THE FUNDAMENTAL DIFFERENTIAL EQUATIONS AND THE BOUNDARY CONDITIONS FOR HIGH SPEED SLIP-FLOW, AND THEIR APPLICATION TO SEVERAL SPECIFIC PROBLEMS

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ABSTRACT

The differential equations of motion and the associated boundary conditions for the slip-flow regime of fluid mechanics are derived from the point of view of the kinetic theory of non-uniform gases. The slip-flow regime comprises the flow of gases whose molecular mean free path is smaller than but not negligible relative to the macroscopic dimension characterizing the gas flow.

A systematic review is presented of the methods of Hilbert and Burnett for obtaining a successive approximation solution to the Boltzmann integro-differential equation. The complete second approximation to the molecular velocity distribution function is calculated for the molecular model of Maxwell. This molecular distribution function is employed for the derivation of the macroscopic differential equations of motion and the associated boundary conditions. It is shown that the same number of boundary conditions are required for slip flows as for gas-dynamical flows, although the differential equations of motion for slip flows are of higher order than those of continuum gas-dynamics. Expressions for the second approximations to the slip velocity and temperature jump are obtained.

The general equations obtained are applied to three specific problems: the propagation of sound waves in rarefied gases, high-speed Couette flow of a rarefied gas, and slip-flow between concentric cylinders in relative rotary motion. It is found that the rarefaction of a gas increases the damping of sound waves, whereas the propagation speed differs from the ordinary adiabatic sound velocity by less than two percent. The Couette flow solution indicates that the slippage of gas and the temperature discontinuity at a solid boundary may reduce the gas-dynamical friction coefficient and heat transfer, respectively, by ten percent under approximate conditions. When applied to the flight of aircraft through the rarefied atmosphere, the theory presented is applicable to an altitude range from 100,000 to 300,000 feet.

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PART I

INTRODUCTION AND SUMMARY

In a recent paper, H. S. Tsien (Reference 1.1) pointed out the need for the further investigation of the mechanics of rarefied gases in order to gain an understanding of the phenomena encountered in the high altitude flight of aircraft and in flow processes of low density gases.

The mechanics of rarefied gases differs from the usual gas-dynamics in that the gas cannot be treated a priori as a continuum, so that the effect of the random motion of individual molecules must be considered. The parameter which measures the relative importance of the microscopic molecular motion to the macroscopic mass motion of the gas is the ratio of the mean free path of the gas, ℓ, to the smallest intrinsic linear dimension L, which characterizes the gas flow. Using this parameter ℓ/L, fluid mechanics is most naturally subdivided into the following "realms" (Reference 1.1):

(1) $\frac{\ell}{L} < \frac{1}{100}$; the molecules are so closely packed that the gas is effectively a continuum. This is the realm of gas-dynamics, in which the fluid adheres to any contacting solid surface.

(2) $\frac{1}{100} < \frac{\ell}{L} < 1$; the mean free path ℓ is small but not negligible relative to the macroscopic dimension L. The appreciable number of collisions between gas molecules and a solid surface cause a slippage of the gas flowing over such a surface. Hence this realm is called the "slip-flow" regime.

(3) $1 < \frac{\ell}{L} < 10$; the collisions between gas molecules and

bounding surfaces are of the same order of frequency as the intermolecular collisions. This region, which has as yet been investigated only experimentally, will be called the "intermediate" realm.

(4) $\frac{l}{L} > 10$; intermolecular collisions are negligible relative to collisions between gas molecules and solid surfaces, hence this realm is designated as the "free molecule flow".

Realms (1), (3), and (4) are also called the Poiseuille, Kundt and Warburg, and Knudsen regions after the physicists who first investigated gas flow through capillaries in these respective realms of fluid mechanics.

The present investigation is confined entirely to the realm of the slip-flow, the primary objective being the derivation and application of the mathematical boundary conditions which are required for the determination of particular flow solutions from the general differential equations of motion for slip flow. The equations of motion for slip flow differ from those of gas-dynamics only in the explicit form the viscous stresses and the heat conduction of the gas. Various approximate expressions for the stresses and heat conduction in a rarefied gas have been deduced by means of the kinetic theory of non-uniform gases by mathematicians and physicists, notably Maxwell, Enskog, Chapman, and Burnett. In fact, Burnett has developed a method which may be used, in principle at least, to deduce the stresses and heat conduction for slip slow to an arbitrary degree of approximation.

The boundary conditions, however, have been deduced only to a first approximation for certain special cases, principally by Maxwell, Knudsen, von Smoluchowski, Basset, and Epstein. A general and more rigorous method for deducing the boundary conditions for slip flow to an arbitrary degree of approximation is presented in Part V of this thesis. Quite analogous to the viscous stresses and the heat conduction, the boundary conditions for the macroscopic motion of the gas are obtained by a process of integration of the momentum and energy exchange between the individual gas molecules and the solid boundaries which confine the gas. This procedure in turn requires the knowledge of the complete molecular velocity distribution function which characterizes the non-uniform gas. Consequently, it is first necessary to calculate a second approximation to the velocity distribution of a non-uniform gas. This is done in Part II using the method developed by D. Burnett.

The kinetic theory of non-uniform gases has been developed during the past eighty years beinning with the mathematical investigations of Maxwell and Boltzmann. The basic method of successive approximation that is employed both for the solution of the differential equations and the evaluation of the boundary conditions for slip flow rests ultimately on the nature of the Boltzmann integro-differential equation whose solution is the desired molecular velocity distribution function of the non-uniform gas. As, to the author's knowledge, there does not exist a unified treatment of the derivation of the complete distribution function from the Boltzmann equation, such an exposition is presented in Part II, where the method of solution discovered by D. Hilbert and its adaption to a practical computation by D. Burnett are explained. The significance of the "molecular model" which is necessarily employed in the calculations of the kinetic theory is also discussed. Part II is concluded with the results of the author's calculation of certain coefficients which completely determine the molecular velocity distribution function, to a second approximation. Certain details of this calculation are given in Appendix III.

The differential equations for the macroscopic motion of a non-uniform gas are given in Part III. The fact that only the stresses and thermal conduction depend on the degree of rarefaction of the gas is emphasized by means of a derivation of these differential equations of motion due to Maxwell and Burnett. It is found that the use of the first approximation to the molecular velocity distribution function in the calculation of the viscous stresses results in the equations of Navier and Stokes of gas-dynamics, whereas the application of the higher orders of approximation to the distribution function yields differential equations of motion which are of increasingly higher order. The validity of the third order partial differential equations that are obtained from the second approximations to the stresses and heat flux, as calculated by Burnett and Chapman, is seen to depend on the relative values of the mean free path of the gas, the characteristic linear dimension, and the Mach number difining a particular slip flow.

Prior to the derivation of the boundary conditions for the slip flow regime, Part IV of this thesis is devoted to the application of the macroscopic differential equations of motion to the particular problem of the propagation of plane sound waves in rarefied gases. This one-dimensional problem, whose solution does not require any boundary conditions, is of interest in connection with the evaluation of atmospheric data at high altitudes by means of the method of anomalous sound propagation. It is shown that the speed of sound propagation and the damping coefficient, for a particular gas depend only on a single parameter, the Reynolds number of sound propagation which is defined in terms of the wave length and the adiabatic propagation speed of the sound wave. For decreasing values of this Reynolds number, a slight increase in the propagation speed, accompanied by a pronounced increase in the damping coefficient, is found. However, even for the extreme conditions to which the theory of Part IV may be applied, such as correspond to the propagation of a wave of 1000 cycles per second at an altitude of sixty miles, the propagation speed differs from the normal adiabatic sound velocity by only two per cent.

It was stated earlier that a decrease in the density of the rarefied gas, requiring for its mathematical description higher order approximations to the viscous stresses, results in an increase in the order of the partial differential equations of motion. This leads one to the expectation that the number of boundary conditions required for the complete evaluation of a slip flow problem should likewise depend on the degree of rarefaction of the gas. However, in the first section of Part V it is shown on both physical and mathematical grounds, that the number of physical boundary conditions required for a slip flow problem is effectively the same as for the corresponding flow in the realm of gas-dynamics.

In the gas-dynamic regime, the flow of a gas over a solid surface is specified by three boundary conditions which require that there shall be no relative normal or tangential velocity between the gas and the solid surface, and that the layer of gas adjacent to the boundary shall have the temperature of the boundary. In the slip-flow regime, the condition of zero relative normal velocity still holds. However the relative tangential or slip velocity is no longer zero but is a definite function of the velocity, temperature, and pressure gradients of the gas layer immediately adjacent to the wall. Similarly the gas temperature differs from the wall temperature by an amount, referred to as the "temperature jump", which is again a function of these velocity, temperature, and pressure gradients.

The determination of the boundary conditions for slip flow is thus reduced to the calculation of appropriate expressions for the slip velocity and the temperature jump. A general method from which such expressions may be calculated, again to an arbitrary degree of approximation, is presented in Part V. This method applies the laws of conservation of mass, momentum,and energy to the infinitesimal layer of gas adjacent to the solid surface, referred to as the "sub-layer", and uses the non-uniform molecular velocity distribution function as developed in Part II.

The first approximations for the slip velocity and temperature jump, as obtained by the use of the first approximation to the velocity distribution agree with the results deduced analytically and verified experimentally by previous investigators for low speed slip flows. The second approximation to the slip velocity and temperature-jump, which are required in conjunction with the second approximation to the viscous stresses and heat conduction for higher speed slip-flows, are also calculated in Part V of this thesis. Certain details of these lengthly calculations are presented in Appendix IV. It is found, as in the case of the viscous stresses and heat flux, that the second approximations to the slip velocity and temperature jump are functions of the second partial derivatives or products of two first order partial derivatives of the velocity, temperature,and pressure of the gas in the immediate vicinity of the solid boundary.

Once the general boundary conditions have been calculated by means of the kinetic theory of non-uniform gases, the method of solution of a particular slip flow problem is quite analogous to the gas-dynamical case, in that the macroscopic velocity, temperature,and pressure fields are obtained by solving the differential equations of motion with a given set of boundary conditions. This process, which does not involve the kinetic theory explicitly, is illustrated by two related concrete examples that are treated as Parts VI and VII of this thesis.

The solution for the Couette flow of a rarefied gas is obtained in Part VI. The arbitrary constants of the general solution of the diferential equations for two-dimensional parallel motion are evaluated from the boundary conditons by an iteration process. The zero order solutions correspond to Couette flow in the gas-dynamic realm, whereas the first and second approximations represent the modification of the basic gas-dynamic flow pattern caused by the slip and temperature jumps occurring at both the fixed and moving plates of the Couette flow. These slip effects, which result in a reduction of both the viscous drag and heat transfer between the parallel plates, are seen to be functions of the two parameters $\frac{M}{Re}$ and M^2, where M and Re are the Mach and Reynolds numbers of the Couette flow respectively. The parameter $\frac{M}{Re}$ is equivalent to the rarefaction parameter $\frac{\ell}{L}$, by means of which the realms of fluid mechanics are distinguished, where in the present case the characteristic length of the problem L represents the distance between the walls of the Couette flow.

The results of the Couette flow computation have been used to obtain an estimate of the skin friction of a flat plate in high speed flight in a rarefied atmosphere. The calculations indicate, for example, that for a chord of one foot, Mach number of 3, and altitude of 250,000 feet, the slip of the air over the plate reduces the gas-dynamic value of the friction coefficient by approximately 10 per cent.

The Couette flow calculation illustrates the application of the slip flow boundary conditions to the case of planar walls. The boundary conditions, deduced in Part V, are however applicable to curved surfaces as well. This is demonstrated in Part VII, where the flow is computed for a rarefied gas confined in the annular gap between two concentric cylinders of which the outer one is stationary, while the inner one is rotating at high speed. The calculation shows that the curvature of the streamlines with the resultant centrifugal pressure field tends to exaggerate the slip and temperature jump effects which are found in the Couette flow under equivalent conditions.

The solutions of the three specific problems, as treated in Parts IV, VI and VII, point to the conclusion that the high speed slip flow of a rarefied gas does not differ materially from the slip flow at lower speeds, which is adequately calculated by means of the first order theory. In fact, it is found, in all of the cases considered, that the second order slip flow theory results in flow patterns which are intermediate to those calculated from the gas-dynamic theory and first order slip-flow theories, respectively. This conclusion is discussed more fully in Part VIII.

An experimental check of these conclusions appears desirable, especially so since the theoretical analysis given here is based on a number of simplifying assumptions such as are generally made in order to make the kinetic theory treatable. One such experiment might consist of a study of the concentric cylinder flow whose analytical treatment is presented in Part VII.

It should be pointed out that the methods of solution presented in this thesis are quite general in that they are not confined to the second order solutions which have been developed in detail. It is believed, however, that the results to be expected from a third order theory, and the simplifying assumptions already made, do not warrant the immense effort required for the preliminary calculation of the third approximation to the molecular velocity distribution function.

The symbols and notations used in the text are listed in Appendix I. The mathematical formulae to which frequent reference is made in the course of the analyses are summarized in Appendix II.

PART II

THE MOLECULAR VELOCITY DISTRIBUTION FUNCTION

The early form of the kinetic theory of gases, as developed by Joule and Clausius was limited by the assumption that all of the molecules composing the gas move with equal speeds. The experimental investigations of the phenomena of heat transfer and viscosity, however, established the existence of intermolecular collisions and hence the finite size of the molecules. A simple argument concerning the conservation of linear momentum during an "elastic collision" of two molecules shows that, even if at one instant of time all of the molecules are moving with equal speeds, the process of elastic collision will immediately produce a variation in the speeds of the molecules. To describe this state of molecular chaos involving a destribution of molecular velocity and of the direction of this velocity, statistical methods must be employed.

2.1 Formulation of the Problem

The following notation will be employed to describe the state of the gas in a statistical manner: Let x,y,z, be coordinates of a point with respect to a set of Cartesian axes fixed in space. The velocity components of a molecule with respect to these axes are ξ , η , ζ (see figure 2.1). The velocity distribution function $f(\xi, \eta, \zeta; x,y,z;t)$ is then defined in such a manner, that the number of molecules per unit volume, dN, which at a time t have coordinates lying between x and x+dx, y and y+dy, z and z+dz, and whose velocity components lie between the limits ξ and $\xi + d\xi$, η and $\eta + d\eta$, ζ and $\zeta + d\zeta$, is given by

$$dN = \nu(x,y,z;t) \cdot f(\xi,\eta,\zeta;x,y,z;t) \cdot (dx\,dy\,dz) \cdot (d\xi\,d\eta\,d\zeta) , \tag{2.1}$$

where ν (x,y,z;t) is the total number of molecules per unit volume at the point (x,y,z) and at time t. From the definition of ν it follows that

$$\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} d\zeta \cdot f(\xi, \eta, \zeta ; x, y, z ; t) \;=\; 1 \tag{2.2}$$

If m is the mass of one molecule of the gas, its density ρ is given by

$$\rho = m\nu \tag{2.3}$$

The problem of finding the mathematical conditions to which the distribution function f must be subjected in order that it should represent the physical behaviour of the gas molecules was investigated successfully by Boltzmann in 1872. He showed that for a gas whose molecules are subjected to the action of (a) the intermolecular forces of binary elastic collision, and (b) an external field of force whose components per unit mass are F_x, F_y, F_z, the velocity distribution function f must be a solution of the integro-differential equation (Reference 2.1)

$$\left[\frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x} + \eta_1 \frac{\partial}{\partial y} + \zeta_1 \frac{\partial}{\partial z} + F_x \frac{\partial}{\partial \xi_1} + F_y \frac{\partial}{\partial \eta_1} + F_z \frac{\partial}{\partial \zeta_1} \right] (\nu f)$$

$$= \nu^2 \int_{-\infty}^{\infty} d\xi_2 \int_{-\infty}^{\infty} d\eta_2 \int_{-\infty}^{\infty} d\zeta_2 \int_{\infty} dg \cdot (f_1' f_2' - f_1 f_2) \cdot G_{12} \,, \tag{2.4}$$

where

$$\begin{aligned} f_1 &= f(\xi_1, \eta_1, \zeta_1 ; x, y, z ; t) \\ f_2 &= f(\xi_2, \eta_2, \zeta_2 ; x, y, z ; t) \\ f_1' &= f(\xi_1', \eta_1', \zeta_1' ; x, y, z ; t) \,, \text{ etc.} \end{aligned}$$

This so-called Boltzmann integral equation is non-linear and has the following physical significance. The differential expression on the left side of equation (2.4) represents the change in the number, (νf_1), of molecules at the point (x,y,z) having velocity components (ξ_1, η_1, ζ_1) as these molecules move to the point (x+dx, y+dy, z+dz) during the infinitesimal time dt. Since the number of molecules having prescribed velocity components can be further altered by the process of elastic collisions, the change in (νf_1) can also be computed by considering the collisions of the molecules (ξ_1 , η_1, ζ_1) with all molecules (ξ_2 , η_2, ζ_2) during the same time interval dt. The net effect of all possible collisions is represented by the integral expression on the right side of equation (2.4). The velocities of the molecules before collision are represented by the unprimed velocity components, whereas the primed quantities are the values of corresponding velocity components, after collision. The form of the quantities G_{12} and dg, which represent the effect of the orientation of the line of centers of the two molecules at the instant of closest approach, depends on the law of interaction between molecules, and will be discussed later. (See also Appendix III, Section 3)

Equation (2.4) was first used by Boltzmann to calculate the distribution function $f_0(\xi, \eta, \zeta)$ for a uniform gas in steady equilibrium and in the absence of external force fields. For this special case the differential expression in (2.4) vanishes and the Boltzmann equation will be satisfied by the sufficient condition

$$(f_1' f_2' - f_1 f_2) = 0 \qquad (2.5)$$

This condition together with the laws of conservation of linear momentum and kinetic energy for elastic impacts results in the well-known Maxwell velocity distribution

$$f_0 (\xi, \eta, \zeta) = \left(\frac{hm}{\pi}\right)^{\frac{3}{2}} e^{-hm(\xi^2+\eta^2+\zeta^2)} \quad , \qquad (2.6)$$

where

$$h = \frac{1}{2kT} \quad . \tag{2.7}$$

Here T is the absolute temperature and k the Boltzmann constant which is the universal gas constant referred to one molecule.

By means of his "H - theorem" Boltzmann then showed that (a) equation (2.5) is also a necessary condition, so that the Maxwell distribution is unique for a uniform gas, and (b) that if a gas is given an arbitrary initial velocity distribution, the gas will return to a Maxwell distribution by means of the process of elastic collisions, if no external influence are present. Thus, Boltzmann succeeded in showing from dynamic considerations that a perfect gas actually satisfies the statistical hypotheses upon which Maxwell's original derivation for f_o had been based.

In the general case, when the distribution function f is a function of the space coordinates and the time, equation (2.5) does not hold, and for the evaluation of the integral appearing in the Boltzmann equation the quantities G_{12} and dg must be known. As mentioned above, this amounts to the selection of a "molecular model" with which is associated a definite mathematical law describing the interaction between molecules.

Throughout the development of the kinetic theory of gases, the molecular models that were used have grown in complexity, ranging from Clausius's smooth, rigid, elastic spherical molecule to models whose mass is concentrated at the origin of a spherically symmetric force field. Whereas the above molecules are "smooth" and capable only of interchanging translational kinetic energy, the "rough" spherical molecule used by Bryan can also account for the exchange of rotational kinetic energy when two molecules collide. A detailed discussion of these and other molecular models may be found in Reference (2.2). The behaviour of actual gas molecules will be, perhaps, most closely approximated by the most complicated molecular models, for which, however, most of the calculations cannot be carried through in practice. Furthermore, from the point of view of modern molecular physics, the validity of models, arbitrarily constructed without proper consideration of the molecular structure and the interaction according to the principles of quantum mechanics, is highly questionable. Fortunately, Bryan's study shows that the introduction of "roughness" of the molecules does not alter essentially the result obtained from smooth molecules. In other words, the difference between the different molecular models lies mainly in the modification of the numerical coefficients in the final equations, but not their form. These numerical coefficients will be, in any case, finally checked and determined by experiments. Therefore, the most fruitful results of the kinetic theory of gases have been and can be obtained with the "smooth" molecular model consisting of a spherically symmetric, repulsive field of force. All of the derivations of this paper will be based on this molecular model.

If the centers of two molecules are separated by a distance r, the force of repulsion, F, will be given by

$$F(r) = K m^2 r^{-n} \quad , \qquad n > 0 \tag{2.8}$$

where m is the mass of the molecule and K is a numerical constant. This force law contains the rigid elastic spherical molecule as the limiting case for $n \rightarrow \infty$. When n = 5 the molecular model is called the "Maxwellian molecule" which was first used by J. C. Maxwell because of its relative mathematical simplicity.

It is well known that the exponent n of the force law is related to the temperature variation of the coefficient of viscosity of the gas.

In particular, it is found experimentally that for many gases the coefficient of viscosity, μ , can be approximated by the expression,

$$\left(\frac{\mu}{\mu_0}\right) = \left(\frac{T}{T_0}\right)^s , \qquad 0.5 < s < 1.0 \tag{2.9}$$

in which case (Reference 2.3) the relation between s and n is simply

$$n = 1 + \frac{4}{2s-1} . \tag{2.10}$$

Evidently the Maxwellian and spherical molecules correspond to the limits on s given in equation (2.9). This is also shown graphically in figure (2.2) where F(r) is plotted to an arbitrary scale for various values of n. The agreement between theory and experiment represented by equation (2.9) is the principal justification for the use of the point center of force molecule defined by equation (2.8).

2.2 Hilbert's Scheme of Solution for the Boltzmann Equation

Before proceeding with the discussion of the general solution of the Boltzmann equation, it is convenient to transform equation (2.4) by referring the velocity components of the molecule to a set of coordinate axes moving with the mean mass velocity of the gas. If u,v,w are the x,y,z components, respectively, of the mean mass velocity of the gas, then by the definition of the velocity distribution function

$$u(x,y,z;t) = \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} d\zeta \cdot \xi \cdot f(\xi,\eta,\zeta;x,y,z;t) , \tag{2.11}$$

with analogous expressions for v and w. The velocity components of the molecules relative to the mean motion of the gas will be denoted by U,V,W (see figure 2.1), so that

$$\left.\begin{aligned} \xi &= u + U \\ \eta &= v + V \\ \zeta &= w + W \end{aligned}\right\} \qquad (2.12)$$

The vector sum of U,V,W is called the "peculiar velocity", C, of a molecule and is, of course, given by

$$C^2 = U^2 + V^2 + W^2. \qquad (2.13)$$

After substituting equation (2.12) into equation (2.4), one may write the Boltzmann equation in the symbolic form

$$\frac{\mathcal{D}}{\mathcal{D}t}(\nu f_1) = \nu^2 \iint (f_1' f_2' - f_1 f_2)\cdot G_{12} \cdot d\underset{\sim}{C}_2 \cdot dg \qquad (2.14)$$

where

$$\frac{\mathcal{D}}{\mathcal{D}t} \equiv \Big[\frac{\partial}{\partial t} + (u+U_1)\frac{\partial}{\partial x} + (v+V_1)\frac{\partial}{\partial y} + (w+W)\frac{\partial}{\partial z}$$

$$+ F_x \frac{\partial}{\partial U_1} + F_y \frac{\partial}{\partial V_1} + F_z \frac{\partial}{\partial W_1} \Big],$$

$$\iint (f_1' f_2' - f_1 f_2)\cdot G_{12} \cdot d\underset{\sim}{C}_2 \cdot dg \equiv \int_{-\infty}^{\infty} dU_2 \int_{-\infty}^{\infty} dV_2 \int_{-\infty}^{\infty} dW_2 \int_{\infty} dg \cdot (f_1' f_2' - f_1 f_2)\, G_{12}, \qquad (2.15)$$

and

$$f_1' = f\,(U_1', V_1', W_1' \,;\; x, y, z \,;\, t), \text{ etc.}$$

The Maxwell distribution function, equation (2.6), can be written as

$$f_0\,(U, V, W) = \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} e^{-\frac{m}{2kT}(U^2+V^2+W^2)}$$

$$= \left(\frac{hm}{\pi}\right)^{\frac{3}{2}} e^{-hmC^2}, \qquad (2.16)$$

because in the derivation of equation (2.6) it was implicitly assumed that the gas be at rest on the average.

The first rigorous mathematical proof for the existence of a unique solution $f(U,V,W;x,y,z;t)$ of the Boltzmann equation (2.14) was given by D. Hilbert (Reference 2.4) in 1912. As the fundamental ideas involved in Hilbert's proof are also required in obtaining the solution f, they will be sketched below.

Hilbert used the following expansion for the distribution function:

$$\nu f = \frac{f_0}{\lambda}\left[1 + \lambda\psi_1 + \lambda^2\psi_2 + \dots + \lambda^n\psi_n + \dots\right], \qquad (2.17)$$

where λ is an arbitrary, but constant parameter, $\psi_n = \psi_n(U,V,W;x,y,z;t)$, for $n = 1,2,\ldots$, and f_0 is the Maxwell distribution of equation (2.16). The basic concept here is thus that of perturbation theory. The terms of the series are perturbations or corrections to the basic Maxwell distribution. Mathematically, the essential improvement to be made is the linearization of the original non-linear integral equation. The resultant set of linear integral equations can then be treated with the aid of the theory of linear integral equations developed by Hilbert himself. Analogous to equation (2.4), the notation

$$\psi'_{n1} = \psi_n(U'_1, V'_1, W'_1; x, y, z; t)$$

will prove useful. Substitution of Hilbert's expansion (2.17) into Boltzmann's equation (2.14) with subsequent collection of terms in ascending powers of the parameter results in

$$\frac{1}{\lambda^2}\left[\,0-\iint\{f_{01}'f_{02}'-f_{01}f_{02}\}\,G_{12}\,d\underset{\sim}{C}_2\,dg\,\right]$$

$$+\frac{1}{\lambda}\left[\,\frac{\mathcal{D}}{\mathcal{D}t}(f_{01})-\iint\{\,f_{01}'f_{02}'(\Psi_{11}'+\Psi_{12}')-f_{01}f_{02}(\Psi_{11}+\Psi_{12})\}\,G_{12}\,d\underset{\sim}{C}_2\,dg\right]$$

$$+\ \left[\,\frac{\mathcal{D}}{\mathcal{D}t}(f_{01}\Psi_{11})-\iint\{(f_{01}'\,f_{02}'\,\Psi_{11}'\,\Psi_{12}'-f_{01}\,f_{02}\,\Psi_{11}\Psi_{12})\right.$$

$$\left.+\,(f_{01}'f_{02}'\,[\,\Psi_{21}'+\Psi_{22}'\,]-f_{01}f_{02}\,[\,\Psi_{21}+\Psi_{22}\,])\}\cdot G_{12}\,d\underset{\sim}{C}_2\,dg\right]$$

$$+\lambda\left[\,\frac{\mathcal{D}}{\mathcal{D}t}(f_{01}\Psi_{21})-\iint\{\ldots\}\,G_{12}\,d\underset{\sim}{C}_2\,dg\right]+\lambda^2[\ \ ]+\ \ldots\ =0\ . \qquad (2.18)$$

Since this equation must be satisfied for arbitrary values of λ , all of the coefficients of λ^n , i.e. the quantities in the square brackets, must vanish identically. The coefficient of $\frac{1}{\lambda^2}$ is automatically zero, since the Maxwell distribution, f_o, satisfies equation (2.5) from which it was derived. Equating the coefficient of $\frac{1}{\lambda}$ to zero results in an integral equation for the unknown function Ψ_1 , since the differential expression in this coefficient is a known function of the velocity components. Assuming that the solution for Ψ_1 can actually be obtained, the coefficient of λ^{o} = 1 then furnishes an integral equation for the unknown function Ψ_2 , and so on. Hence all of the unknown functions Ψ_n of equation (2.17) can be obtained successively provided only that the set of linear integral equations

$$\frac{\mathcal{D}}{\mathcal{D}t}(f_o\,\Psi_{n-1,1})\ -\iint\{\text{funct.}(\Phi_{n-1}\,,\,\Psi_n)\}\,G_{12}\,d\underset{\sim}{C}_2\,dg=0\,,$$

$$n=1,2,3,\ldots \qquad (2.19)$$

has a solution $\Psi_n=\Psi_n(\Phi_{n-1})$. Φ_{n-1} here represents a known function of the form $\Phi_{n-1} = \Phi_{n-1}(f_0, \psi_1, \psi_2, \dots, \psi_{n-1})$.

By means of a number of ingenious transformations Hilbert showed that the set of equations (2.19) could be brought into the form *

$$F_{n-1}(U_1, V_1, W_1) = a\, e^{-(U_1^2+V_1^2+W_1^2)} \cdot \psi_n(U_1, V_1, W_1) + \int K(U_1, V_1, W_1; U_2, V_2, W_2) \cdot \psi_n(U_2, V_2, W_2)\, d\underset{\sim}{C}_2 ,$$

$$n = 1, 2, 3, \dots , \qquad (2.20)$$

where F_{n-1} is a known function, K is a symmetrical kernel, and <u>a</u> is a constant. Equations (2.20) are essentially non-homogeneous, linear integral equations of the second kind with symmetrical kernel, for the unknown functions ψ_n (U,V,W). Hilbert then showed that the associated homogeneous equations, obtained by equating the right side of (2.20) to zero, each have the same five eigen-solutions $\psi_n^{(j)} = \psi^{(j)}$ (j = 1,2,...,5) given explicitly by equation (2.21)

$$\left.\begin{aligned} \psi^{(1)} &= 1 \\ \psi^{(2)} &= U \\ \psi^{(3)} &= V \\ \psi^{(4)} &= W \\ \psi^{(5)} &= U^2 + V^2 + W^2 = C^2 \end{aligned}\right\} \qquad (2.21)$$

---

* The dependence of the functions F_{n-1} and ψ_n on the space and time coordinates x,y,z,t does not affect the nature of the integral equation, and is therefore not shown explicitly.

Physically, these five eigen-functions correspond to the mass, momenta, and kinetic energy of a molecule. By the well known theorems of Fredholm (Reference 2.5), the non-homogeneous equations (2.20) have solutions only for those functions F_{n-1} which satisfy the five associated conditions of integrability (2.22).

$$\int \psi^{(j)}(U,V,W) \cdot F_{n-1}(U,V,W) \cdot d\underset{\sim}{C} = 0$$

$$j = 1, 2, \ldots, 5$$

$$n = 1, 2, 3, \ldots \qquad (2.22)$$

The physical meaning of these conditions will be discussed in the next section. The most general solution of equation (2.20) is then the sum of a linear combination of the eigen-functions $\psi^{(j)}$ of its homogeneous equation and the "particular solution" $\psi_n^{(0)}$, namely

$$\psi_n(U,V,W\,;\,x,y,z\,;t) = \psi_n^{(0)}(U,V,W\,;\,x,y,z\,;t)$$

$$+ \sum_{j=1}^{5} c_n^{(j)}(x,y,z;t) \cdot \psi^{(j)}(U,V,W) \;,$$

$$n = 1, 2, \ldots \qquad (2.23)$$

Consequently, equation (2.17) is solved, in principle at least. Hilbert also showed that the solution (2.23) will be unique if one specifies the values of the following five $\underline{n}$ integrals at some definite time $t = t_o$, and for all points of space x,y,z:

$$\int \psi^{(j)}(U,V,W) \cdot f_o(U,V,W) \cdot \psi_n^{(0)}(U,V,W\,;(x,y,z\,;t)\; d\underset{\sim}{C}$$

$$= H_n^{\prime(j)}(x,y,z\,;t=t_o) \;, \qquad \begin{cases} j = 1, 2, \ldots 5 \\ n = 1, 2, \ldots \end{cases} \qquad (2.24)$$

It should be noted that Hilbert worked out the details of the above existence proof for molecules which are rigid elastic spheres. However, the nature of his arguments was extended to other elastic smooth molecules with spherical symmetry, such as those defined by equation (2.8), by A. Lunn and F. Pidduck (Reference 2.6).

2.3 Procedures of Solution

Before we examine the physical interpretation of Hilbert's solution of the Boltzmann equation, it will be convenient to transform somewhat the pertinent equations (2.17), (2.22), (2.23), and (2.24). The final form of these equations, also leads directly to the methods of Enskog, Chapman, and Burnett for the explicit calculation of the non-uniform velocity distribution function. Substitution of equations (2.23) into the Hilbert expansion (2.17) results in

$$(\nu f) = \frac{1}{\lambda} f_0 \left[1 + \sum_{n=1}^{\infty} \lambda^n \psi_n^{(0)} + \sum_{n=1}^{\infty} \lambda^n \sum_{j=1}^{5} c_n^{(j)} \psi^{(j)} \right] . \qquad (2.25)$$

Let $\phi^{(i)}(U,V,W)$ represent the complete set of orthogonal eigen-functions associated with the symmetric kernel $K(U_1,V_1,W_1;U_2,V_2,W_2)$ of equation (2.20). The particular solutions $\psi_n^{(0)}$, $n = 1,2,\ldots,$ of equation (2.25), as continuous functions of the velocity components, may be expanded in terms of these eigen-functions. The eigen-functions $\psi^{(j)}$ of equation (2.21) belong to the set $\phi^{(i)}$, so that with the notation

$$\phi^{(i)} = \psi^{(j)} \qquad for \qquad i = j = 1,2,\ldots,5, \qquad (2.26)$$

equation (2.25) can be written in the form

$$(\nu f) = \frac{1}{\lambda} f_0 \left[1 + \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \lambda^n a_n^{(i)} \phi^{(i)} \right] , \qquad (2.27)$$

where the coefficients $a_n^{(i)}$ are functions of the space and time variables. This may again be written in the final form

$$\nu(x,y,z;t)\cdot f(U,V,W;\ x,y,z;t)$$
$$= \frac{1}{\lambda} f_0(U,V,W)\left[1+\sum_{i=1}^{\infty} b^{(i)}(x,y,z;t)\cdot\phi^{(i)}(U,V,W)\right] \tag{2.28}$$

where

$$b^{(i)}(x,y,z;t) = \sum_{n=1}^{\infty}\lambda^n a_n^{(i)}(x,y,z;t)\ . \tag{2.29}$$

The deduction of equation (2.28) requires that all of the series involved be uniformly convergent, which fact was established in essence by D. Burnett (Reference 2.7).

As was pointed out by Hilbert, the 5 n conditions of integrability, equation (2.22), are equivalent to the 5 conditions of equation (2.30).

$$\int \psi^{(j)}(U,V,W)\ \frac{D}{Dt}\left[\nu(x,y,z;t)\cdot f(U,V,W;\ x,y,z;t)\right] d\underset{\sim}{c} = 0$$
$$\text{for}\quad j = 1,2,\ldots,5. \tag{2.30}$$

It will appear later on, as first pointed out by Maxwell, that the evaluation of equation (2.30) will result in the hydrodynamic equation of continuity for j = 1, the Eulerian equations of motion in the x,y,z directions for j = 2,3,4 respectively, and the energy equation for a perfect gas for j = 5. The functions $\psi^{(j)}$ were defined by equation (2.21). Physically this means then that any dynamically correct configuration of discrete molecules of a gas - that is one whose velocity distribution function satisfies Boltzmann's equation - must be so constituted that any macroscopic mass of this gas will be subject to the laws of conservation of mass, momentum, and energy which apply to the continuum of classical hydrodynamics.

Finally, it can be shown that the set of 5 n conditions of uniqueness equations (2.24), is equivalent to the following 5 equations:

$$\int \psi^{(j)}(U,V,W)\cdot f(U,V,W\,;\,x,y,z\,;t)\;d\underset{\sim}{C} \;=\; H^{(j)}(x,y,z\,;t)$$

$$\text{for} \quad j = 1, 2, \ldots, 5. \qquad (2.31)$$

Actually, the values of $H^{(j)}$ are not arbitrary, because of the restrictions which, for convenience, have been imposed earlier on the distribution function, and on the velocity coordinate system, namely equations (2.2) and (2.12). Consequently, the following conditions will specify the distribution function uniquely:

$$\begin{aligned} H^{(1)} &= \int f\cdot d\underset{\sim}{C} = 1 \\ H^{(2)} &= \int U\cdot f\cdot d\underset{\sim}{C} \equiv \overline{U} = 0\;;\quad H^{(3)} = \overline{V} = 0\;;\quad H^{(4)} = \overline{W} = 0 \qquad (2.32) \\ H^{(5)} &= \int C^2\cdot f\cdot d\underset{\sim}{C} = \overline{C^2}(x,y,z\,;t)\,. \end{aligned}$$

The bar over a quantity means that this quantity has been averaged over all velocity components of the molecules at the point x,y,z, and time t. It is well known that for a uniform gas the absolute temperature T is related to the mean square of the molecular velocity by

$$\overline{C^2} = \frac{3kT}{m} = \frac{3}{2}\,\frac{1}{hm}\,, \qquad (2.33)$$

where h has already been defined by equation (2.7). It is both convenient and logical to define also the temperature in a non-uniform gas by equation (2.33), where C is now the "peculiar velocity" of a molecule defined by equations (2.12) and (2.13). Physically speaking, this temperature is the one that would be measured by a thermometer moving with the mean mass velocity of the gas. The above definition is evidently consistent with the concept of the static temperature as commonly used in aerodynamics. Therefore, the significance of equations (2.32) is that the state of the gas is uniquely defined by equation (2.28), if five macrospic quantities, - namely the density of the gas ρ, its absolute temperature T, and the three components of the mass velocity u,v,w, - are each known as a function of x,y,z, and t.

Nothing has been said, as yet, about the choice of Hilbert's arbitrary parameter λ, first introduced in equation (2.17) and still present in the final result, equation (2.28). It can be shown (Reference 2.8), that a particular choice of λ merely affects the way in which the various terms of the expansion (2.28) contribute to the sum νf which itself is entirely independent of λ. The fact that the importance of the deviation of the non-uniform velocity distribution f from the uniform distribution function f_o increased with a decrease in the density ρ of the gas was discovered by Maxwell. D. Enskog and S. Chapman (Reference 2.9) used this fact in their selection of the parameter λ, namely

$$\lambda = \frac{1}{\nu} \tag{2.34}$$

where as before ν is the number of molecules per unit volume. For rarefied gases, in which, however, the number of molecules per unit volume is still sufficiently large, the coefficients $b^{(i)}$ of the expansion (2.28) may, therefore, be calculated to any desired degree of accuracy by replacing the infinite series (2.29) by the finite sum of N terms

$$b^{(i)}(x,y,z;t) = \sum_{n=1}^{N} \left(\frac{1}{\gamma}\right)^{n} a_{n}^{(i)}(x,y,z;t) \quad . \tag{2.35}$$

Equation (2.35) also shows that the quantities $b^{(i)}$ can depend explicitly only on the space and time derivatives of the mean motion variables u,v,w, ρ, T since for a uniform gas equation (2.28) must reduce to the Maxwell distribution function.

Both Enskog and Chapmann, using for the functions $\phi^{(i)}$, equation (2.28), the products of various powers of U,V,W and C^2 succeeded in evaluating the coefficients $b^{(i)}$, (2.35), to the first order, N = 1, the labor involved in calculating better approximations being prohibitive.

In 1935, D. Burnett (Reference 2.10, 2.11) discovered that the mathematical complexity of the problem can be considerably reduced by using for the functions $\phi^{(i)}$ the product of spherical harmonics in the peculiar velocity and the so-called Sonine polynomials of the argument C^2. This enabled Burnett to devise a general method for the calculation of the coefficients $b^{(i)}$ for arbitrary values of N. However, the labor involved for N > 2 is still stupendous. Consequently, Burnett calculated only those coefficients $b^{(i)}$, for N = 2, which were required for a calculation of the stresses in the gas, in which Burnett was primarily interested.

It will be seen in Part V of this paper that for the evaluation of the boundary conditions for the slip flow of a rarefied gas, all of the coefficients $b^{(i)}$ must be known. Therefore, the first task of the present study will be the complete computation of all these coefficients. Burnett's form of the distribution function, will be used in the subsequent parts of this investigation, and the method of calculating the unknown coefficients will be outlined presently.

2.4 Method of Computation Devised by Burnett

It is first convenient to express the cartesian velocity components in spherical polar coordinates. According to the notation of figure (2.1), the transformation equations are

$$\left.\begin{aligned} U &= C \sin\theta \cdot \cos\phi \\ V &= C \sin\theta \cdot \sin\phi \\ W &= C \cos\theta \end{aligned}\right\} \qquad (2.36)$$

so that the velocity distribution may now be expressed in the form

$$f(U,V,W;\ x,y,z,t;) = f(C,\theta,\phi;\ x,y,z;t) \qquad (2.37)$$

Burnett writes the expansion for f, equation (2.28), in the form

$$f(C,\theta,\phi;\ x,y,z;t) =$$

$$f_0(hmC^2)\cdot\Bigg\{\sum_{n=0}^{\infty} n!\,\Gamma(n+\tfrac{3}{2})\,A_n(x,y,z;t)\cdot S_{\frac{1}{2}}^{(n)}(hmC^2)$$

$$+\sum_{K=1}^{\infty}(hm)^{\frac{1}{2}K}\Big[\sum_{n=0}^{\infty} n!\,\Gamma(K+n+\tfrac{3}{2})\cdot Y_{Kn}(C,\theta,\phi;\ x,y,z;t)\cdot S_{K+\frac{1}{2}}^{(n)}(hmC^2)\Big]\Bigg\}. \qquad (2.38)$$

The general solid spherical harmonic function Y_{Kn} is defined by equations (A.1) to (A.4) in Appendix II, Section 2. The symbol $S_m^{(n)}$ denotes the Sonine polynomials, which are defined by equations (A.44) or (A.53) of Appendix II, Section 6, where other important properties of these polynomials are given by equations (A.45) to (A.53).

After substitution for f_0 from equation (2.16) and for Y_{kn} from equations

(A.1) to (A.4), equation (2.38) assumes the more explicit form

$$f(C,\theta,\phi\,;\,x,y,z;t) =$$

$$\left(\frac{hm}{\pi}\right)^{\frac{3}{2}} e^{-hmC^2}\Bigg[\!\Bigg[\sum_{n=0}^{\infty} n!\,\Gamma(n+\tfrac{3}{2})\cdot A_n(x,y,z;t)\cdot \overset{(n)}{S}_{\frac{1}{2}}(hmC^2)$$

$$+\sum_{K=1}^{\infty}(hmC^2)^{\frac{1}{2}K}\Bigg\{\sum_{n=0}^{\infty} n!\,\Gamma(K+n+\tfrac{3}{2})\cdot \overset{(n)}{S}_{K+\frac{1}{2}}(hmC^2)\Big[B_{Kn}(x,y,z;t)\cdot P_K(\cos\theta)$$

$$+\,2\sum_{\rho=1}^{K}(-1)^{\rho}\frac{(K-\rho)!}{(K+\rho)!}\left\{\overset{(\rho)}{B}_{Kn}(x,y,z;t)\cdot\cos(\rho\phi)+C_{Kn}(x,y,z;t)\cdot\sin(\rho\phi)\right\}\overset{(\rho)}{P}_K(\cos\theta)\Big]\Bigg\}\Bigg]\!\Bigg]. \quad (2.39)$$

Some important properties of the Legendre polynomials P_K and of the associated Legendre polynomials $\overset{(\rho)}{P}_K$ are summarized in Appendix II, Section 3 and 4, respectively. A comparison of equations (2.39) and (2.28) shows that the non-uniform distribution function f is expanded in terms of the function $\overset{(i)}{\phi}$ of the form *

$$\overset{(i)}{\phi}(C,\theta,\phi) = \overset{(\rho)}{P}_K(\cos\theta)\cdot\begin{Bmatrix}\cos(\rho\phi)\\ \sin(\rho\phi)\end{Bmatrix}\cdot C^K\cdot \overset{(n)}{S}_{K+\frac{1}{2}}(hmC^2) \,. \quad (2.40)$$

The quantities B_{Kn}, $\overset{(\rho)}{B}_{Kn}$, $\overset{(\rho)}{C}_{Kn}$, which we shall call the "Burnett coefficients", correspond to the coefficients $\overset{(i)}{b}$ (x,y,z;t) of equation (2.28). According to the discussion leading to equation (2.35), the Burnett coefficients may then be calculated to the desired degree of accuracy, by choosing for N the value appropriate to the density of the gas, whose state is to be represented by equation (2.39). Moreover, the Burnett coefficients, as functions of the space and time coordinates, will depend explicitly only on the five "variables of state" of the gas,

---

* The notation $\overset{(i)}{\phi}$ for a function will not be used hereafter, so that the use of ϕ for the azimuthal angle will not be confusing.

(ρ, T, u, v, w), and their space and time derivatives, in accordance with the discussion which followed equation (2.33). In the remainder of this section the distribution function will be considered at a fixed point (x, y, z) and a definite time t; the Burnett coefficients can then be treated as "constants", and their dependence on $(x, y, z; t)$ will no longer be indicated explicitly.

Equation (2.39) must satisfy the five uniqueness conditions of equation (2.32). As shown by Burnett, the use of the orthogonality relations (A.16), (A.30), (A.47) reduces equation (2.32) to the five equivalent conditions (2.41), respectively.

$$\left.\begin{aligned} A_0 &= 1 \\ B_{10}^{(1)} &= 0 \\ C_{10}^{(1)} &= 0 \\ B_{10} &= 0 \\ A_1 &= 0 \end{aligned}\right\} \qquad (2.41)$$

In addition, the distribution function must satisfy the five conditions of integrability of Hilbert, equation (2.30). As can be seen from the equations in Appendix II, Sections 2 and 6, the five eigen-functions $\psi^{(i)}$ of equation (2.21) are in Burnett's notation equivalent to equation (2.42).

$$\left.\begin{gathered} \psi^{(1)} = Y_0(C, \theta, \phi) \cdot S_{\frac{1}{2}}^{(0)}(hmC^2) \\ \psi^{(2)} = Y_1^{(1)} S_{\frac{1}{2}}^{(0)} \; ; \quad \psi^{(3)} = Z_1^{(1)} S_{\frac{1}{2}}^{(0)} \; ; \quad \psi^{(4)} = Y_1 S_{\frac{1}{2}}^{(0)} \\ \psi^{(5)} = Y_0 S_{\frac{3}{2}}^{(1)} \; . \end{gathered}\right\} \qquad (2.42)$$

Consequently, the distribution function (2.39) with the restrictions (2.41) is a useful representation of the state of a non-uniform gas, provided only that

(a) the conditions of integrability (2.30) are satisfied by (2.39) and (2.42);

(b) the unknown coefficient A_n, B_{kn}, $B_{kn}^{(\rho)}$, $C_{kn}^{(\rho)}$ can be calculated as functions of $(x,y,z;t)$. Burnett's method for extablishing conditions (a) and (b) will be outlined below.

The point of departure is again the Boltzmann integral equation, which in the notation of equations (2.14) and (2.15) is *

$$\left[\frac{\partial}{\partial t} + (u+U_1)\frac{\partial}{\partial x} + (v+V_1)\frac{\partial}{\partial y} + (w+W_1)\frac{\partial}{\partial z} \right] (\nu f_1)$$

$$= \nu^2 \iint (f_1' f_2' - f_1 f_2) \cdot G_{12} \cdot d\underline{C}_2 \cdot dg \, . \tag{2.43}$$

Multiply equation (2.43) by an arbitrary function Q_1, of the peculiar velocity components,

$$Q_1 = Q(U_1, V_1, W_1) \tag{2.44}$$

and integrate both sides of the resulting equation over all values of the velocity components U_1, V_1, W_1. By means of some partial integrations the result may be written in the most convenient form

---

* The terms containing the body forces F_x, F_y, F_z, have been neglected. This procedure is justified in Appendix III, Section 1.

$$\nu\Big[\frac{\partial\bar{Q}_1}{\partial t} + u\frac{\partial\bar{Q}_1}{\partial x} + v\frac{\partial\bar{Q}_1}{\partial y} + w\frac{\partial\bar{Q}_1}{\partial z} - u\overline{\frac{\partial Q_1}{\partial x}} - v\overline{\frac{\partial Q_1}{\partial y}} - w\overline{\frac{\partial Q_1}{\partial z}}$$

$$- \overline{U_1\frac{\partial Q_1}{\partial x}} - \overline{V_1\frac{\partial Q_1}{\partial y}} - \overline{W_1\frac{\partial Q_1}{\partial z}}\Big] + \frac{\partial}{\partial x}(\nu\overline{U_1Q_1}) + \frac{\partial}{\partial y}(\nu\overline{V_1Q_1}) + \frac{\partial}{\partial z}(\nu\overline{WQ_1})$$

$$= \nu^2\iiint(Q_1' - Q_1)\,f_1f_2\cdot G_{12}\;d\underset{\sim}{C}_1\,d\underset{\sim}{C}_2\,dg\;. \qquad (2.45)$$

The notation used here is analogous to equations (2.4) and (2.15); as in equation (2.32), the bar denotes an average value, namely

$$\bar{F}_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(U_1,V_1,W_1)\cdot f(U_1,V_1,W_1)\cdot dU_1\,dV_1\,dW_1\;. \qquad (2.46)$$

Equation (2.45) was first derived by Maxwell and is frequently called the Maxwell transfer equation. Its physical significance is entirely analogous to the Boltzmann equation (2.4) or (2.43). The left hand side of equation (2.45) represents the net change, $\Delta(\Sigma\, Q)$ in the sum total of the property Q for the $\nu(x,y,z,t)$ dxdydz molecules during the infinitesimal time dt. Since any changes in Q, ΔQ, are brought about by intermolecular collision, $\Delta(\Sigma\, Q)$ can also be determined by summing up the effects of all the collisions which occur during the time dt; the latter idea is represented by the integral on the right side of equation (2.46).

Burnett chooses for Q(U,V,W) functions of the type

$$Q = Q^{(p)}_{kr} = (k+\tfrac{1}{2})\sqrt{\pi}\,(hm)^{\frac{1}{2}k}\cdot Y^{(p)}_{k}(C,\theta,\phi)\cdot S^{(r)}_{k+\frac{1}{2}}(hmC^2)\,, \qquad (2.47)$$

which include the important eigen-functions $\psi^{(i)}$ of equations (2.21) and (2.42) as special cases. The various averages occurring on the left side of equation (2.45) can now be calculated by performing integrations of the type (2.46) by using for f(U,V,W) its value given by equation (2.39). The simplest examples of the expressions thus obtained by Burnett are

$$\overline{Q_{kr}^{(\rho)}} = B_{kr}^{(\rho)}$$

$$\overline{U\,Q_{kr}^{(0)}} = \frac{1}{\sqrt{hm}}\left[\frac{1}{2k+3}\left\{(k+r+\tfrac{3}{2})\,B_{k+1,r}^{(1)} - B_{k+1,r-1}^{(1)}\right\}\right.$$
$$\left. - \frac{1}{2k-1}\left\{B_{k-1,r}^{(1)} - (r+1)\,B_{k-1,r+1}^{(1)}\right.\right] \equiv D_{kr}^{(0)}$$

$$\overline{U\frac{\partial}{\partial x}Q_{kr}^{(0)}} = \frac{2\sqrt{hm}}{2k+3}\left\{\frac{\partial u}{\partial x}\,D_{k+1,r-1}^{(1)} + \dots + \frac{\partial w}{\partial x}\,D_{k-1,r}\right\} + \dots$$
$$+ \frac{1}{h}\frac{\partial h}{\partial x}\left\{(r+\tfrac{1}{2}k)\,D_{k,r} - D_{k,r-1}\right\}. \qquad (2.48)$$

Substitution of the expressions (2.48) into equation (2.45) reduces the Maxwell transfer equation to the form

$$\nu\Big[\operatorname{funct.}_{k,r}^{(\rho)}\Big(u, v, w, h, \nu\,;\ \frac{\partial u}{\partial x}, \dots, \frac{\partial v}{\partial z}, \dots, \frac{\partial w}{\partial t}, \dots, \frac{\partial \nu}{\partial x}, \dots \frac{\partial h}{\partial t}\,;$$
$$B_{\ell s}^{(q)},\ C_{\ell s}^{(q)},\ \frac{\partial}{\partial x}(B_{\ell s}^{(q)}), \dots \frac{\partial}{\partial t}(C_{\ell s}^{(q)})\,;\ \text{etc.}\Big)\Big]$$

$$= \nu^2 \iiint\left\{(Q_{kr}^{(\rho)})_1' - (Q_{kr}^{(\rho)})_1\right\} f_1 f_2 \cdot G_{12} \cdot d\underset{\sim}{C}_1\, d\underset{\sim}{C}_2\, dg\,, \qquad (2.49)$$

where $k-2 \leqq \ell \leqq k+2\,;\ r-2 \leqq s \leqq r+2\,;\ \rho-2 \leqq q \leqq \rho+2\,.$

In order to evaluate the complicated integral of equation (2.49), a specific molecular model must be selected in order to specify the "collision parameters" G_{12} and dg. Burnett succeeded in evaluating this integral for molecules which are point centers of force, and which repel one another according to the law of force of equation (2.8). The details of Burnett's calculation are involved, and even his final results are too lengthy to be reproduced here explicitly. However, certain details of such calculations are given in Appendix III, in connection with the author's evaluation of certain coefficients not previously determined by Burnett.

It can be shown that the integral on the right side of equation (2.49) vanishes for the special molecular properties $Q_{kr}^{(p)}(U,V,W)$, equation (2.47), which are defined by

$$
\begin{aligned}
&(a)\quad k = 0,\ r = 0,\ p = 0 \\
&(b)\quad k = 1,\ r = 0,\ p = 0,1 \qquad\qquad (2.50) \\
&(c)\quad k = 0,\ r = 1,\ p = 0
\end{aligned}
$$

This means mathematically, that the functions $Q_{kr}^{(p)}$ as defined by equations (2.47) and (2.50) correspond to the eigen-functions $\psi^{(j)}$ of equations (2.42) and (2.21), and that Burnett's expansion for the distribution function, equation (2.39), satisfies the conditions of integrability, equations (2.30). It will be seen that the integral of equation (2.30) is an abbreviated notation for the right side of the Maxwell transfer equation, equations (2.45) or (2.49).

The physical interpretation of the above result is, of course, that the total mass, momentum (in three directions) and kinetic energy, are not changed by elastic intermolecular collision. Consequently, the Maxwell tranfer equation reduces to the equations of continuity, momentum and energy for the mean motion of a non-uniform gas, when values of equation (2.50) a,b, and c, respectively are substituted. These fundamental equations are stated explicitly, and discussed in some detail in Part III.

In the general case, however, that is for an arbitrary molecular property Q_{kr}, the integral on the right side of the Maxwell transport equation (2.49), is a non-vanishing function of the Burnett coefficients. The evaluation of this integral is made much simpler by the use of Sonine polynomials. Burnett shows that after evaluation of this integral for arbitrary values of k and r, equation (2.49) takes on the form

$$\left[\text{funct.}\left(u,\dots,h\ ;\ \frac{\partial u}{\partial x},\dots,\frac{\partial h}{\partial t}\ ;\ B^{(q)}_{\ell s}, C^{(q)}_{\ell s}\ ;\ \frac{\partial}{\partial x}\left(B^{(q)}_{\ell s}\right),\dots,\ etc.\right)\right]^{(\rho)}_{k,r}$$

$$= \nu \left[\sum_{\substack{k_1,k_2\\ n_1,n_2\\ \rho_1,\rho_2}} \left(\mathfrak{M}^{(\rho_1,\rho_2)}_{\substack{k,k_1,k_2\\ r,n_1,n_2}} \left\{ B^{(\rho_1)}_{k_1 n_1} B^{(\rho_2)}_{k_2 n_2} + C^{(\rho_1)}_{k_1 n_1} C^{(\rho_2)}_{k_2 n_2} \right\}\right)\right]^{(\rho)}_{kr} . \tag{2.51}$$

The quantities $\mathfrak{M}$ are numerical constants which differ from zero only when its indices satisfy all of the conditions (2.52).

$$\rho_1 = \rho_2 \ ; \qquad \rho_1 \leqq k_1 \geqq 0 \ ; \qquad \rho_2 \leqq k_2 \geqq 0$$

$$k = |k_2 - k_1| + 2n \ , \qquad n = 0, 1, 2, 3, \dots \tag{2.52}$$

In the notation of equation (2.51) it is to be understood that

$$B^{(0)}_{\ell s} \equiv B_{\ell s} \ , \qquad C^{(0)}_{\ell s} = 0 \ , \qquad B_{os} \equiv A_s$$

In the sense of the discussion leading to equation (2.35), the exact equation (2.51) may be approximated as closely as desired by neglecting all terms of order higher than $\left(\frac{1}{\nu}\right)^N$. This means that the summation on the right side of equation (2.51) must contain terms of order $\left(\frac{1}{\nu}\right)^{N+1}$ or less. From equations (2.41), (2.39), and (2.35) it follows that the coefficient $B_{oo} = A_o = 1$ is the only coefficient of order N = 0, since the term in f, equation (2.39), containing A_o corresponds to the Maxwell distribution. Therefore, reduction of equation (2.51) to terms of order N = 0, results in

$$\left[\text{funct.}\left(\frac{\partial u}{\partial x}, \ldots, \frac{\partial h}{\partial t} \right) \right]_{kr} = \gamma \left[\sum_{\substack{k_2 \\ n_2}} \left(M^{(0,0)}_{\substack{k0k_2 \\ r0n_2}} \cdot A_0 \, {}_1B_{k_2 n_2} \right) \right]_{kr} . \quad (2.53)$$

Evidently, if the transport functions Q_{kr} of equation (2.47) are used for all possible values of k and r, equation (2.53) constitutes an infinite set of linear algebraic equations, from which the same infinite number of unknown coefficients, ${}_1B_{k_2 n_2}$ can be calculated. The prescript (1) denotes the fact that the coefficient $B_{k_2 n_2}$, is here accurate only to terms of order $\left(\frac{1}{\gamma}\right)^{N+1}$.

Equation (2.53) leads to the determination of the "basic" Burnett coefficients, ${}_1B_{kr}$, only. However, the "associated" coefficients, ${}_1B^{(\rho)}_{kr}$ and ${}_1C^{(\rho)}_{kr}$ are readily calculated from the basic coefficient, by the following method of rotation of coordinate axes. Since the distribution function, equation (2.39), is valid for any values of (C, θ, ϕ) or (U,V,W), it must also hold for the special case $U' = V' = 0$, $W' = C$. From the equations of Appendix II, Section 2 for the spherical harmonics, it can be seen that

$$Y_{kn} \, (C' = W', \; U' = V' = 0) = B_{kn} \, Y_k \, (C' = W', \; U' = V' = 0) \qquad (2.54)$$

(2.54)

If the coordinate axes are now rotated, equation (2.54) may be expressed in terms of the new coordinates x,y,z;U,V,W with

$$C^2 = C'^2 = U^2 + V^2 + W^2 \, .$$

Equation (2.54), expressed in terms of the primed coordinates can then be arranged in the form of equation (A.1), Appendix II. The associated Burnett coefficients B_{kn}, C_{kn} are then simply the coefficients of the quantities $2 \, \frac{(K-\rho)!}{(K+\rho)!} \cdot Y^{(\rho)}_K (U,V,W)$ and $2 \, \frac{(K-\rho)!}{(K+\rho)!} Z^{(\rho)}_K (U,V,W)$, respectively. The success of this method rests, of course, on the fact that the general form of the expression for the distribution function must be the same for any system of cartesian axes. The above method is illustrated for the case $k = 3$, $n = 1$ in Appendix III.

As all of the Burnett coefficients are thus known up to terms of order , evaluate the next approximation to equation (2.51), which for $N = 1$ is of the form

$$\left[\text{funct.}\left(u, \dots, h \; ; \; \frac{\partial u}{\partial x}, \dots, \frac{\partial h}{\partial t} \; ; \; {}_1B_{\ell s}^{(\wp)} \, , \; {}_1C_{\ell s}^{(\wp)} \; ; \; \frac{\partial}{\partial x}\left({}_1B_{\ell s}^{(\wp)}\right), \dots, \text{etc.} \right) \right]_{kr}$$

$$= \gamma \left[\sum_{\substack{k_2 \\ n_2}} \left(\mathcal{M}_{\substack{k0k_2 \\ r0n_2}}^{(0,0)} A_0 \, {}_2B_{k_2 n_2} \right) + \sum_{\substack{k_1, k_2 \\ n_1, n_2 \\ \wp_1 = \wp_2}} \left(\mathcal{M}_{\substack{kk_1k_2 \\ rn_1n_2}}^{(\wp_1 = \wp_2)} \left\{ {}_1B_{k_1 n_1}^{(\wp_1)} \, {}_1B_{k_2 n_2}^{(\wp_1)} + {}_1C_{k_1 n_1}^{(\wp_1)} \, {}_1C_{k_2 n_2}^{(\wp_1)} \right\} \right) \right]_{kr} . \qquad (2.55)$$

The only unknown quantities appearing in equation (2.55) are the second order, $\left(\frac{1}{\gamma}\right)^2$, approximations to the basic Burnett coefficients, namely ${}_2B_{k_2 n_2}$. Consequently, if equation (2.55) is evaluated for all values of k and r, one has an infinite set of linear algebraic equations for the determination of the coefficients ${}_2B_{kr}$. As before, the associated coefficients ${}_2B_{kr}^{(\wp)}$, ${}_2C_{kr}^{(\wp)}$ are obtained by the method of rotation of axes.

By continuation of this process it is thus possible to calculate all of the Burnett coefficients accurate to terms of order $\left(\frac{1}{\gamma}\right)^{N+1}$ by evaluating the integrated form of the Maxwell transfer equation, equation (2.51), up to terms of order $\left(\frac{1}{\gamma}\right)^{N}$. Thus the calculation scheme of Burnett, as outlined above, is the most convenient method yet devised for actually computing the solution of the Boltzmann equation whose existence was first established by D. Hilbert.

2.5 Results of Burnett's Computation

Actually, the formulae from which the numerical constants $\mathcal{M}_{\substack{kk_1k_2 \\ rn_1n_2}}^{(\wp_1, \wp_2)}$

are calculated are very complicated for molecular models whose force of repulsion varies as an arbitrary power of the reciprocal of the molecular separation, equation (2.8). For this reason, Burnett gives explicit formulae only for the two limiting cases of (a) rigid spherical molecules ($n \rightarrow \infty$) and (b) Maxwellian molecules ($n = 5$).

For spherical molecules, Burnett found that the first order coefficients ${}_1B_{kr}$ differ from zero only for $k = 1$ and 2. The basic coefficients were found to be

$$\left.\begin{aligned} {}_1B_{11} &= -1.009 \times \frac{3}{2}\left(\frac{\mu}{p}\right)\frac{1}{h\sqrt{hm}}\frac{\partial h}{\partial z} \\ {}_1B_{12} &= .048\,{}_1B_{11} \\ {}_1B_{13} &= .004\,{}_1B_{11} \\ {}_1B_{14} &= .0002\,{}_1B_{11} \\ &\text{etc.} \end{aligned}\right\} \qquad (2.56)$$

$$\left.\begin{aligned} {}_1B_{20} &= -\frac{2}{3}\left(\frac{\mu}{p}\right)\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right) \\ {}_1B_{21} &= .062\,{}_1B_{20} \\ {}_1B_{22} &= .0049\,{}_1B_{20} \\ {}_1B_{23} &= .0003\,{}_1B_{20} \\ &\text{etc.} \end{aligned}\right\} \qquad (2.57)$$

In the above equations μ is the coefficient of absolute viscosity of the gas; its significance will be discussed in Part III. The parameter $\frac{1}{\nu}$ is contained in $\frac{1}{p}$, since the pressure p is related to the number density by the equation of state

$$p = \frac{1}{2h} \cdot \nu \,. \qquad (2.58)$$

For spherical molecules, the associated first order coefficients are given by equations (2.59) and (2.60):

$$\left.\begin{array}{ll} {}_1B_{11}^{(1)} = -1.009 \cdot \frac{3}{2}\left(\frac{\mu}{p}\right)\frac{1}{h\sqrt{hm}}\,\frac{\partial h}{\partial x} \; ; & {}_1C_{11}^{(1)} = -1.009 \cdot \frac{3}{2}\left(\frac{\mu}{p}\right)\frac{1}{h\sqrt{hm}}\,\frac{\partial h}{\partial y} \\ {}_1B_{12} = .048\ {}_1B_{11}^{(1)} & {}_1C_{12}^{(1)} = .048\ {}_1C_{11}^{(1)} \\ \text{etc.} & \text{etc.} \end{array}\right\} \quad (2.59)$$

$$\left.\begin{array}{ll} {}_1B_{20}^{(1)} = -2\left(\frac{\mu}{p}\right)\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \; ; & {}_1C_{20}^{(1)} = -2\left(\frac{\mu}{p}\right)\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ {}_1B_{20}^{(2)} = -4\left(\frac{\mu}{p}\right)\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) \; ; & {}_1C_{20}^{(1)} = -4\left(\frac{\mu}{p}\right)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ {}_1B_{21}^{(1)} = .062\ {}_1B_{20}^{(1)} \, , & \text{etc.} \end{array}\right\} \quad (2.60)$$

The rapidity of the convergence of the two sets of coefficients for spherical molecules is apparent.

For Maxwellian molecules, the equations for the calculation of the constants $\mathfrak{M}$ assume the simplest form. In fact for this type of molecule the summation of the unknown coefficients in equations (2.53), (2.55), etc. reduces to a single term of the form $\mathfrak{M}_{kok}\ A_o\ {}_N B_{kr}$, so that each coefficient can be calculated from a single equation. Thus for Maxwell molecules *there are only two basic fist order coefficients, namely* ${}_1B_{11}$ and ${}_1B_{20}$. The complete set of first order coefficients for Maxwellian molecules is given by equations (2.61) and (2.62):

$$\left.\begin{aligned} {}_1B_{11} &= -\frac{3}{2}\left(\frac{\mu}{p}\right)\frac{1}{h\sqrt{hm}}\frac{\partial h}{\partial z} \\ {}_1B_{11}^{(1)} &= -\frac{3}{2}\left(\frac{\mu}{p}\right)\frac{1}{h\sqrt{hm}}\frac{\partial h}{\partial x}\,, \quad {}_1C_{11}^{(1)} = -\frac{3}{2}\left(\frac{\mu}{p}\right)\frac{1}{h\sqrt{hm}}\frac{\partial h}{\partial y} \end{aligned}\right\} \tag{2.61}$$

$$\left.\begin{aligned} {}_1B_{20} &= -\frac{2}{3}\left(\frac{\mu}{p}\right)\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right) \\ {}_1B_{20}^{(1)} &= -2\left(\frac{\mu}{p}\right)\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right), \quad {}_1C_{20}^{(1)} = -2\left(\frac{\mu}{p}\right)\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \\ {}_1B_{20}^{(2)} &= -4\left(\frac{\mu}{p}\right)\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right), \quad {}_1C_{20}^{(2)} = -4\left(\frac{\mu}{p}\right)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \end{aligned}\right\} \tag{2.62}$$

In physical terms, it is apparent and will be shown more explicitly in Part III, that the coefficients ${}_1B_{11}^{(\rho)}$ and ${}_1B_{20}^{(\rho)}$ represent the components of the heat flux vector and of the stress tensor used in the treatment of the hydrodynamics of a continuum. Hence the velocity distribution of a non-uniform gas is completely determined, to a first approximation, by the stresses and heat conduction in the gas. This fact accounts for the great similarity in the expressions for the first order coefficients for the two molecular models discussed above. However, it must be remembered that the explicit expressions for the coefficient of viscosity μ are different for different molecular models, as is implied by equations (2.9) and (2.10).

2.6 Additional Coefficients Computed for Maxwell Molecule

Burnett was primarily interested in calculating the stress tensor accurate to terms of order $\left(\frac{1}{\nu}\right)^2$, or $\left(\frac{\mu}{p}\right)^2$. He therefore evaluated only the coefficients ${}_2B_{20}$, ${}_2B_{20}^{(1)}$, ..., ${}_2C_{20}^{(2)}$ by means of equation (2.55), for both spherical and Maxwellian molecules. For the evaluation of the slip flow boundary conditions, discussed in Part V, all of the basic coefficients ${}_2B_{kr}$ and some of the associated coefficients are required. It was shown by the author that for Maxwell molecules there are only a finite number of coefficients ${}_2B_{kr}$. Since the labor involved in calculating the infinite, though convergent set of second order coefficients for spherical molecules is enormous, and as both the spherical and the Maxwell molecule represent the same order of approximation to the molecule of a real gas such as nitrogen, (see figure 2.2), the author has evaluated these second order coefficients only for the Maxwell molecule.

For the Maxwell molecule, the pertinent equation (2.55) takes on the more explicit form

$$\left(\frac{\mu}{p}\right)\Big[\text{funct.}\,\Big(u,\dots,h\,;\,\frac{\partial u}{\partial x},\dots\frac{\partial h}{\partial t}\,;\,{}_1B_{11}^{(1)},\ {}_1C_{11}^{(1)},\ {}_1B_{11}\,;\ {}_1B_{20},\dots\ {}_1C_{20}^{(1)}\,;$$

$$\frac{\partial}{\partial x}({}_1B_{11}),\dots\ \frac{\partial}{\partial t}({}_1C_{20}^{(2)})\,;\ etc.\Big)\Big]_{kr}$$

$$= \Big[\mathcal{M}^{(0,0)}_{\substack{kok\\ror}}\ {}_2B_{kr} + \sum_{\rho=0}^{\rho=1}\mathcal{M}^{(\rho)}_{\substack{k11\\r11}}\left\{\left[{}_1B_{11}^{(\rho)}\right]^2+\left[{}_1C_{11}^{(\rho)}\right]^2\right\}$$

$$+\sum_{\rho=0}^{\rho=1}\mathcal{M}^{(\rho)}_{\substack{k12\\r10}}\left\{{}_1B_{11}^{(\rho)}\,{}_1B_{20}^{(\rho)} + {}_1C_{11}^{(\rho)}\,{}_1C_{20}^{(\rho)}\right\} + \sum_{\rho=0}^{\rho=2}\mathcal{M}^{(\rho)}_{\substack{k22\\r00}}\left\{\left[{}_1B_{20}^{(\rho)}\right]^2+\left[{}_1C_{20}^{(\rho)}\right]^2\right\}\Big]_{kr}. \tag{2.63}$$

By examining the expressions of the form of equation (2.48), of which the left side of equation (2.63) is composed, one finds that

$$[\text{funct.}(\ )]_{kr} = 0 \qquad \text{for all values} \tag{2.64}$$

of (k,r) except

$(k,r) = (0,2),\ (0,3);\ (1,1),\ (1,2);\ (2,0),\ (2,1),\ (2,2);$

$(3,0),\ (3,1);\ (4,0).$

It can be shown from the general form of the expressions for the constants $\mathcal{M}$, as given by Burnett for Maxwell molecules, that these constants satisfy the sufficient conditions of equations (2.65) :

$$\begin{cases} \mathfrak{M}^{(\rho)}_{\substack{k11\\r11}} = 0 & \text{unless } (k,r) = (0,3),\ (2,2),\ (4,1),\ (6,0). \\ \mathfrak{M}^{(\rho)}_{\substack{k12\\r10}} = 0 & \text{unless } (k,r) = (1,2),\ (3,1),\ (5,0). \\ \mathfrak{M}^{(\rho)}_{\substack{k22\\r00}} & \text{unless } (k,r) = (0,2),\ (2,1),\ (4,0). \end{cases} \tag{2.65}$$

The calculations required for determining the numerical values of the constants $\mathfrak{M}$ is illustrated in Appendix III, Section 2 for $(k,r) = (3,1)$. The values of those constants $\mathfrak{M}$ which according to equation (2.65) may not vanish are listed in Table 2.1. The values of $\mathfrak{M}^{(0,0)}_{\substack{k0k\\r0r}}$ which are required for the calculation of the corresponding coefficients ${}_2B_{kr}$ are also included in Table 2.1.

From equations (2.64) and Table 2.1 it follows that only the ten basic Burnett coefficients of the second order, ${}_2B_{kr}$, which are included in the array (2.66) do not vanish for Maxwellian molecules.

$$\left.\begin{array}{l} {}_2A_2,\ {}_2A_3 \\ {}_2B_{11},\ {}_2B_{12} \\ {}_2B_{20},\ {}_2B_{21},\ {}_2B_{22} \\ {}_2B_{30},\ {}_2B_{31} \\ {}_2B_{40} \end{array}\right\} \tag{2.66}$$

From equation (2.63) and Table 2.1, all of the second order coefficients (2.66) can then be expressed interms of the variables (u,v,w,h,ν), their space and time derivatives, and the first order coefficients ${}_1B_{11}, \dots, {}_1C^{(2)}_{20}$. By the use of equations (2.61) and (2.62), the coefficients ${}_2B_{kr}$ are then finally expressed explicitly in terms of the mean motion variables (u,v,w,h,ν) and their space and time derivatives. The associated coefficients ${}_2B^{(\rho)}_{kr}$ are again calculated by the method of rotation of axes.

The values of the 46 second order coefficients, ${}_2B^{(\rho)}_{kr}$, which form the complete set for Maxwellian molecules are given on the following pages, by equations (2.68) to (2.113). A comparison of these equations with the expressions (2.61) and (2.62) for the first order coefficients indicates how rapidly the algebraic complexity of the calculations increases with each successive approximation, even for the relatively simple Maxwell molecule. For this reason the next approximation, represented by the coefficients ${}_3B^{(\rho)}_{kr}$ has not been determined.

The following standard notation has been used to abbreviate the expressions for the coefficients:

$$\frac{D}{Dt} \equiv \left\{ \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \right\}. \tag{2.67}$$

$$\begin{aligned} {}_2A_2 = \Bigg[& -\frac{3}{2}\left(\frac{\mu}{p}\right)^2 \frac{1}{h} \left\{ \frac{\partial h}{\partial x}\frac{Du}{Dt} + \frac{\partial h}{\partial y}\frac{Dv}{Dt} + \frac{\partial h}{\partial z}\frac{Dw}{Dt} \right\} \\ & + \frac{27}{8}\left(\frac{\mu}{p}\right)^2 \frac{1}{h^3 m} \left\{ \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 + \left(\frac{\partial h}{\partial z}\right)^2 \right\} \\ & + \frac{3}{4}\frac{\mu}{p}\frac{1}{m\nu} \left\{ \frac{\partial}{\partial x}\left(\frac{\mu\nu}{ph^2}\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\mu\nu}{ph^2}\frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\mu\nu}{ph^2}\frac{\partial h}{\partial z}\right) \right\} \\ & + \frac{1}{3}\left(\frac{\mu}{p}\right)^2 \left\{ \frac{1}{3}\left(2\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 \right. \\ & \qquad \left. + \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 \right\} \Bigg] \end{aligned} \tag{2.68}$$

$${}_2A_3 = -\frac{1}{2}\left(\frac{\mu}{p}\right)^2 \frac{1}{h^3 m} \left\{ \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 + \left(\frac{\partial h}{\partial z}\right)^2 \right\} \tag{2.69}$$

$$
\begin{aligned}
{}_2B_{11} = & -\frac{3}{2}\left(\frac{\mu}{p}\right)\frac{1}{h\sqrt{hm}}\frac{\partial h}{\partial z} + \frac{9}{4}\left(\frac{\mu}{p}\right)^2\Big[\nu\frac{D}{Dt}\left(\frac{1}{\nu h\sqrt{hm}}\frac{\partial h}{\partial z}\right) \\
& - \frac{8}{15}\sqrt{hm}\left\{\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\frac{Du}{Dt} + \left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\frac{Dv}{Dt} + \frac{2}{3}\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)\frac{Dw}{Dt}\right\} \\
& - \frac{4}{15}h\left\{\frac{\partial}{\partial x}\left(\frac{1}{h\sqrt{hm}}\left[\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right]\right) + \frac{\partial}{\partial y}\left(\frac{1}{h\sqrt{hm}}\left[\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right]\right)\right. \\
& \left. + \frac{2}{3}\frac{\partial}{\partial z}\left(\frac{1}{h\sqrt{hm}}\left[2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right]\right)\right\} + \frac{17}{15}\frac{1}{h\sqrt{hm}}\frac{\partial h}{\partial z}\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right) \\
& + \frac{6}{5}\frac{1}{h\sqrt{hm}}\left\{\frac{\partial h}{\partial x}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) + \frac{\partial h}{\partial y}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right\} \\
& + \frac{1}{h\sqrt{hm}}\left(\frac{\partial h}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial h}{\partial y}\frac{\partial w}{\partial y}\right)\Big]
\end{aligned}
\tag{2.70}
$$

${}_2B^{(1)}_{11}$ is obtained from ${}_2B_{11}$ by cyclic substitution,

$$x \rightarrow y,\ y \rightarrow z,\ z \rightarrow x;\ u \rightarrow v,\ v \rightarrow w;\ w \rightarrow u; \tag{2.71}$$

${}_2C^{(1)}_{11}$ is obtained from ${}_2B_{11}$ by cyclic substitution,

$$x \rightarrow z,\ y \rightarrow x,\ z \rightarrow y;\ u \rightarrow w,\ v \rightarrow u,\ w \rightarrow v. \tag{2.72}$$

$$
\begin{aligned}
{}_2B_{12} = & -\frac{3}{10}\left(\frac{\mu}{p}\right)^2\frac{1}{h\sqrt{hm}}\Big[2\frac{\partial h}{\partial z}\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right) + 3\frac{\partial h}{\partial x}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \\
& + 3\frac{\partial h}{\partial y}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\Big]
\end{aligned}
\tag{2.73}
$$

$$
\begin{aligned}
{}_2B^{(1)}_{12} = & -\frac{3}{10}\left(\frac{\mu}{p}\right)^2\frac{1}{h\sqrt{hm}}\Big[2\frac{\partial h}{\partial x}\left(2\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}-\frac{\partial w}{\partial z}\right) + 3\frac{\partial h}{\partial y}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) \\
& + 3\frac{\partial h}{\partial z}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\Big]
\end{aligned}
\tag{2.74}
$$

$$
\begin{aligned}
{}_2C^{(1)}_{12} = & -\frac{3}{10}\left(\frac{\mu}{p}\right)^2\frac{1}{h\sqrt{hm}}\Big[2\frac{\partial h}{\partial y}\left(2\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}-\frac{\partial w}{\partial z}\right) + 3\frac{\partial h}{\partial x}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \\
& + 3\frac{\partial h}{\partial z}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\Big]
\end{aligned}
\tag{2.75}
$$

$$
\begin{aligned}
{}_2B_{20} = {} & -\frac{2}{3}\left(\frac{\mu}{p}\right)\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+\Big[\frac{2}{3}\left(\frac{\mu}{p}\right)\frac{D}{Dt}\left(\frac{\mu}{p}\left[2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right]\right) \\
& +\frac{1}{9}\left(\frac{\mu}{p}\right)^2\Big\{2\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)^2+3\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2+3\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2 \\
& -6\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)^2-6\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\Big\}-\left(\frac{\mu}{p}\right)^2\left\{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2-\left(\frac{\partial w}{\partial x}\right)^2-\left(\frac{\partial w}{\partial y}\right)^2\right\} \\
& -\frac{\mu}{p}\,\frac{h}{\gamma m}\left\{2\frac{\partial}{\partial z}\left(\frac{\mu}{h^2}\frac{\partial h}{\partial z}\right)-\frac{\partial}{\partial x}\left(\frac{\mu}{h^2}\frac{\partial h}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{\mu}{h^2}\frac{\partial h}{\partial y}\right)\right\}\Big]
\end{aligned}
\tag{2.76}
$$

$$
\begin{aligned}
{}_2B^{(1)}_{20} = {} & -2\left(\frac{\mu}{p}\right)\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\Big[2\frac{\mu}{p}\frac{D}{Dt}\left(\frac{\mu}{p}\left[\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right]\right) \\
& +\frac{2}{3}\left(\frac{\mu}{p}\right)^2\left\{3\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-2\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\left(2\frac{\partial v}{\partial y}-\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}\right)\right\} \\
& -2\left(\frac{\mu}{p}\right)^2\left\{\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial z}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial z}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial z}\right)-\left(\frac{\partial u}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial w}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\right)\right\} \\
& -3\,\frac{\mu}{p}\,\frac{h}{\gamma m}\left\{\frac{\partial}{\partial z}\left(\frac{\mu}{h^2}\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial x}\left(\frac{\mu}{h^2}\frac{\partial h}{\partial z}\right)\right\}\Big]
\end{aligned}
\tag{2.77}
$$

${}_2C^{(1)}_{20}$ is obtained from ${}_2B^{(1)}_{20}$ by making the substitutions,

$x \to y,\ y \to x;\ u \to v,\ v \to u.$ (2.78)

$$\begin{aligned}
{}_2B_{20}^{(2)} = & -4\left(\frac{\mu}{p}\right)\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right) + \Bigg[4\left(\frac{\mu}{p}\right)\frac{D}{Dt}\left(\frac{\mu}{p}\left[\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right]\right) \\
& + \frac{2}{3}\left(\frac{\mu}{p}\right)^2\Bigg\{3\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2 - 3\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2 \\
& \qquad - 4\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)\Bigg\} \\
& - 2\left(\frac{\mu}{p}\right)^2\left\{\left(\frac{\partial w}{\partial x}\right)^2-\left(\frac{\partial u}{\partial z}\right)^2-\left(\frac{\partial w}{\partial y}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2-2\left(\frac{\partial u}{\partial y}\right)^2+2\left(\frac{\partial v}{\partial x}\right)^2\right\} \\
& - 6\left(\frac{\mu}{p}\right)\frac{h}{\gamma m}\left\{\frac{\partial}{\partial x}\left(\frac{\kappa}{h^2}\frac{\partial h}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{\kappa}{h^2}\frac{\partial h}{\partial y}\right)\right\}\Bigg]
\end{aligned} \tag{2.79}$$

$$\begin{aligned}
{}_2C_{20}^{(2)} = & -4\left(\frac{\mu}{p}\right)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) + \Bigg[4\frac{\mu}{p}\frac{D}{Dt}\left(\frac{\mu}{p}\left[\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right]\right) \\
& + \frac{4}{3}\left(\frac{\mu}{p}\right)^2\left\{3\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)-2\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)\right\} \\
& - 4\left(\frac{\mu}{p}\right)^2\left\{\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)-\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\right)\right\} \\
& - 6\frac{\mu}{p}\frac{h}{\gamma m}\left\{\frac{\partial}{\partial y}\left(\frac{\kappa}{h^2}\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial x}\left(\frac{\kappa}{h^2}\frac{\partial h}{\partial y}\right)\right\}\Bigg]
\end{aligned} \tag{2.80}$$

$$\begin{aligned}
{}_2B_{21} = & \Bigg[\frac{6}{7}\left(\frac{\mu}{p}\right)^2\frac{1}{h}\left\{2\frac{\partial h}{\partial z}\frac{Dw}{Dt}-\frac{\partial h}{\partial x}\frac{Du}{Dt}-\frac{\partial h}{\partial y}\frac{Dv}{Dt}\right\} \\
& + \frac{6}{7}\frac{\mu}{p}\frac{1}{\gamma m}\left\{2\frac{\partial}{\partial z}\left(\frac{\kappa}{h}\frac{\partial h}{\partial z}\right)-\frac{\partial}{\partial x}\left(\frac{\kappa}{h}\frac{\partial h}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{\kappa}{h}\frac{\partial h}{\partial y}\right)\right\} \\
& - \frac{20}{147}\left(\frac{\mu}{p}\right)^2\Bigg\{\frac{2}{3}\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2 \\
& \qquad - 2\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)^2-2\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\Bigg\}\Bigg]
\end{aligned} \tag{2.81}$$

$$\begin{aligned}{}_2B_{21}^{(1)} = \Big[& \frac{18}{7}\left(\frac{\mu}{p}\right)\frac{1}{\nu m}\left\{\frac{\partial}{\partial x}\left(\frac{\mu}{h}\frac{\partial h}{\partial z}\right)+\frac{\partial}{\partial z}\left(\frac{\mu}{h}\frac{\partial h}{\partial x}\right)\right\} \\ & + \frac{18}{7}\left(\frac{\mu}{p}\right)^2\frac{1}{h}\left\{\frac{\partial h}{\partial z}\frac{Du}{Dt}+\frac{\partial h}{\partial x}\frac{Dw}{Dt}\right\} \\ & - \frac{40}{147}\left(\frac{\mu}{p}\right)^2\Big\{3\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right) \\ & - 2\left(2\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}-\frac{\partial w}{\partial z}\right)\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\Big\}\Big]\end{aligned}$$

(2.82)

${}_2C_{21}^{(1)}$ is obtained from ${}_2B_{21}$ by making the substitutions,

$x \rightarrow y,\ y \rightarrow x;\ u \rightarrow v,\ v \rightarrow u.$ (2.83)

$$\begin{aligned}{}_2B_{21}^{(2)} = \Big[& \frac{36}{7}\frac{\mu}{p}\frac{1}{\nu m}\left\{\frac{\partial}{\partial x}\left(\frac{\mu}{h}\frac{\partial h}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{\mu}{h}\frac{\partial h}{\partial y}\right)\right\} \\ & + \frac{36}{7}\left(\frac{\mu}{p}\right)^2\frac{1}{h}\left\{\frac{\partial h}{\partial x}\frac{Du}{Dt}-\frac{\partial h}{\partial y}\frac{Dv}{Dt}\right\} \\ & - \frac{40}{147}\left(\frac{\mu}{p}\right)^2\Big\{4\left[\left(\frac{\partial u}{\partial x}\right)^2-\left(\frac{\partial v}{\partial y}\right)^2\right]-3\left[\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2-\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)^2\right] \\ & + 8\frac{\partial w}{\partial z}\left(\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\right)\Big\}\Big]\end{aligned}$$

(2.84)

$$\begin{aligned}{}_2C_{21}^{(2)} = \Big[& \frac{36}{7}\frac{\mu}{p}\frac{1}{\nu m}\left\{\frac{\partial}{\partial x}\left(\frac{\mu}{h}\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{\mu}{h}\frac{\partial h}{\partial x}\right)\right\} \\ & + \frac{36}{7}\left(\frac{\mu}{p}\right)^2\frac{1}{h}\left\{\frac{\partial h}{\partial x}\frac{Dv}{Dt}+\frac{\partial h}{\partial y}\frac{Du}{Dt}\right\} \\ & - \frac{80}{147}\left(\frac{\mu}{p}\right)^2\Big\{3\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) \\ & - 2\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\Big\}\Big]\end{aligned}$$

(2.85)

$$ {}_2B_{22} = -0.4166 \left(\frac{\mu}{p}\right)^2 \frac{1}{mh^3}\left[\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 - 2\left(\frac{\partial h}{\partial z}\right)^2\right] \tag{2.86} $$

$$ {}_2B^{(1)}_{22} = 2.500 \left(\frac{\mu}{p}\right)^2 \frac{1}{mh^3} \frac{\partial h}{\partial z}\frac{\partial h}{\partial x} \tag{2.87} $$

$$ {}_2C^{(1)}_{22} = 2.500 \left(\frac{\mu}{p}\right)^2 \frac{1}{mh^3} \frac{\partial h}{\partial z}\frac{\partial h}{\partial y} \tag{2.88} $$

$$ {}_2B^{(2)}_{22} = 2.500 \left(\frac{\mu}{p}\right)^2 \frac{1}{mh^3}\left[\left(\frac{\partial h}{\partial x}\right)^2 - \left(\frac{\partial h}{\partial y}\right)^2\right] \tag{2.89} $$

$$ {}_2C^{(2)}_{22} = 4.999 \left(\frac{\mu}{p}\right)^2 \frac{1}{mh^3} \frac{\partial h}{\partial x}\frac{\partial h}{\partial y} \tag{2.90} $$

$$ \begin{aligned} {}_2B_{30} = {} & \frac{4}{15}\left(\frac{\mu}{p}\right)^2 \Bigg[2\sqrt{hm}\left\{ \frac{Dw}{Dt}\left(2\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) - \frac{Du}{Dt}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \right. \\ & \qquad \left. - \frac{Dv}{Dt}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right\} \\ & - 3\frac{1}{h\sqrt{hm}}\left\{ \frac{\partial h}{\partial z}\left(2\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) - \frac{\partial h}{\partial x}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \right. \\ & \qquad \left. - \frac{\partial h}{\partial y}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right\} \\ & + \frac{1}{\nu}\left\{ \frac{\partial}{\partial z}\left[\frac{\nu}{\sqrt{hm}}\left(2\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\right] - \frac{\partial}{\partial x}\left[\frac{\nu}{\sqrt{hm}}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] \right. \\ & \qquad \left. - \frac{\partial}{\partial y}\left[\frac{\nu}{\sqrt{hm}}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right]\right\}\Bigg] \end{aligned} \tag{2.91} $$

$$ \begin{aligned} {}_2B^{(1)}_{30} = {} & \frac{4}{15}\left(\frac{\mu}{p}\right)^2 \Bigg[2\sqrt{hm}\left\{ \frac{Du}{Dt}\left(4\frac{\partial w}{\partial z} - 3\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) - \frac{Dv}{Dt}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \right. \\ & \qquad \left. + 4\frac{Dw}{Dt}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right\} \\ & - 3\frac{1}{h\sqrt{hm}}\left\{ \frac{\partial h}{\partial x}\left(4\frac{\partial w}{\partial z} - 3\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) - \frac{\partial h}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \right. \\ & \qquad \left. + 4\frac{\partial h}{\partial z}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right\} \\ & + \frac{1}{\nu}\left\{ \frac{\partial}{\partial x}\left[\frac{\nu}{\sqrt{hm}}\left(4\frac{\partial w}{\partial z} - 3\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\right] - \frac{\partial}{\partial y}\left[\frac{\nu}{\sqrt{hm}}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right. \right. \\ & \qquad \left.\left. + 4\frac{\partial}{\partial z}\left[\frac{\nu}{\sqrt{hm}}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right]\right\}\Bigg] \end{aligned} \tag{2.92} $$

${}_2C_{30}^{(1)}$ is obtained from ${}_2B_{30}^{(1)}$ by making the substitutions

$$x \rightarrow y,\ y \rightarrow x;\ u \rightarrow v,\ v \rightarrow u. \tag{2.93}$$

$$\begin{aligned}
{}_2B_{30}^{(2)} = \frac{8}{3}\left(\frac{\mu}{p}\right)^2 \Big[& 2\sqrt{hm}\left\{ \frac{Du}{Dt}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) - \frac{Dv}{Dt}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right) + \frac{Dw}{Dt}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)\right\} \\
& - 3\frac{1}{h\sqrt{hm}}\left\{ \frac{\partial h}{\partial x}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) - \frac{\partial h}{\partial y}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right) + \frac{\partial h}{\partial z}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)\right\} \\
& + \frac{1}{\gamma}\left\{ \frac{\partial}{\partial x}\left[\frac{\gamma}{\sqrt{hm}}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right] - \frac{\partial}{\partial y}\left[\frac{\gamma}{\sqrt{hm}}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right] \right. \\
& \left. + \frac{\partial}{\partial z}\left[\frac{\gamma}{\sqrt{hm}}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)\right]\right\}\Big]
\end{aligned} \tag{2.94}$$

$$\begin{aligned}
{}_2C_{30}^{(2)} = \frac{8}{3}\left(\frac{\mu}{p}\right)^2 \Big[& 2\sqrt{hm}\left\{ \frac{Du}{Dt}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right) + \frac{Dv}{Dt}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) + \frac{Dw}{Dt}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right\} \\
& - 3\frac{1}{h\sqrt{hm}}\left\{ \frac{\partial h}{\partial x}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right) + \frac{\partial h}{\partial y}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) + \frac{\partial h}{\partial z}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right\} \\
& + \frac{1}{\gamma}\left\{ \frac{\partial}{\partial x}\left[\frac{\gamma}{\sqrt{hm}}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\frac{\gamma}{\sqrt{hm}}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right] \right. \\
& \left. + \frac{\partial}{\partial z}\left[\frac{\gamma}{\sqrt{hm}}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]\right\}\Big]
\end{aligned} \tag{2.95}$$

$$\begin{aligned}
{}_2B_{30}^{(3)} = 8\left(\frac{\mu}{p}\right)^2 \Big[& 2\sqrt{hm}\left\{ \frac{Du}{Dt}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right) - \frac{Dv}{Dt}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\} \\
& - 3\frac{1}{h\sqrt{hm}}\left\{ \frac{\partial h}{\partial x}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right) - \frac{\partial h}{\partial y}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\} \\
& + \frac{1}{\gamma}\left\{ \frac{\partial}{\partial x}\left[\frac{\gamma}{\sqrt{hm}}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)\right] \right. \\
& \left. - \frac{\partial}{\partial y}\left[\frac{\gamma}{\sqrt{hm}}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]\right\}\Big]
\end{aligned} \tag{2.96}$$

$$
{}_2C^{(3)}_{30} = 8\left(\frac{\mu}{p}\right)^2 \Big[2\sqrt{hm}\left\{ \frac{Du}{Dt}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) + \frac{Dv}{Dt}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)\right\}
$$

$$
- 3\,\frac{1}{h\sqrt{hm}}\left\{ \frac{\partial h}{\partial x}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) + \frac{\partial h}{\partial y}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)\right\}
$$

$$
+ \frac{1}{\nu}\left\{ \frac{\partial}{\partial x}\left[\frac{\nu}{\sqrt{hm}}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\frac{\nu}{\sqrt{hm}}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)\right]\right\}\Big]
$$

(2.97)

$$
{}_2B_{31} = -0.6000\left(\frac{\mu}{p}\right)^2 \frac{1}{h\sqrt{hm}}\Big[\frac{\partial h}{\partial z}\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)
$$

$$
- \frac{\partial h}{\partial x}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) - \frac{\partial h}{\partial y}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\Big]
$$

(2.98)

$$
{}_2B^{(1)}_{31} = -0.6000\left(\frac{\mu}{p}\right)^2 \frac{1}{h\sqrt{hm}}\Big[\frac{\partial h}{\partial x}\left(4\frac{\partial w}{\partial z}-3\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)
$$

$$
- \frac{\partial h}{\partial y}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) + 4\frac{\partial h}{\partial z}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\Big]
$$

(2.99)

$$
{}_2C^{(1)}_{31} = -0.6000\left(\frac{\mu}{p}\right)^2 \frac{1}{h\sqrt{hm}}\Big[\frac{\partial h}{\partial y}\left(4\frac{\partial w}{\partial z}-3\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\right)
$$

$$
- \frac{\partial h}{\partial x}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) + 4\frac{\partial h}{\partial z}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\Big]
$$

(2.100)

$$ {}_2B_{31}^{(2)} = -6.000 \left(\frac{\mu}{p}\right)^2 \frac{1}{h\sqrt{hm}} \left[\frac{\partial h}{\partial x}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) - \frac{\partial h}{\partial y}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) + \frac{\partial h}{\partial z}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\right] \tag{2.101} $$

$$ {}_2C_{31}^{(2)} = -6.000 \left(\frac{\mu}{p}\right)^2 \frac{1}{h\sqrt{hm}} \left[\frac{\partial h}{\partial x}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) + \frac{\partial h}{\partial y}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) + \frac{\partial h}{\partial z}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] \tag{2.102} $$

$$ {}_2B_{31}^{(3)} = -18.000 \left(\frac{\mu}{p}\right)^2 \frac{1}{h\sqrt{hm}} \left[\frac{\partial h}{\partial x}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) - \frac{\partial h}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \tag{2.103} $$

$$ {}_2C_{31}^{(3)} = -18.000 \left(\frac{\mu}{p}\right)^2 \frac{1}{h\sqrt{hm}} \left[\frac{\partial h}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \frac{\partial h}{\partial y}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\right] \tag{2.104} $$

$$ {}_2B_{40} = .06486 \left(\frac{\mu}{p}\right)^2 \left[2\left(2\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)^2 - 4\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 - 4\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 \right] \tag{2.105} $$

$$
{}_2B_{40}^{(1)} = -0.6486\left(\frac{\mu}{p}\right)^2 \left[\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) + \left(3\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - 4\frac{\partial w}{\partial z}\right)\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right]
$$

(2.106)

$$
{}_2C_{40}^{(1)} = -0.6486\left(\frac{\mu}{p}\right)^2 \left[\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) + \left(\frac{\partial u}{\partial x} + 3\frac{\partial v}{\partial y} - 4\frac{\partial w}{\partial z}\right)\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right]
$$

(2.107)

$$
{}_2B_{40}^{(2)} = 3.892\left(\frac{\mu}{p}\right)^2 \left[\left(\frac{\partial v}{\partial y}\right)^2 - \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2 - \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + 2\frac{\partial w}{\partial z}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\right]
$$

(2.108)

$$
{}_2C_{40}^{(2)} = 3.892\left(\frac{\mu}{p}\right)^2 \left[\left(2\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + 2\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right]
$$

(2.109)

$$
{}_2B_{40}^{(3)} = 27.24\left(\frac{\mu}{p}\right)^2 \left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) - \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right]
$$

(2.110)

$$ {}_2C_{40}^{(3)} = 27.24 \left(\frac{\mu}{p}\right)^2 \left[\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \right] \tag{2.111} $$

$$ {}_2B_{40}^{(4)} = 54.48 \left(\frac{\mu}{p}\right)^2 \left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)^2 - \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)^2 \right] \tag{2.112} $$

$$ {}_2C_{40}^{(4)} = 108.96 \left(\frac{\mu}{p}\right)^2 \left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \right] . \tag{2.113} $$

2.7 Discussion of the Results

The result of the preceding calculations may be summarized as follows:

If one substitutes the values of the coefficients as given by equations (2.41) and (2.68) to (2.113) into Burnett's form of the velocity distribution function, equation (2.39), setting all other coefficients which are not listed equal to zero, the expression so obtained is the approximation to the distribution function for Maxwell molecules which is accurate to terms of order $\left(\frac{1}{\nu}\right)^2$ or $\left(\frac{\mu}{p}\right)^2$.

It is evident from equations (2.68) to (2.113), that the terms of the distribution function which are of order $\left(\frac{\mu}{p}\right)$ contain only first derivatives of the mean motion variables (u, v, w, h, ν), whereas the terms of the order $\left(\frac{\mu}{p}\right)^2$ contain either second derivatives of these variables, or the products of two first derivatives. The reason for this lies, of course, in the basic structure of the Boltzmann integro-differential equation. It was seen that a consideration of the terms of order $\left(\frac{\mu}{p}\right)^0$ results in the Maxwell distribution function f_o, for a uniform gas. Terms of the order $\left(\frac{\mu}{p}\right)$ will account for "small deviations" from this basic distribution, and since small deviations from a uniform state of a gas, can be completely described by the gradients of the variables of state, the inherent association of the factor $\left(\frac{\mu}{p}\right)$ with first order derivatives is evident. The extension of this argument to the higher order terms of the distribution function coefficients is obvious. It follows, furthermore, that the pressure regime for which the second order terms of the distribution function are of significance depends on the magnitude of the velocity and thermal gradients which are present in the non-uniform gas.

PART III

THE MACROSCOPIC DIFFERENTIAL EQUATIONS OF MOTION

The laws of the classical dynamics are based on the concept of the conservation of three fundamental properties of a system, its mass, momentum, and energy. When these laws of conservation are applied to a continuum, it is found that its motion is completely described mathematically by one vector and two scalar partial differential equations, which are referred to as the continuity, momentum, and energy equations, respectively.

The same result can be deduced directly from the kinetic theory of non-uniform gases as was first demonstrated by J. C. Maxwell. The method of this derivation has already been mentioned in Part II in connection with Hilbert's conditions of integrability, equation (2.30), for the Boltzmann integral equation, and the Maxwell transport equation, equation (2.45), which was there used for the calculation of the Burnett coefficients. The derivation sketched below is that given by D. Burnett, (Reference 3.1).

3.1 Deduction of Equations of Macroscopic Motion from Kinetic Theory

It will be recalled that the integral on the right side of equation (2.49) is zero for those molecular properties, Q, defined by equations (2.47) and (2.50), which correspond to the mass, momentum components, and kinetic energy of translation of a molecule. In particular, for case (a) of equation (2.50), (k,r) = (0,0), one has from equations (2.47), (A.3), and (A.54)

$$Q_{oo} = 1 \qquad (3.1)$$

which is a constant and hence proportional to the mass m of a molecule.

If one evaluates the expressions of the type (2.48), of which the left side of the Maxwell transfer equation (2.49) is composed, one finds that for the function Q_{oo} the transfer equation reduces to

$$\frac{\partial \nu}{\partial t} + \frac{\partial}{\partial x}(\nu u) + \frac{\partial}{\partial y}(\nu v) + \frac{\partial}{\partial z}(\nu w) = 0 \quad . \tag{3.2}$$

By multiplying this equation by the molecular mass m, one obtains the standard form of the hydrodynamic equation of continuity.

In similar fashion, the molecular property corresponding to equation (2.50) (b) is

$$Q_{10}^{(0)} = 2\sqrt{hm}\ W \tag{3.3}$$

which is proportional to the component of molecular momentum in the z direction. When the function $Q_{10}^{(0)}$ is used, the Maxwell transfer equation is reduced to the form

$$\frac{Dw}{Dt} + \frac{1}{\nu m}\frac{\partial}{\partial z}\left(\frac{\nu}{2h}\right) + \frac{1}{4\nu m}\left[\frac{\partial}{\partial x}\left(\frac{\nu}{h}B_{20}^{(1)}\right)\right.$$
$$\left. + \frac{\partial}{\partial y}\left(\frac{\nu}{h}C_{20}^{(1)}\right) + 2\frac{\partial}{\partial z}(\nu h\, B_{20})\right] = 0 \tag{3.4}$$

by means of equations (2.3) and (2.58), the above can be expressed in terms of the pressure p and density ρ of the gas to give

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{1}{\rho}\left\{\frac{\partial}{\partial x}\left(\frac{1}{2}B_{20}^{(1)}\cdot p\right) + \frac{\partial}{\partial y}\left(\frac{1}{2}C_{20}^{(1)}\cdot p\right) + \frac{\partial}{\partial z}(B_{20}\,p)\right\} \tag{3.5}$$

Since equation (3.5) has the form of the Eulerian momentum equation in the z direction it is now possible to identify the Burnett coefficient B_{20} with the normal stress τ_{zz}, and the coefficients $B_{20}^{(1)}$ and $C_{20}^{(1)}$ with the shear stress components τ_{zx} and τ_{zy} of the gas, as indicated by equation (3.6).

$$\mathcal{T}_{zz} = B_{20}\cdot p \qquad \left.\begin{aligned} \mathcal{T}_{zx} &= \tfrac{1}{2}\, B_{20}^{(1)}\cdot p \\ \mathcal{T}_{zy} &= \tfrac{1}{2}\, C_{20}^{(1)}\cdot p \end{aligned}\right\} \tag{3.6}$$

From equation (2.62) it may be seen that the first approximation of these coefficients corresponds to the ordinary viscous stresses which are the result of a generalization of the Newtonian friction law to three dimensions. This generalization of the Newtonian law is usually obtained by a formal argument based on the invariance of this law under coordinate transformations for isotropic media. However, it should be remarked at this point that the explicit relationship between the coefficient of viscosity and the parameters which define the law of force for a particular molecular model is evaluated from equations (2.62) and (3.6). For example, for the Maxwell molecule one obtains the expression given by equation (A.96) of Appendix III, Section 3. Therefore the kinetic theory does actually give something more: The explicit relation between the viscosity coefficient and the molecular properties.

If the first approximations for the stresses are substituted into the exact equation of motion, (3.5), one obtains the familiar Navier-Stokes differential equation applicable to a viscous compressible fluid, namely

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{\mu}{\rho}\left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \frac{1}{3}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\}. \tag{3.7}$$

In the previous part of this thesis, it was shown that the Burnett coefficients $B_{kr}^{(\ell)}$ are approximated satisfactorily by the first approximations ${}_1B_{kr}^{(\ell)}$ only if (a) the velocity gradients are so small that the second derivatives of the velocity components may be neglected, or if (b) the pressure of the gas is sufficiently great, so that $\left(\frac{\mu}{p}\right)^2$ is negligible. Consequently, the Navier-Stokes equation, equation (3.7), is not applicable to the high speed flow of a rarefied gas or to the nearly discontinuous motion, such as exists within a shockwave, even at atmospheric pressure. A better approximation for the equation of motion required to describe such phenomena, is obtained by substituting the second approximations, ${}_2B_{20}$, ${}_2B^{(1)}_{20}$, ${}_2C^{(1)}_{20}$, as given by equations (2.76) to (2.78) for Maxwellian molecules, into the exact momentum equation (3.4).

If one evaluates the Maxwell transfer equation for the functions

$$Q^{(1)}_{10} = 2\sqrt{hm}\; U \qquad \text{or} \qquad 2\sqrt{hm}\; V \tag{3.8}$$

the momentum equations in the x and y directions, respectively, are obtained. However, the expressions so obtained will not have as simple a form as equation (3.4), due to the fact that the stress components $\mathcal{T}_{xx}$, $\mathcal{T}_{xy}$, $\mathcal{T}_{yy}$ are not proportional to a single Burnett coefficient, but are represented by linear combinations of the coefficients B_{20}, $B^{(1)}_{20}$, $B^{(2)}_{20}$, $C^{(1)}_{20}$, $C^{(2)}_{20}$. This lack of symmetry is caused by the fact that the velocity distribution function, equation (2.39), was expanded in terms of the spherical harmonics Y_k for which the z-axis is a preferred direction. However, the dynamic equations in the x and y directions and the stress components $\mathcal{T}_{xx}$, $\mathcal{T}_{yy}$, $\mathcal{T}_{xy}$, are readily evaluated by cyclic substitution from equations (3.5) and (3.6) respectively. The resulting equations are most conveniently expressed in the cartesian tensor notation. Equation (3.15) below is thus the exact dynamic equation for the motion of a gas. The best approximation available for the stresses $\mathcal{T}_{ij}$ is that calculated by D. Burnett accurate to terms of order

$\left(\frac{\mu}{p}\right)^2$ and given by equation (3.17).

The exact expression for the conservation of energy of the gas is obtained by evaluating the Maxwell transfer equation (2.49), for the function

$$Q_{01}^{(0)} = 1 - \tfrac{2}{3} hm C^2 \tag{3.9}$$

This function, except for an additive constant, represents the peculiar kinetic energy of translation of a molecule. The additive constant introduces into the energy equation certain terms which are eliminated from the continuity equation (3.2). The energy equation is then found to be

$$\begin{aligned}
-\frac{1}{h}\frac{Dh}{Dt} + \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) &= \\
\frac{5}{6}\frac{h}{\nu\sqrt{m}} \left\{ \frac{\partial}{\partial x}\left(\frac{\nu B_{11}^{(1)}}{h^{3/2}}\right) + \frac{\partial}{\partial y}\left(\frac{\nu C_{11}^{(1)}}{h^{3/2}}\right) + \frac{\partial}{\partial z}\left(\frac{\nu B_{11}}{h^{3/2}}\right)\right\} \\
- \frac{1}{6}\left\{ 2 B_{20}\left(2\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + 2 B_{20}^{(1)}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \right. \\
\left. + 2 C_{20}^{(1)}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + B_{20}^{(2)}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + C_{20}^{(2)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right\}.
\end{aligned} \tag{3.10}$$

The first bracket on the right side of equation (3.10) represents the net flux of thermal energy which is transported into an element of gas from adjoining elements by the process of heat conduction. The second bracket on the right side represents the viscous dissipation, that is the transformation of kinetic energy of the mean motion into kinetic energy of random motion, or thermal energy. The terms on the left side of equation (3.10) state, that this total increase in the thermal energy of the element raises the intermal energy of the gas and performs the work of expansion, according to the first law of thermodynamics.

From equation (3.10), the Burnett coefficients B_{11}, $B_{11}^{(1)}$, $C_{11}^{(1)}$ are identified with the components q_z, q_x, q_y of the heat flux vector as shown explicitly by equations (3.11).

$$B_{11} = -\frac{2}{5}\,\frac{1}{p}\sqrt{\frac{2Tm}{k}}\;q_z$$

$$B_{11}^{(1)} = -\frac{2}{5}\,\frac{1}{p}\sqrt{\frac{2Tm}{k}}\;q_x \quad ; \qquad C_{11}^{(1)} = -\frac{2}{5}\,\frac{1}{p}\sqrt{\frac{2Tm}{k}}\;q_y \tag{3.11}$$

If one uses the first approximations for these Burnett coefficients from equations (2.59) or (2.61), the ordinary expressions for the heat flux are obtained, namely

$$q_x = -\lambda\,\frac{\partial T}{\partial x} \quad , \quad etc. \tag{3.12}$$

According to the kinetic theory the coefficient of heat conduction, λ, for a monatomic gas is related to the coefficient of vicosity μ by equation (3.13) (Reference 3.2).

$$\lambda = \frac{5}{2}\,\varepsilon \cdot \mu\, c_v \tag{3.13}$$

c_v is the specific heat at constant volume, and the numerical constant ε has the value 1 for Maxwell molecules and 1.009 for elastic spherical molecules. (see equation 2.56)

The energy equation (3.10) may be brought into a more symmetric form by means of appropriate reductions from the continuity and momentum equations. Thus equation (3.16) below, again in cartesian tensor notation, is the equivalent equation of energy of a viscous compressible fluid. This form, of the energy equation is, of course, that obtained by direct application of the first law of thermodynamics to the motion of a continuum.

The exact differential equations for the mean motion of the non-uniform gas are summarized below in the notation used by H. S. Tsien (Reference 3.3):

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0^* \qquad (3.14)$$

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} - \frac{\partial \mathcal{T}_{ij}}{\partial x_j} , \quad (i = 1,2,3) \qquad (3.15)$$

$$\left(\rho \frac{\partial}{\partial t} + \rho u_i \frac{\partial}{\partial x_i}\right)\left(\frac{1}{2} u_i u_i + c_p T\right) = \frac{\partial p}{\partial t} - \frac{\partial q_i}{\partial x_i} - \frac{\partial}{\partial x_i}(\mathcal{T}_{ij} u_j) . \qquad (3.16)$$

3.2 Formulae for the Stress Tensor and Heat Flux Vector

For a particular molecular model, the components of the heat flux vector, q_i, and the components of the viscous stress tensor, $\mathcal{T}_{ij}$, are calculated from the Burnett coefficients B_{11} and B_{20} by means of equations (3.11) and (3.6), respectively. The best approximations available for the coefficients B_{20}, are those calculated by D. Burnett accurate to terms of order $\left(\frac{\mu}{p}\right)^2$, for both spherical and Maxwell molecules. The expressions for Maxwell molecules are given by equations (2.76) to (2.78); those for spherical molecules are similar but somewhat longer. The second approximation to the stress tensor $\mathcal{T}_{ij}$, as given by equation (3.17), may be evaluated for both Maxwell and spherical molecules by choosing the appropriate values of the numerical constants $K_1, K_2, \ldots, K_6$ from Table 3.1. Equation (3.17) was reduced to the Cartesian tensor notation from the vector-dyadic-tensor notation of Chapman and Cowling (Reference 3.4), by H. S. Tsien (Reference 3.3).

Similarly, the second approximation to the heat flux vector q_i can be

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* The summation convention is used; for example

$$u_j \frac{\partial u_i}{\partial x_j} \equiv u_1 \frac{\partial u_i}{\partial x_1} + u_2 \frac{\partial u_i}{\partial x_2} + u_3 \frac{\partial u_i}{\partial x_3} , \quad (i = 1,2,3) .$$

obtained from the coefficients $_2B_{11}$, $_2B_{11}^{(1)}$, $_2C_{11}^{(1)}$ given by equations (2.70) to (2.72), for Maxwell molecules. Equation (3.16) for the heat flux, was, however, deduced from the more symmetrical expressions of Chapman and Cowling. The numerical constants $\theta_1, \ldots, \theta_5$, have been calculated only for the Maxwell molecules and are given in Table 3.1.

$$
\begin{aligned}
{}_2\tau_{ij} = -2\mu\, e_{ij} + \Big[& K_1 \frac{\mu^2}{p} \frac{\partial u_k}{\partial x_k} e_{ij} + K_2 \frac{\mu^2}{p} \Big\{ -\overline{\frac{\partial}{\partial x_i}\Big(\frac{1}{\rho} \frac{\partial p}{\partial x_j}\Big)} \\
& - \overline{\frac{\partial u_k}{\partial x_i} \frac{\partial u_j}{\partial x_k}} - 2\, \overline{e_{ik} \frac{\partial u_j}{\partial x_k}} \Big\} + K_3 \frac{\mu^2}{\rho T} \overline{\frac{\partial^2 T}{\partial x_i \partial x_j}} \\
& + K_4 \frac{\mu^2}{p\rho T} \overline{\frac{\partial p}{\partial x_i} \frac{\partial T}{\partial x_j}} + K_5 \frac{\mu^2}{\rho T^2} \overline{\frac{\partial T}{\partial x_i} \frac{\partial T}{\partial x_j}} + K_6 \frac{\mu^2}{p} \overline{e_{ik} e_{kj}} \Big]
\end{aligned}
$$

(i,j = 1,2,3) (3.17)

$$
\begin{aligned}
{}_2q_i = -\lambda \frac{\partial T}{\partial x_i} + \Big[& \theta_1 \frac{\mu^2}{\rho T} \frac{\partial u_j}{\partial x_j} \frac{\partial T}{\partial x_i} + \theta_2 \frac{\mu^2}{\rho T} \Big\{ \frac{2}{3} \frac{\partial}{\partial x_i}\Big(T \frac{\partial u_j}{\partial x_j}\Big) \\
& + 2 \frac{\partial u_j}{\partial x_i} \frac{\partial T}{\partial x_j} \Big\} + \Big\{ \theta_3 \frac{\mu^2}{\rho p} \frac{\partial p}{\partial x_j} + \theta_4 \frac{\mu^2}{\rho} \frac{\partial}{\partial x_j} + \theta_5 \frac{\mu^2}{\rho T} \frac{\partial T}{\partial x_j} \Big\} e_{ij} \Big]
\end{aligned}
$$

(i = 1,2,3) (3.18)

In the above equations, the rate of strain tensor e_{ij} is defined by

$$
e_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \Big) - \frac{1}{3} \frac{\partial u_k}{\partial x_k} \cdot \delta_{ij} \quad ,
$$

where $\delta_{ij} = 1$ if $i = j$

$= 0$ if $i \neq j$. (3.19)

The bar over any tensor A_{ij} denotes the associated symmetrical and non-divergent tensor defined by equation (3.20)

$$\overline{A_{ij}} = \frac{1}{2}\left(A_{ij} + A_{ji} \right) - \frac{1}{3} A_{KK} \cdot \delta_{ij} \quad . \tag{3.20}$$

It should be noted that the form of equations (3.17) and (3.18), as derived in a very general manner by S. Chapman, is valid for any spherically symmetric molecular model possessing only translational kinetic energy, irrespective of the law of force which determines the details of intermolecular collision. However, the only numerical values of the constants K and θ are those resulting from the calculations of D. Burnett, as given in Table 3.1.

3.3 Discussion of Results

It is evident from equation (3.17), that the viscous stresses occurring in a gas may be caused by pressure and temperature gradients as well as by velocity gradients. Similarly equation (3.18) shows that a transfer of thermal energy by conduction may result from pressure and velocity gradients as well as from temperature gradients. However, if the pressure of the gas is "sufficiently large" and the velocity, pressure and temperature gradients are "sufficiently small", then the ordinary expressions for the viscous stresses and heat flux, namely

$$_{1}\mathcal{T}_{ij} = -2\mu e_{ij} \qquad \text{and} \qquad _{1}q_{i} = -\lambda \frac{\partial T}{\partial x_i} \quad , \tag{3.21}$$

will correctly represent the state of the non-uniform gas. The above condition can be formulated more precisely by examining the order of magnitude of the ratio B, of a typical additional term of the stress and heat flux to the first approximations of equation (3.21). Thus

$$B \sim \frac{\mu}{p} \frac{\partial u}{\partial y} \sim \frac{\mu}{p} \frac{U}{L} \quad ,$$

where L and U are the characteristic length and velocity, respectively, which determine the velocity gradient in a particular case. The following well known relationships are useful in transforming the ratio $\frac{\mu}{p}$.

$$\frac{p}{\rho} = RT = \frac{a^2}{\gamma} = \frac{\pi}{8}\bar{c}^2 \qquad (3.22)$$

$$\mu = 0.499\,\rho\,\bar{c}\,\ell \qquad (3.23)$$

Here R denotes the gas constant *, γ the ratio of specific heats, a the adiabatic speed of sound propagation, $\bar{c}$ the mean peculiar speed of the molecules, and ℓ the mean free path of the gas. Consequently

$$\frac{\mu}{p} \sim \frac{\mu}{\rho}\cdot\frac{1}{a^2} \sim \bar{c}\,\ell\cdot\frac{1}{a^2} \sim \frac{\ell}{a},$$

so that the ratio B is given by

$$B \sim \frac{\mu}{p}\cdot\frac{U}{L} \sim \frac{U}{a}\cdot\frac{\ell}{L} = M\cdot\frac{\ell}{L}, \qquad (3.24)$$

where M is the characteristic Mach number. The characteristic length L may represent, for example, the wave length of a sound wave, the boundary layer thickness, or the gap between two parallel plates which form the boundaries for Couette flow. Since at a pressure of one atmosphere the mean free path, ℓ, for air is of the order of 6 x 10^{-6}cm., the ratio in that case is negligibly small, in general, so that the effects of the additional stress and heat flux terms are entirely negligible at normal pressures even for Mach numbers of order 10. As the mean free path is inversely proportional to the density ρ, it is evident that the ratio B will have an appreciable magnitude for the high speed flow of a rarefied gas. Hence the complicated expressions (3.17) and (3.18) must be used for the stresses in a rarefied gas.

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* The gas constant R is that commonly employed in engineering; its numerical value depends on the molecular weight of the gas. $R = \frac{k}{m}$ where k is the universal Boltzmann constant and m is the mass of one molecule.

It follows from the nature of the successive approximations to the Burnett coefficients of the molecular velocity distribution function that the third approximation to the stress tensor, ${}_3\tau_{ij}$ will contain terms of the form $\left(\frac{\mu}{p}\right)^3\left(\frac{\partial u_i}{\partial x_j}\right)^3$ in addition to the terms of equation (3.17). Hence equation (3.17) ceases to be a valid approximation for the stress tensor if, for a given Mach number, the gas is so rarefied that $\left(M\frac{\ell}{L}\right)^2$ is not negligible compared to unity. This limitation applies, of course, also to the heat flux vector as given by equation (3.18).

Throughout the derivation of the equations of motion and of the expressions for the stress tensor and heat flux vector, a number of simplifying assumptions had to be made, thus imposing certain restrictions on equations (3.14) to (3.18). These limitations are summarized below:

(1) The equation of continuity (3.14) and the dynamic equations (3.15) can be applied to the motion of any non-uniform gas, (provided only that there are enough molecules per unit volume to permit a statistical analysis).

(2) The energy equation, (3.14) requires, in addition, that the gas be a perfect gas.

(3) Equations (3.17) for the second approximation to the stress tensor, ${}_2\tau_{ij}$, and the heat flux ${}_2q_{i}$ assume that

  (a) the gas is a simple, perfect monatomic gas composed of spherically symmetric molecules;

  (b) the length L and the Mach number M which characterize the flow of a non-uniform gas are so related to its mean free path , that $\left(M\frac{\ell}{L}\right)^2 \ll 1$.

(4) From Table 3.1, the numerical values of the constants θ are known accurately only for Maxwell molecules; the values of the constants K are known for both rigid elastic spheres and Maxwell molecules.

When equations (3.17) and (3.18) are applied to problems of high speed slip flows in rarefied air, the conditions (3a) above, which deal with the molecular structure of the gas, will, of course, be violated. The fact that air is a gas mixture consisting principally of oxygen and nitrogen should not cause any appreciable errors, on account of the small difference in the structure and molecular weight of the N_2 and O_2 molecules. However both of these diatomic molecules posses effectively two rotational degrees of freedom and one scarcely excited vibrational degree of freedom, in addition to the three translational degrees of freedom which alone have been accounted for by the kinetic theory. This fact may be partially accounted for by using for the ratio of the specific heats γ, the appropriate value of $\frac{7}{5}$, whenever γ appears explicitly in the equations. If the relaxation time in highly rarefied air is sufficiently long, so that the rotational energy cannot be excited effectively, a value of γ intermediate to $\frac{5}{3}$ and $\frac{7}{5}$ will be appropriate.

The above effect overshadows those inaccuracies introduced into calculations for air which arise from uncertainties in the values of the constants K and θ. In the light of equations (2.9), (2.10) and figure (1.2) it may, however, be appropriate to use for $s = \frac{T}{\mu}\frac{d\mu}{dT}$ the experimental value for air of 0.77, in an interpolation of the values of Table 3.1.

PART IV

PROPAGATION OF PLANE SOUND WAVES IN RAREFIED GASES. *

The general differential equations of macroscopic motion of a rarefied gas, as developed in Part III, are here applied to the solution of a particular problem of practical interest. The one dimensional problem of the propagation of a plane wave in a medium of infinite extent does not involve any boundary conditions and suitably illustrates the nature of the differential equations under geometrically simple conditions.

4.1 Introduction

The problem of the propagation of plane sound waves in a viscous fluid was treated as early as 1845 by G. G. Stokes, who investigated the effect of viscous action but neglected heat conduction. The effect of viscosity was found to consist of a frequency dependent damping of the amplitudes of the sound waves whose velocity of propagation, to a first approximation, was equal to the adiabatic propagation speed in a frictionless, compressible fluid. As was first pointed out by G. Kirchhoff in 1866, the influence of heat conduction is of the same order of magnitude as that of the viscosity, so that for a consistent solution both factors must be accounted for. This was done in the solution of H. Lamb (Reference 4.2) and resulted in a more highly damped motion than that predicted by Stokes, whereas the speed of propagation remained approximately equal to that for the frictionless fluid.

The validity of these results is dependent, of course, on the validity of the fundamental Navier-Stokes equations which are the basis of these calculations. No question needs to be raised against the general laws of dynamics and kinematics and the only point of doubt is the correctness of

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* The calculations and results of this investigation have been published jointly by Dr. H. S. Tsien and the author in Reference 4.1.

the viscous stresses and of the heat flux used in the Navier-Stokes equations. In the Navier-Stokes equations, the viscous stresses are taken as the product of the viscosity coefficient and the linear combination of the first order space derivatives of the velocity components, and the heat flux is taken as the product of the coefficient of heat conductivity and the gradient of the temperature of the fluid. Results of calculations using these equations agree very well with experimental observations. This fact may be used as the empirical justification of the Navier-Stokes equations.

The results of the kinetic theory of non-uniform gases, as presented in Parts II and III of this thesis show, however, that the viscous stresses and the heat flux as used in the Navier-Stokes equation are only first order approximations. If the number of gas molecules contained in a cube of dimension intrinsic to the problem, such as the wave length of sound propagation, is small, then the first order approximation is no longer sufficient. This means that if the wave length is very small, as in the case of ultrasonic waves, or if the density of gas is very low, the Navier-Stokes equation is no longer valid. Since there are many assumptions introduced in the kinetic theory to make the calculation treatable, one may question the reliability of the theoretical results. However, the success of the kinetic theory in explaining many phenomena and the prediction of the first order viscous stresses and heat flux which agrees with the observational data seem to indicate the reliability os such a theory.

In the present treatment the second approximations to the viscous stresses and the heat flux of Chapman and Burnett, as given by the equations of Section 3.2, are used to calculate the propagation of plane sound waves in rarefied gases or of plane sound waves of very small wave lengths in gases of normal density. This investigation is prompted by the fact that the present knowledge of the state of the atmosphere at high altitudes is almost exclusively obtained through the measurement of the anomalous sound propagation. A knowledge of the effect of low density of the medium on the propagation velocity and the damping of waves will be, perhaps, useful in a critical examination of this method of obtaining data for high altitudes. The result of this investigation is most reassuring as it shows that even under extreme conditions, the increase in the propagation velocity from the normal value at high density is less than 2%. In fact, the effect of the additional terms to the viscous stresses and the heat flux tends to maintain the constancy of the sound velocity with respect to the density of the medium.

4.2 Basic Equations

The fundamental differential equations required to describe the propagation of plane sound waves express the conservation of *mass*, *momentum*, *and energy* of an element of a compressible fluid which is in one-dimensional unsteady motion. Choosing the x-axis as the direction of propagation of the plane wave front, the differential equations of motion are obtained from equations (3.14) to (3.16) of Part III by means of the following reductions:

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial x} \quad ; \quad \frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_3} = 0$$

$$u_1 = u \quad ; \quad u_2 = u_3 = 0.$$

The appropriate continuity, momentum, and energy equations are then given by equations (4.1), (4.2), and (4.3), respectively, where all of the symbols are defined as in Part III.

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \tag{4.1}$$

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} = - \frac{\partial p}{\partial x} - \frac{\partial}{\partial x}(\mathcal{T}_{xx}) \tag{4.2}$$

$$\left(\rho \frac{\partial}{\partial t} + \rho u \frac{\partial}{\partial x}\right)\left(\frac{1}{2} u^2 + c_p T\right) = \frac{\partial p}{\partial t} - \frac{\partial}{\partial x}(q_x) - \frac{\partial}{\partial x}(\mathcal{T}_{xx} \cdot u) \; . \tag{4.3}$$

For purposes of subsequent calculations, it is convenient to separate the dependent variables into their "undisturbed" and "perturbation" components according to equations (4.4); the undisturbed quantities are denoted by the subscript o and the perturbations by a prime:

$$\left.\begin{aligned} p &= p_o + p' \\ T &= T_o + T' \\ \rho &= \rho_o + \rho' \\ u &= \phantom{\rho_o +{}} u' \end{aligned}\right\} \tag{4.4}$$

Since a sound wave is by definition a disturbance of infinitesimal amplitude, any quantities involving squares or products of the perturbation variables or their devivatives can be neglected in comparison to these variables themselves. For instance, from the equation of state of a perfect gas

$$\frac{p}{\rho T} = \text{const.} \tag{4.5}$$

one has

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho'}{\partial t} = \rho_o \left[\frac{1}{p_o} \frac{\partial p'}{\partial t} - \frac{1}{T_o} \frac{\partial T'}{\partial t} \right] \; . \tag{4.6}$$

The general form of the heat flux vector and the stress tensor is given by equations (3.18) and (3.17), respectively, correct to terms of order $\frac{\mu^2}{p}$. The following equations give the expressions for $_2q_x$ and $_2\Upsilon_{xx}$ appropriate to the problem at hand by neglecting second order quantitites in the perturbation variables:

$$_2q_x = -\lambda_0 \frac{\partial T}{\partial x} + \frac{2}{3}(\theta_2+\theta_4)\frac{\mu_0^2}{\rho_0}\frac{\partial^2 u'}{\partial x^2} \tag{4.7}$$

$$_2\Upsilon_{xx} = -\frac{4}{3}\mu_0 \frac{\partial u'}{\partial x} - \frac{2}{3}\mu_0^2\left[\frac{K_2}{p_0\rho_0}\frac{\partial^2 p'}{\partial x^2} - \frac{K_3}{\rho_0 T_0}\frac{\partial^2 T'}{\partial x^2}\right], \tag{4.8}$$

where μ_0 and λ_0 are the viscosity and heat conduction coefficients at the temperature T_0. It will be recalled that θ_2, θ_4, K_2, K_3 are constants whose exact numerical values depend on the intramolecular structure of the molecules composing the gas (see Table 3.1).

Substitution of equations (4.4), (4.6), (4.7), and (4.8) into the exact differential equations (4.1), (4.2) and (4.3), and subsequent ommission of all second order terms in the dependent variables, result in the linearized partial differential equations (4.9), (4.10), and (4.11).

$$\frac{1}{p_0}\frac{\partial p'}{\partial t} - \frac{1}{T_0}\frac{\partial T'}{\partial t} + \frac{\partial u'}{\partial x} = 0 \tag{4.9}$$

$$\rho_0 \frac{\partial u'}{\partial t} = -\frac{\partial p'}{\partial x} + \frac{4}{3}\mu_0 \frac{\partial^2 u'}{\partial x^2} + \frac{2}{3}\mu_0^2\left[\frac{K_2}{\rho_0 p_0}\frac{\partial^3 p'}{\partial x^3} - \frac{K_3}{\rho_0 T_0}\frac{\partial^3 T'}{\partial x^3}\right] \tag{4.10}$$

$$\rho_0 C_p \frac{\partial T'}{\partial t} = \frac{\partial p'}{\partial t} + \lambda_0 \frac{\partial^2 T'}{\partial x^2} - \frac{2}{3}(\theta_2+\theta_4)\frac{\mu^2}{\rho_0}\frac{\partial^3 u'}{\partial x^3} \quad . \tag{4.11}$$

It is convenient to reduce the above equations to dimensionless form by means of the non-dimensional parameters defined below:

$$p^* = \frac{p'}{p_0} \quad , \quad T^* = \frac{T'}{T_0} \quad , \quad u^* = \frac{u'}{c_0} \quad ; \tag{4.12}$$

$$x^* = \frac{x}{L} \quad , \quad t^* = \frac{t}{(L/c_0)} \quad , \tag{4.13}$$

where L is the wave length and c_0 the adiabatic speed of sound propagation defined by

$$c_0 = \sqrt{\gamma \frac{p_0}{\rho_0}} \quad . \tag{4.14}$$

γ here is the ratio of specific heats.

The physical constants of the gas may be expressed in terms of the two dimensionless parameters, β and R

$$\beta = \frac{\lambda_0}{\mu_0 C_p} \quad , \qquad R = \left[\frac{\rho_0 c_0}{\mu_0} \cdot \frac{L}{2\pi} \right] \quad . \tag{4.15}$$

$\frac{1}{\beta} = \frac{\mu_0 C_p}{\lambda_0}$ is the Prandtl number and is a measure of the relative importance of the viscosity and the heat conductivity of the gas. From the point of view of the kinetic theory, viscosity is the result of the transfer of the momentum of the molecules, and heat conduction is the result of the transfer of the energy of the molecules. Both thus must be of the same order of magnitude. This is in agreement with experiment, because the Prandtl number is found to be of the order of unity. The Reynold's number is a measure of the relative importance of the inertia forces and the dissipative forces. If the dissipative forces are very small in comparison with the inertia forces, the Reynold's number will be very large. With increasing importance of the dissipative forces, the Reynold's number decreases. In the propagation of sound waves, the dissipative forces are measured by gradients of velocity and temperature and are inversely proportional to the wave length L for any given amplitude. Therefore in this case, the Reynold's number R is directly proportional to the wave length as shown by equation (4.15).

Substitution of equations (4.12) to (4.15) into equations (4.9) to (4.11) results in the set of three simultaneous partial differential equations, (4.16)

$$\left.\begin{aligned}
&\frac{\partial p^*}{\partial t^*} - \frac{\partial T^*}{\partial t^*} + \frac{\partial u^*}{\partial x^*} = 0 \\
&\left[\frac{\partial p^*}{\partial x^*} - \frac{2}{3}K_2\frac{\gamma}{(2\pi R)^2}\frac{\partial^3 p^*}{\partial x^{*3}}\right] + \frac{2}{3}K_3\frac{\gamma}{(2\pi R)^2}\frac{\partial^3 T^*}{\partial x^{*3}} \\
&\qquad + \left[\gamma\frac{\partial u^*}{\partial t^*} - \frac{4}{3}\frac{\gamma}{(2\pi R)}\frac{\partial^2 u^*}{\partial x^{*2}}\right] = 0 \\
&-\frac{\partial p^*}{\partial t^*} + \frac{\gamma}{\gamma-1}\left[\frac{\partial T^*}{\partial t^*} - \frac{\beta}{(2\pi R)}\frac{\partial^2 T^*}{\partial x^{*2}}\right] + \frac{2}{3}(\theta_2+\theta_4)\frac{\gamma}{(2\pi R)^2}\frac{\partial^3 u^*}{\partial x^{*3}} = 0
\end{aligned}\right\} \tag{4.16}$$

If the properties of the gas are known, the values of γ, β, K_2, K_3, θ_2 and θ_4 are fixed. Then the only remaining parameter of the problem is R which is really a measure of the wave length. Therefore, the solutions of the problem should be expressed as a function of R.

4.3 Solution of the Differential Equations.

The general solution of this set of linear partial differential equations (4.16) is evidently of the form

$$\left.\begin{aligned}
p^* &= A_1 e^{2\pi[(i-b)x^* - i\alpha t^*]} \\
T^* &= A_2 e^{2\pi[(i-b)x^* - i\alpha t^*]} \\
u^* &= A_3 e^{2\pi[(i-b)x^* - i\alpha t^*]}
\end{aligned}\right\} \tag{4.17}$$

,

where the A's represent arbitrary constants. The coefficients b and α are both real quantities, to be determined as functions of the Reynold's number R. It follows from the definitions (4.13) and (4.14) that the damping coefficient per wave length is $2\pi b$, and the physical velocity of propagation of the disturbance is given by $c = \alpha c_0$. By substituting equations (4.17) into equations (4.16), the following linear homogeneous equations for A_1 and A_3 are obtained:

$$\left[-i\alpha\right] A_1 + \left[i\alpha\right] A_2 + \left[i-b\right] A_3$$

$$\left[(i-b) - \frac{2}{3} K_2 \frac{\gamma}{R^2} (i-b)^3\right] A_1 + \left[\frac{2}{3} K_3 \frac{\gamma}{R^2} (i-b)^3\right] A_2 + \left[-i\gamma\alpha - \frac{4}{3}\frac{\gamma}{R}(i-b)^2\right] A_3 = 0$$

$$\left[i\alpha\right] A_1 + \left[\frac{\gamma}{\gamma-1}\left\{-i\alpha - \frac{\beta}{R}(i-b)^2\right\}\right] A_2 + \left[\frac{2}{3}(\theta_2+\theta_4)\frac{\gamma}{R^2}(i-b)^3\right] A_3 = 0$$

In order that the values of A_1, A_2, and A_3 should not be simultaneously equal to zero and thus give a trivial solution, the determinant formed by the coefficients of the A's must be equal to zero. Thus

$$\begin{vmatrix} \left[-i\alpha\right] & \left[i\alpha\right] & \left[i-b\right] \\ \left[(i-b) - \frac{2}{3}\frac{K_2\gamma}{R^2}(i-b)^3\right] & \left[\frac{2}{3}\frac{K_3\gamma}{R^2}(i-b)^3\right] & \left[-i\gamma\alpha - \frac{4}{3}\frac{\gamma}{R}(i-b)^2\right] \\ \left[i\alpha\right] & \left[-\frac{\gamma}{\gamma-1}\left\{i\alpha + \frac{\beta}{R}(i-b)^2\right\}\right] & \left[\frac{2}{3}(\theta_2+\theta_4)\frac{\gamma}{R^2}(i-b)^3\right] \end{vmatrix} = 0$$

By separating the real and imaginary parts of this determinant, one has the following two simultaneous equations (4.18) and (4.19) from which b and α can be determined.

$$A\left[\frac{\alpha}{R^4}(1-b^2)(1-14b^2+b^4)\right] + B\left[\frac{\alpha}{R^2}(1-6b^2+b^4)\right] + C\left[\frac{2b}{R^3}(3-10b^2+3b^4)\right] + D\left[\frac{4b}{R}(1-b^2)\right] - E\left[\frac{2b}{R}\alpha^2\right] + \alpha(\alpha^2+b^2-1) = 0 \qquad (4.18)$$

$$A\left[\frac{2\alpha b}{R^4}(3-10b^2+3b^4)\right]+B\left[\frac{4b\alpha}{R^2}(1-b^2)\right]-C\left[\frac{1}{R^3}(1-b^2)(1-14b^2+b^4)\right]$$

$$-D\left[\frac{1}{R}(1-6b^2+b^4)\right]+E\left[\frac{\alpha^2}{R}(1-b^2)\right]-2\alpha b=0 \qquad (4.19)$$

where

$$\left.\begin{aligned}
A &= \frac{4}{9}\gamma(\gamma-1)(\theta_2+\theta_4)(K_2-K_3)\\
B &= \frac{2}{3}(\gamma-1)(\theta_2+\theta_4)-\frac{4}{3}\gamma\beta-\frac{2}{3}\gamma(K_2-K_3)-\frac{2}{3}K_3\\
C &= \frac{2}{3}\gamma\beta K_2\\
D &= \beta\\
E &= \frac{4}{3}+\gamma\beta
\end{aligned}\right\} \qquad (4.20)$$

For the complete solution of the problem it would be necessary to find the nine roots of equations (4.18) and (4.19). However, from the nature of the classical solution of the problem it is known that the pair of roots of b and α which is of the greatest significance is the one with the least damping. In other words, for vanishing viscosity, or $R \to \infty$, $\alpha \to 1$ and $\beta \to 0$, so that the wave should be undamped and propagate with the normal adiabatic speed c_0. Therefore, the appropriate forms for b and α are the following:

$$b = \frac{b_1}{R}+\frac{b_3}{R^3}+\frac{b_5}{R^5}+\dots \qquad (4.21)$$

$$\alpha = 1+\frac{\alpha_2}{R^2}+\frac{\alpha_4}{R^4}+\frac{\alpha_6}{R^6}+\dots \qquad (4.22)$$

Equations (4.21) and (4.22) may be substituted into (4.18) and (4.19), and the resulting equations may be arranged in order of ascending powers of $\frac{1}{R}$.

In order that the resulting equations are satisfied for arbitrary values of R, each of the coefficients of $\frac{1}{R^n}$ must be equal to zero. This results in a set of algebraic equations from which the coefficients b_1, b_3,..., and α_2, α_4,... are determined successively in the following order: b_1, α_2, b_3, α_4,... .

Equations (4.23) and (4.24) give the final result of this solution. It should be noted that each coefficient is a function only of the previously determined coefficients and of the physical constants A, B, C, D, and E, of equation (4.20).

$$\left.\begin{aligned} b_1 &= \tfrac{1}{2}(E-D) \\ b_3 &= \left[-\tfrac{1}{2}C + 2Bb_1 + (3D-\tfrac{1}{2}E)b_1^2 + E\alpha_2 - b_1\alpha_2\right] \\ b_5 &= \left[3Ab_1 + \tfrac{15}{2}Cb_1^2 - 2Bb_1^3 - \tfrac{1}{2}Db_1^4 + 2Bb_3 + (6D-E)b_1b_3\right. \\ &\quad \left. + \tfrac{1}{2}E\alpha_2^2 + E\alpha_4 - Eb_1^2\alpha_2 + 2Bb_1\alpha_2 - b_1\alpha_4 - b_3\alpha_2\right] \end{aligned}\right\} \quad (4.23)$$

$$\left.\begin{aligned} \alpha_2 &= -\left[\tfrac{1}{2}B + (2D-E)b_1 + \tfrac{1}{2}b_1^2\right] \\ \alpha_4 &= -\left[\tfrac{1}{2}A + 3Cb_1 - 3Bb_1^2 - 2Db_1^3 + (2D-E)b_3 + b_1b_3\right. \\ &\quad \left. + \tfrac{1}{2}B\alpha_2 + \tfrac{3}{2}\alpha_2^2 - 2Eb_1\alpha_2 + \tfrac{1}{2}\alpha_2b_1^2\right] \\ \alpha_6 &= -\left[-\tfrac{15}{2}Ab_1^2 - 10Cb_1^3 + \tfrac{1}{2}Bb_1^4 + 3Cb_3 + \tfrac{1}{2}b_3^2 + (2D-E)b_5\right. \\ &\quad -6Bb_1b_3 - 6Db_1^2b_3 + b_1b_5 + \tfrac{1}{2}A\alpha_2 + \tfrac{1}{2}\alpha_2^3 + \tfrac{1}{2}B\alpha_4 + 3\alpha_4\alpha_2 \\ &\quad \left. - Eb_1\alpha_2^2 - 3Bb_1^2\alpha_2 - 2Eb_1\alpha_4 + \tfrac{1}{2}b_1^2\alpha_4 - 2Eb_3\alpha_2 + b_1b_3\alpha_2\right] \end{aligned}\right\} \quad (4.24)$$

It is interesting to note that the first order solution

$$\left.\begin{aligned} b &= \frac{b_1}{R} = \frac{1}{2R}\left[\tfrac{4}{3} + (\gamma-1)\beta\right] \\ \alpha &= 1 \end{aligned}\right\} \quad (4.25)$$

is identical with the solution given by H. Lamb (Reference 4.2).

4.4 Numerical Calculations

The constants A, B, C, D, E, appearing in equations (4.23) and (4.24), depend on the two physical constants of the gas β and γ and on the constants θ_2, θ_4, K_2, K_3. According to the discussion of Section 3.2, the latter four constants depend on the nature of the force field which surrounds the molecule, that is on the choice of the "molecular model" utilized in the calculations of the kinetic theory.

For monatomic gases, and also for air, the *rigid spherical molecule* appears to be a closer approximation to the reality than the Maxwell molecule (see figure 2.2). Hence the values of K_2 and K_3 as given for spherical molecules by Table 3.1 will be used. For θ_2 and θ_4 the only values available are those given by Table 3.1 for the Maxwell molecule. As the variation in the constants K is only of the order of 10 percent for the two limiting cases considered, and as the variations in the constants θ must be of the same order of magnitude, it can be shown a posteriori that the results obtained are not materially dependent on the precise values of the K's and θ's.

In accordance with the above discussion, the numerical values for the molecular constants are then given by equations (4.26).

$$\left.\begin{array}{ll} \theta_2 = \frac{45}{8} & \theta_4 = 3 \\ K_2 = 2.028\ , & K_3 = 2.418 \end{array}\right\} \qquad (4.26)$$

Equations (4.23), (4.24) and (4.20) will now be applied to calculate the damping coefficient, b, and the dimensionless velocity of propagation, α, in the following three gaseous media, as characterised by the values of the physical constants γ and β:

1. Air at normal atmospheric temperature. This also corresponds closely to the theoretical diatomic gas.
2. Air at 400°C
3. Theoretical monatomic gas. This corresponds very closely to real monatomic gases, such as helium, argon and neon, at normal temperatures.

Table 4.1 gives the values of γ and β for each of the three media, as well as the values of the derived constants A, B, C, D, E, as calculated from (4.20) and (4.26), and the values of the coefficients b_1, b_3, b_5, and α_2, α_4, α_6, as calculated from equations (4.23) and (4.24), respectively. The results of Table 4.1 are then used to calculate the damping coefficient b, and the propagation coefficient α as a function of the Reynold's number R, by means of equations (4.21) and (4.22), respectively.

The nature of the convergence of the expansions (4.21) and (4.22) is shown for the typical case of medium 1 in figures 4.1 and 4.2. It should be noted that the solutions $b = \frac{b_1}{R} + \frac{b_3}{R^3}$, $\alpha = 1 + \frac{\alpha_2}{R^2} + \frac{\alpha_4}{R^4}$ are not necessarily accurate to terms of order $\frac{1}{R^3}$, and $\frac{1}{R^4}$, respectively, since the original differential equations (4.16) are correct only to order $\frac{1}{R^2}$.

4.5 Results

The values of α and b as functions of the Reynold's number for each of the three gaseous media considered are shown in figures (4.3) and (4.4) the effect of decreasing Reynold's number is to increase the damping coefficient in nearly inverse ratio, whereas the speed of propagation, αc_0, increases only slightly. Figures 4.3 and 4.4 also show that the molecular structure of the gas influences the magnitude of the propagation parameters to a much larger extent than does the mean temperature of the gas. This is apparently due to the fact that equations (4.20) depend more critically on the value of

γ, which occurs also as $(\gamma - 1)$, than on β. Both γ and β depend on the number of atoms which compose the molecule, whereas only β is significantly temperature dependent.

The magnitude of the damping effect is more readily ascertained in terms of the amplitude ratio, r_a, which is defined as the ratio of two successive maximum amplitudes of the sound wave. Hence $r_a = e^{-2\pi b} < 1$. Figure 4.5 shows the amplitude ratio as a function of the Reynold's number for the monatomic and the diatomic gas.

For practical applications of the results, one must first calculate the Reynold's number R corresponding to the wave length L. Since for sound propagation problems the given physical parameter is the frequency ν instead of the wave length L, R should be calculated in terms of ν in the following way:

$$R = \frac{c}{c_o} \cdot \frac{\gamma p_o}{\mu_o} \times \frac{1}{2\pi\nu} \quad . \tag{4.27}$$

However, $\frac{c}{c_o} = \alpha$ is very closeto unity, so that

$$R \cong \frac{\gamma p_o}{\mu_o} \cdot \frac{1}{2\pi\nu} \quad . \tag{4.28}$$

To facilitate numerical calculations, Table 4.2 is constructed for R_o which gives the value of R, at various temperatures, for air at the standard pressure of one atmosphere, for $\gamma = 1.4$, and for the frequency $\nu = 1000$ cycles per second. For any other values of the pressure p_o in atmospheres, and of the frequency ν, the Reynold's number R is then given by

$$R = R_o \alpha \left(\frac{p_o}{1}\right)\left(\frac{1000}{\nu}\right) \cong R_o\left(\frac{p_o}{1}\right)\left(\frac{1000}{\nu}\right) \quad . \tag{4.29}$$

In general, the results of this investigation show that although the damping of the waves is greatly increased by the decrease in the Reynold's number R, the speed of propagation is practically unaltered. This justifies the procedure adopted in the analysis of anomalous sound propagation where the normal adiabatic propagation speed c_o is used throughout. However, one must be aware of the fact that kinetic theory uses the smooth spherical model for the molecules and thus does not allow the interchange between translational kinetic energy with the vibrational and the rotational energies of the molecules. Due to the greatly decreased number of molecular collisions in rarefied conditions corresponding to small values of R, it will be difficult to excite the vibrational and the rotational degrees of freedom and the gas tends to behave more closely like a monatomic gas with a corresponding increase in the value of γ. This change in the properties of the gas is not directly taken into account in the calculations presented, as it is beyond the framework of the usual kinetic theory. On the other hand, this effect can be easily accounted for by an appropriate change in the value of γ as the Reynold's number decreases.

PART V

THE BOUNDARY CONDITIONS FOR SLIP FLOW

5.1 General Considerations

When the differential equations of motion, which were discussed in Part III, are applied to that realm of fluid mechanics known as gas-dynamics, the associated boundary conditions are well known. In that case the general solution of equations (3.14) to (3.16) is reduced to the particular solution which corresponds to any geometrical situation of interest by means of the following physical conditions:

(1) If the viscous, compressible fluid extends to infinity, the values of the pressure, temperature, and vector velocity must be specified "at infinity".

(2) All of the "fluid particules" which are adjacent to any solid surface have the same absolute vector velocity and temperature as the corresponding element of the solid boundary.

The second condition requires that there be no flow normal to any solid surface, and that there be no slipping between the fluid and the wall.

The experiments of Knudsen, Kundt and Warburg, and many others, on the flow of rarefied gases through capillaries, have established that this "non-slip" condition is valid only if the mean free path of the gas, ℓ, is completely negligible relative to the characteristic macroscopic dimension L (such as the diameter of the capillary). Hence, for a rarefied gas, the "non-slip condition" must be replaced by some relation which specifies the slip velocity of the gas relative to the solid wall. Thus, for example, the phenomenological assumption of A. Basset specifies (Reference 5.1) that the slip velocity be proportional to the shearing stress at the wall. Similarly, the experiments of von Smoluchowski have shown that in a rarefied gas there exists a "temperature jump" between the wall temperature and the temperature of the gas layer immediately adjacent to the wall, and that, to a first approximation, this temperature jump is proportional to the temperature gradient, (normal to the wall) which exists in the gas at the vicinity of the wall.

Therefore, on purely physical grounds, one would expect that for a rarefied gas, the boundary conditions stated under (2) above would have to be modified to account for the effects of slip and temperature jump, but that the total number of such conditions should be the same for both gas-dynamical or slip flows. This conclusion appears to be at variance with the mathematical fact, established in Part III, that for a rarefied gas the partial differential equations of motion are of a higher order than those of the gas dynamics. Thus, if the stresses ${}_1\mathcal{T}_{ij}$ and the heat flux ${}_1q_i$ (equation 3.21), which are appropriate for the realm of gas dynamics, are substituted into the dynamic equation (3.15) and energy equation (3.16), both of these equations contain at most second order partial derivatives. For a rarefied gas the required expressions ${}_2\mathcal{T}_{ij}$ and ${}_2q_i$, are given by equations (3.17) and (3.18), so that in this case the momentum and energy equations are both partial differential equations of the third order. Better approximations for the stresses and heat flux will, of course, result in equations of motion of still higher order. It will now be shown that the number of boundary conditions associated with the second order differential equations of motion actually suffices for the solution of the equations of motion of the third order, or higher order.

Suppose first that $u_i(x,y,z,t)$, $p(x,y,z,t)$, $\rho(x,y,z,t)$, $T(x,y,z,t)$ represent the solutions of the "exact" equations of motion of a rarefied gas, and that $\mathcal{T}_{ij}(x,y,z,t)$ is the exact expression for the stresses. If p_0 represents the lowest pressure, in the three dimensional flow field, and μ_o the viscosity at the same point, the exact solution can be expanded as in equations (5.1).

$$\left.\begin{aligned} u_i &= {}_1u_i + \left(\frac{\mu_o}{p_o}\right) {}_2u_i^* + \left(\frac{\mu_o}{p_o}\right)^2 {}_3u_i^* + \dots \\ \rho &= {}_1\rho + \left(\frac{\mu_o}{p_o}\right) {}_2\rho^* + \left(\frac{\mu_o}{p_o}\right)^2 {}_3\rho^* + \dots \\ \tau_{ij} &= {}_1\tau_{ij} + \left(\frac{\mu_o}{p_o}\right) {}_2\tau_{ij} + \left(\frac{\mu_o}{p_o}\right)^2 {}_3\tau_{ij}^* + \dots \end{aligned}\right\} \qquad (5.1)$$

If the expressions (5.1) are substituted into the momentum equation (3.15) one obtains after collecting the terms in ascending powers of $\left(\frac{\mu_o}{p_o}\right)$

$$\begin{aligned} &\left[{}_1\rho \frac{\partial\, {}_1u_i}{\partial t} + {}_1\rho\, {}_1u_j \frac{\partial\, {}_1u_i}{\partial x_j} + \frac{\partial\, {}_1p}{\partial x_i} + \frac{\partial\, {}_1\tau_{ij}}{\partial x_j} \right] \\ &+ \left(\frac{\mu_o}{p_o}\right) \left[{}_1\rho \frac{\partial\, {}_2u_i^*}{\partial t} + {}_2\rho^* \frac{\partial\, {}_1u_i}{\partial t} + {}_1\rho\, {}_1u_j \frac{\partial\, {}_2u_i^*}{\partial x_j} + {}_1\rho\, {}_2u_j^* \frac{\partial\, {}_1u_i}{\partial x_j} \right. \\ &\qquad\qquad \left. + {}_2\rho^*\, {}_1u_j \frac{\partial\, {}_1u_i}{\partial x_j} + \frac{\partial\, {}_2p^*}{\partial x_i} + \frac{\partial\, {}_2\tau_{ij}}{\partial x_j} \right] \\ &+ \left(\frac{\mu_o}{p_o}\right)^2 \left[{}_1\rho \frac{\partial\, {}_3u_i^*}{\partial t} + \dots + \frac{\partial\, {}_3p^*}{\partial x_i} + \frac{\partial\, {}_3\tau_{ij}}{\partial x_j} \right] + \dots = 0 \end{aligned} \qquad (5.2)$$

A similar form of the energy equation can be obtained from equation (3.16). Equation (5.2) is valid for arbitrary values of the "pressure level" p_o. In particular, for the realm of gas dynamics, (5.2) reduces to the Navier-Stokes equation, from which, in conjunction with the energy equation of

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* The star indicates that ${}_2u_i^*$ is a function of (x,y,z,t) which does not have the dimensions of a velocity.

gas dynamics and the continuity equation, the general solution for ${}_1u_i$, ${}_1\rho$, ${}_1p$, ${}_1T$ is obtained. The solution of interest is then obtained by applying the boundary conditions of p. 5.1.

If the gas is rarefied such that the terms in $\left(\frac{\mu_0}{P_0}\right)$ are of significance, but those of order $\left(\frac{\mu_0}{P_0}\right)^2$ are negligible, equation (5.2) reduces to the sum of the first two brackets equated to zero, which means that the second bracket must itself vanish identically. It follows from equations (5.1) and (3.17) that the quantity ${}_2\mathcal{T}_{ij}{}^*$ is of the form

$$
{}_2J_{ij}^* \sim -\mu \frac{\partial}{\partial x_j}\left({}_2u_i^*\right) + \left\{ K_1\, \mu\, \frac{\partial\, {}_1u_k}{\partial x_k} \cdot \frac{\partial\, {}_1u_i}{\partial x_j} + \dots \right.
$$

$$
\left. + K_3\, \mu\, R\, \frac{\partial^2\, {}_1T}{\partial x_i\, \partial x_j} + \dots \right\} .
$$

Consequently, the highest derivative appearing in the second bracket of equation (5.2) is $\frac{\partial^3\, {}_1T}{\partial x_i\, \partial^2 x_j}$, which, however, is merely a known function of (x,y,z,t). It follows, therefore, that the second bracket of equation (5.2), when equated to zero, is a partial differential equation of second order in the dependent variables ${}_2u_i^*$, ${}_2p^*$, ${}_2\rho^*$, ${}_2T^*$. The same result can be shown to apply to the energy equation by an anologons argument involving also the heat flux ${}_2q_i$ of equation (3.18).

By an extension of this argument it follows then that one requires as many boundary conditions to determine each group of functions (${}_2u_i^*$, ${}_2p^*$, ${}_2\rho^*$, ${}_2T^*$), (${}_3u_i^*$, ${}_3p^*$, ${}_3\rho^*$, ${}_3T^*$), etc., as is required for the determination of (${}_1u_i$, ${}_1p$, ${}_1\rho$, ${}_1T$) from the equations of motion of gas-dynamics. Since however, a condition on each of ${}_1u_i$, ${}_2u_i^*$, ${}_3u_i^*$, etc., is equivalent to a single condition on the exact velocity component

$$u_i = {}_1u_i + \left(\frac{\mu_0}{p_0}\right) {}_2u_i^* + \left(\frac{\mu_0}{p_0}\right)^2 {}_3u_i^* + \cdots \,, \qquad \text{etc.},$$

it follows that the total number of boundary conditions required for the complete solution of the equations of motion (equations (3.14) to (3.16)) is independent of the degree of rarefaction of the gas.

This method of solution means, essentially, that no radical change of the flow pattern is expected when proceeding from viscous gas-dynamical flow to slip flow, so that one can use the iteration process as explained above. This situation is materially different from the case of transition from non-viscous gas-dynamical flow to viscous gas-dynamical flow, where a radical change occurs in the appearance of the boundary layer, and where an asymptotic integration process is required with a resultant increase in the number of boundary conditions. If it were also possible to use the iteration process for the viscous gas-dynamical flow, then the number of boundary conditions required would be exactly the same as for the non-viscous gas-dynamical flow.

The above application of the iteration method to the solution of the differential equations for slip-flow can be considered from another point of view. Suppose that it were possible to obtain the complete, general solution of the equations of motion, equations (3.14) to (3.16). This general solution will include solutions of the form of equations (5.1) in addition to other solutions which represent radical departures from the gas-dynamical flow pattern. If one imposes now the physical condition that the flow pattern for slip-flow should go over "smoothly" into the viscous gas-dynamical flow pattern, this requires that the appropriate solutions must satisfy equations (5.1). Hence from this more formal point of view, equations (5.1) represent the additional boundary conditions corresponding to the higher order of the differential equations for slip flow.

It follows from the preceding discussion that the problem of the boundary conditions for the high speed flow of a rarefied gas is reduced to finding expressions for the slip velocity and the temperature jump at a solid boundary of the gas. The degree of approximation to which these boundary conditions are evaluated must, of course, be consistent with the order of the approximation that is employed for the stresses $\mathcal{T}_{ij}$ and the heat flux vector q_i.

5.2 Physical Aspects of the Problem

Any theoretical deduction of the boundary conditions must be based on the kinetics of the interaction between the gas molecules and the solid wall. This problem was first investigated by J. C. Maxwell. His calculations are based on the classical concept that even the most highly polished solid surface appeares on a molecular scale extremely rough and jagged. In general, a gas molecule approaching a solid wall will have many collisions with the molecules of the wall, so that the identification of the molecule in terms of its velocity and angle of incidence is partially or completely obliterated. Maxwell approximated this complicated state of affairs by assuming that a certain fraction σ of the incident molecules are temporarily absorbed by the wall and are then reemitted diffusely in all directions, whereas the remaining $(1 - \sigma)$ molecules reflect elastically from the wall like light rays from a plane mirror. Using the notation of figure (5.1), one readily obtains an approximation for the slip velocity of the gas in terms of its mean free path (Reference 5.2):

$$(u)_{z=0} - u_0 = 0.998 \left(\frac{2-\sigma}{\sigma}\right) \left(\frac{\partial u}{\partial z}\right)_{z=0} \cdot \ell \quad . \qquad (5.3)$$

Equation (5.3) can be used to calculate the apparent decrease in viscosity with decrease of pressure that is obtained when viscosity measurements are not corrected for slippage of the gas. This in turn permits an experimental evaluation of the fraction σ for any combination of gas and wall material. Such experiments were performed by R. A. Millikan (Reference 5.3) and resulted in the values of σ as given in Table 5.1. The percentage of molecules which are reflected specularly is seen to be very small.

The values of σ defined by equation (5.3) are average values since no allowance was made for a variation of σ with the molecular velocity or with the angle of incidence of the molecule. In the work that follows, the quantity σ will also, be considered to have a constant value, which depends only on the material of the wall and the gas (Table 5.1), for molecules of all speeds and angles of incidence.

Of course, Maxwell's considerations are based on the concept of classical physics. More recent experiments by O. Stern and his collaborators on the diffraction and reflection of molecular beams by crystal surfaces (Reference 5.4 to 5.6) have demonstrated the wave properties of such molecular beams. In accordance with the De Broglie principle the effective wave length λ' of the molecular beam is given by

$$\lambda' = \frac{h'}{mv} \tag{5.4}$$

where m and v are the mass and speed of a molecule, respectively, and h' is Planck's constant (6.55×10^{-27} erg. sec.). Thus, for nitrogen at 0° C the wave length λ' is of the order of 0.3 A°, which is of the same magnitude as the lattice spacing of crystals. Hence, from the point of view of wave mechanics a very strong diffraction of the molecular beams by the potential field of the crystal surface is expected. Figure 5.2, as given by Frisch and Stern (Reference 5.6) for the diffraction of a beam of helium molecules by a lithium fluoride crystal of low helium absorbtion, shows that, even of those molecules which are not absorbed by the solid, only a small fraction experiences strictly specular reflection. Since engineering materials are generally poly-crystalline with different crystals having different orientations, the diffracted molecules originating from incident molecules of different directions must have a distribution which is tantamount to that of diffuse reflection. This then explains the closeness of σ to unity.

Thus even the diffusely reemitted molecules do not necessarily come from the adsorbed molecules. If they are not first adsorbed, but are simply diffracted, then the "reemission" is certainly instantaneous. However, even for the absorbed molecules, the mean life time of the molecule on the solid wall is very short. According to the calculations of A. F. Devonshire (Reference 5.7) the mean life time of hydrogen on solid surfaces is only of the order of 10^{-10} seconds at ordinary temperatures. Hence the incidence and reemission process may be treated as instantaneous in all of the cases which are of interest here.

The constant σ evidently represents the fraction of the tangential momentum of the gas molecules which is transmitted to the solid wall. Similarly the problem of energy transfer between the gas molecules and those composing the gas is of interest. The experiments of M. Knudsen and von Smoluchowski on the heat transfer between gases and solid surfaces have shown that the σ molecules which are assumed to be temporarily absorbed by the wall are, in general, not reemitted with the temperature of the wall. Instead the temperature of reemission is intermediate to the wall temperature and the temperature of the incident molecules. This phenomenon is most conveniently represented quantitatively in terms of the "accomodation coefficient", a, as defined by Knudsen (Reference 5.8). Since it is not possible to distinguish experimentally between molecules which have been reemitted from the wall and those which have been reflected, the coefficient α is defined by equation (5.5) in terms of the total energy E_i which arrives at the wall, and E_r, the total energy leaving the wall.

$$(E_r - E_i) = \alpha(E_w - E_i) \tag{5.5}$$

Here E_w is the hypothetical energy which would be carried away from the wall if all of the incident molecules were to leave the wall with the Maxwell distribution of velocities (equation 2.6) that corresponds to the temperature of the wall, T_w. Experiments seem to indicate (Reference 5.9) that the accomodation coefficient α is approximately the same for the rotational and translational energies of the molecules, so that equation (5.9) may be used to refer to the total energy of a polyatomic molecule.

The expression for a first approximation to the temperature discontinuity at the wall (Reference 5.10) is quite analogous to equation (5.3), namely

$$(T)_{z=0} - T_w = 0.998 \cdot \left(\frac{2}{\gamma+1}\,\frac{\lambda}{\mu c_v}\right) \cdot \left(\frac{2-\alpha}{\alpha}\right)\left(\frac{\partial T}{\partial z}\right)_{z=0} \cdot \ell\,, \tag{5.5a}$$

where λ is the coefficient of heat conduction, and c_v is the specific heat of the gas at constant volume. From equation (5.5a) one can calculate the experimentally observed decrease in heat conduction with decreasing density, and so obtain experimental values of the accomodation coefficient α, such as are given in Table 5.2.

The expressions (5.3) and (5.5a) for the slip velocity and temperature jump in a rarefied gas are valid only when the temperature and velocity gradients are small, that is for low Mach number gas flows. The physical concepts of specular reflection, absorption and diffuse reemission, and thermal accomodation, as discussed above, will now be applied to a rigorous, systematic deduction of the boundary conditions which are required for the solution of the partial differential equations of motion of a non-uniform, rarefied gas.

5.3 Mathematical Formulation of the Problem

The coordinate system employed in the derivation of the boundary conditions is shown in figure (5.3a). Let x,y,z be a cartesian coordinate system which is fixed in space. The figure shows a portion of a plane wall whose surface moves in the x,y plane with the absolute velocity components u_o and v_o. The upper half of the space $(z > 0)$ is filled with a non-uniform gas which is in motion relative to the wall. As before, (see Figure 2.1) the components of the absolute velocity of the molecules are denoted by ξ, η, ζ respectively. The components of the mass velocity of the gas in the x,y,z directions are again u,v,w, respectively. The object of this investigation is then to determine the mass velocity $u(x,y,o)$, $v(x,y,o)$, $w(x,y,o)$ and the temperature $T(x,y,o)$ of the gas at the solid surface in terms of the velocity of the wall $u_o(x,y)$, $v_o(x,y)$ and the wall temperature $T_w(x,y)$.

The equilibrium of a stable mass of gas is defined by the laws of conservation of mass, momentum, and energy. The differential equations of motion express the equilibrium of any "interior" infinitesimal volume element of the gas, that is one which is completely surrounded by other similar elements of the gas. Consider the element of volume (dx dy dz) represented by the rectangular parallelepiped ABCDEFGH in figure (5.3a). Since its face ABEF is adjacent to the solid wall, the equations of motion do not apply to this particular element of the gas, so that a special form of the conservation laws must be deduced which will be applicable to this "sublayer" of height dz. It will be shown that the assumption that the height of the sublayer dz, which is much smaller than the mean free path ℓ, can be considered as an infinitesimal, leads to consistent results.

In the derivation which follows, the element of the sublayer (dx dy dz) is to be considered as fixed in space. Let the total number of molecules per unit volume at the interior point (x,y, $\frac{1}{2}$ dz) be ν, with the distribution of velocities among these molecules defined by the distribution function

$$f = f(\xi, \eta, \zeta;\ x, y, \tfrac{1}{2}dz;\ t)$$

in the sense of equation (2.1). The number of molecules having specified velocity components (ξ, η, ζ) at the various faces which bound the element (dx dy dz) is proportional to the quantities indicated in figure (5.3b).

The conservations of mass, momentum and energy can be investigated simultaneously by means of the generalized molecular property $Q_j(\xi, \eta, \zeta)$, j = 1,2,...,5, where the individual functions are defined by

$$\left.\begin{aligned} &Q_1 = m \\ Q_2 = m\zeta, \quad &Q_3 = m\xi, \quad Q_4 = m\eta \\ &Q_5 = \tfrac{1}{2} m(\xi^2+\eta^2+\zeta^2) . \end{aligned}\right\} . \qquad (5.6)$$

The generalized conservation law then requires that the net flux in unit time of the property Q_j into the element (dx dy dz) equals the increase in unit time of the total amount of Q_j inside the volume (dx dy dz). It will now be assumed that the total quantity of Q_j inside the element of the sublayer does not change with time. This means that the external conditions which determine the gas flow, such as u_o, v_o, T_w, ect., are either independent of time, or change very slowly relative to the time required for a molecule to traverse the distance dz. With this assumption, the fundamental conservation law requires then that the net flux of Q_j into the element (dx dy dz) be equal to zero.

5.4 Deduction of "Equation of Conservation" for Sub-layer

Consider first the flux of the quantity Q_j into the face AEGD of the element (dx dy dz). From figure (5.3) and equation (2.1), the number of molecules per unit volume having velocity components "close to" ξ, η, ζ is

$$\left[\nu f + \frac{\partial}{\partial x}(\nu f)\cdot\frac{1}{2}\,dx\right]\cdot(d\xi\, d\eta\, d\zeta).$$

The number of these molecules which in unit time will flow into the element through AEGD is given by

$$-\xi\,(dy\,dz)\;\left[\nu f + \frac{\partial}{\partial x}(\nu f)\,\frac{dx}{2}\right](d\xi\, d\eta\, d\zeta)\,.$$

Since each of these molecules transports the quantity $Q_j(\xi, \eta, \zeta)$ into the element, the total amount of Q_j transported through the surface by all of the molecules having all possible velocity components is given by the integral

$$-\,dy\,dz\;\left[\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\eta\int_{-\infty}^{\infty}d\zeta\cdot Q_j(\xi,\eta,\zeta)\cdot\xi\left\{\nu f+\frac{\partial}{\partial x}(\nu f)\,\frac{dx}{2}\right\}\right]. \qquad (5.7)$$

Similarly, the flux of Q_j into the element through the face BFHC is equal to

$$+\,dy\,dz\left[\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\eta\int_{-\infty}^{\infty}d\zeta\cdot Q_j(\xi,\eta,\zeta)\cdot\xi\cdot\left\{\nu f-\frac{\partial}{\partial x}(\nu f)\,\frac{dx}{2}\right\}\right]. \qquad (5.8)$$

The flux of Q_j through the faces ABCD, and EFGH is obtained by replacing x by y in the expressions (5.7) and (5.8), respectively. Consequently the net flux of Q_j which is carried into (dx dy dz) through its four vertical faces is equal to

$$-\,dx\,dy\left[dz\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\eta\int_{-\infty}^{\infty}d\zeta\cdot Q_j\cdot\left\{\xi\,\frac{\partial}{\partial x}(\nu f)+\eta\,\frac{\partial}{\partial y}(\nu f)\right\}\right]. \qquad (5.9)$$

The flux of Q_j through the top face CDGH, analogous to equation (5.7), is found to be equal to

$$-dx\,dy\left[\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\eta\int_{-\infty}^{\infty}d\zeta\cdot Q_j(\xi,\eta,\zeta)\cdot\zeta\left\{\nu f+\frac{\partial}{\partial z}(\nu f)\frac{dz}{2}\right\}\right]. \qquad (5.10)$$

The bottom face, ABFE, of the element of the sublayer requires special consideration, since here the interaction between the molecules of the gas and the wall must be accounted for. This interaction will be represented by the Maxwell hypothesis, which was discussed earlier, namely that of N molecules which strike the wall, the number $(1-\sigma)N$ are reflected specularly whereas the other σ N molecules are temporarily absorbed by the wall and then reemitted diffusely, relative to the wall. This means that the velocity distribution function of the molecules which are reemitted diffusely, which will be represented for the present by $f'(\xi,\eta,\zeta)$, is different from the velocity distribution of the molecules which impinge on the wall, namely $(1-\frac{1}{2}dz\frac{\partial}{\partial z})\,f(\xi,\eta,\zeta)$. Consequently the function $(1-\frac{1}{2}dz\frac{\partial}{\partial z}\cdot)\cdot f(\xi,\eta,\zeta)$ has a physical significance only when ζ is negative, whereas the diffuse distribution function $f'(\xi,\eta,\zeta)$ is defined only for positive values of ζ. In view of the above, the net flux of the molecular property Q_j which flows into the bottom face of the element can then be written as

$$\begin{aligned}dx\,dy\Bigg[&\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\eta\int_{-\infty}^{0}d\zeta\cdot Q_j(\xi,\eta,\zeta)\cdot\zeta\cdot\left\{\nu f-\frac{\partial}{\partial z}(\nu f)\frac{dz}{2}\right\}\\&+(1-\sigma)\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\eta\int_{-\infty}^{0}d\zeta\cdot Q_j(\xi,\eta,-\zeta)\cdot(-\zeta)\cdot\left\{\nu f-\frac{\partial}{\partial z}(\nu f)\cdot\frac{dz}{2}\right\}\\&+\sigma\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\eta\int_{0}^{\infty}d\zeta\cdot Q_j(\xi,\eta,\zeta)\cdot\zeta\cdot\left\{\nu-\frac{\partial\nu}{\partial z}\cdot\frac{dz}{2}\right\}\cdot f'(\xi,\eta,\zeta)\Bigg]\end{aligned} \qquad (5.11)$$

The first integral above represents the amount of the property Q_j which flows out of the bottom face of the element of the sublayer, whereas the second and third integrals represent the amount of Q_j carried back into the element through the bottom face by the specularly reflected and diffusely reemitted molecules, respectively.

If the three expressions (5.9), (5.10) and (5.11) are now added, one obtains the net flux of the quantity Q_j into the element (dx dy dz) through all of its six faces. When this sum is equated to zero one obtains for the fundamental law of conservation for the sublayer,

$$
\begin{aligned}
\Big[&- \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} d\zeta \cdot Q_j(\xi,\eta,\zeta) \cdot \zeta, \cdot f(\xi,\eta,\zeta) \\
&+ \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{0} d\zeta \cdot Q_j(\xi,\eta,\zeta) \cdot \zeta \cdot f(\xi,\eta,\zeta) \\
&+ (1-\sigma) \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{0} d\zeta \cdot Q_j(\xi,\eta,-\zeta) \cdot (-\zeta) \cdot f(\xi,\eta,\zeta) \\
&+ \sigma \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{0}^{\infty} d\zeta \cdot Q_j(\xi,\eta,\zeta) \cdot \zeta \cdot f'(\xi,\eta,\zeta) \Big] \\
&+ \frac{1}{\nu}\, dz \Big[\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} d\zeta \cdot Q_j(\xi,\eta,\zeta) \cdot \Big\{ \Big(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \frac{1}{2} \zeta \frac{\partial}{\partial z} \Big) (\nu f) \Big\} \\
&- \frac{1}{2} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{0} d\zeta \cdot \Big\{ Q_j(\xi,\eta,\zeta) - (1-\sigma) \cdot Q_j(\xi,\eta,-\zeta) \Big\} \cdot \zeta \frac{\partial}{\partial z} (\nu f) \\
&- \frac{1}{2} \sigma \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{0}^{\infty} d\zeta \cdot Q_j(\xi,\eta,\zeta) \cdot \zeta \cdot f'(\xi,\eta,\zeta) \Big] = 0 \qquad (5.12)
\end{aligned}
$$

The above equation is valid provided the point $(x, y, \frac{1}{2} dz)$, figure 5.3a, is an "interior" point of the gas. No difficulty is encountered when the height of the sublayer is allowed to approach zero as the limiting case. It will therefore be assumed that dz = 0; if equation (5.12) then leads to a consistent set of boundary conditions, this assumption may be considered as justified. Since the first two integrals in equation (5.12) may be combined, the condition for the conservation of the molecular property Q_j reduces to *

$$\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{0}^{\infty} d\zeta \cdot \left\{ Q_j(\xi,\eta,\zeta) \cdot \zeta \cdot f(\xi,\eta,\zeta) \right\}$$

$$-(1-\sigma) \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{0} d\zeta \left\{ Q_j(\xi,\eta,-\zeta) \cdot (-\zeta) \cdot f(\xi,\eta,\zeta) \right\}$$

$$= \sigma \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{0}^{\infty} d\zeta \left\{ Q_j(\xi,\eta,\zeta) \cdot \zeta \cdot f'(\xi,\eta,\zeta) \right\}, \qquad (5.13)$$

$$(j = 1,2,3,4,5;\ dz \rightarrow 0).$$

In order to perform the integrations indicated above for the five functions Q_j defined by equation (5.6), the explicit form of the two distribution functions f and f' must be known. The function $f(\xi,\eta,\zeta)$ which holds at the "interior point" of the sublayer is just the general distribution function for a non-uniform gas, as given in Burnett's form by equation (2.39), evaluated at the point $(x,y,z \rightarrow 0)$. All that can be said about the distribution $f'(\xi,\eta,\zeta)$, is that, in accordance with Maxwell's hypothesis, those molecules which have been absorbed by the wall

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* This set of five equations is similar in form to that deduced by P. S. Epstein (Reference 5.11). Epstein's method of derivation differs from the one used above in that he imposed one condition on each of the six stress components of the gas instead of requiring the conservation of the three momentum components. The eight equations so obtained lead to contradictions, in general. However, for the special case of low speed flows without heat transfer to the wall, (which alone was treated in Reference 5.11) Epstein's result agrees with that deduced here from this more general theory.

shall be reemitted from the wall uniformly in all azimuthal directions, if viewed by an observer stationed on the moving wall. Mathematically this means that the distribution function of the diffusely reemitted molecules, $f'(\xi,\eta,\zeta)$ must satisfy the conditions

$$f'(\xi,\eta,\zeta) = f'[(\xi-u_0),(\eta-v_0),\zeta] = f'[-(\xi-u_0),(\eta-v_0),\zeta]$$

$$= f'[(\eta-v_0),(\xi-u_0),\zeta] = f'[-(\eta-v_0),(\xi-u_0),\zeta]$$

$$(\zeta \geqq 0). \tag{5.14}$$

It is now convenient to introduce the abbreviated notations $\overset{(1)}{L}_j$, $\overset{(2)}{L}_j$, J_j, defined by equations (5.15) to (5.17).

$$\overset{(1)}{L}_j \equiv \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{0}^{\infty} d\zeta \cdot Q_j(\xi,\eta,\zeta)\cdot\zeta\cdot f(\xi,\eta,\zeta) \tag{5.15}$$

$$\overset{(2)}{L}_j \equiv \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{0} d\zeta \cdot Q_j(\xi,\eta,-\zeta)\cdot(-\zeta)\cdot f(\xi,\eta,\zeta) \tag{5.16}$$

$$J_j \equiv \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{0}^{\infty} d\zeta \cdot Q_j(\xi,\eta,\zeta)\cdot\zeta\cdot f'(\xi,\eta,\zeta) \; . \tag{5.17}$$

Because the function $f'(\xi,\eta,\zeta)$ must satisfy the conditions (5.14), there exists a convenient relation between the integrals J_3, J_4, and J_1, as will now be shown. For the function $Q_3 = m\xi$ one has

$$J_3 = m\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} d\zeta \cdot \zeta\xi \cdot f'[(\xi-u_0),(\eta-v_0),\zeta]$$

$$= \Big[m\int_{-\infty}^{\infty} d(\xi-u_0) \int_{-\infty}^{\infty} d(\eta-v_0) \int_{0}^{\infty} d\zeta\cdot\zeta\,(\xi-u_0)\cdot f'[(\xi-u_0),(\eta-v_0),\zeta]$$

$$+ m\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{0}^{\infty} d\zeta\cdot\zeta\cdot u_0\cdot f'(\xi,\eta,\zeta)\Big] = m\cdot 0 + u_0 J_1 \, .$$

A similar result holds for the integral J_4 so that

$$J_3 = u_0 J_1 \; ; \quad J_4 = v_0 J_1 \, . \tag{5.18}$$

The fundamental equation (5.13) which expresses the equilibrium of the sublayer may be written more compactly as

$$L_j^{(1)} - (1-\sigma)\, L_j^{(2)} = \sigma\, J_j \; ; \quad (j = 1,2,3,4,5)\,, \tag{5.19}$$

where the various quantities are defined by equations (5.15), (5.16), (5.17) and (5.6).

5.5 Conservation of Mass and Energy at Wall

Thus far only the equilibrium of the sublayer has been considered. Two additional facts must be accounted for, namely that (a) no molecules can accumulate on the wall and (b), there exists an energy balance at the wall which is defined by the "accomodation condition" of Knudsen, equation (5.5), which has been discussed earlier. Fact (a) is expressed mathematically by

$$L_1^{(1)} = (1-\sigma)\, L_1^{(2)} + \sigma\, J_1 \tag{5.20}$$

which states that the total mass of gas which arrives at the wall equals the mass of gas which leaves the wall.

The energy balance at the wall is from equation (5.5)

$$(E_i - E_r) = \alpha (E_i - E_w). \tag{5.12}$$

It follows from Knudsen's definition of the accomodation coefficient α, that the various energies E_i, E_r and E_w must be evaluated in terms of the molecular kinetic energy of translation relative to an observer stationed on the wall. Thus E_i, the total energy which is incident on unit area of the wall per unit time, is given by

$$E_i = \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{0} d\zeta \cdot (-\zeta) \cdot \tfrac{1}{2} m \left\{ (\xi - u_0)^2 + (\eta - v_0)^2 + \zeta^2 \right\} \cdot f(\xi, \eta, \zeta) \, .$$

By expanding the integrand of the above equation, E_i may be expressed in terms of integrals defined by equations (5.16) and (5.6), namely

$$E_i = L_5^{(2)} - (u_o L_3^{(2)} + v_o L_2^{(2)}) + (u_o^2 + v_o^2) L_1^{(2)}. \qquad (5.22)$$

The energy carried away from the wall, E_r, is the sum of the kinetic energies of the $(1 - \sigma)$ specularly reflected molecules, and that of the σ diffusely reemitted molecules, so that

$$E_r = (1-\sigma) E_i$$

$$+ \sigma \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{0}^{\infty} d\zeta \cdot \zeta \cdot \tfrac{1}{2} m \cdot \left\{ (\xi - u_o)^2 + (\eta - v_o)^2 + \zeta^2 \right\} \cdot f' \left[(\xi - u_o), (\eta - v_o), \zeta \right] .$$

When the integrand is expanded one obtains in view of equations (5.17), (5.18) and (5.6)

$$E_r = (1 - \sigma) E_i + \sigma \left[J_5 - \tfrac{1}{2}(u_o^2 + v_o^2) \; J_1 \right]. \qquad (5.23)$$

In accordance with Knudsen's definition, the hypothetical energy E_w that would be carries away from the wall if all of the molecules were reemitted at the temperature of the wall T_w, is given by

$$E_w = \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{0}^{\infty} d\zeta \cdot \zeta \cdot \tfrac{1}{2} m \left\{ (\xi - u_o)^2 + (\eta - v_o)^2 + \zeta^2 \right\} \cdot f''(\xi, \eta, \zeta) . \qquad (5.24)$$

The function $f''(\xi, \eta, \zeta)$ is the Maxwell distribution function for a gas which is uniform with respect to an observer on the wall, namely

$$f''(\xi, \eta, \zeta) = \Lambda'' e^{-\frac{m}{2kT_w} \left[(\xi - u_o)^2 + (\eta - v_o)^2 + \zeta^2 \right]} . \qquad (5.25)$$

The constant Λ'' must be so chosen, that the total number of molecules which are hypothetically emitted by the wall with the velocity distribution $f''(\xi, \eta, \zeta)$, is equal to the total number of molecules which arrive at the wall with the distribution $f(\xi, \eta, \zeta)$. Hence Λ'' is found from the equation

$$L_1^{(2)} = m\Lambda'' \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{0}^{\infty} d\zeta \cdot \zeta \cdot e^{-\frac{m}{2kT_w}\left[(\xi-u_0)^2+(\eta-v_0)^2+\zeta^2\right]} .$$

The integral is readily evaluated in terms of the elementary integrals listed in Table A.1 of Appendix II. The value of Λ'' is thus found to be

$$\Lambda'' = \frac{2}{\pi}\frac{1}{m}\left(\frac{m}{2kT_w}\right)^2 L_1^{(2)} . \tag{5.26}$$

Equation (5.25) for $f''(\xi, \eta, \zeta)$ is now substituted into the integral of equation (5.24). The integration is again performed with the aid of Table A.1, so that one obtains for the hypothetical energy the simple expression

$$E_w = \left(\frac{2kT_w}{m}\right) \cdot L_1^{(2)} . \tag{5.27}$$

The energies E_i, E_r, and E_w have now been expressed explicitly in terms of the integrals defined by equations (5.16) and (5.17). When the expressions (5.22), (5.23) and (5.27) are substituted into the accomodation condition (5.21), one obtains an equivalent expression given by equation (5.34) below.

5.6 The Complete Set of Boundary Conditions

The seven boundary conditions which have been deduced are summarized below:

$$L_1^{(1)} - (1-\sigma)\, L_1^{(2)} = \sigma J_1 \tag{5.28}$$

$$L_2^{(1)} - (1-\sigma)\, L_2^{(2)} = \sigma\, J_2 \tag{5.29}$$

$$L_3^{(1)} - (1-\sigma)\, L_3^{(2)} = \sigma\, u_o J_1 \tag{5.30}$$

$$L_4^{(1)} - (1-\sigma)\, L_4^{(2)} = \sigma\, v_o J_1 \tag{5.31}$$

$$L_5^{(1)} - (1-\sigma)\, L_5^{(2)} = \sigma\, J_5 \tag{5.32}$$

$$L_1^{(2)} = (1-\sigma)\, L_1^{(2)} + \sigma\, J_1 \tag{5.33}$$

$$(\sigma-\alpha)\left[L_5^{(2)} - \left(u_o L_3^{(2)} + v_o L_4^{(2)} \right) + \frac{1}{2}\left(u_o^2 + v_o^2 \right) L_1^{(2)} \right]$$

$$- \sigma J_5 + \frac{\sigma}{2}\left(u_o^2 + v_o^2 \right) J_1 = -\alpha \left(\frac{2kT_w}{m} \right) L_1^{(2)} . \tag{5.34}$$

Equations (5.28) to (5.32) are the five conditions required by the equilibrium of the sublayer as given by equation (5.19), with the modification of equation (5.18). In particular, equation (5.28) expresses the conservation of mass; equations (5.29), (5.30), (5.31) are the conservation of momentum in the z,x, and y directions, respectively; and equation (5.32) represents the conservation of energy of the element (dx dy dz) of the sublayer (figure 5.3a). Equation (5.33) states that no mass of the gas can accumulate on the wall, and equation (5.34) expresses the balance of energy at the wall in terms of the accomodation coefficient α.

The seven boundary conditions summarized above can be consistent only if they involve seven unknowns. This is actually the case. It was stated earlier that the form of the non-uniform distribution fuction $f(\xi, \eta, \zeta)$ (equation 2.39) is known, so that all of the quantities L_j, which are definite integrals of a known function, are themselves known quantities. However, the distribution function $f'(\xi, \eta, \zeta)$ for the diffusely reemitted molecules, is not known (except for the condition (5.14)), so that the integrals J_1, J_2, J_5 appearing above represent three unknowns of the problem. The other four unknowns are just the values of the three mass velocity components u,v,w, and the value of the tmeperature T of the gas, at the point $(x,y,z \rightarrow 0)$. The formulation of the boundary conditions (5.28) to (5.34) thus appears to be consistent. Moreover, if the sublayer had been assumed of finite thickness, say Δz, then there would be eight unknowns, yet only seven physical conditions.

Since we are interested only in finding the values of u,v,w, and T, the remaining unknowns J_1, J_2, and J_5 may be eliminated and the number of equations reduced to four. Thus from equation (5.33) one finds, obviously, that

$$J_1 = L_1^{(2)} \tag{5.35}$$

J_2 appears only in equation (5.33) which may simply be omitted. From equations (5.34) and (5.35) one obtains for J_5 the equation

$$\sigma J_5 = \alpha\left(\frac{2kT_w}{m}\right) L_1^{(2)} + (\sigma-\alpha) L_5^{(2)} - (\sigma-\alpha)\left(u_0 L_3^{(2)} + v_0 L_4^{(2)}\right)$$

$$- \frac{1}{2}\alpha\left(u_0^2+v_0^2\right) L_1^{(2)} \, . \tag{5.36}$$

Equation (5.35) is now substituted into equations (5.28), (5.30), and (5.31), and equation (5.36) into (5.32). This results in the four pertinent boundary conditions given by equations (5.37) to (5.40),

$$L_1^{(1)} - L_1^{(2)} = 0 \tag{5.37}$$

$$L_3^{(1)} - (1-\sigma) L_3^{(2)} = u_0 L_1^{(2)} \tag{5.38}$$

$$L_4^{(1)} - (1-\sigma)\, L_4^{(2)} = \sigma\, v_o L_1^{(2)} \tag{5.39}$$

$$L_5^{(1)} - (1-\alpha)\, L_5^{(2)} = \alpha \left\{ \left(\frac{2kT_w}{m} \right) + \left(\sigma - \frac{\alpha}{2} \right)\left(u_o^2 + v_o^2 \right) \right\} L_1^{(2)}$$

$$- (\sigma - \alpha)\left(u_o L_3^{(2)} + v_o L_4^{(2)} \right) . \tag{5.40}$$

The meaning of the first condition, equation (5.37), is evident without further calculation. From the defining equations (5.15), (5.16) and (5.6), one obtains

$$L_1^{(1)} - L_1^{(2)} = m \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{0}^{\infty} d\eta \cdot \zeta \cdot f(\xi,\eta,\zeta) + m \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{0} d\zeta \cdot \zeta \cdot f(\xi,\eta,\zeta)$$

$$= m \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} d\zeta \cdot \zeta \cdot f(\xi,\eta,\zeta) = 0$$

The last integral is precisely w, the z-component of the mass velocity of the gas (see equation (2.11), Part II), so that equation (5.37) states the somewhat trivial fact that the velocity of the gas is tangential to the wall. Hence equation (5.37) reduces to

$$w = 0 \text{ at the point } (x, y, z \to 0) . \tag{5.41}$$

It will now be shown, with a great deal more labor, that equations (5.38), (5.39) and (5.40) will lead to expressions for the tangential mass velocity components u, v, and the temperature T of the gas near the wall, respectively.

5.7 Reduction of Boundary Conditions

The further reduction of equations (5.38) to (5.40) requires the evaluation of integrals of the type of equations (5.15) and (5.16). As was pointed out before, the function $f(\xi, \eta, \zeta)$ which appears in the integrand of these integrals is given explicitly by equation (2.39) of Part II. Since this distribution function is expressed in terms of the variables C, θ, ϕ it is most convenient to carry out all of the integrations in the spherical polar coordinates of the peculiar velocity of the molecules. Since in the present application w = 0, the coordinate tranformations of equations (2.12) and (2.36), (see also figure 2.1) reduce to the form

$$\left.\begin{aligned} \xi &= u + U = u + C \sin\theta \cos\phi \\ \eta &= v + V = v + C \sin\theta \sin\phi \\ \zeta &= W = C \cos\theta \quad . \end{aligned}\right\} \qquad (5.41)$$

The transformation of the integrals is given by equations (5.42).

$$\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{0}^{\infty} d\zeta \cdot [\;] \cdot f(\xi, \eta, \zeta) = \int_{-\pi}^{\pi} d\phi \int_{0}^{\infty} C^2 dC \int_{0}^{\frac{\pi}{2}} \sin\theta \cdot [\;] \cdot f(C, \theta, \phi)\, d\theta$$

$$\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{0} d\zeta \cdot [\;] \cdot f(\xi, \eta, \zeta) = \int_{-\pi}^{\pi} d\phi \int_{0}^{\infty} C^2 dC \int_{\frac{\pi}{2}}^{\pi} \sin\theta\, [\;] \cdot f(C, \theta, \phi)\, d\theta \; .$$

$$(5.42)$$

Since equations (5.15) and (5.16) must be evaluated for j = 1,3,4,5, it follows from the definition of the functions Q_j, equations (5.6), that the portion of the integrand indicated by the square bracket above, will assume the various forms

$$\left.\begin{aligned} \zeta &= W \\ \zeta\xi &= WU + uW \\ \zeta\eta &= WV + vW \\ \zeta(\xi^2 + \eta^2 + \zeta^2) &= WC^2 + 2uWU + 2vWV + (u^2 + v^2)\, W \; . \end{aligned}\right\} (5.43)$$

It is now convenient to express the peculiar velocity components appearing above in terms of the spherical harmonic functions defined by equations (A.6) and (A.7) of Appendix II, Section 2. Equations (5.43) then assume the form

$$\left.\begin{aligned} f &= Y_1 \\ f\xi &= \tfrac{1}{3} Y_2^{(1)} + u Y_1 \\ f\eta &= \tfrac{1}{3} Z_2^{(1)} + v Y_1 \\ f(\xi^2+\eta^2+\zeta^2) &= C^2 Y_1 + \tfrac{2}{3} u Y_2^{(1)} + \tfrac{2}{3} v Z_2^{(1)} + (u^2+v^2) Y_1 \; . \end{aligned}\right\} \qquad (5.44)$$

The integrations are, of course, performed at a fixed point (x,y,z → 0), so that the mass velocity components u and v are constants. The desired integrals, equations (5.15), (5.16) can therefore be expressed as linear combinations of certain basic integrals of the form

$$\left.\begin{aligned} I_i^{(1)} &= \int_{-\pi}^{\pi} d\phi \int_0^{\infty} C^2 dC \int_0^{\frac{\pi}{2}} \sin\theta \, d\theta \cdot M_i \cdot f(C,\theta,\phi) \\ I_i^{(2)} &= \int_{-\pi}^{\pi} d\phi \int_0^{\infty} C^2 dC \int_{\frac{\pi}{2}}^{\pi} \sin\theta \, d\theta \cdot M_i \cdot f(C,\theta,\phi) \, , \end{aligned}\right\} \qquad (5.45)$$

where $M_i = C^n Y_k^{(p)}$, $C^n Z_k^{(p)}$.

Two sample integrations of the type (5.45), for $M_2 = Y_1$ and $M_7 = Y_2^{(1)}$, are reproduced in Appendix IV, Section 2. It is found, in general, that the integrals $I_i^{(1)}$ and $I_i^{(2)}$ consist of an "even component" $\mathcal{A}_i$, and and "odd component" $\mathcal{B}_i$ such that

$$\left.\begin{aligned} I_i^{(1)} &= \mathcal{A}_i + \mathcal{B}_i \\ I_i^{(2)} &= \mathcal{A}_i - \mathcal{B}_i \; . \end{aligned}\right\} \qquad (5.46)$$

The values of $\mathcal{A}_i$ and $\mathcal{B}_i$, corresponding to a number of functions M_i are tabulated in Section 1 of Appendix IV*. The expressions (A.99) to (A.115) are infinite series of the exact Burnett coefficients of the distribution function, and are applicable to any "smooth" molecular model having spherical symmetry. In the case of Maxwell molecules, however, these infinite series reduce to finite sums. The values of ${}_2\mathcal{A}_i$ and ${}_2\mathcal{B}_i$ given by equations (A.116) to(A.127) are accurate to second order terms, $\left(\frac{\mu}{p}\right)^2$, for Maxwell molecules, in the sense of equation (2.66) of Part II.

It follows from equations (5.6), (5.15), (5.16), (5.44), and from the definition of the functions M_i by equation (A.97), that the seven integrals $L_j^{(1)}$, $L_j^{(2)}$, which are required for the evaluation of the boundary conditions, equations (5.38) to (5.40), are expressed in terms of the fundamental integrals $I_i^{(1)}$, $I_i^{(2)}$, of equations (5.45) as shown below:

$$\left.\begin{aligned}
L_1^{(2)} &= -m\, I_2^{(2)} \\
L_3^{(1)} &= m \left\{ \tfrac{1}{3} I_7^{(1)} + u\, I_2^{(1)} \right\} \\
L_3^{(2)} &= -m \left\{ \tfrac{1}{3} I_7^{(2)} + u\, I_2^{(2)} \right\} \\
L_4^{(1)} &= m \left\{ \tfrac{1}{3} I_9^{(1)} + v\, I_2^{(1)} \right\} \\
L_4^{(2)} &= -m \left\{ \tfrac{1}{3} I_9^{(2)} + v\, I_2^{(2)} \right\} \\
L_6^{(1)} &= \tfrac{1}{2} m \left\{ I_{11}^{(1)} + \tfrac{2}{3} u I_7^{(1)} + \tfrac{2}{3} v I_9^{(1)} + (u^2+v^2)\, I_2^{(1)} \right\} \\
L_5^{(2)} &= -\tfrac{1}{2} m \left\{ I_{11}^{(2)} + \tfrac{2}{3} u\, I_7^{(2)} + \tfrac{2}{3} v\, I_9^{(2)} + (u^2+v^2)\, I_2^{(2)} \right\} .
\end{aligned}\right\} \qquad (5.47)$$

---

* Only four of the integrals calculated in Appendix IV are required for the present purpose of evaluating the boundary conditions. Since the remaining seven integrals may prove useful in other calculations involving the non-uniform velocity distribution function, they have been included here for future reference.

When the expressions (5.47) are substituted into equations (5.38) to (5.40), the resulting equations may then be expressed in terms of $\mathcal{A}_i$ and $\mathcal{B}_i$ by means of equations (5.46) and the following identity:

$$I_j^{(1)} + (1-\sigma)\, I_j^{(2)} = (2-\sigma)\,\mathcal{A}_i + \sigma\,\mathcal{B}_i \quad . \tag{5.48}$$

Thus equation (5.38) becomes

$$(2-\sigma)\left\{\tfrac{1}{3}\mathcal{A}_7 + u\,\mathcal{A}_2\right\} + \sigma\left\{\tfrac{1}{3}\mathcal{B}_7 + u\,\mathcal{B}_2\right\} = -\sigma u_0\left\{\mathcal{A}_2 - \mathcal{B}_2\right\}.$$

Since from equation (A.100) $\mathcal{A}_2 = 0$, equation (5.38) is then equivalent to

$$(u-u_0)\,\mathcal{B}_2 + \tfrac{1}{3}\left(\tfrac{2-\sigma}{\sigma}\right)\mathcal{A}_7 + \tfrac{1}{3}\mathcal{B}_7 = 0 \tag{5.49}$$

Similarly equation (5.39) results in

$$(v-v_0)\,\mathcal{B}_2 + \tfrac{1}{3}\left(\tfrac{2-\sigma}{\sigma}\right)\mathcal{A}_9 + \tfrac{1}{3}\mathcal{B}_9 = 0 \tag{5.50}$$

It is evident, therefore, that the slip velocities $(u - u_o)$ and $(v - v_o)$ can be calculated from the above equations which, it will be recalled, represent the conservation of momentum of the sublayer in the directions tangential to the wall. Moreover, the parameter σ appears in the same form as in the expression (5.3) which was derived from simplified considerations.

The energy condition of equation (5.40), when similarly expressed in terms of $\mathcal{A}_i$ and $\mathcal{B}_i$, results in

$$\frac{1}{2}(2-\alpha)\left\{\mathcal{A}_{11} + \frac{2}{3}u\,\mathcal{A}_7 + \frac{2}{3}v\,\mathcal{A}_9 + (u^2+v^2)\,\mathcal{A}_2\right\}$$

$$+\frac{1}{2}\alpha\left\{\mathcal{B}_{11} + \frac{2}{3}u\,\mathcal{B}_7 + \frac{2}{3}v\,\mathcal{B}_9 + (u^2+v^2)\,\mathcal{B}_2\right\}$$

$$= \left\{\alpha\left(\frac{2kT_w}{m}\right) + \left(\sigma - \frac{\alpha}{2}\right)(u_0^2 + v_0^2)\right\}(\mathcal{B}_2 - \mathcal{A}_2)$$

$$+ (\sigma-\alpha)\left\{\frac{1}{3}u_0(\mathcal{A}_7 - \mathcal{B}_7) + \frac{1}{3}v_0(\mathcal{A}_9 - \mathcal{B}_9) + (uu_0 + vv_0)(\mathcal{A}_2 - \mathcal{B}_2).\right. \qquad (5.51)$$

It is now convenient to replace u, v, and $(u^2 + v^2)$ by identities of the form

$$u = (u - u_0) + u_0$$
$$(u^2 + v^2) = (u - u_0)^2 + (v - v_0)^2 + 2u_0(u - u_0) + 2v_0(v - v_0) + (u_0^2 + v_0^2), \qquad (5.52)$$

Using equations (5.52) and the fact that $\mathcal{A}_2 = 0$, equation (5,51) can be brought into the following form:

$$\frac{1}{2}(2-\alpha)\,\mathcal{A}_{11} + \frac{1}{2}\alpha\,\mathcal{B}_{11} + \left[-\alpha\left(\frac{2kT_w}{m}\right) + \frac{1}{2}\alpha\left\{(u-u_0)^2 + (v-v_0)^2\right\}\right]\mathcal{B}_2$$

$$+ \frac{1}{3}(2-\alpha)\left\{(u-u_0)\,\mathcal{A}_7 + (v-v_0)\,\mathcal{A}_9\right\} + \frac{1}{3}\alpha\left\{(u-u_0)\,\mathcal{B}_7 + (v-v_0)\,\mathcal{B}_9\right\}$$

$$+ \sigma u_0\left[(u-u_0)\,\mathcal{B}_2 + \frac{1}{3}\left(\frac{2-\sigma}{\sigma}\right)\mathcal{A}_7 + \frac{1}{3}\mathcal{B}_7\right]$$

$$+ \sigma v_0\left[(v-v_0)\,\mathcal{B}_2 + \frac{1}{3}\left(\frac{2-\sigma}{\sigma}\right)\mathcal{A}_9 + \frac{1}{3}\mathcal{B}_9\right] = 0 \qquad (5.53)$$

Because of equations (5.49), and (5.50) the last two brackets of the above equation vanish. In addition, the quantities $\mathcal{B}_7$ and $\mathcal{B}_9$ may be eliminated from equations (5.49) and (5.50) respectively, so that equation (5.53) and hence the original energy condition of equation (5.40) is reduced to the relatively simple form of equation (5.54).

$$\left(\frac{2-\alpha}{\alpha}\right)\mathcal{A}_{11} + \mathcal{B}_{11} - \left\{2\left(\frac{2kT_w}{m}\right) + (u-u_0)^2 + (v-v_0)^2\right\}\mathcal{B}_2$$
$$+ \frac{2}{3}\left\{\left(\frac{2-\alpha}{\alpha}\right) - \left(\frac{2-\sigma}{\sigma}\right)\right\}\left\{(u-u_0)\mathcal{A}_7 + (v-v_0)\mathcal{A}_9\right\} = 0 \; . \qquad (5.54)$$

As was to be expected, this condition, from which the temperature T of the gas near the wall will be determined, is a function of the relative velocity between the wall and the gas. The accomodation coefficient α appears in the form $\left(\frac{2-\alpha}{\alpha}\right)$ which is in accordance with equation (5.5).

Equation (5.49), (5.50) and (5.54) are then the exact boundary conditions from which the slip velocities $(u - u_0)$, $(v - v_0)$, and the temperature jump $(T - T_W)$ can be evaluated, for arbitrary molecular models. From here on, the calculation will be restricted to Maxwell molecules. The second approximation to the Burnett coefficients, ${}_2A_2, \ldots, {}_2B_{40}$, as derived in Part II will be employed.

5.8 Slip-velocity for Maxwell Molecule

We shall first derive the explicit expression for the slip velocity $(u - u_0)$ from equation (5.49). Substituting for $\mathcal{B}_2$, $\mathcal{A}_7$, and $\mathcal{B}_7$ the quantities ${}_2\mathcal{B}_2$, ${}_2\mathcal{A}_2$ and ${}_2\mathcal{B}_7$, - as given by equations (A.118), (A.102) and (A.123), respectively - , one obtains from (5.49):

$$(u-u_0)\sqrt{hm}\left[4 - {}_2A_2 - \frac{3}{2}\,{}_2A_3 + 2\,{}_2B_{20} + {}_2B_{21} + \frac{3}{2}\,{}_2B_{22} - {}_2B_{40}\right]$$
$$+ \left(\frac{2-\sigma}{\sigma}\right)\sqrt{\pi}\,{}_2B_{20}^{(1)} + \left[-{}_2B_{11}^{(1)} - \frac{1}{2}\,{}_2B_{12}^{(1)} + {}_2B_{30}^{(1)} + \frac{1}{2}\,{}_2B_{31}^{(1)}\right] = 0. \qquad (5.55)$$

It is now convenient to determine the first approximation to the slip velocity, ${}_1(u - u_0)$, which is to be accurate up to terms of order $\left(\frac{\mu}{p}\right)$. It will be recalled that the only first order Burnett coefficients for Maxwell molecules are those given by equations (2.61) and (2.62) of Part II. Consequently equation (5.55) reduces as a first approximation to

$$ {}_1(u-u_0)\sqrt{hm}\left[4+2\,{}_1B_{20}\right]+\left(\frac{2-\sigma}{\sigma}\right)\sqrt{\pi}\,{}_1B_{20}^{(1)}-{}_2B_{11}^{(1)}=0\,. \tag{5.56} $$

This given for the first approximation to the slip velocity

$$ {}_1(u-u_0) = -\frac{\sqrt{\pi}}{4}\left(\frac{2-\sigma}{\sigma}\right)\frac{1}{\sqrt{hm}}\,{}_1B_{20}^{(1)}+\frac{1}{4}\frac{1}{\sqrt{hm}}\,{}_1B_{11}^{(1)}\,. \tag{5.57} $$

With the aid of equations (2.61) and (2.62) one obtains for the above the more explicit expression

$$ {}_1(u-u_0) = \frac{\sqrt{\pi}}{2}\left(\frac{2-\sigma}{\sigma}\right)\frac{1}{\sqrt{hm}}\left(\frac{\mu}{p}\right)\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)-\frac{3}{8}\left(\frac{\mu}{p}\right)\frac{1}{h^2m}\frac{\partial h}{\partial x}\,. \tag{5.58} $$

The first term is a slight generalization of equation (5.3) and agrees with the result derived by P. S. Epstein (Reference 5.11). The first term of equation (5.58) thus justifies, from the kinetic point of view, the phenomenological assumption of Basset that the slip velocity is proportional to the shearing stress at the wall, to a first approximation. The second term of equation (5.58) represents the phenomenon known as "thermal creep" which is the relatively slow motion of the gas over the surface of a wall on which a temperature gradient exists. It follows from the definition of h, equation (2.7), that the gas moves from the colder to the warmer portion of the wall. The numerical value for the thermal creep as given above agrees with that derived by Kennard (Reference 5.12) by a more direct calculation.

From the boundary condition (5.50) one obtains by a similar analysis the following result for the slip velocity ${}_1(v - v_0)$:

$$ {}_1(v-v_0) = -\frac{\sqrt{\pi}}{4}\left(\frac{2-\sigma}{\sigma}\right)\frac{1}{\sqrt{hm}}\,{}_1C_{20}^{(1)} + \frac{1}{4}\frac{1}{\sqrt{hm}}\,{}_1C_{11}^{(1)} \; . \tag{5.59} $$

As is to be expected from the symmetry of the problem, this result can also be obtained by interchanging x and y, and u and v in equation (5.58) above.

In order to calculate the second approximation to the slip velocity ${}_2(u - u_0)$ from equation (5.55), it is convenient to use the notation of equation (5.60).

$$ {}_2(u - u_0) = {}_1(u - u_0) + {}_{(2)}(u - u_0) \; , \tag{5.60} $$

so that all terms of order $\left(\frac{\mu}{p}\right)^2$ are contained in ${}_{(2)}(u - u_0)$. Likewise, those of the Burnett coefficients which contain both first and second order terms may be written in a similar form as for example

$$ {}_2B_{20} = {}_1B_{20} + {}_{(2)}B_{20} \tag{5.61} $$

When equation (5.55) is thus expanded accurate to terms of order one obtains

$$ \left[4\sqrt{hm}\;{}_1(u-u_0) + \left(\frac{2-\sigma}{\sigma}\right)\sqrt{\pi}\;{}_1B_{20}^{(1)} - {}_1B_{11}^{(1)} \right] $$
$$ + \left[4\sqrt{hm}\;{}_{(2)}(u-u_0) + \left(\frac{2-\sigma}{\sigma}\right)\sqrt{\pi}\;{}_{(2)}B_{20}^{(1)} + 2\;{}_1(u-u_0)\;{}_1B_{20} - {}_{(2)}B_{11}^{(1)} \right. $$
$$ \left. -\frac{1}{2}\;{}_2B_{12}^{(1)} + {}_2B_{30}^{(1)} + \frac{1}{2}\;{}_2B_{31}^{(1)} \right] = 0 \; . \tag{5.62} $$

From this one obtains for the second approximation to the slip velocity, ${}_2(u-u_0)$,

$$
{}_2(u-u_0) = {}_1(u-u_0) + {}_{(2)}(u-u_0) = -\frac{1}{4\sqrt{hm}}\Big[\sqrt{\pi}\left(\frac{2-\sigma}{\sigma}\right){}_2B_{20}^{(1)} - {}_2B_{11}^{(1)}
$$

$$
+ 2\,{}_1(u-u_0)\,{}_1B_{20} - \frac{1}{2}\,{}_2B_{12}^{(1)} + {}_2B_{30}^{(1)} + \frac{1}{2}\,{}_2B_{31}^{(1)}\Big]. \qquad (5.63)
$$

Since ${}_1(u-u_0)$ is known from equation (5.57), the second approximation ${}_2(u-u_0)$ is now expressed entirely in terms of the Burnett coefficients, and hence in terms of the derivatives of the mean motion, density, and temperature of the gas.

Fortunately, it is not necessary to substitute for the Burnett coefficients appearing in equation (5.63) the lengthy expressions given by equations (2.68) to (2.113) of Part II. For the present application, the expressions for the Burnett coefficients may be simplified appreciably by considering that very close to the wall the normal velocity component $\underline{w}$ vanishes, and that the tangential derivatives of the tangential velocity components, i.e. $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, etc., are themselves of order $\left(\frac{\mu}{p}\right)$ as follows from equation (5.58). The details of this simplification in the form of the Burnett coefficients ${}_2B_{kr}^{(p)}$ are carried out in Appendix IV, Section 3, and the expressions for these coefficients which are appropriate to the present application are given by equations (A.154) to (A.163).

One now substitutes into equation (5.63) for the quantities ${}_2B_{20}$, ${}_2B_{11}^{(1)}$, ${}_1(u-u_0)$, ${}_1B_{20}$, ${}_2B_{12}^{(1)}$, ${}_2B_{30}^{(1)}$, ${}_2B_{31}^{(1)}$ from equations (A.161), (A.159), (5.58), (2.62), (A.160), (A.162), (A.163), respectively. After some algebraic reductions, the following expression for the second approximation to the slip velocity is obtained:

$$
\begin{aligned}
{}_{2}(u-u_0) = {} & \left(\frac{\mu}{p}\right)\left[\frac{\sqrt{\pi}}{2}\left(\frac{2-\sigma}{\sigma}\right)\frac{1}{\sqrt{hm}}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)-\frac{3}{8}\frac{1}{h^2 m}\frac{\partial h}{\partial x}\right] \\
& +\left(\frac{\mu}{p}\right)^2\left[2.225\,\frac{1}{h^2 m}\frac{\partial h}{\partial z}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)-\frac{9}{16}\frac{1}{h^2 m}\frac{\partial h}{\partial z}\frac{\partial w}{\partial x}\right. \\
& \quad -\frac{4}{15}\frac{1}{\gamma\sqrt{hm}}\frac{\partial}{\partial z}\left(\frac{\gamma}{\sqrt{hm}}\left[\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right]\right)-\frac{3}{20}\frac{h}{\sqrt{hm}}\frac{\partial}{\partial z}\left(\frac{1}{h\sqrt{hm}}\left[\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right]\right) \\
& \quad \left.+\frac{3}{8}\sqrt{\pi}\left(\frac{2-\sigma}{\sigma}\right)\frac{h}{\gamma m}\frac{\rho}{\mu}\frac{1}{\sqrt{hm}}\left\{\frac{\partial}{\partial z}\left(\frac{\mu}{p}\frac{\gamma}{h^3}\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial x}\left(\frac{\mu}{p}\frac{\gamma}{h^3}\frac{\partial h}{\partial z}\right)\right\}\right] \\
& +\left(\frac{\mu}{p}\right)^2\left[-\frac{3}{4}\sqrt{\pi}\left(\frac{2-\sigma}{\sigma}\right)\frac{1}{h\sqrt{hm}}\frac{Dh}{Dt}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\frac{3}{2}\sqrt{\pi}\left(\frac{2-\sigma}{\sigma}\right)\frac{1}{h\sqrt{hm}}\frac{Dh}{Dt}\frac{\partial w}{\partial x}\right. \\
& -\frac{\sqrt{\pi}}{2}\left(\frac{2-\sigma}{\sigma}\right)\frac{\rho}{\mu}\frac{1}{\sqrt{hm}}\frac{D}{Dt}\left(\frac{\mu}{p}\left[\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right]\right)+\frac{9}{16}\frac{\gamma}{\sqrt{hm}}\frac{D}{Dt}\left(\frac{1}{\gamma h\sqrt{hm}}\frac{\partial h}{\partial x}\right) \\
& +\frac{3}{20}\frac{h}{\sqrt{hm}}\frac{\partial}{\partial x}\left(\frac{1}{h^2\sqrt{hm}}\frac{Dh}{Dt}\right)-\frac{2}{5}\frac{1}{\gamma\sqrt{hm}}\frac{\partial}{\partial x}\left(\frac{\gamma}{\sqrt{hm}}\frac{1}{h}\frac{Dh}{Dt}\right) \\
& \left.+\,0.4312\,\frac{1}{h^3 m}\frac{Dh}{Dt}\frac{\partial h}{\partial x}-\frac{5}{6}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\frac{Dw}{Dt}\right]. \qquad (5.64)
\end{aligned}
$$

The first bracket on the right side of equation (5.64) is just the first approximation ${}_{1}(u-u_0)$ as already found by equation (5.58). The additional terms are again of the form $\frac{\mu^2}{p^2}\frac{\partial^2}{\partial z\,\partial x}(u,v,h,\gamma)$ and represent the effect of the second order terms in the stress tensor and heat flux vector of the gas on its slip velocity. When the flow conditions are steady, $\frac{\partial}{\partial t} = 0$, so that time derivative $\frac{D}{Dt}$ contained in the third bracket of equation (5.64) reduces to $(u_0\frac{\partial}{\partial x} + v_0\frac{\partial}{\partial y})$, according to equation (A.151).

It will be recalled that in the initial setting up of the boundary conditions the assumption was made that the parameters u,v,w,T, ρ which describe the mean motion of the gas should change only by a negligible amount during the time required by the molecules to traverse the sublayer. Subsequently, however the thickness of the sublayer reduced to zero and hence also the transit time of the molecules. Consequently no particular restriction need be applied to the order of magnitude of the time derivatives

$\frac{\partial}{\partial t}$ appearing in the general form of equation (5.64).

The expression for the slip velocity in the y-direction, ${}_2(v - v_0)$, is, of course, easily obtained from equation (5.64) by interchanging x and y and u and v, respectively. Furthermore, it is desirable to express the slip velocity in terms of the more common variables of state of the gas ρ and T, instead of ν and h. This transformation is readily carried out with the aid of equations (A.166) to (A.176) given in Section 4 of Appendix IV. The final form for the second approximation to the slip velocity is given, in a general notation, by equation (5.79) below.

5.9 Temperature-- Jump for Maxwell Molecule

The same method of calculation that was used in obtaining the slip velocity from equations (5.49) and (5.50), will now be applied to the derivation of the second approximation of the temperature jump ${}_2(T - T_w)$ from equation (5.54). In the evaluation of equation (5.54) it is proper to replace terms of the form $(u - u_0)^2\, \mathcal{B}_2$ and $(u - u_0)\, \mathcal{A}_7$ by their respective second approximations, ${}_1(u - u_0)^2\, {}_0\mathcal{B}_2$ and ${}_1(u - u_0)\, {}_1\mathcal{A}_7$. The first approximation to the slip velocity is given by equation (5.57), and the values of ${}_0\mathcal{B}_2$ and ${}_1\mathcal{A}_7$ are obtained from equations (A.118) and (A.102) of Appendix IV, respectively. Thus

$${}_0\mathcal{B}_2 = \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{\sqrt{hm}} \cdot 1 \quad ; \qquad {}_1\mathcal{A}_7 = \frac{3}{8}\, \frac{1}{hm}\, {}_1B_{20}^{(1)} \quad . \tag{5.65}$$

One now substitutes equations (5.56), and (A.104) and (A.127), for ${}_2\mathcal{A}_{11}$ and ${}_2\mathcal{B}_{11}$ respectively, into equation (5.54). The equation so obtained is multiplied by $\left[-(hm)^{3/2}\sqrt{\pi}\right]$, and as by definition

$$h = \frac{1}{2kT} \quad ,$$

one finally obtains

$$\frac{5}{8}\left(\frac{2-\alpha}{\alpha}\right)\sqrt{\pi}\;{}_{2}B_{11} - \Big[\,1 + \frac{3}{4}\,{}_{2}A_{2} + \frac{3}{8}\,{}_{2}A_{3} + \frac{3}{4}\,{}_{2}B_{20} - \frac{3}{8}\,{}_{2}B_{21} - \frac{3}{16}\,{}_{2}B_{22}$$

$$- \frac{1}{2}\,{}_{2}B_{40}\Big] + \frac{T_w}{T}\Big[\,1 - \frac{1}{4}\,{}_{2}A_{2} - \frac{3}{8}\,{}_{2}A_{3} + \frac{1}{2}\,{}_{2}B_{20} + \frac{1}{4}\,{}_{2}B_{21}$$

$$+ \frac{3}{8}\,{}_{2}B_{22} - \frac{1}{4}\,{}_{2}B_{40}\Big] + \frac{1}{2}(hm)\left\{{}_{1}(u-u_0)^2 + {}_{1}(v-v_0)^2\right\}$$

$$- \frac{\sqrt{\pi}}{4}\left\{\left(\frac{2-\alpha}{\alpha}\right) - \left(\frac{2-\sigma}{\sigma}\right)\right\}\sqrt{hm}\,\left\{{}_{1}(u-u_0)\,{}_{1}B^{(1)}_{20} + {}_{1}(v-v_0)\,{}_{1}C^{(1)}_{20}\right\} = 0\,. \qquad (5.66)$$

This equation is to be solved for the unknown $\frac{T_w}{T}$ which, being the ratio of the temperature of the wall to the temperature of the layer of gas adjacent to the wall, is equivalent to the temperature jump $(T - T_w)$.

Analogous to equations (5.60) and (5.61), it is now convenient to seperate the temperature ratio $\frac{T_w}{T}$ into components consisting of terms of equal order in powers of $\left(\frac{\mu}{p}\right)$, as shown by equation (5.67).

$$ {}_{2}\left(\frac{T_w}{T}\right) = {}_{0}\left(\frac{T_w}{T}\right) + {}_{(1)}\left(\frac{T_w}{T}\right) + {}_{(2)}\left(\frac{T_w}{T}\right)\,. \qquad (5.67)$$

By means of equations (5.67), (5.61), and a similar expression for ${}_{2}B_{11}$ it is now possible to rewrite equation (5.66) by grouping together terms containing equal powers of $\left(\frac{\mu}{p}\right)$. This results in

$$\left[{}_{0}\left(\frac{T_w}{T}\right) - 1\right] + \left[\frac{5}{8}\left(\frac{2-\alpha}{\alpha}\right)\sqrt{\pi}\;{}_{1}B_{11} - \frac{3}{4}\,{}_{1}B_{20} + \frac{1}{2}\,{}_{0}\left(\frac{T_w}{T}\right){}_{1}B_{20} + {}_{(1)}\left(\frac{T_w}{T}\right)\right]$$

$$+ \Big[\,\frac{5}{8}\left(\frac{2-\alpha}{\alpha}\right)\sqrt{\pi}\;{}_{(2)}B_{11} - \left\{\frac{3}{4}\,{}_{2}A_{2} + \frac{3}{8}\,{}_{2}A_{3} + \frac{3}{4}\,{}_{(2)}B_{20} - \frac{3}{8}\,{}_{2}B_{21} - \frac{3}{16}\,{}_{2}B_{22} - \frac{1}{2}\,{}_{2}B_{40}\right\}$$

$$+ {}_{(2)}\left(\frac{T_w}{T}\right) + {}_{0}\left(\frac{T_w}{T}\right)\left\{-\frac{1}{4}\,{}_{2}A_{2} - \frac{3}{8}\,{}_{2}A_{3} + \frac{1}{2}\,{}_{(2)}B_{20} + \frac{1}{4}\,{}_{2}B_{21} + \frac{3}{8}\,{}_{2}B_{22} - \frac{1}{4}\,{}_{2}B_{40}\right\}$$

$$+ \frac{1}{2}\,{}_{(1)}\left(\frac{T_w}{T}\right)\,{}_{1}B_{20} + \frac{1}{2}(hm)\left\{{}_{1}(u-u_0)^2 + {}_{1}(v-v_0)^2\right\}$$

$$- \frac{\sqrt{\pi}}{4}\sqrt{hm}\left\{\left(\frac{2-\alpha}{\alpha}\right) - \left(\frac{2-\sigma}{\sigma}\right)\right\}\left\{{}_{1}(u-u_0)\,{}_{1}B^{(1)}_{20} + {}_{1}(v-v_0)\,{}_{1}C^{(1)}_{20}\right\}\Big] = 0 \qquad (5.68)$$

Since equation (5.68) must hold for arbitrary values of the parameter $\left(\frac{\mu}{p}\right)$, each of the three square brackets above must vanish separately. From the first bracket one obtains

$$ {}_{0}\left(\frac{T_w}{T}\right) = 1 \tag{5.69} $$

Hence to the zeroth approximation the temperature of the gas is equal to the wall temperature. Equation (5.67) is now easily transformed into an expression for the second approximation for the temperature jump, ${}_2(T - T_w)$. Thus

$$ T_w = {}_2T \cdot {}_2\left(\frac{T_w}{T}\right) = {}_2T \cdot {}_0\left(\frac{T_w}{T}\right) + {}_2T\left\{ {}_{(1)}\left(\frac{T_w}{T}\right) + {}_{(2)}\left(\frac{T_w}{T}\right) \right\} $$

From equation (5.69) this results in

$$ {}_{(2)}(T - T_w) = -\,{}_2T\left\{ {}_{(1)}\left(\frac{T_w}{T}\right) + {}_{(2)}\left(\frac{T_w}{T}\right) \right\} . \tag{5.70} $$

When the second bracket of equation (5.69) is equated to zero, one readily obtains the first approximation ${}_{(1)}\left(\frac{T_w}{T}\right)$ given by equation (5.71).

$$ {}_{(1)}\left(\frac{T_w}{T}\right) = -\frac{5}{8}\left(\frac{2-\alpha}{\alpha}\right)\sqrt{\pi}\,{}_1B_{11} + \frac{1}{4}\,{}_1B_{20} . \tag{5.71} $$

The above may be expressed directly in terms of the derivatives of the mean motion of the gas. Thus, from equations (2.61), (2.62) and (A.153), one obtains

$$ {}_{(1)}\left(\frac{T_w}{T}\right) = \left(\frac{\mu}{p}\right)\left\{ \frac{15}{16}\sqrt{\pi}\left(\frac{2-\alpha}{\alpha}\right)\frac{1}{\sqrt{hm}}\,\frac{1}{h}\,\frac{\partial h}{\partial z} - \frac{1}{2}\,\frac{1}{h}\,\frac{Dh}{Dt} \right\} , \tag{5.72} $$

where from equation (A.151)

$$ \frac{Dh}{Dt} = \frac{\partial h}{\partial t} + u_0\frac{\partial h}{\partial x} + v_0\frac{\partial h}{\partial y} . $$

The first term of equation (5.72) agrees exactly with the expression derived by von Smoluchowski, equation (5.5a), with the value of γ equal to $\frac{5}{3}$, which is appropriate for a monatomic gas for which equation (5.72) was derived. The equivalence of equation (5.5a) and the first term of (5.72) is readily shown by means of equations (2.7), (3.13), (3.22), and (3.23). The second term in equation (5.72) represents the effect of both unsteady temperature conditions and of thermal gradients in the plane of a wall which is in relative motion with respect to the gas. Neither of these effects were accounted for in the more elementary derivations of Knudsen and von Smoluchowski.

Similarly one obtains the value of ${}_{(2)}\left(\frac{T_w}{T}\right)$ by equating to zero the third square bracket of equation (5.68). When the values of ${}_{(1)}\left(\frac{T_w}{T}\right)$, ${}_{(1)}\left(\frac{T_w}{T}\right)$, ${}_1(u - u_0)$, ${}_1(v - v_0)$ are substituted from equations (5.69), (5.71), (5.57), (5.59), respectively, one obtains after some algebraic reduction

$$\begin{aligned}
{}_{(2)}\left(\frac{T_w}{T}\right) = \Big[& -\frac{5}{8}\left(\frac{2-\alpha}{\alpha}\right)\sqrt{\pi}\,{}_{(2)}B_{11} + \frac{1}{4}\,{}_{(2)}B_{20} + {}_2A_2 + \frac{3}{4}\,{}_2A_3 - \frac{5}{8}\,{}_2B_{21} - \frac{9}{16}\,{}_2B_{22} \\
& -\frac{1}{4}\,{}_2B_{40} - \frac{1}{8}\left({}_1B_{20}\right)^2 + \frac{5}{16}\sqrt{\pi}\left(\frac{2-\alpha}{\alpha}\right)\,{}_1B_{20}\,{}_1B_{11} \\
& -\frac{1}{32}\left\{\left({}_1B_{11}^{(1)}\right)^2 + \left({}_1C_{11}^{(1)}\right)^2\right\} - \frac{\pi}{32}\left(\frac{2-\sigma}{\sigma}\right)\left\{2\left(\frac{2-\alpha}{\alpha}\right) - \left(\frac{2-\sigma}{\sigma}\right)\right\}\left\{\left({}_1B_{20}^{(1)}\right)^2 + \left({}_1C_{20}^{(1)}\right)^2\right\} \\
& + \frac{\sqrt{\pi}}{16}\left(\frac{2-\alpha}{\alpha}\right)\left\{{}_1B_{11}^{(1)}\,{}_1B_{20}^{(1)} + {}_1C_{11}^{(1)}\,{}_1C_{20}^{(1)}\right\}\Big] . \qquad (5.73)
\end{aligned}$$

From the above one obtains the explicit relation between ${}_{(2)}\left(\frac{T_w}{T}\right)$ and the derivatives of the mean motion variables of the gas by substituting the values of the various Burnett coefficients from equations (A.154) to (A.158), (2.61), and (2.62). After a considerable amount of algebraic reduction, equation (5.73) thus becomes

$$
\begin{aligned}
{}_{(2)}\left(\frac{T_w}{T}\right) = \left(\frac{\mu}{p}\right)^2 \Big[& \left\{ 0.31655 + \frac{\pi}{8}\left(\frac{2-\sigma}{\sigma}\right)^2 - \frac{\pi}{4}\left(\frac{2-\sigma}{\sigma}\right)\left(\frac{2-\alpha}{\alpha}\right)\right\}\left\{\left(\frac{\partial u_j}{\partial z} + \frac{\partial w}{\partial x_j}\right)^2\right\} \\
& + \frac{1}{2}\left\{ \frac{\partial w}{\partial x_j}\left(\frac{\partial u_j}{\partial z} + \frac{\partial w}{\partial x_j}\right)\right\} + \frac{3}{8}\sqrt{\pi}\left(\frac{2-\alpha}{\alpha}\right) h \left\{\frac{\partial}{\partial x_j}\left(\frac{1}{h\sqrt{hm}}\left[\frac{\partial u_j}{\partial z} + \frac{\partial w}{\partial x_j}\right]\right)\right. \\
& + \frac{\sqrt{\pi}}{4}\left\{\left(\frac{2-\sigma}{\sigma}\right) - \left(\frac{2-\alpha}{\alpha}\right)\right\}\frac{1}{\sqrt{hm}}\left\{\frac{\partial}{\partial x_j}\left[\frac{\partial u_j}{\partial z} + \frac{\partial w}{\partial x_j}\right]\right\} + \frac{\sqrt{\pi}}{4}\left(\frac{2-\alpha}{\alpha}\right)\frac{1}{\sqrt{hm}}\left\{2\frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial x_j \partial x_j}\right\} \\
& - \frac{3}{2}\sqrt{\pi}\left(\frac{2-\alpha}{\alpha}\right)\frac{1}{h\sqrt{hm}}\left\{\frac{\partial h}{\partial x_j}\left(\frac{\partial u_j}{\partial z} + \frac{\partial w}{\partial x_j}\right)\right\} - \frac{45}{32}\sqrt{\pi}\left(\frac{2-\alpha}{\alpha}\right)\frac{1}{h\sqrt{hm}}\left\{\frac{\partial w}{\partial x_j}\frac{\partial h}{\partial x_j}\right\} \\
& + \frac{\sqrt{\pi}}{6}\left(\frac{2-\sigma}{\sigma}\right)\frac{1}{\nu\sqrt{hm}}\left\{\frac{\partial \nu}{\partial x_j}\left(\frac{\partial u_j}{\partial z} + \frac{\partial w}{\partial x_j}\right)\right\} + \frac{\sqrt{\pi}}{4}\left(\frac{2-\sigma}{\sigma}\right)\frac{p}{\mu}\left\{\frac{\partial}{\partial x_j}\left(\frac{\mu}{p}\frac{1}{\sqrt{hm}}\right)\cdot\left(\frac{\partial u_j}{\partial z} + \frac{\partial w}{\partial x_j}\right)\right\} \\
& - \frac{3}{16}\frac{1}{h^2 m}\left\{\frac{\partial^2 h}{\partial x_j \partial x_j}\right\} + 4.1819\frac{1}{h^3 m}\left\{\frac{\partial h}{\partial x_j}\frac{\partial h}{\partial x_j}\right\} + 2.7456\frac{1}{h^3 m}\left(\frac{\partial h}{\partial z}\right)^2 - \frac{1}{8}\frac{1}{\nu h^2 m}\left\{\frac{\partial h}{\partial x_j}\frac{\partial \nu}{\partial x_j}\right\} \\
& - \frac{3}{16}\frac{p}{\mu}\left\{\frac{\partial h}{\partial x_j}\frac{\partial}{\partial x_j}\left(\frac{\mu}{p}\frac{1}{h^2 m}\right)\right\} + \frac{8}{7}\frac{p}{\mu}\frac{h}{m\nu}\left\{\frac{\partial}{\partial x_j}\left(\frac{\mu}{p}\frac{\nu}{h^3}\frac{\partial h}{\partial x_j}\right)\right\} - \frac{1}{28}\frac{p}{\mu}\frac{h}{m\nu}\frac{\partial}{\partial z}\left(\frac{\mu}{p}\frac{\nu}{h^3}\frac{\partial h}{\partial z}\right)\Big] \\
& + \left(\frac{\mu}{p}\right)^2\Big[1.2183\left(\frac{1}{h}\frac{Dh}{Dt}\right)^2 + \frac{1}{2}\frac{p}{\mu}\frac{D}{Dt}\left(\frac{\mu}{p}\frac{1}{h}\frac{Dh}{Dt}\right) \\
& - \frac{123}{32}\sqrt{\pi}\left(\frac{2-\alpha}{\alpha}\right)\frac{1}{h^2\sqrt{hm}}\frac{\partial h}{\partial z}\frac{Dh}{Dt} - \frac{45}{32}\sqrt{\pi}\left(\frac{2-\alpha}{\alpha}\right)\nu\frac{D}{Dt}\left(\frac{1}{\nu h\sqrt{hm}}\frac{\partial h}{\partial z}\right) \\
& + \frac{3}{4}\sqrt{\pi}\left(\frac{2-\alpha}{\alpha}\right)\frac{Dh}{Dt}\frac{\partial}{\partial z}\left(\frac{1}{h\sqrt{hm}}\right) + \frac{3}{2}\sqrt{\pi}\left(\frac{2-\alpha}{\alpha}\right)\sqrt{hm}\,\frac{1}{h}\frac{Dh}{Dt}\frac{Dw}{Dt} - \frac{18}{7}\frac{1}{h}\frac{\partial h}{\partial z}\frac{Dw}{Dt}\Big]
\end{aligned}
\tag{5.74}
$$

$(j = 1,2)$; $z = 0$.

In equation (5.47) the summation convention over two indices is used. This notation has the meaning

$$
\begin{aligned}
x_1 &\equiv x \; ; \quad x_2 \equiv y \\
u_1 &\equiv u \; ; \quad u_2 \equiv v
\end{aligned}
\tag{5.75}
$$

so that, for example,

$$\left\{ \frac{\partial h}{\partial x_j}\left(\frac{\partial u_j}{\partial z} + \frac{\partial w}{\partial x_j} \right) \right\} \equiv \left\{ \frac{\partial h}{\partial x}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) + \frac{\partial h}{\partial y}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right\}. \tag{5.76}$$

The time derivative, $\frac{D}{Dt}$, is that defined by equation (A.151) of Appendix IV which may be expressed in the above notation as

$$\frac{D}{Dt} \equiv \left[\frac{\partial}{\partial t} + u_{o_j} \frac{\partial}{\partial x_j} \right] . \tag{5.77}$$

The second approximation to the temperature jump, ${}_2(T - T_w)$ can now be calculated from equation (5.70) by substituting for ${}_{(1)}\left(\frac{T_w}{T}\right)$ and ${}_{(2)}\left(\frac{T_w}{T}\right)$ the values given by equations (5.72) and (5.74), respectively.

5.10 Results of Investigation

The complete set of boundary conditions for the high speed flow of a rarefied gas over a solid surface has thus been calculated, accurate to the second approximation. The four boundary conditions consist of equations (5.41) for $w = 0$, (5.64) for ${}_2u$, with a similar expression for ${}_2v$, and (5.70) for ${}_2T$, the temperature of the gas.

For aerodynamic applications it is desirable to replace the variables $\underline{h}$ and ν, that were used throughout the preceding analysis, by the temperature T, and density ρ of the gas, respectively. The pertinent equations, (5.64), (5.72) and (5.74) are then readily transformed by means of formulae (A.166) to (A.174) given in Section 4 of Appendix IV. This results in the final form of the boundary conditions which are given below by equations (5.78) to (5.80). The index notation of equations (5.75) to (5.77) is again employed.

$$w = 0 \tag{5.78}$$

$$\begin{aligned}
{}_2u_j\,(x,y,z\to 0) = {}& u_{o_j} + \left(\frac{\mu}{p}\right)\left[a_1\sqrt{RT}\left(\frac{\partial u_j}{\partial z}+\frac{\partial w}{\partial x_j}\right) + \frac{3}{4}RT\frac{\partial T}{\partial x_j}\right] \\
& + \left(\frac{\mu}{p}\right)^2\left[-\frac{5}{6}RT\frac{\partial}{\partial z}\left(\frac{\partial u_j}{\partial z}+\frac{\partial w}{\partial x_j}\right) + b_1 R\frac{\partial T}{\partial z}\left(\frac{\partial u_j}{\partial z}+\frac{\partial w}{\partial x_j}\right) + \frac{9}{8}R\frac{\partial T}{\partial z}\frac{\partial w}{\partial x_j}\right. \\
& \qquad - \frac{8}{15}\frac{RT}{\rho}\frac{\partial \rho}{\partial z}\left(\frac{\partial u_j}{\partial z}+\frac{\partial w}{\partial x_j}\right) - 3a_1 R\sqrt{RT}\frac{\partial^2 T}{\partial x_j\,\partial z} \\
& \qquad \left. - \frac{3}{2}a_1 R\frac{\sqrt{RT}}{\mu}\left\{\frac{\partial T}{\partial x_j}\frac{\partial \mu}{\partial z} + \frac{\partial T}{\partial z}\frac{\partial \mu}{\partial x_j}\right\}\right] \\
& + \left(\frac{\mu}{p}\right)^2\left[-a_1\sqrt{RT}\frac{D}{Dt}\left(\frac{\partial u_j}{\partial z}+\frac{\partial w}{\partial x_j}\right) + \frac{3}{2}a_1\sqrt{RT}\frac{1}{T}\frac{DT}{Dt}\left(\frac{\partial u_j}{\partial z}+\frac{\partial w}{\partial x_j}\right)\right. \\
& \qquad - 3a_1\sqrt{RT}\frac{1}{T}\frac{DT}{Dt}\frac{\partial w}{\partial x_j} - a_1\sqrt{RT}\left(\frac{\partial u_j}{\partial z}+\frac{\partial w}{\partial x_j}\right)\frac{D}{Dt}\left(\log\frac{\mu}{p}\right) \\
& \qquad - \frac{5}{8}R\frac{D}{Dt}\left(\frac{\partial T}{\partial x_j}\right) + b_2\frac{R}{T}\frac{DT}{Dt}\frac{\partial T}{\partial x_j} + \frac{9}{8}\frac{R}{\rho}\frac{D\rho}{Dt}\frac{\partial T}{\partial x_j} \\
& \qquad \left. + \frac{4}{5}\frac{R}{\rho}\frac{DT}{Dt}\frac{\partial \rho}{\partial x_j} - \frac{5}{6}\left(\frac{\partial u_j}{\partial z}+\frac{\partial w}{\partial x_j}\right)\frac{Dw}{Dt}\right]
\end{aligned} \tag{5.79}$$

$(j = 1,2)$

$$
\begin{aligned}
{}_2T\,(x,y,z\rightarrow 0) = {} & T_w + \left(\frac{\mu}{p}\right)\left[c_1\sqrt{RT}\,\frac{\partial T}{\partial z} - \frac{1}{2}\,\frac{DT}{Dt}\right] \\
& + \left(\frac{\mu}{p}\right)^2\Bigg[e_1 T\left\{\left(\frac{\partial u_j}{\partial z} + \frac{\partial w}{\partial x_j}\right)^2\right\} - \frac{1}{2}\,T\left\{\frac{\partial w}{\partial x_j}\left(\frac{\partial u_j}{\partial z} + \frac{\partial w}{\partial x_j}\right)\right\} + e_2 T\sqrt{RT}\left\{\frac{\partial}{\partial x_j}\left(\frac{\partial u_j}{\partial z} + \frac{\partial w}{\partial x_j}\right)\right\} \\
& \quad + e_3 T\sqrt{RT}\left\{2\,\frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial x_j\,\partial x_j}\right\} + e_4\sqrt{RT}\left\{\frac{\partial T}{\partial x_j}\left(\frac{\partial u_j}{\partial z} + \frac{\partial w}{\partial x_j}\right)\right\} \\
& \quad + e_5\sqrt{RT}\left\{\frac{\partial T}{\partial x_j}\,\frac{\partial w}{\partial x_j}\right\} + \frac{1}{6}\,a_1\sqrt{RT}\,\frac{T}{\rho}\left\{\frac{\partial \rho}{\partial x_j}\left(\frac{\partial u_j}{\partial z} + \frac{\partial w}{\partial x_j}\right)\right\} \\
& \quad - \frac{1}{2}\,a_1\sqrt{RT}\,\frac{T}{\mu}\left\{\frac{\partial \mu}{\partial x_j}\left(\frac{\partial u_j}{\partial z} + \frac{\partial w}{\partial x_j}\right)\right\} + e_6\,RT\left\{\frac{\partial^2 T}{\partial x_j\,\partial x_j}\right\} \\
& \quad + e_7\,R\left\{\frac{\partial T}{\partial x_j}\,\frac{\partial T}{\partial x_j}\right\} + e_8\,R\left(\frac{\partial T}{\partial z}\right)^2 + \frac{1}{8}\,\frac{RT}{\rho}\left\{\frac{\partial T}{\partial x_j}\,\frac{\partial \rho}{\partial x_j}\right\} \\
& \quad + e_6\,\frac{RT}{\mu}\left\{\frac{\partial \mu}{\partial x_j}\,\frac{\partial T}{\partial x_j}\right\} - \frac{1}{14}\,RT\,\frac{\partial^2 T}{\partial z^2} - \frac{1}{14}\,\frac{RT}{\mu}\,\frac{\partial T}{\partial z}\,\frac{\partial \mu}{\partial z}\Bigg] \\
& + \left(\frac{\mu}{p}\right)^2\Bigg[e_9\,\frac{1}{T}\left(\frac{DT}{Dt}\right)^2 + \frac{1}{2}\,\frac{DT}{Dt}\cdot\frac{D}{Dt}\left(\log\frac{\mu}{p}\right) + e_{10}\sqrt{RT}\,\frac{1}{T}\,\frac{DT}{Dt}\,\frac{\partial T}{\partial z} \\
& \quad + e_5\sqrt{RT}\,\frac{D}{Dt}\left(\frac{\partial T}{\partial z}\right) - e_5\sqrt{RT}\,\frac{\partial T}{\partial z}\,\frac{D}{Dt}\left(\log\{\rho\sqrt{T}\}\right) + \frac{1}{2}\,\frac{D^2 T}{Dt^2} \\
& \quad - 3e_3\,\frac{1}{\sqrt{RT}}\,\frac{DT}{Dt}\,\frac{Dw}{Dt} - \frac{18}{7}\,\frac{\partial T}{\partial z}\,\frac{Dw}{Dt}\Bigg].
\end{aligned}
$$

$$(j = 1,2),\quad (z = 0) \qquad (5.80)$$

The quantities a_1, b_1, b_2, c_1, $e_1, \ldots, e_{10}$ which appear in the above equations are numerical constants whose values are given in Table (5.3). Many of these constants depend on the fraction σ and the accomodation coefficient α which were defined by equations (5.3) and (5.5), respectively. In the boundary conditions (5.79) and (5.80) it is understood, of course, that all of the derivatives are to be evaluated at a point $(x,y,z \rightarrow 0)$. The peculiar symmetry of the equations (5.79) and (5.80) is a consequence of the choice of coordinates, figure (5.3), where the z-axis, representing the direction normal to the wall, is a distinquished direction, while the x and y directions are completely equivalent.

5.11 Discussion of Results

As was stated earlier, the two terms in the expression for the slip velocity, equation (5.79), which are of order $\left(\frac{\mu}{p}\right)$ agree with the values for the fist order slip velocity and thermal creep as obtained by References (5.8) and (5.9), respectively. Likewise the first order term $\frac{\mu}{p} c_1 \sqrt{RT} \frac{\partial T}{\partial z}$ of the temperature condition (5.80) agrees with the value obtained by von Smoluchowski, equation (5.5). As is to be expected, however, the value of the constant c_1, Table 5.3) as derived here agrees with c_1' calculated by von Smoluchowski only for monatomic gases for which $\gamma = c_p/c_v = 5/3$, and $\lambda/\mu c_v = 5/2$, according to the kinetic theory.

The various assumptions which were made during the course of the derivation of equations (5.78) to (5.80) impose certain restrictions on the validity of these boundary conditions, as summarized below:

1. The vanishing of the normal velocity component (equation 5.78) holds true without limitation.
2. The expressions for the tangential velocity component u_j (equation 5.79), and for the temperature T, (equation 5.80) are applicable provided that

    (a) The gas is a simple, monatomic, perfect gas of the

Maxwell type (fifth power law of repulsion);

(b) The pressure level and the motion of the gas are such that the second approximation to the stress tensor (equation 3.17) and heat flux vector (equation 3.18) are applicable. This means that the Mach number M, and linear dimension L, which characterize the flow, bear to the mean free path ℓ, the relation $(M \frac{\ell}{L})^2 \ll 1$.

It should be noted that most of the above restrictions arise from the fact that the complete set of Burnett coefficients of the velocity distribution function has been calculated only for the Maxwell molecule to terms of order $\left(\frac{\mu}{p}\right)^2$. However, the basic method used in the deduction of the boundary conditions is valid for any "smooth" spherically symmetric molecular model as well as for an arbitrary degree of approximation $\left(\frac{\mu}{p}\right)^N$. Thus equations (5.49), (5.50) and (5.54) are valid generally, whereas equations (5.55) and (5.66) are restricted to the second approximation for the Maxwell molecule.

However, it is possible to calculate the first approximation to the slip velocity and temperature jump using the rigid elastic spherical molecule, since the appropriate first order Burnett coefficients are known; (see equations (2.56), (2.57), (2.59), (2.60) of Part II). The final result of this calculation is

$$ {}_1u_j = u_{o_j} + \left(\frac{\mu}{p}\right)\left[a_1\sqrt{RT}\left(\frac{\partial u_j}{\partial z} + \frac{\partial w}{\partial x_j}\right) + 1.036 \times \frac{3}{4} R \frac{\partial T}{\partial x_j} \right] \qquad (5.81) $$

$$ (j = 1,2) $$

$$ {}_1T = T_w + \left(\frac{\mu}{p}\right)\left[c_1\sqrt{RT}\,\frac{\partial T}{\partial z} - 0.968 \times \frac{1}{2}\frac{DT}{Dt} \right], \qquad (5.82) $$

where the numerical constants a_1 and c_1 have the values given in Table 5.3. Evidently, the effect of a change in molecular model on the first approximation to the boundary conditions is quite insiginficant.

The validity of the first approximation to the slip velocity and temperature jump has been confirmed experimentally by many observers, (References 5.3, 5.8, 5.10). It would seem desirable to perform similar experiments, at lower pressure levels and higher relative velocities between the gas and the solid surface, in order to test the quantitative validity of the results obtained from the second order kinetic theory that are represented by equations (5.79) and (5.80), as well as by the additional terms of the viscous stresses and heat flux. This seems particularly desirable, since the first order boundary conditions contain the two experimental constants σ and α, whereas no additional experimental constants are contained in the second order terms of the boundary conditions.

PART VI

HIGH SPEED COUETTE FLOW OF A RAREFIED GAS

It is desirable to illustrate the application of the general boundary conditions, as deduced in the preceding part, by means of a geometrically simple example. For this purpose consider the situation depicted by figure 6.1. Two flat plates of infinite extent are separated by a distance h, the intermediate space being filled with a rarefied gas. The lower plate, coincident with the x-axis, has the uniform temperature Tw_o and is held fixed in space, whereas the upper plate, having the temperature Tw_1 moves with the constant velocity U in the x-direction. The problem then consists of determining the distributions of the macroscopic velocity, temperature, pressure, and density of the gas as a function of the y-coordinate.

The problem under consideration thus extends the well known results of the Couette flow of an incompressible, viscous fluid to a gaseous substance which is both compressible and viscous. With such a substance, the effects of thermal conduction, which is of little consequence in the incompressible case, plays an important role. In fact, the general solution to the problem obtained below contains the special case of thermal conduction between infinite plane walls when the velocity U of the upper plate vanishes.

6.1 The Differential Equations of Motion

The differential equations which describe the gas flow of figure 6.1 are obtained from the general equations of macroscopic motion, equations (3.14) to (3.16) Part III, by means of the following reductions:

$$\left.\begin{array}{lll} x_1 = x & \frac{\partial}{\partial x_1} = 0 & u_1 = u(y) \\ x_2 = y & \frac{\partial}{\partial x_2} = \frac{d}{dy} & u_2 = 0 \\ x_3 = z & \frac{\partial}{\partial x_3} = 0 & u_3 = 0 \\ & \frac{\partial}{\partial t} = 0 & \end{array}\right\} \qquad (6.1)$$

Equations (6.1) express the restrictions that

(a) the flow is steady

(b) the flow is two-dimensional

(c) the mass velocity is everywhere parallel to the x-axis

(d) all of the parameters describing the flow are functions of the transverse y-coordinate only.

In view of equation (6.1), the continuity equation, equation (3.14), is satisfied identically and makes no explicit contribution to the problem. The momentum equations in the x and y directions, which are obtained from equation (3.15) with $i = 1$ and 2, similarly reduce to the simple form given by equations (6.2) and (6.3) below, respectively. Finally, the energy relation, equation (3.16), is reduced by equations (6.1) to the form of equation (6.4). The differential equations of motion for Couette flow thus are given by

$$\frac{d}{dy}(\mathcal{T}_{12}) = 0 \qquad (6.2)$$

$$\frac{d}{dy}(P + \mathcal{T}_{22}) = 0 \qquad (6.3)$$

$$\frac{d}{dy}(q_2 + u\,\mathcal{T}_{12}) = 0 \qquad (6.4)$$

The relevant components τ_{12}, and τ_{22} of the viscous stress tensor τ_{ij} are evaluated to the second approximation means of equation (3.17) of Part III with the restrictions of equations (6.1). Similarly, the y-component, q_2, of the heat flux vector q_i is obtained from equation (3.18). These reductions result in

$${}_2\tau_{12} = {}_2\tau_{xy} = -\mu \frac{du}{dy} \tag{6.5}$$

$${}_2\tau_{22} = {}_2\tau_{yy} = p + \frac{\mu^2}{p}\Big[\left(\tfrac{1}{3}K_2 + \tfrac{1}{12}K_6\right)\left(\frac{du}{dy}\right)^2 + \tfrac{2}{3}K_3 R \frac{d^2T}{dy^2}$$
$$+ \tfrac{2}{3}K_4 \frac{R}{p}\frac{dT}{dy}\frac{dp}{dy} + \tfrac{2}{3}K_5\frac{R}{T}\left(\frac{dT}{dy}\right)^2\Big] \tag{6.6}$$

$${}_2q_2 = {}_2q_y = -\lambda \frac{dT}{dy}\,, \tag{6.7}$$

where the various symbols are defined as in Part III. It is to be understood, of course, that in the above expressions for the heat flux and stress components, terms of order $\frac{\mu^3}{p^2}$ or higher have been neglected. Due to the simplicity of the geometry in the present case, only the normal stress component τ_{yy} contains "additional terms" of order $\frac{\mu^2}{p}$.

By substituting the expressions for the stresses and heat flux, equations (6.5) to (6.7), into the differential equations of motion, equations (6.2) to (6.4), one obtains the following three total differential equations:

$$\frac{d}{dy}\left(\mu\frac{du}{dy}\right) = 0 \tag{6.8}$$

$$\frac{d}{dy}\Big[p + \frac{\mu^2}{p}\Big\{\left(\tfrac{1}{3}K_2 + \tfrac{1}{12}K_6\right)\left(\frac{du}{dy}\right)^2 + \tfrac{2}{3}K_3 R\frac{d^2T}{dy^2}$$
$$+ \tfrac{2}{3}K_4\frac{R}{p}\frac{dT}{dy}\frac{dp}{dy} + \tfrac{2}{3}K_5\frac{R}{T}\left(\frac{dT}{dy}\right)^2\Big] = 0 \tag{6.9}$$

$$\frac{d}{dy}\left(\lambda \frac{dT}{dy}\right) + \mu\left(\frac{du}{dy}\right)^2 = 0 \tag{6.10}$$

The solution of equations (6.8) to (6.10) for the dependent variables U, T, p, with the appropriate boundary conditions, to be discussed later, represents then the desired velocity, temperature, and pressure distributions for the Couette flow. The distribution of density is, of course, obtained from the equation of state of a perfect gas.

6.2 Solution of the Differential Equations

The differential equations (6.8) to (6.10) are evidently non-linear total differential equations. The non-linearity of equation (6.8) is due to the fact that, in general, the viscosity of the gas, μ , is a function of the unknown temperature distribution.

In order to facilitate the first attempt to solve these equations, it will now be assumed that the coefficient of viscosity μ has a constant value throughout the gas, independent of the local temperature. This assumption effectively results in the linearization of the differential equations, since the solutions for the unknowns U, T, p may be obtained successively from equations (6.8), (6.9), and (6.10), respectively. The influence of this simplification will be appreciable only when the variation of temperature is relatively large. Even in the latter case, it is believed, that the results so obtained will be of qualitative significance since the nature of the phenomena of slip and temperature jump for a rarefied gas flowing past a solid boundary, does not depend directly on the temperature variation of the viscosity. In fact, it will be recalled that a change in the temperature dependence of the viscosity coefficient is equivalent to a change in the molecular model assumed in the calculations of the kinetic theory. Since the coefficient of thermal conduction, λ, is proportional to the viscosity coefficient μ, (see equation (3.13), Part III), both λ and μ will be treated as constants throughout the following analysis.

Thus with μ constant, equation (6.8) is easily integrated resulting in the linear velocity distribution

$$u = Ay + B \quad , \tag{6.11}$$

where A and B are arbitrary constants. When the quantity $\frac{du}{dy}$ of equation (6.10) is replaced by the constant A of equation (6.11), one obtains a simple differential equation for the temperature T

$$\frac{d^2T}{dy^2} = -\frac{\mu}{\lambda} A^2 . \tag{6.12}$$

The integration of equation (6.12) results in the parabolic temperature distribution

$$T = -\frac{\mu}{2\lambda} A^2 y^2 + Cy + D \quad , \tag{6.13}$$

where C and D are additional constants of integration.

The pressure distribution will now be obtained from equation (6.9). Since the general form of the boundary conditions (Part V) has been evaluated explicitly only for the Maxwell molecules, the values used for the constants K_2, K_3, K_4, K_5, K_6, which appear in equation (6.9), should also be those appropriate to the Maxwell molecule. According to Table 3.1 of Part III, one has

$$\left.\begin{array}{ll} K_2 = 2 & K_4 = 0 \\ K_3 = 3 & K_5 = 3\frac{T}{\mu}\frac{d\mu}{dT} = 0 \\ K_6 = 8 & \end{array}\right\} \tag{6.14}$$

where the constant K_5 vanishes due to the assumption of constant viscosity. With the values of the constants K as given by equation (6.14), and the derivatives $\frac{du}{dy}$ and $\frac{d^2T}{dy^2}$ as obtained from equations (6.11) and (6.12), respectively, equation (6.9) yields upon integration with respect to <u>y</u>

$$p\left[1+\frac{\mu^2}{p^2}\left\{\frac{4}{3}A^2-2\frac{R\mu}{\lambda}A^2\right\}\right]=E\ , \qquad (6.15)$$

where E is again a constant of integration. It follows from equation (6.15) that in the limit $\left(\frac{\mu}{p}\right)\longrightarrow 0$, ${}_{o}p = {}_{o}E$ = constant. Since, however, in the present approximation $\frac{\mu^2}{p^2}$ may be replaced by $\frac{\mu^2}{{}_{o}p^2}$, this means that the pressure is independent of the y-coordinate up to terms of order $\left(\frac{\mu}{p}\right)^2$. Thus the solution of the differential equations (6.8) to (6.10) has been obtained, as summarized below:

$$\left.\begin{aligned} U &= Ay + B \\ T &= -\frac{\mu}{2\lambda}A^2y^2 + Cy + D \\ p &= E = \text{const.} \end{aligned}\right\} \qquad (6.16)$$

It should be noted that the specialization of equation (6.14) for the Maxwell molecule affects only that portion of the pressure distribution which is of order $\frac{\mu^2}{{}_{o}p^2}$, and does not affect the velocity and temperature distributions at all.

6.3 <u>The Boundary Conditions</u>

The general solution, equations (6.16), of the differential equations contain the five arbitrary constants A, B, C, D, E which will now be determined by applying the boundary conditions which correspond to the Couette flow of figure 6.1.

The constant E merely defines the uniform pressure level of the flow and is therefore a known or preassigned quantity. To determine A, B, C, D four conditions are required. These four conditions are obtained by applying the general boundary conditions for the gas velocity tangent to the wall, equation (5.79), and for the temperature of the gas at the wall, equation (5.80), to both the stationary plate at y = 0 and the moving plate at y = h.

Because of the simplicity of the geometry of the Couette flow, equations (5.79) and (5.80) can be reduced considerably. The reductions of equation (6.1), when expressed in the coordinate system used in Part V, (see figure 5.3a and equation (5.75)) are equivalent to

$$\left.\begin{array}{lll} x_1 = x & \frac{\partial}{\partial x_1} = 0 & u_1 = u \\ x_2 = z & \frac{\partial}{\partial x_2} = 0 & u_2 = 0 \\ z = \pm y & \frac{\partial}{\partial z} = \pm \frac{d}{dy} & w = 0 \\ & & \\ & \frac{D}{Dt} = 0 & \end{array}\right\} \qquad (6.17)$$

where the (+) sign applies to the stationary wall at y = 0, and the (-) sign to the moving wall at y = h. In the analysis that follows, the subscript o will denote that a particular quantity is to be evaluated at y = 0, whereas the subscript 1 denotes evaluation at y = h. The four boundary conditions thus obtained from equations (5.79), (5.80) and (6.17) are:

(1) at y = 0

$$u_o = 0 + \left(\frac{\mu}{p_o}\right)\left[a_1\sqrt{RT_o}\left(\frac{du}{dy}\right)_o\right] + \left(\frac{\mu}{p_o}\right)^2\left[-\frac{5}{6}RT_o\left(\frac{d^2u}{dy^2}\right)_o - 4.634\,R\left(\frac{dT}{dy}\right)_o\left(\frac{du}{dy}\right)_o - \frac{8}{15}\frac{RT_o}{p_o}\left(\frac{dp}{dy}\right)_o\left(\frac{du}{dy}\right)_o\right] \qquad (6.18)$$

(2) at $y = h$

$$u_1 = U + \frac{\mu}{p_1}\left[-a_1\sqrt{RT_1}\left(\frac{du}{dy}\right)_1\right] + \left(\frac{\mu}{p_1}\right)^2\left[-\frac{5}{6}RT_1\left(\frac{d^2u}{dy^2}\right)_1 - 4.634\,R\left(\frac{dT}{dy}\right)_1\left(\frac{du}{dy}\right)_1 - \frac{8}{15}\,\frac{RT_1}{p_1}\left(\frac{dp}{dy}\right)_1\left(\frac{du}{dy}\right)_1\right] \tag{6.19}$$

(3) at $y = 0$

$$T_0 = T_{w_0} + \left(\frac{\mu}{p_0}\right)\left[c_1\sqrt{RT_0}\left(\frac{dT}{dy}\right)_0\right] + \left(\frac{\mu}{p_0}\right)^2\left[e_1 T_0\left(\frac{du}{dy}\right)_0^2 - 5.4912\,R\left(\frac{dT}{dy}\right)_0^2 - \frac{1}{14}RT_0\left(\frac{d^2T}{dy^2}\right)_0\right] \tag{6.20}$$

(4) at $y = h$

$$T_1 = T_{w_1} + \left(\frac{\mu}{p_1}\right)\left[-c_1\sqrt{RT_1}\left(\frac{dT}{dy}\right)_1\right] + \left(\frac{\mu}{p_1}\right)^2\left[e_1 T_1\left(\frac{du}{dy}\right)_1^2 - 5.4912\,R\left(\frac{dT}{dy}\right)_1^2 - \frac{1}{14}RT_1\left(\frac{d^2T}{dy^2}\right)_1\right]. \tag{6.21}$$

The numerical values of the constants a_1, c_1, e_1, are computed from the formulae of Table 5.3.

The various derivatives appearing in the above boundary conditions are readily evaluated in terms of the arbitrary constants A, B, C, D and the plate spacing h. The boundary conditions, equations (6.18) to (6.21) can thus be reduced to the following four explicit algebraic relations between the constants A, B, C, D:

$$B = \left(\frac{\mu}{p}\right) a_1 \sqrt{RT_0}\, A - \left(\frac{\mu}{p}\right)^2 \left[\, 4.634\, R C A \right] \tag{6.22}$$

$$B + Ah = U - \left(\frac{\mu}{p}\right) a_1 \sqrt{RT_1}\, A - \left(\frac{\mu}{p}\right)^2 \left[4.634\, R\left(C - \frac{\mu}{\lambda} A^2 h\right) A\right] \tag{6.23}$$

$$D = T_{w_0} + \left(\frac{\mu}{p}\right) c_1 \sqrt{RT_0}\, C - \left(\frac{\mu}{p}\right)^2 \left[- e_1 T_0 A^2 + 5.4912\, R C^2 - \frac{1}{14} R T_0 \frac{\mu}{\lambda} A^2 \right] \tag{6.24}$$

$$D + Ch - \frac{\mu}{2\lambda} A^2 h^2 = T_{w_1} - \left(\frac{\mu}{p}\right) c_1 \sqrt{RT_1}\, \left(C - \frac{\mu}{\lambda} A^2 h\right) - \left(\frac{\mu}{p}\right)^2 \left[- e_1 T_1 A^2 + 5.4912\, R \left(C - \frac{\mu}{\lambda} A^2 h\right)^2 - \frac{1}{14} R T_1 \frac{\mu}{\lambda} A^2 \right]. \tag{6.25}$$

The evaluation of the constants A, B, C, D from the above set of simultaneous algebraic equations is carried out in the next section.

6.4 Calculation of the Constants, A, B, C, D.

The four simultaneous algebraic equations, (6.22) to (6.25) are seen to be non-linear. The appropriate roots of the constants A, B, C, D are again obtained most easily by expanding each of them in ascending powers of the parameter $\theta \equiv \left(\frac{\mu}{p}\right)$. One may thus write

$$\left.\begin{aligned} A &= {}_0A \left[1 + \theta\, {}_1A^* + \theta^2\, {}_2A^* \right] \\ B &= \left[{}_0B + \theta\, {}_1B + \theta^2\, {}_2B \right] \\ C &= {}_0C \left[1 + \theta\, {}_1C^* + \theta^2\, {}_2C^* \right] \\ D &= {}_0D \left[1 + \theta\, {}_1D^* + \theta^2\, {}_2D^* \right] \end{aligned}\right\} \tag{6.26}$$

The stars indicate that a quantity such as $\theta^2\,{}_2A^*$ is dimensionless. It is also convenient to expand the temperature of the gas at $y = h$, T_1, in a similar manner, namely

$$T_1 = {}_0T_1\left[1 + \theta\,{}_1T_1^* + \theta^2\,{}_2T_1^*\right]. \tag{6.27}$$

From equations (6.26) one obtains, accurate to terms of order θ^2, the expansion

$$A^2 = {}_0A^2\left[1 + 2\theta\,{}_1A^* + \theta^2\left({}_1A^{*2} + 2\,{}_2A^*\right)\right]. \tag{6.28}$$

Similarly one obtains the expressions (6.29) which are accurate to terms of order θ :

$$\begin{aligned} {}_1(\sqrt{T_1}) &= \sqrt{{}_0T_1}\left[1 + \tfrac{1}{2}\theta\,{}_1T_1^*\right] \\ {}_1(\sqrt{T_0}) = {}_1(\sqrt{D}) &= \sqrt{{}_0D}\left[1 + \tfrac{1}{2}\theta\,{}_1D^*\right] \end{aligned} \tag{6.29}$$

The various expansions, equations (6.26) to (6.29) are now substituted into equations (6.22) to (6.25) and only those terms which are of order θ^2 or lower are retained. In each equation the terms are then grouped in ascending powers of θ, which results in equations (6.30) to (6.33).

$$\begin{aligned} [{}_0B] + \theta\left[{}_1B - a_1\sqrt{R\,{}_0D}\,{}_0A\right] + \theta^2\Big[{}_2B - a_1\sqrt{R\,{}_0D}\,{}_0A\left({}_1A^* + \tfrac{1}{2}\,{}_1D^*\right) \\ + 4.634\,R\,{}_0C\,{}_0A\Big] = 0 \end{aligned} \tag{6.30}$$

$$\left[{}_0B + {}_0A h - U\right] + \theta\left[{}_1B + {}_0A\,{}_1A^* h + a_1\sqrt{R\,{}_0T_1}\,{}_0A\right]$$

$$+ \theta^2\left[{}_2B + {}_0A\,{}_2A^* h + a_1\sqrt{R\,{}_0T_1}\,{}_0A\left({}_1A^* + \tfrac{1}{2}\,{}_1T_1^*\right)\right.$$

$$\left. + 4.634\,R\,{}_0A\left({}_0C - \frac{\mu}{\lambda} h\,{}_0A^2\right)\right] = 0 \qquad (6.31)$$

$$\left[{}_0D - T_{w_0}\right] + \theta\left[{}_0D\,{}_1D^* - c_1\sqrt{R\,{}_0T_0}\,{}_0C\right]$$

$$+ \theta^2\left[{}_0D\,{}_2D^* - c_1\sqrt{R\,{}_0T_0}\,{}_0C\left({}_1C^* + \tfrac{1}{2}\,{}_1T_0^*\right) - e_1\,{}_0T_0\,{}_0A^2\right.$$

$$\left. + 5.4912\,R\,{}_0C^2 - \frac{1}{14} R\frac{\mu}{\lambda}\,{}_0T_0\,{}_0A^2\right] = 0 \qquad (6.32)$$

$$\left[{}_0T_1 - T_{w_1}\right] + \theta\left[{}_0T_1\,{}_1T_1^* + c_1\sqrt{R\,{}_0T_1}\left({}_0C - \frac{\mu}{\lambda}\,{}_0A^2 h\right)\right]$$

$$+ \theta^2\left[{}_0T_1\,{}_2T_1^* + c_1\sqrt{R\,{}_0T_1}\left(\tfrac{1}{2}\,{}_1T_1^*\left\{{}_0C - \frac{\mu}{\lambda}\,{}_0A^2 h\right\} + {}_0C\,{}_1C^* - 2\frac{\mu}{\lambda}\,{}_0A^2\,{}_1A^* h\right)\right.$$

$$\left. + e_1\,{}_0T_1\,{}_0A^2 + 5.4912\,R\left({}_0C - \frac{\mu}{\lambda}\,{}_0A^2 h\right)^2 - \frac{1}{14} R\,{}_0T_1\frac{\mu}{\lambda}\,{}_0A^2\right] = 0\,. \qquad (6.33)$$

Since the above equations must hold for arbitrary, small values of θ, one finds, as was observed in the preceding Parts, that each of the square brackets above must vanish identically. Thus the values of ${}_0B$, ${}_0A$, ${}_0D$, ${}_0T$, are found immediately by equating to zero the first brackets of equations (6.30), (6.31), (6.32), (6.33), respectively. This yields the "zero order" solutions given by equation (6.34)

$$
{}_0B = 0 \qquad {}_0D = {}_0T_0 = Tw_0
$$
$$
{}_0A = \frac{U}{h} \qquad {}_0T_1 = Tw_1 \tag{6.34}
$$

In order to determine the constant ${}_0C$, evaluate equation (6.13) at the point y = h, retaining only zero order terms. On thus obtains

$$
{}_0T_1 = -\frac{\mu}{2\lambda}\,{}_0A^2 h^2 + {}_0C h + {}_0D \tag{6.35}
$$

from which, using (6.34), ${}_0C$ is found to be

$$
{}_0C = \frac{T_{w_1} - T_{w_0}}{h} + \frac{\mu}{2\lambda}\,\frac{U^2}{h} . \tag{6.36}
$$

The zero order solution of equations (6.34) and (6.36) represent Couette flow in the pressure regime of gas dynamics where there is no slipping of the gas on the walls, and where the layer of gas adjacent to the wall has the temperature of the wall.

With the zero order solution determined, the first order constants ${}_1B$, ${}_1A^*$, ${}_1C^*$, ${}_1T_1{}^*$ can be calculated sucessively by equating to zero the coefficients of θ in equations (6.30) to (6.33) respectively. The result of this purely algebraic calculation is given by equations (6.37) to (6.40).

$$
{}_1B = a_1 \sqrt{RT_{w_0}}\; \frac{U}{h} \tag{6.37}
$$

$$
{}_1A^* = -a_1 \cdot \frac{1}{h}\left(\sqrt{RT_{w_0}} + \sqrt{RT_{w_1}}\right) \tag{6.38}
$$

$$ {}_1D^* = c_1 \sqrt{\frac{R}{T_{w_0}}} \left\{ \frac{T_{w_1} - T_{w_0}}{h} + \frac{\mu}{2\lambda} \frac{U^2}{h} \right\} \tag{6.39} $$

$$ {}_1T_1^* = - c_1 \sqrt{\frac{R}{T_{w_1}}} \left\{ \frac{T_{w_1} - T_{w_0}}{h} - \frac{\mu}{2\lambda} \frac{U^2}{h} \right\}. \tag{6.40} $$

The expression for ${}_1C^*$ is obtained in a manner analogous to equation (6.35), the result being

$$ {}_0C_1C^* = \Big[- c_1 \left(\sqrt{RT_{w_0}} + \sqrt{RT_{w_1}}\right) \frac{T_{w_1} - T_{w_0}}{h^2} - \left\{ (c_1 + 2a_1)\sqrt{RT_{w_0}} + (2a_1 - c_1)\sqrt{RT_{w_1}} \right\} \frac{\mu}{2\lambda} \frac{U^2}{h^2} \Big]. \tag{6.41} $$

The expressions (6.37) to (6.41) represent the effect of the slip velocity and temperature discontinuity at the wall to a first approximation, that is for small values of θ , and will be discussed in detail in section 6.8 below.

Finally, the second order constants ${}_2B$, ${}_2A^*$, ${}_2D^*$, ${}_2T^*$, remain to be calculated. An inspection of equation (6.30) shows that ${}_2B$ is the only unknown quantity contained in the coefficient of θ^2, so that ${}_2B$ can be evaluated by equating the former to zero. With ${}_2B$ known, ${}_2A^*$ can be calculated by equating to zero the coefficient of θ^2 of equation (6.31). Likewise ${}_2D^*$ and ${}_2T_1^*$ are obtained in turn from equations (6.32) and (6.33), respectively. Equations (6.42) to (6.45) give the result of this calculation.

$$ {}_2B = \frac{U}{h} \left[- a_1^2 \frac{R}{h} \left(T_{w_0} + \sqrt{T_{w_0} T_{w_1}}\right) - (4.634 - \tfrac{1}{2} a_1 c_1) R \left\{ \frac{T_{w_1} - T_{w_0}}{h} + \frac{\mu}{2\lambda} \frac{U^2}{h} \right\} \right] \tag{6.42} $$

$$ {}_2A^* = \left[a_1^2 \frac{1}{h^2} \left(\sqrt{RT_{w_0}} + \sqrt{RT_{w_1}} \right)^2 + \left(9.268 - a_1 c_1 \right) \frac{\mu R}{2\lambda} \frac{\mathcal{U}^2}{h^2} \right] \tag{6.43} $$

$$ \begin{aligned} {}_2D^* = \Bigg[& - c_1^2 \left(1 + \sqrt{\frac{T_{w_1}}{T_{w_0}}} \right) \frac{R}{h^2} \left(T_{w_1} - T_{w_0} \right) \\ & + \left\{ e_1 + \frac{R\mu}{2\lambda} \left[\left(\frac{1}{7} - 2 a_1 c_1 - c_1^2 \right) + \left(c_1^2 - 2 c_1 a_1 \right) \sqrt{\frac{T_{w_1}}{T_{w_0}}} \right] \right\} \frac{\mathcal{U}^2}{h^2} \\ & + \left(\frac{1}{2} c_1^2 - 5.4912 \right) \frac{R}{h^2 T_{w_0}} \left\{ \left(T_{w_1} - T_{w_0} \right) + \frac{\mu}{2\lambda} \mathcal{U}^2 \right\}^2 \Bigg] \end{aligned} \tag{6.44} $$

$$ \begin{aligned} {}_0C\,{}_2C^* = \Bigg[& c_1^2 \left(\sqrt{RT_{w_0}} + \sqrt{RT_{w_1}} \right)^2 \frac{T_{w_1} - T_{w_0}}{h^3} + \left(18.536 - 2 a_1 c_1 \right) R \left(\frac{\mu}{2\lambda} \right)^2 \frac{\mathcal{U}^4}{h^3} \\ & + 3\, a_1^2 \left(\sqrt{RT_{w_0}} + \sqrt{RT_{w_1}} \right)^2 \frac{\mu}{2\lambda} \frac{\mathcal{U}^2}{h^3} \\ & + \left\{ e_1 + \frac{R\mu}{2\lambda} \left(-3 c_1^2 - 2 a_1 c_1 + 22.1077 \right) \right\} \left(T_{w_1} - T_{w_0} \right) \frac{\mathcal{U}^2}{h^3} \Bigg] \end{aligned} \tag{6.45} $$

Returning to equation (6.26), it will be seen that it is now possible to calculate the constants A, B, C, D accurate to terms of order $\theta^2 = \left(\frac{\mu}{p}\right)^2$, by use of equations (6.34), (6.36) to (6.39), and (6.41) to (6,45). Consequently, particular velocity and temperature distributions may be computed from equation (6.16) for any combination of plate velocity U, plate spacing h, wall temperatures Tw_0 and Tw_1, and pressure level p for which the expansions of equation (6.26) converge sufficiently rapidly.

6.5 Friction Coefficient and Slip Velocity

It is of interest to calculate the friction force exerted by the rarefied gas on the fixed and moving plates. This calculation will be carried out in dimensionless form. The standard definition of the friction coefficient C_F is

$$C_F = \frac{2\tau}{\rho \mathcal{U}^2} , \tag{6.46}$$

where from equations (6.5) and (6.8) the shear stress τ at either wall is given by

$$\tau = \mu A . \tag{6.47}$$

It is convenient to define a Reynolds number Re, based on the plate spacing h, as

$$Re = \frac{\mathcal{U} \rho h}{\mu} . \tag{6.48}$$

Equation (6.46) may then be written in the form

$$C_F = \frac{2}{Re} A^* , \tag{6.49}$$

where A* is defined by

$$A^* = \frac{h}{\mathcal{U}} A = 1 + \theta_1 A^* + \theta_2^2 A^* . \tag{6.50}$$

The last equality of equation (6.50) results from equations (6.26) and (6.34). Thus the parameter A* is a measure of the influence of the rarefaction of the gas on the friction coefficient.

In order to replace the plate velocity U by a dimensionless quantity, we introduce the plate Mach number M, based on the temperature Tw_1 of the moving plate, as

$$M = \frac{\mathcal{U}}{\sqrt{\gamma R T_{w_1}}} . \tag{6.51}$$

The coefficients of viscosity μ, and thermal conduction λ are replaced by β, the reciprocal of the Prandtl number,

$$\beta = \frac{\lambda}{\mu C_p} , \tag{6.52}$$

where C_p is the specific heat at constant pressure of the gas.

From the equation of state of the gas and from equations (6.48) and (6.51) it is easily shown that

$$\theta \frac{\sqrt{RT_{w_1}}}{h} = \left(\frac{\mu}{p}\right) \frac{\sqrt{RT_{w_1}}}{h} = \sqrt{\gamma}\left(\frac{M}{\mathcal{R}e}\right) . \tag{6.53}$$

The ratio of Mach number to Reynolds number is independent of the plate velocity U and may also be expressed in terms of the mean free path, ℓ, of the gas. Thus from equations (6.53), (3.22) and (3.23) one obtains

$$\left(\frac{M}{\mathcal{R}e}\right) = \sqrt{\frac{2}{\pi\gamma}}\left(\frac{\ell}{h}\right) . \tag{6.54}$$

The quantities ${}_1A^*$ are obtained in dimensionless form by applying equations (6.51), (6.52) and (6.53) to equations (6.38) and (6.43), respectively. The final expression for A^* obtained in this manner from equation (6.50) is

$$A^* = \left[1 - \left\{ a_1\left(1+\sqrt{\frac{T_{w_0}}{T_{w_1}}}\right)\sqrt{\gamma}\left(\frac{M}{\mathcal{R}e}\right)\right\} + \left\{ a_1^2\left(1+\sqrt{\frac{T_{w_0}}{T_{w_1}}}\right)^2\gamma \right.\right.$$
$$\left.\left. + \left(4.634 - \frac{1}{2}a_1c_1\right)\frac{\gamma(\gamma-1)}{\beta}M^2\right\}\left(\frac{M}{\mathcal{R}e}\right)^2\right] . \tag{6.55}$$

It is seen that to a first approximation A^* depends only on the wall temperature ratio $\frac{T_{w_0}}{T_{w_1}}$, and the "rarefaction parameter" $\left(\frac{M}{\mathcal{R}e}\right)$, whereas the second order term involves, in addition, the Mach number explicitly.

Since the influence of the temperature ratio $\frac{T_{w_0}}{T_{w_1}}$ in equation (6.55) is a secondary one, it will suffice for illustrative purposes to calculate A* for the special case of equal wall temperatures and for a particular combination of gas and wall material. Assuming that the gas is air and that the walls are made of metal such as aluminum or brass, the required physical constants have the values given by equations (6.56).

$$\gamma = 1.40 \qquad \sigma = 1.00$$
$$\beta = \frac{4}{3} \qquad \alpha = 0.90 \tag{6.56}$$

The values of the molecular reflection parameter σ and accomodation coefficient α are average values obtained from Tables 5.1 and 5.2, respectively. The constants a_1, c_1, and e_1, (required later) are then calculated using the formulae of Table 5.3, resulting in

$$a_1 = 1.253, \qquad c_1 = 2.377, \qquad e_1 = 0.2507. \tag{6.57}$$

Hence, for the special case under consideration, equation (6.55) is reduced to

$$A^* = \left[1 - 2.97\left(\frac{M}{Re}\right) + \left\{ 8.80 + 1.32\, M^2 \right\}\left(\frac{M}{Re}\right)^2 \right] . \tag{6.58}$$

Equation (6.58) is plotted in figure (6.2) for several constant values of the Mach number M. The "first approximation" to A*, represented by $A^* = 1 - 2.97\,(\frac{M}{Re})$, is also shown for comparison. The nature and significance of the above result will be discussed in section 6.8.

For the case of equal wall temperatures, it follows from the symmetry of the linear velocity profile, that the "slip velocities" between the gas and the fixed plate and the gas and the moving plate are equal. Denoting this relative or slip velocity by Δu, it follows from equations (6.11) and (6.50) that

$$\frac{\Delta u}{U} = \frac{1}{2}(1 - A^*) \quad ; \quad T_{w_0} = T_{w_1} \quad . \tag{6.59}$$

Consequently, figure (6.2) also gives the "percent slip" as a function of Mach number and the rarefaction parameters $\left(\frac{M}{Re}\right)$ or $\left(\frac{\ell}{h}\right)$.

6.6 Temperature Discontinuity at Wall and Heat Transfer

The temperature discontinuities at the two walls, $(Tw_0 - T_0)$ and $(Tw_1 - T_1)$ may be calculated directly from equations (6.13), (6.26) and (6.28). In the general case when $Tw_0 \neq Tw_1$ and $U \neq 0$, it is found that the temperature profile is assymmetric with different temperature jumps at the two walls. In order to illustrate the nature of the phenomenan of temperature jump, the discontinuity in temperature, $Tw_0 - T_0$, occurring at the "fixed" wall, $y = 0$ will be calculated for the following two special cases:

(a) $Tw_0 = Tw_1$ $\quad U \neq 0$; $\quad (Tw_0 - T_0) = (Tw_1 - T_1)$

(b) $Tw_0 \neq Tw_1$ $\quad U = 0$ $\quad (Tw_0 - T_0) \neq (Tw_1 - T_1)$.

Due to the complete symmetry of case (a), the temperature jumps occurring at the two walls are equal in this instance.

From equation (6.13) one has

$$Tw_0 - T_0 = Tw_0 - D.$$

In view of equations (6.26) and (6.34) this may be written in the dimensionless form

$$\frac{T_{w_0} - T_0}{T_{w_0}} = -\left[\theta_1 D^* + \theta_2^2 D^*\right] . \tag{6.60}$$

Expressing equations (6.39) and (6.44) in terms of the dimensionless parameters of equations (6.51) to (6.53), one obtains for case (a) above

$$\frac{T_0 - T_w}{T_w} = \Big[\Big\{ \tfrac{1}{2} c_1 \sqrt{\gamma}\, \frac{\gamma-1}{\beta} M^2 \Big\} \Big(\frac{M}{\mathcal{R}e}\Big) + \Big\{ \Big[\big(\tfrac{1}{14} - 2 c_1 a_1\big) \frac{\gamma(\gamma-1)}{\beta} + e_1 \gamma^2 \Big] M^2$$

$$+ \Big[\frac{\gamma}{4} \Big(\frac{\gamma-1}{\beta}\Big)^2 \big(\tfrac{1}{2} c_1^2 - 5.4912\big)\Big] M^4 \Big\} \Big(\frac{M}{\mathcal{R}e}\Big)^2 \Big] . \qquad (6.61)$$

Considering the numerical example defined by equations (6.56) and (6.57) of the preceding section, one obtains from equation (6.61) the expression for the temperature jump when the space between two metal plates is filled with air, namely

$$\frac{T_0 - T_w}{w} = \Big[0.422\, M^2 \Big(\frac{M}{\mathcal{R}e}\Big) - \Big\{ 1.98\, M^2 + .0840\, M^4 \Big\} \Big(\frac{M}{\mathcal{R}e}\Big)^2 \Big] . \qquad (6.62)$$

Equation (6.62) is represented graphically for various constant values of M in figure 6.3.

We now turn to case (b), above, where both plates are at rest but at different temperatures. This is merely the problem of heat transfer between two infinite parallel plates by thermal conduction of a rarefied gas. In this case it is of interest to calculate the fraction of the total temperature difference occurring as a quasi-discontinuous "temperature jump". Thus, from equation (6.60)

$$\frac{T_{w_0} - T_0}{T_{w_0} - T_{w_1}} = - \frac{1}{1 - \frac{T_{w_1}}{T_{w_0}}} \Big[\theta\, {}_1D^* + \theta\, {}_2D^* \Big] . \qquad (6.63)$$

Since in the present case the velocity U is zero, neither Mach nor Reynolds number are appropriate parameters of the problem, so that the ratio $\left(\frac{\ell}{h}\right)$, equation (6.54), will be used to obtain the dimensionless expressions for ${}_1D^*$ and ${}_2D^*$. Setting $U = 0$, equation (6.63) is reduced to the form

$$\frac{T_{w_0} - T_0}{T_{w_0} - T_{w_1}} = \left[\sqrt{\frac{2}{\pi}}\, c_1 \left(\frac{\ell_0}{h}\right) + \left\{ -\frac{2}{\pi}\, c_1^2 \left(1 + \sqrt{\frac{T_{w_1}}{T_{w_0}}}\right) - \frac{2}{\pi}\left(\frac{1}{2} c_1^2 - 5.4912\right)\left(1 - \frac{T_{w_1}}{T_{w_0}}\right) \right\} \left(\frac{\ell_0}{h}\right)^2 \right], \tag{6.64}$$

where ℓ_0 is the mean free path corresponding to the density, ρ_0, of the gas adjacent to the wall $y = 0$. In particular, for air and metal plates the value of c_1, is given by equation (6.57), which reduces equation (6.64) to

$$\frac{T_{w_0} - T_0}{T_{w_0} - T_{w_1}} = \left[1.90 \left(\frac{\ell_0}{h}\right) - \left\{ 3.60 \left(1 + \sqrt{\frac{T_{w_1}}{T_{w_0}}}\right) - 1.70 \left(1 - \frac{T_{w_1}}{T_{w_0}}\right) \right\} \left(\frac{\ell_0}{h}\right)^2 \right] \tag{6.65}$$

$$(U = 0)$$

This equation is plotted in figure (6.3) for various values of the plate temperature ratio $0 \leqq \frac{T_{w_1}}{T_{w_0}} \leqq 1$.

The heat transfer q can be calculated directly from equation (6.7). Thus, from equation (6.13) one finds for the amount of heat transferred from the gas to the fixed wall, at $y = 0$,

$$q_0 = \lambda C \tag{6.65}$$

As may be seen from equations (6.36), (6.41) and (6.45), the above will, in general, result in a lengthy expression, which will not be given here explicitly. However, for the two important cases treated previously equation (6.65) leads to simple results.

For case (a), with non-vanishing velocity U and equal wall temperatures, the result is conveniently represented by the dimensionless heat transfer q_a* defined as

$$q_a^* = \frac{q_0}{\frac{1}{2}\mu\frac{U^2}{h}} \quad . \tag{6.66}$$

The denominator of this expression is one half of the mechanical energy which is converted into thermal energy by viscous action, when there is no slip, i.e. $\frac{\ell}{h} \rightarrow 0$. The final result of this calculation is

$$q_a^* = \left[1 - 4a_1\sqrt{\gamma}\left(\frac{M}{Re}\right) + \left\{ 12\gamma a_1^2 + (9.266 - a_1c_1)\frac{\gamma(\gamma-1)}{\beta} M^2 \right\} \left(\frac{M}{Re}\right)^2 \right] . \tag{6.67}$$

Using, for air, the values of the constants given by equations (6.56) and (6.57), one obtains from the above

$$q_a^* = \left[1 - 5.94\left(\frac{M}{Re}\right) + \left\{ 26.4 + 2.64 M^2 \right\} \left(\frac{M}{Re}\right)^2 \right] \tag{6.68}$$

$$(T_{w_1} = T_{w_0}) \; .$$

Comparing equation (6.68) with equation (6.58) for A* one observes that the curves of q_a* versus $\left(\frac{M}{Re}\right)$ at constant Mach number M, have the same general shape as those shown in figure 6.2.

For case (b) of pure heat transfer between two fixed walls, the appropriate dimensionless quantity to describe the reduction in heat transfer due to the rarefaction of the gas q_b^*, defined as

$$q_b^* = \frac{q_0}{\lambda\left(\frac{T_{w_0}-T_{w_1}}{h}\right)} \quad . \tag{6.69}$$

From equations (6.65), (6.36), (6.41), (6.45), (6.53) and (6.54) one readily obtains the explicit result

$$q_b^* = \left[1 - \sqrt{\tfrac{2}{\pi}}\, c_1 \left(1+\sqrt{\tfrac{T_{w_1}}{T_{w_0}}}\right)\left(\frac{\ell_0}{h}\right) + \frac{2}{\pi}\, c_1^2 \left(1+\sqrt{\tfrac{T_{w_1}}{T_{w_0}}}\right)^2 \left(\frac{\ell_0}{h}\right)^2 \right] \tag{6.70}$$

$$(K = 0) .$$

6.7 Temperature Profiles

It follows from the linear velocity variation of equation (6.11), that the velocity profile is always completely determined by the slip velocity at both walls. In the case of the temperature distribution, however, the shape of the temperature profile may vary between wide limits, depending the relative magnitudes of the plate Mach number M and the plate temperature ratio T_{w_1}/T_{w_0} . This will now be illustrated by several numerical examples.

By means of equations (6.26), (6.28), (6.51) and (6.52), the temperature variation of equation (6.13) can be expressed in the dimensionless form

$$\begin{aligned} \frac{T}{T_{w_0}} = {} & \left\{ -\frac{\gamma-1}{2\beta}\, \frac{T_{w_1}}{T_{w_0}}\, M^2 \left[1 + 2\theta\, {}_1A^* + \theta^2\left({}_1A^{*^2} + 2\, {}_2A^*\right)\right]\right\} \eta^2 \\ & + \left\{\left[\frac{\gamma-1}{2\beta}\, \frac{T_{w_1}}{T_{w_0}}\, M^2 - \left(1 - \frac{T_{w_1}}{T_{w_0}}\right)\right]\left[1 + \theta\, {}_1C^* + \theta^2\, {}_2C^* \right]\right\} \eta \\ & + \left\{ 1 + \theta\, {}_1D^* + \theta^2\, {}_2D^* \right\} \quad , \end{aligned} \tag{6.71}$$

where $\eta = \frac{y}{h}$.

Equation (6.71) has been evaluated for air by use of the physical constants of equations (6.56). For all of the three cases considered below, the air density corresponds to a value of $\frac{\ell_1}{h} = 0.10$ which is equivalent to $\left(\frac{M}{Re}\right) = 0.0675$:

(1) for $\frac{T_{w_1}}{T_{w_0}} = 1.0, \quad M = 2.0$

$$\frac{T}{T_{w_0}} = 1.072 + 0.462\,\eta(1-\eta) \qquad (6.72)$$

(2) for $\frac{T_{w_1}}{T_{w_0}} = 0.5, \quad M = 0$

$$\frac{T}{T_{w_0}} = 0.919 - 0.376\,\eta \qquad (6.73)$$

(3) for $\frac{T_{w_1}}{T_{w_0}} = 0.5, \quad M = 2.0$

$$\frac{T}{T_{w_0}} = 0.971 - 0.167\,\eta - 0.222\,\eta^2 \,. \qquad (6.74)$$

The three temperature profiles, equations (6.72) to (6.74) are plotted in figure (6.5) for comparison. The significance of these as well as the previous results are discussed in the next section.

6.8 Discussion of Results

The geometry of the Couette flow is sufficiently simple so that the terms of order $\frac{\mu^2}{p}$ occur only in the differential equation from which the pressure distribution is determined, i.e. equation (6.9). This means that the form of the general solution for the velocity and temperature distribution does not depend on the "degree of rarefaction" of the gas. Instead, the degree of rarefaction, which enters this problem through the boundary conditions, merely determines the explicit value of the arbitrary constants of the general solution of the differential equations. This result is quite independent of the assumption of constant coefficients of viscosity μ, and thermal conduction λ.

On the other hand, the linear form of the velocity distribution, equation (6.11), and the parabolic form of the temperature distribution, equation (6.13), are a direct consequence of the assumption of constant viscosity coefficient. The physical reason for the linear velocity distribution is, of course, that the shearing stress τ, as the only force acting in the x-direction, must be a constant. Since in the steady state case treated here the temperature at any ordinate y must also remain constant, the gradient of the thermal conduction, $\frac{dq}{dy}$, is just equal to the energy permit volume which is converted into heat by viscous dissipation, $(\tau \cdot \frac{du}{dy})$. This fact is expressed mathematically by equation (6.12) and accounts for the parabolic form of equation (6.13).

As may be seen from figure 6.5, the general temperature profile, curve (3), is essentially a combination of the symmetric parabolic profile (1) for equal wall temperatures, and the linear profile (2) of thermal conduction without fluid motion. However, curve (3) of figure 6.5 cannot be obtained by a direct superposition of curves (1) and (2), as an inspection of equation (6.71) will show; this is due to the non linearity of the boundary conditions.

The difference in magnitude of the various temperature jumps shown in figure 6.5 is easily accounted for by a consideration of the terms of order $(\frac{\mu}{p})$ in the temperature boundary conditions. Thus equations (6.18) or (6.20) show that, to a first approximation, the temperature jump is proportional to the product of $\sqrt{T_w}$ and the temperature gradient, at the wall. The effect of the second order corrections does not change the above qualitatively as may be seen from the temperature jumps at $\eta = 0$ and $\eta = 1.0$ of curves (1), (2) and (3) of figure (6.5). By an analogous argument, it follows that when the two wall temperatures are unequal, the slip velocity occurring at the warmer wall will be greater than the slip velocity at the colder wall.

In the preceding discussion the various types of temperature and velocity profiles have been compared at a fixed value of the "rarefaction parameter" $\left(\frac{M}{Re}\right)$ or $\left(\frac{\ell}{h}\right)$. The results of sections (6.5) and (6.6), and in particular figures 6.2 and 6.4, show that both the relative slip-velocity $\frac{\Delta u}{U}$ and the relative temperature jump $\frac{T_{w_0}-T_0}{T_{w_0}-T_{w_1}}$ depend on the ratio of the mean free path ℓ to the plate spacing h in a similar manner. From equation (6.58), or figure 6.2, the first approximation to the relative slip $\frac{\Delta u}{U}$ does not depend on the Mach number explicitly, which is a consequence of the linearity of the first order terms of the boundary conditions. The term $1.32\, M^2 \left(\frac{M}{Re}\right)^2$ of equation (6.58) can be traced back to the non-linear term $-\left(\frac{\mu}{p}\right)^2 R \frac{dT}{dy}\frac{du}{dy}$ which appears in the boundary condition of equation (6.18). As mentioned above,

$$\frac{dT}{dy} \sim q \sim \text{dissipation} \sim \tau \frac{du}{dy} \sim U^2$$

which accounts for the explicit appearance of the quantity M^2 in the second order term of equation (6.58).

On the other hand, equation (6.62) shows that the temperature jump is, to a first approximation, proportional to $M^2\left(\frac{M}{Re}\right)$ when the plate temperatures are equal. This is again due to the above stated fact that the temperature gradient at the wall is proportional to M^2. The term proportional to $M^4\left(\frac{M}{Re}\right)^2$ in equation (6.62) is attributable to the term of the form $\left(\frac{\mu}{p}\right)^2\left(\frac{dT}{dy}\right)^2$ of the temperature boundary condition, equation (6.20).

From equation (6.65) and figure (6.4) it appears that the wall temperature ratio $\frac{T_{w_1}}{T_{w_0}}$, for the case of temperature jump in pure heat transfer without gas flow, plays a role analogous to that of the Mach number in the case of slip velocity; that is the first approximation to equation (6.65) depends only on the rarefaction parameter $(\frac{\ell}{h})$ whereas the second order correction term is in addition a function of the wall temperature ratio.

As a consequence of the method of expansion in ascending powers of the "rarefaction parameter" $(\frac{\mu}{p})$, $(\frac{M}{\mathcal{R}e})$ or $(\frac{\ell}{h})$, all of the results obtained in the preceding sections are valid only for "sufficiently small" values of these rarefaction parameters. For this reason, the maxima and minima of the curves of figures 6.2 to 6.4 do not have any physical significance, but indicate that when the deviation of the second approximations from the fist order solutions are large, third and higher order terms in the viscous stresses, heat flux, and boundary conditions must be used.

The region of values of $(\frac{\ell}{h})$ or $(\frac{M}{\mathcal{R}e})$ for which the second order solutions are inadequate is indicated by the dashed curves of figures 6.2 to 6.4. It is evident from figures 6.2 and 6.3 that the maximum value of $(\frac{\ell}{h})$, for which the second order theory is adequate, decreases with increasing Mach number. This result agrees with the more general criteria that were established in Section 3.3. It was shown there from dimensional considerations that the second approximations to the stresses and the heat flux become inadequate when $(M \frac{\ell}{h})^2$ is not negligible compared to unity.

Moreover, the results represented by figures 6.2 to 6.4 indicate that in all cases calculated the second order corrections, proportional to $(\frac{M}{\mathcal{R}e})^2$, diminish the values of slip-velocity and temperature jump that are predicted by the first order theory.

6.9 Application of Result to Laminar Boundary Layer

The calculations of Section 6.5 on the friction coefficient and slip velocity of the Couette flow may be utilized to obtain a rough estimate of the effect of slip flow on the skin friction of a flat plate. If the true velocity profile of the boundary layer is replaced by a linear velocity profile, the boundary layer thickness δ is equivalent to the plate spacing h of the Couette flow, the influence of the increase of the boundary layer thickness in the x-direction being neglected. Equation (6.49) may then be given the more general meaning

$$C_F(R_x, M) = C_{F_0}(R_x, 0) \cdot A^*(Re, M) , \qquad (6.75)$$

where R_x is the Reynolds number based on the distance from the leading edge of the plate, that is

$$R_x = \frac{u \rho x}{\mu} . \qquad (6.76)$$

The quantity Re appearing in equation (6.75) is the Reynold's number based on the boundary layer thickness δ, as defined by equation (6.48) with h replaced by δ.

The meaning of equation (6.75) is then, that the friction coefficient for a given slip-flow condition defined by M and R_x, is obtained by multiplying the friction coefficient of the gas-dynamical flow at the same value of R_x by the factor A^*, equation (6.55), which represents the reduction in the skin friction caused by the slippage of the gas over the surface. The relationship between the two Reynolds numbers Re and R_x is known only in the two limiting cases of vanishing and infinite Reynolds number. For large values of R_x, one may use the well known law of Blasius which results in (Reference 6.1)

$$\frac{Re}{\sqrt{R_x}} = \frac{\delta}{x}\sqrt{R_x} = 3.65 \qquad (6.77)$$

$$(R_x \gg 1)$$

For very small values of the Reynolds number, δ and x are of the same order of magnitude so that

$$\mathrm{Re} = R_x \qquad (R_x \ll 1) \tag{6.78}$$

It is of interest to determine the realm of Mach and Reynolds numbers to which the value of A^* as calculated from equation (6.55) may be applied. The calculations for air, figure 6.2, will be used. It will be assumed, that the curves of figure 6.2 give results of sufficient accuracy in the regions where they are drawn solid. The end points of the solid curve have been chosen, somewhat arbitrarily, such that for a given value of M,

$$1 - A^* = \frac{2}{3}\,(1 - A^* \text{ First Approx.})$$

To every value of M there corresponds thus a maximum permissible value of $\left(\frac{M}{\mathcal{R}e}\right)$ and hence a minimum value of the Reynolds number Re. The corresponding value of R_x is obtained from equations (6.77) or (6.78). For application of the result to the flight of an aircraft through the rarefied atmosphere, it is most convenient here to assume a standard chord length x of one foot, so that the Reynolds number R_x becomes a unique function of the standard altitude H and the Mach number M.

The final result of this calculation is shown in the Mach number altitude plane of figure 6.6. Curve (1) represents the maximum altitude at which, for a given Mach number, the second order theory of figure 6.2 gives valid results, the intermediate calculation being based on equation (6.78). Curve (2) results when the relation (6.77) is used for the calculation. Curve (3) has been faired in to indicate that curves (1) and (2) are more appropriate for the low Mach number and high Mach number regions, respectively.

Curve (4) of figure 6.6 represents the locus of points where the slip velocity, Δu is equal to one percent of the free stream velocity U. At altitudes lying below curve (4), $\frac{\Delta u}{U} < .01$, so that the effects of slip will be negligible. Hence the region in the Mach number-altitude plane where the present approximation leads to significant results is included between curves (3) and (4), figure 6.6. For example, for $M = 3$ at an altitude of 250,000 feet, figures 6.2 and 6.6 give approximately a 10% reduction in the friction coefficient.

Figure 6.6 emphasizes the previously established fact that for very high Mach numbers the second order slip flow theory yields useful results only for a small region of Reynolds numbers.

PART VII

SLIP FLOW BETWEEN CONCENTRIC CYLINDERS

The results obtained in the preceding part for the Couette flow of a rarefied gas will now be extended to the case of slip flow between concentric cylinders of finite radius of curvature. This particular problem illustrates the application of the general boundary conditions for a curved wall, as well as the method of solution when the pressure varies throughout the flow field.

The notation and coordinate system to be employed are shown in figure 7.1. The two dimensional flow field is described by the plane-polar coordinates r and ϕ . The gas is confined between the inner cylinder of radius a and the outer cylinder of radius b. The outer cylinder is assumed to be at rest, whereas the inner cylinder is rotating at constant angular velocity w, its surface velocity being denoted by $U = aw$. As for the Couette flow, the gap between the cylinders is denoted by h. Tw_a and Tw_b represent the wall temperatures of the inner and outer cylinders, respectively.

Due to the axial sysmmetry of this problem the tangential velocity u_ϕ , temperature T, pressure p and density ρ will be functions of the radial distance r only.

7.1 The Differential Equations of Motion

The appropriate equations of motion are obtained by expressing equations (3.14) to (3.16) of Part III in plane-polar coordinates. Since the motion is assumed to be steady, and as from the symmetry of figure 7.1 the radial velocity component u_r vanishes everywhere, the restrictions imposed by equation (7.1) apply to the case at hand.

$$u_r = 0, \quad \frac{\partial}{\partial\phi} = 0, \quad \frac{\partial}{\partial t} = 0 \qquad (7.1)$$

The coordinate transformations are performed easily, using the formulae of References (7.1) and (7.2). It is found that the equation of continuity is automatically satisfied in view of equation (7.1). The momentum equations in the tangential and radial directions are given by equations (7.2) and (7.3), respectively. The energy equation (3.16) transforms to equation (7.4) below. The appropriate differential equations of motion are, therefore:

$$\frac{1}{\varrho}\left[\frac{d}{dr}(\mathcal{T}_{r\phi}) + \frac{2}{r}\,\mathcal{T}_{r\phi}\right] = 0 \tag{7.2}$$

$$-\frac{u_\phi^2}{r} + \frac{1}{\varrho}\left[\frac{dp}{dr} + \frac{d}{dr}(\mathcal{T}_{rr}) + \frac{1}{r}(\mathcal{T}_{rr} - \mathcal{T}_{\phi\phi})\right] = 0 \tag{7.3}$$

$$\frac{d}{dr}(q_r) + \frac{1}{r}\,q_r + \mathcal{T}_{r\phi}\left(\frac{d}{dr}u_\phi - \frac{1}{r}u_\phi\right) = 0\,. \tag{7.4}$$

In the above equations, $\mathcal{T}_{r\phi}$ is the viscous shearing stress, $\mathcal{T}_{rr}$ and $\mathcal{T}_{\phi\phi}$ are the normal stresses in the radial and tangential directions, respectively, and q_r is the radial component of the heat flux vector.

The explicit expressions for the required components of the viscous stress tensor and the heat flux vector are obtained by transforming the general expressions (3.17) and (3.18) of Part III, respectively, into plane polar coordinates and making the reductions required by equation (7.1). The results of this transformation are given by equations (7.5) to (7.8):

$$\mathcal{T}_{r\phi} = -\mu\left(\frac{du_\phi}{dr} - \frac{u_\phi}{r}\right) \tag{7.5}$$

$$\begin{aligned}\mathcal{T}_{rr} = \frac{\mu^2}{p}\Big[&\left(\tfrac{1}{12}K_6 + \tfrac{1}{3}K_2\right)\left(\frac{du_\phi}{dr}\right)^2 + \left(\tfrac{1}{3}K_2 - \tfrac{1}{6}K_6\right)\frac{u_\phi}{r}\frac{du_\phi}{dr} \\ &+ \left(\tfrac{1}{12}K_6 - \tfrac{2}{3}K_2\right)\left(\frac{u_\phi}{r}\right)^2 + \tfrac{2}{3}K_4\frac{1}{\varrho T}\frac{dp}{dr}\frac{dT}{dr} + \tfrac{2}{3}K_3 R\frac{d^2T}{dr^2} \\ &+ \tfrac{2}{3}K_5\frac{R}{T}\left(\frac{dT}{dr}\right)^2 - \tfrac{1}{3}K_3 R\frac{1}{r}\frac{dT}{dr}\Big]\end{aligned} \tag{7.6}$$

$$\tau_{\phi\phi} = \frac{\mu^2}{p}\Big[\left(\tfrac{1}{12}K_6 - \tfrac{2}{3}K_2\right)\left(\frac{du_\phi}{dr}\right)^2 + \left(\tfrac{1}{3}K_2 - \tfrac{1}{6}K_6\right)\frac{u_\phi}{r}\frac{du_\phi}{dr}$$

$$+ \left(\tfrac{1}{12}K_6 + \tfrac{1}{3}K_2\right)\left(\frac{u_\phi}{r}\right)^2 + \tfrac{2}{3}K_3 R\,\frac{1}{r}\frac{dT}{dr} - \tfrac{1}{3}K_3 R\,\frac{d^2T}{dr^2}$$

$$- \tfrac{1}{3}K_4\,\frac{1}{\rho T}\,\frac{dp}{dr}\,\frac{dT}{dr} - \tfrac{1}{3}K_5\,\frac{R}{T}\left(\frac{dT}{dr}\right)^2\Big] \tag{7.7}$$

$$q_r = -\lambda\,\frac{dT}{dr}\,. \tag{7.8}$$

When the expressions (7.5) to (7.8) are substituted into the equations of motion (7.2) to (7.4), one obtains a set of three non-linear total differential equations for the unknown tangential velocity, temperature and pressure distributions. The radial density distribution is obtained from the equation of state of the gas.

7.2 Solution of the Differential Equations

As in the case of Couette flow, Part VI, it is found that the above differential equations are linearized effectively by the assumption that the coefficients of viscosity μ , and thermal conduction λ are constants throughout the flow field. The solution obtained below is based on this assumption, whose significance has already been discussed in Section 6.2.

Substitution of equation (7.5) into the tangential momentum equation (7.2) results in a total differential equation for the velocity u_ϕ

$$\frac{d^2u_\phi}{dr^2} + \frac{1}{r}\frac{du_\phi}{dr} - \frac{u_\phi}{r^2} = 0\,. \tag{7.9}$$

The general solution of equation (7.9) is found to be of the form

$$u = Ar + B\,\frac{1}{r}\,, \tag{7.10}$$

where A and B are arbitrary constants.

When the expressions for u_ϕ and q_r are substituted into the energy equation (7.4) from equations (7.8) and (7.10), respectively, one obtains the following total differential equation for the temperature T:

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = -4\frac{\mu}{\lambda}B^2\frac{1}{r^4} \quad . \qquad (7.11)$$

The general solution of equation (7.11) is

$$T = D + C \log r - \frac{\mu}{\lambda} B^2 \frac{1}{r^2} \quad , \qquad (7.12)$$

where C and D are additional constants of integration *.

With the general form of the velocity and temperature distributions known from equations (7.10) and (7.11), respectively, the radial momentum equation (7.3) becomes a complicated total differential equation for the pressure distribution. In order to simplify the application equations (7.6) and (7.7) for the normal stresses, the numerical values of the constants K_2, K_3, K_4, K_5, K_6 as given by equation (6.14) for the Maxwell molecule will be used here.

Using equations (7.6), (7.7), (7.10), and (7.12), one may rewrite the tangential momentum equation (7.3) in the form

$$\frac{1}{p}\frac{dp}{dr}\left\{1 - \frac{\mu^2}{p^2}Y_1\right\} + \left(\frac{\mu}{p}\right)^2\left\{\frac{d}{dr}Y_1 + Y_2\right\} = \frac{1}{RT}\frac{u_\phi^2}{r} \quad , \qquad (7.13)$$

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* The arbitrary constants A, B, C, D above must be distinguished from those of the Couette flow, Section 6.3. The latter cannot be obtained directly from the former by letting the radius of curvature become infinite.

where

$$Y_1 = -(4\ AB + 3\ RC)\frac{1}{r^2} + \left(\frac{4}{3} - 14\frac{R\lambda}{\mu}\right)B^2\frac{1}{r^4} \qquad (7.14)$$

$$Y_2 = -(8\ AB + 6\ RC)\frac{1}{r^3} - 24\frac{R\lambda}{\mu}B^2\frac{1}{r^5}\,. \qquad (7.15)$$

At this point it is advantageous to obtain the solution for the pressure distribution by the method of successive approximation in power series expansion of the "rarefaction parameter" $(\frac{\mu}{p})$. Since the pressure p is itself a function of the radius, it is necessary to define the rarefaction parameter in terms of a fixed pressure, preferably the lowest pressure occurring in the flow field. Due to the nature of the centrifugal pressure field, the lowest pressure p_a will occur at the inner cylinder, that is at $r = a$ (figure 7.1). We therefore define the rarefaction parameter θ as

$$\theta = \frac{\mu}{p_a}\,, \qquad (7.16)$$

so that

$$\frac{\mu}{p} = \theta \cdot \left(\frac{p_a}{p}\right). \qquad (7.17)$$

Since the stresses and the heat flux are known only up to terms of order θ^2, equation (7.13) will likewise be reduced to the same degree of accuracy. Equation (7.13) may thus be written as

$$\frac{1}{p}\frac{dp}{dr} = \frac{1}{RT}\frac{u_\phi^2}{r} - \theta^2\,{}_o\!\left(\frac{p_a}{p}\right)^2\left\{{}_oY_2 + \frac{d}{dr}\,{}_oY_1 - {}_o\!\left(\frac{1}{p}\frac{dp}{dr}\right){}_oY_1\right\}, \qquad (7.18)$$

where the prescript o denotes the zeroth approximation, that is the value of the particular quantity in the gas dynamical case where $\theta \rightarrow 0$.

To obtain the zero order pressure distribution, one sets Θ equal to zero in equation (7.18), whence upon integration

$$\log \left(\frac{{}_0p}{p_a} \right) = \frac{1}{R} \int_{r=a}^{r} \frac{1}{r} \frac{{}_0u_\phi^2}{{}_0T} \, dr. \qquad (7.19)$$

This integral may be evaluated numerically using the zeroth approximations to the velocity and temperature distributions of equations (7.10) and (7.12). The second approximation to the pressure distribution can then be obtained by means of a similar numerical integration of equation (7.18).

It is to be noted that the constant of integration so obtained is the reference pressure p_a which defines the pressure level of the flow. If desired, p_a can be expressed in terms of the total mass of gas contained in the annular space between the cylinders.

The general solution of the differential equations has thus been obtained. In the following sections the slip-flow boundary conditions are used to evaluate the four constants of integration, A, B, C, D.

7.3 The Boundary Conditions

It was seen above that four boundary conditions, in addition to the pressure level, are required for a complete determination of the particular solution for the concentric cylinder gas flow. These four conditions consist of the slip-velocity and temperature-jump relations applied to both the inner and outer cylindrical walls.

It is first necessary to express the general boundary conditions, as given by equations (5.79) and (5.80) of Part V in the cartesian coordinate system of figure 5.3a, in terms of the polar coordinate system of figure 7.1. This transformation is performed with the aid of figure 7.2. Let x and y be the cartesian axes associated with the polar coordinates r and ϕ of figure 7.1. The two auxiliary coordinate systems, x_a - z_a, and x_b - z_b of figure 7.2 represent the coordinate system of figure 5.3a, and equations (5.79) and (5.80), as applied to the inner wall ($r = a$) and outer wall ($r = b$), respectively. The relationship between the various coordinate systems of figure 7.2 is given by equations (7.20) and (7.12):

$$\left.\begin{aligned} x &= a + z_a = r\cos\phi \\ y &= \phantom{a +{}} x_a = r\sin\phi \end{aligned}\right\} \qquad (7.20)$$

$$\left.\begin{aligned} x &= b - z_b = r\cos\phi \\ y &= - x_b = r\sin\phi \ . \end{aligned}\right\} \qquad (7.21)$$

The velocity components u_a, w_a, and u_b, w_b of the auxiliary coordinate system are related to the tangential velocity u_ϕ as shown by figure 7.2 and equations (7.22) and (7.23).

$$u_a = u_\phi \cos\phi \ , \quad w_a = -u_\phi \sin\phi \qquad (7.22)$$

$$u_b = -u_\phi \cos\phi \ , \quad w_b = u_\phi \sin\phi \qquad (7.23)$$

The partial derivatives with respect to the auxiliary coordinates are expressed in terms of the partial derivatives with respect to r and ϕ by means of formulae (7.24) and (7.25), which are easily obtained from equations (7.20) and (7.21), respectively.

$$\left.\begin{aligned} \frac{\partial}{\partial x_a} &= \sin\phi \frac{\partial}{\partial r} + \cos\phi \cdot \frac{1}{r}\frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial z_a} &= \cos\phi \frac{\partial}{\partial r} - \sin\phi \cdot \frac{1}{r}\frac{\partial}{\partial \phi} \end{aligned}\right\} \qquad (7.24)$$

$$\left.\begin{aligned} \frac{\partial}{\partial x_b} &= -\sin\phi \frac{\partial}{\partial r} - \cos\phi \cdot \frac{1}{r}\frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial z_b} &= -\cos\phi \frac{\partial}{\partial r} + \sin\phi \cdot \frac{1}{r}\frac{\partial}{\partial \phi} \end{aligned}\right\} \qquad (7.25)$$

All of the first and second order partial derivatives appearing in equations (5.79) and (5.80) for the boundary conditions can now be transformed into polar coordinates by means of equations (7.22) to (7.25). After all of the differentiations with respect to ϕ have been performed, ϕ is put equal to zero in the resultant expressions, in accordance with figure 7.2. Due to the condition of axial symmetry, equation (7.1), the partial derivatives of u_ϕ, T, p, ρ with respect to ϕ all vanish. The results of this calculation for both the convex (r = a) and concave (r = b) cylindrical surfaces are summarized in Table 7.1. The transformations for the derivative $\frac{D}{Dt}$ are obtained from equations (5.77) and (7.1), using the fact that the velocity of the wall, u_o has the values U and zero, at a and b, respectively.

With the substitutions of Table 7.1 one finally obtains from equations (5.79), (5.80) and (7.17) the four appropriate boundary conditions given by equations (7.26) to (7.29) below. As before, the subscripts a and b denote the evaluation of a particular quantity at r = a, and r = b, respectively:

$$\text{(1) at } r = a\,, \quad (u_\phi)_a = U + \theta a_1 \sqrt{RT_a}\left\{ r\frac{d}{dr}\left(\frac{u_\phi}{r}\right)\right\}_a + \theta^2 X_a \qquad (7.26)$$

$$\text{(2) at } r = b\,, \quad -(u_\phi)_b = 0 + \theta\left(\frac{p_a}{p_b}\right) a_1 \sqrt{RT_b}\left\{ r\frac{d}{dr}\left(\frac{u_\phi}{r}\right)\right\}_b + \theta^2\left(\frac{p_a}{p_b}\right)^2 X_b \qquad (7.27)$$

$$\text{(3) at } r = a\,, \quad T_a = T_{w_a} + \theta c_1 \sqrt{RT_a}\left(\frac{dT}{dr}\right)_a - \theta^2 Z_a \qquad (7.28)$$

$$\text{(4) at } r = b\,, \quad T_b = T_{w_b} - \theta\left(\frac{p_a}{p_b}\right) c_1 \sqrt{RT_b}\left(\frac{dT}{dr}\right)_b - \theta^2\left(\frac{p_a}{p_b}\right)^2 Z_b\,. \qquad (7.29)$$

The quantities X_a, X_b, Z_a, and Z_b are defined by equations (7.30) to (7.33) respectively:

$$X_a = \Big[-\frac{5}{6} R T_a \left(\frac{d}{dr}\left\{ r \frac{d}{dr}\left(\frac{u_\phi}{r} \right) \right\} \right)_a - 5.167\, R \left(\frac{dT}{dr} \right)_a \cdot \left(r \frac{d}{dr}\left\{ \frac{u_\phi}{r} \right\} \right)_a$$

$$- \frac{9}{8} R \left(\frac{dT}{dr} \frac{u_\phi}{r} \right)_a - \frac{8}{15} R T_a \left(\frac{1}{p} \frac{dp}{dr} \cdot r \frac{d}{dr}\left\{ \frac{u_\phi}{r} \right\} \right)_a$$

$$+ \frac{8}{15} R \left(\frac{dT}{dr} \right)_a \cdot \left(r \frac{d}{dr}\left\{ \frac{u_\phi}{r} \right\} \right)_a - \frac{5}{8} R U \left(\frac{1}{r} \frac{dT}{dr} \right)_a + \frac{5}{6} U \cdot \left(u_\phi \frac{d}{dr}\left\{ \frac{u_\phi}{r} \right\} \right)_a \Big] \tag{7.30}$$

$$X_b = \Big[\frac{5}{6} R T_b \left(\frac{d}{dr}\left\{ r \frac{d}{dr}\left(\frac{u_\phi}{r} \right) \right\} \right)_b + 5.167\, R \left(\frac{dT}{dr} \right)_b \cdot \left(r \frac{d}{dr}\left\{ \frac{u_\phi}{r} \right\} \right)_b + \frac{9}{8} \left(\frac{dT}{dr} \frac{u_\phi}{r} \right)_b$$

$$+ \frac{8}{15} R T_b \left(\frac{1}{p} \frac{dp}{dr} \right)_b \cdot \left(r \frac{d}{dr}\left\{ \frac{u_\phi}{r} \right\} \right)_b - \frac{8}{15} R \left(\frac{dT}{dr} \right)_b \cdot \left(r \frac{d}{dr}\left\{ \frac{u_\phi}{r} \right\} \right)_b \Big] \tag{7.31}$$

$$Z_a = T_a \Big[-e_1 \left\{ r \frac{d}{dr}\left(\frac{u_\phi}{r} \right) \right\}_a^2 - \frac{1}{2} \left(u_\phi \frac{d}{dr}\left\{ \frac{u_\phi}{r} \right\} \right)_a - e_6 R \left(\frac{1}{r} \frac{dT}{dr} \right)_a$$

$$- e_8 \frac{R}{T_a} \left(\frac{dT}{dr} \right)_a^2 + \frac{1}{14} R \left(\frac{d^2T}{dr^2} \right)_a - \frac{U^2}{T_a} \left\{ \left(\frac{1}{2} + \frac{18}{7} \frac{u_\phi}{U} \right) \frac{1}{r} \frac{dT}{dr} \right\}_a \Big] \tag{7.32}$$

$$Z_b = T_b \Big[-e_1 \left\{ r \frac{d}{dr}\left(\frac{u_\phi}{r} \right) \right\}_b^2 - \frac{1}{2} \left(u_\phi \frac{d}{dr}\left\{ \frac{u_\phi}{r} \right\} \right)_b - e_6 R \left(\frac{1}{r} \frac{dT}{dr} \right)_b$$

$$- e_8 \frac{R}{T_b} \left(\frac{dT}{dr} \right)_b^2 + \frac{1}{14} R \left(\frac{d^2T}{dr^2} \right)_b \Big]. \tag{7.33}$$

The values of the numerical constants a_1, c_1, e_1, e_6, and e_8 appearing in equations (7.26) to (7.33) are given by Table 5.3 of Part V.

7.4 Calculation of the Constants A, B, C, D.

Before substituting the expressions for the velocity, pressure, and temperature distribution into the derivatives of the boundary-conditions, it is desirable to transform equation (7.12) for the temperature distribution. The arbitrary constants C and D may be expressed in terms of T_a and T_b, which are the gas temperatures at $r = a$ and $r = b$, respectively. The temperature distribution can then be written in the form

$$T = \frac{1}{\log(\frac{b}{a})} \left\{ (T_b + \frac{B^2}{c_p \beta} \cdot \frac{1}{b^2}) \cdot \log(\frac{r}{a}) - (T_a + \frac{B^2}{c_p \beta^2} \cdot \frac{1}{a^2}) \log(\frac{r}{b}) \right\} - \frac{B^2}{c_p \beta} \cdot \frac{1}{r^2} \quad . \tag{7.34}$$

In the above equation, λ and μ have been expressed in terms of the specific heat of constants pressure c_p, and the reciprocal β of the Prandtl number, according to equation (6.52).

As in the case of the Couette flow, it is convenient here to expand the arbitrary constants in terms of the rarefaction parameter θ of equation (7.16). Analogous to equations (6.26), we define:

$$A = {}_oA(1 + \theta\, {}_1A^* + \theta^2\, {}_2A^*) \tag{7.35}$$

$$B = {}_oB(1 + \theta\, {}_1B^* + \theta^2\, {}_2B^*) \tag{7.36}$$

$$\left. \begin{aligned} T_a &= {}_oT_a(1 + \theta\, {}_1T_a^* + \theta^2\, {}_2T_a^*) \\ T_b &= {}_oT_b(1 + \theta\, {}_1T_b^* + \theta^2\, {}_2T_b^*) \quad . \end{aligned} \right\} \tag{7.37}$$

In addition, it is convenient to expand the pressure ratio $(\frac{p_a}{p_b})$ and the temperature gradients at a and b in powers of θ , namely

$$(\frac{p_a}{p_b}) = {}_o(\frac{p_a}{p_b}) \left[1 + \theta\, {}_1(\frac{p_a}{p_b})^* + \theta^2\, {}_2(\frac{p_a}{p_b})^* \right] \quad , \tag{7.38}$$

and

$$\left(\frac{dT}{dr}\right)_a = {}_0\left(\frac{dT}{dr}\right)_a \left[1 + \theta\, {}_1\left(\frac{dT}{dr}\right)_a^* + \theta^2\, {}_2\left(\frac{dT}{dr}\right)_a^*\right] \tag{7.39}$$

The various derivatives occurring in the four boundary conditions are evaluated from equations (7.10) and (7.34) in terms of the expansions (7.35) to (7.39). The resulting expressions are then substituted into the boundary conditions (7.26) to (7.29), which after arrangement of all terms in ascending powers of assume the following form:

(1)

$$\left[U - {}_0A\,a - \frac{{}_0B}{a}\right] + \theta\left[-{}_0A\,{}_1A^* - \frac{{}_0B\,{}_1B^*}{a} - 2a_1\sqrt{R\,{}_0T_a}\,\frac{{}_0B}{a^2}\right]$$

$$+\,\theta^2\left[-{}_0A\,{}_2A^* - \frac{{}_0B\,{}_2B^*}{a} - 2a_1\sqrt{R\,{}_0T_a}\,\frac{{}_0B}{a^2}\left\{\frac{1}{2}\,{}_1T_a^* + {}_1B^*\right\} + {}_0\overline{X}_a\right] = 0$$

(7.40)

(2)

$$\left[{}_0A\,b + \frac{{}_0B}{b}\right] + \theta\left[{}_0A\,{}_1A^* + \frac{{}_0B\,{}_1B^*}{b} - 2a_1\sqrt{R\,{}_0T_b}\,\frac{{}_0B}{b^2}\,{}_0\left(\frac{p_a}{p_b}\right)\right]$$

$$+\,\theta^2\left[{}_0A\,{}_2A^* + \frac{{}_0B\,{}_2B^*}{b} - 2a_1\sqrt{R\,{}_0T_b}\,\frac{{}_0B}{b^2}\,{}_0\left(\frac{p_a}{p_b}\right)\left\{\frac{1}{2}\,{}_1T_b^* + {}_1B^* + {}_1\left(\frac{p_a}{p_b}\right)^*\right\}\right.$$

$$\left. +\,{}_0\left(\frac{p_a}{p_b}\right)^2 {}_0\overline{X}_b\right] = 0 \tag{7.41}$$

(3)

$$\left[{}_0T_a - T_{w_a}\right] + \theta\left[{}_0T_a\,{}_1T_a^* - c_1\sqrt{R\,{}_0T_a}\,{}_0\left(\frac{dT}{dr}\right)_a\right]$$

$$+\,\theta^2\left[{}_2T_a - c_1\sqrt{R\,{}_0T_a}\,{}_0\left(\frac{dT}{dr}\right)_a\left\{\frac{1}{2}\,{}_1T_a^* + {}_1\left(\frac{dT}{dr}\right)_a^*\right\} + {}_0Z_a\right] = 0$$

(7.42)

(4)

$$\left[{}_0T_b - T_{w_b} \right] + \theta \left[{}_0T_b \, {}_2T_b^{*} + c_1 \sqrt{R \, {}_0T_b} \; {}_0\!\left(\frac{dT}{dr} \right)_b \, {}_0\!\left(\frac{p_a}{p_b} \right) \right]$$

$$+ \theta^2 \left[{}_0T_b \, {}_2T_b^{*} + c_1 \sqrt{R \, {}_0T_b} \; {}_0\!\left(\frac{dT}{dr} \right)_b \, {}_0\!\left(\frac{p_a}{p_b} \right) \left\{ \frac{1}{2} \, {}_1T_b^{*} + {}_1\!\left(\frac{dT}{dr} \right)_b^{*} + {}_1\!\left(\frac{p_a}{p_b} \right)^{*} \right\} \right.$$

$$\left. + {}_0\!\left(\frac{p_a}{p_b} \right)^2 {}_0Z_b \right] = 0 \, . \tag{7.43}$$

The quantities ${}_0X_a$, ${}_0X_b$, ${}_0Z_a$, ${}_0Z_b$, appearing in the above equations, are the terms of zero order when the expressions (7.30), (7.31), (7.32), (7.33), respectively, are expanded in powers of θ.

Proceeding as in Section 6.4, one obtains the zero order values of the constants A, B, T_a, T_b, by equating to zero the first brackets of equations (7.40) to (7.43). This results in

$${}_0A = - \frac{\mathcal{U}}{b} \cdot \left(\frac{a}{b} \right) \cdot \frac{1}{1 - \left(\frac{a}{b} \right)^2} \tag{7.44}$$

$${}_0B = \mathcal{U} a \cdot \frac{1}{1 - \left(\frac{a}{b} \right)^2} \tag{7.45}$$

$${}_0T_a = Tw_a ; \quad {}_0T_b = Tw_b \, . \tag{7.46}$$

Due to the algebraic complexity of the first and second order terms of the various constants, it is desirable to introduce dimensionless parameters at this point. The basic geometry (see figure 7.1) of the problem is determined either by the dimensionless "gap" h^*, or the diameter ratio **k**, where by definition

$$h^* = \frac{h}{a} \; ; \quad k = \frac{a}{b} = \frac{1}{1 + h^*} \, . \tag{7.47}$$

The variable radius r is replaced by the dimensionless variable ξ, where

$$\xi = \frac{r - a}{h} \quad ; \quad \left(\frac{r}{a} \right) = 1 + h^{*} \xi \, , \tag{7.48}$$

such that ξ always varies from zero to unity as r varies from a to b, independent of the ratio k.

The surface velocity U is replaced by the Mach number M based on the wall temperature T_{w_a}, namely

$$M = \frac{U}{\sqrt{\gamma R T_{w_a}}} \,. \tag{7.49}$$

The Reynolds number of the flow, Re, is defined again by equation (6.48), as for the Couette flow. Analogous to equation (6.53), the appropriate rarefaction parameter, $\left(\frac{M}{Re}\right)$ is defined by equation (7.50):

$$\theta \, \frac{\sqrt{RT_{w_a}}}{h} = \frac{\mu}{p_a} \cdot \frac{\sqrt{RT_{w_a}}}{h} = \sqrt{\gamma}\left(\frac{M}{Re}\right) . \tag{7.50}$$

It may be shown from equations (7.10), (7.35), (7.36), (7.44) and (7.45), that the dimensionless form of the second approximation to the velocity distribution is given by

$${}_{(2)}\left(\frac{u_\phi}{U}\right) = \left[\frac{1}{1-k^2}\left(1+\theta\,{}_1B^* + \theta^2\,{}_2B^*\right)\frac{1}{1+h^*\xi} - \frac{k^2}{1-k^2}\left(1+\theta\,{}_1A^* + \theta^2\,{}_2A^*\right)\left(1+h^*\xi\right)\right] . \tag{7.51}$$

Similarly, from equation (7.34), one finds for the second approximation to the dimensionless temperature distribution

$$\begin{aligned}
{}_{(2)}\left(\frac{T}{T_{w_a}}\right) = \Big[& \left(1+\theta\,{}_1T_a^* + \theta^2\,{}_2T_a^*\right) \\
& + \left\{\left(\frac{T_{w_b}}{T_{w_a}}-1\right) + \theta\left({}_1T_b^* - {}_1T_a^*\right) + \theta^2\left({}_2T_b^* - {}_2T_a^*\right)\right\} \cdot \frac{\log(1+h^*\xi)}{\log\frac{1}{k}} \\
& + \frac{\gamma-1}{\beta} M^2 \cdot \frac{1}{(1-k^2)^2}\left\{1 + 2\theta\,{}_1B^* + \theta^2\left({}_1B^{*2} + 2\,{}_2B^*\right)\right\} \times \\
& \times \left\{1 - (1-k^2)\frac{\log(1+h^*\xi)}{\log\frac{1}{k}} - \frac{1}{(1+h^*\xi)^2}\right\}\Big] .
\end{aligned} \tag{7.52}$$

The zeroth approximation to the pressure distribution, which is valid in the pressure regime of gas dynamics is obtained from equation (7.19), whence

$$\log {}_{o}\left(\frac{p}{p_a}\right) = \gamma M^2 h^* \int_{\xi=0}^{\xi} \frac{{}_{o}\left(\frac{u_\phi}{U}\right)^2}{(1+h^*\xi)\,{}_{o}\left(\frac{T}{T_{w_a}}\right)}\, d\xi \quad ; \tag{7.53}$$

${}_{o}\left(\frac{u_\phi}{U}\right)$ and ${}_{o}\left(\frac{T}{T_{w_a}}\right)$ are obtained by setting θ equal to zero in equations (7.51) and (7.52), respectively. Since equation (7.18) does not explicitly contain terms of order θ, the first approximation ${}_1\left(\frac{p}{p_a}\right)$ can be obtained from equation (7.53) by replacing the prescript o by (1), ${}_{(1)}\left(\frac{u}{U}\right)$ and ${}_{(1)}\left(\frac{T}{T_{w_a}}\right)$ being evaluated from equations (7.51) and (7.52), respectively, with the terms of order θ^2 ommitted. The quantity ${}_1\left(\frac{p_a}{p_b}\right)^*$ will be required presently. From equation (7.38) one has

$$\theta\,{}_1\left(\frac{p_a}{p_b}\right)^* = \left\{{}_{(1)}\left(\frac{p_a}{p_b}\right)\cdot{}_{o}\left(\frac{p_b}{p_a}\right)\right\} - 1 \quad , \tag{7.54}$$

where ${}_{(1)}\left(\frac{p_a}{p_b}\right)$ and ${}_{o}\left(\frac{p_b}{p_a}\right)$ are obtained from equation (7.53) with the upper limit of integration equal to unity.

The second approximation to the pressure distribution is obtained by integrating equation (7.18). In view of equations (7.14) and (7.15), this results in

$$\log {}_{(2)}\left(\frac{p}{p_a}\right) = \Big[\gamma M^2 h^* \int_{\xi=0}^{\xi} \frac{{}_{(2)}\left(\frac{u_\phi}{U}\right)}{(1+h^*\xi)\,{}_{(2)}\left(\frac{T}{T_{w_a}}\right)}\, d\xi$$

$$- \gamma \left(\frac{M}{Re}\right)^2 \frac{\gamma M^2 {h^*}^3}{(1-K^2)^2} \int_{\xi=0}^{\xi} {}_{o}\left(\frac{p_a}{p}\right)^2 \cdot f(\xi)\, d\xi \Big] \quad , \tag{7.55}$$

where

$$f(\xi) = \left[\frac{\left(32\,\frac{\gamma-1}{\gamma\beta} - \frac{16}{3}\right)}{(1+h^*\xi)^5} - \gamma M^2 \frac{{}_0\!\left(\frac{u_\phi}{U}\right)^2}{(1+h^*\xi)\left(\frac{{}_0T}{T_{w_a}}\right)} \times \right.$$

$$\left. \times \left\{ \left(4K^2 + 3\,\frac{\gamma-1}{\gamma\beta}\,\frac{1-K^2}{\log\frac{1}{K}}\right) \cdot \frac{1}{(1+h^*\xi)^2} + \left(\frac{4}{3} - 14\,\frac{\gamma-1}{\gamma\beta}\right) \cdot \frac{1}{(1+h^*\xi)^4} \right\} \right] .$$

(7.56)

We return now to the calculation of the first and second order components of the four constants A, B, T_a, and T_b, from the four boundary conditions, equations (7.40) to (7.43). Equating to zero the coefficient of θ in each of equations (7.40) to (7.43), one obtains four linear algebraic equations for the determination of ${}_1A^*$, ${}_1B^*$, ${}_1T_a{}^*$, ${}_1T_b{}^*$, in terms of the known zero order coefficients. The results of this calculation are given in dimensionless form by equations (7.57) to (7.60):

$$\theta\,{}_1A^* = -2a_1\sqrt{\gamma}\left(\frac{M}{Re}\right)\frac{h^*}{1-K^2}\left\{1 + K\,{}_0\!\left(\frac{P_a}{P_b}\right)\sqrt{\frac{T_{w_b}}{T_{w_a}}}\right\} \tag{7.57}$$

$$\theta\,{}_1B^* = -2a_1\sqrt{\gamma}\left(\frac{M}{Re}\right)\frac{h^*}{1-K^2}\left\{1 + K^3\,{}_0\!\left(\frac{P_a}{P_b}\right)\sqrt{\frac{T_{w_b}}{T_{w_a}}}\right\} \tag{7.58}$$

$$\theta\,{}_1T_a{}^* = c_1\sqrt{\gamma}\left(\frac{M}{Re}\right)h^*\left[\frac{\left(\frac{T_{w_b}}{T_{w_a}} - 1\right)}{\log\frac{1}{K}} + \frac{2(\gamma-1)}{\beta}M^2\left\{\frac{1 - \frac{1-K^2}{2\log\frac{1}{K}}}{(1-K^2)^2}\right\}\right] \tag{7.59}$$

$$\theta\,{}_1T_b{}^* = -c_1\sqrt{\gamma}\left(\frac{M}{Re}\right)h^*\left[\frac{K}{\log\frac{1}{K}}\left(1 - \frac{T_{w_a}}{T_{w_b}}\right)\right.$$

$$\left. + \frac{2(\gamma-1)}{\beta}M^2\,\frac{T_{w_a}}{T_{w_b}}\,\frac{K^3}{(1-K^2)^2}\left\{1 - \frac{1-K^2}{2K^2\log\frac{1}{K}}\right\}\right] .$$

(7.60)

Having determined the first order constants one similarly obtains expressions for the second order constants ${}_2A^*$, ${}_2B^*$, ${}_2T_a{}^*$, ${}_2T_b{}^*$ by equating to zero the coefficients of θ^2 in the four equations (7.40) to (7.43). The final result of this calculation is given by equations (7.61) to (7.64):

$$\theta^2\,{}_2A^* = \Big[-2a_1\sqrt{\gamma}\left(\frac{M}{Re}\right)\frac{h^*}{1-K^2}\Big\{\frac{1}{2}\,\theta\,{}_1T_a{}^* + \theta\,{}_1B^* + K\sqrt{\frac{T_{w_b}}{T_{w_a}}}\;{}_0\!\left(\frac{P_a}{P_b}\right)\left(\frac{1}{2}\,\theta\,{}_1T_b{}^* + \theta\,{}_1B^* + \theta\,{}_1\!\left(\frac{P_a}{P_b}\right)^*\right)\Big\} + \Big\{\frac{\theta^2}{U}\,{}_0X_a + \frac{1}{K}\,{}_0\!\left(\frac{P_a}{P_b}\right)^2\frac{\theta^2}{U}\,{}_0X_b\Big\}\Big] \tag{7.61}$$

$$\theta^2\,{}_2B^* = \Big[-2a_1\sqrt{\gamma}\left(\frac{M}{Re}\right)\frac{h^*}{1-K^2}\Big\{\frac{1}{2}\,\theta\,{}_1T_a{}^* + \theta\,{}_1B^* + K^3\sqrt{\frac{T_{w_b}}{T_{w_a}}}\;{}_0\!\left(\frac{P_a}{P_b}\right)\left(\frac{1}{2}\,\theta\,{}_1T_b{}^* + \theta\,{}_1B^* + \theta\,{}_1\!\left(\frac{P_a}{P_b}\right)^*\right\} + \Big\{\frac{\theta^2}{U}\,{}_0X_a + K\,{}_0\!\left(\frac{P_a}{P_b}\right)^2\frac{\theta^2}{U}\,{}_0X_b\Big\}\Big] \tag{7.62}$$

$$\theta^2\,{}_2T_a{}^* = \Big[\frac{1}{2}\left(\theta\,{}_1T_a{}^*\right)^2 + c_1\sqrt{\gamma}\left(\frac{M}{Re}\right)\Big\{\frac{\theta h}{T_{w_a}}\,{}_1\!\left(\frac{dT}{dr}\right)_a\Big\} - \frac{\theta^2}{T_{w_a}}\,{}_0Z_a\Big] \tag{7.63}$$

$$\theta^2\,{}_2T_b{}^* = \Big[\frac{1}{2}\left(\theta\,{}_1T_b{}^*\right)^2 + \theta\,{}_1\!\left(\frac{P_a}{P_b}\right)^*\cdot\left(\theta\,{}_1T_b{}^*\right) - c_1\sqrt{\gamma}\left(\frac{M}{Re}\right)\sqrt{\frac{T_{w_b}}{T_{w_a}}}\;{}_0\!\left(\frac{P_a}{P_b}\right)\Big\{\frac{\theta h}{T_{w_b}}\,{}_1\!\left(\frac{dT}{dr}\right)_b\Big\} - {}_0\!\left(\frac{P_a}{P_b}\right)^2\frac{\theta^2}{T_{w_b}}Z_b\Big]. \tag{7.64}$$

The above equations contain several quantities which are of zero or the first order, and for which explicit expressions will now be obtained. The quantities ${}_oX_a$, ${}_oX_b$, ${}_oZ_a$, ${}_oZ_b$ are obtained by evaluating equations (7.30), (7.31), (7.32), (7.33), respectively, using the zero order values for the constants A, B, T_a, T_b, as given by equations (7.44) to (7.46). This results in equations (7.65) to (7.68), below.

$$\frac{\theta^2}{\mathcal{U}}\,{}_o\bar{X}_a = \gamma\left(\frac{M}{\mathcal{R}e}\right)^2 h^{*2}\Bigg[\left(-\frac{3}{5}\gamma M^2 + \frac{10}{3}\right)\frac{1}{1-K^2}$$

$$+\left(\frac{9.268}{1-K^2} - \frac{7}{4}\right)\left\{\frac{\frac{T_{w_b}}{T_{w_a}} - 1}{\log\frac{1}{K}} + 2\,\frac{\gamma-1}{\beta}M^2\cdot\frac{1 - \frac{1-K^2}{2\log\frac{1}{K}}}{(1-K^2)^2}\right\}\Bigg] \tag{7.65}$$

$$\frac{\theta^2}{\mathcal{U}}\,{}_o\bar{X}_b = \gamma\left(\frac{M}{\mathcal{R}e}\right)^2 h^{*2}\,\frac{K^3}{1-K^2}\Bigg[\frac{10}{3}\,\frac{\frac{T_{w_b}}{T_{w_a}} - 1}{\log\frac{1}{K}}$$

$$-\,9.268\left\{\frac{\frac{T_{w_b}}{T_{w_a}} - 1}{\log\frac{1}{K}} + 2\,\frac{\gamma-1}{\beta}M^2\cdot\frac{K^2}{(1-K^2)^2}\left(1 - \frac{1-K^2}{2K^2\log\frac{1}{K}}\right)\right\}\Bigg] \tag{7.66}$$

$$\frac{\theta^2}{T_{w_a}}\,{}_oZ_a = \gamma\left(\frac{M}{\mathcal{R}e}\right)^2 h^{*2}\Bigg[\frac{M^2}{1-K^2}\left\{\gamma - \frac{1}{1-K^2}\left[4\gamma e_1 + \frac{\gamma-1}{\beta}\left(2e_6 + \frac{3}{7} + \frac{43}{7}\gamma M^2\right)\right]\right.$$

$$\left. - \frac{4e_8}{(1-K^2)^3}\left(\frac{\gamma-1}{\beta}\right)^2 M^2\right\} - \Lambda\left\{e_6 + \frac{1}{14} + \left(4e_8\,\frac{\gamma-1}{\beta} + \frac{43}{14}\gamma\right)M^2\right\}$$

$$-\,e_8\,\Lambda^2\Bigg] \tag{7.67}$$

$$\frac{\theta^2}{T_{w_b}}\,{}_0Z_b = \gamma\left(\frac{M}{Re}\right)^2 h^{*^2} K^2\Big[-\frac{K^2}{(1-K^2)^2}\,M^2\Big\{4\gamma e_1 + \left(2e_6 - \frac{3}{7}\,\frac{\gamma-1}{\beta}\right)$$

$$+\,4e_8\,\frac{T_{w_a}}{T_{w_b}}\left(\frac{\gamma-1}{\beta}\right)^2 M^2\,\frac{K^2}{(1-K^2)^2}\Big\}$$

$$-\,\Lambda\left\{e_6 + \frac{1}{14} + 4e_8\,\frac{\gamma-1}{\beta}\,M^2\,\frac{T_{w_a}}{T_{w_b}}\,\frac{K^2}{(1-K^2)^2}\right\} - e_8\,\Lambda^2\Big]\,, \qquad (7.68)$$

where

$$\Lambda = \frac{1}{\log\frac{1}{K}}\left\{\left(\frac{T_{w_b}}{T_{w_a}} - 1\right) - \frac{1}{1-K^2}\,\frac{\gamma-1}{\beta}\,M^2\right\}. \qquad (7.69)$$

The quantity ${}_1\left(\frac{dT}{dr}\right)_a$ appearing in equation (7.63) is defined, in the notation of equation (7.39), as

$${}_1\left(\frac{dT}{dr}\right)_a = {}_0\left(\frac{dT}{dr}\right)_a \cdot {}_1\left(\frac{dT}{dr}\right)_a^* . \qquad (7.70)$$

Hence ${}_1\left(\frac{dT}{dr}\right)_a$, and similarly ${}_1\left(\frac{dT}{dr}\right)_b$ can be found from equation (7.52) by differentiation. This results in the expressions (7.71) and (7.72):

$$\left\{\left(\frac{\theta h}{T_{w_a}}\right){}_1\left(\frac{dT}{dr}\right)_a\right\} = \Big[\frac{h^*}{\log\frac{1}{K}}\left\{\frac{T_{w_b}}{T_{w_a}}\cdot\left(\theta\,{}_1T_b^*\right) - \theta\,{}_1T_a^*\right\}$$

$$+\,4\,\frac{\gamma-1}{\beta}\,M^2\,\frac{h^*}{(1-K^2)^2}\left\{1 - \frac{1-K^2}{2\log\frac{1}{K}}\right\}\cdot\left(\theta\,{}_1B^*\right)\Big] \qquad (7.71)$$

$$\left\{\left(\frac{\theta h}{T_{w_b}}\right){}_1\left(\frac{dT}{dr}\right)_b\right\} = \Big[\frac{h^*K}{\log\frac{1}{K}}\left\{\theta\,{}_1T_b^* - \frac{T_{w_a}}{T_{w_b}}\cdot\left(\theta\,{}_1T_a^*\right)\right.$$

$$\left.+\,4\,\frac{\gamma-1}{\beta}\,M^2\,\frac{h^*K^3}{(1-K^2)^2}\left\{1 - \frac{1-K^2}{2\log\frac{1}{K}}\right\}\cdot\left(\theta\,{}_1B^*\right)\frac{T_{w_a}}{T_{w_b}}\right]. \qquad (7.72)$$

All of the first and second order components of the constants A, B, T_a, and T_b have thus been determined from the boundary conditions. Consequently, it is now possible to calculate the velocity, temperature, and pressure distribution for the concentric cylinder flow from equations (7.51), (7.52) and (7.55), respectively. The details of this method of calculation are illustrated in the next section by means of a specific numerical example.

Due to the algebraic complexity of the formulae given above, it is not pratical to present the results in a closed form for arbitrary values of the four principal parameters, h^*, M, $\frac{Tw_a}{Tw_b}$, and $(\frac{M}{Re})$.

For small values of the dimensionless gap h^*, one may expand all functions of k and h^* occurring in the above equations, in powers of h^*, and by neglecting powers of h^* higher than the first arrive at a solution which is linearized with respect to the effects of curvature. It was found, however, that the application of such a linearized theory results in little, if any, saving of labor relative to the "exact" method presented above. The linearized theory was used, however, to check that for the limiting case of $h^* \rightarrow 0$, the results of this section reduce to the Couette flow solution obtained independently in Part VI.

In concluding this section, the significance of the constant B should be pointed out. From equation (6.46), the friction coefficient for the present case is defined as

$$C_F = \frac{2\,\tau_{r\phi}}{\rho\, U^2} \qquad (7.73)$$

From equations (7.5) and (7.10), the shear stress $\tau_{r\phi}$ is given by

$$\tau_{r\phi} = 2\mu B \cdot \frac{1}{r^2}\,. \qquad (7.74)$$

Using equations (7.36), (7.45), (7.47), (7.73) and (7.74), one obtains for the friction coefficients at the inner and outer walls, C_{F_a} and C_{F_b}, respectively, the expressions

$$C_{F_a} = \frac{4}{\mathcal{R}e} \cdot \frac{1}{K(1+K)} \left[1 + \theta B_1^* + \theta^2 B_2^* \right] \qquad (7.75)$$

$$C_{F_b} = \frac{4}{\mathcal{R}e} \cdot \frac{K}{1+K} \left[1 + \theta B_1^* + \theta^2 B_2^* \right], \qquad (7.76)$$

where $$\mathcal{R}e = \frac{Uh\rho}{\mu} .$$

Since there must be equal torques on both the inner and outer cylinders, $C_{F_b}/C_{F_a} = k^2$, as shown by equations (7.75) and (7.76).

7.5 Numerical Example

The equations of Section 7.4 will now be applied to calculate the velocity, temperature, pressure, and density distribution when the gas is air, and the basic parameters have the following values:

$$\begin{aligned} h^* &= 1.0 \qquad (k = \tfrac{1}{2}) \\ M &= 2.0 \\ \left(\frac{M}{Re}\right) &= .035 \qquad Tw_a = Tw_b = Tw. \end{aligned} \qquad (7.77)$$

The large value of h^* has been chosen, in order to demonstrate the effect of curvature on the characteristics of the flow more clearly.

The values of the physical and numerical constants required for this calculation are obtained from equations (6.56) and Table 5.3, as summarized below:

$$\gamma = 1.40 \qquad a_1 = 1.253 \qquad e_1 = 0.2507$$

$$\beta = \frac{4}{3} \qquad c_1 = 2.377 \qquad e_6 = 1.911 \qquad (7.78)$$

$$e_8 = -5.491$$

The zero order, or gasdynamical velocity distribution is obtained from equation (7.51), with $\theta = 0$, which reduces to

$${}_o\left(\frac{u_\phi}{U}\right) = \frac{4}{3}\,\frac{1}{1+\xi} - \frac{1}{3}(1+\xi) . \qquad (7.79)$$

Similarly one obtains for the zero order temperature distribution, from equation (7.52),

$${}_o\left(\frac{T}{T_W}\right) = \left[1 + 2.133\left\{1 - 1.082\log(1+\xi) - \frac{1}{(1+\xi)^2}\right\}\right] . \qquad (7.80)$$

Equations (7.79) and (7.80) are shown graphically by the solid curves of figure 7.3. The dashed curves are the velocity and temperature distribution of the corresponding gasdynamical Couette flow ($h^* = 0$), as calculated from the equations of Part V. These curves are discussed in the next section.

From equation (7.53), one obtains for the zero order pressure distribution

$$\log {}_o\left(\frac{p}{p_a}\right) = 5.6\int_{\xi=0}^{\xi} {}_o\Psi(\xi)\, d\xi , \qquad (7.81)$$

where the function ${}_o\Psi(\xi)$ is computed from equations (7.79) and (7.80), and is plotted in figure 7.4. The integration of equation (7.81) is performed graphically. The pressure distribution ${}_o\left(\frac{p}{p_a}\right)$, is plotted in figure 7.5. The density distribution, also shown in figure 7.5, is the quotient of $_{o}\left(\frac{p}{p_a}\right)$ and $_{o}\left(\frac{T}{T_w}\right)$. The value of $_{o}\left(\frac{p_a}{p_b}\right)$, which is required later, is obtained from figure 7.5 for $\xi = 1.0$, and is recorded in Table 7.2.

The first order constants are now calculated from equations (7.57) to (7.60), the results being given in Table 7.2. The first order velocity and temperature distributions are obtained, by omitting all terms in θ^2, from equations (7.51) and (7.53). This results in equations (7.82) and (7.83):

$$_{(1)}\left(\frac{u_\phi}{U}\right) = 0.8555 \times \frac{4}{3}\frac{1}{1+\xi} - 0.8374 \times \frac{1}{3}(1+\xi) \tag{7.82}$$

$$\left(\frac{T}{T_w}\right) = \Big[1.1928 - 0.2474 \log (1+\xi)$$

$$+ 0.7110 \times 2.133 \left\{1 - 1.082 \log (1+\xi) - \frac{1}{(1+\xi)^2}\right\}\Big] \tag{7.83}$$

For the calculation of the second order constants one requires the ratio $_{1}\left(\frac{p_a}{p_b}\right)$. This is obtained as follows: From equation (7.53)

$$\log_{(1)}\left(\frac{p_b}{p_a}\right) = 5.6 \int_{\xi=0}^{\xi=1.0} {}_{1}\psi(\xi)\, d\xi \; , \tag{7.84}$$

where

$${}_{1}\psi(\xi) = \frac{_{(1)}\left(\frac{u_\phi}{u}\right)}{(1+\xi)\,_{(1)}\left(\frac{T}{T_w}\right)} \; . \tag{7.85}$$

The function $_{1}\psi(\xi)$ is plotted in figure 7.5. The value of $_{1}\left(\frac{p_a}{p_b}\right)$ as obtained by a graphical integration of equation (7.84) is again listed in Table 7.2.

The various quantities required for the determination of the second approximation to the velocity and temperature distributions are computed from equations (7.61) to (7.72) in the order indicated in Table 7.2. Using the results of Table 7.2, one obtains for the final velocity and temperature distributions from equations (7.51) and (7.52), respectively,

$$ {}_{(2)}\left(\frac{u_\phi}{U}\right) = 0.8812 \times \frac{4}{3}\,\frac{1}{1+\xi} - 0.8592 \times \frac{1}{3}\,(1+\xi) \qquad (7.86) $$

$$ {}_{(2)}\left(\frac{T}{Tw}\right) = \Big[1.0839 - .0812 \log\,(1+\xi) $$

$$ + 0.7833 \times 2.133 \left\{ 1 - 1.082 \log\,(1+\xi) - \frac{1}{(1+\xi)^2} \right\}\Big] . \qquad (7.87) $$

Equations (7.86) and (7.87) are shown graphically by the solid curves of figure 7.5. The dashed curves represent again the equivalent results for the Couette flow ($h^* = 0$) at Mach number 2, and equal wall temperatures.

The pressure distribution corresponding to the velocity and temperature distributions of equations (7.86) and (7.87) is calculated from equation (7.55) which reduces to the form

$$ \log {}_{(2)}\left(\frac{p}{p_a}\right) = 5.6 \int_{\xi=0}^{\xi} {}_2\psi(\xi)\, d\xi \quad . \qquad (7.88) $$

The function ${}_2\psi(\xi)$ is evaluated from equations (7.55) and (7.56), and is plotted in figure 7.5. The final pressure distribution ${}_{(2)}\left(\frac{p}{p_a}\right)$ obtained from equation (7.88) is plotted in figure 7.7, where the corresponding density distribution ${}_{(2)}\left(\frac{\rho}{\rho_a}\right)$ is also shown. Due to the presence of the temperature jump at $r = a$ (see figure 7.6 at $\xi = 0$), the density distribution is calculated from equation (7.89).

$$ {}_{(2)}\left(\frac{\rho}{\rho_a}\right) = {}_{(2)}\left(\frac{p}{p_a}\right) \cdot {}_{(2)}\left(\frac{Tw_a}{T}\right) \cdot {}_{(2)}\left(\frac{T_a}{Tw_a}\right) \quad . \qquad (7.89) $$

The density ρ_a is easily referred to the average density ρ_i, based on the total mass of gas contained in the annular space, by means of the formula

$$\frac{\rho_i}{\rho_a} = \frac{2K^2 h^*}{1-K^2} \int_{\xi=0}^{\xi=1.0} \left(\frac{\rho}{\rho_a}\right)(1+h^*\xi)\, d\xi \quad . \tag{7.90}$$

The values of $\left(\frac{\rho_i}{\rho_a}\right)$, as well as the values of the friction coefficients C_{F_a} and C_{F_b} from equations (7.75) and (7.76) are given in Table 7.2.

The results of the preceding calculation, as represented by figures 7.3, 7.4, 7.6, and 7.7, are discussed in the next section.

7.6 Discussion of the Results

The basic characteristics of the flow of a rarefied gas between concentric cylinders are interpreted most conveniently by considering the deviation from the characteristics of the Couette flow that is produced by the curvature of the streamlines. Since this curvature is the only difference in the two cases treated in Part VI and VII, the discussion of Section 6.8 in regard to the assumption of constant viscosity, the general nature of the differential equations and boundary conditions, and the convergence of the expansion in powers of the rarefaction parameter, $\left(\frac{M}{Re}\right)$, applies to the concentric cylinder flow as well as to the Couette flow.

Consider first the effect of curvature on the gasdynamical velocity profile, as shown by figure 7.3. Since the gas exerts equal torques on both the inner cylinder ($\xi = 0$) and the outer cylinder ($\xi = 1.0$), the shear stress $\tau_{r\phi}$ must be higher at $\xi = 0$ than at $\xi = 1.0$, as shown by equation (7.74). This in turn requires that the slope of the velocity profile be steeper at $\xi = 0$ than at $\xi = 1.0$, as shown by the solid curve of figure (7.3) for $h^* = 1.0$. Evidently, even for the large curvature of $h^* = 1.0$, the gasdynamical velocity profile does not depart radically from that of the Couette flow.

As was explained in Section 6.8, the temperature distribution, for the special case when both wall temperatures are equal, is determined entirely by the viscous dissipation which depends on the square of the velocity gradient. Since the dissipation is greater near the inner, rotating cylinder, than near the outer one, the effect of the curvature tends both to increase the maximum temperature, and to shift the point of maximum temperature towards lower values of ξ , as shown in figure 7.3.

As shown by equation (7.18), the gas-dynamical pressure distribution is determined by the fact that the pressure gradient must balance the centrifugal force experienced by an element of the gas. Hence the pressure gradient must decrease from a maximum at $\xi = 0$, where the tangential velocity u_ϕ and radius of curvature r have their respective maximum and minimum values, to a value of zero at the stationary wall, $\xi = 1.0$. The pressure distribution ${}_0(\frac{p}{p_a})$ of figure 7.4 conforms to the preceding discussion. The density distribution ${}_0(\frac{\rho}{\rho_a})$, also shown in figure 7.4, is an immediate consequence of the shape of the pressure and temperature distributions.

It will be recalled from the discussion of Section 6.8, that the rapidity of convergence of the expansions in powers of the rarefaction parameter$(\frac{M}{Re})$depends, essentially, on the product of $(\frac{M}{Re})$ and the maximum value of the velocity or temperature gradients. A comparison of the slopes of the solid and dashed curves of figure 7.3, leads to the expectation that, for a fixed Mach number M, and a given value of $(\frac{M}{Re})$, the rapidity of convergence of the solutions decreases with increasing values of the curvature h^*. This expectation was verified by a preliminary calculation, which showed that the temperature solution converges more slowly than the corresponding velocity distribution.

In accordance with the above, the value of $(\frac{M}{Re}) = .035$ was chosen for the numerical example of the preceding section, as this value corresponds, for $h^* = 1.0$, to a rapidity of convergence which is of the same magnitude as that of the Couette flow for $(\frac{M}{Re}) = .0675$, used in the computation of figure 6.5.

Figure 7.6 shows the velocity and temperature distributions, as calculated from the second order theory, for both Couette flow and concentric cylinder flow of $h^* = 1.0$, for equal wall temperatures, Mach number 2.0, and $(\frac{M}{Re})$ of .035. The various velocity and temperature discontinuities at the stationary and moving walls shown in figure 7.6 are seen to be proportional, to a first approximation, to the corresponding slopes of the gasdynamical velocity and temperature profiles of figure 7.3. The slip velocity and temperature jump at $\xi = 0$ are thus considerably larger than those at $= 1.0$. In figure 7.3, the slopes of the temperature profiles at $\xi = 1.0$ are approximately the same for $h^* = 1.0$ and $h^* = 0$. In figure 7.6, however, the temperature jump at $\xi = 1.0$ for $h^* = 1.0$ is seen to be less than that for the Couette flow. This is caused by the fact that the second order term of the temperature jump increases negatively with increasing values of h^*.

The pressure and density distributions as calculated from the second order theory are shown in figure 7.7. The shape of the curves of figure 7.7 is similar to those for the gasdynamic case of figure 7.4. It is seen that the rarefaction of the gas decreases the pressure gradient near the inner cylinder $(\xi = 0)$ and the pressure difference across the curved channel. This is, of course, due to the slippage of the gas over the inner rotating cylinder, thus resulting in a reduction in the centrifugal force.

The curves of $_{(2)}(\frac{p}{p_a})$ and $_{(2)}(\frac{\rho}{\rho_a})$ are seen to cross in figure 7.7, because the gas temperatures at $\xi = 0$ and $\xi = 1.0$ are different due to the unequal temperature jumps shown in figure 7.6.

From equations (7.75) and (7.76) and the results of Table 7.2, one finds that the slippage of the gas over the walls reduces the friction coefficients C_{F_a} and C_{F_b} by 11.9 per cent of their value in the gas-dynamic regime. The corresponding reduction for the Couette flow, at $M = 2$ and $(\frac{M}{Re}) = .035$, is found to be 8.7 per cent from figure 6.2. The similarity of these results, even for the large curvature of $h^* = 1.0$ emphasizes the conclusion to be drawn from the above discussion, namely that the Couette flow solution of Part VI will represent a very satisfactory approximation to the concentric cylinder flow with small relative curvature of $h^* = 0.1$ or less, such as would be used in the practical design of an apparatus for the experimental investigation of the slip phenomena. In such an approximation, the pressure distribution may be calculated from equation (7.53), using for ${}_{(2)}(\frac{u_\phi}{U})$ and ${}_{(2)}(\frac{T}{T_{w_a}})$ the second approximation to the velocity and temperature profiles of the Couette flow.

PART VIII

GENERAL CONCLUSIONS

Using the results obtained from the solution of the three specific problems treated in this thesis as a basis for discussion, one may draw certain conclusions concerning the nature of the slip flow realm of fluid mechanics.

The most important physical parameter which characterizes a particular slip flow is the magnitude of the gradient of the macroscopic velocity per mean free path of the gas. This parameter, whose dimensionless equivalent is $(M \frac{\ell}{L})$ in the notation previously employed, measures the degree of non-uniformity of the gas, that is the deviation of its molecular velocity distribution from the Maxwell distribution of a uniform gas. The mean free path ℓ is the only distinguished linear microscopic property of the gas, so that the macroscopic gradients must properly be referred to the former. The characteristic macroscopic length L must always be that linear dimension which determines the order of magnitude of the velocity gradients in a particular case. Thus, for example, the "gap" h was seen to be the characteristic length for the Couette and cylinder flows, whereas the slip flow effects in boundary layer phenomena are determined by the relation of the boundary layer thickness δ to the mean free path of the gas.

In Part IV it was shown that for the propagation of a sound wave through a gas the characteristic length L, which determines the velocity, temperature, and pressure gradients in the gas, is the wave length of sound. The results of that investigation are therefore equally applicable to the propagation of a sound wave of normal wave length through a rarefied gas, and to the propagation of an ultrasonic wave through a gas of normal atmospheric density.

The analysis of Part IV applies only to the case of a plane wave of infinitesimal strength. Another related case, that of a plane shock-wave has already been treated by L. H. Thomas (Reference 9.1). In a shock-wave the magnitude of the velocity, pressure, and temperature gradients is determined by the thickness of the shock-wave. Thomas found that the second order terms of the viscous stresses and heat flux, Part III, tend to increase the value of the shock-wave thickness that is calculated by use of the first order stresses and heat flux alone, so that the shock-wave thickness is always of the order of several mean free paths, even for strong shocks.

The various examples mentioned above emphasize the fact that the concepts characteristic of the slip flow regime are by no means restricted to rarefied gases.

Although the slip flow parameter ℓ/L is most useful for purposes of visualization of the phenomena involved, the equivalent parameter M/Re is more convenient for the application of the slip flow theory to aerodynamic calculations. It is interesting to note here the dual role played by these two basic parameters of fluid mechanics, the Mach and Reynold's numbers. The Mach and Reynolds numbers, as individual parameters, account for the effects of the compressibility and viscosity, respectively, of the gaseous medium which determine the basic gasdynamical flow pattern. The ratio of Mach to Reynolds number, on the other hand, determines the order of magnitude of the deviation of the slip flow pattern from the basic gasdynamical flow caused by the additional viscous stresses and heat flux, and by the slip velocity and temperature jump at the boundaries of the gas.

The numerical calculations of Parts IV, VI, and VII indicate what appears to be a general trend regarding the nature of the successive approximation procedure of the slip flow theory. For the cases considered, it appears that the second approximations reduce the slip flow effects predicted by a first order theory resulting in a flow pattern intermediate to that of gas-dynamics and the first order slip flow theory. From the nature of the curves of Part VI, it is to be expected that, if third and higher order approximations were calculated, the terms of the power series solution in $^M/Re$ might have alternating signs and that this series might converge very slowly, especially for high Mach numbers. In fact, the convergence of such series has not been proven.

In view of the above, it is believed that the extremely laborious preliminary calculation of third and higher approximation to the molecular velocity distribution function is not worth while. Furthermore, it must be remembered that all such calculations are necessarily based on a number of simplifying assumptions which are not in accordance with all of the physical facts. The most important such simplification is that the kinetic theory calculations are based on classical particle mechanics, so that the wave mechanical interactions among the gas molecules, as well as between gas molecules and the molecules composing the solid bounding surfaces are not considered. In addition, the kinetic theory merely accounts for the translational motion of the molecules, so that the rotational and vibrational degrees of freedom of molecules of poly-atomic gases are but inadequately accounted for by adjusting the value of the ratio of specific heats. The errors entailed in these approximations to the physical facts become of increasing significance for the higher order approximations.

In validity of the first order slip flow theory has been tested experimentally by many observers. These experiments actually verify only the relatively simple functional relationships between the first approximation to the slip velocity, the rarefaction parameter $\frac{l}{L}$, and the velocity gradient, since the constant of proportionality involves the molecular fraction σ which is determined by the experiment. The analogous situation exists with respect to the temperature jump, where the accomodation coefficient α is determined experimentally.

An extension of the experiments of R. A. Millikan and his collaborators on the concentric cylinder flow into the realm of higher rotative speeds, would be valuable in order to test, at least qualitatively the results predicted by the second order slip flow theory. For this purpose, the solutions of the preceding parts for the Couette and concentric cylinder flows should be modified to include the effect of the variation of the coefficients of viscosity and thermal conduction with temperature. A close numerical agreement between the results of such a modified theory and experimental tests should, perhaps, not be expected since certain numerical constants occurring in the second order slip flow theory depend on the choice of the molecular model of the kinetic theory; furthermore, the second order slip flow theory does not contain any constants, in addition to σ and α, to be determined from the experiment.

In conclusion, the slip flow regime will be considered briefly relative to some of the other realms of fluid mechanics. For this purpose consider only the viscous drag or friction coefficient, characteristic of a particular geometry. The friction coefficient may be expressed as the product of the friction coefficient for the gas dynamic realm and a factor A^* which is a function of the basic parameter $\frac{\ell}{L}$, as in section 6.5. The value of A^* depends on the particular realm of fluid mechanics. Figure 9.1 shows how A^* decreases from a value of unity in the gas dynamic realm, approaching zero in the limit of infinite mean free path in the free molecule flow regime, for the special case of the motion of a sphere at very low speeds. Figure 9.1 has been plotted from the experimental results of R. A. Millikan (Reference 9.2) on the free fall of small oildrops through rarefied air. The empirical relation given in figure 9.1, representing the extension of Stokes' law of gas dynamics to the other realms of fluid mechanics, agrees with the theoretical calculation of P. S. Epstein for the limiting cases $\frac{\ell}{L} \ll 1$ and $\frac{\ell}{L} \gg 1$. For the case of the sphere, the characteristic length L is, of course, its radius.

For comparison the results of the solution for the low Mach number Couette flow in the slip flow regime are also shown in figure 9.1, L being now the plate spacing h. It is seen how the second approximation for A^*, as obtained from figure 6.2, reduces the too rapid decrease of A^* with increasing $\frac{\ell}{L}$ of the first order solution of the Couette flow. It is expected that the general shape of the A^* versus $\frac{\ell}{L}$ curve for the Couette flow, and for other cases as well, should resemble that for the sphere with a point of inflection somewhere in the "intermediate" regime.

APPENDIX I

LIST OF SYMBOLS AND NOTATIONS

The numerous symbols used throughout the text and the successive Appendices are summarized below. The symbols are arranged in two groups, the first consisting of the Roman and the second of the Greek letters. Each group is ordered alphaletically.

The Roman numerals in brackets indicate the parts of the text or appendix in which a particular symbol is used.

1. Roman Symbols

a	adiabatic speed of sound (II, III, VI)
a	radius of inner cylinder (VII)
a	constant (II)
a_1	numerical constant in boundary conditions (V, VI, VII)
$a_n^{(i)}$	coefficient, function of x,y,z,t (II)
a_n	coefficient in Sonine polynomial expansion (A II)
A	arbitrary constant of integration (VI, VII)
A	numerical constant (IV)
A_1, A_2, A_3	arbitrary constants (IV)
A_n	basic Burnett coefficient (II, III, V)
A_{ij}	tensor component (III)
a_i	even component of boundary condition integral (V)
b	damping coefficient (IV)
b	radius of outer cylinder (VII)
b	characteristic distance of molecular orbit (A III)
b_1, b_2	numerical constant in boundary conditions (V)
$b^{(i)}$	coefficient, function of x,y,z,t, (II)

B	ratio of terms in stress tensor (III)
B	numerical constant (IV)
B	arbitrary constant of integration (VI, VII)
B_{kr}	basic Burnett coefficient (II, III, V)
$B_{kr}^{(\rho)}$	associated Burnett coefficient (II, III, V)
$\mathcal{B}_i$	odd component of boundary condition integral (V)

c	propagation speed of sound wave (IV)
c_o	adiabatic speed of sound in perfect fluid (IV)
c_v	specific heat at constant volume
c_p	specific heat at constant pressure
c_1	numerical constant in boundary condition (V, VI, VII)
C	peculiar velocity of molecule (II, V)
C	numerical constant (III)
C	arbitrary constant of integration (VI, VII)
$C_{kr}^{(\rho)}$	associated Burnett coefficient (II, III, V)
$C_n^{(j)}$	coefficient, function of x,y,z,t (II)

D	numerical constant (IV)
D	arbitrary constant of integration (VI, VII)
$D_{kr}^{(\rho)}$	combination of Burnett coefficients (II)
$\frac{D}{Dt}$	convective time derivative of hydrodynamics (II, III, V, VII)
$\frac{\mathcal{D}}{\mathcal{D}t}$	molecular convective time derivative of Boltzmann's equation (II)

e	base of natural logarithm
$e_1, e_2, \ldots, e_{10}$	numerical constants in boundary conditions (V, VI, VII)
e_{ij}	rate of strain tensor (III)
E	numerical constant (IV)

E	arbitrary constant of integration (VI)
E_i	total incident energy (V)
E_r	total reemitted energy (V)
E_w	hypothetical wall energy (V)

f	non-uniform velocity distribution (II, V)
f_o	Maxwell distribution (II, V)
f'	distribution function of reemitted molecules (V)
f''	hypothetical distribution function corresponding to wall energy (V)
F	function of U,V,W (II)
F_{n-1}	known function (II)
F, F_o	functions of collision variable (A III)
F_x, F_y, F_z	components of external body force (II, A III)

g	acceleration of gravity (A III)
g	molecular collision parameter (II)
G_{12}	molecular collision parameter (II)

h	reciprocal of absolute temperature, $h = \frac{1}{2kT}$ (II, III, V)
h	plate or cylinder spacing (gap) (VI, VII)
h^*	dimensionless gap (VII)
h'	Planck's constant (V)
H	altitude (VI)
$H_n^{\prime(j)}$	specified function of x,y,z,t (II)
$H^{(j)}$	specified function of x,y,z,t (II)

i	summation index
i	square root of minus one (IV)
$I_i^{(1)}$, $I_i^{(2)}$	definite integrals (V)

j	summation index
J	Maxwell's definite integral (A III)
J_j	boundary condition integrals (V)
k	summation index
k	Boltzmann constant (II, III, V)
k	diameter ratio (VII)
k	modulus of complete elliptic integral (A III)
K	constant of proportionality of force law (II)
K	complete elliptic integral of first kind (A III)
K	symmetrical kernel of integral equation (II)
K_1 $K_2, \ldots, K_6$	numerical constants of stress tensor (III, IV, VI, VII)
L	characteristic length
L	wave length of sound wave (IV)
L_4	definite integral (A III)
$L_j^{(1)}, L_j^{(2)}$	boundary condition integrals (V)
ℓ	mean free path of gas
ℓ	summation index
m	mass of molecule (II, III, V)
m	summation index
M	Mach number
M_i	molecular functions (V)
$M_{\substack{k\,k_1k_2\\ r\,n_1n_2}}$	numerical constant (II)
n	summation index
n	exponent in molecular force law (II)

N	total number of molecules (II)
N	order of approximation
p	static pressure of gas
P_k	Legendre polynomial (II, V)
$P_k^{(\rho)}$	associated Legendre polynomial (II, V)
ρ	summation index
q	summation index
q_i	component of heat flux vector
Q	molecular property (II)
Q_{kr}	molecular property (II)
Q_j	molecular property (V)
r	summation index
r	molecular separation (II)
r	radial coordinate (VII)
r	radius vector for relative orbit (A III)
r_a	amplitude ratio (IV)
R	gas constant (III, V, VI, VII)
R	Reynold's number of sound propagation (IV)
R_x	Reynold's number based on chor length (VI)
Re	Reynolds number based on "gap" (VI, VII)
s	exponent in viscosity-temperature relation
s	summation index
s	collision parameter (A III)
$S_m^{(n)}$	Sonine polynomial (II, III, V)

t	time
t	variable of integration (A II, A III)
T	absolute temperature of gas
T_w	absolute temperature of wall

u	x-component of macroscopic velocity
u'	x-component of macroscopic perturbation velocity (IV)
u_i	component of macroscopic velocity vector
u_r	radial component of macroscopic velocity (VII)
u_ϕ	tangential component of macroscopic velocity (VII)
u_o	absolute wall velocity in x-direction (V)
U	x-component of peculiar velocity of molecule (II, III, V)
U	characteristic macroscopic velocity
U	absolute tangential velocity of moving wall (VI, VII)

v	y-component of macroscopic velocity
v_o	absolute wall velocity in y-direction (V)
V	y-component of peculiar velocity of molecule (II, III, V)

w	z-component of macroscopic velocity
W	z-component of peculiar velocity of molecule (II, III, V)

x	cartesian coordinate of physical space
x	direction of propagation of plane wave (IV)
x	real variable (A II)
X_a, X_b	sums of boundary condition derivatives (VII)

y	cartesian coordinate of physical space
Y_{kn}	generalized spherical harmonic (II, III, V)
Y_k, $Y_k^{(\rho)}$	spherical harmonics (II, III, V)
z	cartesian coordinate of physical space
$Z_k^{(\rho)}$	spherical harmonic (II, III, V)
Z_a, Z_b	sums of boundary condition derivatives (VII)

2. Greek Symbols

α	dimensionless speed of sound propagation (IV)
α	accomodation coefficient (V, VI, VII)
α	dimensionless collision parameter (A III)
α	ratio of gravitational to dynamic forces (A III)
β	reciprocal of Prandtl number (IV, VI, VII)
γ	ratio of specific heats
Γ	gamma function
δ	boundary layer thickness (VI)
ε	numerical constant (III)
ε	arbitrarily small positive number (A II)
ζ	z-component of absolute molecular velocity
η	y-component of absolute molecular velocity (II, V)
η	dimensionless y-coordinate (VI)

θ	polar angle of peculiar velocity of molecule in spherical polar coordinates (II, V)
θ	angular coordinate of relative molecular orbit (A III)
θ	rarefaction parameter, $\theta = \frac{\mu}{p}$ (VI, VII)
$\theta_1, \theta_2, \ldots, \theta_5$	numerical constants in heat flux vector (III, IV)
λ	coefficient of thermal conduction (III, IV, VI, VIII)
λ	arbitrary parameter (II)
λ'	de Broglie wave length of molecular beam (V)
Λ''	constant of Maxwell distribution function f'' (V)
μ	coefficient of absolute viscosity
μ	argument of Legendre polynomials (A II, A IV)
ν	number of molecules per unit volume (II, III, V)
ν	frequency of sound wave in cycles per second (IV)
ξ	x-component of absolute velocity of molecule (II, V)
ξ	dimensionless radial coordinate (VII)
ρ	mass density of gas
σ	fraction of molecules diffusely reemitted
τ	shear stress (VI, VII)
τ_{ij}	component of viscous stress tensor
ϕ	azimuthal angle of peculiar velocity of molecule in spherical polar coordinates (II, V)

ϕ	angular coordinate in plane polar coordinates (VII)
$\phi^{(i)}$	orthogonal eigen-functions (II)
Φ_{n-1}	known function (II)
χ	deflection angle of molecular orbit (A III)
ψ	apse line angle of relative molecular orbit (A III)
ψ_n	Hilbert expansion functions (II)
$\psi_j^{(j)}$	eigen-functions of homogeneous integral equation (II)
$\psi_n^{(0)}$	particular solutions of integral equation (II)
${}_0\psi,\ {}_1\psi,\ {}_2\psi$	functions for determining pressure destribution by numerical integration (VII)
ω	angular velocity (VII)
Ω	positive constant (A II)

APPENDIX II

SUMMARY OF MATHEMATICAL FORMULAE

1. Values of the Definite Integral $\int_0^\infty x^n e^{-\Omega x^2} dx$

Ω is a positive constant

$n = 0, 1, 2, 3, \ldots$

Table A.1

n	$\int_0^\infty x^n e^{-\Omega x^2} dx$
0	$\frac{1}{2}\sqrt{\frac{\pi}{\Omega}}$
1	$\frac{1}{2}\cdot\frac{1}{\Omega}$
2	$\frac{1}{4}\sqrt{\frac{\pi}{\Omega^3}}$
3	$\frac{1}{2}\cdot\frac{1}{\Omega^2}$
4	$\frac{3}{8}\sqrt{\frac{\pi}{\Omega^5}}$
5	$\frac{1}{\Omega^3}$
6	$\frac{15}{16}\sqrt{\frac{\pi}{\Omega^7}}$

2. The Solid Spherical Harmonics

The general spherical harmonic function is defined by

$$Y_{kn}(c,\theta,\phi) = B_{kn} Y_k(c,\theta,\phi)$$

$$+ 2\sum_{p=1}^{k} \frac{(k-p)!}{(k+p)!}\left\{ B_{kn}^{(p)} Y_k^{(p)}(c,\theta,\phi) + C_{kn}^{(p)} Z_k^{(p)}(c,\theta,\phi)\right\}. \quad (A.1)$$

B_{kn}, $B_{kn}^{(p)}$, $C_{kn}^{(p)}$ are constants. B_{kn} will be called a "basic Burnett coefficient". B_{kn}, C_{kn} are called the "associated Burnett coefficients". The indices k,n,p can take on the values:

$$\left.\begin{array}{l} k = 1,2,3,\ldots \\ n = 0,1,2,3,\ldots \\ p = 1,2,3,\ldots,k \end{array}\right\} \qquad (A.2)$$

The (2k+1) linearly independent harmonic functions of degree k of which Y_{kn} is composed are defined by equations (A.3) to (A.5):

$$Y_K = C^K P_K(\cos\theta) \qquad (A.3)$$

$$Y_K^{(\rho)} = (-1)^{\rho} C^K P_K^{(\rho)}(\cos\theta)\cdot\cos(\rho\phi) \qquad (A.4)$$

$$Z_K^{(\rho)} = (-1)^{\rho} C^K P_K(\cos\theta)\cdot\sin(\rho\phi)\ . \qquad (A.5)$$

The Legendre polynomials $P_k(\cos\theta)$, and the associated Legendre polynomials $P_k^{(\rho)}(\cos\theta)$ are defined respectively in sections 3 and 4 below.

The individual harmonic functions up to degree 4 are listed below in terms of the cartesian velocity components of equation (2.36):

for k = 1: $$Y_1 = W\ ,\quad Y_1^{(1)} = U\ ,\quad Z_1^{(1)} = V\ . \qquad (A.6)$$

for k = 2: $$Y_2 = \tfrac{1}{2}(3W^2 - C^2)$$

$$Y_2^{(1)} = 3WU\ ,\quad Z_2^{(1)} = 3WV$$

$$Y_2^{(2)} = 3(U^2 - V^2)\ ,\quad Z_2^{(2)} = 6UV\ . \qquad (A.7)$$

for k = 3: $$Y_3 = \tfrac{1}{2}W(5W^2 - 3C^2)$$

$$Y_3^{(1)} = \tfrac{3}{2}U(5W^2 - C^2)\ ,\quad Z_3^{(1)} = \tfrac{3}{2}V(5W^2 - C^2)$$

$$Y_3^{(2)} = 15W(U^2 - V^2)\ ,\quad Z_3^{(2)} = 30\,UVW$$

$$Y_3^{(3)} = 15U(U^2 - 3V^2)\ ,\quad Z_3^{(3)} = 15V(3U^2 - V^2)\ . \qquad (A.8)$$

for k = 4: $Y_4 = \frac{1}{8}(35W^4 - 30C^2W^2 + 3C^4)$

$$Y_4^{(1)} = \frac{5}{2}UW(7W^2 - 3C^2)\ ; \qquad Z_4^{(1)} = \frac{5}{2}VW(7W^2 - 3C^2)$$

$$Y_4^{(2)} = \frac{15}{2}(U^2 - V^2)(7W^2 - C^2)\ ; \qquad Z_4^{(2)} = 15UV(7W^2 - C^2)$$

$$Y_4^{(3)} = -105UW(3V^2 - U^2)\ ; \qquad Z_4^{(3)} = -105VW(V^2 - 3U^2)$$

$$Y_4^{(4)} = 105\{8U^2(U^2 - C^2) + (C^2 - W^2)^2\}\ ; \quad Z_4^{(4)} = 420UV(U^2 - V^2) \tag{A.9}$$

By substituting the equations (A.6) to (A.9) into the general expression (A.1), the special formulae (A.10) to (A.13) are obtained.

$$Y_{1n} = [B_{1n} \cdot W + B_{1n}^{(1)} U + C_{1n}^{(1)} V] \tag{A.10}$$

$$Y_{2n} = \Big[B_{2n} \cdot \tfrac{1}{2}(3W^2 - C^2) + B_{2n}^{(1)} WU + C_{2n}^{(1)} WV + B_{2n}^{(2)} \cdot \tfrac{1}{4}(U^2 - V^2) + C_{2n}^{(2)} \cdot \tfrac{1}{2} UV\Big] \tag{A.11}$$

$$\begin{aligned} Y_{3n} = \Big[& B_{3n} \cdot \tfrac{1}{2} W(5W^2 - 3C^2) + B_{3n}^{(1)} \cdot \tfrac{1}{4} U(5W^2 - C^2) \\ & + C_{3n}^{(1)} \cdot \tfrac{1}{4} V(5W^2 - C^2) + B_{3n}^{(2)} \cdot \tfrac{1}{4} W(U^2 - V^2) + C_{3n}^{(2)} \cdot \tfrac{1}{2} UVW \\ & + B_{3n}^{(3)} \cdot \tfrac{1}{24} U(U^2 - 3V^2) + C_{3n}^{(3)} \cdot \tfrac{1}{24} V(3U^2 - V^2)\ \Big] \end{aligned} \tag{A.12}$$

$$\begin{aligned} Y_{4n} = \Big[& B_{4n} \cdot \tfrac{1}{8}(35W^4 - 30C^2W^2 + 3C^4) + B_{4n}^{(1)} \cdot \tfrac{1}{4}(7W^2 - 3C^2) UW \\ & + C_{4n}^{(1)} \cdot \tfrac{1}{4}(7W^2 - 3C^2) VW + B_{4n}^{(2)} \cdot \tfrac{1}{24}(U^2 - V^2)(7W^2 - C^2) \\ & + C_{4n}^{(2)} \cdot \tfrac{1}{12} UV(7W^2 - C^2) + B_{4n}^{(3)} \cdot \tfrac{1}{24} UW(U^2 - 3V^2) \\ & + C_{4n}^{(3)} \cdot \tfrac{1}{24} VW(3U^2 - V^2) + B_{4n}^{(4)} \cdot \tfrac{1}{192}\{U^2(U^2 - 8W^2) + V^2(V^2 - 6U^2)\} \\ & + C_{4n}^{(4)} \cdot \tfrac{1}{48} UV(U^2 - V^2)\Big]. \end{aligned} \tag{A.13}$$

3. Properties of the Legendre Polynomials $P_k(\mu)$

The argument of the Legendre polynomials is the real variable which is restricted by $-1 \leqq \mu \leqq +1$. In the present application, the argument occurs as $\mu = \cos\theta$, see equation (A.3). The following well known definite integrals are useful in the integrations performed in the evaluation of the boundary conditions (Part V ; see also Appendix IV).

The indices k,q,m,n are all positive integers or zero.

$$\int_0^1 P_{2q}(\mu)\, d\mu = 0 \qquad \text{for } q > 0 \tag{A.14}$$

$$\int_0^1 P_{2q+1}(\mu)\, d\mu = (-1)^q \frac{(2q)!}{2^{(2q+1)} \cdot q! \cdot (q+1)!} \tag{A.15}$$

$$\int_{-1}^1 \mu^k P_n(\mu)\, d\mu = 0 \qquad \text{for } k = 0,1,2,\ldots,n-1. \tag{A.16}$$

$$\int_{-1}^1 \mu^n P_n(\mu)\, d\mu = \frac{2^{n+1} \cdot (n!)^2}{(2n+1)!} \tag{A.17}$$

$$\int_0^1 [P_n(\mu)]^2 d\mu = \frac{1}{2n+1} \tag{A.18}$$

$$\int_0^1 P_m(\mu) P_n(\mu)\, d\mu = \frac{m\, P_n(0)\, P_{m-1}(0) - n\, P_m(0)\, P_{n-1}(0)}{(m-n)\ (m+n+1)}, \ (m \neq n) \tag{A.19}$$

$$\int_0^1 P_m(\mu) P_n(\mu)\, d\mu = 0,$$

if m = n, and m and n are either both even, or both odd.

$$\int_0^1 P_{2k}(\mu)\, P_{2q+1}(\mu)\, d\mu = (-1)^{k+q+1} \frac{(2k)!\ (2q+1)!}{4^{k+q}\ (2k-2q-1)(2+2k+2q)(k!)^2(q!)^2} \tag{A.20}$$

$$\int_0^1 P_{2q+1}(\mu)\,d\mu = (-1)^q \frac{(2q)!}{2^{2q+1}\,(q!)^2\,(q+1)} \quad . \tag{A.21}$$

A few of the Legendre polynomials, and special values which occur frequently are listed below.

$$\left.\begin{aligned} P_0(\mu) &= 1 \\ P_1(\mu) &= \mu \\ P_2(\mu) &= \tfrac{1}{2}(3\mu^2-1) \\ P_3(\mu) &= \tfrac{1}{2}(5\mu^3-3\mu) \\ P_4(\mu) &= \tfrac{1}{8}(35\mu^4-30\mu^2+3) \\ P_5(\mu) &= \tfrac{1}{8}(63\mu^5-70\mu^3+15\mu) \\ P_6(\mu) &= \tfrac{1}{16}(231\mu^6-315\mu^4+105\mu^2-5) \end{aligned}\right\} \tag{A.22}$$

$$P_{2q+1}(0) = 0 \qquad \text{for } k = 2q+1 = \text{odd} \tag{A.23}$$

$$\begin{aligned} P_0(0) &= 1 & P_6(0) &= -\tfrac{5}{16} \\ P_2(0) &= -\tfrac{1}{2} & P_8(0) &= \tfrac{35}{128} \\ P_4(0) &= \tfrac{3}{8} \end{aligned} \tag{A.24}$$

$$P_k(1) = 1 \qquad \text{for all } k \tag{A.25}$$

4. The Associated Legendre Polynomials $P_k^{(\rho)}(\mu)$

The real variable μ is restricted as in section 3. The index p can assume the values p = 1,2,...,k. The associated Legendre polynomials, $P_k^{(\rho)}(\mu)$ are derivable from the polynomials $P_k(\mu)$ by successive differentiation.

$$P_K^{(\rho)}(\mu) = (-1)^{\rho} (1-\mu^2)^{\frac{1}{2}\rho} \frac{d^{\rho}}{d\mu^{\rho}} \left\{ P_K(\mu) \right\} \tag{A.26}$$

The formulae (A.27) to (A.29) are direct consequences of the definition (A.26).

$$P_K^{(\rho)}(\mu) = \frac{(2K)!\,(-1)^{\rho}}{2^K \cdot K! \cdot (K-\rho)} (1-\mu^2)^{\frac{1}{2}\rho} \left\{ \mu^{K-\rho} - \frac{(K-\rho)(K-\rho-1)}{2(2K-1)} \mu^{K-\rho-2} + \frac{(K-\rho)(K-\rho-1)(K-\rho-2)(K-\rho-3)}{2 \cdot 4 \cdot (2K-1)(2K-3)} \mu^{K-\rho-4} - \dots \right\} \tag{A.27}$$

$$P_K^{(K)}(\mu) = \frac{(2K)!\,(-1)^K}{2^K \cdot K!} (1-\mu^2)^{\frac{1}{2}K} \tag{A.28}$$

$$P_K^{(\rho)}(\mu) = (-1)^{K-\rho} P_K^{(\rho)}(-\mu) \quad . \tag{A.29}$$

The orthogonality of the associated Legendre polynomials is expressed by equation (A.30):

$$\begin{aligned} \int_{-1}^{1} P_K^{(\rho)}(\mu)\, P_m^{(\rho)}(\mu)\, d\mu &= 0 && \text{if } k \neq m \\ &= \frac{2}{2K+1} \frac{(K+\rho)!}{(K-\rho)!} && \text{if } k = m \end{aligned} \tag{A.30}$$

The functions $P_k(\quad)$ satisfy the differential equation

$$\frac{d}{d\mu} \left\{ (1-\mu^2) \frac{d}{d\mu} P_K^{(\rho)}(\mu) \right\} + \left\{ K(K+1) - \frac{\rho^2}{1-\mu^2} \right\} P_K^{(\rho)}(\mu) = 0 . \tag{A.31}$$

Equation (A.31) is used to derive formula (A.32).

$$\int_0^1 P_K^{(\rho)}(\mu)\, P_m^{(\rho)}(\mu)\, d\mu \;= \tag{A.32}$$

$$\frac{1}{K(K+1) - m(m+1)} \left\{ P_m^{(\rho)}(0) \left[\frac{d}{d\mu} P_m^{(\rho)}(\mu)\right]_{\mu=0} - P_K^{(\rho)}(0) \left[\frac{d}{d\mu} P_K^{(\rho)}(\mu)\right]_{\mu=0} \right\}$$

If k = m, equations (A.29) and (A.30) give

$$\int_0^1 \left[P_K^{(\rho)}(\mu)\right]^2 d\mu \;=\; \frac{1}{2K+1}\,\frac{(K+\rho)!}{(K-\rho)!}\;. \tag{A.33}$$

The application of formula (A.32) requires equations (A.34) and (A.35).

$$P_K^{(\rho)}(0) = (-1)^{\frac{1}{2}(\rho+K)} \frac{(K+\rho)!}{2^K \left(\frac{K-\rho}{2}\right)! \left(\frac{K+\rho}{2}\right)!} \qquad \text{if } (k-p) \text{ is even} \tag{A.34}$$

$$= \;0 \qquad \text{if } (k-p) \text{ is odd}$$

$$\left[\frac{d}{d\mu} P_K^{(\rho)}(\mu)\right]_{\mu=0} = (-1)^{\frac{1}{2}(\rho+K+3)} \frac{(K+\rho+1)}{2^K \left(\frac{K-\rho-1}{2}\right)! \left(\frac{K+\rho+1}{2}\right)!}\;, \quad (k-p) \text{ odd} \tag{A.35}$$

$$= \;0 \qquad , \; (k-p) \text{ even}$$

5. <u>Formulae involving Gamma Functions</u>

Only Gamma functions of the real variable x occur in the calculations of this paper. In the formulae summarized below, n is a positive integer.

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt \tag{A.36}$$

$$\Gamma(x+1) = x\,\Gamma(x) \tag{A.37}$$

$$\Gamma(n) = (n-1)! \tag{A.38}$$

$$\Gamma(n+\tfrac{1}{2}) = \frac{(2n)!}{2^{2n}\cdot n!}\sqrt{\pi} \tag{A.39}$$

The values of $\Gamma(x)$ for common values of x are listed below.

$$\Gamma(1) = \Gamma(2) = 1 \tag{A.40}$$

$$\frac{1}{\Gamma(0)} = \frac{1}{\Gamma(-n)} = 0 \qquad n = 1,2,3,\ldots \tag{A.41}$$

$$\frac{\Gamma(0)}{\Gamma(-1)} = -1 \tag{A.42}$$

$$\left.\begin{aligned}
\Gamma(\tfrac{1}{2}) &= \sqrt{\pi}\\
\Gamma(\tfrac{3}{2}) &= \tfrac{1}{2}\sqrt{\pi}\\
\Gamma(\tfrac{5}{2}) &= \tfrac{3}{4}\sqrt{\pi}\\
\Gamma(\tfrac{7}{2}) &= \tfrac{15}{8}\sqrt{\pi}\\
\Gamma(\tfrac{9}{2}) &= \tfrac{105}{16}\sqrt{\pi}\\
\Gamma(\tfrac{11}{2}) &= \tfrac{945}{32}\sqrt{\pi}
\end{aligned}\right\} \tag{A.43}$$

$$\left.\begin{aligned}
\Gamma(-\tfrac{1}{2}) &= -2\sqrt{\pi}\\
\Gamma(-\tfrac{3}{2}) &= \tfrac{4}{3}\sqrt{\pi}\\
\Gamma(-\tfrac{5}{2}) &= -\tfrac{8}{15}\sqrt{\pi}
\end{aligned}\right\} \tag{A.44}$$

6. The Sonine Polynomials, $S_m^{(n)}(x)$

These polynomials are defined as the coefficients of the power series expansion in t of the generating function $e^{-\frac{xt}{1-t}}\cdot(1-t)^{-m-1}$, namely

$$e^{-\frac{\chi t}{1-t}}(1-t)^{-m-1} = \sum_{n=0}^{\infty}\left\{\Gamma(m+n+1)\, S_m^{(n)}(\chi)\right\} t^n \quad . \qquad \text{(A.44)}$$

In the applications in this paper, the argument x is associated with the peculiar energy of a molecule, (see equations (2.7), (2.15)), by equation (A.45).

$$x = \text{hm } C^2 \qquad (m = \text{mass of molecule}) \qquad \text{(A.45)}$$

Therefore, x is restricted to positive values. The indices m and n occur as in (A.46).

$$\begin{aligned} m &= K + \tfrac{1}{2} \\ n &= 0,1,2,\ldots \\ k &= 0,1,2,\ldots \end{aligned} \qquad \text{(A.46)}$$

The functions $S_m^{(n)}(x)$ satisfy the orthogonality relation

$$\int_0^\infty \chi^m e^{-\chi} S_m^{(p)}(\chi)\, S_m^{(q)}(\chi)\, d\chi = \frac{1}{\Gamma(m+p+1)\cdot p!} \qquad \text{for } p = q \qquad \text{(A.47)}$$

$$= 0 \qquad \text{for } p \neq q$$

The generating function, equation (A.44), may be used to derive the relations (A.48) to (A.49a).

$$\int_0^\infty \chi^{K+\frac{3}{2}} e^{-\chi} S_{K+\frac{1}{2}}^{(r)}(\chi)\, S_{K+\frac{3}{2}}^{(s)}(\chi)\, d\chi = \frac{(-1)^{r-s}}{\Gamma(K+r+\frac{3}{2})\cdot s!} \qquad \text{for} \quad r = \left\{ {s \atop s+1} \right\} \qquad \text{(A.48)}$$

$$= 0 \qquad \text{for} \quad r \neq \left\{ {s \atop s+1} \right\}$$

$$\int_0^\infty \chi^\alpha e^{-\chi} S_\beta^{(n)}(\chi)\, d\chi = \frac{\Gamma(\alpha+1)\,\Gamma(\beta-\alpha+n)}{\Gamma(\beta-\alpha)\,\Gamma(n+1)\,\Gamma(\beta+n+1)} \qquad \text{(A.49)}$$

$$\frac{d}{d\chi} S_m^{(n)}(\chi) = -S_{m+1}^{(n-1)}(\chi) \quad . \qquad \text{(A.49a)}$$

It was shown by D. Burnett, (Reference 2.10, p.403) that an arbitrary function f(x) may be expanded in a convergent series of Sonine polynomials of the form

$$f(x) = \sum_{n=0}^{\infty} \Gamma(n+m+1)\, a_n\, S_m^{(n)}(x) \;, \tag{A.50}$$

provided only that conditions (A.52) are satisfied. The coefficients a_n are given by

$$a_n = n!\int_0^{\infty} e^{-t}\, t^m\, S_m^{(n)}(t)\cdot f(t)\, dt \;. \tag{A.51}$$

The function f(x) must satisfy the two conditions:

(i) $\displaystyle\int_0^{\infty} e^{-\frac{t}{1-s}}\, t^m\, f(t)\, dt$ converges for $|s| < 1$;

(ii) $|a_n|\cdot n^{m+1+\varepsilon}$ tends to a finite limit or zero as n tends to infinity. (ε is an arbitrarily small positive number). (A.52)

The Sonine polynomials may be calculated explicitly from equation (A.53) which follows directly from (A.44).

$$S_m^{(n)}(x) = \frac{1}{\Gamma(m+n+1)} \sum_{j=0}^{n} (-1)^j \frac{(m+n)(m+n-1)\ldots(m+n-\{n-j-1\})}{j!\;(n-j)!}\, x^j \tag{A.53}$$

In particular,

$$S_m^{(0)} = \frac{1}{\Gamma(m+1)} \tag{A.54}$$

$$S_m^{(1)} = \frac{1}{\Gamma(m+2)}\,(m+1-x) \;. \tag{A.55}$$

APPENDIX III

DETAILS PERTAINING TO THE CALCULATION OF THE BURNETT COEFFICIENTS

1. The Neglection of Gravitational Forces

The calculation of the Burnett coefficients was based on the Boltzmann equation in the form of equation (2.43). In this equation, terms of the form $F_x \frac{\partial}{\partial u}(\nu f)$, which appeared in equation (2.15), were neglected. As was shown by Burnett (Reference 2.11), the presence of body forces F_x, F_y, F_z, does not affect the expressions for the first order coefficients of equation (2.61) and (2.62). However, certain of the second order coefficients will contain additional terms involving these body forces.

In particular, for Maxwell molecules, only four of the basic Burnett coefficients, and their respective associated coefficients are so affected. The coefficients ${}_2A_2$ (equation 2.68) and ${}_2B_{11}$ (equation 2.70) will each contain additional terms of the form $\left(\frac{\mu}{p}\right)^2 F_x \frac{1}{h}\frac{\partial h}{\partial x}$. The coefficients ${}_2B_{21}$ (equation 2.81) and ${}_2B_{30}$ will contain additional terms of the form $\left(\frac{\mu}{p}\right)^2 F_x \sqrt{hm}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$. From equations (2.68) to (2.113) it may be seen that a typical term in most of the second order coefficients is of the form $\left(\frac{\mu}{p}\right)^2 \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2$. Consequently, the ratio α of the magnitude of the terms neglected to those which have been retained is

$$\alpha \sim \frac{F_x\sqrt{hm}}{\left(\frac{\partial u}{\partial z}\right)} . \tag{A.56}$$

In the applications of aerodynamic interest, the only body force to be considered is the gravitational acceleration, g. From the definition of h, equation (2.7)

$$\sqrt{hm} = \frac{1}{\sqrt{2RT}} \sim \frac{1}{a} , \tag{A.57}$$

where a is the velocity of sound in the gas. Thus

$$\alpha \sim \frac{g}{a} \cdot \frac{1}{\left(\frac{\partial u}{\partial z}\right)} \sim \frac{g\,L}{a^2 M} \quad ,$$

where L and M are a typical length, and Mach number of the problem, respectively. Consequently in the foot-pound-second system of units,

$$\alpha \sim 3 \times 10^{-5} \frac{L}{M} \, . \tag{A.58}$$

It is evident that, in general, the effect of gravitational forces on the velocity distribution of a non-uniform gas is entirely negligible.

2. Sample Calculation of the Coefficients ${}_2B_{31}^{(p)}$

The general method used for the calculation of the second order Burnett coefficients was outlined in Part II of the text. For Maxwell molecules the method essentially reduced to the evaluation of equation (2.63) for various values of (k,r). This type of calculation is illustrated below for the particular value (k,r) = (3,1).

It will be recalled that the left side of equation (2.63) resulted from the substitution of expressions of the type (2.48) into the left side of equation (2.45). In particular, for (k,r) = (3,1), the equations of the form (2.48) reduce to

$$\left.\begin{aligned}
&\bar{Q}_{31} = {}_1B_{31} = 0 \\
&\frac{D\bar{Q}}{Dt} = \frac{\partial \bar{Q}}{\partial t} + u\frac{\partial \bar{Q}}{\partial x} + v\frac{\partial \bar{Q}}{\partial y} + w\frac{\partial \bar{Q}}{\partial z} = 0 \\
&\overline{\frac{DQ}{Dt}} = \overline{\frac{\partial Q}{\partial t}} + u\,\overline{\frac{\partial Q}{\partial x}} + v\,\overline{\frac{\partial Q}{\partial y}} + w\,\overline{\frac{\partial Q}{\partial z}} = 0 \\
&\left[\overline{U\frac{\partial Q}{\partial x}} + \overline{V\frac{\partial Q}{\partial y}} + \overline{W\frac{\partial Q}{\partial z}}\right] = \left[-\frac{2}{5}\,{}_1B_{11}\left(2\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\right. \\
&+ \frac{2}{5}\,{}_1B_{11}^{(1)}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) + \frac{2}{5}\,{}_1C_{11}^{(1)}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) - \frac{1}{5}\,\frac{1}{h\sqrt{hm}}\left\{3\,{}_1B_{20}\frac{\partial h}{\partial z}\right. \\
&\qquad\qquad\qquad\qquad \left.\left. - {}_1B_{20}^{(1)}\frac{\partial h}{\partial x} - {}_1C_{20}^{(1)}\frac{\partial h}{\partial y}\right\}\right] \\
&\left[\frac{\partial}{\partial x}\left(\nu\,\overline{UQ}\right) + \frac{\partial}{\partial y}\left(\nu\,\overline{VQ}\right) + \frac{\partial}{\partial z}\left(\nu\,\overline{WQ}\right)\right] = 0
\end{aligned}\right\} \tag{A.59}$$

Consequently, the left side of equation (2.63) reduces to the expression (A.60).

$$\Big[\text{left side of equation (2.63)}\Big]_{\substack{K=3\\r=1}} = -\left(\frac{\mu}{p}\right)\left\{\overline{u\frac{\partial Q}{\partial x}} + \overline{V\frac{\partial Q}{\partial y}} + \overline{W\frac{\partial Q}{\partial z}}\right\}_{\substack{K=3\\r=1}} \qquad \text{(A.60)}$$

In order to evaluate the right hand side of equation (2.63), it is first necessary to calculate the numerical values of the eight constants

$$\mathcal{M}^{(0,0)}_{\substack{303\\101}}\ ;\ \mathcal{M}^{(0)}_{\substack{311\\111}},\ \mathcal{M}^{(1)}_{\substack{311\\111}}\ ;\ \mathcal{M}^{(0)}_{\substack{312\\110}},\ \mathcal{M}^{(1)}_{\substack{312\\110}}\ ;\ \mathcal{M}^{(0)}_{\substack{322\\100}},\ \mathcal{M}^{(1)}_{\substack{322\\100}},\ \mathcal{M}^{(2)}_{\substack{322\\100}}\ ;$$

by means of the formulae given by D. Burnett (Reference 2.11) for Maxwell molecules. Because of the orthogonality properties of the Sonine polynomials and of the spherical harmonics of which the distribution function (2.39) is composed, the constants $\mathcal{M}^{(\ell_1=\ell_2)}_{\substack{K\,K_1K_2\\r\,n_1\,n_2}}$, - which are the results of integrations of the form of the right side of equation (2.49) - differ from zero only when k = $|k_2 - k_1| + 2n$, where n is a positive integer or zero. Consequently

$$\mathcal{M}^{(0)}_{\substack{311\\111}} = \mathcal{M}^{(1)}_{\substack{311\\111}} = \mathcal{M}^{(0)}_{\substack{322\\100}} = \mathcal{M}^{(1)}_{\substack{322\\100}} = \mathcal{M}^{(2)}_{\substack{322\\100}} = 0\,, \qquad \text{(A.61)}$$

and for $\mathcal{M}^{(0,0)}_{\substack{303\\101}}$ n = 0, whereas for $\mathcal{M}^{(0)}_{\substack{312\\110}}$ and $\mathcal{M}^{(1)}_{\substack{312\\110}}$ n = 1.

For the particular case when n = 0, that is when k = $|k_2 - k_1|$, the expressions given by Burnett are equivalent to equations (A.62) to (A.64):

$$\mathcal{M}^{(\ell_1=\ell_2)}_{\substack{K\,K_1K_2\\r\,n_1n_2}} = \left[\frac{4}{3}\sqrt{\pi}\ C\ n_1!\ n_2!\ (K+\tfrac{1}{2})\frac{\Gamma(K+\frac{1}{2})\cdot\Gamma(K_1+\frac{1}{2})}{K!\ \Gamma(K+r+\frac{3}{2})}\cdot\frac{(K+K_1+\ell_1)!}{(K-\ell_1)!}\right.$$

$$(K=|K_2-K_1|) \qquad \left.\times\ \frac{1}{J}\int_0^\infty\left\{F(\chi)-F(\chi=0)\right\}\cdot\alpha\,d\alpha\right], \qquad \text{(A.62)}$$

where J is a numerical constant, see equation (A.67), and C and $F(\chi)$ are defined by equations (A.63) and (A.64), respectively.

$$
\begin{aligned}
C &= 1 && \text{for } p_1 = p_2 = 0 \\
&= 2\,\frac{(k_1-p_1)!}{(k_1+p_1)!}\cdot\frac{(k_2-p_1)!}{(k_2+p_1)!} && \text{for } p_1 = p_2 \neq 0
\end{aligned}
\qquad (A.63)
$$

$$
\begin{aligned}
F(\chi) &= F_0(\chi) + F_0(\pi-\chi) && \text{if } k_2 \neq k_1 \text{, or } n_2 \neq n_1 \\
&= F_0(\chi) && \text{if } k_2 = k_1 \text{, and } n_2 = n_1
\end{aligned}
\qquad (A.64)
$$

where $F_0(\chi)$ is the coefficient of $(s^{n_1}\, t^{n_2}\, u^r)$ in the power series expansion of

$$
\Bigg[\frac{u^{k_1}\left(\sin\tfrac{1}{2}\chi\cdot\cos\tfrac{1}{2}\chi\right)^{k+k_1}}{\left\{1-us\cdot\cos^2(\tfrac{1}{2}\chi)-ut\cdot\sin^2(\tfrac{1}{2}\chi)\right\}^{k+k_1+\frac{3}{2}}}\cdot\Big\{P_{k_1}(0)\cdot\tan^k(\tfrac{1}{2}\chi)
$$

$$
-k\,P_{k_1+1}(0)\tan^{k-1}(\tfrac{1}{2}\chi)+\frac{k(k-1)}{2!}\,P_{k_1+2}(0)\tan^{k-2}(\tfrac{1}{2}\chi)-\ldots\pm P_{k_1+k}(0)\Big\}\Bigg]
$$

The quantities χ and α are equivalent to the "collision parameters" G_{12}, dg of equation (2.49) for Maxwell molecules. Actually χ is the angle through which a molecule is deflected by an elastic collision with another molecule. The explicit form of the functional relation $\chi = \chi(\alpha)$ is given in Section 3 of this Appendix. The quantities $P_k(0)$ are the Legendre polynomials of argument zero given by equations (A.23) and (A.24) of Appendix II.

The constant $\mathcal{M}^{(0,0)}_{\substack{303\\101}}$ is obtained from equations (A.62) to (A.64) by making the substitutions $k = 3$, $k_1 = 0$, $k_2 = 3$; $r = 1$, $n_1 = 0$, $n_2 = 1$;

$p_1 = p_2 = 0$. Thus the expression for $F_0(\chi)$ reduces to

$$F_0(\chi) = \tfrac{9}{2}\sin(\tfrac{1}{2}\chi)\,(\sin\tfrac{1}{2}\chi\,\cos\tfrac{1}{2}\chi)^3\left\{\tan^3(\tfrac{1}{2}\chi) - \tfrac{3}{2}\tan(\tfrac{1}{2}\chi)\right\}. \qquad (A.65)$$

The expression for $F(\chi)$ obtained from equation (A.64) can be reduced to the form

$$\left\{F(\chi) - F(0)\right\} = \left\{-\tfrac{99}{4}\sin^2(\tfrac{1}{2}\chi)\cdot\cos^2(\tfrac{1}{2}\chi) + \tfrac{45}{2}\sin^4(\tfrac{1}{2}\chi)\cos^4(\tfrac{1}{2}\chi)\right\}. \qquad (A.66)$$

It is convenient to define the two definite integrals

$$J = 4\pi\int_0^\infty\left\{\sin^2(\tfrac{1}{2}\chi)\cos^2(\tfrac{1}{2}\chi)\right\}\alpha\,d\alpha \;, \qquad (A.67)$$

$$L_4 = 4\pi\int_0^\infty\left\{\sin^4(\tfrac{1}{2}\chi)\cos^4(\tfrac{1}{2}\chi)\right\}\alpha\,d\alpha \;. \qquad (A.68)$$

The numerical evaluation of the integral J, which was carried out by J. C. Maxwell, S. Chapmann and others, and of the integral L_4, performed by the author, is discussed in Section 3 of this Appendix. Substitution of equations (A.66) to (A.68) into equation (A.62) yields for the desired constant

$$\mathcal{M}^{(0,0)}_{\substack{303\\101}} = -\tfrac{1}{6}\cdot\tfrac{1}{J}\,(11\,J - 10\,L_4) \qquad (A.69)$$

With the value of $\frac{L_4}{J} = .1577$, as calculated in Section 3 below, one obtains for the constant $\mathcal{M}^{(0,0)}_{\substack{303\\101}}$ the value -1.5704 as given in Table (2.1).

The constants $\mathcal{M}^{(0)}_{\substack{312\\110}}$ and $\mathcal{M}^{(1)}_{\substack{312\\110}}$ are calculated from a formula which is valid for n = 1, that is $k = |k_1 - k_2| + 2$, and which is of the same form as equations (A.62) to (A.64), although somewhat longer. The results analogous to equation (A.69) are

$$\mathcal{M}^{(0)}_{\substack{312\\110}} = \frac{1}{J}\left(L_4 - \frac{1}{10} J\right) \; ; \quad \mathcal{M}^{(1)}_{\substack{312\\110}} = -\frac{1}{3}\, \mathcal{M}^{(0)}_{\substack{312\\110}} \; . \tag{A.70}$$

The values of the eight M constants required for the evaluation of the coefficient ${}_2B_{31}$ are thus given by equations (A.61), (A.69), and (A.70) and are also summarized in Table 2.1.

All of the quantities which are required for the calculation of the basic coefficient ${}_2B_{31}$ from equation (2.63) have now been determined. Thus, by means of equations (A.60), (A.59), and the constants M of Table 2.1, equation (2.63), for (k,r) = (3,1), can be written explicitly in the form

$$\begin{aligned} -\left(\frac{\mu}{p}\right)\Big[-\frac{2}{5}\, {}_1B_{11}\left(2\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{2}{5}\, {}_1B^{(1)}_{11}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) + \frac{2}{5}\, {}_1C^{(1)}_{11}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\ - \frac{1}{5}\frac{1}{h\sqrt{hm}}\left\{3\, {}_1B_{20}\frac{\partial h}{\partial z} - {}_1B^{(1)}_{20}\frac{\partial h}{\partial x} - {}_1C^{(1)}_{20}\frac{\partial h}{\partial y}\right\}\Big] \\ = -1.5704\, {}_2B_{31} + \left[.0577\, {}_1B_{11}\, {}_1B_{20} - .0192\left\{ {}_1B^{(1)}_{11}\, {}_1B^{(1)}_{20} + {}_1C^{(1)}_{11}\, {}_1C^{(1)}_{20}\right\}\right]. \end{aligned} \tag{A.71}$$

Equation (A.71) is now solved for ${}_2B_{31}$, and the values of the first order coefficients are substituted from equations (2.61) and (2.62). This results in equations (2.98) which expresses the second order coefficient ${}_2B_{31}$ explicityly in terms of the derivatives of the mean motion variables of the gas.

The expressions for the six associated coefficients ${}_2B^{(1)}_{31}, \ldots, {}_2C^{(3)}_{31}$ will now be obtained from the basic coefficient ${}_2B_{30}$, equation (2.98), by means of the method of rotation of coordinate axes, which was discussed in the text in connection with equation (2.54). For the case at hand, (k,n) = (3,1), equation (2.54) can be evaluated by means of equation (A.12) of Appendix II, so that

$$Y_{31}(c = w',\ u' = v' = 0) = {}_2B_{31} \cdot c^3 \qquad (A.72)$$

where the coefficient ${}_2B_{31}$ is expressed in terms of the primed coordinates by equations (A.73) and (A.74).

$$ {}_2B_{31} = -0.600\left(\frac{\mu}{p}\right)^2 \bar{X}_{31}\,, \qquad (A.73)$$

where

$$\bar{X}_{31} = \frac{\partial h}{\partial z'}\left(2\frac{\partial w'}{\partial z'} - \frac{\partial u'}{\partial x'} - \frac{\partial v'}{\partial y'}\right) - \frac{\partial h}{\partial x'}\left(\frac{\partial w'}{\partial x'} + \frac{\partial u'}{\partial z'}\right) - \frac{\partial h}{\partial y'}\left(\frac{\partial w'}{\partial y'} + \frac{\partial v'}{\partial z'}\right). \qquad (A.74)$$

The rotation of the coordinate axes is defined by equations (A.75).

$$\left.\begin{aligned} x' &= l_1x + m_1y + n_1z \\ y' &= l_2x + m_2y + n_2z \\ z' &= l_3x + m_3y + n_3z \end{aligned}\right\} \qquad (A.75)$$

The nine direction cosines $l_1, \ldots, n_3$ satisfy the various orthogonality relationships of equations (A.76).

$$\left.\begin{aligned} &l_i^2 + m_i^2 + n_i^2 = 1 && i = 1,2,3 \\ &l_1^2 + l_2^2 + l_3^2 = 1 && \text{etc.} \\ &l_1l_2 + m_1m_2 + n_1n_2 = 0 && \text{etc.} \\ &l_1m_1 + l_2m_2 + l_3m_3 = 0 && \text{etc.} \end{aligned}\right\} \qquad (A.76)$$

The expressions for the transformations of the velocity components (u', v', w') and for the components of the gradients ($\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial y_1}$, $\frac{\partial}{\partial z_1}$) are analogous to equations (A.75). The peculiar velocity components in the unprimed coordinate system are given by equations (A.77).

$$\left.\begin{aligned} U &= l_3 C \\ V &= m_3 C \qquad C = C' \\ W &= n_3 C \end{aligned}\right\} \qquad (A.77)$$

Equation (A.74) is now multiplied by C^3 and all of the derivatives appearing in X_{31} are expressed in terms of the unprimed coordinate system by means of equations of the type (A.75). By means of equations (A.76) and (A.77), the result so obtained can be juggled into the form

$$\begin{aligned} -0.600\left(\frac{\mu}{p}\right)^2\left\{X_{31}C^3\right\} = \Big[& {}_2B_{31}\left\{\tfrac{1}{2}W(5W^2-3C^2)\right\} \\ & + {}_2B_{31}^{(1)}\left\{\tfrac{1}{4}U(5W^2-C^2)\right\} + {}_2C_{31}^{(1)}\left\{\tfrac{1}{4}V(5W^2-C^2)\right\} \\ & + {}_2B_{31}^{(2)}\left\{\tfrac{1}{4}W(U^2-V^2)\right\} + {}_2C_{31}^{(2)}\left\{\tfrac{1}{2}UVW\right\} \\ & + {}_2B_{31}^{(3)}\left\{\tfrac{1}{24}U(U^2-3V^2)\right\} + {}_2C_{31}^{(3)}\left\{\tfrac{1}{24}V(3U^2-V^2)\right\}\Big] , \end{aligned} \qquad (A.78)$$

where the quantities which precede the curly brackets are those given by equations (2.98) to (2.104). It will be seen that equation (A.78) is identically equal to the general spherical harmonic function Y_{31} as obtained from equation (A.12) of Appendix II. Consequently, the quantities given by equations (2.98) to (2.104) really are the associated Burnett coefficients which appear in the distribution function, equation (2.39), when (k,r) = (3,1).

The remainder of the second order Burnett coefficients were calculated by the author in a similar manner. In most cases the algebraic detail of the calculations was more complex than that for the case (k,r) = (3,1) given here, as may be seen by examining equations (2.68) to (2.113).

3. Evaluation of the Definite Integral L_4

It was seen in Section 2, above, that the calculation of the Burnett coefficients requires the evaluation of the two definite integrals, J, and L_4, of equations (A.67) and (A.68), respectively. The significance of the quantities χ and α, which describe the details of the process of intermolecular collision, can be ascertained from a study of the molecular orbit.

As was shown by Maxwell, (Reference 2.12) the encounter between two molecules A, and B is most easily described in terms of the "relative orbit" of figure A.1. The molecule A is considered as the stationary origin of the plane polar coordinates r and θ. The relative motion of molecule B with respect to A is then the same as that of a particle of mass m under the influence of a central force field located at A. For Maxwell molecules, the equivalent repulsive force is then $\frac{2Km^2}{r^5}$, in view of equation (2.8). If χ is the total change in the direction of the relative motion of molecule B produced by an elastic encounter with molecule A, the magnitude of the relative velocity before and after the encounter will have the same value, say $\underline{q}$. Since the orbit is completely defined in terms of the distance $\underline{b}$ (figure A.1), and the initial relative velocity $\underline{q}$, the deflection angle can be calculated from an expression of the form $\chi = \chi\,(b,q)$.

From the laws of conservation of angular momentum and energy it is easy to derive the differential equation of the relative orbit for Maxwell molecules

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{r^4}{b^2} - \frac{Km}{q^2 b} - r^2 \qquad (A.79)$$

This can be integrated to give

$$\theta = \left| \int_{\infty}^{r} \frac{dr}{\sqrt{\frac{r^4}{b^2} - r^2 - \frac{Km}{q^2 b^2}}} \right| \qquad (A.80)$$

As was discovered by Maxwell, the equation of the relative orbit can be changed from a two parameter curve to one containing only one parameter by means of the substitutions

$$t = \frac{b}{r} \quad , \quad \alpha = b\left(\frac{q^2}{2Km}\right)^{\frac{1}{4}} \tag{A.81}$$

Equation (A.80) then becomes

$$\theta = \int_0^t \frac{dt}{\sqrt{1 - t^2 - \frac{1}{2}(t/\alpha)^4}} \quad . \tag{A.82}$$

We are interested in finding the deflection angle χ in terms of the dimensionless collision parameter α. Consequently

$$\pi - \chi = 2\psi = 2\int_0^{t_0} \frac{dt}{\sqrt{1 - t^2 - \frac{1}{2}(t/\alpha)^4}} \quad , \tag{A.83}$$

where t_0 corresponds to the intersection of the orbit with the apse line, $\theta = \psi$, or $\frac{dr}{d\theta} = 0$ (see figure A.1). The denominator of the integrand of equation (A.83) has only one real and positive root which is t_0, namely

$$t_0 = \alpha\left[\sqrt{\alpha^4 + 2} - \alpha^2\right]^{\frac{1}{2}} \tag{A.84}$$

By means of a number of transformations the definite integral (A.83) can be reduced to the standard form of the complete elliptic integral of the first kind, namely

$$(\pi - \chi) = \frac{2\alpha}{(\alpha^4+2)^{1/4}} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\left[1 - \left(\frac{\sqrt{\alpha^4+2} - \alpha^2}{2\sqrt{\alpha^4+2}}\right)\sin^2\phi\right]^{\frac{1}{2}}} \tag{A.85}$$

Finally, it is convenient to make the substitution $\alpha^2 = s$, so that the deflection angle χ is now a function of the single collision parameter s, that is

$$(\pi - \chi) = 2\left(\frac{s}{\sqrt{s^2+2}}\right)^{\frac{1}{2}} \cdot K(k) \qquad (A.86)$$

$$k = \frac{1}{\sqrt{2}}\left(1 - \frac{s}{\sqrt{s^2+2}}\right)^{\frac{1}{2}} .$$

where K(k) is the complete elliptic integral of the first kind with the modulus k.

Values of $(\pi - \chi)$ have been computed by the author for the range of $0 \leqq s \leqq 10$. The result of this calculation is summarized in Table A.2. This table also includes the values of the functions $\sin\chi$, $\sin^2\chi$, $\sin^4\chi$, which have in turn been plotted in figure A.2 as functions of the collision parameter s.

In terms of the notation used above, the two integrals of equations (A.67) and (A.68) may be written as in equations (A.87) and (A.88) respectively.

$$J = \frac{\pi}{2}\int_0^\infty \sin^2\chi \, ds \qquad (A.87)$$

$$L_4 = \frac{\pi}{8}\int_0^\infty \sin^4\chi \, ds \quad . \qquad (A.88)$$

In view of equation (A.81), and the definition of s, the integration over all positive values of s amounts to the consideration of all possible orbits of the molecule B with respect to the Molecule A. A comparison with equation (2.15) indicates that the differential ds is equivalent to $dC_2\, dg$, and that the deflection angle χ is closely associated with the encounter function G_{12}, for Maxwell molecules.

The integral J of equation (A.87) has been evaluated numerically by several authors, as shown in Table A.3. The value of J to be used hereafter will be taken as

$$J = 1.370 \ . \tag{A.89}$$

Disregarding the somewhat inaccurate calculation of Maxwell, the above should be accurate to four significant figures.

The integral L_4 had not been evaluated previously and was calculated by the author as follows. From (A.88)

$$L_4 = \frac{\pi}{8}\Big[\int_0^{.10} \sin^4\chi \, ds + \int_{.10}^{.30} \sin^4\chi \, ds + \int_{.30}^{.80} \sin^4\chi \, ds + \int_{.80}^{1.60} \sin^4\chi \, ds + \int_{1.60}^{3.60} \sin^4\chi \, ds + \int_{3.60}^{\infty} \sin^4\chi \, ds \Big] . \tag{A.90}$$

The first five of these integrals were evaluated numerically by means of the ordinates given in Table A.2 and Simpson's rule. For the last integral of equation (A.90), the asymptotic expansion of the integrand was used. Thus for large values of s, equation (A.86) reduces to

$$(\pi - \chi) = \frac{3\pi}{8} \cdot \frac{1}{s^2} \Big[1 - \frac{35}{32} \frac{1}{s^2} + \frac{199}{768} \frac{1}{s^4} - \dots \Big] \tag{A.91}$$

From this one obtains

$$\sin^4\chi = \left(\frac{3\pi}{8}\right)^4 \cdot \frac{1}{s^8} \Big[1 - \frac{35}{8} \cdot \frac{1}{s^4} + \frac{12{,}617 - 144\pi^2}{1536} \cdot \frac{1}{s^4} - \dots \Big] , \tag{A.92}$$

and finally

$$\int_{s_1}^{\infty} \sin^4\chi \, ds = \frac{1}{7} \left(\frac{3\pi}{8}\right)^4 \cdot \frac{1}{s_1^7} \Big[1 - \frac{245}{72} \cdot \frac{1}{s_1^2} + \frac{7}{11} \cdot \frac{12{,}617 - 144\pi^2}{1536} \cdot \frac{1}{s_1^4} - \dots \Big] . \tag{A.93}$$

Equation (A.90) then reduced to

$$L_4 = \frac{\pi}{8} \Big[.018620 + .168670 + .301801 + .057775 + .003410 + .000028 \Big] = 0.21610 \ . \tag{A.94}$$

As was seen in Section 2 of this Appendix, the numerical factor required for the evaluation of the quantities M is the ratio $\frac{L_4}{J}$ which has been determined accurate to four significant figures.

$$\frac{L_4}{J} = +\,0.1577 \tag{A.95}$$

Although the constant J does not appear explicitly in the expressions for the Burnett coefficients, equations (2.68) to (2.113), the value of this integral is, nevertheless, contained in the coefficient of viscosity, μ. As was first shown by Maxwell, the viscosity coefficient for the moelcular model which bears his name is given by

$$\mu = \frac{\sqrt{2}}{3J} \cdot \frac{kT}{\sqrt{Km}} \,, \tag{A.96}$$

where k is again the Boltzmann constant and K is the constant of proportionality in the force law of equation (2.8).

APPENDIX IV

DETAILS PERTAINING TO THE EVALUATION OF THE BOUNDARY CONDITIONS

1. Values of the Definite Integrals $I_i^{(1)}$ and $I_i^{(2)}$

The integrals are defined by equation (5.45), Part V. The functions M_i are defined by equations (A.97):

$$\begin{array}{lll} M_1 = 1 & M_5 = C^2 & M_9 = Z_2^{(1)} \\ M_2 = Y_1 & M_6 = Y_2 & M_{10} = Z_2^{(2)} \\ M_3 = Y_1^{(1)} & M_7 = Y_2^{(1)} & M_{11} = C^2 Y_1 \ . \\ M_4 = Z_1^{(1)} & M_8 = Y_2^{(2)} & \end{array} \tag{A.97}$$

The spherical harmonic functions $Y_1, \ldots, Z_2$ are defined by equations (A.3) to (A.7) of Appendix II.

From equation (5.46) Part V

$$\begin{aligned} I_i^{(1)} &= \mathcal{A}_i + \mathcal{B}_i \\ I_i^{(2)} &= \mathcal{A}_i - \mathcal{B}_i \end{aligned} \tag{A.98}$$

The values of $\mathcal{A}_i$ and $\mathcal{B}_i$ for $i = 1,2,\ldots, 11$ are given by equations (A.99) to (A.115), in terms of the exact Burnett coefficients of the velocity distribution function, equation (2.39).

$$\mathcal{A}_1 = \frac{1}{2} \tag{A.99}$$

$$\mathcal{A}_2 = \mathcal{A}_3 = \mathcal{A}_4 = 0 \tag{A.100}$$

$$\mathcal{A}_5 = \frac{3}{4} \, \frac{1}{hm} \tag{A.101}$$

$$a_6 = \frac{3}{8}\frac{1}{hm} B_{20} \;;\quad a_7 = \frac{3}{8}\frac{1}{hm} B^{(1)}_{20} \;;\quad a_8 = \frac{3}{8}\frac{1}{hm} B^{(2)}_{20} \tag{A.102}$$

$$a_9 = \frac{3}{8}\frac{1}{hm} C^{(1)}_{20} \;;\quad a_{10} = \frac{3}{8}\frac{1}{hm} C^{(2)}_{20} \tag{A.103}$$

$$a_{11} = -\frac{5}{8}\frac{1}{(hm)^{3/2}} B_{11} \tag{A.104}$$

$$\mathcal{B}_1 = \frac{1}{\sqrt{\pi}} \sum_{q=0}^{\infty}\sum_{n=0}^{\infty} (-1)^q \frac{1\cdot 3\ldots(2q-3)(2q-1)}{2\cdot 4\ldots(2q)(2q+2)}\cdot\frac{\Gamma(q+2)\,\Gamma(q+n+\frac{1}{2})}{\Gamma(q+\frac{1}{2})} B_{2q+1,n} \tag{A.105}$$

$$\mathcal{B}_2 = \frac{1}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{hm}}\Bigg[-\frac{1}{4\sqrt{\pi}}\sum_{n=0}^{\infty}\Gamma(n-\tfrac{1}{2})\cdot A_n$$

$$+\sum_{q=1}^{\infty}\sum_{n=0}^{\infty}(-1)^{q+1}\frac{\Gamma(2q+1)\,\Gamma(q+2)\,\Gamma(q+n-\frac{1}{2})}{2^{2q}(2q-1)(2q+2)\{\Gamma(q+1)\}^2\,\Gamma(q-\frac{1}{2})} B_{2q,n}\Bigg] \tag{A.106}$$

$$\mathcal{B}_3 = \frac{1}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{hm}} \sum_{q=1}^{\infty}\sum_{n=0}^{\infty}(-1)^{q+1}\frac{\Gamma(2q)\,\Gamma(q+n-\frac{1}{2})\,\Gamma(2q+3)}{\Gamma(2q+2)\,\Gamma(q-\frac{1}{2})\cdot[2q(2q+1)-2]\cdot 2^{2q}\,\Gamma(q)} B^{(1)}_{2q,n} \tag{A.107}$$

$\mathcal{B}_4$ is obtained from (A.107) by replacing $B^{(1)}_{2q,n}$ by $C^{(1)}_{2q,n}$ (A.108)

$$\mathcal{B}_5 = \frac{1}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{hm}} \sum_{q=0}^{\infty}\sum_{n=0}^{\infty}(-1)^q \frac{1\cdot 3\ldots(2q-3)(2q-1)}{2\cdot 4\ldots(2q)(2q+2)}\cdot\frac{\Gamma(q+3)\,\Gamma(q+n-\frac{1}{2})}{\Gamma(q-\frac{1}{2})} B_{2q+1,n} \tag{A.109}$$

$$\mathcal{B}_6 = \frac{1}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{hm}} \sum_{q=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{q+1}\frac{(2q+1)\cdot\Gamma(2q+1)\cdot\Gamma(q+3)\,\Gamma(q+n-\frac{1}{2})}{(2q-1)(2q+4)\,2^{2q+1}\cdot\{\Gamma(q+1)\}^2\,\Gamma(q-\frac{1}{2})} B_{2q+1,n} \tag{A.110}$$

$$\mathcal{B}_7 = \frac{3}{\sqrt{\pi}}\frac{1}{hm} \sum_{q=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{q+1}\frac{\Gamma(2q+1)\cdot(q+2)\,\Gamma(q+n-\frac{1}{2})}{[(2q+1)(2q+2)-6]\cdot 2^{2q+1}\,\Gamma(q+1)\,\Gamma(q-\frac{1}{2})} B^{(1)}_{2q+1,n} \tag{A.111}$$

$$\mathcal{B}_8 = \frac{3}{\sqrt{\pi}} \frac{1}{hm} \sum_{q=0}^{\infty} \sum_{n=0}^{\infty} (-1)^{q+1} \frac{(2q+4) \quad \Gamma(2q) \quad \Gamma(q+n-\frac{1}{2})}{2^{2q+1} \cdot \Gamma(q) \left[(2q+1)(2q+2)-6\right] \cdot \Gamma(q-\frac{1}{2})} \cdot B^{(2)}_{2q+1,n} \tag{A.112}$$

$\mathcal{B}_9$ is obtained from equation (A.111) by substituting $C^{(1)}_{2q+1,n}$ for $B^{(1)}_{2q+1,n}$. (A.113)

$\mathcal{B}_{10}$ is obtained from equation (A.112) by substituting $C^{(2)}_{2q+1,n}$ for $B^{(2)}_{2q+1,n}$. (A.114)

$$\mathcal{B}_{11} = \frac{1}{\sqrt{\pi}} \cdot \frac{1}{(hm)^{3/2}} \left[\frac{3}{4} \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} A_n \cdot \Gamma(n-\tfrac{3}{2}) \right.$$
$$\left. + \sum_{q=1}^{\infty} \sum_{n=0}^{\infty} (-1)^{q+1} \frac{\Gamma(2q+1) \, \Gamma(q+3) \, \Gamma(q+n-\frac{3}{2})}{2^{2q} (2q-1)(2q+2) \left\{ \Gamma(q+1) \right\}^2 \Gamma(q-\frac{3}{2})} \; B_{2q,n} \right] . \tag{A.115}$$

The above expressions are valid for any "smooth" molecule having spherical symmetry. However, the complete set of Burnett coefficients has been calculated only for the Maxwell molecule, and there only up to terms of order $\left(\frac{\mu}{p}\right)^2 \frac{\partial^2}{\partial x^2}$. For this case the general expressions for $\mathcal{A}_i$ and $\mathcal{B}_i$ reduce to those given below.

Equations (A.99) to (A.104) for the quantities $\mathcal{A}_i$ remain unchanged, except that the prescript 2 is affixed to all of the Burnett coefficients; for example

$${}_2\mathcal{A}_7 = \frac{3}{8} \frac{1}{hm} \, {}_2B^{(1)}_{20} \, , \qquad \text{etc.} \tag{A.116}$$

The quantities ${}_2\mathcal{B}_i$ reduce to the finite sums (cf. equation 2.66) given by equations (A.117) to (A.127):

$${}_2\mathcal{B}_1 = \frac{1}{8\sqrt{\pi}}\left[2\,{}_2B_{11} + 3\,{}_2B_{12} - 2\,{}_2B_{30} - 3\,{}_2B_{31}\right] \qquad (A.117)$$

$${}_2\mathcal{B}_2 = \frac{1}{2\sqrt{\pi}}\left[1 - \frac{1}{4}\,{}_2A_2 - \frac{3}{8}\,{}_2A_3 + \frac{1}{2}\,{}_2B_{20} + \frac{1}{4}\,{}_2B_{21} + \frac{3}{8}\,{}_2B_{22} - \frac{1}{4}\,{}_2B_{40}\right] \qquad (A.118)$$

$${}_2\mathcal{B}_3 = \frac{1}{4\sqrt{\pi}\sqrt{hm}}\left[{}_2B_{20}^{(1)} + \frac{1}{2}\,{}_2B_{21}^{(1)} + \frac{3}{4}\,{}_2B_{22}^{(1)} - \frac{1}{2}\,{}_2B_{40}^{(1)}\right] \qquad (A.119)$$

$${}_2\mathcal{B}_4 = \frac{1}{4\sqrt{\pi}\sqrt{hm}}\left[{}_2C_{20}^{(1)} + \frac{1}{2}\,{}_2C_{21}^{(1)} + \frac{3}{4}\,{}_2C_{22}^{(1)} - \frac{1}{2}\,{}_2C_{40}^{(1)}\right] \qquad (A.120)$$

$${}_2\mathcal{B}_5 = -\frac{1}{2\sqrt{\pi}}\cdot\frac{1}{hm}\left[{}_2B_{11} + \frac{1}{2}\,{}_2B_{12} + \frac{3}{2}\,{}_2B_{30} + \frac{3}{4}\,{}_2B_{31}\right] \qquad (A.121)$$

$${}_2\mathcal{B}_6 = -\frac{1}{8\sqrt{\pi}}\cdot\frac{1}{hm}\left[{}_2B_{11} + \frac{1}{2}\,{}_2B_{12} - 6\,{}_2B_{30} - 3\,{}_2B_{31}\right] \qquad (A.122)$$

$${}_2\mathcal{B}_7 = -\frac{3}{8\sqrt{\pi}}\cdot\frac{1}{hm}\left[{}_2B_{11}^{(1)} + \frac{1}{2}\,{}_2B_{12}^{(2)} - {}_2B_{30}^{(1)} - \frac{1}{2}\,{}_2B_{31}^{(1)}\right] \qquad (A.123)$$

$${}_2\mathcal{B}_8 = \frac{3}{8\sqrt{\pi}}\cdot\frac{1}{hm}\left[{}_2B_{30}^{(2)} + \frac{1}{2}\,{}_2B_{31}^{(2)}\right] \qquad (A.124)$$

$${}_2\mathcal{B}_9 = -\frac{3}{8\sqrt{\pi}}\cdot\frac{1}{hm}\left[{}_2C_{11}^{(1)} + \frac{1}{2}\,{}_2C_{12}^{(1)} - {}_2C_{30}^{(1)} - {}_2C_{31}^{(1)}\right] \qquad (A.125)$$

$${}_2\mathcal{B}_{10} = \frac{3}{8\sqrt{\pi}}\cdot\frac{1}{hm}\left[{}_2C_{30}^{(2)} + \frac{1}{2}\,{}_2C_{31}^{(2)}\right] \qquad (A.126)$$

$$ {}_2\mathcal{B}_{11} = \frac{1}{\sqrt{\pi}} \cdot \frac{1}{(hm)^{3/2}} \Big[1 + \tfrac{3}{4}\,{}_2A_2 + \tfrac{3}{8}\,{}_2A_3 + \tfrac{3}{4}\,{}_2B_{20} - \tfrac{3}{8}\,{}_2B_{21} $$

$$ - \tfrac{3}{16}\,{}_2B_{22} - \tfrac{1}{2}\,{}_2B_{40} \Big] . \qquad \text{(A.127)} $$

The second approximations to the Burnett coefficients, ${}_2A_2, \ldots, {}_2B_{40}$, which appear in the above equations, are given explicitly by equations (2.68) to (2.105) of Part II.

2. Sample Calculation of the Definite Integrals $\overset{(1)}{I}_2$, $\overset{(2)}{I}_2$, $\overset{(1)}{I}_7$, $\overset{(2)}{I}_7$

(a) The integrals $\overset{(1)}{I}_2$ and $\overset{(2)}{I}_2$

From the defining equations (5.45) and (A.97) one has

$$ \overset{(1)}{I}_2 = \int_{-\pi}^{\pi} d\phi \int_0^{\frac{\pi}{2}} \sin\theta \cdot d\theta \int_0^{\infty} C^2 dC \cdot f(C,\theta,\phi) \cdot Y_1 \, , \qquad \text{(A.128)} $$

where $f(C,\theta,\phi)$ is given by equation (2.39) and from equation (A.3),

$$ Y_1 = C\,P_1\,(\cos\theta) . $$

By means of the conventional change of variable

$$ \cos\theta = \mu \qquad \text{(A.129)} $$

one obtains

$$ \int_0^{\frac{\pi}{2}} [\,] \cdot \sin\theta \, d\theta = \int_0^1 [\,] \, d\mu \;\; ; \;\; \int_{\frac{\pi}{2}}^{\pi} [\,] \cdot \sin\theta \cdot d\theta = \int_{-1}^{0} [\,] \, d\mu \, . \qquad \text{(A.130)} $$

Equation (2.39) for $f(C,\theta,\phi)$ is now substituted into the integral (A.128). The integration with respect to the azimuthal angle ϕ is performed first, so that all those terms in $f(C,\theta,\phi)$ which contain $\sin(\rho\phi)$ or $\cos(\rho\phi)$ integrate to zero. The integral (A.128) then reduces to

$$ \overset{(1)}{I}_2 = 2\pi \left(\frac{hm}{\pi}\right)^{3/2} \int_0^{\infty} C^3 dC \cdot e^{-hmC^2} \cdot \Big[\sum_{n=0}^{\infty} n!\,\Gamma(n+\tfrac{3}{2}) \cdot A_n \cdot \overset{(n)}{S}_{\frac{1}{2}}(hmC^2) \int_0^1 P_1(\mu)\, d\mu $$

$$ + \sum_{K=1}^{\infty} (hmC^2)^{\frac{K}{2}} \Big\{ \sum_{n=0}^{\infty} n!\,\Gamma(K+n+\tfrac{3}{2}) \cdot B_{Kn} \cdot \overset{(n)}{S}_{K+\frac{1}{2}}(hmC^2) \int_0^1 P_K(\mu) \cdot P_1(\mu)\, d\mu \Big\} \Big] . \qquad \text{(A.131)} $$

The integration with respect to μ is performed next. From equation (A.15) of Appendix II, Section 3, one obtains

$$\int_0^1 P_1(\mu)\, d\mu = \frac{1}{2}. \tag{A.132}$$

Similarly, from equations (A.18) to (A.20), one finds

$$\begin{aligned}
\int_0^1 P_K(\mu)\, P_1(\mu)\, d\mu &= \frac{1}{2} && \text{for } k = 0 \\
&= \frac{1}{3} && \text{for } k = 1 \\
&= (-1)^{q+1} \frac{\Gamma(2q+1)}{2^{2q}\, \{\Gamma(q+1)\}^2 (2q-1)(2q+2)}, && \begin{cases} K = 2q \\ q \gtreqless 1 \end{cases} \\
&= 0 && \text{for } k = 2q+1,\ q \gtreqless 1. \qquad \text{(A.133)}
\end{aligned}$$

Equations (A.132) and (A.133) are substituted into (A.131), and with the change of variable defined by

$$x = hm\, c^2 \tag{A.134}$$

the integral $I_2^{(1)}$ becomes

$$\begin{aligned}
I_2^{(1)} = \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{\sqrt{hm}} \Big[&\sum_{n=0}^{\infty} n!\, \Gamma(n+\tfrac{3}{2})\, A_n \int_0^\infty x\, e^{-x}\, S_{\frac{1}{2}}^{(n)}(x)\, dx \\
&+ \frac{2}{3} \sum_{n=0}^{\infty} n!\, \Gamma(n+\tfrac{5}{2})\, B_{1n} \int_0^\infty x^{\frac{3}{2}} e^{-x}\, S_{\frac{3}{2}}^{(n)}(x)\, dx \\
+ 2 \sum_{q=1}^{\infty} \sum_{n=0}^{\infty} n!\, \Gamma(2q+n+\tfrac{3}{2}) (-1)^{q+1} &\frac{\Gamma(2q+1)}{2^{2q}(2q-1)(2q+2)\{\Gamma(q+1)\}^2} \cdot B_{2q,n} \int_0^\infty x^{q+1} e^{-x} S_{2q+\frac{1}{2}}^{(n)}(x)\, dx \Big]. \qquad \text{(A.135)}
\end{aligned}$$

The three integrals involving the Sonine polynomials $S_m(x)$ are each evaluated from the general formula of equation (A.49), Appendix II, Section 6. Thus one obtains

$$\begin{aligned}
\int_0^\infty x^{3/2} e^{-x} S_{\frac{3}{2}}^{(n)}(x)\, dx &= 1 \text{ if } n = 0 \\
&= 0 \text{ if } n \neq 0 \qquad \text{(A.136)}
\end{aligned}$$

$$\int_0^\infty x\, e^{-x}\, S_{\frac{1}{2}}^{(n)}(x)\, dx = -\frac{1}{2}\, \frac{\Gamma(n-\frac{1}{2})}{\Gamma(\frac{1}{2})\, \Gamma(n+\frac{3}{2})\, \Gamma(n+1)} \qquad (A.137)$$

$$\int_0^\infty x^{q+1}\, e^{-x}\, S_{2q+\frac{1}{2}}^{(n)}(x)\, dx = \frac{\Gamma(q+2)\, \Gamma(q+n-\frac{1}{2})}{\Gamma(q-\frac{1}{2})\, \Gamma(n+1)\, \Gamma(2q+n+\frac{3}{2})} \;. \qquad (A.138)$$

It follows from equation (A.136) and from the fact that $B_{10} = 0$, (see equation (2.41), Part II) that the second summation of equation (A.135) vanishes. Hence when equations (A.137) and (A.138) are substituted into the expression (A.135) the integral I_2 is found to be an infinite series containing all of the Burnett Coefficients which are of the forms A_n and $B_{2q,n}$.

The only difference between the integrals $I_2^{(2)}$ and $I_2^{(1)}$, - as can be seen from equations (5.45) and (A.130) -, is that for $I_2^{(2)}$ the limits of integration over the variable $\mu = \cos\theta$ are -1 and 0, instead of 0 and +1 for $I_2^{(1)}$. Since in equation (A.132) $P_1(\mu) = \mu$ is an odd function of μ, and since also the integral of (A.133) vanishes unless $\{P_k(\mu) \cdot P_1(\mu)\}$ is an odd function of μ, it follows that

$$I_2^{(2)} = -I_2^{(1)} \qquad (A.139)$$

Consequently, from equations (A.98) one obtains

$$\mathcal{A}_2 = 0\, ; \quad \mathcal{B}_2 = I_2^{(1)} \;. \qquad (A.140)$$

The value of $\mathcal{B}_2$ which is thus obtained from equations (A.135) to (A.138) is given by equation (A.106) of Section 1 of this Appendix.

(b) The integrals $I_7^{(1)}$ and $I_7^{(2)}$

It follows from equations (A.97) and (A.4) that the integral $I_7^{(1)}$

involves the function

$$M_7 = Y_2^{(1)} = -C^2 \, P_2^{(1)}(\mu) \cos\phi$$

The integral which is to be evaluated is, therefore,

$$I_7^{(1)} = -\int_{-\pi}^{\pi} d\phi \int_0^1 d\mu \int_0^\infty dC \cdot f(C,\mu,\phi)\cdot C^4 \cdot P_2^{(1)}(\mu)\cdot \cos\phi \, . \qquad \text{(A.141)}$$

When the distribution function $f(C,\mu,\phi)$ is substituted into the above from equation (2.39) and the integration over ϕ is performed, all terms of f will integrate to zero except those containing the factor $\left\{B_{kn}^{(1)} \cos\phi \, P_k^{(1)}(\mu)\right\}$ as a consequence of the orthogonality properties of the circular functions $\sin(\rho\phi)$ and $\cos(\rho\phi)$. The integral $I_7^{(1)}$ is then reduced to

$$I_7^{(1)} = 2\pi \left(\frac{hm}{\pi}\right)^{3/2} \int_0^\infty C^4 dC \, e^{-hmC^2} \Big[\sum_{K=1}^{\infty} (hmC^2)^{\frac{K}{2}} \cdot$$

$$\cdot \left\{ \sum_{n=0}^{\infty} n!\,\Gamma(K+n+\tfrac{3}{2})\cdot S_{K+\frac{1}{2}}^{(n)}(hmC^2)\cdot \frac{(K-1)!}{(K+1)!}\cdot B_{Kn}^{(1)} \int_0^1 P_K^{(1)}(\mu)\,P_2^{(1)}(\mu)\,d\mu \right\} \Big]$$

(A.142)

The integration of the product of two associated Legendre polynomials is performed with the aid of formulae (A.32) to (A.35) of Appendix II, Section 4. The result is thus found to be

$$\int_0^1 P_K^{(1)}(\mu)\,P_2^{(1)}(\mu)\,d\mu = \frac{6}{5} \qquad \text{for } k = 2$$

$$= 0 \qquad \text{for } k = 2q,\ q = 0,2,3,4,\ldots$$

$$= (-1)^{q+1}\cdot \frac{3}{2^{2q+1}} \cdot \frac{1}{[(2q+1)(2q+2)-6]} \, \frac{\Gamma(2q+3)}{\Gamma(q+1)\,\Gamma(q+2)} \, . \qquad \text{(A.143)}$$

for $k = 2q + 1$, $q = 0,1,2,3,\ldots$

Substitution of (A.143) into (A.142) with the change of variable defined by equation (A.134) results in

$$I_7^{(1)} = \frac{3}{\sqrt{\pi}} \cdot \frac{1}{hm} \Big[\frac{2}{5} \sum_{n=0}^{\infty} n! \, \Gamma(n+\tfrac{7}{2}) \cdot \frac{1}{6} \, B_{2,n}^{(1)} \int_0^{\infty} x^{5/2} e^{-x} S_{5/2}^{(n)}(x) \, dx$$

$$+ \sum_{q=0}^{\infty} \sum_{n=0}^{\infty} \Big\{ n! \, (-1)^{q+1} \frac{\Gamma(2q+n+\frac{5}{2}) \cdot \Gamma(2q+1)}{[(2q+1)(2q+2)-6] \cdot 2^{2q+1} \, \Gamma(q+1) \Gamma(q+2)} \cdot$$

$$\cdot B_{2q+1,n}^{(1)} \int_0^{\infty} x^{q+2} e^{-x} S_{2q+\frac{3}{2}}^{(n)}(x) \, dx \Big\} \Big] . \qquad (A.144)$$

The value of the two integrals involving the Sonine polynomials $S_m^{(n)}(x)$ is again obtained from formula (A.49), so that

$$\int_0^{\infty} x^{5/2} e^{-x} S_{\frac{5}{2}}^{(n)}(x) \, dx \qquad = 1 \text{ if } n = 0$$

$$= 0 \text{ if } n = 0 \qquad (A.145)$$

$$\int_0^{\infty} x^{q+2} e^{-x} S_{2q+\frac{3}{2}}^{(n)}(x) \, dx = \frac{\Gamma(q+3) \cdot \Gamma(q+n-\frac{1}{2})}{\Gamma(q-\frac{1}{2}) \cdot \Gamma(n+1) \Gamma(n+2q+\frac{5}{2})} . \qquad (A.146)$$

When the expressions (A.145), and (A.146) are substituted into equation (A.144), it appears that the integral $I_7^{(2)}$ is an infinite series of all of the associated Burnett coefficients which are of the form $B_{2,n}^{(1)}$ and $B_{2q+1,n}^{(1)}$.

The integral $I_7^{(2)}$ differs from $I_7^{(1)}$ only in the limits of integration of the variable μ. Thus, the integral analogous to equation (A.143) results in (see formulae A.30, A.32 and A.33)

$$\int_{-1}^{0} P_k^{(1)}(\mu) \, P_2^{(1)}(\mu) \, d\mu \quad = + \int_0^1 P_k^{(1)}(\mu) \, P_2^{(1)}(\mu) \, d\mu \qquad \text{for } k = 2$$

$$= - \int_0^1 P_k^{(1)}(\mu) \, P_2^{(1)}(\mu) \, d\mu \qquad \text{for } k \neq 2 \qquad (A.147)$$

Consequently, the first summation in equation (A.144) corresponds to the even component $\mathcal{A}_7$ and the second summation to the odd component $\mathcal{B}_7$ according to the definition of equation (A.98). The explicit values of $\mathcal{A}_7$ and $\mathcal{B}_7$ are given by equations (A.102) and (A.111), respectively.

3. The Values of the Burnett Coefficients Close to a Solid Wall

In the deduction of the boundary conditions one requires the values which certain of the Burnett coefficients take on close to a solid surface, that is at a point $(x, y, z, \to 0)$ in the coordinate system of figure (5.3a). The general expressions for the Burnett coefficients, equations (2.68) to (2.113), may then be simplified somewhat by the following considerations.

Consider a term of the form $\left(\frac{\mu}{p}\right)^2 \frac{\partial u}{\partial x}$. This may be written as

$$\left(\frac{\mu}{p}\right)^2 \frac{\partial}{\partial x}\left[(u-u_0)+u_0\right] = \left(\frac{\mu}{p}\right)^2 \frac{\partial}{\partial x}(u-u_0)$$

Since the differentiation is in a direction parallel to the wall, one may use for $(u - u_0)$ the first approximation, ${}_1(u - u_0)$, given by equation (5.58), so that

$$\left(\frac{\mu}{p}\right)^2 \frac{\partial u}{\partial x} \doteq \left(\frac{\mu}{p}\right)^3 \frac{\partial}{\partial x}\left[\frac{\sqrt{\pi}}{2}\left(\frac{2-\sigma}{\sigma}\right)\frac{1}{\sqrt{hm}}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) - \frac{3}{8}\,\frac{1}{h^2 m}\,\frac{\partial h}{\partial x}\right] \doteq 0 \qquad \text{(A.148)}$$

because in the present approximation the highest order of terms retained is $\left(\frac{\mu}{p}\right)^2$. By the analogous argument for the v-component of the velocity, and differentiation in the y-direction one obtains,

$$\left(\frac{\mu}{p}\right)^2 \left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}\right) \doteq 0 \qquad (z = 0) \quad \text{(A.149)}$$

The same result does not hold for differentiation in the z-direction, i.e. $\left(\frac{\mu}{p}\right)^2 \frac{\partial u}{\partial z}$, since equation (5.58) is valid only at $z = 0$.

Many of the Burnett coefficients contain terms of the form

$$\left(\frac{\mu}{p}\right)^2 \frac{D}{Dt} = \left(\frac{\mu}{p}\right)^2 \left[\frac{\partial}{\partial t} + \left\{(u-u_o)+u_o\right\}\frac{\partial}{\partial x} + \left\{(v-v_o)+v_o\right\}\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \right] \qquad (A.150)$$

It was whown, equation (5.41), that the normal velocity w is zero everywhere on the wall. Since $(u - u_o)$ is of order $\frac{\mu}{p}$, equation (A.150) then reduces to

$$\left(\frac{\mu}{p}\right)^2 \frac{D}{Dt} \doteq \left(\frac{\mu}{p}\right)^2 \left[\frac{\partial}{\partial t} + u_o\frac{\partial}{\partial x} + v_o\frac{\partial}{\partial y} \right] \qquad (z = 0) \qquad (A.151)$$

On a plane wall w vanishes everywhere, so that $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = 0$. On a curved surface, however, $\frac{\partial w}{\partial x} = \frac{u_\phi}{r} \neq 0$, where u_ϕ is the tangential velocity and r is the radius of curvature of the wall. In order to preserve the maximum generality inherent in this derivation of the boundary conditions, the derivatives $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ will be retained.

Another frequently occurring term is of the form $\left(\frac{\mu}{p}\right)^2 \frac{\partial w}{\partial z}$. Again it is sufficient to use the zeroth approximation for $\frac{\partial w}{\partial z}$. From the energy equation (3.10) of Part III one obtains, correct to terms of zero order

$${}_o\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \doteq \frac{3}{2}\,\frac{1}{h}\,\frac{Dh}{Dt} \qquad (A.152)$$

Taking into account equation (A.149), the above yields,

$$\left(\frac{\mu}{p}\right)^2 \frac{\partial w}{\partial z} \doteq \frac{3}{2}\left(\frac{\mu}{p}\right)^2 \frac{1}{h}\,\frac{Dh}{Dt} \qquad (z = 0). \qquad (A.153)$$

Equations (A.149), (A.151) and (A.153) have been used to reduce those Burnett coefficients which are required for the evaluation of equations (5.63) and (5.73) to the form given below. Only those coefficients which are affected by the above transformations are listed.

$$
\begin{aligned}
{}_2A_2 = \Big[& -\frac{3}{2}\left(\frac{\mu}{p}\right)^2 \frac{\partial h}{\partial z}\frac{Dw}{Dt} + \frac{27}{8}\left(\frac{\mu}{p}\right)^2 \cdot \frac{1}{h^3 m}\left\{\left(\frac{\partial h}{\partial x}\right)^2+\left(\frac{\partial h}{\partial y}\right)^2+\left(\frac{\partial h}{\partial z}\right)^2\right\} \\
& + \frac{3}{4}\,\frac{\mu}{p}\,\frac{1}{m\nu}\left\{\frac{\partial}{\partial x}\left(\frac{\mu\nu}{ph^2}\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\mu\nu}{ph^2}\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\mu\nu}{ph^2}\frac{\partial h}{\partial z}\right)\right\} \\
& + \left(\frac{\mu}{p}\right)^2\left(\frac{1}{h}\frac{Dh}{Dt}\right)^2 + \frac{1}{3}\left(\frac{\mu}{p}\right)^2\left\{\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2\right\}\Big]
\end{aligned}
$$

(A.154)

$$
\begin{aligned}
{}_2B_{11} = & -\frac{3}{2}\,\frac{\mu}{p}\,\frac{1}{h\sqrt{hm}}\frac{\partial h}{\partial z} + \left(\frac{\mu}{p}\right)^2\Big[\frac{9}{4}\nu\frac{D}{Dt}\left(\frac{1}{\nu h\sqrt{hm}}\frac{\partial h}{\partial z}\right) - \frac{12}{5}\,\frac{1}{h}\frac{Dh}{Dt}\frac{Dw}{Dt} \\
& - \frac{3}{5}h\left\{\frac{\partial}{\partial x}\left(\frac{1}{h\sqrt{hm}}\left[\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right]\right)+\frac{\partial}{\partial y}\left(\frac{1}{h\sqrt{hm}}\left[\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right]\right)\right\} - \frac{2}{5}h\frac{\partial}{\partial z}\left(\frac{1}{h\sqrt{hm}}\left[2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right]\right) \\
& + \frac{153}{20}\,\frac{1}{h\sqrt{hm}}\frac{\partial h}{\partial z}\frac{1}{h}\frac{Dh}{Dt} + \frac{27}{10}\,\frac{1}{h\sqrt{hm}}\left\{\frac{\partial h}{\partial x}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)+\frac{\partial h}{\partial y}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right\} \\
& + \frac{9}{4}\,\frac{1}{h\sqrt{hm}}\left(\frac{\partial h}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial h}{\partial y}\frac{\partial w}{\partial y}\right)\Big]
\end{aligned}
$$

(A.155)

$$
\begin{aligned}
{}_2B_{20} = & -\frac{2}{3}\left(\frac{\mu}{p}\right)\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right) \\
& + \Big[2\left(\frac{\mu}{p}\right)\frac{D}{Dt}\left(\frac{\mu}{p}\,\frac{1}{h}\frac{Dh}{Dt}\right) + 2\left(\frac{\mu}{p}\right)^2\left(\frac{1}{h}\frac{Dh}{Dt}\right)^2 \\
& + \frac{1}{3}\left(\frac{\mu}{p}\right)^2\left\{\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2\right\} - \left(\frac{\mu}{p}\right)^2\left\{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right\} \\
& - \left(\frac{\mu}{p}\right)\frac{h}{\nu m}\left\{2\frac{\partial}{\partial z}\left(\frac{\mu}{h^2}\frac{\partial h}{\partial z}\right)-\frac{\partial}{\partial x}\left(\frac{\mu}{h^2}\frac{\partial h}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{\mu}{h^2}\frac{\partial h}{\partial y}\right)\right\}\Big]
\end{aligned}
$$

(A.156)

$$
\begin{aligned}
{}_2B_{21} = \Big[& \frac{6}{7}\left(\frac{\mu}{p}\right)\frac{1}{m\nu}\left\{2\frac{\partial}{\partial z}\left(\frac{\mu}{h}\frac{\partial h}{\partial z}\right)-\frac{\partial}{\partial x}\left(\frac{\mu}{h}\frac{\partial h}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{\mu}{h}\frac{\partial h}{\partial y}\right)\right\} \\
& + \frac{12}{7}\left(\frac{\mu}{p}\right)^2\frac{1}{h}\frac{\partial h}{\partial z}\frac{Dw}{Dt} - \frac{40}{99}\left(\frac{\mu}{p}\right)^2\left(\frac{1}{h}\frac{Dh}{Dt}\right)^2 \\
& - \frac{20}{147}\left(\frac{\mu}{p}\right)^2\left\{\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2\right\}\Big]
\end{aligned}
$$

(A.157)

$$ {}_2B_{40} = .06486\left(\frac{\mu}{\rho}\right)^2\left[18\left(\frac{1}{h}\frac{Dh}{Dt}\right)^2 - 4\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2 - 4\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2\right] \tag{A.158} $$

$$ \begin{aligned} {}_2B^{(1)}_{11} = {} & -\frac{3}{2}\left(\frac{\mu}{\rho}\right)\frac{1}{h\sqrt{hm}}\frac{\partial h}{\partial x} + \left(\frac{\mu}{\rho}\right)^2\Big[\frac{9}{4}\gamma\frac{D}{Dt}\left(\frac{1}{\gamma h\sqrt{hm}}\frac{\partial h}{\partial x}\right) - \frac{6}{5}\sqrt{hm}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\frac{Dw}{Dt} \\ & + \frac{3}{5}h\frac{\partial}{\partial x}\left(\frac{1}{h^2\sqrt{hm}}\frac{Dh}{Dt}\right) - \frac{3}{5}h\frac{\partial}{\partial z}\left(\frac{1}{h\sqrt{hm}}\left[\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right]\right) - \frac{153}{40}\frac{1}{h^2\sqrt{hm}}\frac{\partial h}{\partial x}\frac{Dh}{Dt} \\ & + \frac{27}{10}\frac{1}{h\sqrt{hm}}\frac{\partial h}{\partial z}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) + \frac{9}{4}\frac{1}{h\sqrt{hm}}\frac{\partial h}{\partial z}\frac{\partial u}{\partial z}\Big] \end{aligned} \tag{A.159} $$

$$ {}_2B^{(1)}_{12} = \frac{9}{10}\left(\frac{\mu}{\rho}\right)^2\left[\frac{1}{h^2\sqrt{hm}}\frac{Dh}{Dt}\frac{\partial h}{\partial x} - \frac{\partial h}{\partial z}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right] \tag{A.160} $$

$$ \begin{aligned} {}_2B^{(1)}_{20} = {} & -2\left(\frac{\mu}{\rho}\right)\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) + \Big[2\frac{\mu}{\rho}\frac{D}{Dt}\left(\frac{\mu}{\rho}\left\{\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right\}\right) \\ & + 2\left(\frac{\mu}{\rho}\right)^2\frac{1}{h}\frac{Dh}{Dt}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) - 3\left(\frac{\mu}{\rho}\right)^2\left(\frac{\partial w}{\partial x}-\frac{\partial u}{\partial z}\right)\frac{1}{h}\frac{Dh}{Dt} \\ & - 3\left(\frac{\mu}{\rho}\right)\frac{h}{\gamma m}\left\{\frac{\partial}{\partial z}\left(\frac{\mu}{h^2}\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\mu}{h^2}\frac{\partial h}{\partial z}\right)\right\}\Big] \end{aligned} \tag{A.161} $$

$$ \begin{aligned} {}_2B^{(1)}_{30} = {} & \left(\frac{\mu}{\rho}\right)^2\Big[\frac{32}{15}\sqrt{hm}\frac{Dw}{Dt}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) - \frac{24}{5}\frac{1}{h^2\sqrt{hm}}\frac{\partial h}{\partial x}\frac{Dh}{Dt} \\ & - \frac{16}{5}\frac{1}{h\sqrt{hm}}\frac{\partial h}{\partial z}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) + \frac{8}{5}\frac{1}{\gamma}\frac{\partial}{\partial x}\left(\frac{\gamma}{\sqrt{hm}}\frac{1}{h}\frac{Dh}{Dt}\right) \\ & + \frac{16}{15}\frac{1}{\gamma}\frac{\partial}{\partial z}\left(\frac{\gamma}{\sqrt{hm}}\left[\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right]\right)\Big] \end{aligned} \tag{A.162} $$

$$ {}_2B^{(1)}_{31} = -0.600\left(\frac{\mu}{\rho}\right)^2\frac{1}{h\sqrt{hm}}\left[6\frac{1}{h}\frac{\partial h}{\partial x}\frac{Dh}{Dt} + 4\frac{\partial h}{\partial z}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]. \tag{A.163} $$

The corresponding expressions for the coefficients ${}_2C_{11}^{(1)}, \ldots, {}_2C_{31}^{(1)}$ are obtained by replacing x by y and u by v in equations (A.159) to (A.163), respectively.

4. Formulae for Expressing the Boundary Conditions in Terms of Temperature T and Density .

The equation of state of a perfect gas can be written in the two alternate forms

$$p = k\nu T \tag{A.164}$$

$$\frac{p}{\rho} = RT \tag{A.165}$$

where k is the Boltzmann constant, R the gas constant per unit mass, ν the number density and ρ the mass density of the gas. By definition

$$h = \frac{1}{2kT} \tag{A.166}$$

$$\rho = m\nu \tag{A.167}$$

where m is the mass of one molecule. From equations (A.164) to (A.167) one has

$$mh = \frac{1}{2RT} \tag{A.168}$$

Equations (A.167) and (A.168) are used to derive the formulae given below, which are in turn used in expressing the boundary conditions in terms of ρ and T.

$$\frac{1}{h^2 m}\frac{\partial h}{\partial x} = -2R\frac{\partial T}{\partial x} \tag{A.169}$$

$$\frac{1}{\gamma\sqrt{hm}}\ \frac{\partial}{\partial z}\left(\frac{\gamma}{\sqrt{hm}}\ \frac{\partial u}{\partial z}\right) = 2RT\ \frac{\partial^2 u}{\partial z^2} + 2RT\ \frac{\partial u}{\partial z}\ \frac{\partial}{\partial z}\left\{\log(\rho\sqrt{T})\right\} \tag{A.170}$$

$$\frac{p}{\mu}\ \frac{h}{\gamma m\sqrt{hm}}\ \frac{\partial}{\partial z}\left(\frac{\kappa}{p}\ \frac{\gamma}{h^3}\ \frac{\partial h}{\partial x}\right) = -(2R)^{3/2}\sqrt{T}\left\{\frac{\partial^2 T}{\partial z\,\partial x} + \frac{\partial T}{\partial x}\ \frac{\partial}{\partial z}(\log\mu)\right\} \tag{A.171}$$

$$\frac{p}{\mu}\ \frac{\partial}{\partial x}\left(\frac{\kappa}{p}\ \frac{1}{\sqrt{hm}}\right) = \sqrt{2RT}\left\{-\frac{1}{2T}\ \frac{\partial T}{\partial x} - \frac{1}{\rho}\ \frac{\partial \rho}{\partial x} + \frac{1}{\mu}\ \frac{\partial \mu}{\partial x}\right\} \tag{A.172}$$

$$\frac{1}{h^2 m}\ \frac{\partial^2 h}{\partial x^2} = -2R\left[\frac{\partial^2 T}{\partial x^2} - \frac{2}{T}\left(\frac{\partial T}{\partial x}\right)^2\right] \tag{A.173}$$

$$\frac{1}{\gamma h^2 m}\ \frac{\partial h}{\partial x}\ \frac{\partial \gamma}{\partial x} = -2R\cdot\frac{1}{\rho}\ \frac{\partial \rho}{\partial x}\ \frac{\partial T}{\partial x} \tag{A.174}$$

$$\gamma\,\frac{D}{Dt}\left(\frac{1}{\gamma h\sqrt{hm}}\ \frac{\partial h}{\partial z}\right) = -\sqrt{2RT}\cdot\frac{1}{T}\left\{\frac{D}{Dt}\left(\frac{\partial T}{\partial z}\right) - \frac{\partial T}{\partial z}\ \frac{D}{Dt}\left(\log\{\rho\sqrt{T}\}\right)\right\} \tag{A.175}$$

$$\frac{Dh}{Dt}\cdot\frac{\partial}{\partial z}\left(\frac{1}{h\sqrt{hm}}\right) = -\frac{3}{\sqrt{2}}\sqrt{RT}\cdot\frac{1}{T^2}\ \frac{DT}{Dt}\ \frac{\partial T}{\partial z}\ , \tag{A.176}$$

etc.

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Table 2.1

Values of the Constants $\mathfrak{M}^{(\rho_1=\rho_2)}_{\substack{k\,k_1\,k_2\\ r\,n_1\,n_2}}$ *

$_2B_{kr}$	k	r	$M^{(0,0)}_{\substack{kok\\ror}}$	$M^{(0)}_{\substack{k11\\r11}}$	$M^{(1)}_{\substack{k11\\r11}}$	$M^{(0)}_{\substack{k12\\r10}}$	$M^{(1)}_{\substack{k12\\r10}}$	$M^{(0)}_{\substack{k22\\r00}}$	$M^{(1)}_{\substack{k22\\r00}}$	$M^{(2)}_{\substack{k22\\r00}}$
$_2A_2$	0	2	$-\frac{2}{3}$	0	0	0	0	$-\frac{1}{30}$	$-\frac{1}{90}$	$-\frac{1}{360}$
$_2A_3$	0	3	-1	0	0	0	0	0	0	0
$_2B_{11}$	1	1	$-\frac{2}{3}$	0	0	0	0	0	0	0
$_2B_{12}$	1	2	-1	0	0	$\frac{1}{15}$	$\frac{1}{30}$	0	0	0
$_2B_{20}$	2	0	-1	0	0	0	0	0	0	0
$_2B_{21}$	2	1	$-\frac{7}{6}$	0	0	0	0	$\frac{1}{21}$	$\frac{1}{126}$	$-\frac{1}{252}$
$_2B_{22}$	2	2	-1.3423	.0526	-.0263	0	0	0	0	0

* The decimal values are the consequence of numerical integration. (See Appendix III, Section 3)

Table 2.1 (cont'd)

${}_2B_{kr}$	k	r	$M^{(0,0)}_{\substack{kok\\ror}}$	$M^{(0)}_{\substack{k11\\r11}}$	$M^{(1)}_{\substack{k11\\r11}}$	$M^{(0)}_{\substack{k12\\r10}}$	$M^{(1)}_{\substack{k12\\r10}}$	$M^{(0)}_{\substack{k22\\r00}}$	$M^{(1)}_{\substack{k22\\r00}}$	$M^{(2)}_{\substack{k22\\r00}}$
${}_2B_{30}$	3	0	$-\frac{3}{2}$	0	0	0	0	0	0	0
${}_2B_{31}$	3	1	-1.5704	0	0	.0577	-.0192	0	0	0
${}_2B_{40}$	4	0	-1.3734	0	0	0	0	.0326	-.00732	.00045
${}_2B_{41}$	4	1	not re- quired	0	0	0	0	0	0	0
${}_2B_{50}$	5	0		0	0	0	0	0	0	0
${}_2B_{60}$	6	0		0	0	0	0	0	0	0

Table 3.1

Molecular Model	Maxwell molecule	Rigid elastic spherical molecule
Exponent of force law, n.	5	∞
$s = \frac{T}{\mu}\frac{d\mu}{dT}$	1	$\frac{1}{2}$
K_1	$\frac{4}{3}(\frac{7}{2} - s)$	$1.014 \times \frac{4}{3}(\frac{7}{2} - s)$
K_2	2	1.014×2
K_3	3	0.86×3
K_4	0	0.681
K_5	$3s$	$0.806 \times 3s - 0.990$
K_6	8	0.928×8
θ_1	$\frac{15}{4}(\frac{7}{2} - s)$	
θ_2	$\frac{45}{8}$	
θ_3	-3	
θ_4	3	
θ_5	$\frac{3}{2}(5 - s)$	

Table 4.1

Medium	1	2	3
$\beta = D$		1.393 1.610	
A	-0.83720	-0.81644	-1.6611
B	-1.4369	-1.9804	-0.67867
C	2.5237	3.0322	3.3800
E	3.2000	3.5761	3.8333
b_1	0.9333	0.9830	1.1667
b_3	-0.0839	-0.2477	2.4253
b_5	3.0195	2.5156	22.434
α_2	0.7807	0.8570	0.6310
α_4	-4.2314	-5.7015	-4.9838
α_6	-2.8344	-6.9062	-2.5438

Table 4.2

Temperature °C	$R_o \times 10^{-6}$
0	1.321
20	1.248
40	1.186
60	1.130
80	1.081
100	1.037
150	0.947
200	0.874
250	0.815
300	0.766
350	0.725
400	0.689
450	0.658
500	0.630

Table 5.1

Values of the Molecular Fraction σ

(Reference 5.3)

Gas and Surface	σ
Air or CO_2 on machined brass	1.00
Air or CO_2 on old shellac	1.00
Air on oil	0.895
CO_2 on oil	0.92
Hydrogen on oil	0.925
Air on glass	0.89
Helium on oil	0.874
Air on fresh shellac	0.79

Table 5.2

Values of the Accomodation Coefficient α for Air

(Reference 5.13)

Surface Description	Accomodation Coefficient	
	Minimum	Maximum
Flat black lacquer on bronze	0.881	0.894
Bronze, polished	0.91	0.94
Bronze, machined	0.89	0.93
Bronze, etched	0.93	0.95
Cast iron, polished	0.87	0.93
Cast iron, machined	0.87	0.88
Cast iron, etched	0.89	0.96
Aluminum, polished	0.87	0.95
Aluminum, machined	0.95	0.97

Table (5.3)

$$a_1 = \sqrt{\frac{\pi}{2}}\left(\frac{2-\sigma}{\sigma}\right)$$

$$b_1 = -5.167$$

$$b_2 = 0.8749$$

$$c_1 = \frac{15}{8}\sqrt{\frac{\pi}{2}}\left(\frac{2-\alpha}{\alpha}\right) \quad ; \qquad c_1' = \left(\frac{2}{\gamma+1}\cdot\frac{\lambda}{\mu c_v}\right)\sqrt{\frac{\pi}{2}}\left(\frac{2-\alpha}{\alpha}\right)$$

$$e_1 = -\left\{0.31655 + \frac{\pi}{8}\left(\frac{2-\sigma}{\sigma}\right)^2 - \frac{\pi}{4}\left(\frac{2-\sigma}{\sigma}\right)\left(\frac{2-\alpha}{\alpha}\right)\right\}$$

$$e_2 = -\sqrt{\frac{\pi}{2}}\left\{\frac{1}{4}\left(\frac{2-\alpha}{\alpha}\right) + \frac{1}{2}\left(\frac{2-\sigma}{\sigma}\right)\right\}$$

$$e_3 = -\frac{1}{2}\sqrt{\frac{\pi}{2}}\left(\frac{2-\alpha}{\alpha}\right)$$

$$e_4 = -\sqrt{\frac{\pi}{2}}\left\{\frac{33}{8}\left(\frac{2-\alpha}{\alpha}\right) - \frac{1}{4}\left(\frac{2-\sigma}{\sigma}\right)\right\}$$

$$e_5 = -\frac{45}{16}\sqrt{\frac{\pi}{2}}\left(\frac{2-\alpha}{\alpha}\right)$$

$$e_6 = \frac{107}{56}$$

$$e_7 = -7.9888$$

$$e_8 = -5.4912$$

$$e_9 = -1.7183$$

$$e_{10} = \frac{159}{16}\sqrt{\frac{\pi}{2}}\left(\frac{2-\alpha}{\alpha}\right) .$$

Table 7.1

Boundary Condition Derivatives for Axial Symmetry

Cartesian Derivative	Equivalent Radial Derivative at Convex Surface (r = a)	Concave Surface (r = b)
$\frac{\partial T}{\partial x}$	0	0
$\frac{\partial T}{\partial z}$	$\frac{dT}{dr}$	$-\frac{dT}{dr}$
$\frac{\partial^2 T}{\partial x^2}$	$\frac{1}{r}\frac{dT}{dr}$	$\frac{1}{r}\frac{dT}{dr}$
$\frac{\partial^2 T}{\partial z^2}$	$\frac{d^2 T}{dr^2}$	$\frac{d^2 T}{dr^2}$
$\frac{\partial^2 T}{\partial x \partial z}$	0	0
$\frac{\partial u}{\partial x}$	0	0
$\frac{\partial u}{\partial z}$	$\frac{du_\phi}{dr}$	$\frac{du_\phi}{dr}$
$\frac{\partial^2 u}{\partial x^2}$	$\frac{d}{dr}\left(\frac{u_\phi}{r}\right)$	$-\frac{d}{dr}\left(\frac{u_\phi}{r}\right)$
$\frac{\partial^2 u}{\partial z^2}$	$\frac{d^2 u_\phi}{dr^2}$	$-\frac{d^2 u_\phi}{dr^2}$
$\frac{\partial^2 u}{\partial x \partial z}$	0	0
$\frac{\partial w}{\partial x}$	$-\frac{u_\phi}{r}$	$-\frac{u_\phi}{r}$
$\frac{\partial w}{\partial z}$	0	0
$\frac{\partial^2 w}{\partial x^2}$	0	0
$\frac{\partial^2 w}{\partial z^2}$	0	0
$\frac{\partial^2 w}{\partial x \partial z}$	$-\frac{d}{dr}\left(\frac{u_\phi}{r}\right)$	$\frac{d}{dr}\left(\frac{u_\phi}{r}\right)$

Table 7.1 (continued)

Cartesian Derivative	Equivalent Radial Derivative at	
	Convex Surface (r = a)	Concave Surface (r = b)
u	u_ϕ	$-u_\phi$
$\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)$	$r\frac{d}{dr}\left(\frac{u_\phi}{r}\right)$	$r\frac{d}{dr}\left(\frac{u_\phi}{r}\right)$
$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)$	$\frac{d}{dr}\left\{r\frac{d}{dr}\left(\frac{u_\phi}{r}\right)\right\}$	$-\frac{d}{dr}\left\{r\frac{d}{dr}\left(\frac{u_\phi}{r}\right)\right\}$
$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)$	0	0
$\frac{DT}{Dt}$	0	0
$\frac{D^2T}{Dt^2}$	$U^2\frac{1}{r}\frac{dT}{dr}$	0
$\frac{D}{Dt}\left(\frac{\partial T}{\partial x}\right)$	$U\frac{1}{r}\frac{dT}{dr}$	0
$\frac{D}{Dt}\left(\frac{\partial T}{\partial z}\right)$	0	0
$\frac{D}{Dt}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)$	0	0
$\frac{Dw}{Dt}$	0	0

Table 7.2

Results of Numerical Calculation

for $h^* = 1.0$, $M = 2.0$ $\left(\frac{M}{Re}\right) = .035$, $\frac{Tw_a}{Tw_b} = 1.0$

Quantity	Numerical Value	Computed from equation
${}_0\left(\frac{p_a}{p_b}\right)$	.3488	(7.53)
$\theta_1 A^*$	-.1626	(7.57)
$\theta_1 B^*$	-.1445	(7.58)
$\theta_1 T_a{}^*$	.1928	(7.59)
$\theta_1 T_b{}^*$	.7110	(7.60)
${}_{(1)}\left(\frac{p_a}{p_b}\right)$	.4858	(7.84)
$\theta_1\left(\frac{p_a}{p_b}\right)^*$	.3928	(7.54)
$\frac{\theta^2}{U}\, {}_0X_a$	.02038	(7.65)
$\frac{\theta^2}{U}\, {}_0X_b$	.00424	(7.66)
$\theta^2{}_2A^*$	.02161	(7.61)
$\theta^2{}_2B^*$	.02573	(7.62)

Table 7.2 (continued)

Quantity	Numerical Value	Computed from equation
$\frac{\theta h}{T_{w_a}}\,{}_1\left(\frac{dT}{dr}\right)_a$	-.08224	(7.71)
$\frac{\theta h}{T_{w_b}}\,{}_1\left(\frac{dT}{dr}\right)_b$	.05564	(7.72)
$\frac{\theta^2}{T_{w_a}}\,{}_0Z_a$	.04749	(7.67)
$\frac{\theta^2}{T_{w_b}}\,{}_0Z_b$	.00355	(7.68)
$\theta^2\,{}_2T_a^*$	-.1089	(7.63)
$\theta^2\,{}_2T_b^*$	.00625	(7.64)
${}_{(2)}\left(\frac{T_a}{T_{w_a}}\right)$	1.0839	(7.87)
${}_{(2)}\left(\frac{\rho_i}{\rho_a}\right)$	1.876	(7.90)
C_{Fa}	.08224	(7.75)
C_{Fb}	.02056	(7.76)

Table A.2

S	k	K	$(\pi - \chi)$	$\sin \chi$	$\sin^2 \chi$	$\sin^4 \chi$
0	.707107	1.85407	0	0	0	0
.01	.704603	1.85116	.311323	.306318	.093831	.008804
.02	.702090	1.84820	.439556	.425538	.181083	.032791
.03	.699542	1.84516	.537425	.511926	.262068	.068680
.04	.697039	1.84228	.619540	.580661	.337167	.113682
.05	.694503	1.83942	.691519	.637708	.406671	.165381
.06	.691958	1.83645	.756191	.686156	.470810	.221662
.07	.689407	1.83364	.815398	.727998	.529981	.280880
.08	.686848	1.83085	.870210	.764464	.584405	.341529
.09	.684213	1.82794	.921337	.796411	.634270	.402298
.100	.681713	1.82532	.969548	.824630	.680015	.462420
.125	.675262	1.81841	1.079128	.881546	.777123	.603920
.150	.668778	1.81177	1.176810	.923386	.852642	.726998
.175	.662267	1.80524	1.265249	.953683	.909511	.827210
.200	.655734	1.79892	1.346323	.974911	.950451	.903357
.225	.649183	1.79269	1.421198	.988831	.977787	.956061
.250	.652620	1.78656	1.490802	.996802	.993614	.987269
.275	.636051	1.78072	1.555982	.999891	.999782	.999564
.300	.629479	1.77490	1.617065	.998930	.997861	.995727
.350	.616345	1.76382	1.729045	.987510	.975176	.950968
.400	.603256	1.75326	1.829330	.966766	.934636	.873544
.450	.590671	1.74360	1.920234	.939564	.882781	.779302
.500	.577351	1.73387	2.002100	.908422	.825231	.681006
.550	.564597	1.72499	2.077064	.874561	.764857	.585006
.600	.552012	1.71661	2.145605	.839297	.704419	.496206
.650	.539620	1.70879	2.208560	.803430	.645500	.416670
.700	.527443	1.70133	2.266461	.767628	.589253	.347219
.750	.515455	1.69437	2.319765	.732392	.536398	.287723
.800	.503803	1.68781	2.369632	.698258	.487564	.237719
.900	.481200	1.67594	2.456030	.633109	4.00827	.160662
1.000	.459701	1.66539	2.530847	.573478	.328877	.108160
1.100	.439340	1.65611	2.599310	.516093	.266352	.070943
1.200	.420121	1.64759	2.651093	.471067	.221904	.049241
1.300	.402053	1.64081	2.699625	.427710	.182936	.033466
1.400	.385016	1.63440	2.741755	.389269	.151530	.022961
1.500	.369048	1.62889	2.778886	.354807	.125888	.015848
1.600	.354070	1.62385	2.811219	.324775	.105479	.011126

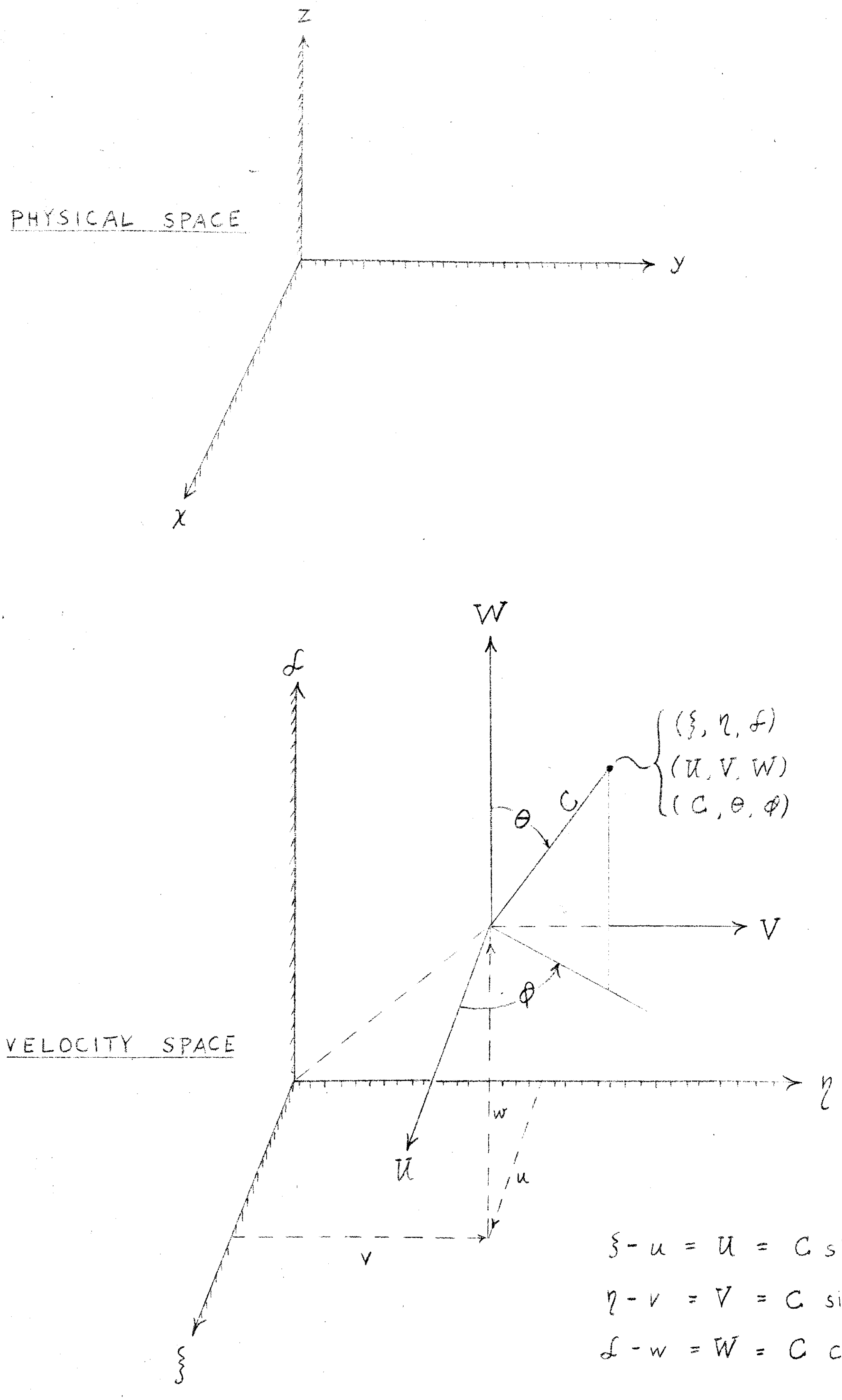
Table A.2 (Cont'd)

S	k	K	$(\pi - \chi)$	$\sin \chi$	$\sin^2 \chi$	$\sin^4 \chi$
1.800	.326853	1.61548	2.865070	.273012	.074536	.005556
2.000	.302906	1.60881	2.907448	.232012	.053830	.002898
2.200	.281788	1.60346	2.941275	.198981	.039593	.001568
2.400	.263107	1.59908	2.968523	.172207	.029655	.000879
2.600	.246518	1.59551	2.990816	.150203	.022560	.000509
2.800	.231724	1.59255	3.009216	.131991	.017422	.000304
3.000	.217441	1.58988	3.024178	.117146	.013723	.000188
3.200	.206568	1.58797	3.037399	.104006	.010817	.000117
3.400	.195815	1.58618	3.048295	.093163	.008679	.000075
3.600	.186069	1.58465	3.057608	.083886	.007036	.000050
4.000	.169102	1.58220	3.072582	.068956	.004755	.000022
4.400	.154866	1.58035	3.083965	.057596	.003317	.000011
4.800	.142771	1.57890	3.092765	.048809	.002382	.000006
5.200	.132382	1.57745	3.099118	.042462	.001803	.000003
5.600	.123367	1.57683	3.105292	.036293	.001317	.000002
6.000	.115480	1.57607	3.109819	.031769	.001009	.000001
6.400	.108524	1.57544	3.113548	.028041	.000786	.000000
6.800	.102345	1.57491	3.116652	.024938	.000622	.000000
7.200	.096822	1.57450	3.119340	.022251	.000495	.000000
7.600	.091859	1.57412	3.121559	.020033	.000401	.000000
8.000	.087373	1.57380	3.123480	.018112	.000328	.000000
8.400	.083301	1.57353	3.125247	.016445	.000270	.000000
8.800	.079586	1.57329	3.126587	.015005	.000225	.000000
9.200	.076187	1.57309	3.127866	.013727	.000188	.000000
9.600	.073065	1.57290	3.128961	.012632	.000160	.000000
10.000	.070189	1.57274	3.129948	.011645	.000136	.000000

TABLE A.3

Values of $J = 4\pi \int_0^\infty \left\{ \sin^2(\tfrac{1}{2}\chi) \cos^2(\tfrac{1}{2}\chi) \right\} \cdot \alpha \, d\alpha$

Computed by	Value of J
J. C. Maxwell	1.3682
Aichi and Tanakadate	1.3704
S. Chapmann	1.3700
Author	1.3696



$$\xi - u = U = C \sin\theta \cos\phi$$

$$\eta - v = V = C \sin\theta \sin\phi$$

$$\zeta - w = W = C \cos\theta$$

Fig. 2·1

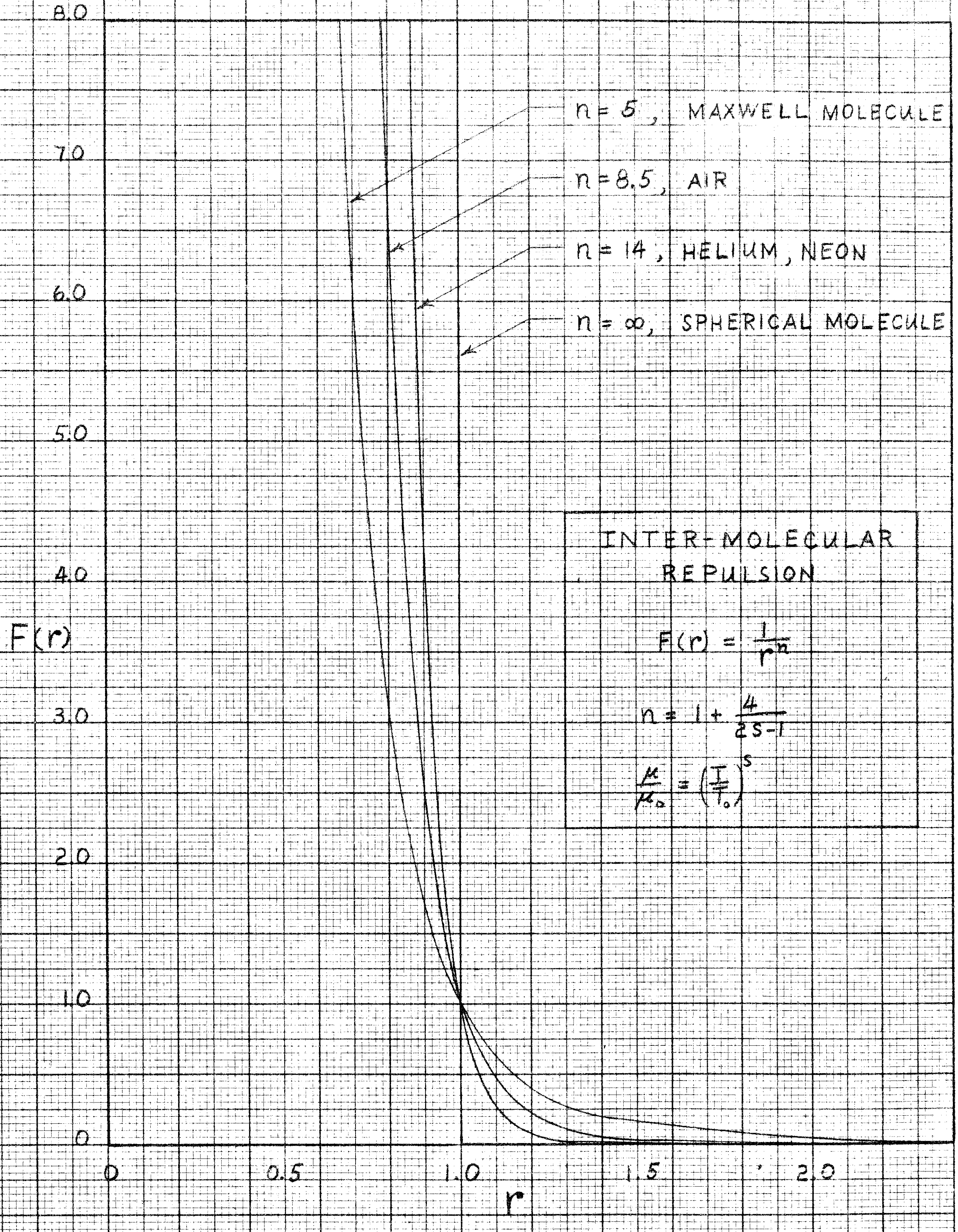

8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0
F(r)
0
0.5
1.0
1.5
2.0
r
n = 5 , MAXWELL MOLECULE
n = 8.5 , AIR
n = 14 , HELIUM, NEON
n = ∞, SPHERICAL MOLECULE
INTER-MOLECULAR REPULSION
$F(r) = \frac{1}{r^n}$
$n = 1 + \frac{4}{2s-1}$
$\frac{\mu}{\mu_0} = \left(\frac{T}{T_0}\right)^s$

Fig. 2.2

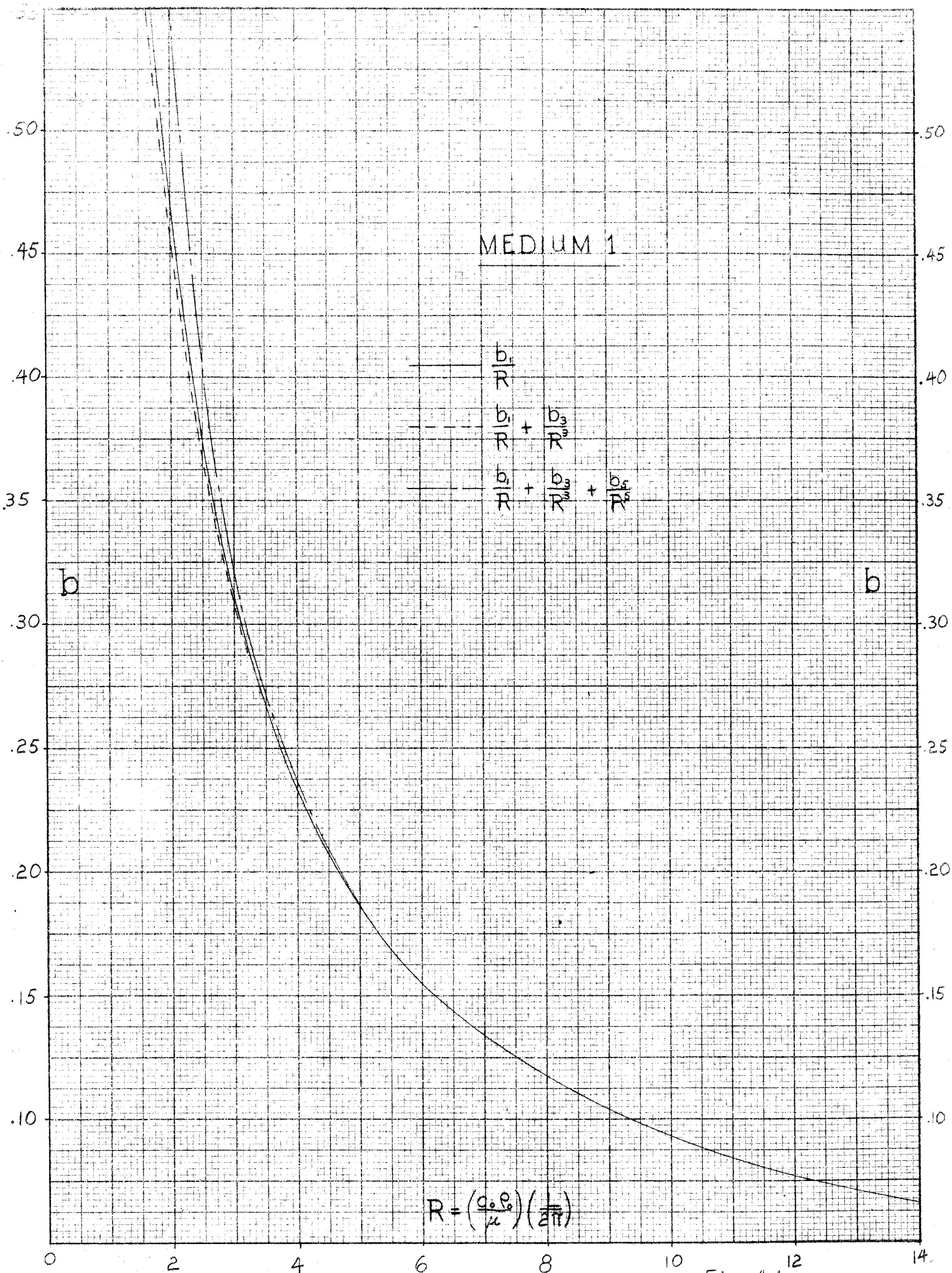

.55
.50
.45
.40
.35
.30
.25
.20
.15
.10
MEDIUM 1
$\frac{b_1}{R}$
$\frac{b_1}{R} + \frac{b_3}{R^3}$
$\frac{b_1}{R} + \frac{b_3}{R^3} + \frac{b_5}{R^5}$
b
b
$R = \left(\frac{c_0 \rho_0}{\mu}\right)\left(\frac{1}{2\pi}\right)$
0
2
4
6
8
10
12
14

Fig. 4.1

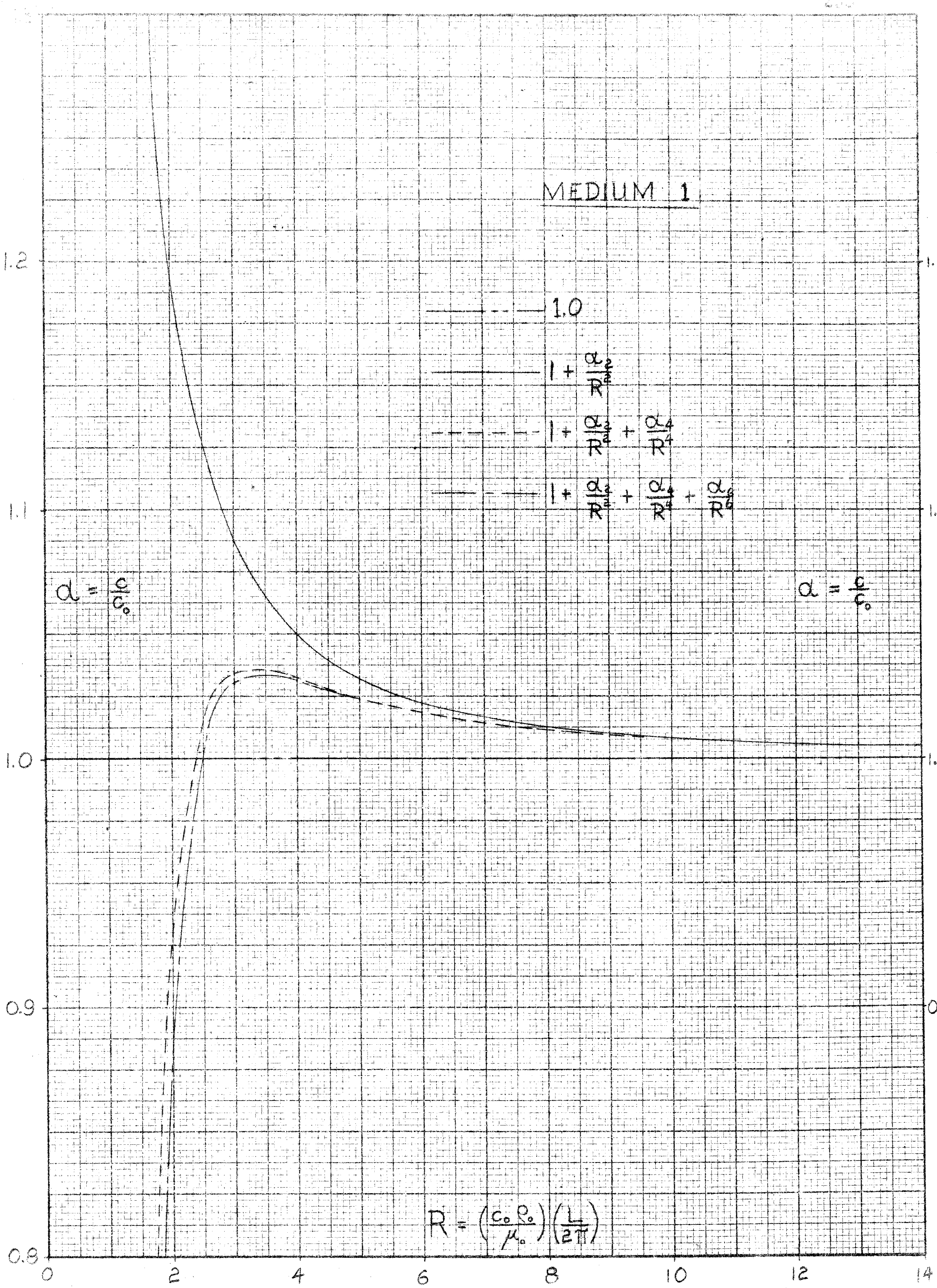

MEDIUM 1
1.0
$1 + \frac{\alpha_2}{R^2}$
$1 + \frac{\alpha_2}{R^2} + \frac{\alpha_4}{R^4}$
$1 + \frac{\alpha_2}{R^2} + \frac{\alpha_4}{R^4} + \frac{\alpha_6}{R^6}$
$\alpha = \frac{c}{c_0}$
$\alpha = \frac{c}{c_0}$
1.2
1.1
1.0
0.9
0.8
1.2
1.1
1.0
0.9
$R = \left(\frac{c_0 \rho_0}{\mu_0}\right)\left(\frac{L}{2\pi}\right)$
0
2
4
6
8
10
12
14

Fig. 4.2

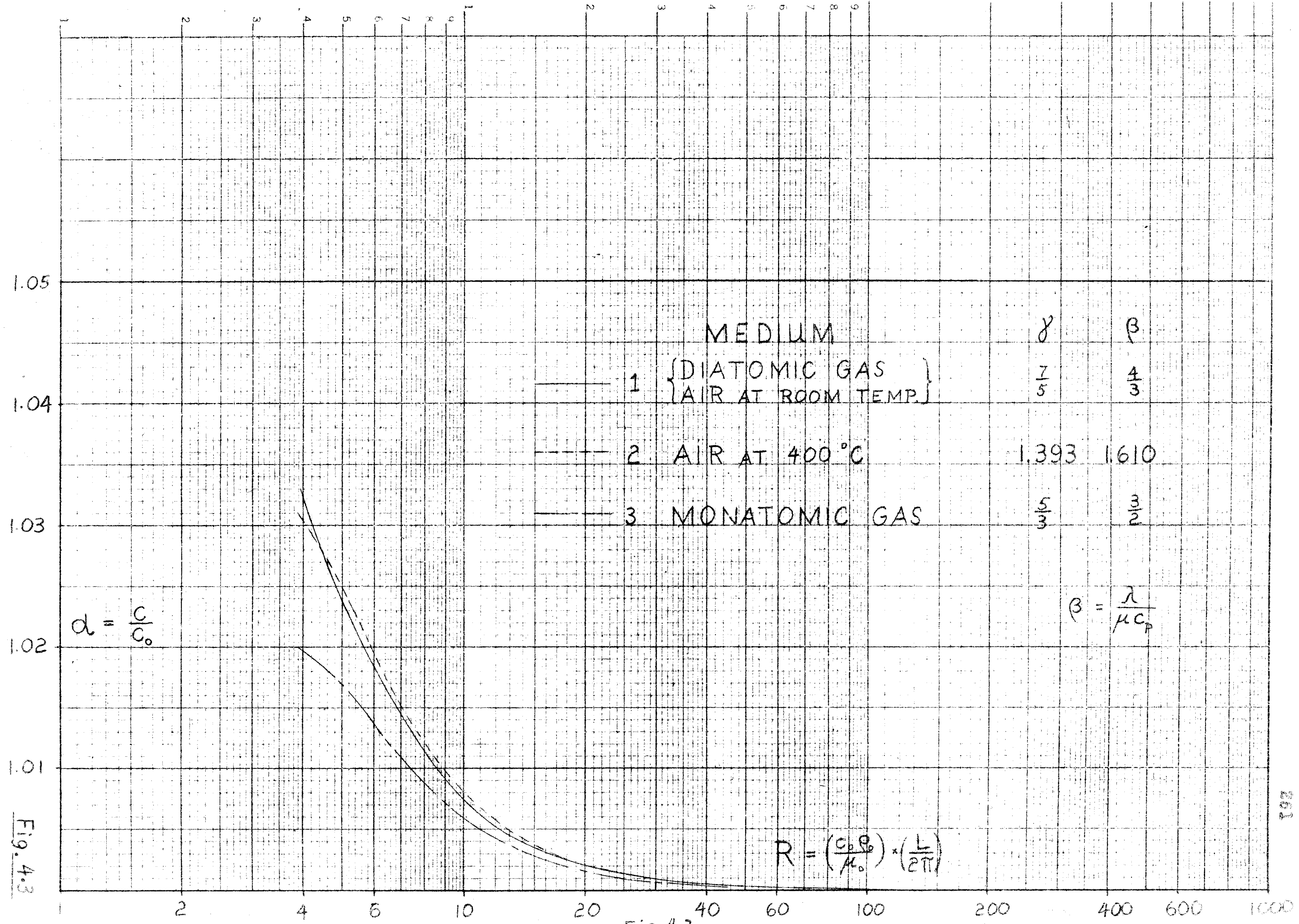

MEDIUM
γ
β
1 {DIATOMIC GAS / AIR AT ROOM TEMP.}
$\frac{7}{5}$
$\frac{4}{3}$
2 AIR AT 400 °C
1.393
1.610
3 MONATOMIC GAS
$\frac{5}{3}$
$\frac{3}{2}$
$\beta = \frac{\lambda}{\mu c_p}$
$\alpha = \frac{c}{c_0}$
$R = \left(\frac{c_0 \rho_0}{\mu_0}\right) \times \left(\frac{L}{2\pi}\right)$
1.05
1.04
1.03
1.02
1.01
1
2
4
6
10
20
40
60
100
200
400
600
1000
Fig. 4.3

Fig. 4.3

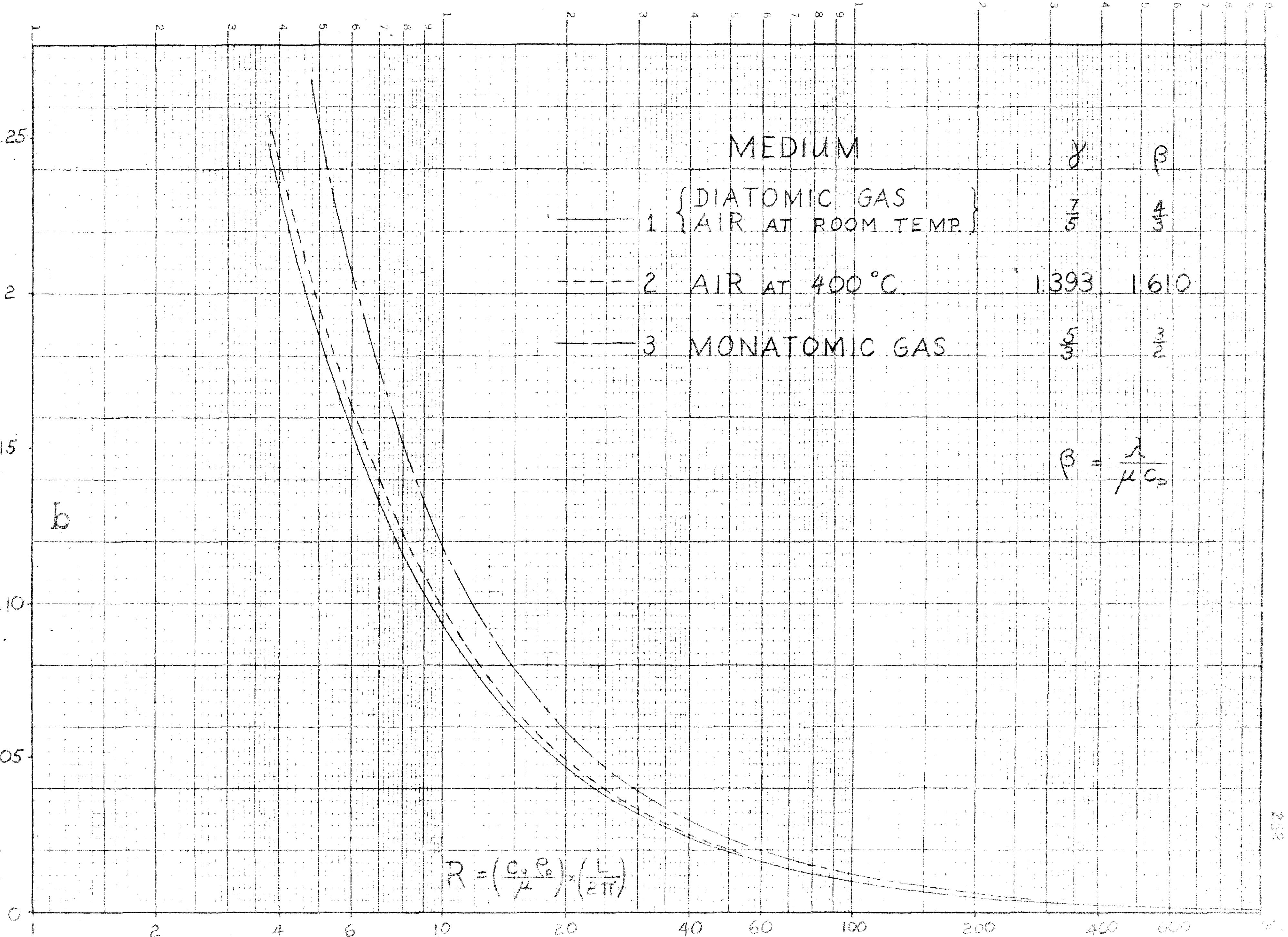

MEDIUM
γ
β
1 {DIATOMIC GAS / AIR AT ROOM TEMP.}
7/5
4/3
2 AIR AT 400 °C
1.393
1.610
3 MONATOMIC GAS
5/3
3/2
β = λ/(μ c_p)
b
R = (c₀ ρ₀/μ) × (L/2π)
0.25
0.2
0.15
0.10
0.05
0
1
2
4
6
10
20
40
60
100
200
400
600

Fig. 4.4

Fig 4.4

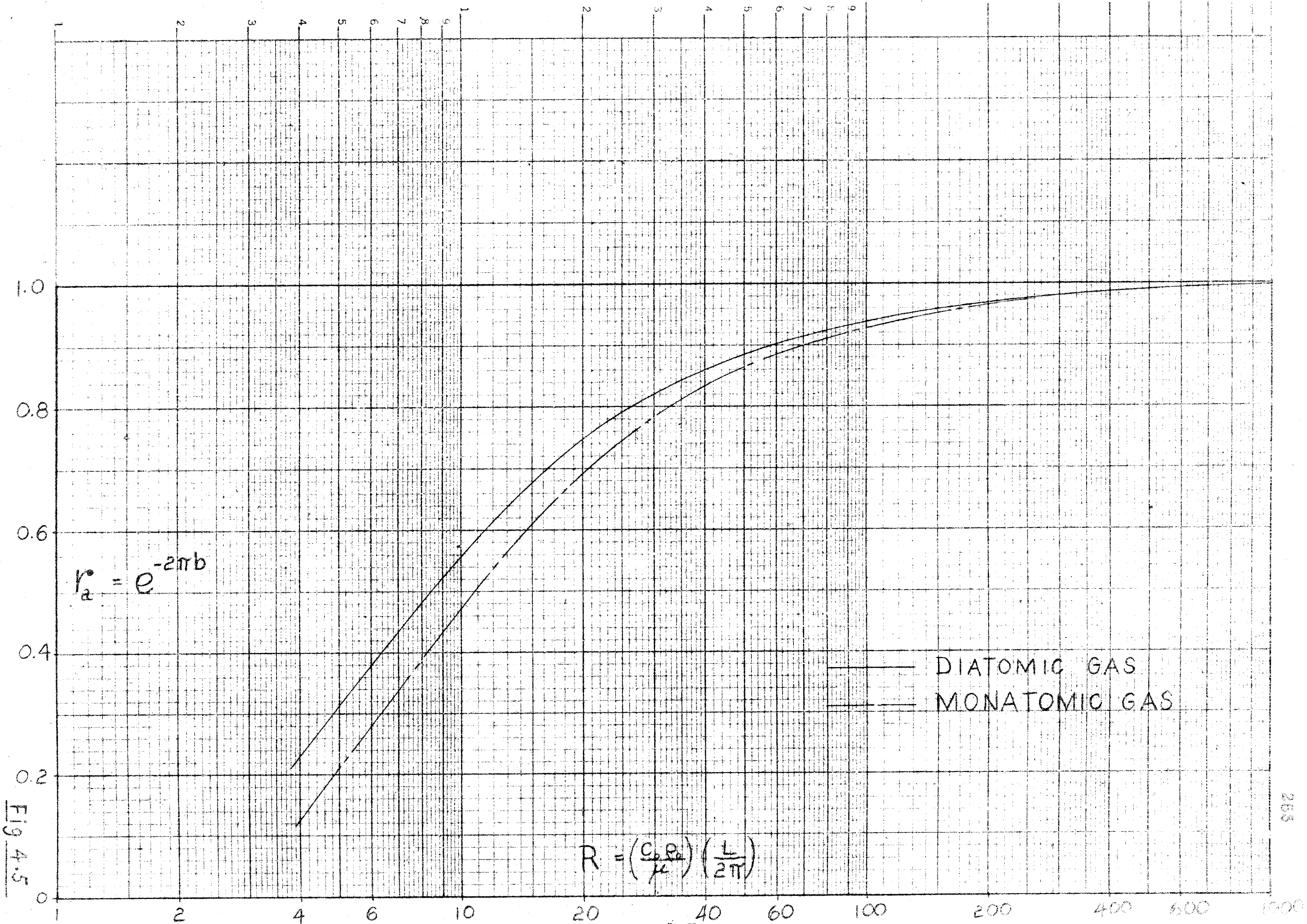

1.0
0.8
0.6
0.4
0.2
0
1
2
4
6
10
20
40
60
100
200
400
600
1000
$r_a^{\circ} = e^{-2\pi b}$
DIATOMIC GAS
MONATOMIC GAS
$R = \left(\frac{C_b \rho_a}{\mu}\right)\left(\frac{L}{2\pi}\right)$

Fig 4.5

Fig.4.5

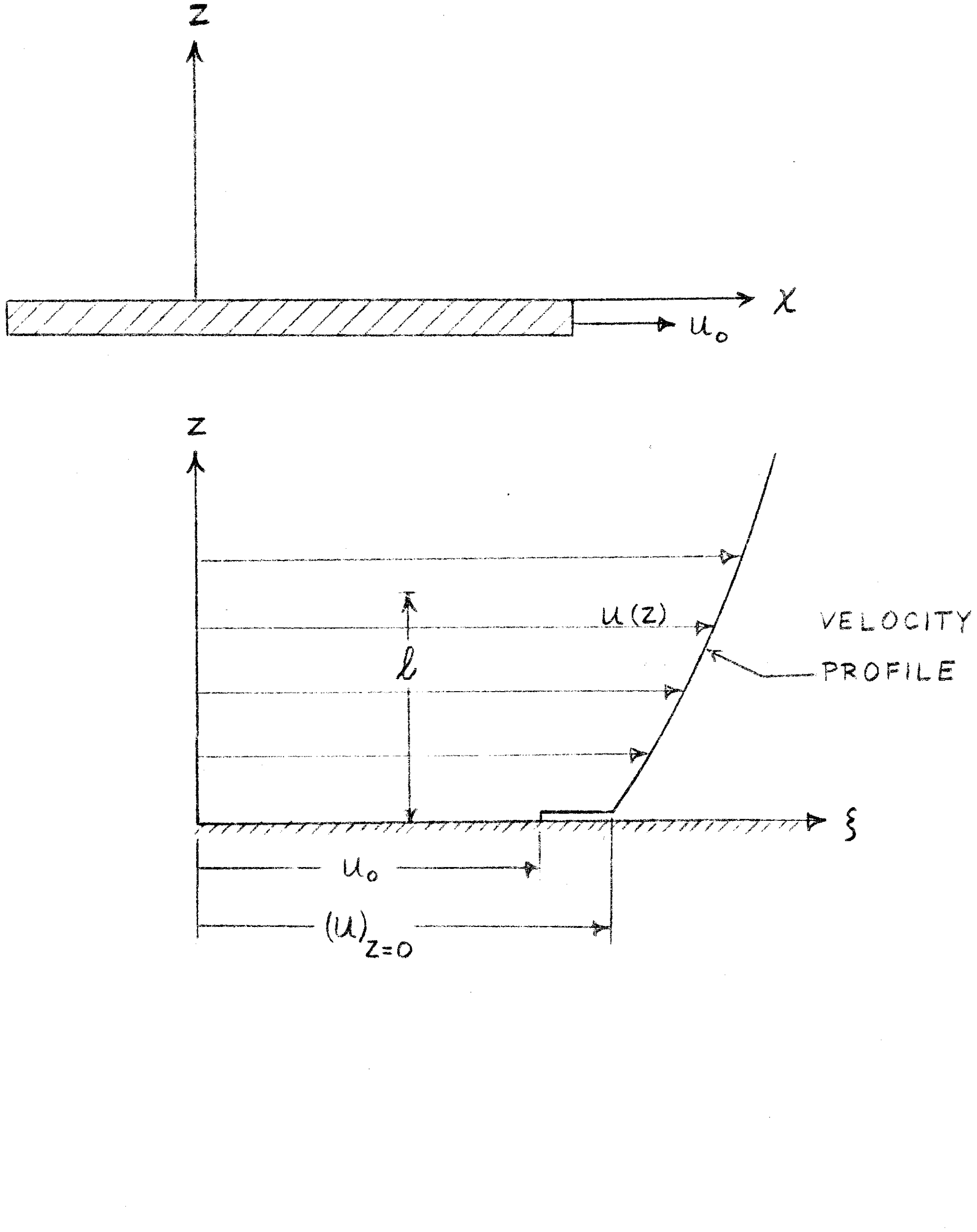
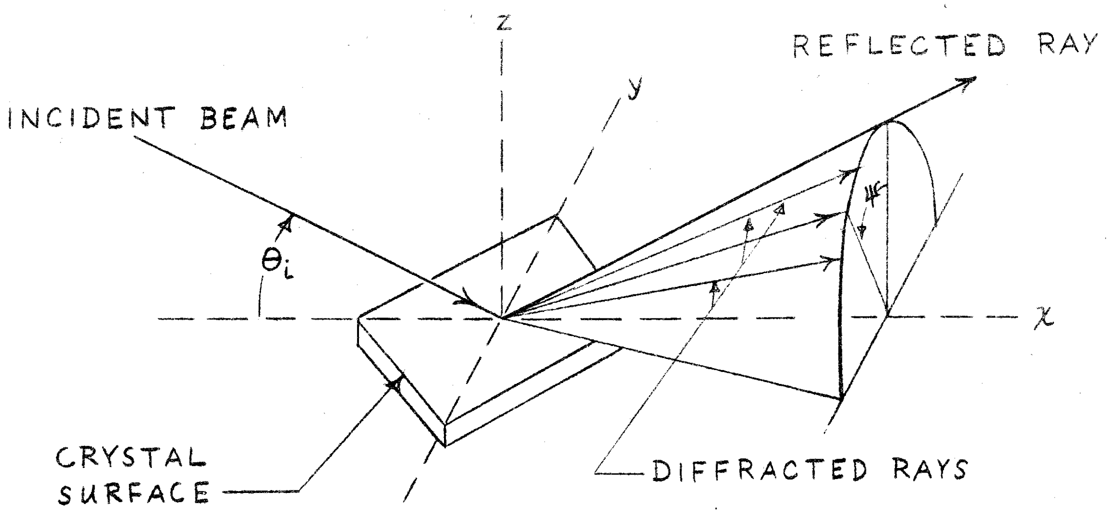


Fig. 5.1



I = INTENSITY OF DIFFRACTED BEAM AT ANGLE ψ

θ_i = ANGLE OF INCIDENCE

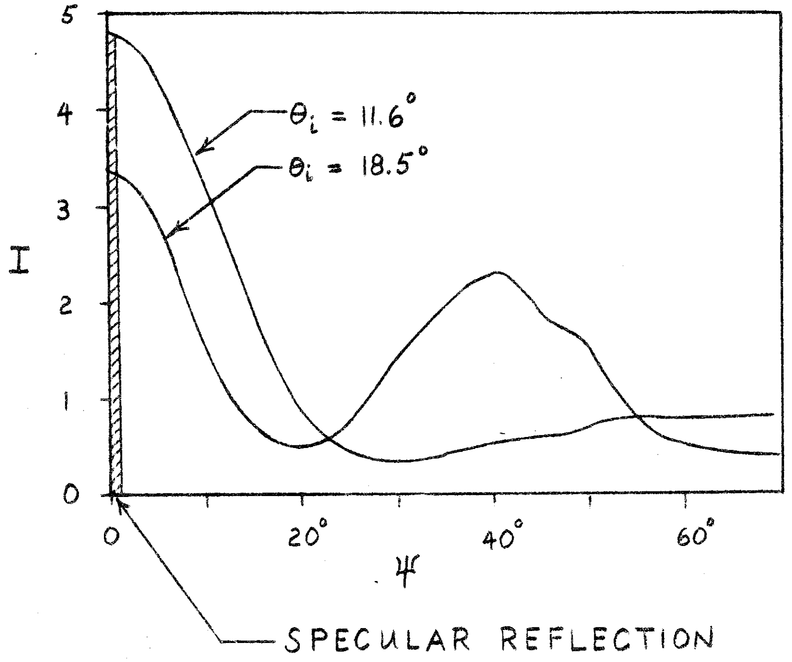


Fig. 5·2

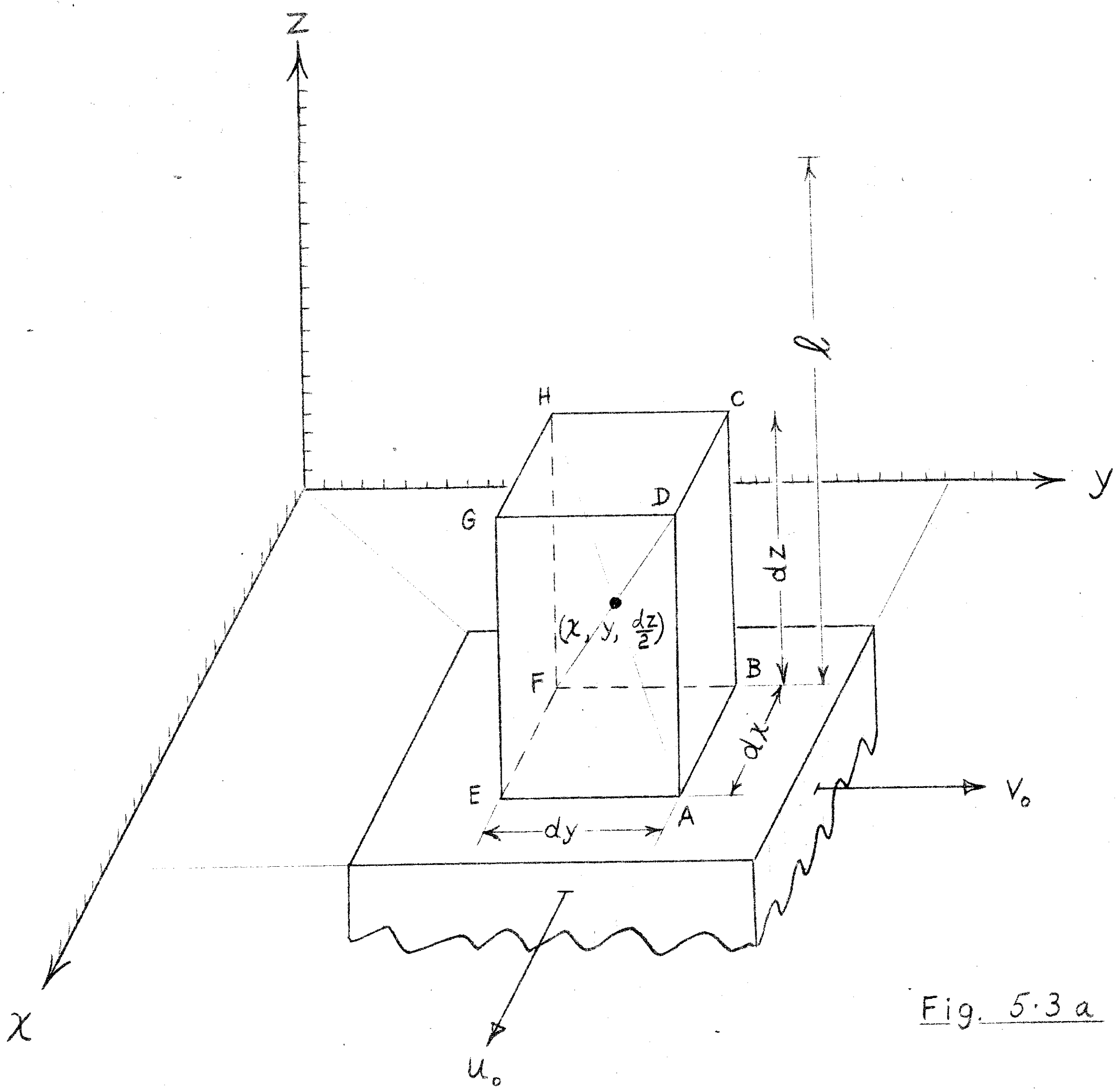


Fig. 5·3 a

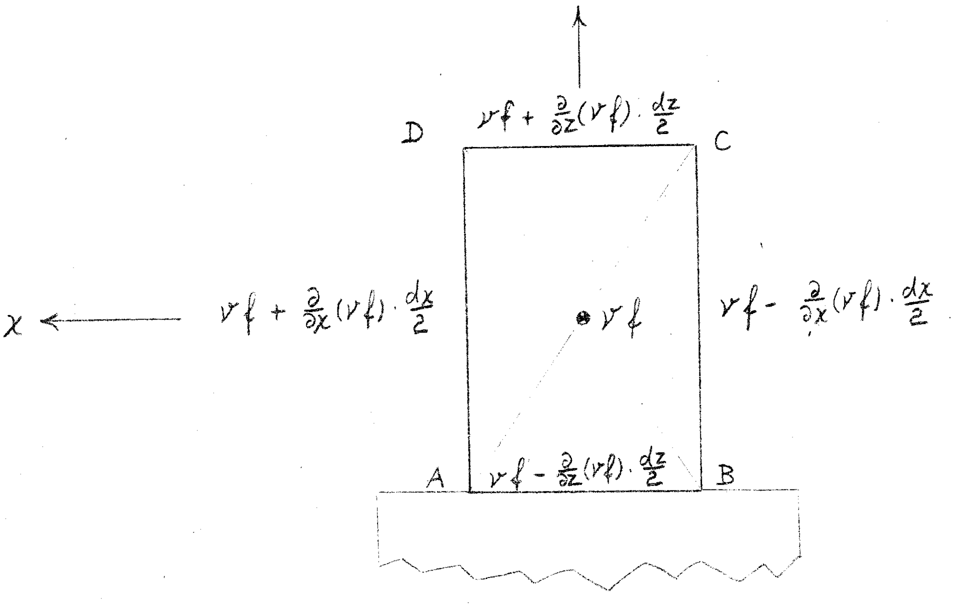


Fig. 5·3 b

Fig 5·3

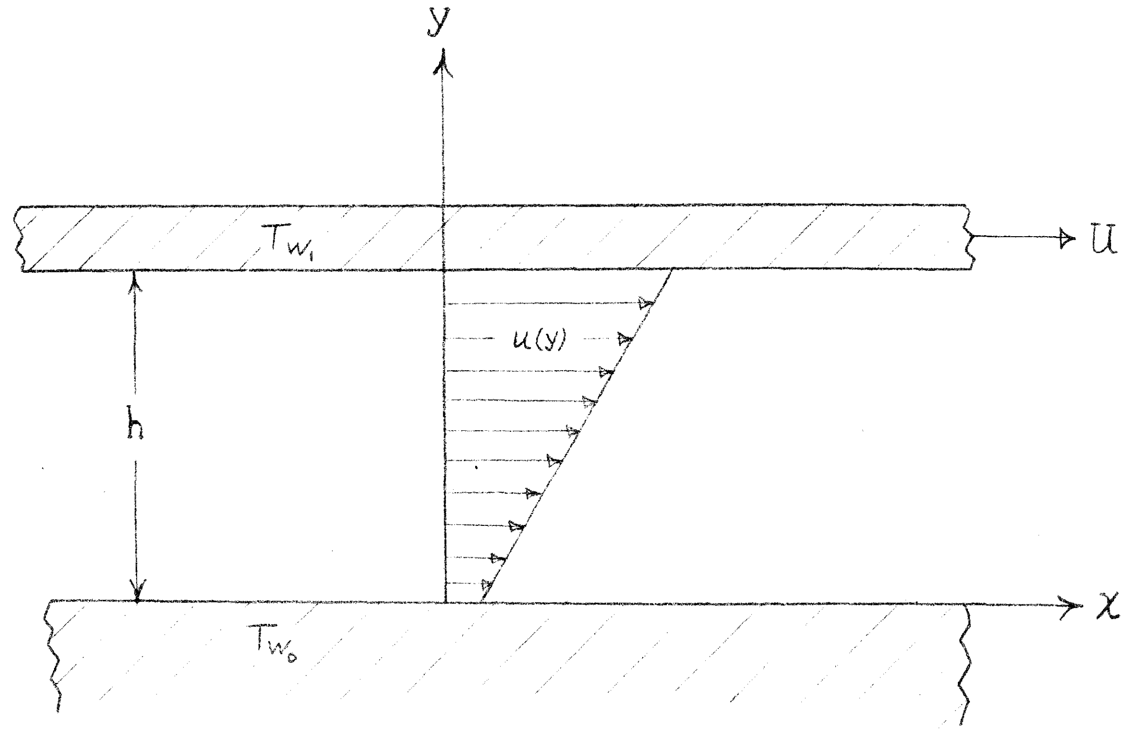


Fig. 6·1

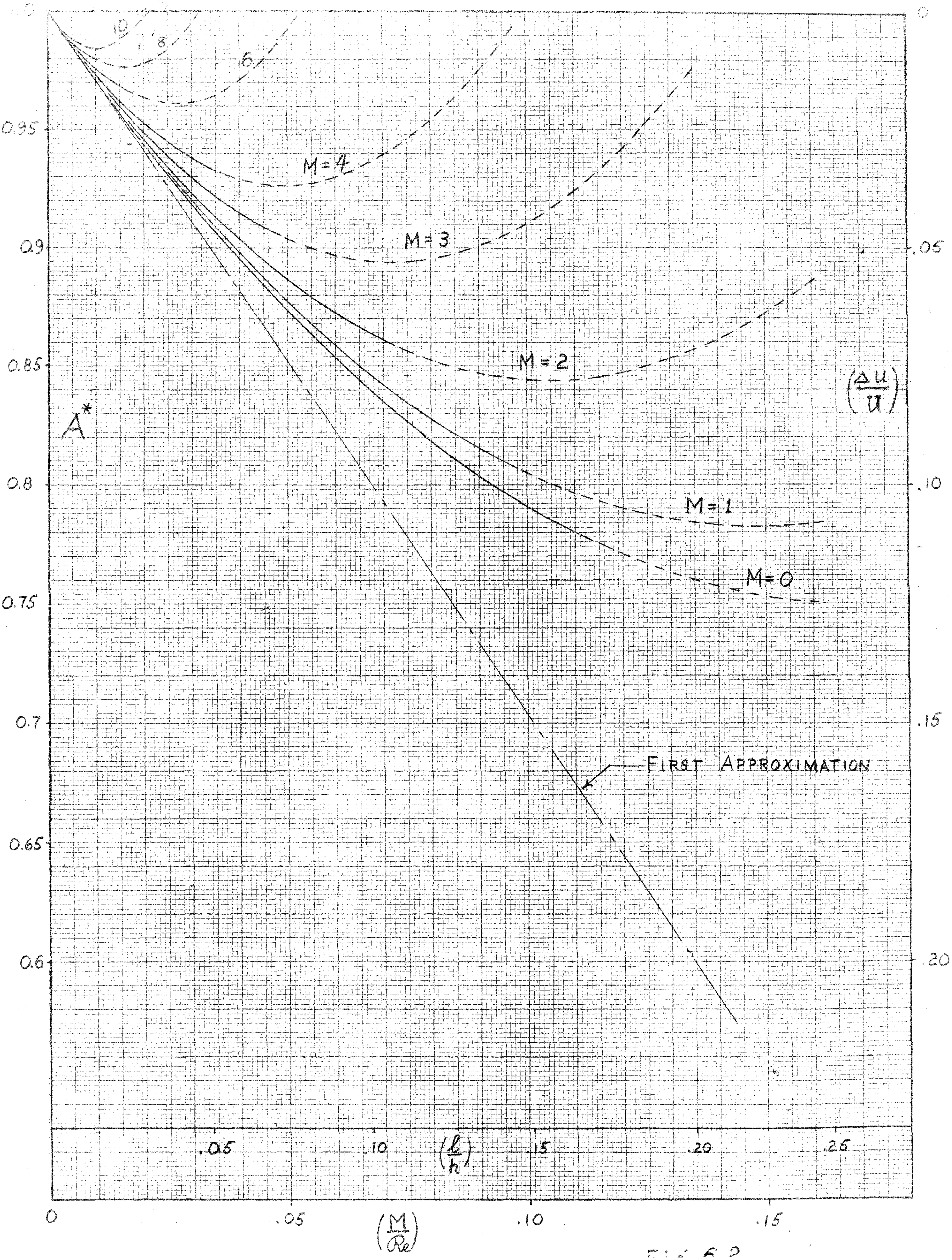

1.0
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
A^*
0
.05
.10
.15
.20
$\left(\frac{\Delta u}{U}\right)$
10
8
6
M=4
M=3
M=2
M=1
M=0
FIRST APPROXIMATION
.05
.10
$\left(\frac{l}{h}\right)$
.15
.20
.25
0
.05
.10
.15
$\left(\frac{M}{Re}\right)$

Fig. 6.2

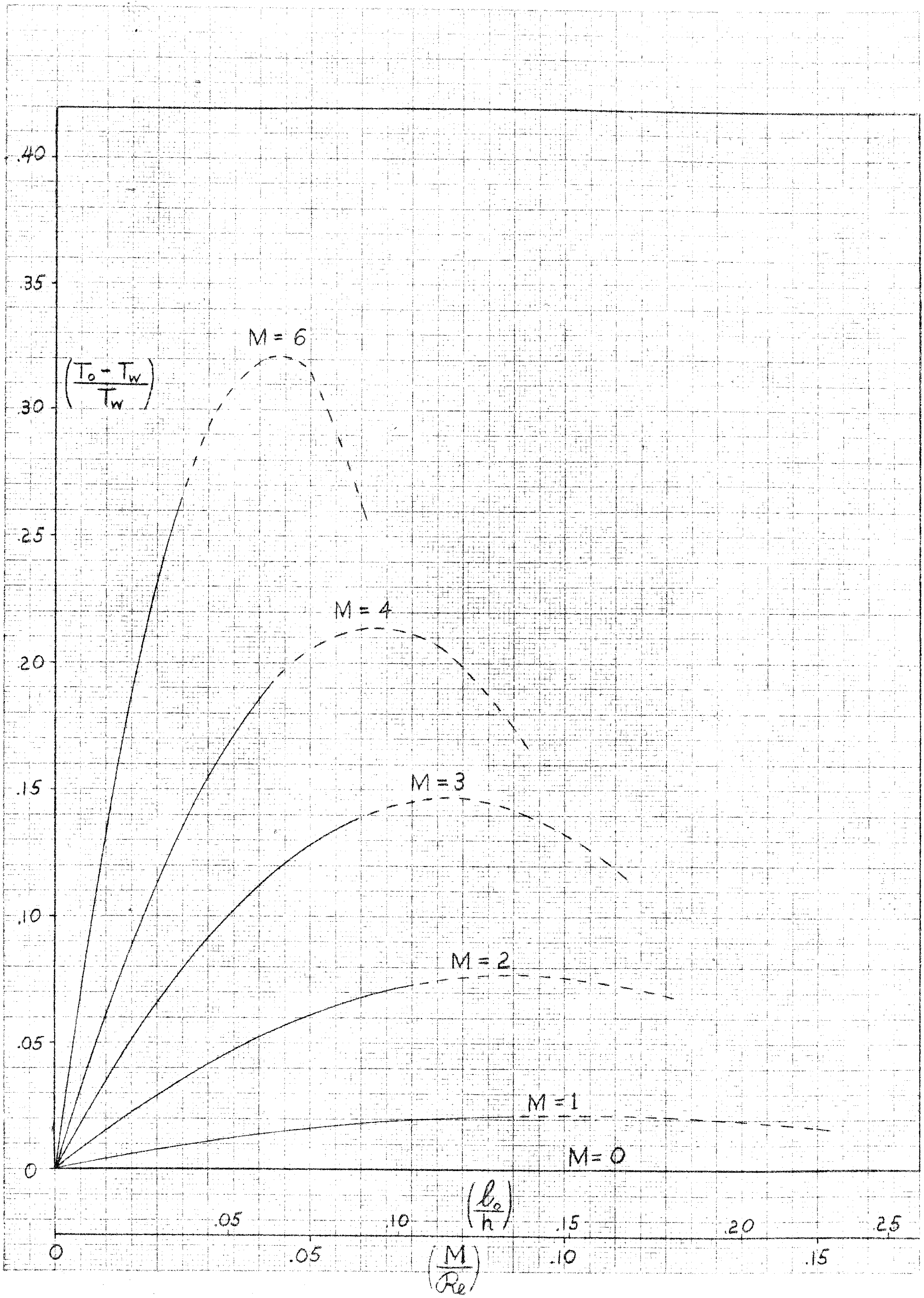

.40
.35
.30
.25
.20
.15
.10
.05
0
$\left(\frac{T_o - T_w}{T_w}\right)$
M = 6
M = 4
M = 3
M = 2
M = 1
M = 0
$\left(\frac{\ell_o}{h}\right)$
.05
.10
.15
.20
.25
0
.05
.10
.15
$\left(\frac{M}{Re}\right)$

Fig. 6·3

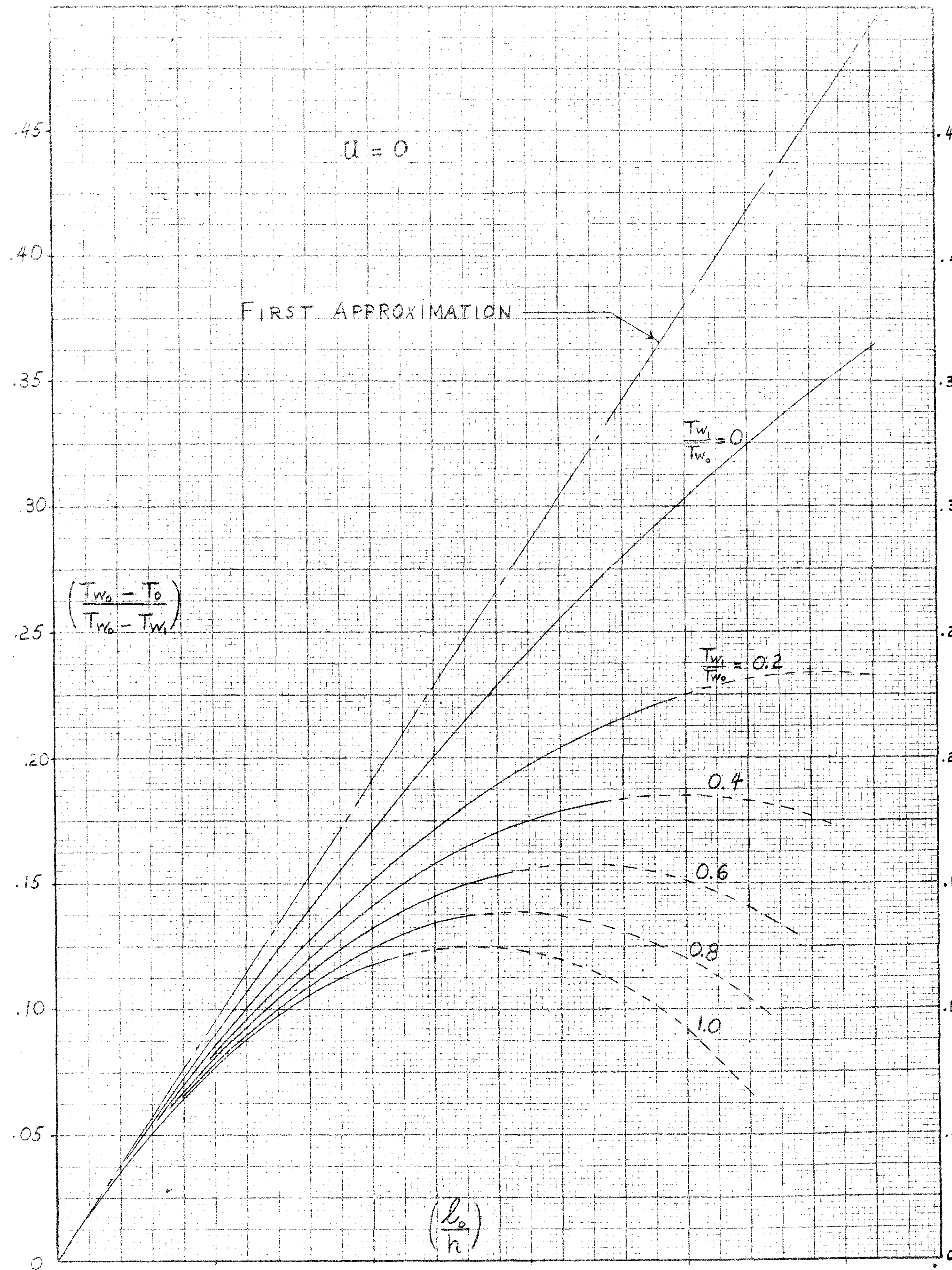

U = 0
FIRST APPROXIMATION
$\frac{T_{W_1}}{T_{W_0}} = 0$
$\frac{T_{W_1}}{T_{W_0}} = 0.2$
0.4
0.6
0.8
1.0
$\left(\frac{T_{W_0} - T_0}{T_{W_0} - T_{W_1}}\right)$
$\left(\frac{l_0}{h}\right)$
0
.05
.10
.15
.20
.25
.30
.35
.40
.45
.05
.10
.15
.20
.25

Fig. 6.4

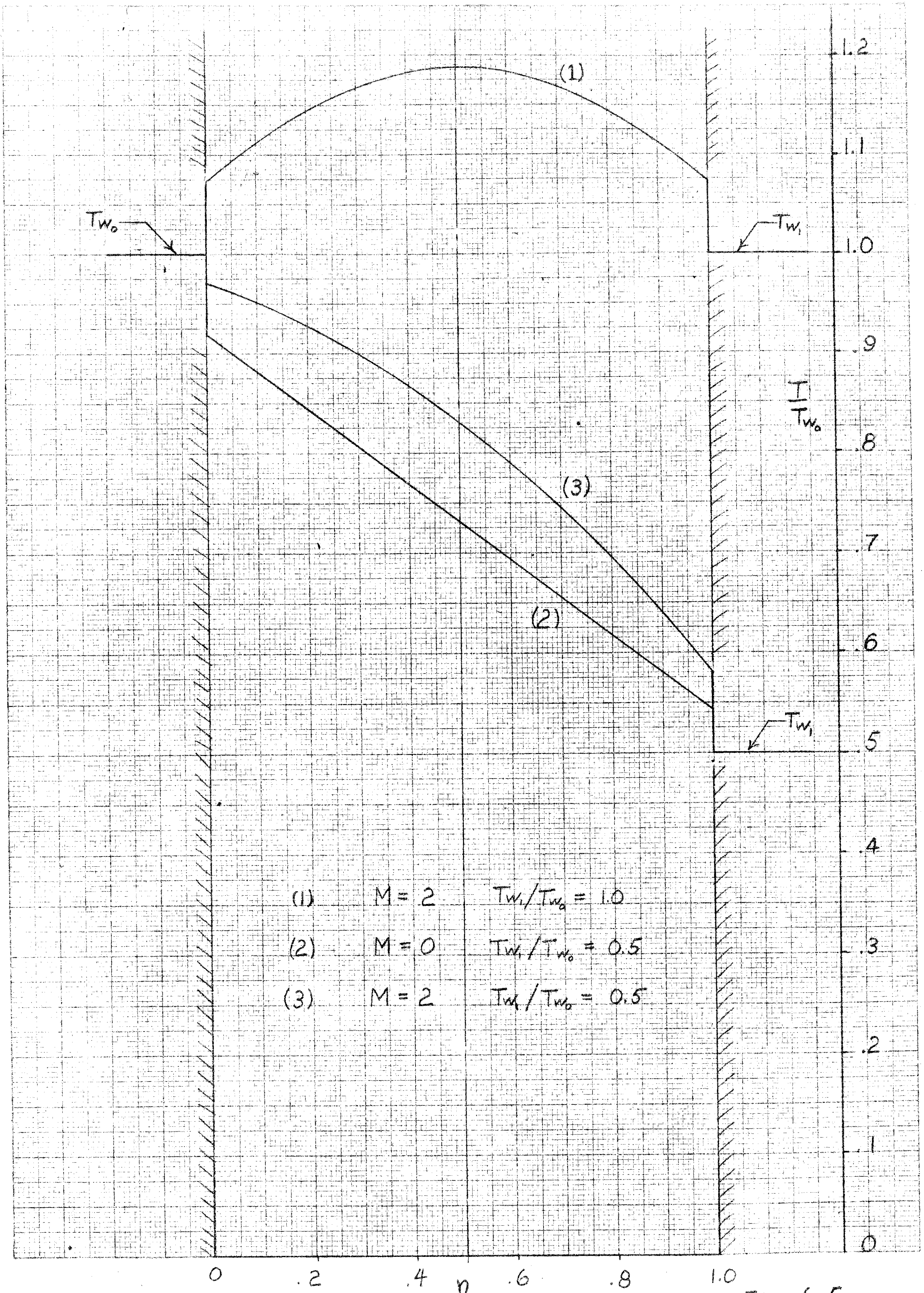

(1)
(2)
(3)
T_{W_0}
T_{W_1}
T_{W_1}
$\frac{T}{T_{W_0}}$
1.2
1.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
0
.2
.4
η
.6
.8
1.0
(1) M = 2 T_{W_1}/T_{W_0} = 1.0
(2) M = 0 T_{W_1}/T_{W_0} = 0.5
(3) M = 2 T_{W_1}/T_{W_0} = 0.5

Fig. 6·5

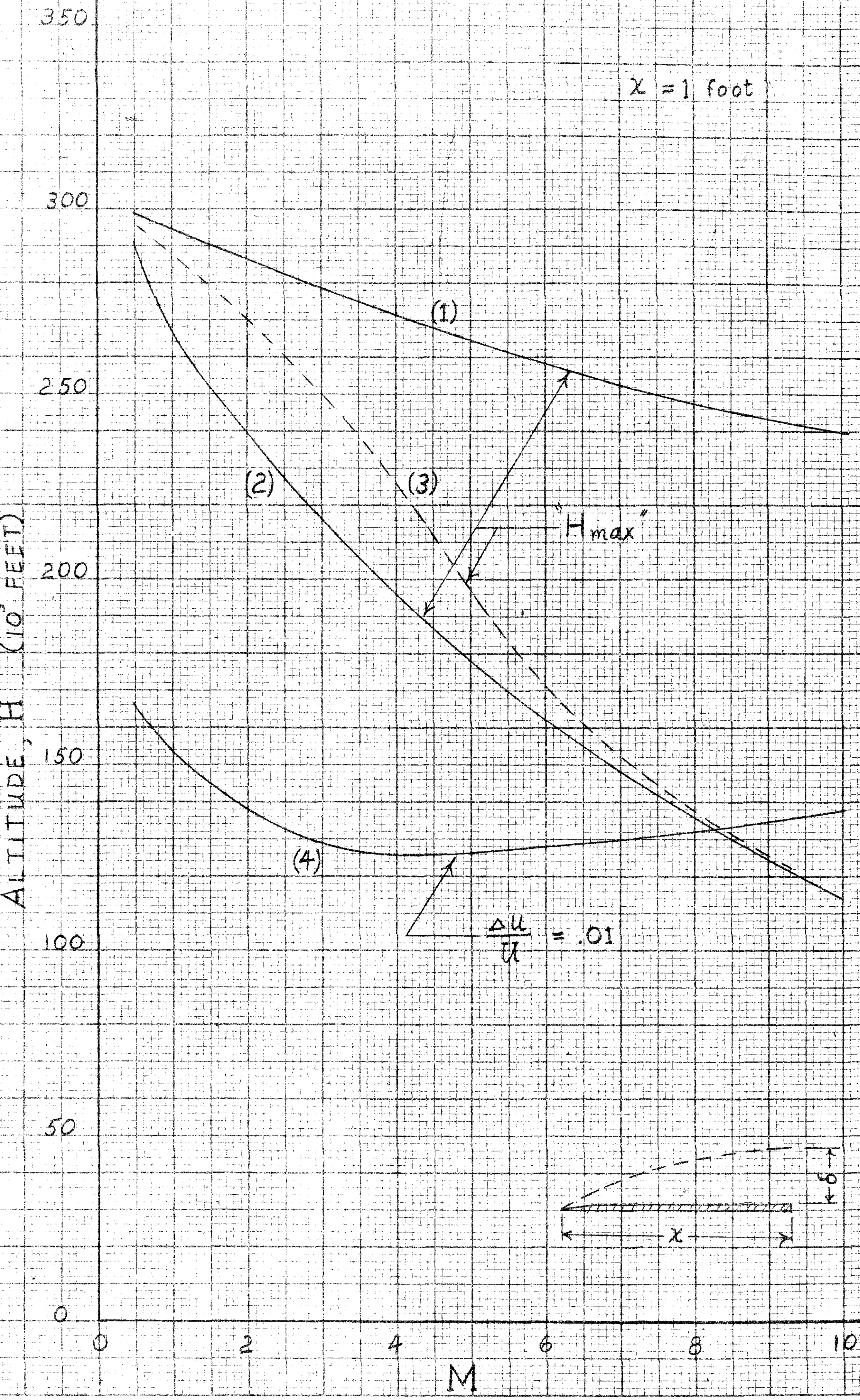

350
300
250
200
150
100
50
0
ALTITUDE, H (10^3 FEET)
0
2
4
6
8
10
M
x = 1 foot
(1)
(2)
(3)
(4)
"H_{max}"
$\frac{\Delta u}{U}$ = .01
δ
x

Fig. 6·6

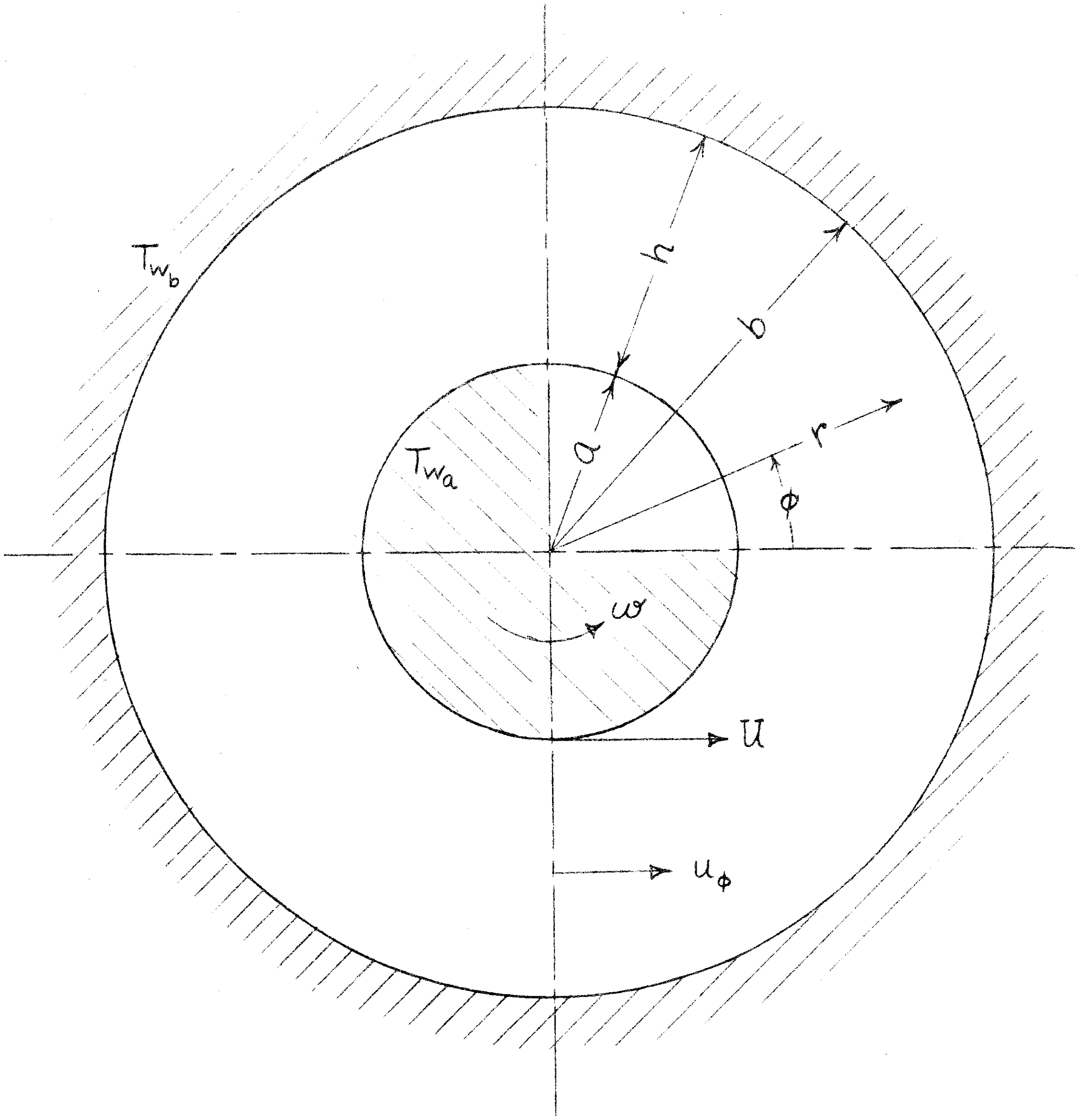


Fig. 7.1

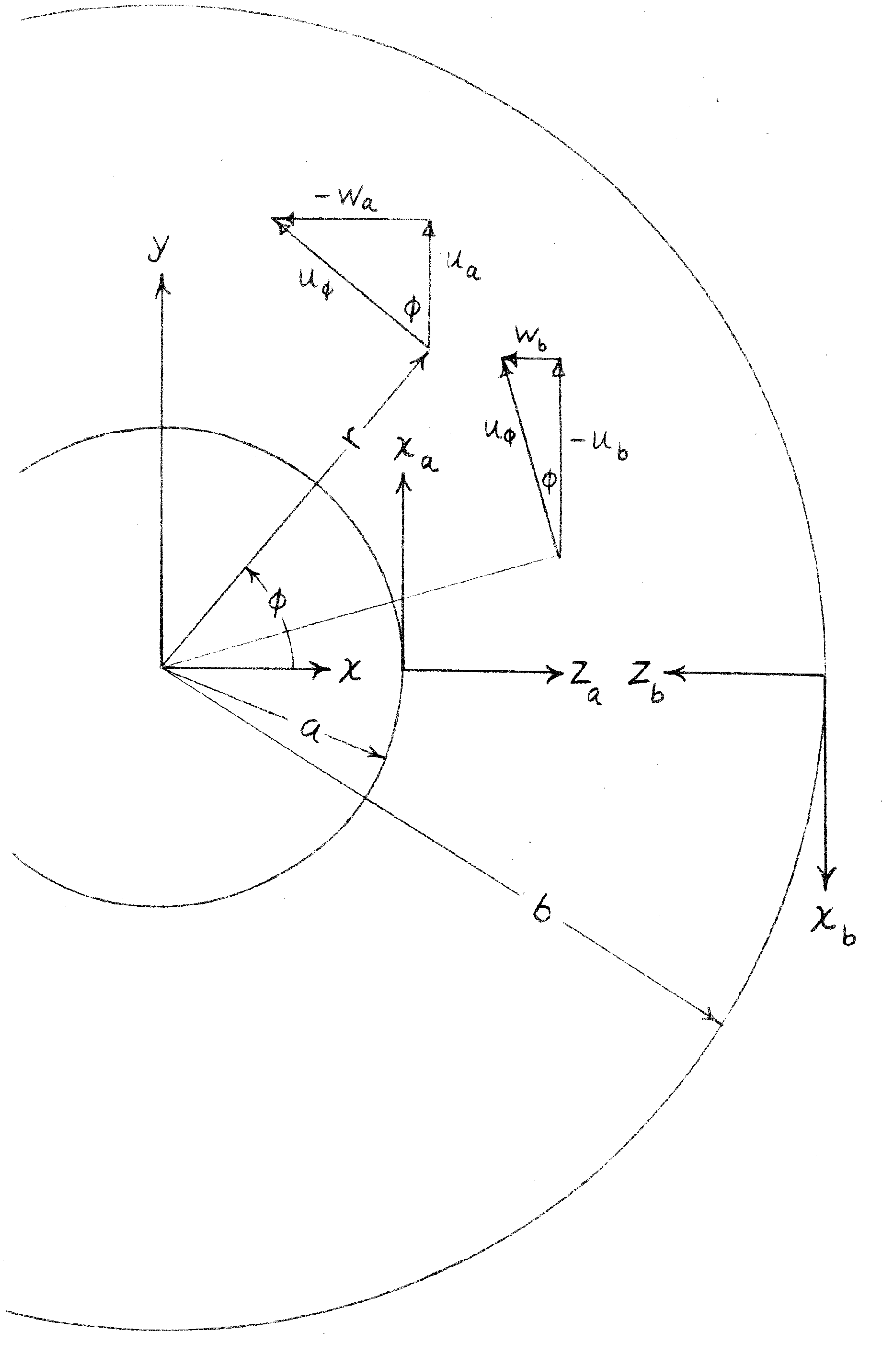


Fig. 7·2

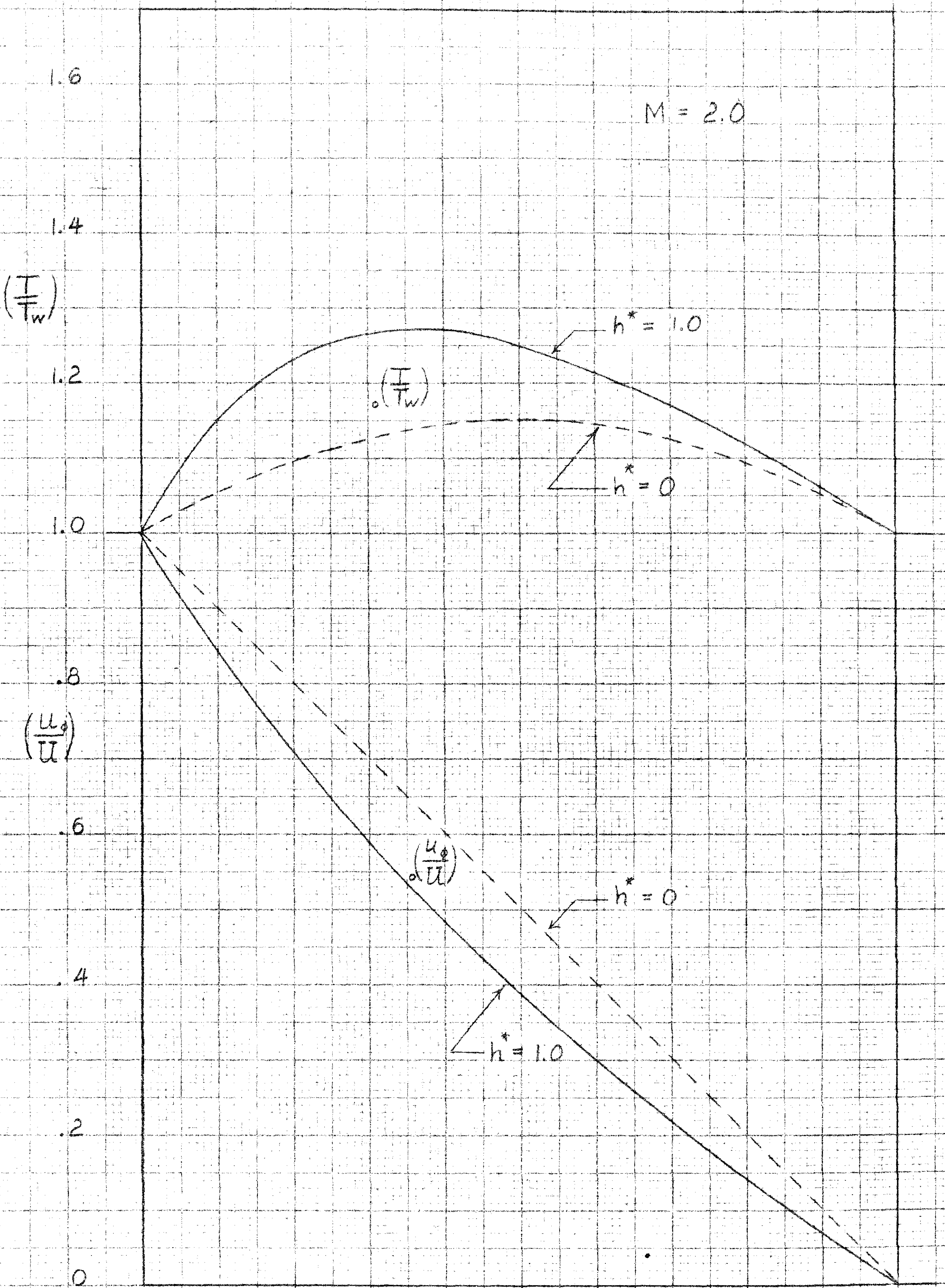

M = 2.0
$\left(\frac{T}{T_w}\right)$
$\left(\frac{u_\phi}{U}\right)$
$_0\left(\frac{T}{T_w}\right)$
$_0\left(\frac{u_\phi}{U}\right)$
$h^* = 1.0$
$h^* = 0$
$h^* = 0$
$h^* = 1.0$
1.6
1.4
1.2
1.0
.8
.6
.4
.2
0
0
.2
.4
.6
.8
1.0
ξ

Fig. 7.3

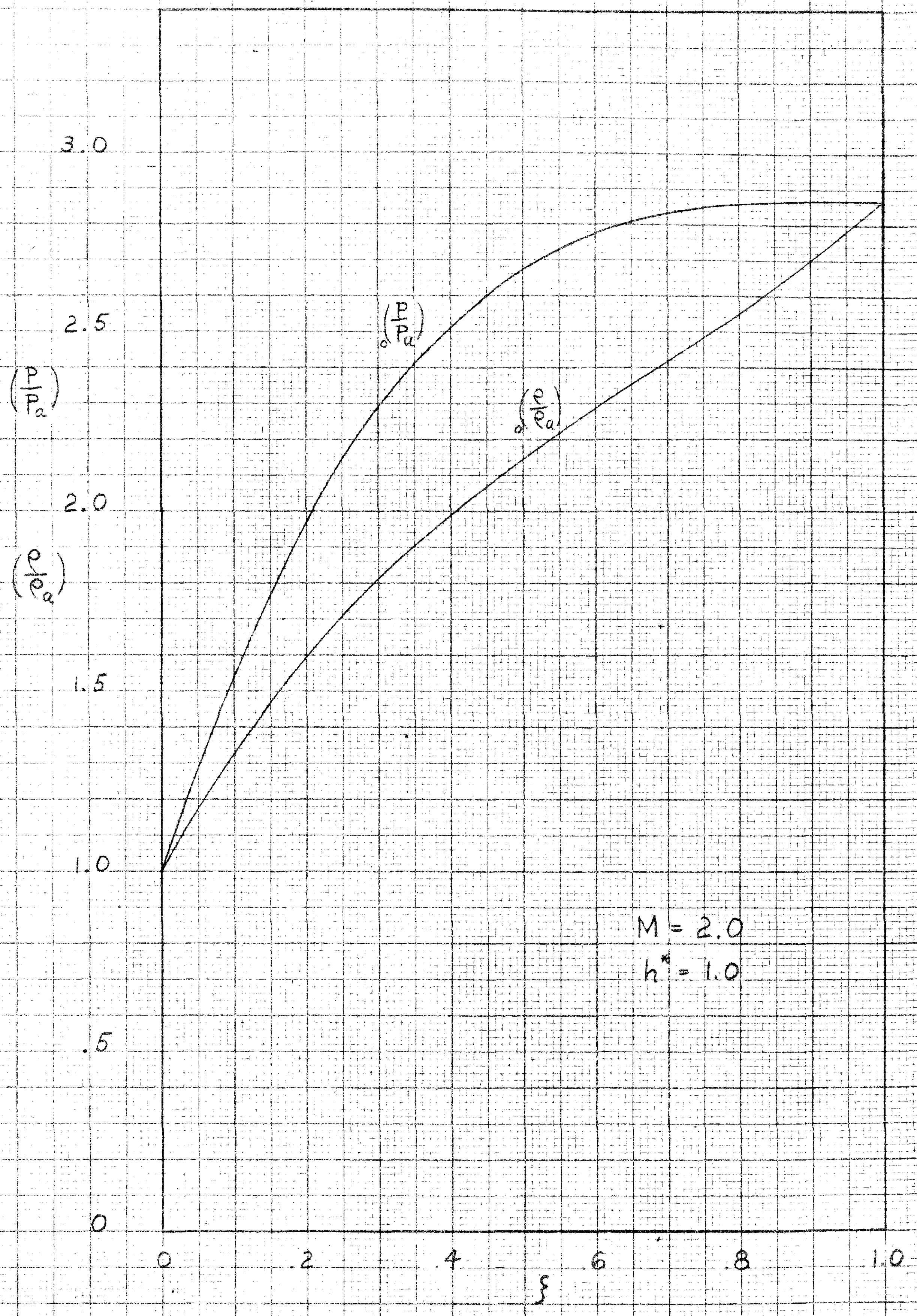

3.0
2.5
2.0
1.5
1.0
.5
0
$\left(\frac{p}{p_a}\right)$
$\left(\frac{\rho}{\rho_a}\right)$
$_o\left(\frac{p}{p_a}\right)$
$_o\left(\frac{\rho}{\rho_a}\right)$
M = 2.0
h^* = 1.0
0
.2
.4
.6
.8
1.0
ξ

Fig. 7.4

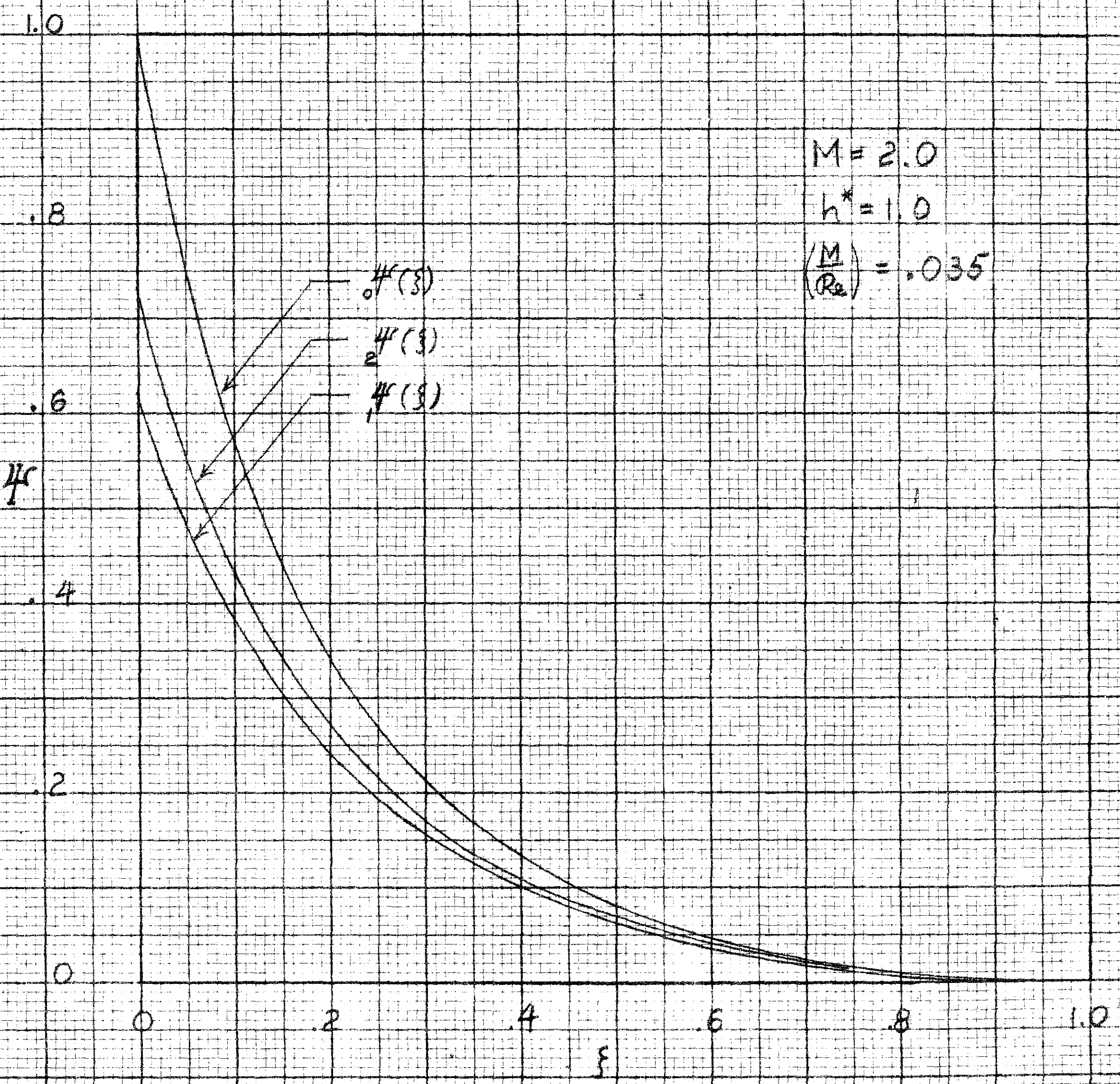

1.0
.8
.6
.4
.2
0
ψ
0
.2
.4
.6
.8
1.0
ξ
${}_0\psi(\xi)$
${}_2\psi(\xi)$
${}_1\psi(\xi)$
M = 2.0
$h^* = 1.0$
$\left(\frac{M}{Re}\right) = .035$

Fig. 7.5

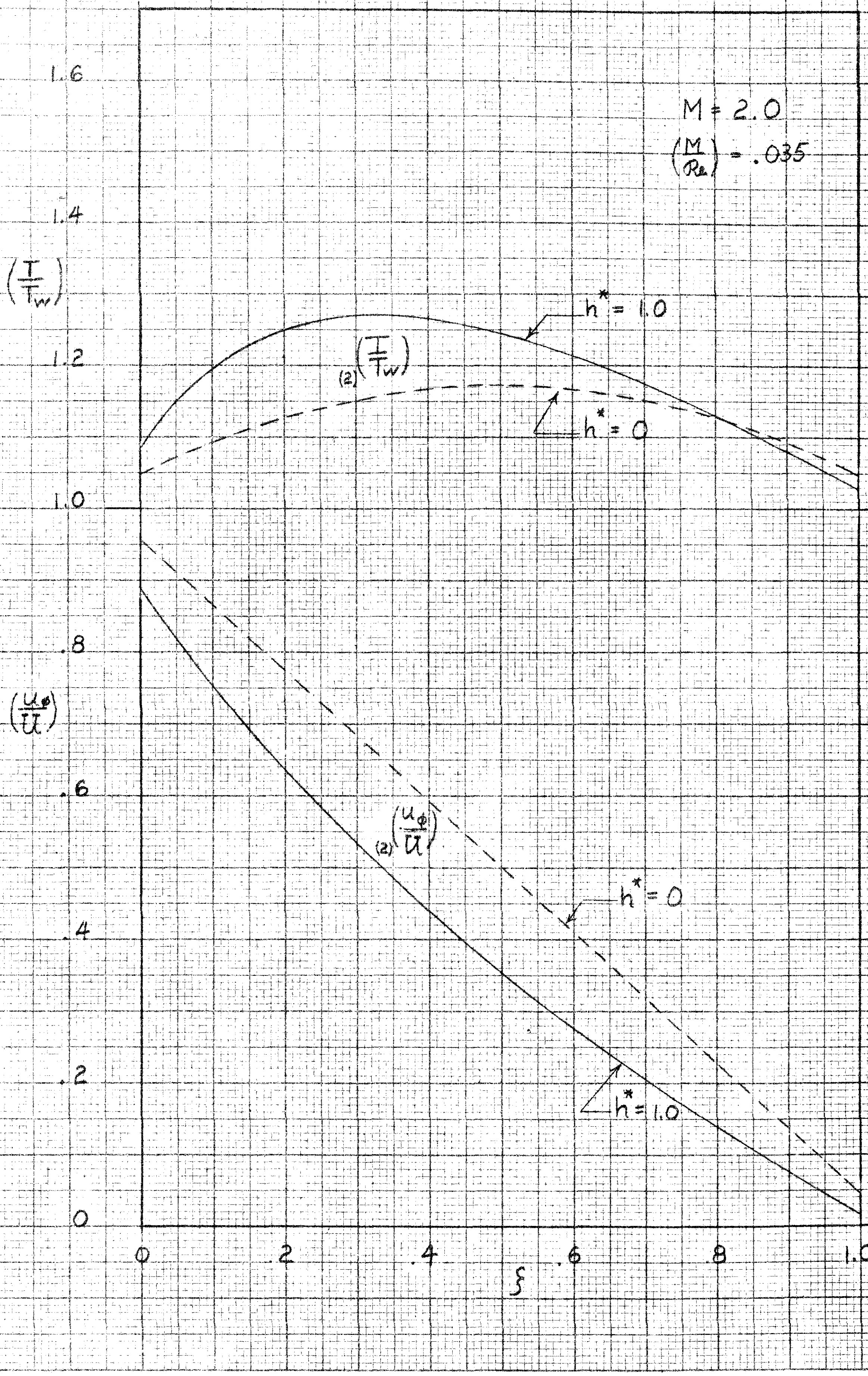

1.6
1.4
1.2
1.0
.8
.6
.4
.2
0
$\left(\frac{T}{T_w}\right)$
$\left(\frac{u_\phi}{U}\right)$
M = 2.0
$\left(\frac{M}{Re}\right) = .035$
$h^* = 1.0$
$_{(2)}\left(\frac{T}{T_w}\right)$
$h^* = 0$
$_{(2)}\left(\frac{u_\phi}{U}\right)$
$h^* = 0$
$h^* = 1.0$
0
.2
.4
.6
.8
1.0
ξ

Fig. 7.6

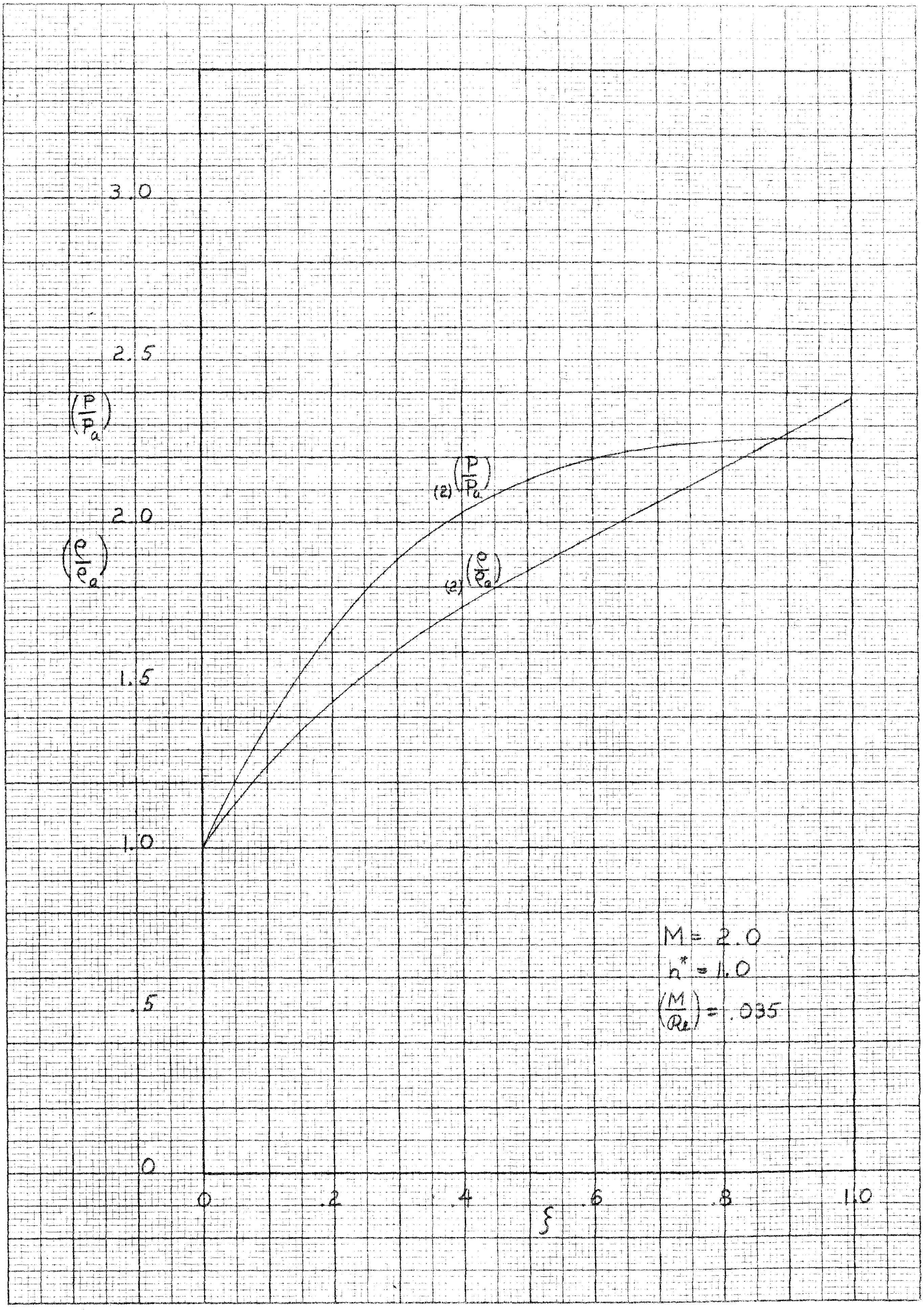

3.0
2.5
2.0
1.5
1.0
.5
0
$\left(\frac{p}{p_a}\right)$
$\left(\frac{\rho}{\rho_a}\right)$
$(2)\left(\frac{p}{p_a}\right)$
$(2)\left(\frac{\rho}{\rho_a}\right)$
0
.2
.4
.6
.8
1.0
ξ
$M = 2.0$
$h^* = 1.0$
$\left(\frac{M}{Re}\right) = .035$

Fig. 77

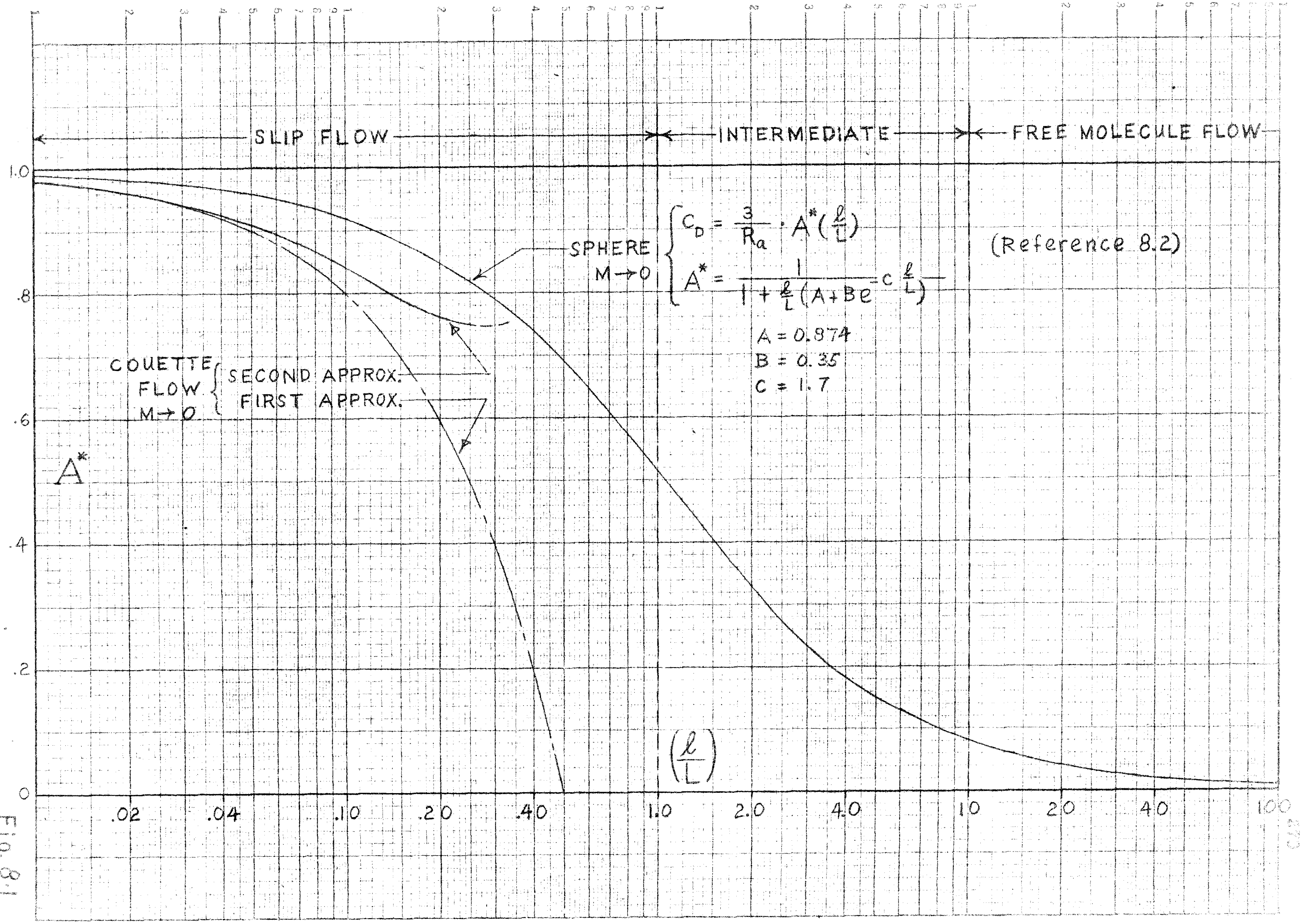

SLIP FLOW
INTERMEDIATE
FREE MOLECULE FLOW
SPHERE
M → 0
$C_D = \frac{3}{R_a} \cdot A^*\left(\frac{\ell}{L}\right)$
$A^* = \frac{1}{1 + \frac{\ell}{L}\left(A + Be^{-c\frac{\ell}{L}}\right)}$
(Reference 8.2)
A = 0.874
B = 0.35
C = 1.7
COUETTE FLOW M → O
SECOND APPROX.
FIRST APPROX.
A^*
$\left(\frac{\ell}{L}\right)$
1.0
.8
.6
.4
.2
0
.02
.04
.10
.20
.40
1.0
2.0
4.0
10
20
40
100

Fig 8·1

Fig. 8·1

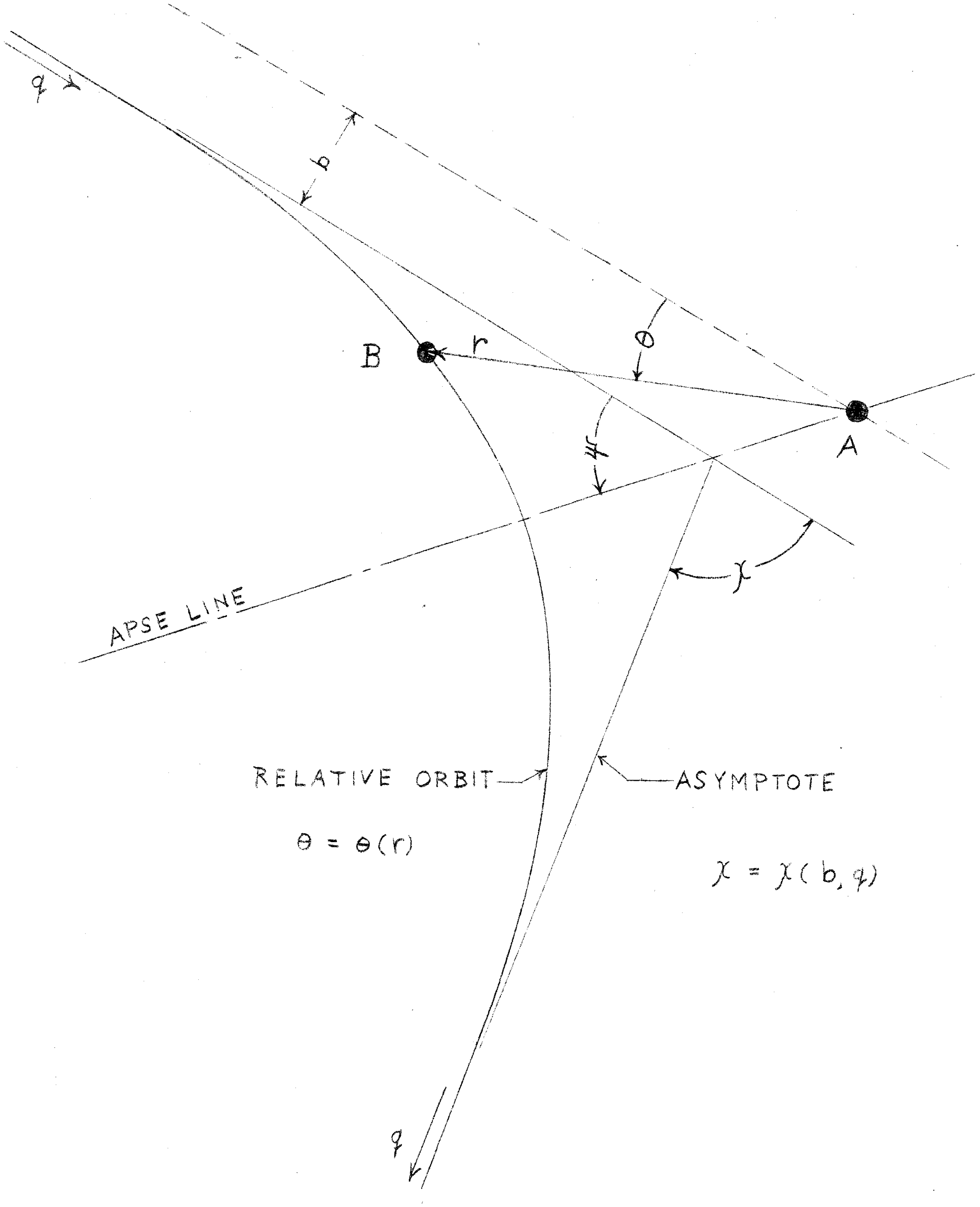


Fig. A·1

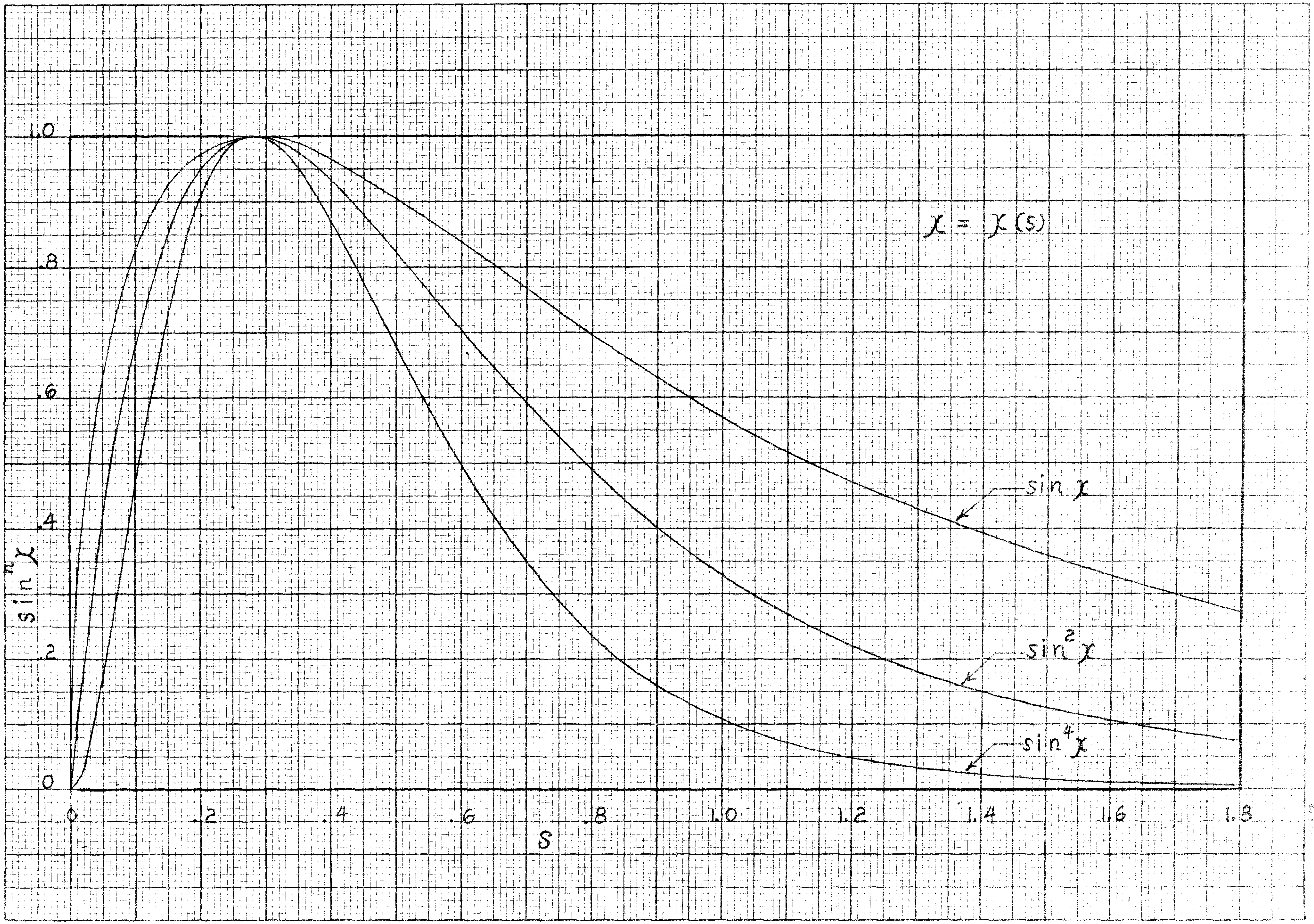


Fig. A.2

Fig. A.2