraw-Hill Book Company, 1934.

(3) H. Bateman "Partial Differential Equations", Cambridge University Press, 1932.

(4) E. E. Sechler "The Ultimate Compressive Strength of Thin Sheet Metal Panels", Ph.D. thesis.

(5) P. Kuhn "Analysis of 2-Spar Cantilever Wings with Special Reference to Torsion and Load Transference", N.A.C.A. Rep. 508.

TABLE I

Three Stiffener Panel

Total Load 2P = 4000 lbs. ; $\sigma_m = \frac{2P}{\text{total area}} = \frac{4000}{.748}$

$\sigma_m = 0.535 \times 10^4$

% of Length	Stiffener A $\sigma_A \times 10^3$	$\frac{\sigma A}{\sigma_m} = \frac{\sigma A}{.535}$	Stiffener B $\sigma_B \times 10^3$	$\frac{\sigma B}{\sigma_m} = \frac{\sigma B}{.535}$
5.8	.690	1.29	.650	1.22
8.5	.690	1.29	--	--
18.7	.745	1.39	.615	1.15
21.0	.705	1.32	--	--
31.5	--	--	.585	1.09
33.5	.66	1.23	--	--
35.5	--	--	.625	1.17
41.8	--	--	.540	1.01
43.7	.700	1.31	--	--
56.0	.650	1.22	.495	0.925
58.0	.675	1.26	--	--
66.5	--	--	.445	0.832
69.0	.725	1.35	--	--
79.0	--	--	.350	0.655
80.7	.700	1.31	--	--
82.5	.770	1.44	--	--
83.5	.72	1.35	--	--
85.0	.77	1.44	--	--
89.0	.725	1.36	--	--
90.0	.815	1.52	.175	0.327

TABLE II

Six Stiffener Panel

Experimental Results for $\frac{\sigma}{\sigma_m}$

Stiffener "A"

% of Length	2P = 2000#	2P = 4000#	2P = 6000#
96.5	3.02	--	--
86.0	2.73	3.36	3.36
79.1	2.66	2.03	2.03
77	2.38	2.38	2.38
66.0	2.10	2.52	2.52
56.2	1.43	1.74	1.74
46.0	1.385	1.50	1.50
43.7	1.32	--	--
18.7	1.29	1.62	1.62
10.4	.70	1.54	1.54
06.2	1.37		

Stiffener "B"

% of Length	2P = 2000#	2P = 4000#	2P = 6000#
90.5	.384	.437	.478
79	.665	.700	.711
67	.805	.822	.875
56.4	.84	.805	.956
44	.945	.98	1.002
33.5	.91	.945	.98
18.5	.98	.98	1.039
5.8	.98	1.015	1.05

Stiffener "C"

% of Length	2P = 2000#	2P = 4000#	2P = 6000#
90.5	.14	.157	--
79	.315	.35	--
67	.525	.525	.583
56.4	.665	.665	.70
44	.770	.77	.816
33.5	.805	.822	.84
18.5	.945	.910	.934
5.8	1.015	.997	1.001

Stiffener "D"

% of Length	2P = 2000#	2P = 4000#	2P = 6000#
90.5	.07	.07	.093
79	.175	.21	.222
67	.315	.315	.362
56.4	.49	.49	.501
44	.63	.612	.654
33.5	.77	.77	.770
18.5	.945	.945	.968
5.8	1.05	1.015	1.001

FIG 9

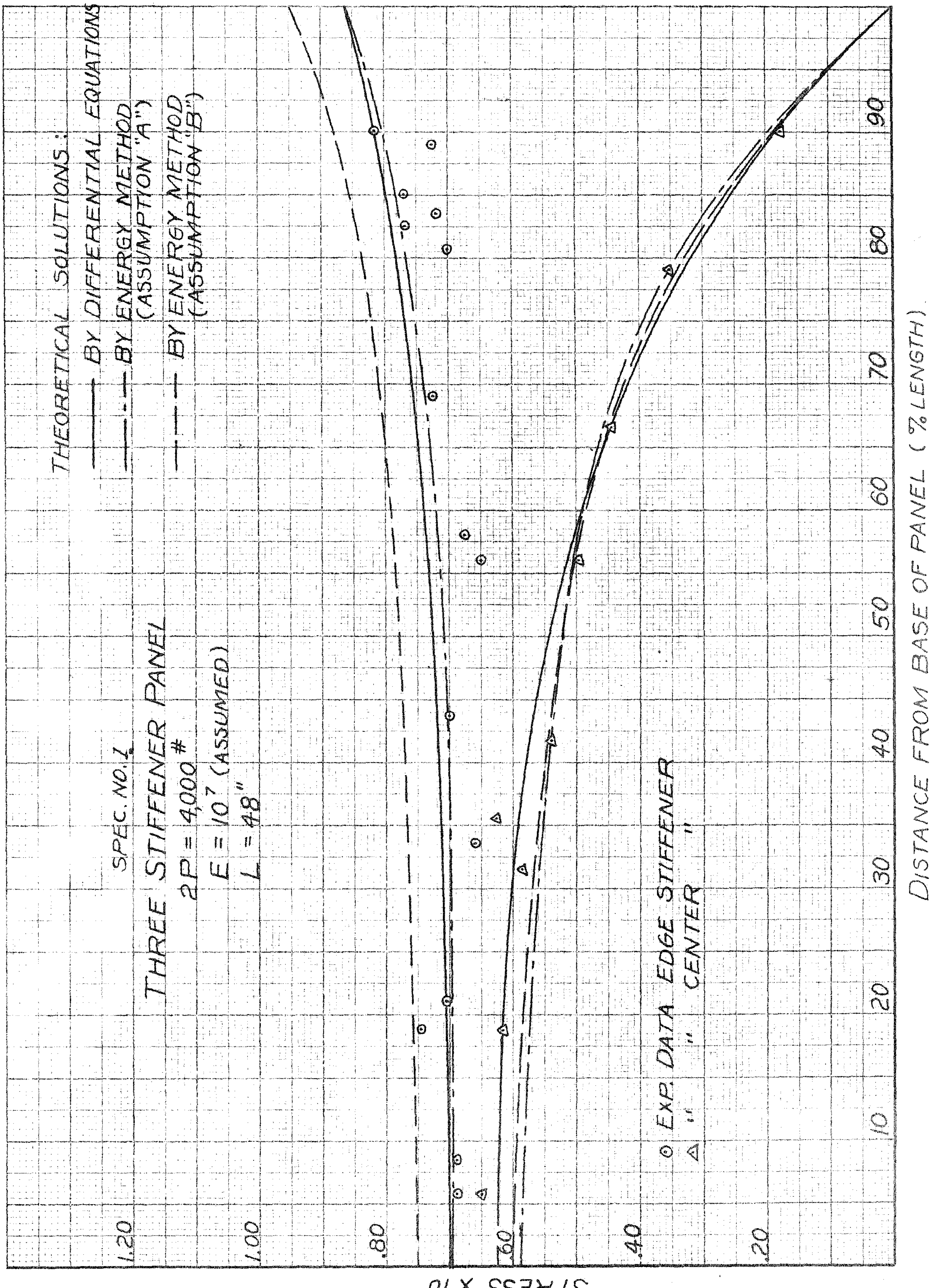
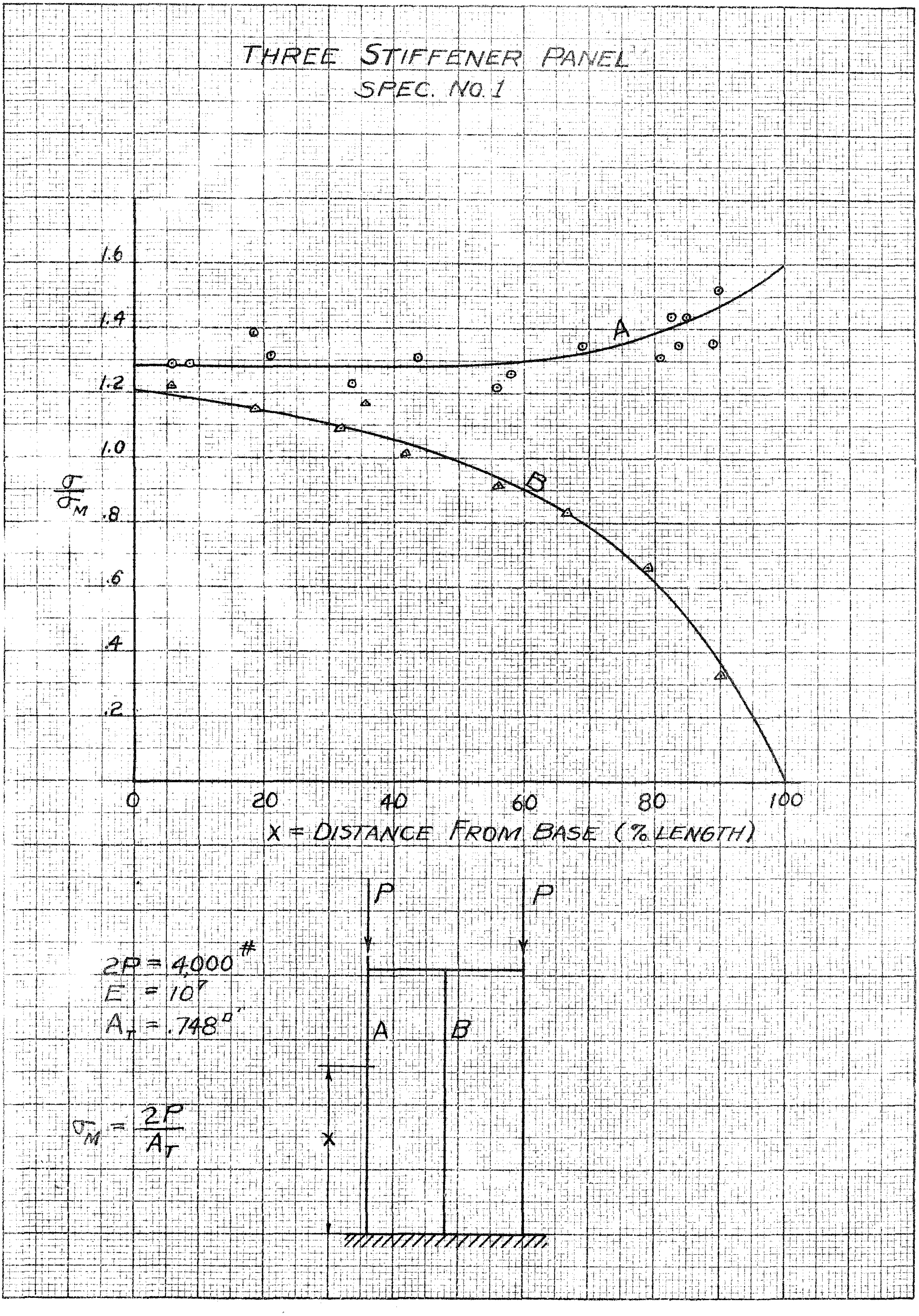
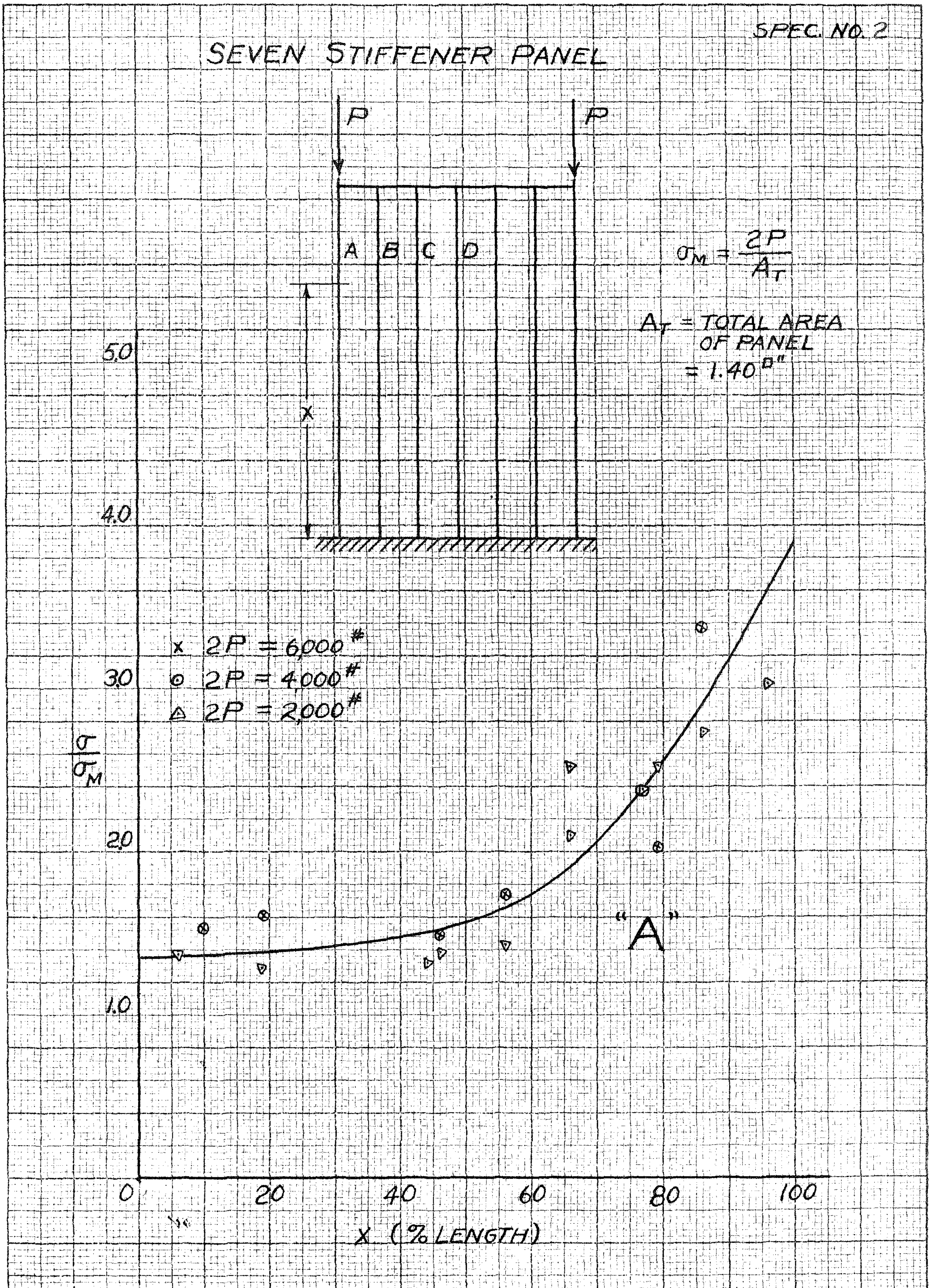

THEORETICAL SOLUTIONS:
BY DIFFERENTIAL EQUATIONS
BY ENERGY METHOD (ASSUMPTION "A")
BY ENERGY METHOD (ASSUMPTION "B")
SPEC. NO. 1
THREE STIFFENER PANEL
2P = 4,000 #
E = 10^7 (ASSUMED)
L = 48"
EXP. DATA EDGE STIFFENER
" " CENTER "
1.20
1.00
.80
.60
.40
.20
10
20
30
40
50
60
70
80
90
DISTANCE FROM BASE OF PANEL (% LENGTH)
STRESS x 10^4

FIG 10


THREE STIFFENER PANEL
SPEC. NO. 1
1.6
1.4
1.2
1.0
.8
.6
.4
.2
σ / σ_M
A
B
0
20
40
60
80
100
X = DISTANCE FROM BASE (% LENGTH)
P
P
2P = 4,000#
E = 10^7
A_T = .748□"
A
B
X
σ_M = 2P / A_T


SPEC. NO. 2
SEVEN STIFFENER PANEL
P
P
A
B
C
D
X
$\sigma_M = \frac{2P}{A_T}$
A_T = TOTAL AREA OF PANEL = 1.40□"
x 2P = 6,000#
⊙ 2P = 4,000#
△ 2P = 2,000#
$\frac{\sigma}{\sigma_M}$
5.0
4.0
3.0
2.0
1.0
0
20
40
60
80
100
"A"
X (% LENGTH)

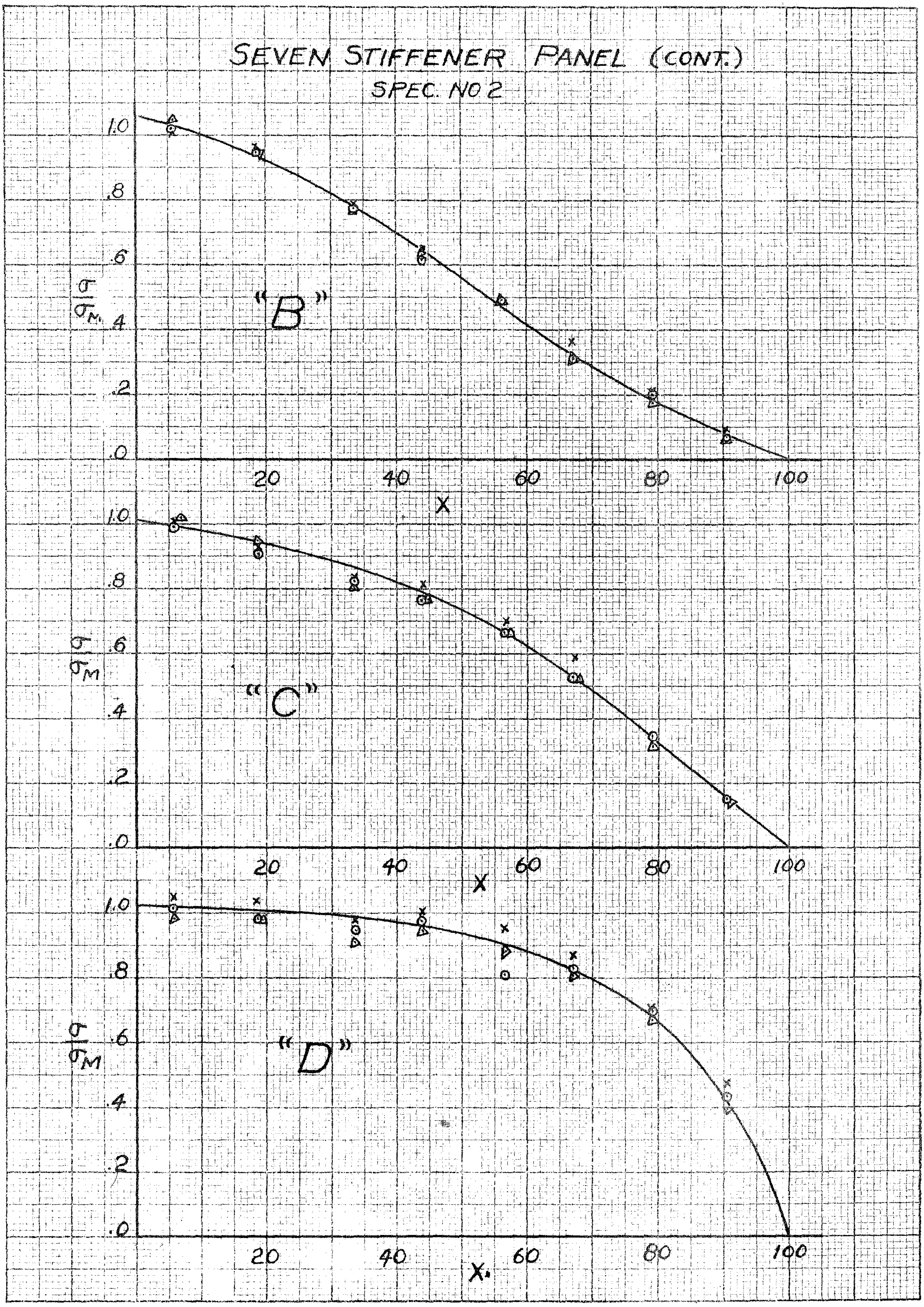

SEVEN STIFFENER PANEL (CONT.)
SPEC. NO 2
"B"
"C"
"D"
σ/σM
X
1.0
.8
.6
.4
.2
.0
20
40
60
80
100

FIG 12

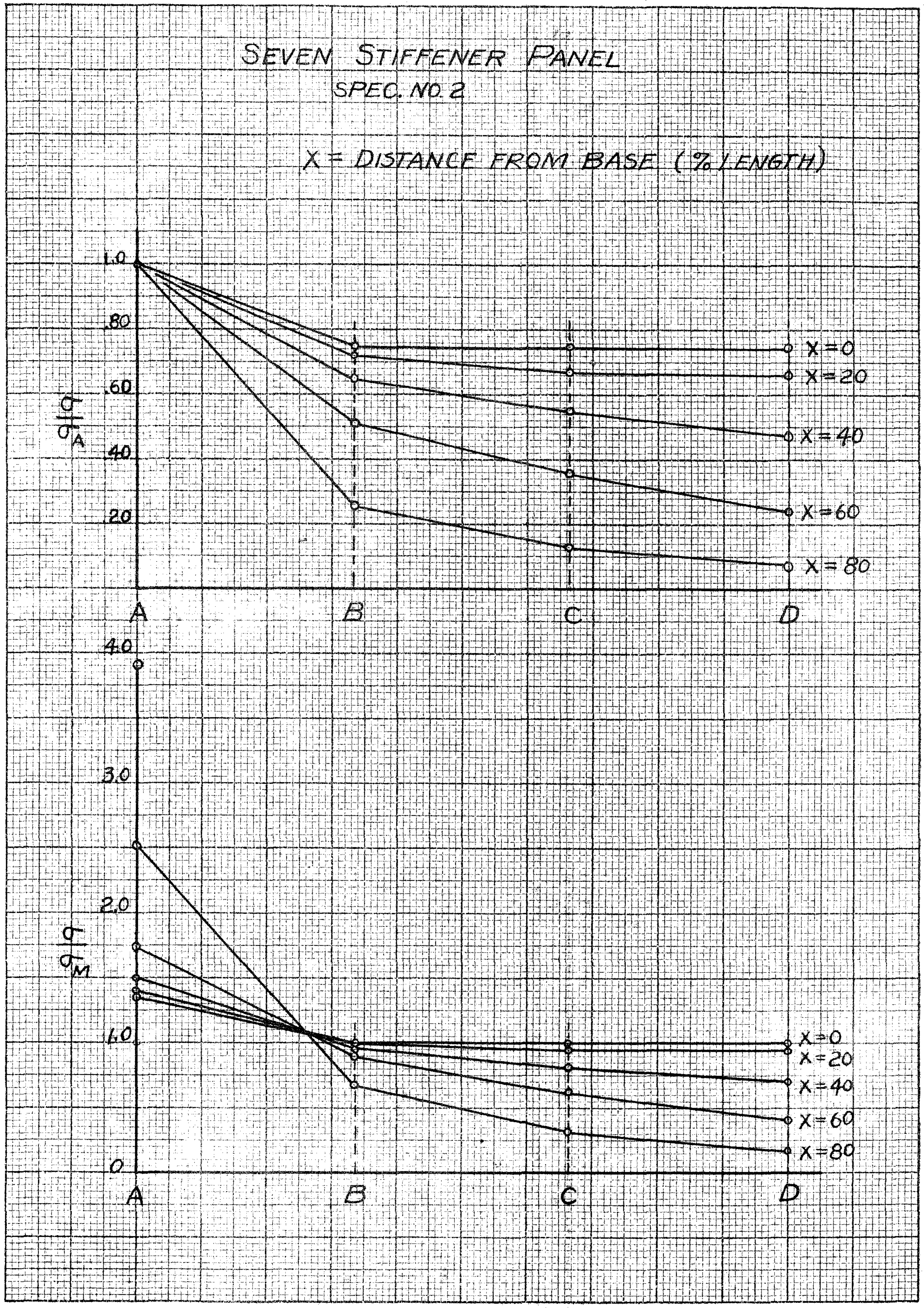

SEVEN STIFFENER PANEL
SPEC. NO. 2
X = DISTANCE FROM BASE (% LENGTH)
1.0
.80
.60
.40
.20
σ/σA
X=0
X=20
X=40
X=60
X=80
A
B
C
D
4.0
3.0
2.0
1.0
0
σ/σM
X=0
X=20
X=40
X=60
X=80
A
B
C
D

Fig. 14

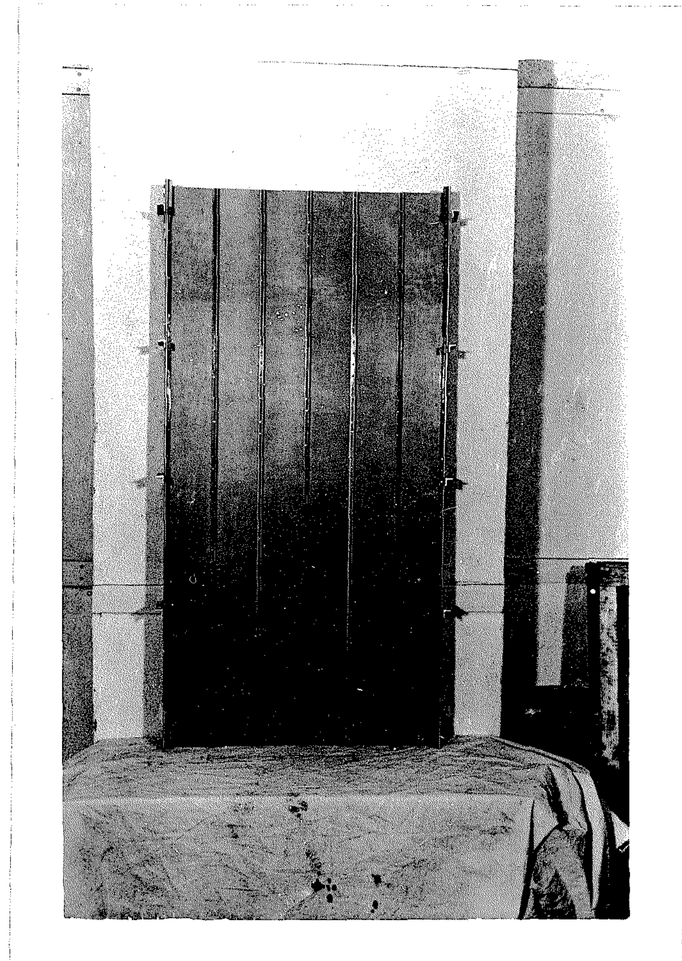


Fig. 15

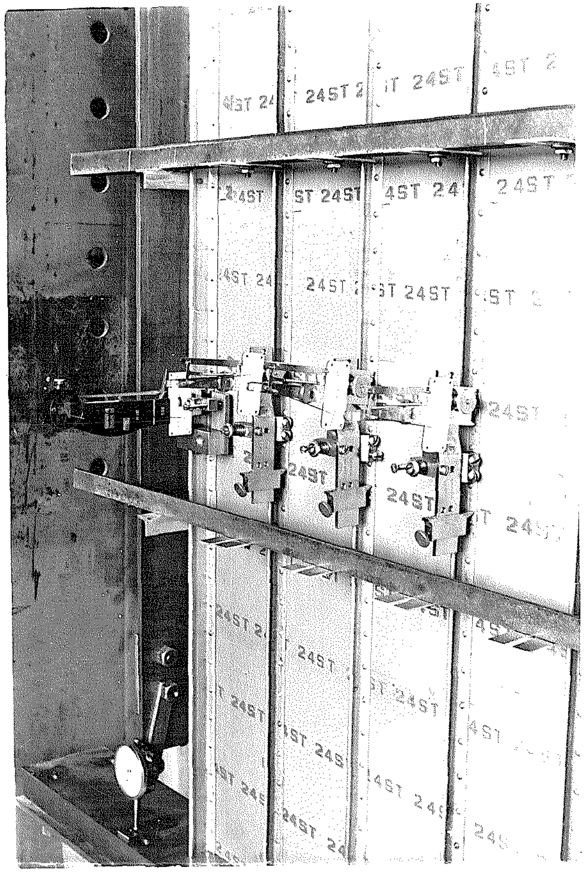


Fig. 17.

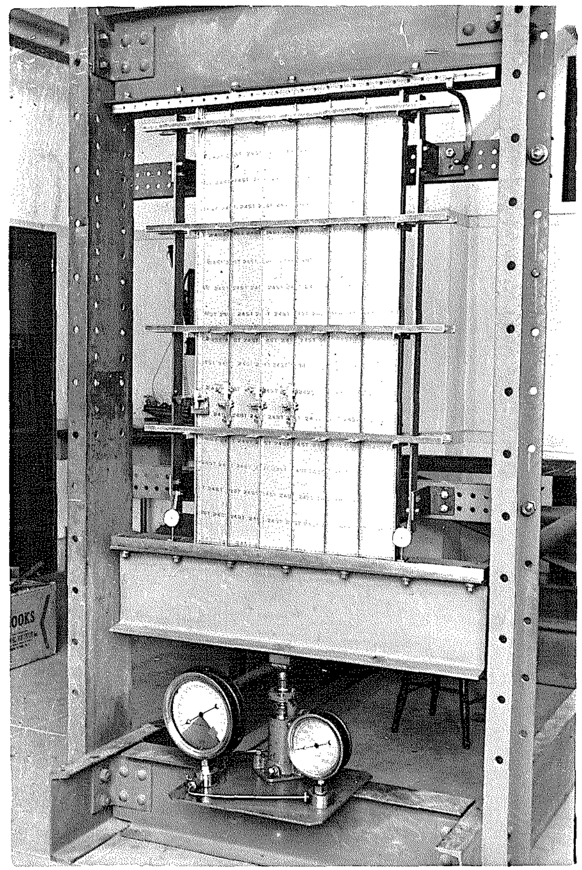


Fig. 16.