UNSYMMETRICAL LIFT DISTRIBUTIONS
ON A STALLED MONOPLANE WING

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SUMMARY

The purpose of the research was to find if it is possible to have an unsymmetrical span load distribution on a monoplane wing at the stall without a rolling velocity. The investigation is based upon the Prandtl first order wing theory. It is shown that is is possible to have an unsymmetrical lift distribution at the stall if a certain shape is assumed for the curve of section lift coefficient vs. effective angle of attack. Two methods of determining such a solution for any particular wing shape are discussed. Curves are presented showing unsymmetrical solutions for a number of different wing shapes and also showing the resulting lift and rolling moment coefficients. The important conclusions are as follows:

1. An unsymmetrical lift distribution across the span of a monoplane wing at the stall is possible.
2. The unsymmetrical solution produces a rolling moment of the same order of magnitude as that caused by a fully deflected aileron.
3. The unsymmetrical distribution can occur only in a very small range of angles of attack after the wing begins to stall.
4. The magnitude of the maximum rolling moment coefficient and the difference between the angle of attack at which the maximum rolling moment occurs and the angle of attack for first stall are nearly the same for all of the wing shapes investigated.

NOTATION

V	Rectilinear velocity far from the wing
x, y, z	Rectangular coordinates
u, v, w	Induced velocities in x, y, z directions
Γ	Circulation around lifting line
b	Span of wing
α	Geometrical angle of attack
α_ε	Effective angle of attack
L'	Lift per unit length of wing
C_L	Lift coefficient of profile section
ρ	Density of the air
t	Wing chord
t_o	Root chord of wing
$s = \frac{2x}{b}$	Dimensionless coordinate along span
C_{L_1}	Lift coefficient above stall
ξ	Fraction of span not stalled
C_r	Rolling moment coefficient of wing
$\Delta\alpha$	Difference between α and geometrical angle of attack of wing at which stall begins.
$\overline{C_L}$	Total lift coefficient of wing

INTRODUCTION

The problem of the stalling of tapered wings is of great importance. One of the greatest disadvantages of the highly tapered wing is its tendency to tip stall. When this tip stall occurs there is not only a loss of lateral control but also there often exists a rolling moment of sufficient magnitude to put the airplane into an undesirable maneuver. This fact led to the belief that it is possible to have an unsymmetrical lift distribution of a wing which is partially stalled even when there is no rolling velocity. This paper is a discussion of the research carried out in looking for such unsymmetrical lift distributions.

Much work has been done on the determination of the lift distribution of a wing which is not stalled. Several good methods for calculating this lift distribution have been developed. These methods can also be used for calculating the unsymmetrical distribution which exists when the wing has a rolling velocity. When the wing is at an angle of attack just above the stall and is given a rolling velocity it is possible to have a rolling moment produced which causes the rolling velocity to continue. This is known as autorotation. All of these cases have been rather thoroughly investigated, both theoretically and experimentally. However, apparently there has been no successful investigation of unsymmetrical lift distributions at the stall without a rolling velocity.

The basis of this work is the Prandtl wing theory. In this theory the wing is represented by a line vortex or a sum of coincident line vortices. This line which represents the wing is called the lifting line. The circulation about the lifting line at any position along the span is equal to the circulation about the wing at the same spanwise location. In general the circulation about the lifting line varies along the span. This circulation is the sum of the strengths of the line vortices which form the lifting line. Since the number of vortices forming the lifting line changes and since by Helmholtz' Theorem a vortex tube never ends in the fluid, it is evident that vortices must be shed from the lifting line. Free vortices move with the fluid, and therefore the vortex lines which are shed from the lifting line trail in the direction of the fluid. These are called the trailing vortices. In the steady state these trailing vortices extend to infinity downstream from the lifting line. One of these vortices which starts at infinity, goes up to the lifting line and then back again to infinity is called a horse-shoe vortex. This assumption of the Prandtl wing theory results in the concept of an infinite number of horse-shoe vortices of infinitesimal strengths, so placed that their central parts coincide to form the lifting line and their trailing parts lie parallel to the fluid velocity.

Another assumption of the Prandtl wing theory is that the circulation around the lifting line is everywhere small. The significance of this assumption is seen from a consideration of the so-called induced velocity of a vortex. Consider an infinitely long straight vortex. At a point not on the line of the vortex there is a velocity which is proportional to the strength of the vortex, inversely proportional to the distance from the vortex, and in a direction perpendicular to the plane containing both the line and the point. Therefore the system of horse-shoe vortices described above has associated with it induced velocities which are perpendicular to the plane of the vortices. If the circulation around the lifting line is large, then these induced velocities are large and the original free stream flow is considerably distorted. The assumption that the circulation is small means that the induced velocities can be considered so small that the trailing vortices follow the streamlines of the original undisturbed flow. The only case to be considered in this work is that of rectilinear flow from infinity. For this case the second assumption gives the result that the free vortices are straight lines in the direction of the flow.

The first assumption requires that the thickness and chord of the wing be small in comparison with the span. In other words, the assumption is not good for wings of small aspect ratio. Also the vortex filaments which represent the wing must be everywhere parallel to each other. This means that a wing with sweepback cannot be represented by a simple lifting line.

For the further considerations it is convenient to set up a coordinate system. Take the rectilinear velocity far from the wing as V. Take the origin of the coordinate system at the center of the span of the lifting line. Let the y axis be parallel to V, the x axis coincident with the lifting line, and the Z axis vertically down. Let u, v, and w be the induced velocities of the vortex system in the x, y, and z directions respectively.

The induced velocities of the vortex system must be found for any point on the lifting line. This is done by applying Biot-Savart's law. For all points in the plane of the horse-shoe vortices the velocities u and v are zero. For points on the x axis, which includes the lifting line, the so-called downwash velocity w is given by the expression

$$w(x,0,0) = \frac{1}{4\pi} \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\frac{d\Gamma}{dx'}\,dx'}{x - x'}$$

where Γ is the circulation around the lifting line and b is the span of the wing.

The problem to be solved is to find the lift distribution for a given wing shape. The wing shape is given when the chord, airfoil section, and geometrical angle of attack are given for each point along the span. The solution of the problem comes from a combination of the three-dimensional analysis just given and the two-dimensional airfoil theory. At this point must be introduced the assumptions of the strip theory. The flow at any section along the span is the sum of the original rectilinear flow, V, and the induced velocities, u, v, and w. The assumption of the strip theory is that the induced velocities, u and v, are small enough to be neglected and therefore that the resultant velocity at any section is the sum of V and w. This means that the flow at any section of the wing is two dimensional and that the two dimensional airfoil theory can be applied.

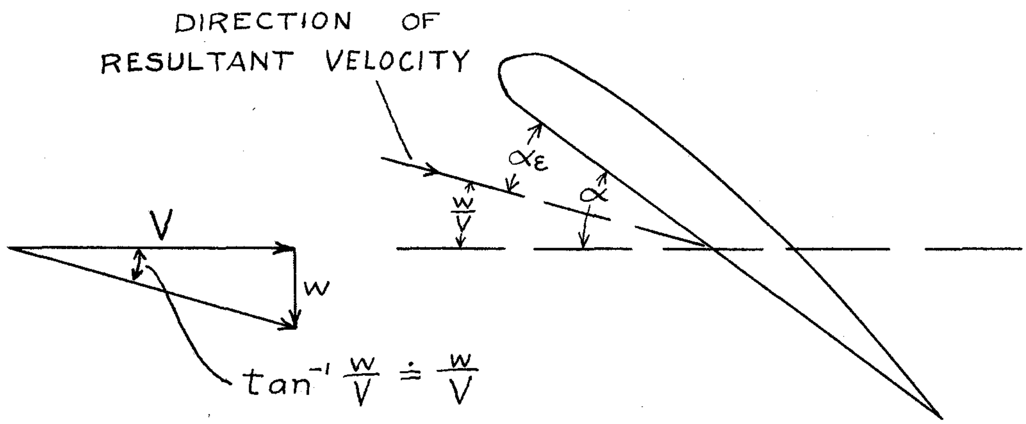
This assumption is fairly good over most of the span. At the wing tips the cross flow, u, is not small and therefore this theory does not give the correct conditions near the tip. For this reason the theory is better for wings of large aspect ratio than for those of small aspect ratio. This paper is concerned with conditions at the stall. At the stall the assumptions that have been made are not so valid as they are below the stall. First of all,at the stall the wing is at a high angle of attack. This throws some doubt on the assumption that the wing can be represented by a lifting line from which are shed trailing vortices all of which lie in the same plane. Also at the stall the wing is acting at a very high lift coefficient. This means that the circulation around the wing is much larger than for conditions well below the stall. Therefore all of the vortex filaments are fairly strong and their induced velocities are proportionately large. This makes the assumption that the trailing vortices follow the rectilinear flow, V, poorer when the wing is at the stall than for any other condition. Also at the stall the cross flow at the wing tips is not small. It is probable that when there is an unsymmetrical lift distribution such as those discussed later there is cross flow even in the regions far from the wing tip. In spite of these objections to the assumptions that have been made, it is considered that the assumptions are good enough for a first order theory.

Consider conditions at a cross section of the wing cut by a plane parallel to the yz plane. The airfoil is at a geometrical angle of attack, α, relative to the rectilinear flow, V. Also acting at the airfoil is the induced downwash velocity, w, which is assumed to be uniform over the chord of the wing. The resultant velocity is the vector sum of V and w. V is taken as very much larger than w so that it is assumed that the magnitude of the resultant velocity is the same as the magnitude of V. The important effect of w is that it changes the direction of the flow. The change in angle is

$$\tan^{-1}\frac{w}{V} \doteq \frac{w}{V}$$

since w is small compared to V. If α_ε is the effective angle of attack, there is the result:

$$\alpha = \alpha_\varepsilon + \frac{w}{V}$$



In the two dimensional airfoil theory the lift per unit length of wing can be written:

$$L' = C_L \tfrac{1}{2}\rho V^2 t$$

where L' is the lift per unit length, C_L is a non-dimensional lift coefficient which is a function of the effective angle of attack of the airfoil, ρ is the fluid density, and t is the chord of the airfoil. According to the Kutta-Joukowsky Law:

$$L' = \rho V \Gamma$$

Therefore

$$\Gamma = \frac{Vt}{2} C_L\left(\alpha - \frac{w}{V}\right)$$

where $C_L\left(\alpha-\frac{w}{V}\right)$ is read: C_L as a function of $\left(\alpha - \frac{w}{V}\right)$. This, combined with the equation for w, gives:

$$\Gamma(x) = \frac{V t(x)}{2} C_L \left(\alpha - \frac{1}{4\pi V} \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{d\Gamma}{dx'} \frac{dx'}{x - x'} \right)$$

This is the integral equation which must be satisfied by any lift distribution. It is convenient to write the equation in the dimensionless form

$$C_L \frac{t}{t_o} = \frac{t}{t_o}(s) \cdot C_L \left(\alpha - \frac{1}{4\pi \frac{b}{t_o}} \int_{-1}^{1} \frac{d(C_L \frac{t}{t_o})}{ds'} \frac{ds'}{s - s'} \right)$$

where t_o is the root chord and $s = \frac{2x}{b}$.

The discussion up to this point gives the basis upon which the research was carried out. The integral equation shown above is the starting point for all lift distribution investigations.

THE DETERMINATION OF UNSYMMETRICAL LIFT DISTRIBUTIONS OF A WING AT THE STALL

Any lift distribution, whether symmetrical or unsymmetrical, must satisfy the equations:

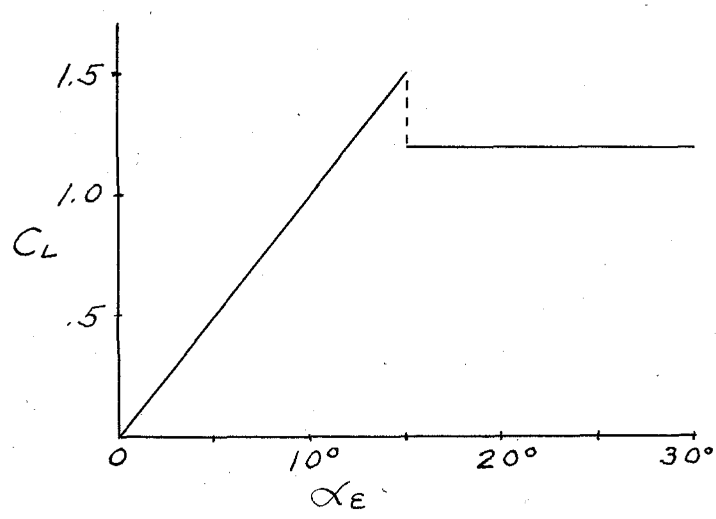
$$C_L \frac{t}{t_0} = \frac{t}{t_0}(s)\, C_L\left(\alpha - \frac{w}{V}\right)$$

$$\frac{w}{V} = \frac{1}{4\pi \frac{b}{t_0}} \int_{-1}^{1} \frac{d(C_L \frac{t}{t_0})}{ds'} \frac{ds'}{s - s'}$$

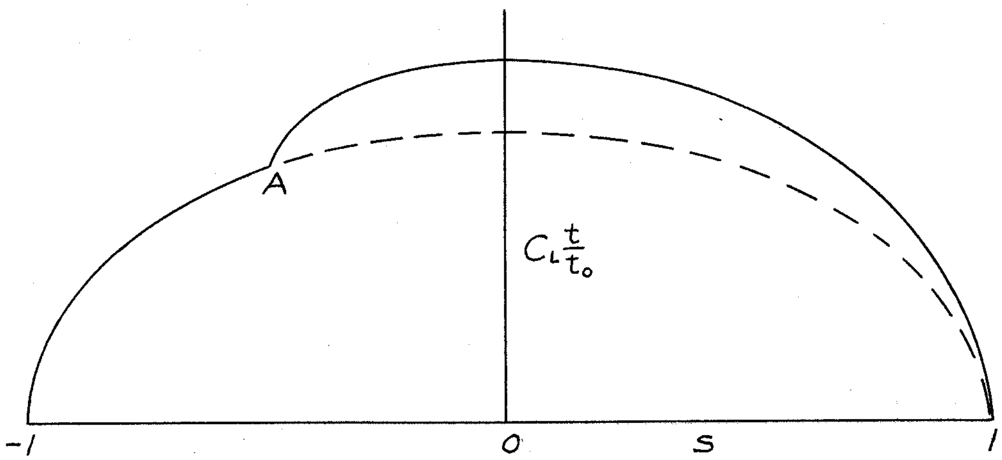
This is the most general form in which the problem can be stated. In order to do any more with the problem it is necessary to express the functional relationships explicitly. The relationship which varies least with the design of the wing is that between the lift coefficient and effective angle of attack. Therefore it is usual to assume an explicit form for this function and to develope on this basis methods for solving the integral equation for any given wing shape. For much of the work that has been done on lift distributions it has been assumed that this airfoil section characteristic does not change as a function of the spanwise position. In considering the lift distribution of a wing which is not stalled it is customary to assume that the lift coefficient is directly proportional to the effective angle of attack above the angle of attack for zero lift. This is a true assumption for almost every airfoil section. It is only in the slope of the curve that the airfoils differ from each other and even this variation is not great.

At the stall the section characteristic is not so well defined. First of all there is no theoretical way of determining what the shape of the C_L vs. α_ϵ curve is at the stall. This means that the curve must be determined experimentally for each airfoil section. In this experimental determination it is almost always true that the measured curve is the average characteristic for the whole wing and not the true property of the two dimensional cross section. Also the value of the maximum lift coefficient obtained depends upon the turbulence in the wind tunnel in which the experiment is made and upon the Reynolds number. Some airfoil sections have a flat topped lift curve which means that they maintain a high lift coefficient over a wide range of angles of attack. Others have curves which break suddenly at the stall and may even be discontinuous. In general the lift coefficient drops slowly back to zero as the angle of attack is increased from that at which the stall occurs to 90°. This discussion shows the difficulty in choosing a lift coefficient-angle of attack curve which is general enough to be useful in a large number of cases.

The shape of the curve that is used in the following work is shown below. The angle of zero lift is taken as zero degrees. The slope of the curve below the stall is 5.73 per radian so that at an angle of ten degrees the lift coefficient is one. For all of the computations that have been made the maximum lift coefficient is 1.5. At the stall the curve is discontinuous. For all angles of attack above the stall the lift coefficient has the same value. For most of the computations this value is 1.2.



With a lift curve of the shape shown it is possible to have unsymmetrical lift distributions in which part of the wing is stalled and the rest of it is unstalled. Such a distribution is sketched below. The wing in this case is of elliptical planform and untwisted. Therefore all parts of the wing reach the peak of the lift curve at the same time. The wing in this case is at an angle of attack very slightly greater than the angle of attack



at which the stall can occur. If the whole wing were stalled, the lift distribution would be that indicated by the broken line. However, it is also possible to have the distribution shown by the full line. In this case there are shed near A trailing vortices of great strength which induce a downwash over the part of the wing to the right of A which is large enough to make the effective angle of attack less than that at which the stall occurs. Just to the left of the point A there is a very large upwash which causes the effective angle of attack to be very great. Since the lift coefficient is independent of the angle of attack above the stall, this upwash does not affect the lift distribution.

In reference five there is a discussion of the behavior of the lift distribution near singular points of the wing. The circulation must vary continuously across the span. At the tip of the wing the circulation must vanish like $\sqrt{1 - s^2}$. At all other points of the wing the slope of the lift distribution curve, $\frac{d(C_L \frac{t}{t_0})}{ds}$, must be continuous if the possibility of infinite upwash is excluded. A discontinuity in this slope means that at the point where there is the discontinuity there is an infinite upwash. The curve shown has such a discontinuity at the point A. However, the airfoil section characteristic which is used does not exclude the possibility of infinite upwash and therefore this limitation is avoided. If an airfoil section characteristic is chosen which does not permit infinite upwash, then an unsymmetrical solution is still possible. In this case it is necessary to have a lift distribution which causes a finite discontinuity in the downwash at the point between the stalled and the unstalled parts of the wing. Betz and Petersohn (reference 3) have determined the lift distribution for a wing having a finite jump in the geometrical angle of attack. This solution gives a finite discontinuity in the downwash and can be used in determining the shape of the lift distribution curve at the point at which the wing stalls. The change in the lift distribution caused by considering this effect is such that there is only a very small change in the rolling moment caused by the unsymmetrical solution. Since it is the rolling moment that causes the unsymmetrical solution to be important, it is considered satisfactory to use the C_L vs. α_ε

curve which permits infinite upwash.

The following equations show a method which can be used for determining an unsymmetrical lift distribution at the stall.

The equations which must be satisfied are:

$$C_L = 5.73\left(\alpha - \frac{w}{V}\right) \qquad \left(\alpha - \frac{w}{V}\right) < \alpha_{STALL}$$

$$C_L = C_{L_1} \text{ (CONSTANT)} \qquad \left(\alpha - \frac{w}{V}\right) > \alpha_{STALL}$$

$$\frac{w}{V} = \frac{1}{4\pi \frac{b}{t_0}} \int_{-1}^{1} \frac{d(C_L \frac{t}{t_0})}{ds'} \frac{ds'}{s - s'}$$

Let $$C_L = C_{L_1} + C_{L_2}$$

where C_{L_2} is not constant

Then

$$\int_{-1}^{1} \frac{d(C_L \frac{t}{t_0})}{ds'} \frac{ds'}{s - s'} = \int_{-1}^{1} \frac{d(C_{L_1} \frac{t}{t_0})}{ds'} \frac{ds'}{s - s'} + \int_{-1}^{1} \frac{d(C_{L_2} \frac{t}{t_0})}{ds'} \frac{ds'}{s - s'}$$

Let

$$\left(\frac{w}{V}\right)_1 = \frac{1}{4\pi \frac{b}{t_0}} \int_{-1}^{1} \frac{d(C_{L_1} \frac{t}{t_0})}{ds'} \frac{ds'}{s - s'}$$

$$\left(\frac{w}{V}\right)_2 = \frac{1}{4\pi \frac{b}{t_0}} \int_{-1}^{1} \frac{d(C_{L_2} \frac{t}{t_0})}{ds'} \frac{ds'}{s - s'}$$

$\left(\frac{w}{V}\right)_1$, is a known function of s as soon as the constant C_{L_1} and the chord function are given.

$$\frac{w}{V} = \left(\frac{w}{V}\right)_1 + \left(\frac{w}{V}\right)_2$$

Substitution into the equation for the unstalled part of the wing gives:

$$C_{L_1} + C_{L_2} = 5.73\left[\alpha - \left(\frac{w}{V}\right)_1 - \left(\frac{w}{V}\right)_2\right]$$

$$C_{L_2} = 5.73\left[\alpha - \left(\frac{w}{V}\right)_1 - \frac{C_{L_1}}{5.73} - \left(\frac{w}{V}\right)_2\right]$$

Let $$\alpha_2 = \alpha - \left(\frac{w}{V}\right)_1 - \frac{C_{L_1}}{5.73}$$

$$\therefore \quad C_{L_2} = 5.73\left[\alpha_2 - \left(\frac{w}{V}\right)_2\right]$$

When the wing shape and C_{L_1} are given, then α_2 is a known function of s. Therefore the equations which must be satisfied by the lift distribution over the unstalled part of the wing are:

$$C_{L_2} = 5.73\left[\alpha_2 - \left(\frac{w}{V}\right)_2\right]$$

$$\left(\frac{w}{V}\right)_2 = \frac{1}{4\pi\frac{b}{t_0}}\int_{-1}^{1}\frac{d(C_{L_2}\frac{t}{t_0})}{ds'}\frac{ds'}{s-s'}$$

The equation for $(\frac{W}{V})_2$ includes an integral over the whole span of the wing. Since C_{L_2} is different from zero only in the unstalled part of the wing, the equations above give a distribution of C_{L_2} which is the same as that obtained by considering a wing having the planform of only the unstalled part of the wing. Therefore the procedure to be followed in determining the lift distribution according to this method is as follows.

The things that are given are the wing shape (chord and twist distribution) and the curve of lift coefficient against effective angle of attack. The latter curve is assumed to hold for all cross sections of the wing. The first thing to compute is the downwash distribution caused by the wing acting everywhere at the lift coefficient beyond the stall. Then the function, α_2, is determined. An arbitrary choice is made of the part of the wing which is not stalled. The lift distribution on this odd shaped piece of the wing is determined by any of the usual methods for a number of different angles of attack. It is assumed that at some point of the wing the maximum lift coefficient is obtained. Therefore the angle of attack is chosen for which the maximum value of C_{L_2} is equal to the difference between $C_{L_{max}}$ and C_{L_1}. The values of C_{L_2} for this angle of attack added to C_{L_1} give the distribution of C_L and from this the circulation or $C_L \cdot \frac{t}{t_o}$ distribution is determined. This method is not fool-proof and must be used with care. Its limitations are seen from the following discussion of its use in particular cases.

The case for which the method is most useful is that of the untwisted wing of elliptical planform. For an elliptical wing which everywhere acts at the same lift coefficient the down wash is the same at all points across the span. Therefore for an untwisted elliptical wing the value of α_2 is the same at all points across the span. This simplifies the problem of finding the distribution of C_{L_2} for the unstalled portion of the wing because for an untwisted wing which has the same section characteristic at all spanwise positions the shape of the lift distribution is the same for all angles of attack and the magnitude is proportional to the angle of attack. For this reason it is necessary to compute only one lift distribution for the unstalled part of the wing in order to determine the angle of attack at which the maximum lift coefficient is reached. In this case the computations are easy and short enough to make the method quite useful.

Condider the case of a tapered wing, either twisted or untwisted. Here the downwash caused by a uniform lift coefficient across the span is not the same for all spanwise positions. Near the center the downwash is very great while at the tips it is very much smaller. This means that α_2 varies across the span unless the wing happens to be twisted so that the difference between α and $(\frac{W}{V})1$ is a constant. This case does not ordinarily occur. Therefore the unstalled part of the wing is treated as twisted in the determination of the distribution of C_{L_2}. Now in order to find the angle of attack at which the maximum lift coefficient occurs it is necessary to determine lift distributions for several angles of attack because for a twisted wing the shape of the lift distribution depends upon the angle of attack. Another difficulty with the use of this method for a tapered wing arises from the fact that in the usual symmetrical solution all parts of the wing do not come to the effective angle of attack for stall at the same time. In the unsymmetrical solution some of the parts of the wing are stalled when the wing is at an angle of attack at which in the symmetrical case these parts are not stalled. This means that at such points the unsymmetrical solution must cause sufficient upwash to increase the effective angle of attack to a value greater than that for the stall. Therefore in finding an unsymmetrical solution it is necessary to check that this does happen. All of this makes this method so complicated that it was not used in the determination of the lift distributions on tapered wings.

Another method which can be used is one shown by Fage (reference 6). It is a simple graphical method of determining the downwash caused by any given lift distribution. The process is one of trial and error. The solution is known when the assumed and calculated lift distribution curves agree. The procedure to be followed in determining the lift distribution is as follows. First assume a lift distribution curve. From this determine by the graphical method the downwash at a number of different sections. From the section characteristic of lift coefficient against effective angle of attack is obtained the lift coefficient which corresponds to the calculated downwash. If the value obtained in this way does not agree with the value originally assumed, it is necessary to repeat the whole process until there is agreement.

Fage's graphical method is as follows. Since the integrand in the downwash equation

$$\frac{w}{V} = \frac{1}{4\pi \frac{b}{t_o}} \int_{-1}^{1} \frac{d(C_L \frac{t}{t_o})}{ds'} \frac{ds'}{s-s'}$$

becomes infinite when s = s', the integral is evaluated in two sections.

$$4\pi \frac{b}{t_o} \frac{w}{V} = \int_0^{\varepsilon_1} \frac{d\,(C_L \frac{t}{t_o})}{s_1 - s'} + \int_{\varepsilon_2}^{0} \frac{d\,(C_L \frac{t}{t_o})}{s_1 - s'} + \int_{s_1 - s_2}^{s_1 + s_2} \frac{d(C_L \frac{t}{t_o})}{ds'} \frac{ds'}{s_1 - s'}$$

where ε_1 is the value of $C_L \frac{t}{t_o}$ at $s = (s_1 - s_2)$, and ε_2 the value at $s = (s_1 + s_2)$. The value of the first two integrals is the area of the diagram obtained when $\frac{1}{s_1 - s'}$ is plotted against $(C_L \frac{t}{t_o})$ as s' varies from -1 to $(s_1 - s_2)$, and from $(s_1 + s_2)$ to 1. The third integral is transformed into the form

$$\int_0^{s_2} \left\{ \left[\frac{d(C_L \frac{t}{t_o})}{ds}\right]_{s_1 - z} - \left[\frac{d(C_L \frac{t}{t_o})}{ds}\right]_{s_1 + z} \right\} \frac{dz}{z}$$

The slopes used in this integral are determined graphically from the assumed lift distribution curve. The integration is performed graphically.

In order to arrive at the solution in a reasonably short time it is necessary to have the first assumed lift curve close to the final result. It is found that the unsymmetrical solutions occur only at angles of attack just slightly above the angle at which stall first happens. Also the lift coefficient over most of the unstalled part of the wing differs very little from the value obtained at the angle for first stall. Therefore the first step in finding an unsymmetrical solution for a tapered wing is to calculate the lift distribution at an angle of attack such that at one or more points there is the maximum lift coefficient and at no point is the wing stalled. This is done by any of the ordinary methods.

In this work the only type of solution that is considered is that in which the wing is stalled from the tip in to some internal point and then the rest of the wing is unstalled. In finding the solution, the point separating the stalled and unstalled portions of the wing is taken arbitrarily. At this point the lift coefficient changes rapidly from that above the stall to some higher value in the unstalled region. Now there is enough material available to draw up the first assumed lift distribution curve so that it is near to the final result. The shape of the curve is known for the stalled region. At the opposite wing tip the curve is almost identical to the calculated curve for the case just before the stall. In the intermediate region as the point of stalling is approached the lift coefficient falls slightly below the above mentioned calculated value. The curve is drawn with a continous slope everywhere except at the point separating the stalled and unstalled parts.

It is found that the value of the downwash at any point depends mostly upon the shape of the lift distribution curve in the immediate meighborhood of the point and upon the maximum value of $C_L \frac{t}{t_0}$ that exists on the wing. Also it is found that the spanwise position of the point of maximum lift coefficient differs very little from the position found for the completely unstalled wing. These facts are useful in determining the angle of attack at which the unsymmetrical solution occurs. Starting with the assumed $C_L \frac{t}{t_o}$ curve, the downwash is calculated by Fage's method at the point of maximum lift coefficient and at the point of maximum $C_L \frac{t}{t_o}$. If the correct solution is assumed at first then the angle of attack is the sum of the effective angle of attack at which the stall occurs and the downwash angle calculated at the point of maximum lift coefficient. This angle of attack and the downwash at the point of maximum $C_L \frac{t}{t_o}$ are used to check this maximum value. If the assumed and calculated values do not agree, the curve must be changed and the process repeated until there is agreement. Once this agreement is reached, any further changes that are necessary in the shape of the curve affect only very slightly the angle of attack and the maximum value of $C_L \frac{t}{t_o}$. The next part of the curve to be brought to the proper shape is the unstalled portion near to the part that is stalled. This is done by the same graphical method. After these three parts of the curve have been made to check, it is probable that the rest of the curve is nearly correct and that only slight changes need to be made. It is advisable to check as soon as possible in the process that points in the stalled region actually are stalled. This consideration is one which limits the amount of the wing that can be stalled in the case of a highly tapered wing.

The process is long and tedious because even with a lot of experience it is difficult to make a very close first approximation to the final result. The method is not one of successive approximations but one of trial and error. For the type of solution discussed here it is usually necessary to make three or four attempts in order to find a curve that is good enough. Unless some such systematic approach as is outlined above is used, it is almost impossible to arrive at the solution. Blind guesses are quite useless because the downwash is quite sensitive to changes in the shape of the lift distribution. Another trouble with the method is that it is difficult to determine accurately the slope of the lift distribution curve. However, apparently this can be done well enough to give sufficient accuracy to the final result.

The method that was used in the computation of the lift distribution below the stall is one developed by Multhopp (reference 2). The method is quick and easy to use and the computations are not long.

One of the most difficult things in the whole problem is to find the angle of attack at which any particular amount of the wing is stalled. It is found that the range of angles of attack in which the unsymmetrical solutions can occur is very small and also that for each solution there is a definite angle of attack. For this reason, a pure guess at the angle of attack is useless. It is an essential point of both of the methods outlined in this paper that the angle of attack is determined without any guessing.

DISCUSSION OF COMPUTATIONS AND RESULTS FOR SEVERAL DIFFERENT WING SHAPES

The first case to be considered is that of an untwisted elliptical wing having an aspect ratio of 10.19. The method of computation is described in the previous chapter. Solutions were found for cases in which different fractions of the wing were unstalled. For each of these fractions of the wing an ordinary lift distribution is obtained and from that the distribution of $\frac{C_L}{\alpha}$ across the span of the piece of the wing. For each piece of the wing there is a maximum value of $\frac{C_L}{\alpha}$.

Let $$K = \left(\frac{C_L}{\alpha}\right)_{MAX}$$

K is a function of the fraction of the wing that is not stalled.

The equation on page 12 for α_2 gives

$$\alpha = \alpha_2 + \left(\frac{W}{V}\right)_1 + \frac{C_{L_1}}{5.73} \qquad (\alpha \text{ IN RADIANS})$$

$$\text{OR} \quad \alpha = \alpha_2 + \left(\frac{W}{V}\right)_1 + 10\, C_{L_1} \qquad (\alpha \text{ IN DEGREES})$$

In this equation

$$\alpha_2 = \frac{C_{L_2\,MAX}}{K}$$

For an untwisted elliptical wing of aspect ratio 10.19

$$\left(\frac{W}{V}\right)_1 = 1.79\, C_{L_1}$$

$$\alpha = \frac{C_{L_2 MAX}}{K} + 11.79\, C_{L_1}$$

$$C_{L\,MAX} = C_{L_1} + C_{L_2\,MAX}$$

$$\therefore \quad \alpha = \frac{C_{L\,MAX}}{K} + \left(11.79 - \frac{1}{K}\right) C_{L_1}$$

Let α_3 = angle of attack at which wing first stalls

$$\Delta\alpha = \alpha - \alpha_3$$

Then

$$\alpha_3 = 11.79\ C_{L\,MAX}$$

$$\therefore \quad \Delta\alpha = \left(\frac{1}{K} - 11.79\right)\left(C_{L\,MAX} - C_{L_1}\right) \qquad (\Delta\alpha \text{ IN DEGREES})$$

Let ξ = fraction of span not stalled.

The computations give the following result.

ξ	$(\frac{1}{K} - 11.79)$
1/8	2.61
1/4	1.23
3/8	.72
1/2	.40
5/8	.23
3/4	.14
1	0

The rolling moment coefficient caused by an unsymmetrical lift distribution is given by the following equation.

$$C_r = \frac{1}{2} \frac{\int_{-1}^{1} C_L \frac{t}{t_0} s\, ds}{\int_{-1}^{1} \frac{t}{t_0} ds}$$

For wings having a symmetrical chord distribution this equation can be transformed into

$$C_r = \frac{1}{2} \frac{\int_{0}^{1} \left[(C_{L2})_s - (C_{L2})_{-s} \right] \frac{t}{t_0} s\, ds}{\int_{-1}^{1} \frac{t}{t_0} ds}$$

where $C_{L_2} = C_L - C_{L_1}$

and C_{L_1} IS CONSTANT

For a given fraction of an untwisted elliptical wing not stalled, C_{L_2} is directly proportional to $C_{L_2 max}$ or in other words to the drop in lift coefficient of the profile section at the stall. Therefore the rolling moment in this case is also proportional to that drop in lift coefficient. It is shown that $\Delta\alpha$ is proportional to this same quantity.

The lift coefficient for the whole wing is given by the following equation

$$\bar{C}_L = \frac{\int_{-1}^{1} C_L \frac{t}{t_o}\, ds}{\int_{-1}^{1} \frac{t}{t_o}\, ds}$$

$$C_L = C_{L_1} + C_{L_2}$$

$$\int_{-1}^{1} C_L \frac{t}{t_o}\, ds = C_{L_1} \int_{-1}^{1} \frac{t}{t_o}\, ds + \int_{-1}^{1} C_{L_2} \frac{t}{t_o}\, ds$$

For the untwisted elliptical wing the second integral on the right is proportional to the drop in lift coefficient at the stall.

The results for the untwisted elliptical wing can be summed up as follows. $\Delta\alpha$, the rolling moment coefficient, and part of the total lift coefficient are proportional to the drop in the infinite aspect ratio lift coefficient at the stall for any given fraction of the wing that is not stalled. These values do not depend upon the absolute magnitude of $C_{L_{max}}$ and C_{L_1} but only upon the difference between them. The variation with the amount of the wing that is not stalled of $\Delta\alpha$, the rolling moment coefficient, and the total lift coefficient is determined by computation. The results are shown in figure 1.

For all of the other planforms considered, the graphical method of solution described previously was used. Besides for the untwisted elliptical wing, computations were carried out for wings with five to one taper untwisted, five to one taper with 3° twist, and three to one taper untwisted. All of the wings have straight taper from the center to the tip except that the tips are rounded off from nine tenths of the semi-span to the tip. All of the wings have the same aspect ratio as the elliptical wing. For the case of a tapered wing, the dependence of $\Delta\alpha$, the rolling moment coefficient, and the total lift coefficient upon the drop in infinite aspect ratio lift coefficient at the stall cannot be stated so elegantly as for the untwisted elliptical wing. These quantities have been computed on the basis that $C_{Lmax} = 1.5$ and $C_{L1} = 1.2$.

The highly tapered wing causes a very large downwash near the center of the span and therefore the effective angle of attack of the center is less than that at the wing tip. For this reason the stall begins near the tip of the wing and works in toward the center as the angle of attack is increased. If there is sufficient taper, the center of the wing may never stall within the usually encountered angles of attack. In the discussion of unsymmetrical lift distributions the term, $\Delta\alpha$, has the same meaning for tapered wings as it has for the elliptical wing. It must be remembered that with the elliptical wing all sections reach the effective angle of attack for the stall at the same time while for the tapered wing the stall starts at a single point somewhere near the wing tip. It is found that $\Delta\alpha$ is usually smaller than half of a degree and that therefore the number of unsymmetrical solutions is limited for the highly tapered wing. For the elliptical wing a solution is possible with any given fraction of the wing stalled. However, for the highly tapered wing it is necessary that more than half of the wing be unstalled. The reason for this is that the downwash near the center is so great that even the unsymmetrical solution does not cause sufficient upwash to bring the effective angle of attack to that above the stall. The fraction of the wing that can be stalled depends upon the amount of taper. A tapered wing which is twisted so that the tips are at a smaller geometrical angle of attack than the center acts more nearly like an elliptical wing. The stall starts nearer the center of the wing and the lift coefficient is more nearly uniform across the span than in the case of an untwisted wing of the same planform. Therefore a larger part of the wing can be stalled in the unsymmetrical solution for the twisted wing than for the untwisted.

It is of interest to follow the sequence of events when a wing stalls unsymmetrically. The wing acts quite normally while the angle of attack is increased until some point reaches the angle of attack of stall. If the angle is made just slightly larger, it is possible for one tip of the wing to stall while all of the rest remains unstalled. As the angle is increased the amount of the wing that can be stalled increases until the limit described above is reached. When this limit is reached the value of $\Delta\alpha$ is of the order of magnitude of only a quarter of a degree. A further increase in the angle of attack eliminates the possibility of an unsymmetrical solution and the rolling moment disappears. The rolling moment is a maximum when the wing is as nearly as possible half stalled. The magnitudes of the rolling moment and of $\Delta\alpha$ for any given amount of the wing stalled depend upon the difference between the maximum lift coefficient and the lift coefficient above the stall for infinite aspect ratio. For a highly tapered wing the amount of the wing that can be stalled in an unsymmetrical solution also depends upon this difference in lift coefficients. As the drop in lift coefficient decreases, the amount of wing that can be stalled decreases. When there is no decrease in the lift coefficient at the stall, the unsymmetrical solution of the type described here is no longer possible.

Figures 2 through 5 show the unsymmetrical solutions calculated for four different wings. It is seen that all of the solutions are quite similar in appearance. The effect of twisting the five to one tapered wing is to increase the total lift coefficient of the wing and the rolling moment. At the stall the distribution of lift at the tips is very little changed but the tip stall is delayed so that the center acts at a higher lift coefficient for the twisted wing than for the flat one. This also permits more of the wing to be stalled. Figure 6 shows the way in which the total lift coefficient of the wing decreases after the wing reaches its maximum lift. If the elliptical wing stalls symmetrically, the lift coefficient drops immediately to 1.2 as soon as $\Delta\alpha$ is greater than zero. Figure 7 shows the variation of the rolling moment coefficient with the total lift coefficient of the wing. The curves for the tapered wings are discontinuous. After the maximum rolling coefficient is reached it drops suddenly back to zero. The lift coefficient of the wing just after the breakdown of the unsymmetrical solution has not been calculated.

Figure 8 shows the variation of the rolling moment coefficient with $\Delta\alpha$. It shows that the magnitude of the maximum rolling moment coefficient is about the same for all of the wings investigated. It shows that the range of angles of attack in which the unsymmetrical solution can occur is very small. The value of $\Delta\alpha$ giving the maximum rolling moment coefficient does not change much with the shape of the wing. The difference between the angle of attack at which the stall begins and the angle for zero lift decreases as the taper ratio increases. Twisting a tapered wing increases this difference in angles. Therefore, although the twisted wing can give just as large a rolling moment as the untwisted wing, the stall does not occur as soon and also a larger total lift coefficient is reached. It seems to be the fact that the stall is reached sooner and at a smaller total lift coefficient that causes the highly tapered wing to be more dangerous than one not so highly tapered.

This theory indicates that if a wing is brought slowly up to the stall, the unsymmetrical solution can cause the wing to roll rather violently. However, if the wing is taken through the stall quickly, it is probably possible to get beyond the range of angles of attack in which the rolling moment is possible before the rolling moment has had time to seriously disturb the wing.

The unsymmetrical lift distributions are possible but not necessary. There is nothing in the theory which explains the cause of such a solution. Also there is nothing to indicate which side of a perfectly symmetrical wing will stall first in producing such an unsymmetrical lift distribution. There has been no consideration of engine nacelles and the fuselage. In any actual airplane there is probably enough dissimilarity between the wings to cause the rolling moment to be always in the same direction.

Because of the great amount of time involved in finding an unsymmetrical solution for any particular case, the number of examples for which the calculations have been carried out is very small. The calculated points show quite a bit of scatter and therefore it would be of interest to make more computations with the same wings in order to find out the true nature of the curves. Wings of other shapes should be investigated. It is probable that for an untwisted rectangular wing solutions of the type described here are not possible. It would be interesting to find the wing shape that is the limit of the possibility of an unsymmetrical solution. Other variations to try are the cases of wings with flaps and of wings with a varying profile characteristic across the span. In these cases the type of profile characteristic is not changed but such things as the slope of the lift coefficient curve below the stall, the maximum lift coefficient, and the value of the lift coefficient above the stall could be changed.

Also work should be done in looking for solutions in the cases of lift coefficient vs. effective angle of attack curves of different shapes than the one used in this work.

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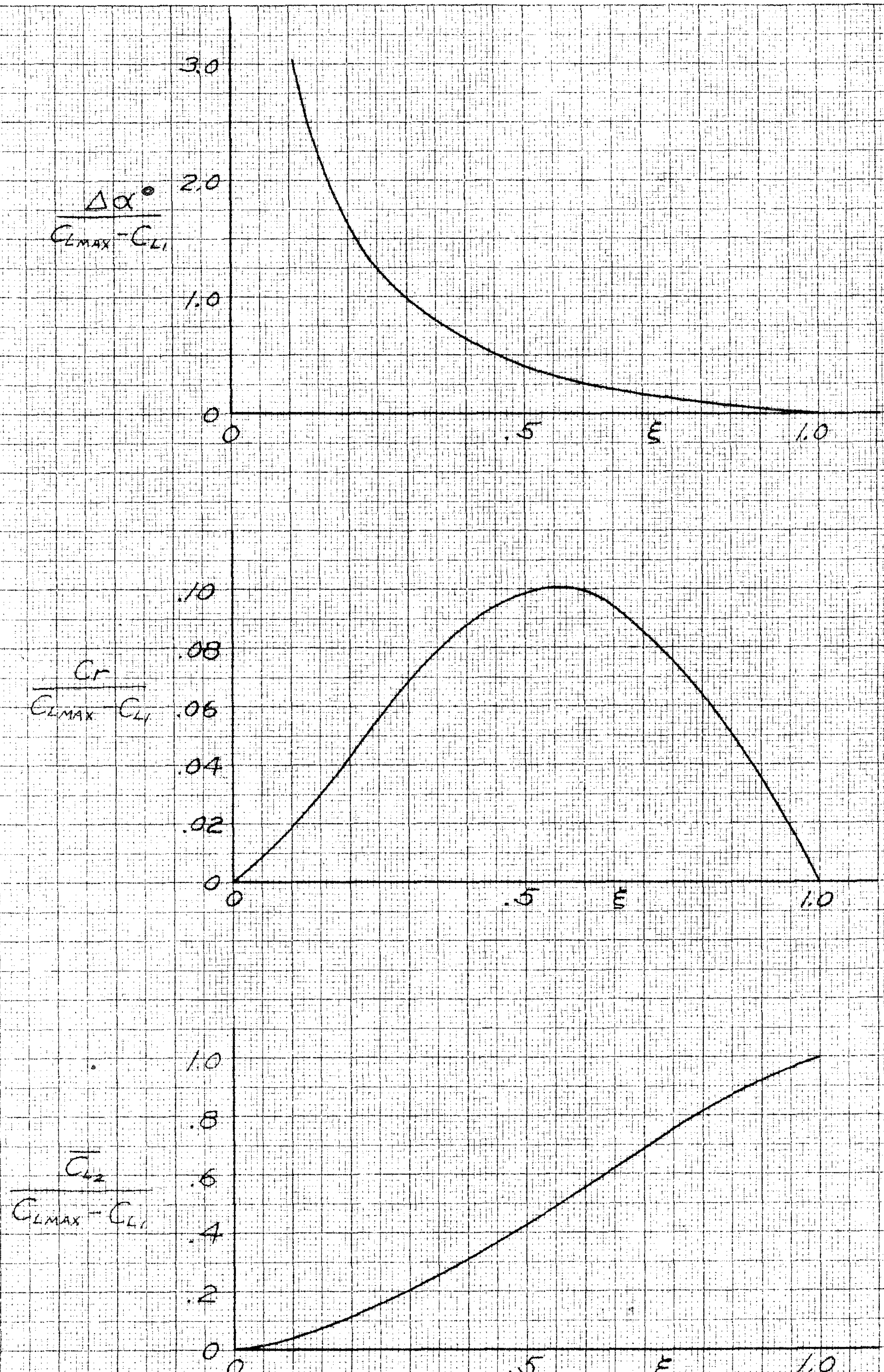


FIG. 1 UNTWISTED ELLIPTICAL WING CHARACTERISTICS

ξ = PART OF WING NOT STALLED

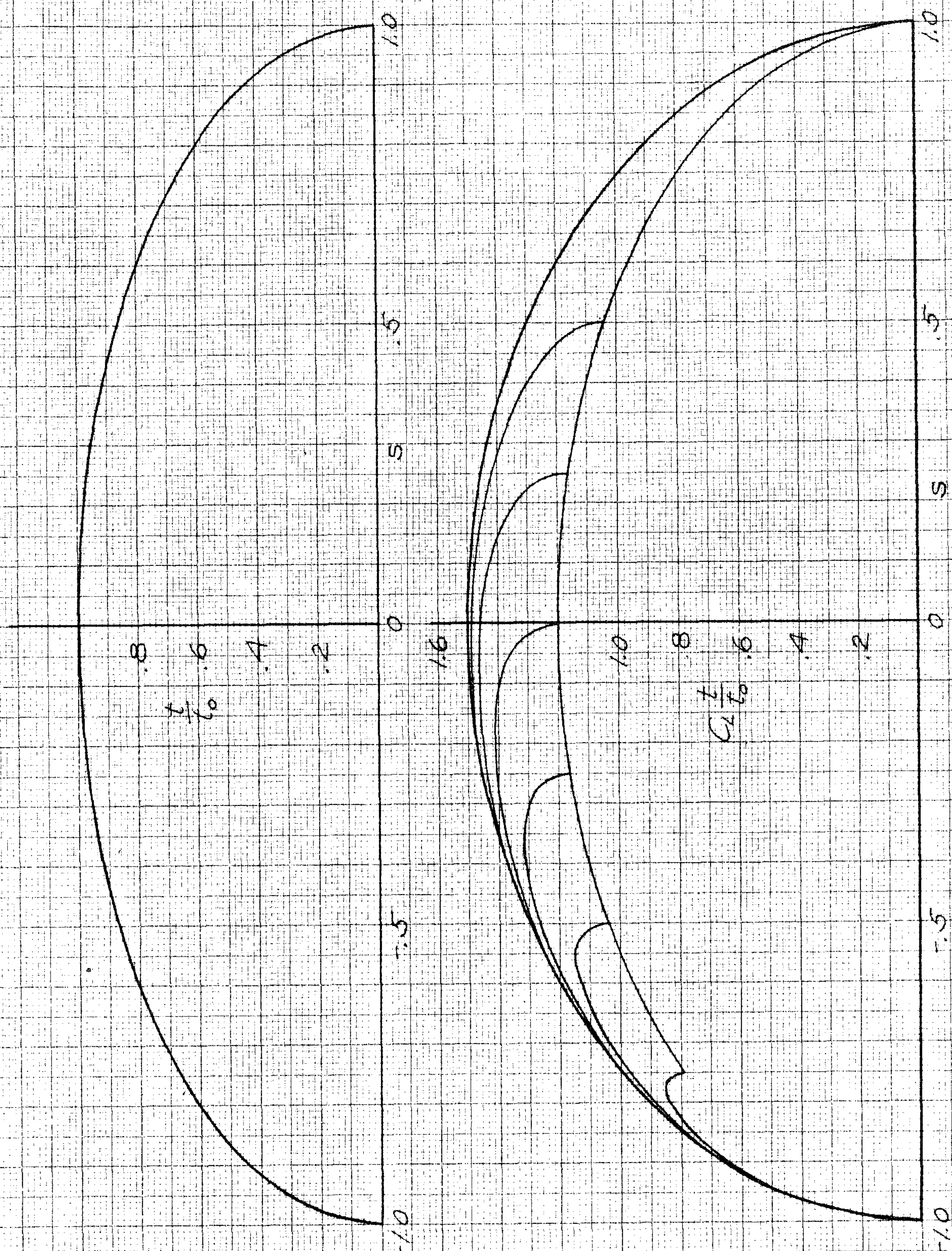


FIG. 2 CHORD AND SPAN LOAD DISTRIBUTIONS

ELLIPTICAL PLANFORM NO TWIST

ASPECT RATIO = 10.19

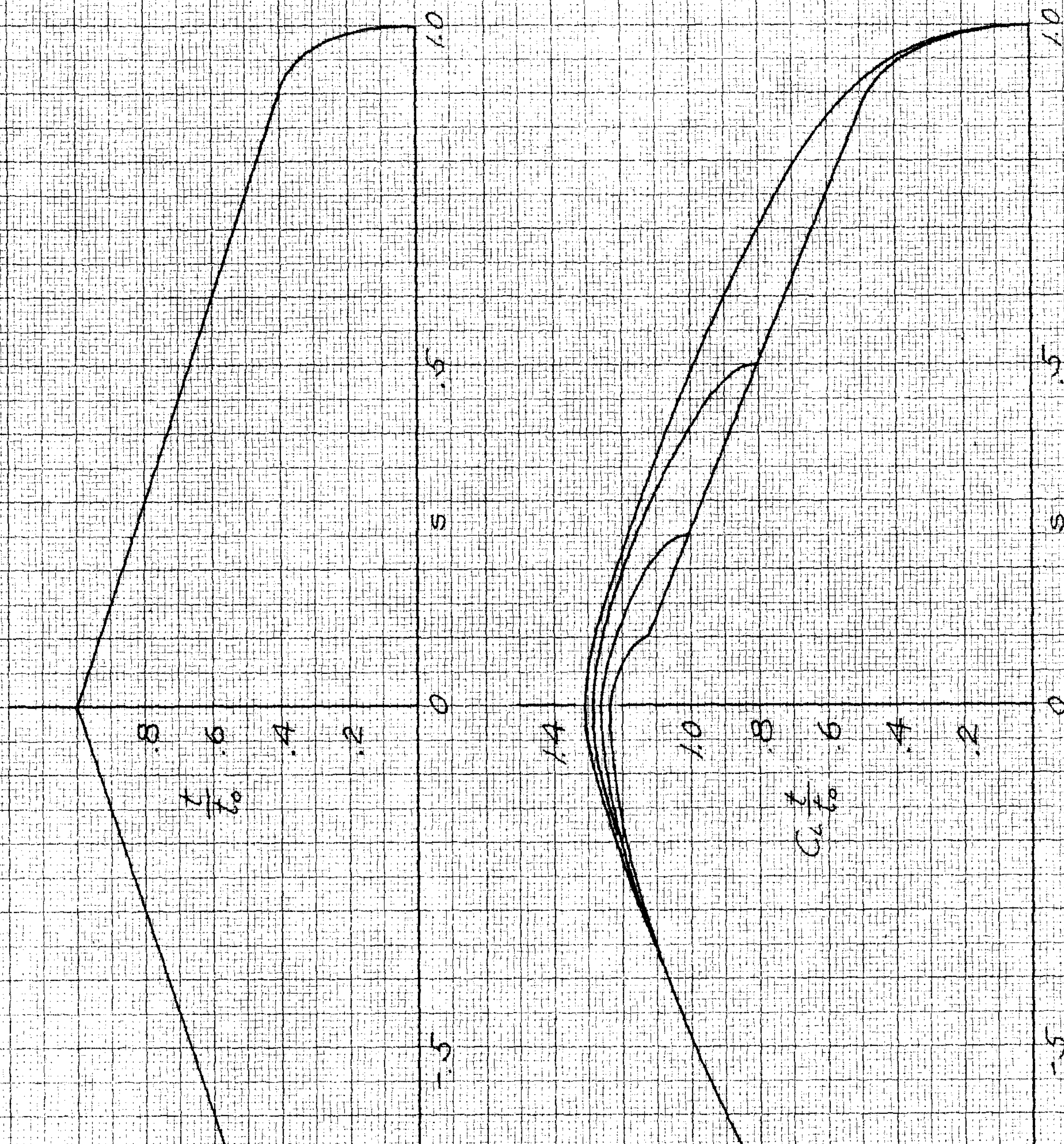


FIG. 3 CHORD AND SPAN LOAD DISTRIBUTIONS

3-1 TAPER NO TWIST

ASPECT RATIO = 10.10

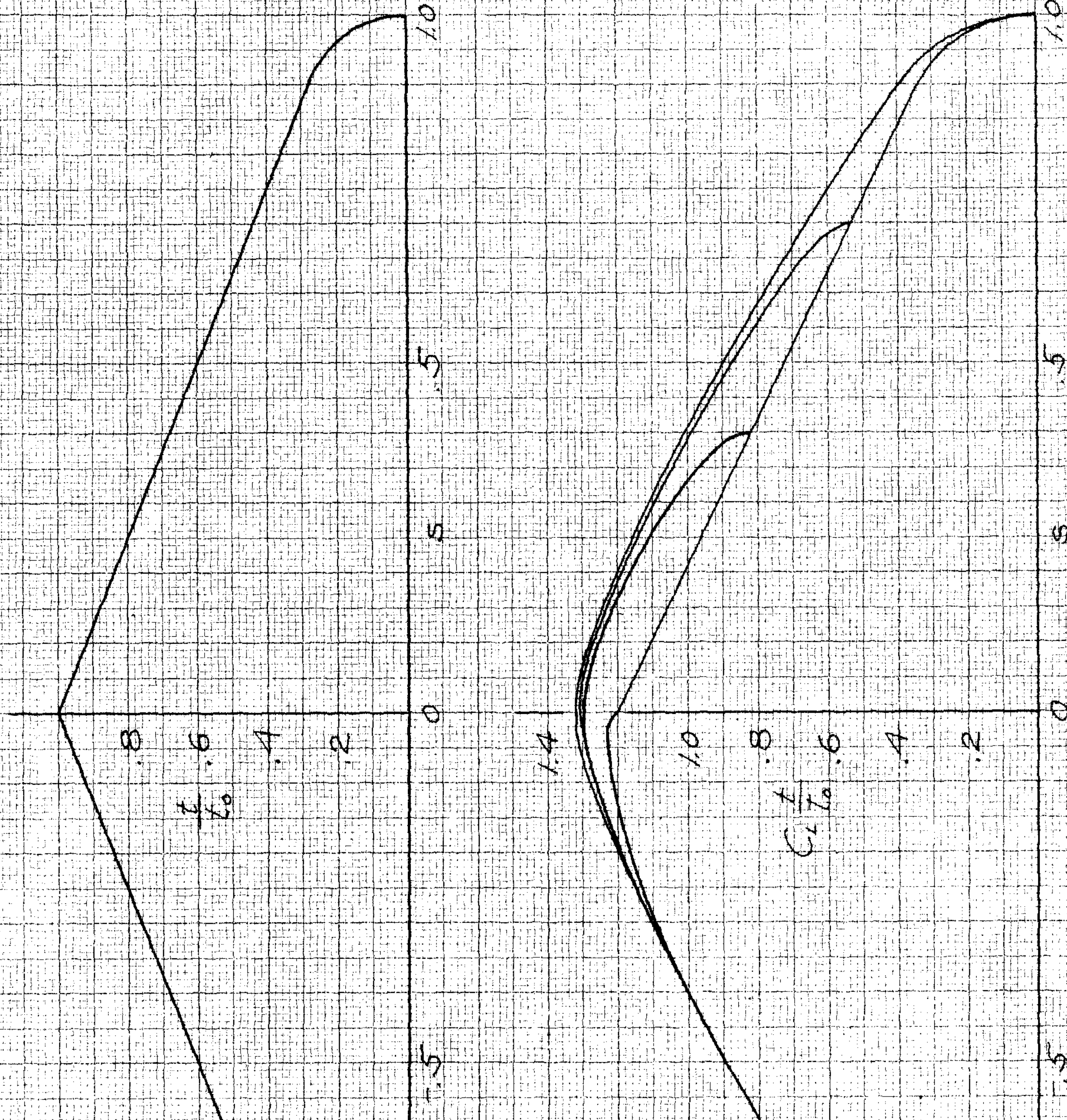


FIG 4 CHORD AND SPAN LOAD DISTRIBUTIONS
5-1 TAPER 3° TWIST
ASPECT RATIO = 10.04

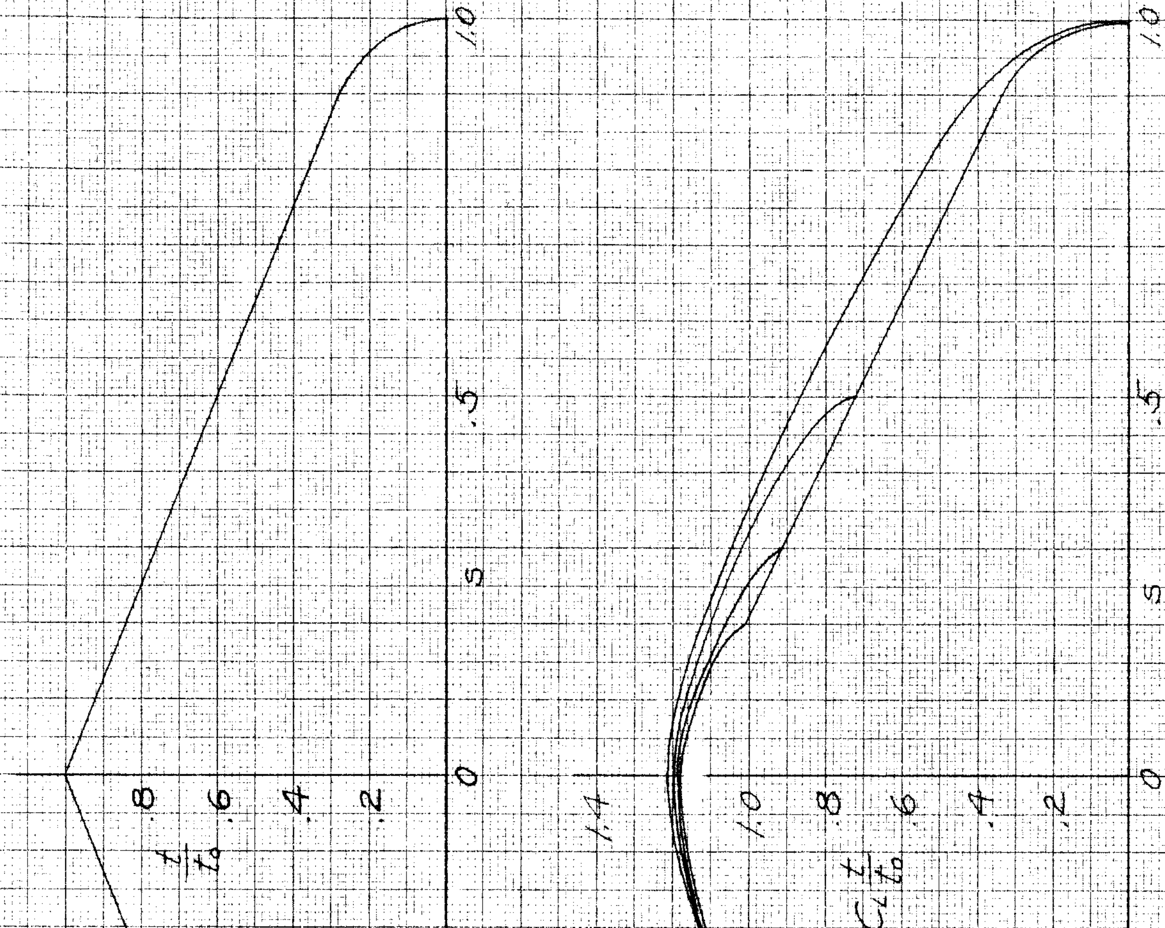
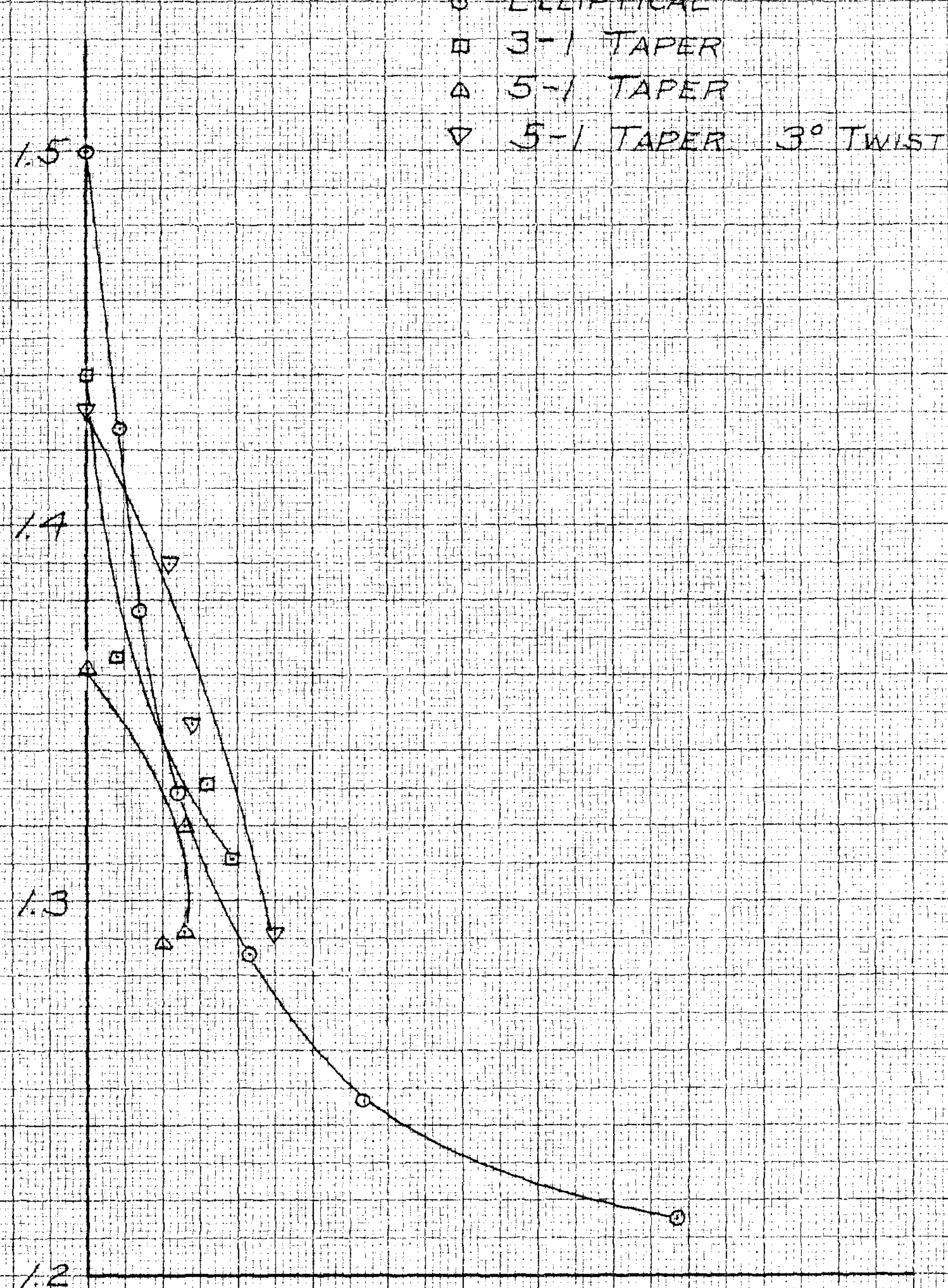


FIG. 5 CHORD AND SPAN LOAD DISTRIBUTIONS
5-1 TAPER NO TWIST
ASPECT RATIO = 10.04



ELLIPTICAL
3-1 TAPER
5-1 TAPER
5-1 TAPER 3° TWIST
1.5
1.4
1.3
1.2
$\overline{C_L}$
0
.5
1.0
$\Delta\alpha°$

FIGURE 6

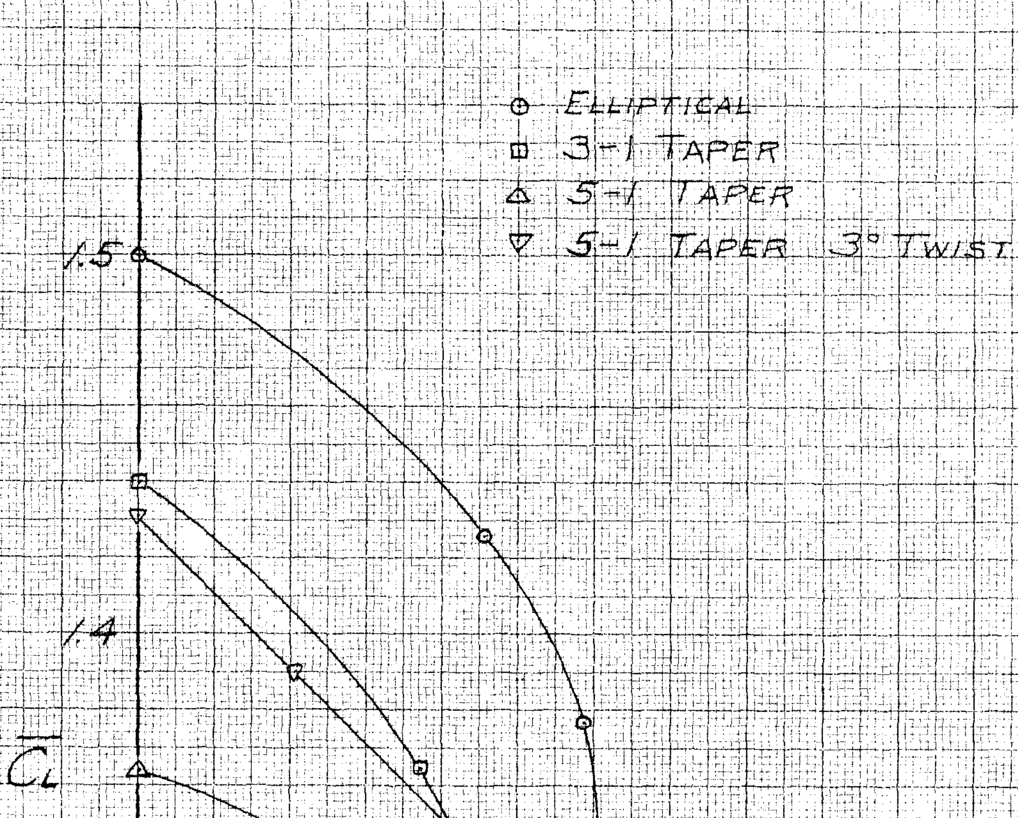

ELLIPTICAL
3-1 TAPER
5-1 TAPER
5-1 TAPER 3° TWIST
1.5
1.4
1.3
1.2
$\overline{C_L}$
0
.01
.02
.03
.04
C_r

FIGURE 7

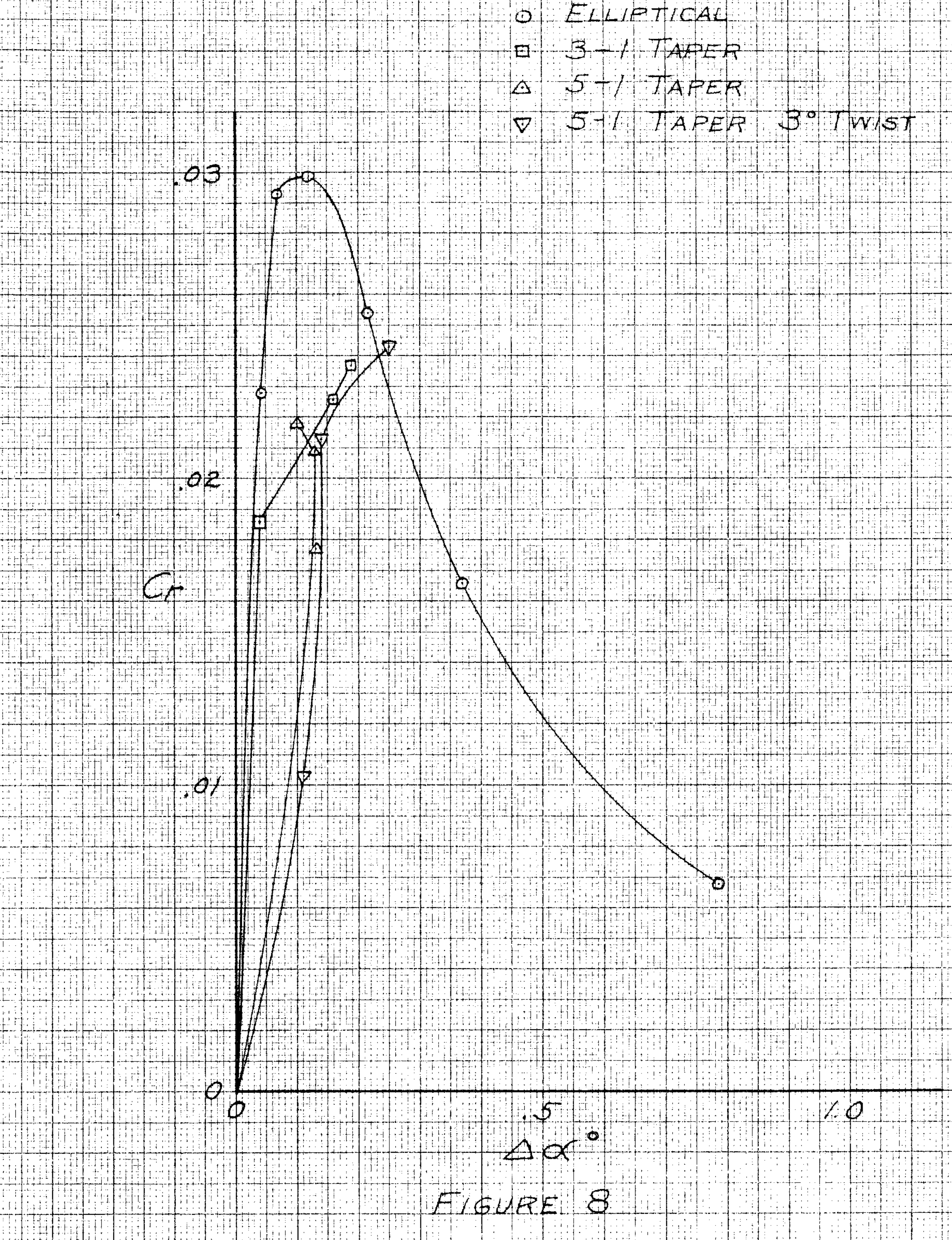

ELLIPTICAL
3-1 TAPER
5-1 TAPER
5-1 TAPER 3° TWIST
.03
.02
.01
0
C_r
0
.5
1.0
$\Delta\alpha°$

FIGURE 8