INVESTIGATION INTO THE FLOW OF A VISCOUS HEAT CONDUCTING COMPRESSIBLE FLUID

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Leon Trilling

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SUMMARY

The present investigation is concerned with the effect of a small viscosity and heat conduction coefficient on the flow of a compressible fluid. It is well known that, in the case of an incompressible fluid, such an investigation leads to the boundary layer theory.

The chief purpose of this paper is to determine whether the main result of boundary layer theory, namely, that viscosity plays a negligible part in the flow outside a very narrow region in the immediate vicinity of any solid boundary of the fluid, is still valid for a compressible fluid. To investigate that point, a very simple type of flow is selected: the flow past a semi-infinite two-dimensional flat plate parallel to the main stream direction. The problem is further simplified as follows: on the basis of experimental results, the existence of a layer influenced by viscosity is assumed, and the boundary conditions are applied near the outer edge of this layer. This allows a linearization of the equations of motion, and gives information on the interaction between the outer edge of this layer and the main field of flow.

The analysis is carried out by methods based on the theory of the Laplace Transformation. The results are essentially, that if the flow is subsonic, the boundary layer theory developed for incompressible fluids may be extended without qualitative changes. However, in a supersonic flow, one must expect two related effects: one finds the boundary layer, which, as a first approximation, is similar to the boundary layer of an incompressible fluid, and a shock-wave along the Mach line which starts at the leading edge of the flat plate, and whose strength is given by the expression:

$$u = \frac{u_0}{M^2-1}\left(\frac{4\bar{\nu}}{c\ell}\right)^{1/4} e^{-\frac{n^2 c}{2\bar{\nu}\frac{M^2}{\sqrt{M^2-1}}\ell}} \qquad (0.1)$$

where $\bar{u}$ is the normal velocity across the shock, M is the free stream Mach Number, ℓ is the distance from the leading edge of the flat plate along the shock, n is the distance normal to the shock, c is sonic velocity of the free stream and $\bar{\nu}$ is the mean effective free stream kinematic viscosity of the fluid.

NOTATION

t time variable

x, y, z, x_i Cartesian space variables

$$\nabla.\bar{A} \equiv \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla B = \bar{i}\frac{\partial B}{\partial x} + \bar{j}\frac{\partial B}{\partial y} + \bar{k}\frac{\partial B}{\partial z}$$

$$\nabla \times \bar{A} = \bar{i}\left(\frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y}\right) + \bar{j}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \bar{k}\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right)$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{u}.\nabla$$

$$\frac{D}{Dt}(\text{Lin.}) = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x}$$

Repeated indices indicate summation

ρ density of fluid

p pressure of fluid

T temperature of fluid

$\bar{u}$ velocity of fluid (u_x, u_y, u_z are x, y, z comp.)

$\bar{F}$ body force

ν kinematic viscosity coefficient

J mechanical equivalent of unit heat energy

$\bar{k}$ heat conduction coefficient

c_p specific heat at constant pressure

c_v specific heat at constant volume

$\gamma = c_p/c_v$ ratio of specific heats

$\bar{R}$ Universal gas constant

σ density fluctuation $\rho = \rho_0(1+\sigma)$

π pressure fluctuation $p = p_0(1+\pi)$

θ temperature fluctuation $T = T_0(1+\theta)$

c Laplacian velocity of sound: $c^2 = \gamma \bar{R} T$

M free stream Mach number: $M = U/c$

u, v, w local velocity fluctuations: $u = u'_x/c$; $v = u'_y/c$ ...

f_i component of body force

κ thermometric heat conduction:

τ dimensionless time variable: $\tau = \frac{tc^2}{4/3\nu}$

ξ, η, ζ, ξ_i dimensionless length variables: $\xi_i = \frac{x_i c}{4/3\nu}$

$\mathcal{U}$ mean stream velocity

$\mathcal{U}$ boundary condition on the velocity u in the s, λ space

V boundary condition on the velocity v in the s, λ space

Σ boundary condition on the density σ in the s, λ space

Π boundary condition on the pressure π in the s, λ space

Θ boundary condition on the temperature θ in the s, λ space

s transformation parameter for the time variable τ

λ transformation parameter for the space variable ξ

φ abbreviation to designate any of the quantities $u, v, \pi, \sigma, \theta$

Φ abbreviation to designate any of the quantities $\mathcal{U}, V, \Pi, \Sigma, \Theta$

α parameter of the solution which depends on s, λ

R_x Reynolds Number based on x

R Reynolds number

φ any disturbance in a viscous fluid field

Φ Transform of φ in the (s, ξ) plane

ψ impulsive disturbance in a viscous fluid flow

k parameter used in the Fourier representation of solutions

a, b hyperbolic characteristic coordinates

r spherical radius

ρ dimensionless spherical radius

$\bar{r}$ cylindrical radius

$\bar{\rho}$ dimensionless cylindrical radius

F_1 inhomogeneous part of the boundary layer equation in (η, λ)

F_2 inhomogeneous part of the shock wave equation in (η, λ)

K eigen parameter

L range of the variable

Φ_n eigen solution

α, β dummy variables of integration

δ Dirac function defined by $\delta(x) = 0 \; (x \neq 0)$; $\int_0^\infty \delta(x)\,dx = 1$

Δ thickness of boundary layer or shock wave

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1. INTRODUCTION

The present paper describes an investigation into the effect of a small viscosity and heat conduction coefficient on the high speed flow of a compressible fluid.

Because of the great difficulties which arise in the solution of the equations of motion of real fluids, it has been found convenient to simplify the problem by assuming that, the viscosity and heat conduction coefficients, being "small", their effect could be neglected, at least as a first approximation. In that manner, an elaborate theory of ideal fluids has been developed, and, in many cases, its predictions have been found to be in good agreement with the results of experiments.

However, as will soon be seen, the terms neglected in the equations of motion contain the derivatives of highest order, and should therefore determine the character of the solution. It is also known that if those terms are neglected, some of the boundary conditions of the problem can no longer be satisfied, since the order of the equations is reduced, and the number of permissible boundary conditions is reduced as a result. It is therefore necessary to investigate how the solutions of the equations of motion of slightly viscous fluids behave when the coefficient of viscosity decreases and approaches zero. In particular, do those solutions then approach solutions of the equations of motion of ideal fluids, and if they do, what boundary conditions do those solutions satisfy?

In the case of incompressible fluids, this problem was first discussed by L. Prandtl in 1904 in a paper which laid the foundations of boundary layer theory. (1). By investigating the order of magnitude of the terms of the equations of motion, Prandtl showed that when the viscosity coefficient is small, or more precisely, when the Reynolds Number of the flow is large, the effect of viscosity is restricted to a very narrow region in the immediate vicinity of the solid boundary past which the fluid flows. The boundary condition to be satisfied is, that on such a surface, the velocity of the fluid with respect to the boundary is zero. Prandtl showed that the solution of the real fluid equations, which satisfies this condition, in many cases approaches the solution of the equations for an ideal fluid, with the boundary condition that the velocity component normal to the boundary vanishes, while there is no condition on the tangential component. This ideal solution holds outside a very thin boundary layer.

In this manner, the question asked above is answered in the case of an incompressible fluid. Since Prandtl's historic paper, the theory of the incompressible boundary layer has been greatly developed. The flow inside the layer was analysed by Blasius (2), von Karman (3), and many other investigators. The question of the separation of the boundary layer from the solid surface where it started, and the effect of this phenomenon on the creation of the vortex sheets so essential to airfoil theory, has also received a great deal of attention. A very complete account of the main results of these investigations is to be found in "Modern Developments in Fluid Dynamics" edited by S. Goldstein (4).

Thus, while the theory of a slightly viscous incompressible fluid is in a fairly advanced stage, and the place of the theory of ideal fluids is defined in the general scheme, the same cannot be said of the theory of compressible fluids. And it is essentially an investigation of the existence and character of the boundary layer in compressible fluids which forms one of the main subjects of the present paper.

Insofar as a distinction of this sort can be made, the theory of compressible fluids has developed in two main branches. On the one hand, the great physicists of the XIXth century have studied the propagation of small disturbances in compressible fluids, the main subject of their interest being the construction of a theory of sound. The first important concept of that theory, namely, that sound is propagated at a finite velocity, dates back to the XVIIth century: Gassendi attempted to measure that velocity in 1635. Newton tried to explain it in terms of an isothermal process in 1687. In the XVIIIth century, fairly accurate measurements of the velocity of sound propagation were obtained, and Laplace constructed a theory which predicted results in good agreement with experiment, on the basis of an adiabatic process (1825). D'Alembert also realized that the phenomenon of sound propagation in an elastic medium was described by the so-called "wave-equation". But it was left to Poisson (1820) and Helmholtz (1860) to put the theory on a sound mathematical and physical basis.

While the theory of sound propagation in an elastic medium was thus completed, it was realized that air was not really elastic, and the effect of the viscosity of the air was first investigated by G. Stokes in 1857 (5). In 1868, Kirchhoff (6) pointed out that the effect of heat conduction is of the same order of magnitude as that of viscosity. He proved that in a perfect real gas, sound waves propagate at the same velocity as in an ideal gas, but their intensity decreases exponentially. All these investigations, so important to the XIXth century physicists because of the place which the sound propagation velocity held in the determination of the fundamental constant γ are summarized in Lord Rayleigh's "Theory of Sound" (7) and in Lamb's "Hydrodynamics" (8).

The theory of the propagation of sound in an ideal and real fluid has thus a long history of research; but the investigation of the dynamics of compressible fluids itself was undertaken much later. Rankine and Hugoniot studied shock-waves as early as 1870 and Chaplygin concerned himself with compressible jets in 1901. But modern theory is still far from complete. Most authors are interested exclusively in ideal fluids (see for instance Sauer's "Einführung in die Theoretische Gas Dynamik") and it was felt that the effect of viscosity was appreciable only in a boundary layer near a solid surface and in the narrow confines of shock waves. While there is good experimental evidence that there is a thin layer along the solid boundaries of a compressible fluid, in which large velocity gradients are observed, for instance by H.W. Liepmann (9), there is no proof that the effect of viscosity is negligible outside this layer.

The present paper discusses the effect of a small viscosity and heat conduction coefficient on the propagation of a distrubance in a compressible fluid. It attempts to determine whether it remains sharply localized in a boundary layer or whether it spreads into the entire field of flow.

To isolate this effect, the cause of the disturbance is taken to be a semi-infinite two-dimensional flat plate parallel to the direction of the undisturbed flow; an ideal fluid would experience no disturbance from such a flat plate.

Since the main problem is to determine whether any distrubances spread outside the boundary layer, the boundary conditions are applied near the outer edge of the boundary layer, and the field outside the layer is investigated. This artifice allows the use of a linearized system of equations, since the velocity along the selected boundary surface is only slightly different from the mean stream velocity.

2. THE EQUATIONS OF MOTION

A moving fluid, like any other physical system, must satisfy the fundamental equations of conservation of mass, momentum and energy. It must also satisfy a thermodynamic equation of state. The investigator thus has four equations available to determine the density, pressure, temperature and velocity which characterize the moving fluid at every instant and point. In all that follows, the fluid will be taken as a perfect gas, so that the equation of state will be the Charles-Boyle equation. In vector notation, the four equations of motion are therefore:

$$\frac{D\rho}{Dt} + \rho\, \nabla\cdot\bar{u} = 0 \tag{2.1}$$

$$\frac{D\bar{u}}{Dt} + \frac{1}{\rho}\nabla p = \bar{F} + \nu\left[\nabla^2\bar{u} + \frac{1}{3}\nabla(\nabla\cdot\bar{u})\right] \tag{2.2}$$

$$\rho J c_p \frac{DT}{Dt} - \frac{Dp}{Dt} = J\bar{k}\,\nabla^2 T + \rho\nu\left\{-\frac{2}{3}(\nabla\cdot\bar{u})^2 + \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2\right\} \tag{2.3}$$

$$p = \rho\bar{R}T \tag{2.4}$$

where the notation is explained on page (i). These equations are essentially the ones derived in "Modern Developments in Fluid Dynamics" (11). It is immediately apparent that the chief difficulty of these equations lies in the fact that they are not linear. It is to avoid that difficulty that the present paper is restricted to problems for which it is reasonable to linearize the system (2.1/2.4). This linearization is carried out, following a scheme due essentially to Oseen (12). If the mean stream velocity is U_0 the x axis is selected parallel to U_0 and the velocity components are written as: $u_x = U_0 + u'_x$; $u_y = u'_y$; $u_z = u'_z$ where the primed quantities have the property $u'_i/U_0 \ll 1$ so that their squares and higher powers may be neglected. In the dimensionless notation defined on page (i), the linearized equations of motion become:

$$\frac{D\sigma}{D\tau} + \nabla\cdot\bar{u} = \frac{\partial\sigma}{\partial\tau} + M\frac{\partial\sigma}{\partial\xi} + \nabla\cdot\bar{u} = 0 \tag{2.5}$$

$$\frac{D\bar{u}}{D\tau} + \frac{1}{\gamma}\nabla\pi = \frac{3}{4}\nabla^2\bar{u} + \frac{1}{4}\nabla(\nabla\cdot\bar{u}) \tag{2.6}$$

$$\gamma\frac{D\theta}{D\tau} - (\gamma-1)\frac{D\pi}{D\tau} = \frac{3}{4}\kappa\nabla^2\theta \tag{2.7}$$

$$\theta = \pi - \sigma \tag{2.8}$$

The remainder of the present investigation concerns itself with a solution of the dimensionless linearized system (2.5/2.8), where the perturbation velocities are the small differences between mean stream velocity and the velocity near the outer edge of the boundary layer.

3. FORMAL SOLUTION OF THE EQUATIONS

It may be noted that the problem discussed in this investigation lies in a two-dimensional space. To simplify the analysis, therefore, it will be convenient to construct only two-dimensional solutions of the system (2.5/2.8). This will not restrict the main results of the investigation seriously, and will avoid extremely cumbersome calculations. In the analysis, extensive use is made of the methods of the Laplace Transformation. A very complete discussion of that subject has been given by G. Doetsch (13) and the main results needed here are summarized in Appendix I. The definition of the transformation is set down here for convenience:

$$\mathcal{L}_t \{\varphi(x,t); s\} = \Phi(x,s) = \int_0^\infty e^{-st}\varphi(x,t)\,dt \tag{3.1}$$

with the condition that the integral exist.

If appropriate initial conditions on the system (2.5/2.8) are given, a Laplace Transformation with respect to time gives a new system:

$$s\bar\sigma + M\frac{\partial\bar\sigma}{\partial\xi} + \frac{\partial\bar u}{\partial\xi} + \frac{\partial\bar v}{\partial\eta} = \sigma(\xi,\eta,0) \tag{3.2}$$

$$s\bar u + M\frac{\partial\bar u}{\partial\xi} + \frac{1}{\gamma}\frac{\partial\bar\pi}{\partial\xi} - \frac{\partial^2\bar u}{\partial\xi^2} - \frac{3}{4}\frac{\partial^2\bar u}{\partial\eta^2} - \frac{1}{4}\frac{\partial^2\bar v}{\partial\xi\partial\eta} = u(\xi,\eta,0) \tag{3.3}$$

$$s\bar v + M\frac{\partial\bar v}{\partial\xi} + \frac{1}{\gamma}\frac{\partial\bar\pi}{\partial\eta} - \frac{\partial^2\bar v}{\partial\eta^2} - \frac{3}{4}\frac{\partial^2\bar v}{\partial\xi^2} - \frac{1}{4}\frac{\partial^2\bar u}{\partial\xi\partial\eta} = v(\xi,\eta,0) \tag{3.4}$$

$$\gamma\left(s\bar\theta + M\frac{\partial\bar\theta}{\partial\xi}\right) - (\gamma-1)\left(s\bar\pi + M\frac{\partial\bar\pi}{\partial\xi}\right) - \frac{3}{4}\kappa\left(\frac{\partial^2\bar\theta}{\partial\xi^2} + \frac{\partial^2\bar\theta}{\partial\eta^2}\right) = \gamma\theta - (\gamma-1)\pi \tag{3.5}$$

$$\bar\theta = \bar\pi - \bar\sigma \tag{3.6}$$

where the barred quantities are the Laplace transforms of the unbarred quantities. The system (3.2/3.6) is a system in two independent variables ξ, η with the parameter s. If appropriate (not necessarily independent) boundary conditions are given along the lines $\xi = 0, \eta = 0$

the system is easily transformed again with respect to ξ with parameter λ :

$$(s+\lambda M)\Sigma' + \lambda \mathcal{U}' + \frac{d\mathcal{V}'}{d\eta} = F_1(s,\lambda,\eta) \tag{3.7}$$

$$(s+\lambda M)\mathcal{U}' + \frac{\lambda}{\gamma}\Pi' - \lambda^2 \mathcal{U}' - \frac{3}{4}\frac{d^2\mathcal{U}'}{d\eta^2} - \frac{1}{4}\lambda\frac{d\mathcal{V}'}{d\eta} = F_2(s,\lambda,\eta) \tag{3.8}$$

$$(s+\lambda M)\mathcal{V}' + \frac{1}{\gamma}\frac{d\Pi'}{d\eta} - \frac{d^2\mathcal{V}'}{d\eta^2} - \frac{3}{4}\lambda^2\mathcal{V}' - \frac{1}{4}\lambda\frac{d\mathcal{U}'}{d\eta} = F_3(s,\lambda,\eta) \tag{3.9}$$

$$\gamma(s+\lambda M)\Theta' - (\gamma-1)(s+\lambda M)\Pi' - \frac{3}{4}\kappa\left(\lambda^2\Theta' + \frac{d^2\Theta'}{d\eta^2}\right) = F_4(s,\lambda,\eta) \tag{3.10}$$

$$\Theta' = \Pi' - \Sigma' \tag{3.11}$$

The primed capitals introduced here refer to the Laplace transforms in ξ of the barred quantities introduced in the system (3.2/3.6). The four functions $F_i(s,\lambda,\eta)$ depend on the initial conditions and on the boundary conditions of the problem. It is known, however, that the quantities $\Sigma', \mathcal{U}'\ldots$ must be analytic functions of the complex variables s,λ in their respective right half planes $R\,|\,s > s_0 ; R\,|\,\lambda > \lambda_0$ If the F_i were arbitrary, then, for certain eigen-values of s,λ, the system (3.7/3.11) would have no solution, so that for those values of s,λ, $\Sigma', \mathcal{U}'\ldots$ would be singular. Since this is impossible, F_i must be so chosen that solutions exist even when s,λ take their eigen-values. A particular solution of this type is obtained when F_i vanish. Then, since the system (3.7/3.11) is a system in one unknown, one can write:

$$\Sigma', \mathcal{U}'\ldots = \Sigma, \mathcal{U}\ldots e^{\alpha\eta} \tag{3.12}$$

and substitute this tentative solution into the system. One then finds the following algebraic relations:

$$(s+\lambda M)\Sigma + \lambda\mathcal{U} + \alpha V = 0 \tag{3.13}$$

$$(s+\lambda M)\mathcal{U}+\frac{\lambda}{\gamma}\Pi-\lambda^2\mathcal{U}-\frac{3}{4}\alpha^2\mathcal{U}-\frac{1}{4}\lambda\alpha V=0 \qquad (3.14)$$

$$(s+\lambda M)V+\frac{\alpha}{\gamma}\Pi-\alpha^2 V-\frac{3}{4}\lambda^2 V-\frac{1}{4}\lambda\alpha\mathcal{U}=0 \qquad (3.15)$$

$$(s+\lambda M)\gamma\Theta-(s+\lambda M)(\gamma-1)\Pi-\frac{3}{4}\kappa(\lambda^2+\alpha^2)\Theta=0 \qquad (3.16)$$

$$\Theta-\Pi+\Sigma=0 \qquad (3.17)$$

The system (3.13/3.17) is therefore the algebraic equivalent of the original system (2.5/2.8) and it may be well to pause and re-examine the steps taken to obtain this system, and the significance of the terms it contains. The system (3.13/3.17) could have been obtained formally by postulating solutions of the form:

$$u=\mathcal{U}e^{s\tau+\lambda\xi+\alpha\eta} \qquad (3.18)$$

and seeking the conditions which s,α,λ must satisfy. The method of the Laplace Transformation used above is merely the mathematically rigorous formulation of that very process, which gives also some information on the type of boundary conditions which may be specified, as the prize of a more elaborate analysis. The parameters designated by unprimed capitals are easily defined in terms of the initial and boundary conditions of the problem; they are calculated as follows:

$$\mathcal{U}=\int_0^\infty\int_0^\infty u(\tau,\xi,0)\,e^{-s\tau}e^{-\lambda\xi}\,d\tau\,d\xi \qquad (3.19)$$

It has therefore been possible to transform the system of partial differential equations (2.5/2.8) and its initial and boundary conditions into a homogeneous system of algebraic equations. It is clear from the analysis up to this point, that s,λ are any complex numbers. Therefore, for the system (3.13/3.17) considered as a system of equations in the unknowns $\mathcal{U},V,\Sigma,\Pi,\Theta$, to have non-trivial solutions, the coefficients of those unknowns must be so chosen that their determinant vanishes, and that gives a relation to define the parameter α as a function of the parameters s, λ. One will return to the discussion of the boundary conditions after this determination of $\alpha(s,\lambda)$.

The parameter α is therefore determined by the relation:

$$\begin{vmatrix} \lambda & \alpha & s+M\lambda & 0 & 0 \\ s+M\lambda-\lambda^2-\frac{3}{4}\alpha^2 & -\frac{\alpha\lambda}{4} & 0 & \frac{\lambda}{\gamma} & 0 \\ -\frac{\alpha\lambda}{4} & s+M\lambda-\frac{3}{4}\lambda^2-\alpha^2 & 0 & \frac{\alpha}{\gamma} & 0 \\ 0 & 0 & 0 & (\gamma-1)(s+M\lambda) & \gamma(s+\lambda M)-\frac{3}{4}K(\lambda^2+\alpha^2) \\ 0 & 0 & 1 & -1 & 1 \end{vmatrix} \qquad (3.20)$$

$$\Delta = 0$$

This determinant is computed out to give:

$$\left\{s+M\lambda-\frac{3}{4}(\lambda^2+\alpha^2)\right\}\left\{(s+\lambda M)\left[(s+\lambda M)-\frac{3}{4}K(\lambda^2+\alpha^2)\right]\right. \qquad (3.21)$$
$$\left.\left[(s+\lambda M)-(\lambda^2+\alpha^2)\right]-(\alpha^2+\lambda^2)\left[(s+\lambda M)-\frac{3}{4}\frac{K}{\gamma}(\alpha^2+\lambda^2)\right]\right\}=0$$

which can be satisfied in the following two ways:

$$s+M\lambda-\frac{3}{4}(\lambda^2+\alpha^2)=0 \qquad (3.22)$$

$$(s+M\lambda)\left[(s+\lambda M)-\frac{3}{4}K(\lambda^2+\alpha^2)\right]\left[(s+\lambda M)-(\lambda^2+\alpha^2)\right] \qquad (3.23)$$
$$-(\alpha^2+\lambda^2)\left[(s+\lambda M)-\frac{3}{4}\frac{K}{\gamma}(\alpha^2+\lambda^2)\right]=0$$

Equations (3.22) and (3.23) can be reinterpreted in terms of partial differential equations. They are made up of combinations of the expressions: $(s+\lambda M)$; $(\alpha^2+\lambda^2)$. But from the solution form (3.18), it follows that:

$$\frac{D\varphi}{D\tau}=\frac{\partial\varphi}{\partial\tau}+M\frac{\partial\varphi}{\partial\xi}=(s+\lambda M)\Phi \qquad (3.24)$$

$$\nabla^2\varphi=\frac{\partial^2\varphi}{\partial\xi^2}+\frac{\partial^2\varphi}{\partial\eta^2}=(\lambda^2+\alpha^2)\Phi \qquad (3.25)$$

so that equation (3.22) is equivalent to the partial differential equation:

$$\frac{D\varphi}{D\tau} = \frac{3}{4}\nabla^2\varphi \tag{3.26}$$

while equation (3.23) is equivalent to the partial differential equation:

$$\frac{D}{D\tau}\left[\frac{D^2\varphi}{D\tau^2}-\nabla^2\varphi\right]-\nabla^2\left[\left(1+\frac{3}{4}\kappa\right)\frac{D^2\varphi}{D\tau^2}-\frac{3}{4}\frac{\kappa}{\gamma}\nabla^2\varphi\right]+\frac{3}{4}\kappa\frac{D}{D\tau}\nabla^4\varphi=0 \tag{3.27}$$

where φ is an abbreviation to represent any of the functions $u, v, \sigma, \pi, \theta$. It has thus been shown that the solution of the system (2.5/2.8) is equivalent to the solution of the partial differential equations of higher order (3.26) and (3.27). It is also possible to deduce equations (3.26) and (3.27) from the system (2.5/2.8).

The role of the solutions of (3.26) among the solutions of the system can be inferred by inspection. For if $\sigma = \pi = \theta = 0$, then, the system is reduced to:

$$\nabla\cdot\bar{u}_1 = 0 \tag{3.28}$$

$$\frac{D\bar{u}_1}{D\tau} = \frac{3}{4}\nabla^2\bar{u}_1 \tag{3.29}$$

$$\theta = \pi = \sigma = 0 \tag{3.30}$$

This part of the solution, therefore, does not involve the dynamical properties of the fluid. It is of purely kinematic character, and can be superimposed on any solution which satisfies the dynamic requirements of the problem.

In order to obtain equation (3.27) from the system, it is necessary to carry out a lengthy calculation which is not reproduced here.

To summarize the results obtained to this point, it was shown that the solution of the system (2.5/2.8) is of the form:

$$\bar{u} = \bar{u}_1 + \bar{u}_2 \qquad ; \ \sigma = \sigma_2 \ ; \ \pi = \pi_2 \ ; \ \theta = \theta_2 \tag{3.31}$$

where:

$$\frac{D\varphi_1}{D\tau} = \frac{3}{4}\nabla^2\varphi_1 \tag{3.26}$$

$$\frac{D}{D\tau}\left[\frac{D^2\varphi_2}{D\tau^2} - \nabla^2\varphi_2\right] - \nabla^2\left[\left(1+\frac{3}{4}K\right)\frac{D^2\varphi_2}{D\tau^2} - \frac{3}{4}\frac{K}{\gamma}\nabla^2\varphi_2\right] + \frac{3}{4}K\frac{D}{D\tau}\nabla^4\varphi_2 = 0 \tag{3.27}$$

The solution of the system is thus replaced by the solution of two higher order equations. But the system (3.13/3.17) and equation (3.21) actually give a great deal more information; they determine the scheme of boundary conditions which makes the problem determinate, and they give the solution in the form of complex contour integrals. Indeed, given a value of $\alpha(\lambda, S)$ which satisfies equation (3.21), the system (3.13/3.17) can be solved excplicitly. If U is given, then one finds:

$$V_2 = \frac{\alpha_2}{\lambda} U_2 \tag{3.32}$$

$$\Sigma_2 = -\frac{\alpha_2^2 + \lambda^2}{\lambda(S+\lambda M)} U_2 \tag{3.33}$$

$$\Pi_2 = -\frac{\gamma}{\lambda}\left[S + \lambda M - (\alpha_2^2 + \lambda^2)\right] U_2 \tag{3.34}$$

$$\Theta_2 = -\frac{\gamma-1}{\lambda}\,\frac{S+\lambda M}{S+\lambda M - \frac{3}{4}\frac{K}{\gamma}(\alpha_2^2+\lambda^2)}\left[S+\lambda M - (\alpha_2^2+\lambda^2)\right] U_2 \tag{3.35}$$

The system (3.32/3.35) is a non-trivial solution of (3.13/3.17) when α_2 satisfies (3.23) while if α_1 satisfies (3.22), then:

$$V_1 = -\frac{\lambda}{\alpha_1} U_1 \tag{3.36}$$

$$\Sigma_1 = \Pi_1 = \Theta_1 = 0 \tag{3.37}$$

Now, the homogeneous part of the solution of the original system

(2.5/2.8) can be written:

$$\varphi = -\frac{1}{4\pi^2}\iint_C \left[\Phi_{0_1} e^{\alpha_1\eta} + \Phi_{0_2} e^{\alpha_2\eta} + \dots\right] e^{s\tau+\lambda\xi}\, ds\, d\lambda \qquad (3.38)$$

where C is the inverse Laplace Transformation path, and will be discussed in greater detail later. It therefore remains to determine the number of acceptable linearly independent values of α which can be substituted into (3.38). One notes that equations (3.22), (3.23) involve only α^2. One obvious condition on α is:

$$Rl\,\{\alpha\eta\} < 0 \qquad (3.39)$$

to insure convergence of the solution at large distances from the source of the disturbance. To each value of α^2 therefore, corresponds one permissible value of α for which the boundary condition as $\eta \to \infty$ is automatically satisfied.

Equation (3.22) gives one value for α :

$$\alpha_1 = \pm\sqrt{\frac{4}{3}(s+\lambda M) - \lambda^2} \qquad (3.40)$$

while equation (3.23) gives two values:

$$\alpha_2 = \pm\sqrt{\frac{(s+\lambda M)\left[1+\left(\frac{3}{4}K+1\right)(s+\lambda M)\right] \pm \sqrt{(s+\lambda M)^2\left[1+\left(\frac{3}{4}K+1\right)(s+\lambda M)\right]^2 - \frac{3}{4}\frac{K}{\gamma}\left[\gamma(s+\lambda M)-1\right](s+\lambda M)^3}}{\frac{3}{4}\frac{K}{\gamma}\left[\gamma(s+\lambda M)-1\right]} - \lambda^2} \qquad (3.41)$$

There are thus only three possible values of α which may be designated $\alpha_1, \alpha_2, \alpha_3$. To each of these values α_i, for a given value of $\mathcal{U}$ or $\mathcal{V}, \Pi \dots$ correspond values for the other parameters, as given by equations (3.32) to (3.37). It follows that three out of the five parameters may be selected arbitrarily, provided that one of them is $\mathcal{U}$ or $\mathcal{V}$. This analytical result is the confirmation of the intuitive feeling that the flow of a viscous heat conducting fluid is determined by the no-slip condition at its solid boundaries ($U = U_0; V = 0$) and by their temperature distribution ($\Theta = \Theta_0$).

It may be noted that in the very important case of the viscous non-conducting fluid, the third condition becomes irrelevant, so that only two conditions are given (no slip). Then, α_2 , instead of having two permissible values, has the single value:

$$\alpha_2 = \pm\sqrt{\frac{(s+\lambda M)^2}{1+(s+\lambda M)} - \lambda^2} \qquad (3.42)$$

4. SPECIAL CASE OF THE INCOMPRESSIBLE FLUID

It may be well to pause at this point of the discussion for a digression intended to throw some light on an important feature brought out in the formal solution above; the split of the solution into a purely kinematic part and a dynamic part.

This is best done by studying the simpler more familiar flow of an incompressible fluid. The problem of the flow of an incompressible fluid about a two-dimensional semi-infinite flat plate parallel to the main stream will therefore be discussed in some detail.

The linearized equations of motion of an incompressible fluid in steady motion are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4.1}$$

$$U_0 \frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{4.2}$$

$$U_0 \frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial y} = \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{4.3}$$

It is known that the pressure satisfies the simple equation:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0 \tag{4.4}$$

as can easily be verified by differentiating equation (4.2) with respect to x, (4.3) with respect to y, and taking the sum of the obtained equations. It is also known that the velocity components u,v, satisfy the following system of equations:

$$\bar{u} = \bar{u}_1 + \bar{u}_2 \tag{4.5}$$

$$U_0 \frac{\partial \bar{u}_1}{\partial x} = \nu\left(\frac{\partial^2 \bar{u}_1}{\partial x^2} + \frac{\partial^2 \bar{u}_1}{\partial y^2}\right) \tag{4.6}$$

$$\frac{\partial^2 \bar{u}_2}{\partial x^2} + \frac{\partial^2 \bar{u}_2}{\partial y^2} = 0 \tag{4.7}$$

A complete discussion of these results is found in Lamb's "Hydrodynamics" (14) or in Oseen's original paper (15) where they are derived in connection with Stokes flow about a sphere.

Now, it is clear that equation (4.6) corresponds to (3.26). Not only can it be seen to correspond to the case $\rho=0$ in the system (4.1/4.3) just as (3.26) corresponds to the case $\sigma=\pi=\theta=0$, but it is also identical in form. On the other hand, equations (4.4), (4.7) correspond to (3.27) and again, the dynamic equations for all the variables take the same form.

With these remarks in mind, consider the flat plate problem in the following precise statement of boundary conditions:

$$\begin{aligned} u(x,0) &= u_0 \qquad 0<x<\infty \\ u_y(x,0) &= 0 \qquad -\infty<x<0 \end{aligned} \tag{4.8}$$

$$\lim_{y\to\infty} (u(x,y)) = 0 \tag{4.9}$$

$$\lim_{x\to-\infty} u(x,y) = 0 \tag{4.10}$$

$$v(x,0) = 0 \tag{4.11}$$

$$\lim_{y\to\infty} v(x,y) = 0 \tag{4.12}$$

$$\lim_{x\to-\infty} v(x,y) = 0 \tag{4.13}$$

where observations are taken near the outer edge of the boundary layer so that the linearized equations have some validity even when the free stream Reynolds number is large.

These boundary conditions imply that the fluid is at rest with respect to a fictitious surface very near the flat plate; and that the disturbance created by the presence of the plate is damped out at large distances away from the plate. Under those conditions, it is possible to solve the problem, as Lord Rayleigh essentially did (16).

A Laplace transformation of the system (4.1/4.3) with respect to x and with parameter λ and the postulation of a solution of the transformed system of the form (3.18) puts the system in the algebraic form:

$$\lambda U + \alpha V = 0 \tag{4.14}$$

$$\left[U_0 \lambda - \nu(\alpha^2+\lambda^2)\right] U + \frac{\lambda}{\rho} P = 0 \tag{4.15}$$

$$\left[U_0 \lambda - \nu(\alpha^2+\lambda^2)\right] V + \frac{\alpha}{\rho} P = 0 \tag{4.16}$$

which is similar to the system (3.13/3.17), discussed previously. It is easily verified that the condition for non-trivial solutions of U, V, P, is that the parameter α satisfy the equation:

$$\left[U_0 \lambda - \nu(\lambda^2+\alpha^2)\right]\left[\alpha^2+\lambda^2\right] = 0 \tag{4.17}$$

which leads to the two possibilities:

$$U_0 \lambda - \nu(\lambda^2+\alpha^2) = 0 \tag{4.18}$$

which corresponds to equation (4.6); and:

$$\lambda^2 + \alpha^2 = 0 \tag{4.19}$$

which corresponds to equation (4.7).

In this simple case, the equivalence of the two methods of solution of the system is demonstrated.

Because of the symmetry of the problem, it is sufficient to solve it for the case $y>0$; the solution in the half-plane $y<0$ is then obtained by reflection. Boundary conditions (4.9), (4.12) are therefore satisfied if the root $\alpha<0$ is selected.

The technique of the Laplace Transformation gives the line $x=0$

a special position which has no physical significance; the problem must therefore first be solved in the quadrant $x>0; y>0$; then in the quadrant $x<0; y>0$ and the two solutions must then be matched.

The boundary conditions (4.8), (4.11) give:

$$U_1 + U_2 = \frac{u_0}{\lambda} \tag{4.20}$$

$$V_1 + V_2 = 0 \tag{4.21}$$

Since both $\bar{u}_1$, and $\bar{u}_2$ satisfy the equation of continuity, one has the further relations:

$$\lambda U_1 + \alpha_1 V_1 = 0 \tag{4.22}$$

$$\lambda U_2 + \alpha_2 V_2 = 0 \tag{4.23}$$

where, from equations (4.18), (4.19), one has:

$$\alpha_1 = -\sqrt{\frac{u_0 \lambda}{\nu} - \lambda^2} \tag{4.24}$$

$$\alpha_2 = i\lambda \tag{4.25}$$

It follows that the transformed boundary conditions in the plane (y, λ) are:

$$U_1 = \frac{u_0}{\lambda} \frac{-i\sqrt{\frac{u_0}{\nu\lambda} - 1}}{1 - i\sqrt{\frac{u_0}{\nu\lambda} - 1}} \quad ; \quad U_2 = \frac{u_0}{\lambda} \frac{1}{1 - i\sqrt{\frac{u_0}{\nu\lambda} - 1}} \tag{4.26}$$

$$V_1 = -V_2 = \frac{u_0}{\lambda} \frac{-i}{1 - i\sqrt{\frac{u_0}{\nu\lambda} - 1}} \tag{4.27}$$

The principal part of the solution is therefore given by an inverse Laplace Transformation:

$$u = \frac{u_0}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{\lambda \xi}}{1 - i\sqrt{\frac{u_0}{\lambda\nu} - 1}} \left\{ -i\sqrt{\frac{u_0}{\nu\lambda} - 1}\; e^{-\lambda y \sqrt{\frac{u_0}{\nu\lambda} - 1}} + e^{i\lambda y} \right\} \frac{d\lambda}{\lambda} \tag{4.28}$$

The integrand of this complex integral has a pole and a branch-point at the origin and a branch-point at $\lambda = \frac{u_0}{\nu}$ which introduces a cut in the right half plane along the real axis $\frac{u_0}{\nu} < \lambda < \infty$ but causes no difficulty. It is therefore possible to replace the path of integration by one which follows the imaginary axis with an indentation to the right to avoid the pole at the origin. Writing $i\omega$ for λ, one finds:

$$u = \frac{u_0}{2\pi i} \int_C \frac{e^{i\omega x}}{1 - i\sqrt{\frac{u_0}{\nu\lambda} - 1}} \left\{ -i\sqrt{\frac{u_0}{\nu i\omega} - 1}\, e^{-i\omega\sqrt{\frac{u_0}{\nu i\omega} - 1}} + e^{-y\omega} \right\} \frac{d\omega}{\omega} \quad (4.29)$$

where the contour C is the one just defined. The following dimensionless parameters are now introduced: $\frac{\nu\omega}{u_0} = \bar{\omega}$; $\frac{u_0 x, y}{\nu} = \bar{x}, \bar{y}$. This transforms the integral (4.29) into:

$$u = \frac{u_0}{2\pi i} \int_C \frac{e^{i\bar{\omega}\bar{x}}}{1 - \sqrt{\frac{1}{i\bar{\omega}} - 1}} \left\{ -i\sqrt{\frac{1}{i\bar{\omega}} - 1}\, e^{-i\bar{\omega}\sqrt{\frac{1}{i\bar{\omega}} - 1}} + e^{-\bar{y}\bar{\omega}} \right\} \frac{d\bar{\omega}}{\bar{\omega}} \quad (4.30)$$

The dimensionless parameters $\bar{x}, \bar{y}$ which are of the nature of Reynolds numbers may be considered to be large, and only asymptotic solutions as $\bar{x} \to \infty$; $\bar{y} \to \infty$ need be found. Then, only that part of the integrand for which $|\bar{\omega}| \to 0$ contributes significantly to the result. The asymptotic solution then comes from the u_1 alone and is given by an integral of the form:

$$u = \frac{u_0}{2\pi i} \int_C e^{i\bar{\omega}\bar{x}}\, e^{-\bar{y}\sqrt{\bar{\omega}}} \frac{d\bar{\omega}}{\bar{\omega}} \quad (4.31)$$

which gives the well-known asymptotic solution:

$$u = u_0 \left(1 - \operatorname{erf} \frac{y\sqrt{R_x}}{2x} \right) \quad (4.32)$$

where R_x is the Reynolds Number $\frac{u_0 x}{\nu}$ and erf represents the error function defined by:

$$\operatorname{erf} a = \int_0^a e^{-\alpha^2} d\alpha \quad (4.33)$$

It is well to point out that to neglect exponentials of order $e^{-\bar{y}\bar{\omega}}$ as compared to $e^{-\bar{y}\sqrt{\bar{\omega}}}$, as was done in evaluating the integral (4.30) approximately, is equivalent to neglecting the term $\frac{\partial^2 u}{\partial x^2}$ as compared to $\frac{\partial u}{\partial x}$ in the equations of motion as Prandtl did in his original formulation of the boundary layer theory.

The second part of the solution valid in the quadrant $X<0$ is asymptotically 0. Thus, all the boundary conditions are asymptotically satisfied.

The investigation of the flow of an incompressible fluid past a flat plate has thus shown that the flow is undisturbed except in a very thin layer of thickness:

$$\Delta = x \, \frac{2.26}{\sqrt{R_x}} \tag{4.34}$$

and that in the disturbed region, there is no pressure gradient. But this is exactly the conclusion which Prandtl's boundary layer theory leads to. It has essentially been shown therefore, that the kinematic part of the solution corresponds to flow in a boundary layer, while the dynamic part of the solution is, at least some distance away from the leading edge, of negligible importance. One of the aims of this investigation is to verify whether the kinematic part of the solution corresponds to a boundary layer in which the fluid is slowed down without pressure gradients, for a compressible fluid also. The other aim will be to examine the dynamic part of the solution and to determine under what conditions it may be expected to play an important role.

5. PROPAGATION OF ONE-DIMENSIONAL SMALL DISTURBANCES IN A REAL FLUID

In order to have an idea of the importance of the dynamic part of the solution, it is well to investigate a physical problem in which it occurs alone. One may thus obtain a feeling for the type of conditions under which it may be of signigicance.

One is thus led to the study of the propagation of small disturbances in an infinite fluid initially at rest. To simplify the analysis, the investigation is restricted to one-dimensional plane, spherical and cylindrical distrubances. The results obtained are for the most part identical with those found by C. Possio(10).

The problem is essentially to solve equation (3.27):

$$\frac{D}{D\tau}\left[\frac{D^2\varphi}{D\tau^2}-\nabla^2\varphi\right]-\nabla^2\left[\left(1+\frac{3}{4}K\right)\frac{D^2\varphi}{D\tau^2}-\frac{3}{4}\frac{K}{\gamma}\nabla^2\varphi\right]-\frac{3}{4}K\frac{D}{D\tau}\nabla^4\varphi=0 \qquad (3.27)$$

In the discussion, only a viscous non-conducting fluid is considered, so that $K=0$. Furthermore, since the disturbance propagates in a fluid initially at rest, one has $M=0$, and the equation takes the simple form:

$$\frac{\partial^2\varphi}{\partial\tau^2}-\nabla^2\varphi=\frac{\partial}{\partial\tau}\nabla^2\varphi \qquad (5.1)$$

This equation is the generalization of the equation derived by Stokes in 1857 (5) and it will be analysed in some detail. It is notable that equation (5.1) differs from the ideal fluid equation only by the last term. But in physical coordinates, equation (5.1) is:

$$\frac{\partial^2\varphi}{\partial t^2}-c^2\nabla^2\varphi=\frac{4}{3}\nu\frac{\partial}{\partial t}\nabla^2\varphi \qquad (5.2)$$

Therefore, the additional viscous term is smaller than the other two terms unless there are large gradients in the solution. But it is well known that any disturbance travels unchanged in an ideal fluid.

It is in the regions of the time-space domain where large gradients would have existed if the fluid were ideal that the additional viscous term becomes significant. Its effect may be surmised qualitatively, on both physical and mathematical grounds. For it is known that a real fluid cannot support any discontinuities, the mechanism of viscosity being present to smooth them out. Also, while the ideal fluid equation is hyperbolic and thus implies a finite signal velocity, equation (5.1) is of parabolic type; the signal velocity is infinite and a diffusive process is taking place, the result of which must be to smear out any discontinuities. The problem is therefore to determine how the "smearing" process affects the solution obtained in the case of the ideal fluid, and more precisely, what role the characteristics of the previous solution play in this case.

The first problem to be discussed is that of a disturbance of strength φ_0 created suddenly at the time $t=0$ and maintained constant through the rest of time. The other problem is that of an instantaneous impulse.

Consider first a one-dimensional space such as would exist inside a long open straight pipe filled with fluid. The equation here takes the form:

$$\frac{\partial^2\varphi}{\partial\tau^2}-\frac{\partial^2\varphi}{\partial\xi^2}-\frac{\partial^3\varphi}{\partial\tau\partial\xi^2}=0 \qquad (5.3)$$

with the initial conditions:

$$\varphi(0,x)=0 \qquad \frac{\partial\varphi}{\partial t}(0,x)=0 \qquad (5.4)$$

and the boundary conditions:

$$\begin{aligned}&\varphi(\tau,0)=0 \qquad -\infty<\tau<0 \\ &\varphi(\tau,0)=\varphi_0 \qquad 0<\tau<\infty\end{aligned} \qquad \lim_{\xi\to\infty}\varphi(\xi,\tau)<\infty \qquad (5.5)$$

Such a set of conditions appears particularly adapted to a solution by means of the Laplace Transformation. The transformed equation is:

$$\frac{d^2\Phi}{d\xi^2} - \frac{s^2}{1+s}\,\Phi = 0 \qquad (5.6)$$

with the boundary conditions:

$$\Phi(0,s) = \frac{\varphi_0}{s} \qquad\qquad \lim_{\xi\to\infty} \Phi(\xi,s) < \infty \qquad (5.7)$$

so that the solution in the (ξ, s) plane is:

$$\Phi = \frac{\varphi_0}{s}\, e^{-\frac{s\xi}{\sqrt{1+s}}} \qquad (5.8)$$

and the solution in the (ξ, τ) plane is obtained by an inverse Laplace Transformation:

$$\varphi(\xi,\tau) = \frac{\varphi_0}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{s\tau}\, e^{-\frac{s\xi}{\sqrt{1+s}}}\,\frac{ds}{s} \qquad (5.9)$$

The integrand of (5.9) has a simple pole of residue $\mathcal{R}=1$ at the origin and a branch point and an essential singularity at the point $s=-1$. c represents any real positive number.

The contour of integration in the s plane is therefore the line AB (see Figure 1). But since the integrand is regular in the region $\mathcal{D}$, the contour AB may be replaced by the contour C.

The integral along the large arcs B'C and G'A is easily evaluated. Along those arcs, $s = Re^{i\theta}$ * where $R \to \infty$. Thus, between B' and C, the integrand is given by:

$$\left| e^{s\tau - s\frac{\xi}{\sqrt{1+s}}} \right| = \left| \exp\left\{ Re^{i\theta}\left(\tau - \frac{\xi}{\sqrt{1+Re^{i\theta}}}\right)\right\}\right| \leq \qquad (5.10)$$
$$\leq \exp\left(R\tau\cos\theta - K\xi\sqrt{R}\cos\frac{\theta}{2}\right)$$

*The symbol R introduced at this point to designate a very large radius in the complex s plane must not be confused with the symbol for the universal gas constant or the Reynolds Number.

Here, K is a numerical constant which approaches 1 uniformly as $R \to \infty$. ξ and τ are positive numbers. Since the contour is on the left of the imaginary axis, $\cos\theta < 0$ and along B'C, the exponent is dominated by $-A\sqrt{R}$ so that the integral along B'C certainly vanishes. A slight difficulty is encountered in the vicinity of A' where $\cos\theta \to 0$ while $\cos\frac{\theta}{2} < 0$. But, for any given set of values ξ, τ and any value of θ it is always possible to select an R(θ) such that the integral converges to 0. Thus, the integral along the arcs B'C, GA' vanishes.

Similarly, along the segments BB', AA', s becomes $Rl\,s + iR$. Then, since Rl s can be made as small as desired, $e^{s\tau}$ can always be dominated by a finite constant while $\frac{-s\xi}{\sqrt{1+s}}$ is of order $-\xi\sqrt{R}$ and the integral along BB', AA' vanishes. Thus:

$$\varphi = \frac{\varphi_0}{2\pi i}\int_C e^{s\tau}\, e^{\frac{-s\xi}{\sqrt{1+s}}}\ \frac{ds}{s} \qquad (5.11)$$

where the contour C is CDEFG on the Figure.

Since this contour plays an important part here and further along in the analysis, it is described in some detail. It runs from $s = -\infty + i\epsilon$ along the top of the cut to the point $-2+i\epsilon$. It then goes along a circle of unit radius of center $s=-1$ with an indentation which leaves the pole at the origin outside the region $\mathcal{D}$ and to the point $-2-i\epsilon$ at the bottom of the cut. It then runs back along the lower edge of the cut from $-2-i\epsilon$ to $-\infty-i\epsilon$ where it joins the arc GA'.

It is convenient to introduce at this point the following transformation, which maps the s plane on the k plane:

$$\frac{-is}{\sqrt{1+s}} = 2k \qquad\qquad s = -2k\left(k \pm \sqrt{k^2-1}\right) \qquad (5.12)$$

$$\frac{ds}{s} = 2i\,dk\left(\frac{1}{k} \pm \frac{1}{\sqrt{k^2-1}}\right) \qquad (5.13)$$

The contour C is mapped as follows: As s runs above the cut from $-\infty$ to -2, k goes from $-\infty$ to -1 along the real k axis, with the transformation $s = -2k(k-\sqrt{k^2-1})$. The first quadrant of the circle corresponds to the segment $-1<k<-\frac{1}{\sqrt{2}}$ with $s=-2k(k+\sqrt{k^2+1})$; the second quadrant maps into $-\frac{1}{\sqrt{2}}<k<0$; the third quadrant into $0<k<\frac{1}{\sqrt{2}}$ and the fourth into $\frac{1}{\sqrt{2}}<k<1$; the path back along the bottom of the cut corresponds to $1<k<\infty$ with the transformation still $s=-2k(k+\sqrt{k^2-1})$

The integral (5.11) thus becomes:

$$\varphi = \frac{1}{2}\varphi_0 + \frac{\varphi_0}{\pi}\int_1^\infty e^{-2k(k+\sqrt{k^2-1})\tau}\left(\frac{\sin 2k\xi}{k} + \frac{\cos 2k\xi}{\sqrt{k^2-1}}\right)dk \qquad (5.14)$$
$$+\frac{\varphi_0}{\pi}\int_{-1}^{1} e^{-2k^2\tau}\, e^{2ik(\xi-\tau\sqrt{1-k^2})}\left(\frac{1}{k}+\frac{i}{\sqrt{1-k^2}}\right)dk$$

When the definition of the dimensionless variables ξ, τ is recalled, it becomes clear that except in the immediate vicinity of the origin of the x,t plane, their value is extremely large. If one takes the values c = 33,100 cm/sec. and ν = 0.15 cm /sec., then, it is found that a time interval of 1 second corresponds to $\tau = 5.44\times10^{10}$ while a distance of 1 cm. corresponds to $\xi = 1.65\times10^{5}$. It is therefore physically reasonable to analyse the solution in the asymptotic case when $\tau\to\infty$. In that case, the first integral gives a negligible contribution, while the second integral may be approximated by:

$$\varphi = \varphi_0\left[\frac{1}{2} - \frac{1}{\pi}\int_0^\infty e^{-2k^2\tau}\sin 2k(\xi-\tau)\frac{dk}{k}\right] \qquad (5.15)$$

and the result of the integration is found to be:

$$\varphi = \frac{1}{2}\varphi_0\left[1 - \operatorname{erf}\frac{\xi-\tau}{\sqrt{2\tau}}\right] \qquad (5.16)$$

or, in terms of physical parameters,

$$\varphi = \frac{1}{2}\varphi_0\left[1 - \operatorname{erf}\frac{x-ct}{\sqrt{\frac{8}{3}\nu t}}\right] \qquad (5.17)$$

in agreement with Possio's result (10). The details of the calculation are omitted here to preserve the continuity of the discussion; they are found in Appendix II.

The thickness of such a perturbation front can be estimated as follows: In dimensionless parameters:

$$\bar{\Delta} = \int_0^\infty \frac{1}{2}\left[1 - \operatorname{erf}\frac{\xi-\tau}{\sqrt{2\tau}}\right] d\xi = 4.52\sqrt{\tau} \tag{5.18}$$

or in physical parameters:

$$\Delta(t) = \frac{\bar{\Delta} c}{\frac{4}{3}\nu} = 5.25\sqrt{\nu t} \tag{5.19}$$

This is essentially a displacement thickness; Possio calculates a thickness defined as the width within which the intensity of the disturbance falls to 1% and his value is $5.8\sqrt{\nu t}$ which agrees in order of magnitude with the present result.

C. Possio obtains his similar results by using a different method of solution. Instead of attacking equation (5.1) by separating the time variable through a Laplace Transformation, he carries out a straightforward separation of variables, writing: $\varphi = T(\tau)\,\Xi(\xi)$, which gives:

$$T''\Xi - T\Xi'' - T'\Xi'' = 0 \tag{5.20}$$

or, when the variables are separated,

$$\Xi'' + 4k^2\,\Xi = 0 \tag{5.21}$$

$$T'' + 4k^2(T' + T) = 0 \tag{5.22}$$

This suggests a solution of the form:

$$\varphi = \int_{-\infty}^{\infty} F(k)\, e^{-2k(k \pm \sqrt{k^2-1})\tau}\, e^{2ik\xi}\, dk \tag{5.23}$$

Now, the solution (5.23) is identical with the solution (5.14). But the determination of the function F(k) to fit the boundary and initial conditions is not simple, because they are given along the τ axis while equation (5.23) is better suited to a problem of boundary conditions along the ξ axis. The transformation (5.12) is the bridge from one method of solution to the other, and demonstrates their equivalence.

The physical meaning of the solution (5.17) is easily interpreted. The disturbance created and maintained at the origin, instead of propagating as a sharp wave front, at a constant signal velocity c:

$$\varphi = \frac{\varphi_0}{2}\left(x - ct - |x - ct|\right) \tag{5.24}$$

now propagates as a smeared front; the center of gravity of the disturbance still travels at constant velocity c; but as time goes on, the sharp wave front spreads about its center, its thickness increasing proportionally to the square root of the elapsed time. As the viscosity coefficient approaches 0 continuously, it is seen that the ideal solution is also approached continuously.

It is instructive to study how the "Fourier components" which remain grouped in the ideal solution, spread because of the distribution of phase velocities in the viscous fluid. Indeed, if one considers the component of the solution which corresponds to the wave-number range $k, k+dk$ in the solution (5.23), the phase velocity of that component is seen to be:

$$v_p = \frac{1}{\sqrt{1-k^2}} \qquad 0 \leq |k| \leq 1 \tag{5.25}$$

Since the range $|k| \leq 1$ contributes the entire asymptotic solution developed above, it follows that the phase velocity is distributed over the entire range between 1 and ∞ in the ξ,τ plane or c,∞ in the x,t plane.

These considerations suggest another method of obtaining the asymptotic solution (5.17). Since one has a phenomenon of diffusion superimposed on a hyperbolic pattern, one is tempted to make the Galilean transformation:

$$a = \xi - \tau \qquad ; \qquad b = \tau \tag{5.26}$$

The fundamental equation (5.1) then becomes:

$$\frac{\partial^2 \varphi}{\partial b^2} - 2\frac{\partial^2 \varphi}{\partial a \partial b} = \frac{\partial^3 \varphi}{\partial a^2 \partial b} - \frac{\partial^3 \varphi}{\partial a^3} \tag{5.27}$$

which can be solved by satisfying simultaneously:

$$\frac{\partial \varphi}{\partial b} = \frac{\partial^2 \varphi}{\partial a^2} \qquad ; \qquad \frac{\partial \varphi}{\partial a} = 0 \tag{5.28}$$

But equation (5.28) is the familiar heat conduction equation. The boundary conditions are:

$$\varphi(b,0) = \varphi_0 \qquad ; \qquad \varphi(0,a) = \tfrac{1}{2}\varphi_0 \tag{5.29}$$

meaning that the disturbance φ_0 is imposed along the b axis and that it is symmetric about the line (o,a) along which its center travels. The second part of (5.28) can be considered as an auxiliary condition to be approximately satisfied. Now it is well known that the solution of the heat conduction equation with boundary conditions (5.29) is given by:

$$\varphi(a,b) = \tfrac{1}{2}\varphi_0 \left(1 - \operatorname{erf}\frac{a}{\sqrt{2b}}\right) \tag{5.30}$$

which is identical with the solution (5.17). It remains to see how nearly the condition (5.28) is satisfied.

$$\frac{\partial \varphi}{\partial a} = \sqrt{\frac{\pi}{2b}}\, e^{-\frac{a^2}{2b}} \tag{5.31}$$

The parameter $2b = 2\tau$ is very large, so that $\frac{\partial\varphi}{\partial a}$ is certainly small everywhere except in the vicinity of $a=0$ where it is given by

$$\frac{\partial\varphi}{\partial a} \simeq \sqrt{\frac{2\pi\nu}{3}}\,\frac{\varphi_0}{c}\,\frac{1}{\sqrt{t}} \tag{5.32}$$

The value of $\frac{\partial\varphi}{\partial a}$ is therefore quite small everywhere and the approximation that it vanishes is a reasonable one, away from the origin. This constitutes a more intuitive derivation of the result obtained formally above. One effect of viscosity, the smearing out of sharp discontinuities, is exhibited in this simple example. Another property of viscous fluids, energy dissipation, is not clearly shown here since energy is fed into the system to keep the wave front expanding, even in the ideal case, and the amount of energy needed to balance viscous dissipation is not clearly in evidence.

To show how dissipation affects the propagation of a disturbance in a viscous fluid, it is more convenient to analyse the propagation of an instantaneous impulse. It is well known that the solution in this case is obtained by differentiating the previous solution with respect to x. One has therefore in this case:

$$\varphi = \frac{3\psi_0 c}{4\nu}\left\{\frac{2}{\pi}\int_1^\infty e^{-2k(k+\sqrt{k^2-1})\tau}\left[\cos 2k\xi - \frac{k}{\sqrt{k^2-1}}\sin 2k\xi\right]dk + \frac{2}{\pi}\int_0^1 e^{-2k^2\tau}\,e^{2ik(\xi-\tau\sqrt{1-k^2})}\left[1+\frac{ik}{\sqrt{1-k^2}}\right]dk\right\} \tag{5.33}$$

which is obtained by differentiating the solution (5.13) under the integral; the integral (5.21) is uniformly convergent; the integral (5.33) must be checked for uniform convergence to justify the differentiation; but it is clearly convergent. When the same asymptotic approximations are made as before, it is found that:

$$\varphi = \frac{3\psi_0 c}{4\nu\pi}\int_0^\infty e^{-2k^2\tau}\cos 2k(\xi-\tau)\,dk \tag{5.34}$$

which is integrated into:

$$\varphi = \frac{\psi_0}{\sqrt{\pi t \nu}} \, e^{-\frac{(x-ct)^2}{8/3\,\nu t}} \tag{5.35}$$

where the details of the limiting processes and integrations which lead from (5.33) to (5.35) are to be found in Appendix II.

Thus, the amplitude of the disturbance some distance away from its origin decreases as the square root of elapsed time. The type of smearing observed previously is present here also. The center of the disturbance still follows the path which it would have taken if the fluid were ideal.

Before leaving the subject of the propagation of small one-dimensional disturbances in slightly viscous fluids, it is easy to discuss spherically and cylindrically symmetric disturbances of the same type as the plane disturbances just described.

In the case of spherically symmetric disturbances, one has:

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial\rho^2} + \frac{2}{\rho}\frac{\partial\varphi}{\partial\rho} \qquad \rho = \frac{r c}{\frac{4}{3}\nu}\,^* \tag{5.36}$$

so that equation (5.1) takes the form:

$$\frac{\partial^2\varphi}{\partial\tau^2} - \left(\frac{\partial^2\varphi}{\partial\rho^2} + \frac{2}{\rho}\frac{\partial\varphi}{\partial\rho}\right) = \frac{\partial^3\varphi}{\partial\tau\partial\rho^2} + \frac{2}{\rho}\frac{\partial^2\varphi}{\partial\tau\partial\rho} \tag{5.37}$$

It is convenient here to separate the variables and write $\varphi = T(\tau)P(\rho)$

One then finds that equation (5.37) leads to:

$$\frac{T''}{T+T'} = \frac{P'' + \frac{2}{\rho}P'}{P} \tag{5.38}$$

so that

$$T = e^{-2ik(k \pm \sqrt{k^2-1})\tau} \tag{5.39}$$

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*The symbol ρ used here is not to be confused with the density which is also designated by ρ in equations (21./2.4), but does not appear anywhere else.

$$P = \frac{1}{\sqrt{\rho}} J_{-1/2}(2k\rho) = \sqrt{\frac{2}{\pi}} \frac{\cos 2k\rho}{\rho} \tag{5.40}$$

A general form of solution is therefore:

$$\varphi = \frac{1}{\rho}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} G(k)\, e^{-2k(k\pm\sqrt{k^2-1})\tau} \cos 2k\rho\, dk \tag{5.41}$$

Let the initial condition in this case be:

$$\begin{aligned} \varphi &= \varphi_0 \qquad (0<\rho<\rho_0) \\ \varphi &= 0 \qquad (\rho_0<\rho<\infty) \end{aligned} \tag{5.42}$$

with the additional condition:

$$\lim_{\rho\to\infty} \varphi(\rho,\tau) < \infty \tag{5.43}$$

Then, when $\tau=0$, the following relation must hold to satisfy the initial condition:

$$\rho\varphi_0 = \sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} G(k) \cos 2k\rho\, dk \tag{5.44}$$

and since φ_0 is known from equation (5.42), the function G(k) is found by an inverse transformation:

$$G(k) = \sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} \rho\varphi_0 \cos 2k\rho\, d\rho = \sqrt{\frac{2}{\pi}}\frac{\varphi_0}{4k^2}\Big[(\cos 2k\rho_0 - 1) + 2k\rho_0 \sin 2k\rho_0\Big] \tag{5.45}$$

The required solution is therefore:

$$\varphi = \frac{\varphi_0}{2\pi\rho}\int_{-\infty}^{\infty} \frac{e^{-2k(k\pm\sqrt{k^2-1})\tau}}{k^2}\Big\{\cos 2k(\rho+\rho_0) + \cos 2k(\rho-\rho_0) - \cos 2k\rho + 2k\rho_0\big[\sin 2k(\rho+\rho_0) - \sin 2k(\rho-\rho_0)\big]\Big\} dk \tag{5.46}$$

The two special cases: the disturbance started at $\tau=0$ and maintained for the rest of time, and the impulse, can be obtained from the solution (5.46) by limiting processes.

If one keeps in mind the definition of the dimensionless radius ρ given in (5.36), one sees that if a finite spherical source of radius r emits a steady signal, its dimensionless radius ρ is necessarily quite large. Then, that solution is given by:

$$\lim_{\rho_0 \to \infty} \varphi = \frac{\varphi_0 \rho_0}{\rho} \cdot \frac{2}{\pi} \int_0^\infty e^{-2k(k \pm \sqrt{k^2-1})\tau} \sin 2k(\rho-\rho_0) \frac{dk}{k} \qquad (5.47)$$

which leads to the solution:

$$\varphi = \frac{\varphi_0 r_0}{2r} \left(1 - \operatorname{erf} \frac{r - r_0 - ct}{\sqrt{\frac{8}{3}\nu t}}\right) \qquad (5.48)$$

On the other hand, an impulse at the origin suggests the limiting process $\rho_0 \to 0$ in which case the solution is of the form:

$$\lim_{\rho \to 0} \varphi = \frac{\varphi_0 \rho_0^2}{\rho} \frac{2}{\pi} \int_0^\infty e^{-2k(k \pm \sqrt{k^2-1})\tau} \cos 2k\rho \, dk \qquad (5.49)$$

and if, somewhat as for the plane impulse, the spherical impulse is defined by $\psi_0 = \varphi_0 r^2$ being now proportional to the surface of emission, the solution is, in physical coordinates:

$$\varphi = \frac{1}{r} \frac{\psi_0}{\sqrt{\pi \nu t}} \, e^{-\frac{(r-ct)^2}{8/3 \, \nu t}} \qquad (5.50)$$

The solutions in the case of spherically symmetric disturbances are therefore very similar to those found for plane disturbances. The only difference is the presence of the factor $\frac{1}{r}$ in the solution, which is introduced by the geometry of the three-dimensional space. It is well known that the factor $\frac{1}{r}$ occurs similarly in the solution of three-dimensional propagation equations for an ideal fluid.

In the case of a two-dimensional space, the equation of motion becomes:

$$\frac{\partial^2 \varphi}{\partial \tau^2} - \left(\frac{\partial^2 \varphi}{\partial \bar{\rho}^2} + \frac{1}{\bar{\rho}} \frac{\partial \varphi}{\partial \bar{\rho}}\right) = \frac{\partial^3 \varphi}{\partial \tau \partial \bar{\rho}^2} + \frac{1}{\bar{\rho}} \frac{\partial^2 \varphi}{\partial \tau \partial \bar{\rho}} \qquad (5.51)$$

where the cylindrical radius symbol carries a bar to distinguish it from the spherical radius. It is found by separating the variables that the general solution of this equation convergent at large distances from the origin, is given by:

$$\varphi = \int_0^\infty \bar{G}(k)\, e^{-2k(k \pm \sqrt{k^2-1})\tau} J_0(2k\bar{\rho})\,dk \tag{5.52}$$

The discussion of the problem, as before, involves the initial conditions:

$$\begin{aligned} \varphi &= \varphi_0 \quad (0<\bar{\rho}<\bar{\rho}_0) \\ \varphi &= 0 \quad (\bar{\rho}_0<\bar{\rho}<\infty) \end{aligned} \tag{5.53}$$

and the function $\bar{G}(k)$ is obtained by means of the Fourier-Bessel expansion theorem:

$$\begin{aligned} \varphi(\bar{\rho},0) &= \int_0^\infty 2k J_0(2k\bar{\rho}) \left[\int_0^\infty 2s\varphi_0(s) J_0(2sk)\,ds\right] dk \\ &= \varphi_0\bar{\rho}_0 \int_0^\infty J_1(2k\bar{\rho}_0) J_0(2k\bar{\rho})\,dk \end{aligned} \tag{5.54}$$

so that the solution is:

$$\varphi = \varphi_0\bar{\rho}_0 \int_0^\infty e^{-2k(k \pm \sqrt{k^2-1})\tau} J_1(2k\bar{\rho}_0) J_0(2k\bar{\rho})\,dk \tag{5.55}$$

In particular, if $\bar{\rho}_0 \to \infty$, the solution is obtained by using the asymptotic expansions of the Bessel functions:

$$J_1(2k\bar{\rho}_0) \sim \frac{1}{\sqrt{\pi k \bar{\rho}_0}} \sin 2k\bar{\rho}_0 \tag{5.56}$$

$$J_0(2k\bar{\rho}) \sim \frac{1}{\sqrt{\pi k \bar{\rho}}} \sin 2k\bar{\rho} \tag{5.57}$$

It follows that the solution is:

$$\varphi = \frac{\varphi_0}{\pi}\sqrt{\frac{\bar{\rho}_0}{\bar{\rho}}} \int_0^\infty e^{-2k(k \pm \sqrt{k^2-1})\tau} \sin 2k(\bar{\rho}-\bar{\rho}_0)\,\frac{dk}{k} \tag{5.58}$$

or in physical coordinates:

$$\varphi = \frac{1}{2}\varphi_0\sqrt{\frac{\bar{r}_0}{\bar{r}}}\left(1 - \operatorname{erf} \frac{\bar{r}-\bar{r}_0-ct}{\sqrt{8/3\,\nu t}}\right) \tag{5.59}$$

Similarly, if $\bar{\rho}_0 \to 0$, one finds that:

$$\varphi = \frac{\varphi_0\bar{\rho}_0}{\sqrt{\bar{\rho}}} \frac{2}{\pi} \int_0^\infty e^{-2k(k \pm \sqrt{k^2-1})\tau} \cos 2k\bar{\rho}\,dk \tag{5.60}$$

which leads to:

$$\varphi = \frac{1}{\sqrt{\bar{r}}} \frac{\psi_0}{\sqrt{\pi \nu t}} \, e^{-\frac{(\bar{r}-ct)^2}{8/3 \, \nu t}} \tag{5.61}$$

The effect of the two-dimensional space, here, is to introduce the factor $\frac{1}{\sqrt{\bar{r}}}$; but otherwise, these solutions are of the same type as the previous ones.

While these remarks conclude the mathematical discussion of the propagation of small one-dimensional disturbances in a viscous fluid, it is necessary to indicate, at least qualitatively, the influence of the heat conduction on the process. Since heat conduction is a diffusive process, it may be expected that the pattern found above, which is the superimposition of a diffusive process on a sharply defined propagation process, will not be radically altered. This rather vague physical inference can be made more precise as follows. In all that follows, only one-dimensional plane disturbances are discussed, but the conclusions are easily extended to other types of such disturbances.

In physical coordinates, the propagation equation is:

$$\frac{\partial}{\partial t}\left[\frac{\partial^2 \varphi}{\partial t^2} - c^2 \frac{\partial^2 \varphi}{\partial x^2}\right] = \frac{\partial^2}{\partial x^2}\left[\left(\frac{4}{3}\nu + \kappa\right)\frac{\partial^2 \varphi}{\partial t^2} - \frac{\kappa c^2}{\gamma}\frac{\partial^2 \varphi}{\partial x^2}\right] - \frac{4}{3}\nu\kappa \frac{\partial^5 \varphi}{\partial t \, \partial x^4} \tag{5.62}$$

If a characteristic lenghth exists in the system, and a Reynolds number based on it and the velocity c is introduced, equation (5.62) becomes:

$$\frac{\partial}{\partial t'}\left[\frac{\partial^2 \varphi}{\partial t'^2} - c^2 \frac{\partial^2 \varphi}{\partial x'^2}\right] = \frac{1}{R}\frac{\partial^2}{\partial x'^2}\left[\left(\frac{4}{3} + \frac{\kappa}{\nu}\right)\frac{\partial^2 \varphi}{\partial t'^2} - \frac{\kappa}{\gamma \nu}\frac{\partial^2 \varphi}{\partial x'^2}\right] - \frac{4}{3}\frac{1}{R^2}\frac{\kappa}{\nu}\frac{\partial^5 \varphi}{\partial t' \, \partial x'^4} \tag{5.63}$$

The ratio $\frac{\kappa}{\nu}$ of thermometric heat conduction to kinematic viscosity is a constant for any given gas, and of order unity. For a monatomic gas, Maxwell gives the value 2.5 for that constant. A somewhat more general expression was proposed by Eucken (17) for a poly-atomic gas:

$$\frac{K}{\nu} = \frac{9\gamma-5}{4} \tag{5.64}$$

For a diatomic gas such as air, this formula gives $\frac{K}{\nu} = 1.9$ which is known to be in good agreement with experimental results.

Under those circumstances, and with the possible exception of regions where φ changes very suddenly (inside the "shock-waves" discussed previously), the last term of equation (5.63) is of an order of magnitude smaller than the other terms and may be neglected.

Furthermore, it is seen that:

$$\frac{\partial^2\varphi}{\partial t'^2} - \frac{\partial^2\varphi}{\partial x'^2} = O\left(\frac{1}{R}\right) \tag{5.65}$$

so that $\frac{1}{R}\left(\frac{\partial^2\varphi}{\partial t'^2} - \frac{\partial^2\varphi}{\partial x'^2}\right) = O\left(\frac{1}{R^2}\right)$. It follows that equation (5.63) can be approximated by replacing the term $\partial^2\varphi/\partial^2 t'^2$ by $\partial^2\varphi/\partial x'^2$ on the right hand side, the approximation being valid to the order $O(1/R^2)$ except inside the "shock-waves". It then becomes:

$$\frac{\partial}{\partial t'}\left[\frac{\partial^2\varphi}{\partial t'^2} - \frac{\partial^2\varphi}{\partial x'^2}\right] = \frac{1}{R}\left[\frac{4}{3} + \frac{K}{\gamma}\,\frac{\gamma-1}{\nu}\right]\frac{\partial^4\varphi}{\partial x'^2\partial t'^2} \tag{5.66}$$

which is, after one integration with respect to t':

$$\frac{\partial^2\varphi}{\partial t^2} - c^2\frac{\partial^2\varphi}{\partial x^2} = \left(\frac{4}{3}\nu + K\,\frac{\gamma-1}{\gamma}\right)\frac{\partial^3\varphi}{\partial t\,\partial x^2} \tag{5.67}$$

the constant of integration being taken as 0 to satisfy the boundary condition as $x \to \infty$. Thus, it is seen that as a first approximation, the effect of heat conduction is to replace the kinematic viscosity coefficient by an effective coefficient given by:

$$\bar{\nu} = \frac{4}{3}\nu\left(1 + \frac{3}{4}\,\frac{\gamma-1}{\gamma}\,\frac{K}{\nu}\right) \tag{5.68}$$

The order of magnitude of this effect is $\frac{\bar{\nu}}{\nu}-1=0.41$. As Kirchhoff had pointed out, it is not negligible. But on the other hand, it introduces no new character into the flow pattern and merely modifies the numerical value of a constant. For instance, the displacement thickness of a wave front becomes:

$$\Delta = 6.33\sqrt{\nu t} \tag{5.69}$$

It has been found, in conclusion, that in a viscous heat-conducting fluid, small disturbances are propagated diffusely; the center of the disturbance is always along the line where the disturbances would have been if the fluid had been ideal. If the equation of motion of the ideal fluid is hyperbolic, the characteristics of that equation are for the real fluid, "quasi-characteristics" along which propagating disturbances are centered. This makes the wave equation such a useful approximation. The effect of viscosity and heat-conduction, with their infinite signal velocity, is essentially the smearing feature. It may be illustrated by the experience of an observer placed far ahead of the disturbance. Well before the main body of the disturbance reaches him, he is aware of a diffusion of energy. Here $c^2\frac{\partial^2\varphi}{\partial x^2} \ll \frac{\partial^2\varphi}{\partial t^2}$ because of the location of the observer, and therefore, he feels a disturbance:

$$\varphi = \frac{\varphi_0}{2}\left(1-\operatorname{erf}\frac{x}{\sqrt{2\nu t}}\right) \tag{5.70}$$

which is the value predicted by the solution (5.17). On the other hand, had the same observer been placed far behind the disturbance, where the conditions are steady again and $\frac{\partial}{\partial t}\to 0$, he would have observed $\varphi=\varphi_0$ which is again the limiting value predicted by (5.17).

6. FLOW PAST A TWO-DIMENSIONAL SEMI-INFINITE FLAT PLATE

Having completed a discussion of the incompressible boundary layer near a semi-infinite flat plate and of the propagation of small disturbances in a real compressible fluid, one is ready to investigate the flow of a real compressible fluid past a semi-infinite flat plate. From the similarity between equation (3.26) and the equation for the incompressible fluid discussed in part 4, one is led to expect a boundary layer, while the similarity between (3.27) and the propagation equation discussed in part 5 suggests that if the second degree terms of (3.27) are of hyperbolic character, there may be some disturbance along the "quasi-characteristics", in this case, the Mach lines.

The first step in the analysis is to show that the flow past the flat plate started impulsively from rest tends to a steady state as time goes on. Then, the steady motion will be investigated in greater detail, for both subsonic and supersonic mean stream velocities.

Let the problem be specified precisely as follows. Consider an undisturbed parallel flow of Mach number M. At the instant $t=0$, let a semi-infinite two-dimensional flat plate be introduced into the stream, parallel to it. The origin of a system of coordinates is then selected at the fixed leading edge of the flat plate; the ξ axis is along the surface of the plate and the η axis is normal to it. The boundary conditions on the surface of the plate, under the linearizing assumption made in this investigation, may be stated as $u=u_0$, where u_0 is then a small velocity difference between a point near the outer edge of the boundary layer, where observations are taken, and some point far removed from the flat plate. The other conditions describe the symmetry of the flow with respect to the ξ axis, the fact that no fluid passes through the surface of the plate, and the damping out of all disturbances far away from the plate. Mathematically, the initial conditions and boundary conditions are:

$$\left.\begin{aligned} v(\tau,\xi,0) &= u(\tau,\xi,0) = 0 \\ v_\tau(\tau,\xi,0) &= u_\tau(\tau,\xi,0) = 0 \end{aligned}\right\} -\infty < \tau \leq 0 \tag{6.1}$$

$$\left.\begin{aligned} u(\tau,\xi,0) &= u_0 \quad 0<\xi<\infty \\ u_\eta(\tau,\xi,0) &= 0 \quad -\infty<\xi<0 \end{aligned}\right\} \quad 0<\tau<\infty \tag{6.2}$$

$$v(\tau,\xi,0) = 0 \qquad\qquad 0<\tau<\infty \tag{6.3}$$

$$\lim_{\xi\to-\infty} u = \lim_{\xi\to-\infty} v = 0 \qquad \lim_{\eta\to\infty} u = \lim_{\eta\to\infty} v = 0 \tag{6.4}$$

As was shown in part 3, if the fluid is viscous and non-conducting, the problem is now specified. If the fluid is also conducting, the temperature distribution at the surface of the plate must still be given. But, as was shown at the end of part 5, the salient features of the flow are brought out if heat-conduction is neglected, and this idealization brings about a considerable simplification in the analysis. From this point on, therefore, the fluid will be considered non-conducting, and the effect of heat conduction may be approximated by substituting $\bar{\nu}$ for $\frac{4}{3}\nu$ in the final equations.

The equations of motion are conveniently written in vector form; in the Cartesian system used here, this is merely a device for combining two identical equations for u and v into one. The equations of motion are:

$$\bar{u} = \bar{u}_1 + \bar{u}_2 \tag{6.5}$$

$$\frac{\partial \bar{u}_1}{\partial \tau} + M\frac{\partial \bar{u}_1}{\partial \xi} = \frac{3}{4}\left(\frac{\partial^2 \bar{u}_1}{\partial \xi^2} + \frac{\partial^2 \bar{u}_1}{\partial \eta^2}\right) \tag{6.6}$$

$$\frac{\partial^2 \bar{u}_2}{\partial \tau^2} + 2M\frac{\partial^2 \bar{u}_2}{\partial \tau \partial \xi} + (M^2-1)\frac{\partial^2 \bar{u}_2}{\partial \xi^2} - \frac{\partial^2 \bar{u}_2}{\partial \eta^2} = \left(\frac{\partial}{\partial \tau} + M\frac{\partial}{\partial \xi}\right)\left(\frac{\partial^2 \bar{u}_2}{\partial \xi^2} + \frac{\partial^2 \bar{u}_2}{\partial \eta^2}\right) \tag{6.7}$$

Following the scheme outlined in part 3, one makes a Laplace Transformation of those equations. Since the initial conditions imply a start from rest, the transformed equations are that much simpler:

$$\bar{u}_1' + \bar{u}_2' = \bar{u}' \tag{6.8}$$

$$s\bar{u}_1' + M\frac{\partial \bar{u}_1'}{\partial \xi} = \frac{3}{4}\left(\frac{\partial^2 \bar{u}_1'}{\partial \xi^2} + \frac{\partial^2 \bar{u}_1'}{\partial \eta^2}\right) \tag{6.9}$$

$$s\bar{u}_2' + 2Ms\frac{\partial \bar{u}_2'}{\partial \xi} + (M^2-1)\frac{\partial^2 \bar{u}_2'}{\partial \xi^2} - \frac{\partial^2 \bar{u}_2'}{\partial \eta^2} = (s + M\frac{\partial}{\partial \xi})\left(\frac{\partial^2 \bar{u}_2'}{\partial \xi^2} + \frac{\partial^2 \bar{u}_2'}{\partial \eta^2}\right) \tag{6.10}$$

The primed quantities here refer to the transforms of the unprimed quantities. The second step involves a Laplace Transformation with respect to ξ and here, greater care must be exercised, since the conditions along the line $\xi = 0$ are not known and must be determined from the regularity of the solution, as a function of the complex variables s, λ in some right half-plane $R|\lambda > \lambda_0$, including eigen values of λ . The details of this procedure are outlined in Doetsch's "Laplace Transformation" and summarized in Appendix I. With this remark in mind, one transforms the system (6.8/6.10) into:

$$\bar{U}_1 + \bar{U}_2 = \bar{U} \tag{6.11}$$

$$\left[s + \lambda M - \frac{4}{3}\lambda^2\right]\bar{U}_1 - \frac{3}{4}\frac{d^2\bar{U}_1}{d\eta^2} = F_1(\eta) \tag{6.12}$$

$$\frac{d^2\bar{U}_2}{d\eta^2} + \left(\lambda^2 - \frac{s+\lambda M}{1+s+\lambda M}\right)U_2 = F_2(\eta) \tag{6.13}$$

The functions F_1, F_2 are known functions of η which need not be calculated explicitly yet. To solve the total differential equations (6.12), (6.13), one follows the usual procedure of writing the solution as the sum of the solution of the homogeneous equation which satisfies the boundary conditions of the problem, and a particular solution of the non-homogeneous equation which staisfies zero boundary conditions. The required solutions are then:

$$U_1 = U_1(0)\, e^{-\eta\sqrt{\frac{4}{3}(s+\lambda M)-\lambda^2}} - e^{\eta\sqrt{\frac{4}{3}(s+\lambda M)-\lambda^2}} \int_0^\eta e^{-2\alpha\sqrt{\frac{4}{3}(s+\lambda M)-\lambda^2}} \cdot \int_0^\alpha e^{\beta\sqrt{\frac{4}{3}(s+\lambda M)-\lambda^2}} F_1(\beta)\, d\beta\, d\alpha \tag{6.14}$$

$$\mathcal{U}_2 = \mathcal{U}_0(0)\, e^{-\eta\sqrt{\frac{(s+\lambda M)^2}{1+s+\lambda M}-\lambda^2}}$$
$$- e^{\eta\sqrt{\frac{(s+\lambda M)^2}{1+s+\lambda M}-\lambda^2}} \int_0^{\eta} e^{-2\alpha\sqrt{\frac{(s+\lambda M)^2}{1+s+\lambda M}-\lambda^2}} \int_0^{\alpha} e^{\beta\sqrt{\frac{(s+\lambda M)^2}{1+s+\lambda M}-\lambda^2}}\, F_2(\beta)\, d\beta\, d\alpha \qquad (6.15)$$

Note that the exponent of (6.15) is that found in the general discussion in equation (3.42).

The boundary condition valid in the quadrant $\xi > 0,\ \eta > 0$ is given by the relations:

$$\mathcal{U}_1(0) + \mathcal{U}_2(0) = \frac{u_0}{\lambda s} \qquad (6.16)$$

$$V_1(0) + V_2(0) = 0 \qquad (6.17)$$

From equations (3.32), (3.36), the relations between $\mathcal{U}_1, \mathcal{U}_2, V_1, V_2$ are also known. They are in this case:

$$\sqrt{\tfrac{4}{3}(s+\lambda M)-\lambda^2}\; V_1 = \lambda\, \mathcal{U}_1 \qquad (6.18)$$

$$\lambda V_2 = -\sqrt{\frac{(s+\lambda M)^2}{1+s+\lambda M}-\lambda^2}\; \mathcal{U}_2 \qquad (6.19)$$

From equations (6.16/6.19), the four constants of integration are calculated as follows:

$$\mathcal{U}_1 = \frac{u_0}{\lambda s}\, \frac{-\sqrt{\left[\frac{(s+\lambda M)^2}{1+s+\lambda M}-\lambda^2\right]\left[\frac{4}{3}(s+\lambda M)-\lambda^2\right]}}{\lambda^2 - \sqrt{\left[\frac{(s+\lambda M)^2}{1+s+\lambda M}-\lambda^2\right]\left[\frac{4}{3}(s+\lambda M)-\lambda^2\right]}} \qquad (6.20)$$

$$\mathcal{U}_2 = \frac{u_0}{\lambda s}\, \frac{\lambda^2}{\lambda^2 - \sqrt{\left[\frac{(s+\lambda M)^2}{1+s+\lambda M}-\lambda^2\right]\left[\frac{4}{3}(s+\lambda M)-\lambda^2\right]}} \qquad (6.21)$$

$$V_1 = \frac{u_0}{\lambda s}\, \frac{+\lambda\sqrt{\frac{(s+\lambda M)^2}{1+s+\lambda M}-\lambda^2}}{\lambda^2 - \sqrt{\left[\frac{(s+\lambda M)^2}{1+s+\lambda M}-\lambda^2\right]\left[\frac{4}{3}(s+\lambda M)-\lambda^2\right]}} \qquad (6.22)$$

$$V_2 = \frac{u_0}{\lambda s}\, \frac{-\lambda\sqrt{\frac{(s+\lambda M)^2}{1+s+\lambda M}-\lambda^2}}{\lambda^2 - \sqrt{\left[\frac{(s+\lambda M)^2}{1+s+\lambda M}-\lambda^2\right]\left[\frac{4}{3}(s+\lambda M)-\lambda^2\right]}} \qquad (6.23)$$

The formal solution for the velocity component u can now be written down as a complex integral:

$$u = -\frac{u_0}{4\pi^2}\int_{c-i\infty}^{c+i\infty}\int_{c'-i\infty}^{c'+i\infty}\frac{\lambda^2 e^{-\eta\sqrt{\frac{(s+\lambda M)^2}{1+s+\lambda M}-\lambda^2}}-\sqrt{\left[\frac{(s+\lambda M)^2}{1+s+\lambda M}-\lambda^2\right]\left[\frac{4}{3}(s+\lambda M)-\lambda^2\right]}\,e^{-\eta\sqrt{\frac{4}{3}(s+\lambda M)-\lambda^2}}}{\lambda^2-\sqrt{\left[\frac{(s+\lambda M)^2}{1+s+\lambda M}-\lambda^2\right]\left[\frac{4}{3}(s+\lambda M)-\lambda^2\right]}}\,e^{\lambda\xi+s\tau}\,\frac{d\lambda\, ds}{\lambda s} \tag{6.24}$$

$$-\frac{u_0}{4\pi^2}\int_{c-i\infty}^{c+i\infty}\int_{c'-i\infty}^{c'+i\infty}\Big\{e^{\eta\sqrt{\frac{4}{3}(s+\lambda M)-\lambda^2}}\int_0^{\eta}e^{-2\alpha\sqrt{\frac{4}{3}(s+\lambda M)-\lambda^2}}\int_0^{\alpha}e^{\beta\sqrt{\frac{4}{3}(s+\lambda M)-\lambda^2}}F_1(\beta)\,d\beta\, d\alpha + e^{\eta\sqrt{\frac{(s+\lambda M)^2}{1+s+\lambda M}-\lambda^2}}\int_0^{\eta}e^{-2\alpha\sqrt{\frac{(s+\lambda M)^2}{1+s+\lambda M}-\lambda^2}}\int_0^{\alpha}e^{\beta\sqrt{\frac{(s+\lambda M)^2}{1+s+\lambda M}-\lambda^2}}F_2(\beta)\,d\beta\, d\alpha\Big\}\,e^{s\tau+\lambda\xi}\,ds\, d\lambda$$

Consider the singularities of the integrand as a function of the complex variable s with the parameters $\tau, \xi, \eta > 0$; λ imaginary.

There is a simple pole at $s=0$; there is a pole at the points:

$$s+\lambda M = 0 \qquad\qquad s+\lambda M = \lambda\left[\frac{7}{8}\lambda \pm\sqrt{1+\frac{49}{64}\lambda^2}\right] \tag{6.25}$$

at which the expression:

$$\lambda^2-\sqrt{\left[\frac{(s+\lambda M)^2}{1+s+\lambda M}-\lambda^2\right]\left[\frac{4}{3}(s+\lambda M)-\lambda^2\right]} = 0 \tag{6.26}$$

has a zero. The first part of (6.25) implies $\mathrm{Rl}\, s = 0$. The second part implies $\mathrm{Rl}\, s < 0$ everywhere except possibly in the vicinity of the origin. Near $\lambda = 0$, λ must make an indentation to the right about the origin, but the radius of that indentation can always be made small enough so that for any given M, $\mathrm{Rl}\, s \leq 0$ still holds. Thus, for all the poles of the integrand, one has the relation: $\mathrm{Rl}\, s \leq 0$. There is also an essential singularity at $1+s+\lambda M = 0$ for which also $\mathrm{Rl}\, s < 0$.

It is clear from the argument just made that the integrand is regular in the half-plane $s > 0$. The contour $c-i\infty < s < c+i\infty$ may therefore be changed into a contour of type C (Figure 1). Furthermore, the integral along the arcs BB'C, GA'A vanishes just as integral (5.9) vanished.

Now, if the contour C is modified into a contour C' to include the origin within region $\mathcal{D}$, the solution (6.24) can be divided into a contour integral about C' and a set of singularities for which

$R\,l\,s<0$ holds, on the one hand, and a pole at the origin on the other hand. The pole $s+\lambda M=0$ has a residue of value 0 as can easily be verified, and consequently contributes nothing to the integral.

But if the solution exists for $\tau=0$, as the problem postulates it to exist, then, the first of the two parts just mentioned, which is proportional to $e^{s\tau}$, must decrease exponentially as time goes on, since at all points along C', the relation $R\,l\,s<0$ holds. Meanwhile, the part of the solution which comes from the pole at the origin is independent of time and therefore represents a steady state solution. The solution (6.24) is therefore broken into a steady and a transient part, and it has been shown that the transient part vanishes as time goes on.

The existence of the steady state solution being thus proved, no more need be said about the transient solution, and the rest of the analysis is concerned with working out a closed asymptotic expression for the steady state solution. This can be done either by returning to the original equations, and solving them with the time-dependent terms omitted, or better, by investigating the pole at the origin in the solution (6.24). That pole can be written as:

$$u=\frac{u_0}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\lambda\xi}\,\frac{e^{-\eta\lambda\sqrt{\frac{M^2}{1+\lambda M}-1}}-\sqrt{\left(\frac{M^2}{1+\lambda M}-1\right)\left(\frac{4}{3}\frac{M}{\lambda}-1\right)}\,e^{-\eta\sqrt{\frac{4}{3}M\lambda-\lambda^2}}}{1-\sqrt{\left(\frac{M^2}{1+\lambda M}-1\right)\left(\frac{4}{3}\frac{M}{\lambda}-1\right)}}\,\frac{d\lambda}{\lambda}$$

$$+\frac{u_0}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\lambda\xi}\frac{d\lambda}{\lambda}\Big\{e^{\eta\sqrt{\frac{4}{3}M\lambda-\lambda^2}}\int_0^{\eta} e^{-2\alpha\sqrt{\frac{4}{3}M\lambda-\lambda^2}}\int_0^{\alpha} e^{\beta\sqrt{\frac{4}{3}M\lambda-\lambda^2}}F_1(\beta)\,d\beta\,d\alpha+e^{\lambda\eta\sqrt{\frac{M^2}{1+\lambda M}-1}}\int_0^{\eta} e^{-2\alpha\lambda\sqrt{\frac{M^2}{1+\lambda M}-1}}\int_0^{\alpha} e^{\beta\lambda\sqrt{\frac{M^2}{1+\lambda M}-1}}F_2\,d\beta\,d\alpha\Big\} \qquad (6.27)$$

Equation (6.27) is now investigated in greater detail. To simplify the discussion, it is convenient to divide it into four parts, concerned respectively with the homogeneous and the non-homogeneous parts of u_1 and u_2, and to discuss each part separately.

The homogeneous part of u_1 is given by the integral:

$$(u_1)_h = \frac{u_0}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\sqrt{\left(\frac{M^2}{1+\lambda M}-1\right)\left(\frac{4}{3}\frac{M}{\lambda}-1\right)}}{1-\sqrt{\left(\frac{M^2}{1+\lambda M}-1\right)\left(\frac{4}{3}\frac{M}{\lambda}-1\right)}}\, e^{-\eta\sqrt{\frac{4}{3}M\lambda-\lambda^2}}\, e^{\lambda\xi}\,\frac{d\lambda}{\lambda} \tag{6.28}$$

The integrand has a pole and a branch-point at the origin and a branch-point at $\lambda = \frac{4}{3}M$. A cut must therefore be introduced along the real axis from $-\infty$ to 0 and from $\frac{4}{3}M$ to $+\infty$. This cut causes no difficulty, and the integrand is regular in a cut right half-plane. There is also a pole at the point

$$\lambda = \frac{4(M^2-1)}{7M} \tag{6.29}$$

where

$$1-\sqrt{\left(\frac{M^2}{1+\lambda M}-1\right)\left(\frac{4}{3}\frac{M}{\lambda}-1\right)} = 0 \tag{6.30}$$

but that pole is of strength $\lambda^{-1/2}$ and therefore contributes nothing to the integral.

The contour along which the integral is to be evaluated is of the type C, with a loop about the origin, and the constant c sufficiently large to include the removable singularity to the left. This is always possible without interfering with the cut introduced into the right half plane, since when $M>0$, it is always true that:

$$\frac{4}{3}M > \frac{4(M^2-1)}{7M} \tag{6.31}$$

There is no difficulty in proving the convergence to 0 of the integrals over the quadrants B'C, GA'. Over the segments, now of finite lenghth, BB', AA', the radical in the η exponential also assures convergence to 0 since

$$|e^{\lambda\xi}| \leq e^{\frac{4(M^2-1)\xi}{7M}} < A \tag{6.32}$$

$$e^{-\eta\sqrt{\frac{4}{3}M\lambda-\lambda^2}} \sim e^{-\eta\sqrt{R}} \tag{6.33}$$

The integral is then of the same type as the significant part of equation (4.29), and is calculated somewhat in the same way. Because of the very large value of the parameters ξ, η, only that part of the integrand which is near $\lambda = 0$ contributes to the asymptotic solution, and therefore, a limiting process, carried out in detail in Appendix II, shows that as $\xi \to \infty; \eta \to \infty$, the integral (6.28) is approximated by:

$$(u_1)_h = \frac{2u_0}{\pi}\int_0^{\infty} e^{-\eta\sqrt{\frac{4}{3}M\lambda}} \; e^{i\lambda\xi} \; \frac{d\lambda}{\lambda} \tag{6.34}$$

which leads to the asymptotic formula:

$$(u_1)_h = u_0\left(1 - \operatorname{erf}\frac{\eta\sqrt{4/3M\pi}}{\sqrt{\xi}}\right) \tag{6.35}$$

It is now necessary to investigate the non-homogeneous part of u_1. This is given by:

$$(u_1)_n = \frac{u_0}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\lambda\xi} e^{\eta\sqrt{\frac{4}{3}M\lambda-\lambda^2}} \int_0^{\eta} e^{-2\alpha\sqrt{\frac{4}{3}M\lambda-\lambda^2}} \int_0^{\alpha} e^{\beta\sqrt{\frac{4}{3}M\lambda-\lambda^2}} F_1 \, d\beta\, d\alpha\, d\lambda \tag{6.36}$$

Before this integral can be discussed in detail, it is necessary to evaluate the function $F_1(\eta)$ which first appears on the right hand side of equation (6.11). In Appendix I, it is shown that must satisfy the relation:

$$\lim_{L\to\infty}\int_0^L F_1(\eta)u_n(\eta)\,d\eta + \frac{K_n}{L}\int_0^L \{u_0(L-\eta)+u_1\eta\}u_n\,d\eta = 0 \tag{6.37}$$

where L denotes the interval over which η ranges, while u_n are the eigen solutions of the homogeneous equation (6.12) and K_n are the eigen-values, which, here, form the continuous spectrum:

$$K = \sqrt{\lambda^2 - \frac{4}{3}M\lambda} \tag{6.38}$$

When the integration in (6.37) is carried out, it is found that:

$$\int_0^\infty F_1(\eta)\, e^{iK\eta}\, d\eta = -i\,U_0(\lambda) \tag{6.39}$$

so that if the Fourier integral is inverted, it follows that:

$$F_1(\eta) = -\frac{2}{\pi}\int_0^\infty U_0(\lambda) \sin K\eta \, dK \tag{6.40}$$

or, in terms of λ :

$$F_1(\eta) = -\frac{2i}{\pi}\int_0^\infty \frac{2\lambda - \frac{4}{3}M}{2\sqrt{\lambda^2 - \frac{4}{3}\lambda M}} \; \frac{\sqrt{\left(\frac{M^2}{1+\lambda M} - 1\right)\left(\frac{4}{3}\frac{M}{\lambda} - 1\right)}\; e^{-\eta\sqrt{\frac{4}{3}M\lambda - \lambda^2}}}{1 - \sqrt{\left(\frac{M^2}{1+\lambda M} - 1\right)\left(\frac{4}{3}\frac{M}{\lambda} - 1\right)}} \; \frac{d\lambda}{\lambda} \tag{6.41}$$

which is most easily estimated asymptotically, for large values of η in which case only the portion of the integrand near the origin contributes significantly. Then, $F_1(\eta)$ is:

$$F_1(\eta) = A\int_0^\infty e^{-\eta\sqrt{\lambda}}\, \frac{d\lambda}{\lambda} = B\delta \tag{6.42}$$

where δ is the Dirac function and B is some finite constant. Then, $(u_1)_\eta$ can be determined asymptotically as follows:

$$(u_1)_\eta = \frac{u_0}{2\pi i}\int_C B e^{\lambda\xi}\, e^{-\eta\sqrt{\frac{4}{3}M\lambda - \lambda^2}} \, \frac{d\lambda}{\sqrt{\frac{4}{3}M\lambda - \lambda^2}} \rightarrow 0 \tag{6.43}$$

as $\xi \rightarrow \infty, \eta \rightarrow \infty$, the approximation being of the same order as in (6.70).

It has thus been shown that the component u_1 of the velocity is given by the expression:

$$u_1 = u_0\left(1 - \operatorname{erf} \frac{y\sqrt{u}}{\sqrt{\xi}}\right) \tag{6.35}$$

which, put in terms of the physical parameters, takes the form:

$$u_1 = u_0\left(1 - \operatorname{erf}\frac{y\sqrt{u}}{\sqrt{2\nu x}}\right) \tag{6.44}$$

This expression is independent of the Mach number, and identical to that found for the incompressible fluid in equation (4.32). It depends only on the Reynolds number referred to free stream velocity and distance downstream from the leading edge. At least as a first approximation, therefore, it is seen that the boundary layer is a purely kinetic phenomenon; there is no pressure, density or temperature gradient in it. For that reason, it is independent of Mach number.

In order to discuss the order of magnitude of the effect discussed here, the displacement thickness of the boundary layer is calculated as a function of distance downstream from the leading edge:

$$\frac{\Delta}{x} = \frac{1}{x}\int_0^\infty \left(1 - \operatorname{erf}\frac{y\sqrt{u}}{2\sqrt{\nu x}}\right)dy = \frac{2.26}{\sqrt{ux/\nu}} = \frac{2.26}{\sqrt{R_x}} \tag{6.45}$$

This result is again identical with that obtained by Lord Rayleigh in the case of an incompressible fluid.

It remains to investigate the portion of the solution which satisfies an equation of the same type as the propagation equation studied in part 5. While the equation is discussed in some detail in Appendix III, the solution is worked out here on the basis of equation (6.27).

Consider first the homogeneous part of the solution:

$$(u_2)_R = \frac{u_0}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-\lambda\eta\sqrt{\frac{M^2}{1+\lambda M}-1}} \frac{e^{\lambda\xi}}{1-\sqrt{\frac{M^2}{1+\lambda M}-1}\left(\frac{3}{4}\frac{M}{\lambda}-1\right)}\frac{d\lambda}{\lambda} \tag{6.46}$$

This contour integral has the removable singularity (6.29) described above, and a pole at the origin. It also has the essential singularity at which played such an important part in the solution of the propagation equation. The contour C used there is used here again. The integrals along BC and GA are easily shown to vanish, just as they did in part 5. And along the contour C, it is convenient to introduce the transformation (5.12) in slightly modified form:

$$\frac{i\lambda M}{\sqrt{1+\lambda M}} = 2k \qquad \lambda M = -2k\left(k \pm \sqrt{k^2-1}\right) \tag{6.47}$$

$$\frac{d\lambda}{\lambda} = 2i\,dk\left[\frac{1}{k} \pm \frac{1}{\sqrt{k^2-1}}\right] \tag{6.48}$$

While the integral (6.46) cannot be carried out explicitly, the same type of asymptotic expansion which was successful with several previous integrals is used here to give an approximate evaluation.

The η exponent can be written as:

$$\alpha = -\eta\lambda\sqrt{\frac{M^2}{1+\lambda M} - 1} = \frac{2ik\eta}{M}\sqrt{(M^2-1) - \frac{\lambda}{M}} \tag{6.49}$$

Since the important part of the integrand is in the vicinity of $\lambda = 0$, the sign of the parameter M^2-1 becomes of great importance in determining the character of the solution, since the exponent is real or imaginary depending on whether M^2-1 is positive or negative. If M^2-1 is positive, equation (6.7) is of quasi-hyperbolic type; this corresponds to the hyperbolic equation of the supersonic flow of an ideal fluid; it is to be expected that the Mach lines of supersonic flow will appear as smeared out quasi-characteristics similar to those encountered in part 5. On the other hand, should M^2-1 be negative, equation (6.7) is of quasi-elliptic type, and, at least in the limit, the disturbance u_2 should disappear as it did in part 4.

These qualitative statements must now be made more precise by the actual asymptotic evaluation of the integral (6.46). When the transformation (6.47) is applied, and the rules as to choice of sign of page 26are followed, the integral becomes:

$$(u_2)_h = \frac{u_0}{\pi}\int_1^{\infty} \frac{e^{-2k(k+\sqrt{k^2-1})\xi/M}}{G(k)} \left[\frac{\sin\frac{2k\eta}{M}\sqrt{M^2-1-\frac{\lambda}{M}}}{k} + \frac{\cos\frac{2k\eta}{M}\sqrt{M^2-1-\lambda M}}{\sqrt{k^2-1}}\right] dk$$

$$+ \frac{u_0}{\pi}\int_{-1}^{1} e^{-2k^2\xi/M}\left[\frac{1}{k}+\frac{i}{\sqrt{1-k^2}}\right] e^{2i\frac{k}{M}(\eta\sqrt{M^2-1-\lambda M}-\xi\sqrt{1-k^2})} \frac{dk}{G(k)} \tag{6.50}$$

$$G(k) = 1 - \sqrt{\left(\frac{M^2}{1+\lambda M}-1\right)\left(\frac{4}{3}\frac{M}{\lambda}-1\right)} \tag{6.51}$$

As in the case of the integral (5.14), the first integral is negligibly small, compared to the second, when $\xi \to \infty$, because of the strong convergence of the exponential. The second integral is evaluated by considering what the contribution near the origin is:

$$\lim_{k\to 0} G(k) = \lim_{\lambda \to 0} 1 - \sqrt{\left(\frac{M^2}{1+\lambda M}-1\right)\left(\frac{4}{3}\frac{M}{\lambda}-1\right)} = \frac{\sqrt{M^2-1}}{M}\left(1+\frac{ik}{M^2-1}\right) \tag{6.52}$$ $$\frac{M\sqrt{M^2-1}}{\sqrt{3k}}(1+i)$$

Similarly:

$$\lim_{k\to 0}\sqrt{(M^2-1)-\lambda M} = \sqrt{M^2-1}\left(1+\frac{ki}{M^2-1}\right) \tag{6.53}$$

It may be noted that the radical in (6.52) forces an extension of the cut from $\frac{-1}{M}$ to 0, which is done without difficulty since the contour C avoids the origin by an indentation to the right.

When the approximate expressions (6.52), (6.53) are introduced into the integral (6.50), it becomes:

$$(u_2)_h = \frac{2u_0}{\pi}\sqrt{\frac{3}{2}}\frac{1}{M\sqrt{M^2-1}}\int_0^{\infty} e^{-\frac{2k^2}{M}\left(\xi+\frac{\eta}{\sqrt{M^2-1}}\right)} \cos 2k(\xi-\eta\sqrt{M^2-1})\frac{dk}{\sqrt{k}} \tag{6.54}$$

which, by considerations of symmetry and asymptotic approximation, is finally put into the form:

$$(u_2)_h = \frac{2u_0}{\pi}\sqrt{\frac{3}{2}}\frac{1}{M\sqrt{M^2-1}}\int_0^{\infty} e^{-\frac{2k^2}{M}\left(\xi+\frac{\eta}{\sqrt{M^2-1}}\right)} \cos 2k(\xi-\eta\sqrt{M^2-1})\frac{dk}{\sqrt{k}} \tag{6.55}$$

This integral is carried out in detail in Appendix II; the result turns out to be:

$$(u_2)_h = u_0\sqrt{\frac{3}{2}}\frac{1}{\sqrt{M(M^2-1)}}\left[\frac{2}{M(\xi+\eta/\sqrt{M^2-1})}\right]^{1/4} e^{-\frac{(\xi-\eta\sqrt{M^2-1})^2}{4M(\xi+\eta/\sqrt{M^2-1})}} \tag{6.56}$$

or, since $\xi = \eta\sqrt{M^2-1}$ along the line where $(u_2)_h$ does not vanish,

$$(u_2)_h = u_0\sqrt{\frac{3}{2}}\frac{1}{\sqrt{M(M^2-1)}}\left[\frac{2(M^2-1)}{M^3\xi}\right]^{1/4} e^{-\frac{(\xi-\eta\sqrt{M^2-1})^2}{\frac{4M^3}{M^2-1}\xi}} \tag{6.57}$$

which, in physical coordinates, gives:

$$(u_2)_h = \frac{u_0}{M^{5/4}(M^2-1)^{1/4}}\left(\frac{6\nu}{xc}\right)^{1/4} e^{-\frac{(x-y\sqrt{M^2-1})^2}{16M^5\nu x/3(M^2-1)c}} \tag{6.58}$$

On the other hand, if the mean flow is subsonic, and M^2-1 is negative, the limits (6.52) and (6.53) are rewritten as follows:

$$\lim_{k\to 0} G(k) = \frac{M(1-M^2)}{\sqrt{3k}}(i-1) \tag{6.59}$$

$$\lim_{k\to 0}\sqrt{M^2-1-\lambda M} = \sqrt{1-M^2}\left(i+\frac{k}{1-M^2}\right) \tag{6.60}$$

so that the integral becomes in this case:

$$(u'_2)_h = \frac{u_0}{\pi}\int_{-1}^{1} e^{-\frac{2k}{M}(k\xi+\eta\sqrt{1-M^2})}\frac{e^{\frac{2ik}{M}\left(\frac{k\eta}{\sqrt{1-M^2}}-\xi\sqrt{1-k^2}\right)}}{M\sqrt{1-M^2}}\frac{i+1}{\sqrt{3}}\frac{dk}{\sqrt{k}} \tag{6.61}$$

and, when $\xi\to\infty; \eta\to\infty$, the limiting process changes this to:

$$(u'_2)_h = \frac{2u_0}{\pi}\sqrt{\frac{3}{2}}\frac{1}{M\sqrt{1-M^2}}\int_0^\infty e^{-2k\frac{\sqrt{1-M^2}}{M}\eta}\, e^{2ik\xi/M}\,\frac{dk}{\sqrt{k}} \tag{6.62}$$

Equation (6.62) may be compared to the corresponding part of the solution found for an incompressible fluid. There, it was found that the asymptotically approximate formula was:

$$(u_2)_{h_i} = \frac{\sqrt{2}}{\pi}u_0\int_0^\infty e^{-\bar{\omega}\bar{y}}\, e^{i\bar{\omega}\bar{x}}\,\frac{d\bar{\omega}}{\bar{\omega}} \tag{6.63}$$

When $M\to 0$, the definition of the parameters ξ, η loses its significance, because the sonic velocity c then becomes infinite. It is therefore necessary to redefine the parameters.

$$\bar{x} = \frac{3\xi}{4M} = \frac{xU}{\nu} \qquad \bar{y} = \frac{3\eta}{4M} = \frac{yU}{\nu} \tag{6.64}$$

When this is done, it is immediately plain that the equations (6.62) and (6.63) become identical, so that the solution for a slightly compressible fluid approaches the solution for an incompressible fluid as $M \to 0$.

It is shown in Appendix II that the integral (6.62) can be evaluated in terms of hypergeometric functions as follows:

$$(u'_2)_h = \frac{u_0}{\sqrt{\pi M(1-M^2)}}\sqrt{\frac{2}{\xi}}\, F\left(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; -\frac{\eta^2(1-M^2)}{\xi^2}\right) \qquad (6.65)$$

so that the velocity decreases as $\frac{1}{\sqrt{\xi}}$ along a radial line from the leading edge of the lat plate. The hypergeometric series converges when $\left|\frac{\eta\sqrt{1-M^2}}{\xi}\right| \leq 1$. It is still convergent when the argument reaches the value -1, where it was summed by Kummer and found to be $\frac{\sin 3\pi/8}{2^{1/4}}$. Beyond that point, the hypergeometric function must be continued analytically, which is done as follows:

$$(u'_2)_h = \frac{u_0}{\sqrt{\pi M(1-M^2)}}\sqrt{\frac{2}{\xi}}\, \frac{\Gamma(1/2)}{\Gamma(1/4)\Gamma(3/4)} \left\{ \frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(1/4)} \sqrt{\frac{\xi}{\eta\sqrt{1-M^2}}} \right.$$
$$\left. F\left(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; -\frac{\xi^2}{\eta^2(1-M^2)}\right) + \frac{\Gamma(3/4)\Gamma(-1/2)}{\Gamma(-1/2)}\left(\frac{\xi}{\eta\sqrt{1-M^2}}\right)^{3/2} F\left(\frac{3}{4}, \frac{5}{4}; \frac{3}{2}; -\frac{\xi^2}{\eta^2(1-M^2)}\right) \right\} \qquad (6.66)$$

This analytic continuation converges when $\eta\sqrt{1-M^2} > \xi$. It is easily verified that for a given value of ξ, $(u_2)_h$ decreases as η increases. To estimate the importance of its contribution, therefore, it is calculated on the surface of the plate, 1 cm behind the leading edge. It is found that for $M = 0.5$,

$$\frac{(u'_2)_h}{u_0} = 5.05 \times 10^{-4} \qquad (6.67)$$

so that it is clear that except very near the leading edge, this contribution is very small.

Before the mathematical discussion of the solution u_2 is completed, it is necessary to examine the non-homogeneous part of the solution. From Equation (6.27), one recalls:

$$(u_2)_n = \frac{u_0}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\lambda\xi} e^{\lambda\eta\sqrt{\frac{M^2-1-\lambda M}{1+\lambda M}}} \int_0^{\eta} e^{-2\lambda\alpha\sqrt{\frac{M^2-1-\lambda M}{1+\lambda M}}} \int_0^{\alpha} e^{\beta\lambda\sqrt{\frac{M^2-1-\lambda M}{1+\lambda M}}} F_2(\beta)\,d\beta\,d\alpha\,d\lambda \qquad (6.68)$$

where the function F_2 first appears on the right hand side of equation (6.12). In order to calculate F_2, the orthogonality property already used in connection with F_1 is used again. The eigen spectrum here is given by:

$$K = \lambda\sqrt{\frac{M^2-1-\lambda M}{1+\lambda M}} \qquad (6.69)$$

and it is shown here, as it was for F_1, that:

$$F_2 = -\frac{2}{\pi}\int_0^{\infty} U(\lambda)\sin K\eta\, dK \qquad (6.70)$$

or, when λ is eliminated by use of (6.69) and the usual asymptotic approximations are made,

$$F_2 = \frac{1}{M^2-1}\int_0^{\infty} e^{-K^2\eta}\cos K\eta\,\frac{dK}{\sqrt{K}} \qquad (6.71)$$

To estimate the effect of this term on the solution, it is sufficient to note that because of its very rapid convergence with η, $F_2(\eta)$ may be approximated by a Dirac function. The inhomogeneous solution is therefore an impulse of some strength A. Such an impulse is damped out as $\frac{1}{\sqrt{\xi}}$, so that its effect on the solution is negligible at distances of the order of 1 cm from the leading edge of the flat plate, where a simple calculation shows $\frac{(u_2)_n}{u_0}$ to be of the order of 3×10^{-3} which is considerably less than the strength of the wave predicted by the homogeneous part of the solution.

The main quantities needed to describe the solution are now available. It is seen that besides the boundary layer, there is in supersonic flow a disturbance which is concentrated in the immediate vicinity of the Mach line, and which is proportional to the coefficient of viscosity and the inverse quarter power of the distance from the leading edge. This disturbance in the velocity field is also connected with density, pressure and temperature fluctuations which are discussed below.

To describe the disturbance precisely, the variation of the velocity vector and of the thermodynamic parameters across it is calculated. The v component of the velocity vector is obtained from boundary condition (6.20).

$$\lambda V_2 = -\sqrt{\frac{(s+\lambda M)^2}{1+s+\lambda M} - \lambda^2}\; U_2 \tag{6.20}$$

which becomes, in the case of steady flow, with $s=0$, and in the vicinity of $\lambda=0$:

$$\lambda V_2 = -\lambda\sqrt{\frac{M^2}{1+\lambda M} - 1} \tag{6.72}$$

$$V_2 = -\sqrt{M^2-1}\; U_2 \tag{6.73}$$

The physical significance of this result is the following: the component of the velocity vector always adjusts itself in such a way with respect to the u_2 component that the resulting velocity vector $\bar{u}$ is perpendicular to the direction of the Mach wave.

In view of this result, the velocity perturbation is best expressed in terms of coordinates l, n along and normal to the Mach line. It then takes the form:

$$\bar{u} = \frac{u_0}{M^2-1}\left(\frac{6\nu}{c\ell}\right)^{1/4} e^{-\frac{n^2 c}{\frac{16}{3}\frac{M^2}{\sqrt{M^2-1}}\nu\ell}} \tag{6.74}$$

The order of magnitude of that wave may be estimated by carrying out the calculation indicated in (6.74) for air under standard conditions of pressure and temperature. Then:

$$\frac{\bar{u}}{u_o} = \frac{1}{M^2-1} \quad \frac{0.065}{\ell^{1/4}} \quad e^{-\frac{3.75\times10^{4}\, n^2}{\frac{M^2}{\sqrt{M^2-1}}\ell}} \tag{6.75}$$

The thickness of this wave is given by:

$$\Delta = 1.07\times10^{-6}\, \frac{M}{(M^2-1)^{3/4}}\, \ell^{3/4} \tag{6.76}$$

The thermodynamic parameters are calculated from equations (3.32/3.35) as follows:

$$\Sigma = -\frac{\alpha^2+\lambda^2}{\lambda(s+\lambda M)}\, \mathcal{U}_2 = -\,\mathcal{U}_2 M \tag{6.77}$$

so that it follows from (6.74) that

$$\sigma = \frac{u_o}{M^{5/4}(M^2-1)^{1/4}}\left(\frac{6\nu}{c\ell}\right)^{1/4} e^{-\frac{n^2 c}{\frac{16 M^2}{3\sqrt{M^2-1}}\nu\ell}} \tag{6.78}$$

Similarly, in the case of the pressure fluctuation:

$$\Pi = -\frac{\gamma}{\lambda}\left[(s+\lambda M)-(\alpha^2+\lambda^2)\right]\mathcal{U}_2 = -\gamma M\mathcal{U}_2 \tag{6.79}$$

and for the temperature fluctuation in the absence of heat conduction:

$$\Theta = -(\gamma-1)\, M\mathcal{U}_2 \tag{6.80}$$

The thermodynamic parameters all vary in the same manner as the velocity vector across the shock, because they are so intimately connected with it.

The solution describes the flow of a viscous non-conducting fluid. Now, the condition satisfied by θ along the surface of the plate is $\theta = 0$. Therefore, if the coefficient $\frac{4}{3}\nu$ is replaced by the effective viscosity coefficient $\bar{\nu}$ defined by equation (5.68), the solution will also be a good approximation to the solution for the flow of a viscous heat-conducting fluid past a flat plate maintained at a constant temperature. To the extent that the present theory is valid, that condition is not physically unreasonable, so that the solution has significance also in the case of a heat conducting fluid.

It is noted at this point that both the subsonic and the supersonic solutions determined in this analysis have a singularity at $M=1$. This simple linearized theory is therefore not sufficiently powerful to show how the damping action of viscosity and heat-conduction removes the well-known singularity in the solution of the linearized equations of flow of an ideal fluid. It is necessary to include some non-linear terms in the equations of motion to assure smooth passage from quasi-elliptic to quasi-hyperbolic equations. Specifically, these solutions break down because, when the integral (6.49) is evaluated asymptotically, the term M^2-1 was considered larger than the range of significant variation of λ. As M approaches 1, that approximation must necessarily break down, and the linearized asymptotic solution does not depend continuously on M in the vicinity of the point $M=1$.

In order to complete the solution of the problem, now that the field of flow is obtained in the quadrant $\xi, \eta > 0$ of the ξ, η plane, it is necessary to discuss the solution in the rest of the plane. Because all the equations and boundary conditions are symmetric with respect to the ξ axis, it is clear that the solution in the half-plane $\eta > 0$ is the reflected image of the solution in the half-plane $\eta < 0$. Therefore, it remains to investigate the solution in the quadrant $\xi < 0 ; \eta > 0$ or, in physical terms, the solution in front of the flat plate.

That solution must satisfy the boundary conditions (6.1/6.4). Actually, it is simpler to discuss the solution for one of the thermodynamic parameters, because of its simpler structure. It is only necessary to satisfy equation (6.12), for which, as will be shown in Appendix III, there exists a uniqueness theorem. Consider for instance the quantity:

$$\bar{\sigma} = \frac{\partial \sigma}{\partial \eta} \qquad (6.81)$$

Because of the symmetry of the flow, $\bar{\sigma}$ must vanish along the negative ξ axis; it must vanish as $\eta \to \infty$ and as $\xi \to -\infty$. It must furthermore, with all its derivatives, match the solution (6.78) along the line $\xi = 0$. But, at least asymptotically, it is easily verified that the solution (6.78) and all its derivatives vanish along the line $\xi = 0$ except in the immediate vicinity of the origin. All the conditions are therefore asymptotically satisfied by a solution which vanishes identically in the quadrant $\xi < 0; \eta > 0$. The same argument may be repeated for all of the physical parameters of the problem. The null solution is in fact the quasi-analytic continuation into the quadrant $\xi < 0; \eta > 0$ of the asymptotic solution discussed here. (For a definition of quasi analytic continuation, see for instance Hadamard (19)).

While the conclusion that there is no disturbance ahead of the flat plate is asymptotically correct, and equivalent to the similar conclusion obtained by Blasius in his solution for the incompressible boundary layer problem, it is necessary to qualify it as follows, for the sake of mathematical completeness.

The conclusion is acceptable insofar as the boundary layer part of the solution is concerned, because the boundary layer equation is approximated by the heat conduction equation, and the line $\xi = 0$ is a characteristic across which a quasi-analytic continuation is possible. However, the shock part of the solution has the characteristics $\eta = \text{const.}$

so that the continuation process cannot be carried out. What actually happens is that along the line $\xi = 0$, the solution and its derivatives do not vanish; they are given by the non-homogeneous part of the solution discussed on page (52). There is a definite non-zero solution in the half-plane $\xi < 0$ but because of the properties of the function F, the values of the physical parameters given by that solution are of an order of magnitude smaller than u_0, as soon as one moves away from the leading edge of the plate by an appreciable amount. This agrees with the concept of infinite signal velocity for the viscous disturbance and gives the order of magnitude of the energy transmitted ahead of the flat plate. In practice, the asymptotic solution which gives no disturbance ahead of the plate is an approximation of the same order as the other approximations made in the analysis.

With this remark, one concludes the investigation of the flow of a real fluid past a flat plate. The results are summarized below.

7. CONCLUSION

The study of the flow of a real compressible fluid past an obstacle with a view to investigating how nearly such a flow can be approximated by the flow of an ideal fluid, is thus completed. It was carried out with the simplest possible flow pattern, so that the analysis could be kept manageable, and so that the effect of viscosity and heat conduction could easily be picked out. The work was further simplified by a linearization process, justified by the known existence of a thin layer where viscosity plays an important part in shaping the flow pattern.

Under those simple conditions, it was found that there always exists a boundary layer, and the effect of the no-slip boundary condition imposed by viscosity is not felt in the main stream. The boundary layer is a purely kinematic phenomenon; its properties are independent of the free stream Mach number and depend only on the free stream Reynolds number. As a first approximation, there is no pressure, density or temperature gradient across the boundary layer, and the methods and results developed for calculating the skin friction of an incompressible fluid have some validity when the fluid is compressible. However, it must be made clear that, particularly in the case of a supersonic mean stream, only the result that a boundary layer exists can be maintained with any rigor. As a matter of fact, since the energy equation must be satisfied in the boundary layer, the detailed results found above cannot hold throughout the layer, but only along its edge. The temperature fluctuation which the energy equation:

$$\frac{1}{2}u^2 + c_p T = \text{const.} \tag{7.1}$$

predicts to correspond to the observed velocity fluctuation, is due to the energy dissipation which was neglected when the equations of motion were linearized. The flow inside the boundary layer has been the subject of much research, and the only statement which can be made here is that by linearizing the equations of motion, one has thrown away many important features of the problem.

The second significant result of this investigation is the sharp difference which was found between the flow patterns associated with subsonic and supersonic mean stream velocities. Actually, the subsonic flow was very similar in character to incompressible flow, and showed no other manifestations of the presence of viscosity than a boundary layer. On the other hand, when the mean stream velocity was supersonic, an entirely new phenomenon was found: the existence of a disturbance, somewhat similar to a shock-wave, starting at the leading edge of the flat plate (or at any discontinuity of the submerged obstacle) was deduced from the equations of motion. That disturbance was situated along the Mach line; the velocity vector through it was normal to it, and a density, pressure and temperature disturbance was associated with it.

The existence of such a quasi-Mach wave, even in the flow past a flat plate at zero angle of attack, is essentially a new feature. Such a quasi-Mach wave can be justified on the grounds that, at the leading edge of the flat plate, the birth of the boundary layer forces the stream near the flat plate to slow down; this disturbance is then propagated as a smeared out Mach wave. It is fairly safe to state that the actual disturbance at the leading edge of the plate, which comes from a rapid velocity drop, not from U to $U-u_0$, but from U to 0, is stronger than the one computed here, and the variation with distance away from the leading edge found in this analysis is the limit of the actual variation as one moves away from the plate. However, equation (6.75) shows that this wave is in general fairly weak, and it may not be easy to observe it, since it is necessarily overshadowed by actual Mach waves, which are due to the fact that in any experimental arrangement, there is no infinitely thin flat plate; these actual waves are not damped out as rapidly as the viscous quasi-Mach waves.

On the whole, it was found that the solutions obtained for slightly viscous fluids converge continuously to the ideal solutions. The asymptotic solutions found under the assumption of small viscosity, which justified considering only the limits as $\xi, \eta \to \infty$, were seen to exhibit the fundamental character of the ideal solutions, so that it seems reasonable to trust the results of the ideal fluid solutions, at least within the limits where small perturbation theory can be used.

The present analysis also serves as a lead, and an indication of the possible results of the investigation of the non-linear equations, which should help elucidate the three main questions raised by this work.

Granted that the existence of an actual boundary layer is proved, how nearly is the condition of no density, pressure and temperature gradient across it satisfied when the Mach number is significantly different from zero?

How does the quasi-elliptic flow of a subsonic real fluid pass into the quasi-hyperbolic flow of a supersonic real fluid, and in particular, how is it possible to construct a flow which passes across $M = 1$?

What becomes of the quasi-Mach wave when u_0 is made to approach U and non-linear terms are taken into account? In particular, what relation does it have to the shape of the boundary layer near the leading edge, and what relation does it have to the disturbance of a perfect fluid which would flow past a boundary layer-shaped body?

APPENDIX I.

CERTAIN PROPERTIES OF THE LAPLACE TRANSFORMATION AND ITS APPLICATION TO THE SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS.

The methods of the Laplace Transformation, which gives a rigorous basis to the Heaviside Operational Calculus, are found very useful in solving linear partial differential equations with constant coefficients, such as the ones which are encountered in this paper. Free use has been made of these methods, throughout the analysis, and while most points can be cleared up by reference to G. Doetsch's standard work on the subject, a summary of the main results which were used is included here for the reader's convenience.

The Laplace Transform of a function $\varphi(x_i,t)$ is defined as follows:

$$\Phi(x_i,s) = \mathcal{L}\{\varphi(x_i,t);s\} = \int_0^\infty e^{-st}\varphi(x_i,t)\,dt \tag{I.1}$$

provided that the integral is convergent for $Rl\, s > s_0$. One of the important properties of the Laplace transform is therefore that $\Phi(x_i,s)$, considered as a function of the complex variable s, is analytic and regular in a right half-plane $Rl\, s > s_0$.

If $\Phi(x_i,s)$ is the Laplace transform of some function $\varphi(x_i,t)$, it can be shown that φ is obtained by the following inversion formula:

$$\varphi(x_i,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\,\Phi(x_i,s)\,ds \tag{I.2}$$

where c is a real constant so chosen that all singularities of the function Φ lie to the left of the path of integration; the existence of c is guaranteed by the fact that Φ is regular and analytic for $s \geqslant s_0$ if it is a Laplace transform; it is therefore sufficient to have $c \geqslant s_0$. It is also shown that the transformations (I.1/I.2) have a uniqueness property; to each function φ corresponds a unique function Φ and vice versa, up to a null function.

Consider now the application of formulas (I.1), (I.2) to a partial differential equation; to the third order equation in two variables encountered in part 5 in connection with the propagation of plane disturbances. The method outlined here is valid for other equations also.

Let the equation be given as:

$$\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} - \frac{\partial^3\varphi}{\partial t\partial x^2} = 0 \qquad (I.3)$$

Let the problem be an initial value problem, so that the solution is defined for $t > 0$. Let the conditions be given as:

$$\text{(a)}\quad \begin{aligned} \varphi(0,x) &= \psi_0 \\ \varphi_t(0,x) &= \psi_1 \end{aligned} \qquad\qquad \text{(b)}\quad \begin{aligned} \varphi(t,0) &= \varphi_0 \\ \varphi(t,\infty) &= 0 \end{aligned} \qquad (I.4)$$

where $\psi_0, \psi_1, \varphi_0$ will be restricted later if the need arises. The purpose of the Laplace Transformation is to replace the problem (I.3/I.4) in two variables (x,t) by a problem in one variable x with parameter s. Indeed, apply transformation (I.1) to each term of (I.3):

$$\mathcal{L}\left\{\frac{\partial^2\varphi}{\partial t^2}; s\right\} = \int_0^\infty e^{-st}\frac{\partial^2\varphi}{\partial t^2}\,dt = e^{-st}\frac{\partial\varphi}{\partial t}\Big|_0^\infty + s\int_0^\infty e^{-st}\frac{\partial\varphi}{\partial t}\,dt = -\psi_1 + s\int_0^\infty e^{-st}\frac{\partial\varphi}{\partial t}\,dt$$

$$= -[\psi_1 + s\psi_0] + s^2\,\Phi(x,s) \qquad (I.5)$$

$$\mathcal{L}\left\{\frac{\partial^2\varphi}{\partial t^2}; s\right\} = \int_0^\infty e^{-st}\frac{\partial^2\varphi}{\partial x^2}\,dt = \frac{\partial^2}{\partial x^2}\int_0^\infty e^{-st}\varphi(x,t)\,dt = \frac{d^2\Phi}{dx^2} \qquad (I.6)$$

$$\mathcal{L}\left\{\frac{\partial^3\varphi}{\partial t\partial x^2}; s\right\} = \int_0^\infty e^{-st}\frac{\partial^3\varphi}{\partial t\partial x^2}\,dt = \frac{\partial^2}{\partial x^2}\int_0^\infty e^{-st}\frac{\partial\varphi}{\partial t}\,dt = \frac{d^2}{dx^2}\left[-\psi_0 + s\Phi\right] \qquad (I.7)$$

Equation (I.5) was obtained by carrying out two integrations by parts and using the initial conditions (I.4a). Equation (I.6) was obtained by interchanging the order of integration and differentiation. It is assumed that this is possible, and the resulting solution is checked to verify that the interchange was indeed permissible. Equation (I.7) is obtained by interchanging the order of differentiation and integration, and integrating by parts. The function Φ must now satisfy the total differential equation:

$$\frac{d^2\Phi}{dx^2} - \frac{s^2}{1+s}\,\Phi = \frac{1}{1+s}\left\{-(\psi_1 + s\psi_0) - \frac{d^2\psi_0}{dx^2}\right\} = \Psi(x,s) \qquad (I.8)$$

and when $x=0$; $x \to \infty$ one must have:

$$\Phi_0 = \int_0^\infty e^{-st}\,\varphi_0\,dt \qquad\qquad \Phi_\infty = 0 \qquad (I.9)$$

The problem (I.3/I.4ab) has therefore been replaced by the simpler problem (I.8/I.9) and the two problems are completely equivalent because of the uniqueness of all the transformations, provided that all the transformation integrals exist. Since the function φ_0 represents a physical disturbance, it may be expected that the integrals will in general converge.

It remains to solve equation (I.8) with boundary (I.9). The solution can be written as:

$$\Phi = \Phi_0\, e^{-\frac{sx}{\sqrt{1+s}}} + e^{\frac{sx}{\sqrt{1+s}}}\int_0^x e^{-\frac{2s\alpha}{\sqrt{1+s}}}\int_0^\alpha e^{\frac{s\beta}{\sqrt{1+s}}}\,\Psi(\beta)\,d\beta\,d\alpha \qquad (I.10)$$

and to complete the solution of the problem, it is sufficient to transform the Φ back into the x,t space.

$$\varphi(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\left\{\Phi_0\, e^{-\frac{sx}{\sqrt{1+s}}} + e^{\frac{sx}{\sqrt{1+s}}}\int_0^x e^{-\frac{2s\alpha}{\sqrt{1+s}}}\int_0^\alpha e^{\frac{s\beta}{\sqrt{1+s}}}\,\Psi(\beta)\,d\beta\,d\alpha\right\} ds \qquad (I.11)$$

However, a point must be raised here: the Laplace transform must be regular and analytic in s in a right half plane. Now, there is a set of eigen-values of s for which equation (I.8), being a non-homogeneous equation, has no solutions if Ψ is arbitrary. One must verify that all the eigen-values are to the left of a vertical line in the complex s plane, or impose such conditions on Ψ that the problem will be possible. In the present case, the eigen-values are given by:

$$k = \frac{-is}{\sqrt{1+s}} \qquad s = -2k\left(k \pm \sqrt{k^2-1}\right) \qquad (I.12)$$

where k is real. But relation (I.12) is familiar to the reader: it is the transformation (5.12) which was used so frequently in the analysis. It is remembered that the contour C in the s plane could be mapped into the real k axis, and contour C is to the left of the line $R\,s > 0$. Therefore, the problem can be solved for an arbitrary choice of Ψ. This is not surprising since the mathematical problem corresponds to a physical initial value problem, which should be possible for any set of reasonable not over-prescribed initial conditions. The particular problem solved in part 5 had the values:

$$\Psi = 0 \qquad \Phi_0 = \frac{\varphi_0}{s} \qquad (I.13)$$

so that the solution was given by the integral:

$$\varphi(x,t) = \frac{\varphi_0}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} e^{\frac{-sx}{\sqrt{1+s}}} \frac{ds}{s} \qquad (I.14)$$

To discuss the problem of singularities at eigen-values of s further, consider now the equation:

$$a\frac{\partial^3\varphi}{\partial x^3} + \frac{\partial^3\varphi}{\partial x \partial y^2} + b\frac{\partial^2\varphi}{\partial y^2} - \frac{\partial^2\varphi}{\partial x^2} = 0 \qquad (a>0) \qquad (I.15)$$

which is essentially the equation of the quasi-Mach wave encountered in part 6 of the analysis. Let the boundary conditions in the quadrant $x>0\,;\ y>0$ be given as follows:

(a) $\varphi(0,y) = \psi_0(y)$
$\varphi_x(0,y) = \psi_1(y)$
$\varphi_{xx}(0,y) = \psi_2(y)$

(b) $\varphi(x,0) = \varphi_0(x)$
$\varphi(x,L) = \varphi_1(x)$ (I.16)

A transformation similar to the one shown in detail in (I.5/I.7) is carried out to eliminate the x variable and replace it by the parameter λ . The equation satisfied by Φ under assumptions of convergence similar to those made previously, is found to be:

$$\frac{d^2\Phi}{dy^2} + \lambda^2 \frac{a\lambda-1}{\lambda+b}\Phi = \frac{1}{\lambda+b}\left\{(a\lambda-1)(\lambda\psi_0+\psi_1) + a\psi_2 + \psi_0''\right\} = \Psi(\lambda, y) \qquad (I.17)$$

and the boundary conditions are:

$$\Phi_0 = \int_0^\infty e^{-\lambda x}\varphi_0\,dx \qquad\qquad \Phi_1 = \int_0^\infty e^{-\lambda x}\varphi_1(x,L)\,dx \qquad (I.18)$$

A solution for the non-homogeneous equation (I.17) with boundary conditions (I.18) can again be found, and again, it is necessary to raise the question whether there are eigen-values of λ in a right half-plane. The eigen-values here form the discrete set:

$$\lambda\sqrt{\frac{a\lambda-1}{\lambda+b}} = \frac{n\pi}{L} \qquad (n = 0, 1, 2 \ldots) \qquad (I.19)$$

Now, the constants a,b, are real and positive, and therefore λ takes eigen-values whenever it is a root of the cubic:

$$\lambda_n^3 - \frac{\lambda_n^2}{a} - \frac{n^2\pi^2}{aL^2}\lambda_n - b\frac{n^2\pi^2}{aL^2} = f.(\lambda_n) = 0 \qquad (I.20)$$

The extrema of this cubic are located at the points:

$$(\lambda_n)_e = \frac{1}{3a} \pm \sqrt{\left(\frac{1}{3a}\right)^2 + \frac{n^2\pi^2}{3aL^2}} \qquad (I.21)$$

and when $n\pi/L$ is very large, $(\lambda_n)_e \to \pm\frac{1}{\sqrt{3a}}\frac{n\pi}{L}$, where $-$ corresponds to the maximum and $+$ to the minimum. When $n\pi/L$ is very large, the minimum can be estimated as:

$$f_{min}(\lambda_n) = 0\left[(\lambda_n)_e^3 - \frac{n^2\pi^2}{aL^2}(\lambda_n)_e\right] = -0\left\{(\lambda_n)_e \frac{2}{3a}\frac{n^2\pi^2}{L^2}\right\} < 0 \qquad (I.22)$$

The minimum, when $n \to \infty$, is therefore always negative. But, for a fixed n, as $\lambda_n \to \infty$, $f_n \to +\infty$. This proves that there is always at least one root of order $\frac{1}{\sqrt{3a}} \, n\pi/L$. Therefore, there exist eigen-values along the real λ axis in any right half of the λ plane. Since the solution of equation (I.17) must be a Laplace transform for the original problem to have a solution, it follows that Ψ is not arbitrary, but must satisfy the condition that the solution still exists when λ takes its eigen-values. This means that Ψ must be orthogonal to the eigen-solutions $\Phi_n(\lambda)$ and must therefore satisfy the relation:

$$\int_0^L \Psi(y)\,\Phi_n(y)\,dy = \frac{\lambda_n}{L}\int_0^L \left[\Phi_0(L-y) + \Phi_1 y\right]\Phi_n(y)\,dy \qquad \text{(I.23)}$$

Since the right hand side of equation (I.23) is known, Ψ can be calculated as the inverse Fourier transform of the right-hand side. In the particular problem encountered in this investigation, the range L is infinite and $\Phi_1(y) = 0$. Under those conditions, equation (I.23) becomes:

$$\int_0^\infty \Psi(y)\,\Phi_K(y)\,dy = \lim_{L\to\infty}\int_0^L \frac{K}{L}\,\Phi_0(L-y)\,\Phi_K(y)\,dy \qquad \text{(I.24)}$$

where K is now the continuous set of eigen-values of equation (I.17):

$$\Phi_K = e^{iKy} \qquad \text{(I.25)}$$

The right hand side of (I.24) is now:

$$\lim_{L\to\infty}\frac{K}{L}\int_0^L \Phi_0(L-y)\,\Phi_K(y)\,dy = \Phi_0 \lim_{L\to\infty}\frac{K}{L}\left\{\frac{L}{iK}\left(e^{iKL}-1\right) - \frac{1}{iK}\left[e^{iKL}\left(L-\frac{1}{iK}\right)+\frac{1}{iK}\right]\right\} \qquad \text{(I.26)}$$

$$= i\Phi_0$$

Therefore, (I.24) becomes:

$$\int_0^\infty \Psi(y)\,\sin Ky\,dy = \Phi_0(\lambda) \qquad \text{(I.27)}$$

which is easily inverted as:

$$\Psi(y) = \frac{2}{\pi}\int_0^{\infty} \Phi_0(\lambda) \sin Ky \, dK \qquad (I.28)$$

Equation (I.28) is all that is required to solve the problem, since λ is a function of K given by the generalization of relation (I.19)

$$\lambda \sqrt{\frac{a\lambda - 1}{\lambda + b}} = K \qquad (I.29)$$

This is essentially what is done at several points of the analysis.

APPENDIX II.

EVALUATION OF CERTAIN INTEGRALS

It is necessary to carry out in detail certain asymptotic integrations which were omitted from the text of this paper for the sake of brevity. The fundamental idea is that since ξ, η, τ are always very large, only the portion of the integrands near the origin is significant.

Consider first the integral which occurs in the investigation of the propagation of small perturbations:

$$\varphi = \frac{\varphi_0}{\pi}\left\{\int_1^\infty e^{-2k(k+\sqrt{k^2-1})\tau}\left[\frac{\sin 2k\xi}{k}+\frac{\cos 2k\xi}{\sqrt{k^2-1}}\right]dk + \int_{-1}^{1} e^{-2k^2\tau}\, e^{2ik(\xi-\tau\sqrt{1-k^2})}\left[\frac{1}{k}+\frac{i}{\sqrt{1-k^2}}\right]dk\right\} + \frac{\varphi_0}{2} \tag{5.14}$$

The first part of the integral can contribute nothing since:

$$\int_1^\infty e^{-2k(k+\sqrt{k^2-1})\tau}\,\frac{\sin 2k\xi}{k}\,dk < \int_1^\infty e^{-2k^2\tau}\,dk < \int_0^\infty e^{-2k^2\tau}\,dk < \frac{1}{\sqrt{2\tau}}\sqrt{\frac{\pi}{2}} \to 0 \tag{II.1}$$

The same holds for the second part of the first integral, although there is a removable singularity at $k=1$. This is best shown by use of the transformation:

$$K=\sqrt{k^2-1} \quad ; \quad k=\sqrt{K^2+1} \quad ; \quad K\,dK = k\,dk \tag{II.2}$$

which changes the integral into:

$$\int_0^\infty e^{-2\tau[K^2+1+K\sqrt{K^2+1}]}\cos 2\xi\sqrt{K^2+1}\,\frac{dK}{\sqrt{K^2+1}} < \int_0^\infty e^{-2\tau K^2}\,dK \to 0 \tag{II.3}$$

The second integral is subdivided into four parts:

$$\int_{-1}^{1} e^{-2k^2\tau}\sin 2k(\xi-\tau\sqrt{1-k^2})\,\frac{dk}{k} + i\int_{-1}^{1} e^{-2k^2\tau}\sin 2k(\xi-\tau\sqrt{1-k^2})\frac{dk}{\sqrt{1-k^2}}$$
$$+ i\int_{-1}^{1} e^{-2k^2\tau}\cos 2k(\xi-\tau\sqrt{1-k^2})\frac{dk}{\sqrt{1-k^2}} + \int_{-1}^{1} e^{-2k^2\tau}\cos 2k(\xi-\tau\sqrt{1-k^2})\,\frac{dk}{k} \tag{II.4}$$

It is immediately obvious that the second and third integrals contribute nothing because the integrand is odd and the range symmetric with respect to the origin.

The fourth integral leads to the following:

$$\frac{1}{2}I_4 = \int_0^1 e^{-2k^2\tau} \cos 2k(\xi-\tau\sqrt{1-k^2})\frac{dk}{\sqrt{1-k^2}}$$

$$= \int_0^\epsilon + \int_\epsilon^{1-\delta} + \int_{1-\delta}^1 e^{-2k^2\tau}\cos 2k(\xi-\tau\sqrt{1-k^2})\frac{dk}{\sqrt{1-k^2}} \tag{II.5}$$

and the three pieces are discussed separately:

$$(I_4)_1 < \int_0^\epsilon e^{-2k^2\tau}dk < \int_0^\infty e^{-2k^2\tau}dk \to 0 \tag{II.6}$$

$$(I_4)_2 < \frac{1}{\sqrt{2\delta}}\int_0^\infty e^{-2k^2\tau}dk \to 0 \tag{II.7}$$

For $(I_4)_3$, use the transformation (II.2) to obtain:

$$(I_4)_3 = \int_{-2\delta-\delta^2}^0 e^{-2(K^2+1)\tau}\cos 2(K^2+1)^{1/2}(\xi-K\tau)\frac{dK}{\sqrt{K^2+1}} < \int_{-\infty}^0 e^{-2K^2\tau}dK \to 0 \tag{II.8}$$

The integral (5.14) is thus reduced to:

$$\varphi = \frac{1}{2}\varphi_0 + \frac{2\varphi_0}{\pi}\int_0^1 e^{-2k^2\tau}\sin 2k(\xi-\tau\sqrt{1-k^2})\frac{dk}{k} \tag{II.9}$$

Consider now the function $A(\xi,\tau)$ defined by:

$$A(\xi,\tau) = \int_0^1 e^{-2k^2\tau}\sin 2k(\xi-\tau\sqrt{1-k^2})\frac{dk}{k} - \int_0^\infty e^{-2k^2\tau}\sin 2k(\xi-\tau)\frac{dk}{k} \tag{II.10}$$

and write:

$$A(\xi,\tau) = \int_0^\epsilon + \int_\epsilon^1 e^{-2k^2\tau}\left[\sin 2k(\xi-\tau\sqrt{1-k^2}) - \sin 2k(\xi-\tau)\right]\frac{dk}{k}$$
$$+ \int_1^\infty e^{-2k^2\tau}\sin 2k(\xi-\tau)\frac{dk}{k} \tag{II.11}$$

The second part of the integral vanishes just as (II.1) vanished. Now define ϵ in such a manner that $\lim\limits_{\tau\to\infty}\epsilon^3\tau = 0$; $\lim\limits_{\tau\to\infty}\epsilon^2\tau \to \infty$. Write:

$$\left|\sin 2k(\xi-\tau\sqrt{1-k^2})-\sin 2k(\xi-\tau)\right|<2k^3\tau \qquad (0<k<1) \tag{II.12}$$

as can be verified by expanding the sines into a power series. When this is substituted into the first integral of (II.11), it is seen that:

$$\int_0^{\epsilon}\frac{e^{-2k^2\tau}}{k}\left[\sin 2k(\xi-\tau\sqrt{1-k^2})-\sin 2k(\xi-\tau)\right]dk<\int_0^{\epsilon}2k^2\tau\,dk=\frac{2}{3}\epsilon^3\tau\to 0 \tag{II.13}$$

$$\int_{\epsilon}^{1}\frac{e^{-2k^2\tau}}{k}\left[\sin 2k(\xi-\tau\sqrt{1-k^2})-\sin 2k(\xi-\tau)\right]dk<2\int_{\epsilon}^{1}\frac{e^{-2k^2\tau}}{k}dk=\int_{2\epsilon^2\tau}^{2\tau}\frac{e^{-\alpha}}{\alpha}d\alpha\to 0 \tag{II.14}$$

It has therefore been proved that the integral (5.14) is asymptotically equal to:

$$\varphi=\frac{1}{2}\varphi_0-\frac{2}{\pi}\int_0^{\infty}e^{-2k^2\tau}\sin 2k(\xi-\tau)\frac{dk}{k} \tag{II.15}$$

This last integral is evaluated by integrating the following uniformly convergent integral under the sign of integration:

$$\int_0^{\infty}e^{-\alpha k^2}\cos\beta k\,dk=\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\,e^{-\beta^2/4\alpha} \tag{II.16}$$

and the required result follows at once.

The integration of equation (5.33) is carried out as follows. Given:

$$\varphi=\frac{\psi_0 c}{4/3\nu}\left\{\frac{1}{\pi}\int_1^{\infty}e^{-2k(k+\sqrt{k^2-1})\tau}\left(\cos 2k\xi-\frac{k}{\sqrt{k^2-1}}\sin 2k\xi\right)dk\right.$$
$$\left.+\frac{1}{\pi}\int_{-1}^{1}e^{-2k^2\tau}e^{2ik(\xi-\tau\sqrt{1-k^2})}\left(1+\frac{ik}{\sqrt{1-k^2}}\right)dk\right\} \tag{5.33}$$

The first integral contributes nothing as can be seen by comparing its first part to $\int_1^{\infty}e^{-2k^2\tau}dk$ and studying its second part after transformation (II.2). The second integral contributes, after the symmetry of the integrand has been taken into account,

$$\frac{2}{\pi}\int_0^1 e^{-2k^2\tau}\left\{\cos 2k(\xi-\tau\sqrt{1-k^2})+\frac{ik}{\sqrt{1-k^2}}\sin 2k(\xi-\tau\sqrt{1-k^2})\right\}dk \tag{II.17}$$

and the first integral, dominated by $\int_0^{\epsilon} 2k\xi\, dk$ in the interval $0<k<\epsilon \rightarrow \lim_{\xi\to\infty} \epsilon\, \xi^2 = 0$; by $A\int_{\epsilon}^{1-\delta} e^{-2k^2\tau} dk$ in the interval $\epsilon<k<1-\delta$ and by $\int_{-\delta}^{0} \sqrt{1+K^2}\, \sin K\, dK$ in the interval $1-\delta<k<1$, contributes no more than the integral I_4 of (II.5/II.7).

Define now the quantity B:

$$B(\xi,\tau) = \int_0^1 e^{-2k^2\tau} \cos 2k(\xi-\tau\sqrt{1-k^2})\, dk - \int_0^{\infty} e^{-2k^2\tau} \cos 2k(\xi-\tau)\, dk \tag{II.18}$$

It can be broken down into three parts, just as A was in (II.11)

$$\begin{aligned} B(\xi,\tau) = \int_0^{\epsilon} + \int_{\epsilon}^{1} e^{-2k^2\tau}\left[\cos 2k(\xi-\tau\sqrt{1-k^2}) - \cos 2k(\xi-\tau)\right] dk \\ - \int_1^{\infty} e^{-2k^2\tau} \cos 2k(\xi-\tau)\, dk \end{aligned} \tag{II.19}$$

The first part of the integral vanishes since $\cos 2k(\xi-\tau\sqrt{1-k^2}) - \cos 2k(\xi-\tau) < A k^3\tau$ if $\lim_{\tau\to\infty} \epsilon^4\tau = 0$ and the second part vanishes if $\lim_{\tau\to\infty} \epsilon^2\tau \to \infty$ The third part of the integral vanishes like the first integral in (5.33). The asymptotic value of (5.33) is therefore:

$$\varphi = \frac{\psi_0 c}{4/3\nu}\, \frac{2}{\pi} \int_0^{\infty} e^{-2k^2\tau} \cos 2k(\xi-\tau)\, dk \tag{II.20}$$

which was integrated in (II.16).

Another integral which requires detailed discussion is equation (6.28).

$$(u_1)_R = \frac{u_0}{2\pi i} \int_c \frac{\sqrt{\left(\frac{M^2}{1+\lambda M}-1\right)\left(\frac{4M}{3\lambda}-1\right)}}{1-\sqrt{\left(\frac{M^2}{1+\lambda M}-1\right)\left(\frac{4M}{3\lambda}-1\right)}}\, e^{-\eta\sqrt{\frac{4}{3}M\lambda-\lambda^2}}\, e^{\lambda\xi}\, \frac{d\lambda}{\lambda} \tag{6.28}$$

where the contour c may be taken along the imaginary axis with an indentation to the right about the origin, since all the poles to the right of the origin have residue 0 as was proved in part 6.

It is convenient to rewrite the integral as follows:

$$(u_1)_R = \frac{u_0}{2\pi i}\left[\int_{-i\infty}^{-i\epsilon} + \int_{\bar{c}} + \int_{i\epsilon}^{+i\infty}\right] \frac{\sqrt{\left(\frac{M^2}{1+\lambda M}-1\right)\left(\frac{4M}{3\lambda}-1\right)}}{1-\sqrt{\left(\frac{M^2}{1+\lambda M}-1\right)\left(\frac{4M}{3\lambda}-1\right)}}\, e^{-\eta\sqrt{\frac{4}{3}M\lambda-\lambda^2}}\, e^{\lambda\xi}\frac{d\lambda}{\lambda} = I_1+I_2+I_3 \tag{II.21}$$

where the contour $\bar{c}$ represents the indentation about the origin.

Consider now I_1, with $\omega = -i\lambda$:

$$I_1 = \int_\epsilon^\infty \frac{\sqrt{\left(\frac{M^2}{1-M\omega i}-1\right)\left(\frac{4}{3}\frac{Mi}{\omega}-1\right)}}{1-\sqrt{\left(\frac{M^2}{1-M\omega i}-1\right)\left(\frac{4}{3}\frac{Mi}{\omega}-1\right)}} e^{-\eta\sqrt{\omega^2-\frac{4}{3}M\omega i}} e^{-i\omega\xi} \frac{d\omega}{\omega} \qquad \text{(II. 22)}$$

$$I_1 < \int_\epsilon^\infty A\epsilon^{-3/2} e^{-\eta\, Rl\sqrt{\omega^2-\frac{4}{3}M\omega i}}\, d\omega < A\epsilon^{-3/2}\int_\epsilon^\infty e^{-\eta\sqrt{\sqrt{\omega^4+\frac{4}{9}M^2}-\omega^2}}\, d\omega \to 0 \qquad \text{(II. 23)}$$

and the same holds for the integral I_3 . Consider now the function $C(\xi,\eta)$ defined by:

$$C(\xi,\eta) = \int_{\bar{c}} \frac{\sqrt{\left(\frac{M^2}{1+\lambda M}-1\right)\left(\frac{4}{3}\frac{M}{\lambda}-1\right)}}{1-\sqrt{\left(\frac{M^2}{1+\lambda M}-1\right)\left(\frac{4}{3}\frac{M}{\lambda}-1\right)}} e^{-\eta\sqrt{\frac{4}{3}M\lambda-\lambda^2}} e^{\lambda\xi}\frac{d\lambda}{\lambda} + \int_c e^{-\eta\sqrt{4/3 M\lambda}} e^{\lambda\xi}\frac{d\lambda}{\lambda} \qquad \text{(II. 24)}$$

If the second integral is divided into two parts, from 0 to ϵ and from ϵ to ∞ , it is easily verified that for a fixed ϵ , the second part of the integral vanishes as $\xi,\eta \to \infty$. It remains to investigate the portion of the integral along $\bar{c}$, which is twice the integral along a quarter-circle of radius ϵ , to avoid the pole at the origin.

Now,

$$e^{-\eta\sqrt{\frac{4}{3}M\lambda-\lambda^2}} = e^{-\eta\sqrt{\frac{4}{3}M\lambda}}\left\{e^{-\eta\sqrt{\frac{4}{3}M\lambda-\lambda^2}+\eta\sqrt{\frac{4}{3}M\lambda}}\right\} \qquad \text{(II. 25)}$$

so that the function $C(\xi,\eta)$ is rewritten as:

$$C(\xi,\eta) = \int_{\bar{c}}\left\{\frac{e^{-\eta\sqrt{\frac{4}{3}M\lambda-\lambda^2}+\eta\sqrt{\frac{4}{3}M\lambda}}\sqrt{\left(\frac{M^2}{1+\lambda M}-1\right)\left(\frac{4}{3}\frac{M}{\lambda}-1\right)}}{1-\sqrt{\left(\frac{M^2}{1+\lambda M}-1\right)\left(\frac{4}{3}\frac{M}{\lambda}-1\right)}}+1\right\} e^{-\eta\sqrt{\frac{4}{3}M\lambda}} e^{\lambda\xi}\frac{d\lambda}{\lambda} \qquad \text{(II. 26)}$$

It is convenient to reduce the quantity in brackets to a common denominator, and since $Rl\,\lambda > 0$, the exponential can be rewritten:

$$e^{-\eta\left[\sqrt{\frac{4}{3}M\lambda-\lambda^2}-\sqrt{\frac{4}{3}M\lambda}\right]} = e^{-\eta\sqrt{\frac{4}{3}M\lambda}\left[-\frac{3}{8}\frac{\lambda}{M}-\cdots\right]} < e^{2\lambda\eta\sqrt{\frac{3\lambda}{16M}}} \qquad \text{(II. 27)}$$

so that:

$$C(\xi,\eta) < \int_{\bar{c}} \frac{1 + 2\lambda\eta\sqrt{\frac{3\lambda}{16M}}\sqrt{\left(\frac{M^2}{1+\lambda M}-1\right)\left(\frac{4}{3}\frac{M}{\lambda}-1\right)}}{1 - \sqrt{\left(\frac{M^2}{1+\lambda M}-1\right)\left(\frac{4}{3}\frac{M}{\lambda}-1\right)}} e^{-\eta\sqrt{\frac{4}{3}M\lambda}} e^{\lambda\xi} \frac{d\lambda}{\lambda} < \int_{\bar{c}} \frac{2B}{1-\frac{A}{\sqrt{\lambda}}} \frac{d\lambda}{\lambda} \qquad \text{(II.28)}$$

from which it follows that :

$$C(\xi,\eta) < \int_{\bar{c}} \frac{2B}{1 - A/\sqrt{\lambda}} \frac{d\lambda}{\lambda} \qquad \text{(II.29)}$$

since the exponentials are bounded. Therefore:

$$C(\xi,\eta) < \int_{\bar{c}} \frac{2B\,d\lambda}{\lambda - A\sqrt{\lambda}} < \int_{\bar{c}} \frac{D\,d\lambda}{\sqrt{\lambda}} = D\int_0^{\pi/2} \frac{\epsilon e^{i\theta}\,d\theta}{\sqrt{\epsilon}\,e^{i\theta/2}} \to 0 \qquad \text{(II.30)}$$

since the radius ϵ of the indentation can be made as small as required. It has been shown that:

$$\lim_{\xi,\eta\to\infty} \frac{u_0}{2\pi i}\int_C \frac{\sqrt{\left(\frac{M^2}{1+\lambda M}-1\right)\left(\frac{4}{3}\frac{M}{\lambda}-1\right)}}{1-\sqrt{\left(\frac{M^2}{1+\lambda M}-1\right)\left(\frac{4}{3}\frac{M}{\lambda}-1\right)}} e^{-\eta\sqrt{\frac{4}{3}M\lambda-\lambda^2}} e^{\lambda\xi}\frac{d\lambda}{\lambda} = \frac{2u_0}{\pi}\int_0^\infty e^{-\eta\sqrt{\frac{4}{3}M\lambda}} e^{i\lambda\xi}\frac{d\lambda}{\lambda} \qquad \text{(II.31)}$$

and this is equation (6.34) which is evaluated by making the simple change of variables $\lambda = \omega^2$. It then takes the form

$$(u_1)_R = \frac{u_0}{\pi}\int_0^\infty e^{-\eta\sqrt{\frac{4}{3}M}\,\omega} e^{i\omega^2\xi}\frac{d\omega}{\omega} \qquad \text{(II.32)}$$

which is similar to (II.16).

The next integral which requires careful discussion is (6.46):

$$(u_2)_R = \frac{u_0}{2\pi i}\int_C e^{-\lambda\eta\sqrt{\frac{M^2}{1+\lambda M}-1}} \frac{e^{\lambda\xi}}{1-\sqrt{\left(\frac{M^2}{1+\lambda M}-1\right)\left(\frac{4}{3}\frac{M}{\lambda}-1\right)}}\frac{d\lambda}{\lambda} \qquad (M^2-1>0) \qquad (6.46)$$

which becomes, after application of the transformation (6.47):

$$(u_2)_R = \frac{u_0}{\pi}\int_1^\infty e^{-2k(k+\sqrt{k^2-1})\xi/M} e^{2ik\,\eta/M\sqrt{M^2-1-M\lambda}} \left(\frac{1}{k}+\frac{1}{\sqrt{k^2-1}}\right)\frac{dk}{G(k)}$$

$$+ \frac{u_0}{\pi}\int_{-1}^1 e^{-2k^2[\xi/M+\eta/M\sqrt{M^2-1}]} e^{\frac{2ik}{M}[\eta\sqrt{M^2-1-M\lambda}-\xi\sqrt{1-k^2}]}\left(\frac{1}{k}+\frac{i}{\sqrt{1-k^2}}\right)\frac{dk}{G(k)} \qquad \text{(II.33)}$$

Since $G(k)$ is bounded, the first integral can be shown to vanish in the same manner as the first part of (II.1). Similarly, if the real and imaginary parts of the second integral are separated, and the symmetry of the integrands is made use of, it becomes evident that:

$$(u_2)_h = \frac{2u_0}{\pi}\int_0^1 e^{-\frac{2k^2}{M}\left(\xi+\frac{\eta}{\sqrt{M^2-1}}\right)} \cos \frac{2k}{M}\left(\eta\sqrt{M^2-1-M\lambda}-\xi\sqrt{1-k^2}\right)\frac{dk}{kG(k)} \qquad \text{(II.34)}$$

To evaluate the integral (II.34), consider the function:

$$D(\xi,\eta)=\int_0^1 e^{-\frac{2k^2}{M}\left(\xi+\frac{\eta}{\sqrt{M^2-1}}\right)}\cos\frac{2k}{M}\left(\eta\sqrt{M^2-1-M\lambda}-\xi\sqrt{1-k^2}\right)\frac{dk}{kG(k)} - \sqrt{\frac{3}{2}}\,\frac{1}{M\sqrt{M^2-1}}\int_0^\infty e^{-\frac{2k^2}{M}\left(\xi+\frac{\eta}{\sqrt{M^2-1}}\right)}\cos\frac{2k}{M}\left(\xi-\eta\sqrt{M^2-1}\right)\frac{dk}{\sqrt{k}} \qquad \text{(II.35)}$$

The function D is broken down in the usual manner:

$$D(\xi,\eta)=\left[\int_0^\epsilon+\int_\epsilon^1\right] e^{-\frac{2k^2}{M}\left(\xi+\frac{\eta}{\sqrt{M^2-1}}\right)}\left\{\cos\frac{2k}{M}\left(\eta\sqrt{M^2-1-\lambda M}-\xi\sqrt{1-K^2}\right)\frac{1}{kG(k)}-\frac{1}{\sqrt{k}}\cos\frac{2k}{M}\left(\xi-\eta\sqrt{M^2-1}\right)\right\}dk + \int_1^\infty e^{-\frac{2k^2}{M}\left(\xi+\frac{\eta}{\sqrt{M^2-1}}\right)}\cos\frac{2k}{M}\left(\xi-\eta\sqrt{M^2-1}\right)\frac{dk}{\sqrt{k}} \qquad \text{(II.36)}$$

It is clear that the third part of the integral contributes nothing to the result; to analyse the first part, one writes:

$$e^{-2k^2\xi/M} \leq 1 \qquad \text{(II.37)}$$

$$\frac{1}{kG(k)}=\frac{1}{k\left[1-\sqrt{\left(\frac{M^2}{1+\lambda M}-1\right)\left(\frac{4}{3}\frac{M}{\lambda}-1\right)}\right]} \leq \frac{1+Ak+\ldots}{\frac{2}{3}\sqrt{M^2-1}\,\sqrt{k}} \qquad \text{(II.38)}$$

$$\cos\frac{2k}{M}\left[\eta\sqrt{M^2-1-\lambda M}-\xi\sqrt{1-k^2}\right]-\cos\frac{2k}{M}\left(\xi-\eta\sqrt{M^2-1}\right)\leq Ak^2\eta \qquad \text{(II.39)}$$

Therefore, if $\lim\limits_{\eta\to\infty} \epsilon^2\eta \to 0$, the first integral vanishes.

The second integral is written:

$$\int_\epsilon^1 e^{-\frac{2k^2}{M}\left(\xi+\frac{\eta}{\sqrt{M^2-1}}\right)}F(k)\,dK < \int_\epsilon^1 A\,e^{-2Bk^2}dk \to 0 \qquad \text{(II.40)}$$

It follows that the integral (6.46) is equivalent to:

(II.41)

Consider therefore the integral:

$$I = \int_0^\infty e^{-Ak^2} \cos Bk \frac{dk}{\sqrt{k}} \qquad \text{(II.42)}$$

This is written as:

$$I = \sqrt{\frac{\pi}{2}} \sqrt{B} \int_0^\infty e^{-Ak^2} J_{-1/2}(kB)\, dk \qquad \text{(II.43)}$$

To evaluate that integral, one uses the following result, due to Hankel:

$$\int_0^\infty J_\nu(at)\, e^{-p^2t^2} t^{\mu-1} dt = \frac{\Gamma(\frac{1}{2}\mu + \frac{1}{2}\nu)\, a^\nu}{\Gamma(\nu+1)\, 2^{\nu+1} p^{\mu+\nu}} \, {}_1F_1\left(\frac{\mu+\nu}{2}; \nu+1; -\frac{a^2}{4p^2}\right) \qquad \text{(II.44)}$$

provided $|\arg p| < \frac{\pi}{4}$ and $Rl(\mu+\nu) > 0$. In the present case, p is real and $\mu = 1;\ \nu = -\frac{1}{2}$.

The above formula gives:

$$I = \sqrt{\frac{\pi}{2}} \frac{\Gamma(1/4)}{\sqrt{2}\,\Gamma(1/2)} \frac{1}{A^{1/4}} \, {}_1F_1\left(\frac{1}{4}; \frac{1}{2}; -\frac{B^2}{4A}\right) \qquad \text{(II.45)}$$

It is now convenient to introduce Kummer's duplication formula:

$$ {}_1F_1(\beta; 2\beta; 2z) = e^z \, {}_0F_1\left(\beta + \frac{1}{2}; \frac{z^2}{4}\right) \qquad \text{(II.46)}$$

which gives here:

$$ {}_1F_1\left(\frac{1}{4}; \frac{1}{2}; -\frac{B^2}{4A}\right) = e^{-\frac{B^2}{8A}} \, {}_0F_1\left(\frac{3}{4}; \left(\frac{B^2}{8A}\right)^2\right) \qquad \text{(II.47)}$$

But, by definition,

$$ {}_0F_1\left(\frac{3}{4}; z\right) = \Gamma(3/4) \sum_0^\infty \frac{z^n}{n!} \frac{1}{\Gamma(n+3/4)} \qquad \text{(II.48)}$$

Therefore, substituting the results of (II.46), (II.48) into (II.45), one finds:

$$I = \frac{\Gamma(1/4)\Gamma(3/4)}{\Gamma(1/2)} \sqrt{\frac{\pi}{2}}\, A^{-1/4} e^{-B^2/8A} [1 + \dots] \qquad \text{(II.49)}$$

where the rapid convergence of the exponential makes it unnecessary to consider more than the first term of the series. It is recalled that:

$$A = \frac{2}{M}\left(\xi + \frac{\eta}{\sqrt{M^2-1}}\right) \qquad B = \frac{2}{M}\left(\xi - \eta\sqrt{M^2-1}\right) \tag{II.50}$$

and the formula (6.57) follows by substitution:

$$(u_2)_R = u_0\sqrt{\frac{3}{2}}\,\frac{1}{\sqrt{M}\sqrt{M^2-1}}\left(\frac{2(M^2-1)}{M^3\xi}\right)^{1/4} e^{-\frac{(\xi-\eta\sqrt{M^2-1})^2}{\left(\frac{4M^3}{M^2-1}\right)\xi}} \tag{6.57}$$

When the flow is subsonic, the same method of approximation gives:

$$(u_2')_R = \frac{2u_0}{\pi}\sqrt{\frac{3}{2}}\,\frac{1}{M\sqrt{1-M^2}}\int_0^\infty e^{-2k\frac{\sqrt{1-M^2}}{M}\eta}\, e^{2ik\xi/M}\,\frac{dk}{\sqrt{k}} \tag{II.51}$$

so that one must investigate:

$$J = \int_0^\infty e^{-kA}\cos kB\,\frac{dk}{\sqrt{k}} \tag{II.52}$$

which can be written:

$$J = \sqrt{\frac{\pi}{2}}\sqrt{B}\int_0^\infty e^{-kA}\,J_{-1/2}(kB)\,dk \tag{II.53}$$

Hankel also evaluated this integral in terms of hypergeometric functions as follows:

$$\int_0^\infty e^{-at}\,J_\nu(bt)\,t^{\mu-1}\,dt = \frac{b^\nu\,\Gamma(\mu+\nu)}{2^\nu a^{\mu+\nu}\,\Gamma(\nu+1)}\,F\left(\frac{\mu+\nu}{2}, \frac{\mu+\nu+1}{2}; \nu+1; -\frac{b^2}{a^2}\right) \tag{II.54}$$

if $R\ell(\mu+\nu) > 0$ and $R\ell(a \pm ib) > 0$. These conditions are satisfied here since $a, b > 0$ are real and $\mu+\nu = \frac{1}{2}$

Thus:

$$J = \sqrt{\frac{\pi}{2}}\,\frac{1}{\sqrt{A}}\,F\left(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; -\frac{A^2}{B^2}\right) \tag{II.55}$$

This leads to the formula (6.65) when their values are substituted for the parameters A, B.

$$(u_x')_h = \frac{u_0}{\sqrt{\pi M(1-M^2)}} \sqrt{\frac{2}{\xi}}\; F\left(\tfrac{1}{4}, \tfrac{3}{4} ; \tfrac{1}{2} ; -\frac{\eta^2(1-M^2)}{\xi^2}\right) \qquad (6.65)$$

This completes the discussion of the more elaborate evaluation of definite integrals which had been omitted from the main body of the text.

APPENDIX III

A CERTAIN PARTIAL DIFFERENTIAL EQUATION

It is necessary to say a few words about the partial differential equation satisfied by the dynamic disturbances. That equation is written:

$$L(\varphi) = a\frac{\partial^3\varphi}{\partial x^3} + \frac{\partial^3\varphi}{\partial x\partial y^2} + \frac{\partial^2\varphi}{\partial y^2} - \frac{\partial^2\varphi}{\partial x^2} = 0 \quad (a \geqslant 0) \qquad \text{(III.1)}$$

When $a=0$, one has the propagation equation (5.2); when $a>0$, one has the two-dimensional steady state equation. Before undertaking an explicit scheme of solution, one may examine the character of the equation a little more closely.

The characteristics of this equation, as can easily be verified, run parallel to the x axis. The solution may therefore have discontinuities across lines parallel to the x axis. But in a given strip $y_1 < y < y_2$ in which the solution is continuous, the equation has an elliptic character.

It follows from the elliptic character of the equation that it will not satisfy a Cauchy set of boundary conditions, since one boundary condition must always specify that the solution is everywhere finite, either inside or outside a certain closed boundary. With this in mind, one attempts to prove a uniqueness theorem for equation (III.1) under the following conditions.

Consider two solutions which satisfy the same boundary conditions φ, φ_x along $x=0$, φ along y_1, y_2, and which converge at $x \to \infty$. Let the difference between two such solutions be $\bar{\varphi}$. It is shown that $\bar{\varphi} \equiv 0$

Construct the expression:

$$\int_{y_1}^{y_2}\int_0^{\infty} L(\bar{\varphi})\frac{\partial\bar{\varphi}}{\partial x}\,dx\,dy = 0 \qquad \text{(III.2)}$$

Expression (III.2) is true since $\bar{\varphi}$ satisfies equation (III.1). Now, examine (III.2) term by term, under the assumption that the integral is uniformly convergent and that the order of integration can be changed.

In view of the previous statements, the boundary conditions satisfied by are:

$$\begin{array}{ll} \bar{\varphi}(0,y)=0 & \bar{\varphi}(x,y_1)=0 \\ \bar{\varphi}_x(0,y)=0 & \bar{\varphi}(x,y_2)=0 \\ \bar{\varphi}_x(\infty,y)=0 & \end{array} \qquad \text{(III.3)}$$

with $\bar{\varphi}$ defined in the region $0<x<\infty$; $y_1<y<y_2$.

Consider now the terms of (III.2) separately:

$$\int_{y_1}^{y_2}\int_0^\infty a\frac{\partial^3\bar{\varphi}}{\partial x^3}\frac{\partial\bar{\varphi}}{\partial x}dxdy = a\int_{y_1}^{y_2}\frac{\partial\bar{\varphi}}{\partial x}\frac{\partial^2\bar{\varphi}}{\partial x^2}\Big|_0^\infty dy - a\int_{y_1}^{y_2}\int_0^\infty\Big(\frac{\partial^2\bar{\varphi}}{\partial x^2}\Big)^2dxdy = -a\int_{y_1}^{y_2}\int_0^\infty\Big(\frac{\partial^2\bar{\varphi}}{\partial x^2}\Big)^2dxdy \qquad \text{(III.4)}$$

$$\int_{y_1}^{y_2}\int_0^\infty\frac{\partial^3\bar{\varphi}}{\partial x\partial y^2}\frac{\partial\bar{\varphi}}{\partial x}dxdy = \int_0^\infty\frac{\partial^2\bar{\varphi}}{\partial x\partial y}\frac{\partial\bar{\varphi}}{\partial x}\Big|_{y_2}^{y_1} - \int_{y_1}^{y_2}\int_0^\infty\Big(\frac{\partial^2\bar{\varphi}}{\partial x\partial y}\Big)^2dxdy = -\int_{y_1}^{y_2}\int_0^\infty\Big(\frac{\partial^2\bar{\varphi}}{\partial x\partial y}\Big)^2dxdy \qquad \text{(III.5)}$$

$$\int_{y_1}^{y_2}\int_0^\infty\frac{\partial^2\bar{\varphi}}{\partial y^2}\frac{\partial\bar{\varphi}}{\partial x}dxdy = \int_0^\infty\frac{\partial\bar{\varphi}}{\partial x}\frac{\partial\bar{\varphi}}{\partial y}\Big|_{y_1}^{y_2}dx - \int_0^\infty\int_{y_1}^{y_2}\frac{\partial\bar{\varphi}}{\partial y}\frac{\partial^2\bar{\varphi}}{\partial x\partial y}dxdy = 0 \qquad \text{(III.6)}$$

$$\int_0^\infty\int_{y_1}^{y_2}\frac{\partial^2\bar{\varphi}}{\partial x^2}\frac{\partial\bar{\varphi}}{\partial x}dxdy = \frac{1}{2}\int_{y_1}^{y_2}\Big(\frac{\partial\bar{\varphi}}{\partial x}\Big)^2_\infty dy = 0 \qquad \text{(III.7)}$$

Expression (III.2) is now changed to:

$$\int_{y_1}^{y_2}\int_0^\infty\Big[a\Big(\frac{\partial^2\bar{\varphi}}{\partial x^2}\Big)^2+\Big(\frac{\partial^2\bar{\varphi}}{\partial x\partial y}\Big)^2\Big]dxdy = 0 \qquad \text{(III.8)}$$

Since the integrand is always non-negative, it must venish in the region of definition. Then,

$$\frac{\partial^2\bar{\varphi}}{\partial x^2}=0 \qquad \frac{\partial^2\bar{\varphi}}{\partial x\partial y}=0 \qquad \text{(III.9)}$$

must hold in the entire domain in which the solution $\bar{\varphi}$ is defined. But relation (III.9) implies that $\bar{\varphi}_x$ is independent of both x and y; it is therefore constant, and since it vanishes along the y axis, it vanishes in the whole region of definition. This completes the proof of the uniqueness theorem for equation (III.1), if the interchange in the order of integration is justified. But the range in y is finite, and from the condition of convergence at infinity, one may expect the integrals to converge.

It may be noted in passing that only two conditions were postulated along the y axis. Thus, if φ, φ_x are given, it is expected that φ_{xx} is determined. This agrees with the result found when this equation was solved by means of the Laplace Transformation in Appendix I. For, there, a function Ψ which involved a linear combination of $\varphi, \varphi_x, \varphi_{xx}$ along the y axis was determined as a function of the other conditions of the problem. φ_{xx} was thus determined when φ and φ_x were known.

Some properties of the solution may be found by separation of the variables in (III.1) and examination of the resulting equations. Let the solution be represented by:

$$\varphi(x,y) = f(x)\,g(y) \qquad \text{(III.10)}$$

Then, (III.1) becomes:

$$a f'''g + f'g'' + fg'' - f''g = 0 \qquad \text{(III.11)}$$

or, if $f'+f \neq 0$; $g \neq 0$:

$$\frac{af''' - f''}{f+f'} = -\frac{g''}{g} \qquad \text{(III.12)}$$

so that the solution can be obtained by writing:

$$g'' + k^2 g = 0 \tag{III.13}$$

$$a f''' - f'' - k^2(f' + f) = 0 \tag{III.14}$$

Since the equation for g is harmonic, as many conditions along the y axis can be specified as there are different convergent solutions for f. Therefore, it becomes important to investigate the cubic indicial equation associated with equation (III.14).

$$F(\alpha) = \alpha^3 - \frac{1}{a}\alpha^2 - \frac{k^2}{a}(\alpha + 1) = 0 \tag{III.15}$$

There always exists one real solution α_1. The other two solutions are either the real numbers α_2, α_3, or the complex numbers $\beta \pm i\gamma$ depending on the value of the parameter k. As a matter of fact, as α goes from $-\infty$ to $+\infty$ $F(\alpha)$ goes from $-\infty$ to a maximum; then to a minimum; then to $+\infty$. There are three real roots if the maximum $F_m(\alpha) > 0$. Now, the extremum of F occurs when:

$$\alpha_m^2 - \frac{2}{3a}\alpha_m - \frac{k^2}{3a} = 0 \tag{III.16}$$

or:

$$\alpha_m = -\frac{1}{3a} \pm \sqrt{\left(\frac{1}{3a}\right)^2 + \frac{k^2}{3a}} \tag{III.17}$$

Clearly, the two extrema always exist; and $\alpha_{max} < 0$; $\alpha_{min} > 0$ for all values of the parameter k, as $a > 0$. The value of F_m is obtained by substituting the result of (III.17) into (III.15). One then finds that $F_m = 0$ if the following cubic is satisfied by k^2:

$$k_m^2 \left\{ \frac{k_m^4}{3a} + k_m^2\left[\frac{1}{9a^2} - \frac{2}{9a} - (3a+1)^2\right] + \left[\frac{1}{27a} - \frac{2}{27a^2} + \frac{1}{9a}(3a+1)\right] \right\} = 0 \tag{III.18}$$

the solutions of which are:

$$k_m=0 \quad ; \quad k_m^2 = \frac{3a}{2}\left\{-\left[\frac{1}{9a^2}-\frac{2}{9a}-(3a+1)^2\right] \pm \sqrt{\left[\frac{1}{9a}-\frac{2}{9a}-(3a+1)^2\right]^2-\frac{4}{27a^2}\left[\frac{1}{3}-\frac{2}{3a}+3a+1\right]}\right\} \qquad \text{(III.19)}$$

Equation (III.19) thus gives three values of k_m at which $F_m=0$.
An examination of these and the observation that:

$$\begin{aligned} k^2 &\to 0 \quad \Rightarrow \quad 3\,Rl\ \text{sol.} \\ k^2 &\to \infty \quad \Rightarrow \quad \text{"} \quad \text{"} \end{aligned} \qquad \text{(III.20)}$$

indicate that the range for imaginary solutions is:

$$A+\sqrt{A^2+B^2} \geqslant k^2 \geqslant A-\sqrt{A^2+B^2} \qquad \text{(III.21)}$$

provided it exists. The three roots in the vicinity of $k=0$ are $\frac{1}{a}, 0, 0$, while it was shown in (I.22) that there are three roots as $k\to\infty$.

Consider now the case of three real roots $\alpha_1, \alpha_2, \alpha_3$. The following relations must hold:

$$\alpha_1\alpha_2\alpha_3 = \frac{k^2}{a} > 0 \qquad \text{(III.22)}$$

$$\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1 = -\frac{k^2}{a} < 0 \qquad \text{(III.23)}$$

From (III.22) it is clear that there are one or three positive roots; and it follows from (III.24) that there can be only one.

Consider now the case of one real root α_1 and the two complex roots $\beta \pm i\gamma$. Then,

$$\alpha_1(\beta^2+\gamma^2) = \frac{k^2}{a} > 0 \qquad \text{(III.24)}$$

$$2\alpha_1\beta + (\beta^2+\gamma^2) = -\frac{k^2}{a} < 0 \qquad \text{(III.25)}$$

From the first relation, it is clear that $\alpha_1 > 0$; it follows from the second that $\beta < 0$.

The following general result is therefore proved; the equation

(III.15) always has a real positive root; the other two roots always have a negative real part. The solution thus takes the form:

$$\varphi = \int_{-\infty}^{\infty} F(k) e^{iky} e^{\alpha_1 x} dk \qquad (-\infty < x < 0) \tag{III.26}$$

$$\varphi = \int_{-\infty}^{\infty} [G(k) e^{\alpha_2 x} + H(k) e^{\alpha_3 x}] e^{iky} dk \quad (0 < x < \infty) \tag{III.27}$$

The solution is therefore asymmetric in x, which is essentially a consequence of the fact that viscous flow is irreversible, and the x axis has a definite direction which cannot be changed arbitrarily.

Insofar as the solution of the present problem is concerned, the condition satisfied along the y axis is the eigen-condition obtained in the theory of the Laplace transformation. The eigen-values are the α_i of the present solution; therefore, the eigen solution provides that if φ is continuously joined along the y axis between the $x > 0$ and $x < 0$ half planes, the higher derivatives are also continuously joined, so that a solution valid in the entire plane is obtained.

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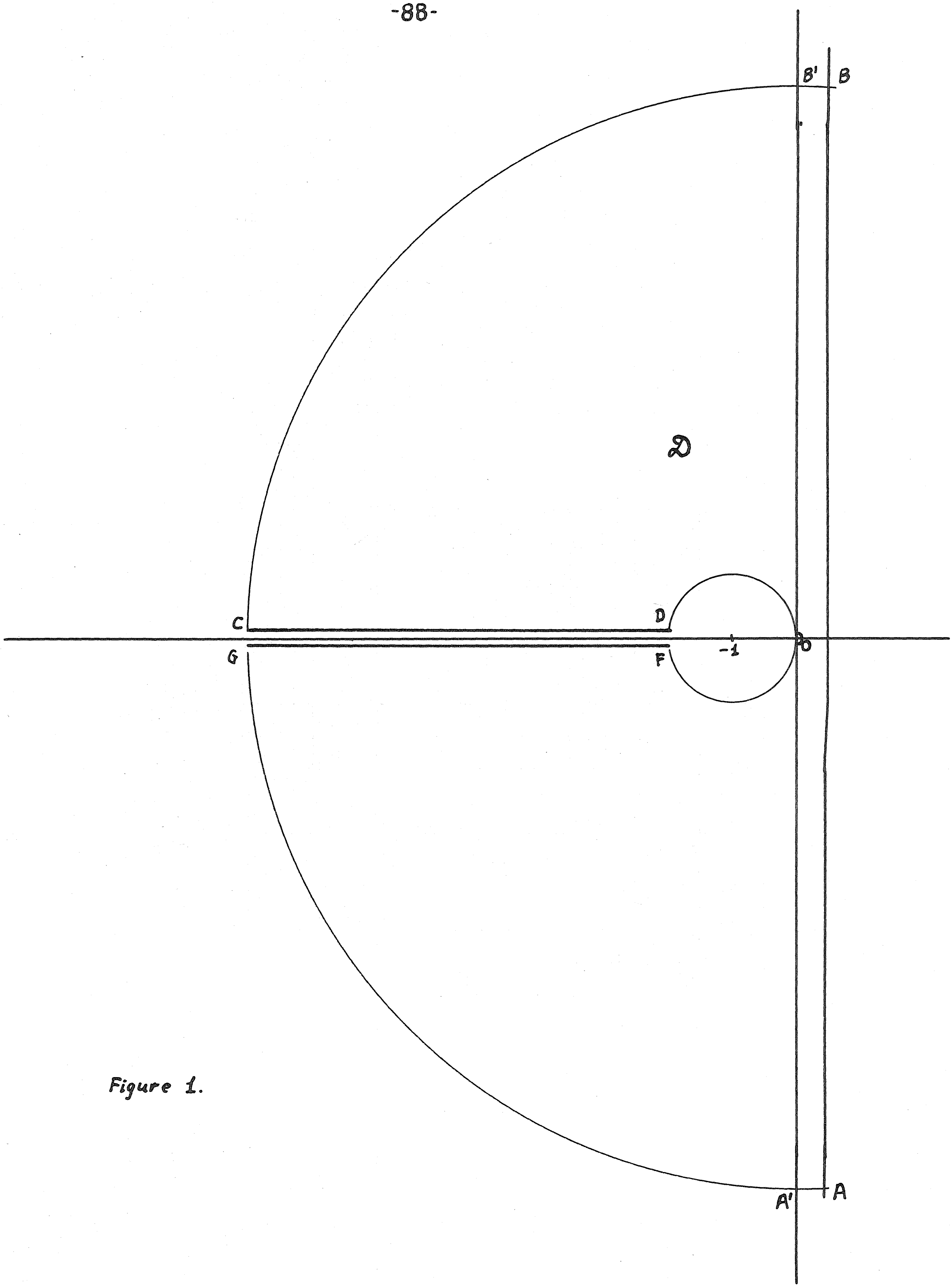

Figure 1.