TRANSIENT ANALYSIS METHODS FOR DETERMINING THE LONGITUDINAL STABILITY DERIVATIVES OF A SUBMERGED BODY FROM FREE FLIGHT TESTS

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I. ABSTRACT

The stability derivatives obtainable from dynamic free flight tests are determined. Methods for reducing flight test data to the form of stability derivatives using the Fourier integral and the Laplace transform are developed.

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II. SUMMARY

Fundamentally only three lift derivatives and three moment derivatives can be determined from a dynamic flight test where the forward velocity is held constant and the test is made from level flight. This is shown in Sections 9.1 and 9.2.

For the general case of a submerged vehicle, five lift derivatives $(C_{L_\alpha} + C_{D_o})$, $C_{L_{\dot\alpha}}$, C_{L_q}, $C_{L_{\dot q}}$, and C_{L_δ} and five moment derivatives C_{M_α}, $C_{M_{\dot\alpha}}$, C_{M_q}, $C_{M_{\dot q}}$, and C_{M_δ} are considered. To evaluate these stability derivatives from flight tests, two lift and two moment derivatives must be known from other sources.

For the specific case of a dirigible or torpedo $C_{L_{\dot q}}$ and $C_{M_{\dot\alpha}}$ are neglected, leaving four unknown lift and four unknown moment derivatives. In tests with locked control surfaces the derivatives C_{L_δ} and C_{M_δ} do not appear in the equations. Thus, the three remaining lift and three remaining moment derivatives can be determined. To perform such a test it is necessary to disturb the test vehicle with a non-hydrodynamic force such as that produced by a rocket fired from the test vehicle.

For the specific case of the airplane the lift derivatives $C_{L_{\dot\alpha}}$, C_{L_q}, $C_{L_{\dot q}}$ and the moment derivative $C_{M_{\dot q}}$ are usually neglected. In addition C_{D_o} is small compared to C_{L_α}. This leaves two unknown lift derivatives C_{L_α} and C_{L_δ} and the four unknown moment derivatives C_{M_α}, $C_{M_{\dot\alpha}}$, C_{M_q}, C_{M_δ}. The derivatives C_{L_α} and C_{L_δ} can be determined from a flight test where a control surface motion is used to disturb the test vehicle, but only ratios of the moment derivatives can be obtained. However, by using a non-aerodynamic disturbance C_{M_α}, $C_{M_{\dot\alpha}}$ and C_{M_q} can be determined.

It is interesting to note that for the case of the dirigible, measurement of the pitching velocity and the force are sufficient to determine the derivative $C_{L_{\dot\alpha}}$ or the apparent transverse mass. For the air-

plane, the term $C_{M_{\dot{\alpha}}}$ associated with downwash can be determined from the same measurements and they are the only two derivatives that may be explicitly determined from these two measurements.

The transient measurements required in flight are pitching velocity, control surface displacement and normal acceleration. If, in place of a control surface displacement, a non-hydrodynamic force is used as the disturbance, the transient force must be measured. Angle of attack may be substituted for normal acceleration.

Measurements are also required of the steady state values of the forward velocity and angle of attack. It is desirable to measure the transient values of angle of attack so that the range of this angle may be known.

In section X it is shown that the problem of reducing the flight test data to the form of stability derivatives is essentially that of determining the transfer function constants from the flight test data. Two methods for computing these constants are given. They are the Fourier Integral Method and the LaPlace Transform Method.

In these data analysis procedures the flight test data are integrated to perform the transformation of the variables, q, δ , a_z, from time dependent quantities to quantities depending upon a new independent variable, s. This integration tends to smooth the data and avoids the difficulties encountered in differentiating test data in other reduction methods.

The Fourier integral method in which the test data are transformed by the unilateral Fourier integral

$$\bar{x}(\omega) = \int_0^\infty e^{-i\omega t} \quad x(t)\, dt$$

is used for tests of dynamically stable systems, that is, systems in which x(t) reaches a steady state value.

The LaPlace transform method may be used to study unstable systems. In this method the data are transformed by the LaPlace Transform

$$x(s) = \int_0^{\infty} x(t)\, e^{-st}\, dt$$

where s is taken to be a real positive quantity large enough to insure convergence of the integral.

The final step in the data reduction is the computation of the stability derivatives from the transfer function constants. This is accomplished by the simultaneous solution of equations 9.2, 9.3 and 9.4 for the lift derivatives and equations 9.5, 9.6 and 9.7 for the moment derivatives.

III. INTRODUCTION

This paper discusses the determination of the longitudinal stability derivatives of a submerged body from an analysis of the transient motion following a disturbance in free flight. It attempts to answer two questions: (1) What are the stability derivatives that are determinable from dynamic flight tests? (2) How are the flight test data reduced to the form of these derivatives?

The problem here is essentially that of servomechanism synthesis, that is, given the response of a system to a certain input, what are the elements that comprise the system, or mathematically stated, what are the coefficients in the differential equations of motion?

Accordingly, the transient analysis methods of servomechanism synthesis using the Laplace transform and Fourier integral are used here.

The transient flight testing technique is explained in Section V. Following this in Section VI the stability derivatives for a submerged body are developed in terms of wind axes.

The linearized equations of motion are written in operational form in Section VII by use of the Laplace transform. They are based on the assumptions of constant coefficients and small disturbances from steady symmetric flight. The assumption of constant forward velocity is further made and the longitudinal set reduces to two equations. Since angle of attack is difficult to measure in free flight, the equations are written in terms of pitching velocity and normal acceleration.

The concept of the transfer function is taken from servomechanism theory and is defined as the ratio of the Laplace transform of the response to the Laplace transform of the input to a system. The transfer functions are written for a submerged body in terms of the stability derivatives in section VIII.

In section IX the derivatives obtainable from dynamic flight tests are determined. Two types of disturbances are considered: (1) a control surface displacement, and (2) a non-hydrodynamic force such as a rocket fired from the test vehicle. The derivatives obtainable from the special case of the airplane and the dirigible are discussed in this section.

In section X the the flight test data reduction procedures are developed. The analysis procedure is outlined in section 10.1. In this data reduction procedure the data is transformed by the Fourier integral or the LaPlace transform. In section 10.2 transformation of test data by the Fourier integral is explained, and in section 10.3 the transformation of test data by the LaPlace transform is presented.

Both of these procedures are based on the fact that the equations of motion are linear and have constant coefficients. They are used to determine the transfer function constants from the test data.

Finally, in section 10.4 the procedure is presented for computation of the stability derivatives from the transfer function constants.

In Appendix A, the well-known Euler's equations for a rigid body in terms of moving coordinates are developed for completeness of the work; in Appendix B, the linearized equations of motion for a submerged body are developed from Euler's equations.

IV. NOMENCLATURE

A_1, A_2, etc. Transfer function constants

a_z Component of acceleration along z axis, ft/sec^2

$\bar{a}_z$ LaPlace transform of a_z

B Buoyancy force, lbs.

c Characteristic length, ft.

D Drag force, lbs.

$\bar{e}_z$ Unit vector along z axis

F Force, lbs.

$\bar{F}$ LaPlace transform of force

G W-B, lbs.

G_x, G_y, G_z Components of G along x, y, z axes respectively, lbs.

g Acceleration of gravity, ft/sec^2

$\bar{H}$ Angular momentum vector, slug ft/sec

H_x, H_y, H_z Component H along x, y, z axes respectively

$\underset{\sim}{I}$ Dyad of inertia

$$I_{xx} = \int_{vol} (y^2 + z^2) \, d\tau \quad \text{slug ft}^2$$

$$I_{yy} = \int_{vol} (x^2 + z^2) \, d\tau \quad \text{slug ft}^2$$

$$I_{zz} = \int_{vol} (x^2 + y^2) \, d\tau \quad \text{slug ft}^2$$

$$I_{xy} = I_{yx} = - \int_{vol} xy \, d\tau \text{ slug ft}^2$$

$$I_{xz} = I_{zx} = - \int_{vol} xz \, d\tau \text{ slug ft}^2$$

$$I_{yz} = I_{zy} = - \int_{vol} yz \, d\tau \text{ slug ft}^2$$

$\bar{i}, \bar{j}, \bar{k}$ Unit vectors along x, y, z axes respectively

L Lift force, lbs.

M Moment, ft lbs.

M_u $\frac{\partial M}{\partial u}$, M_w $\frac{\partial M}{\partial w}$, etc.

L, M, H Components of moment about x, y, z axes respectively, ft lbs

L_H, M_H, N_H Components of hydrodynamic moment about x, y, z axes, ft lbs

L_B, M_B, N_B Components of buoyancy moment about x, y, z axes, ft lbs

L_T, M_T, N_T Components of thrust moment about x, y, z axes, ft lbs

$\bar{\mathfrak{m}}$ Moment vector, ft lbs

m Mass, slugs

P, Q, R Components of angular velocity about x, y, z axes respectively

$\dot{P} = \frac{dP}{dt}, \dot{Q} = \frac{dQ}{dt}, \dot{R} = \frac{dR}{dt}$, radians/sec^2

p, q, r small pertubations of P, Q, R respectively, radians/sec

$\dot{p} = \frac{dp}{dt}, \dot{q} = \frac{dq}{dt}, \dot{r} = \frac{dr}{dt}$, radians/sec^2

$\bar{q}$ LaPlace transform of q

$\bar{r}_B$ Vector from center of gravity to center of buoyancy, ft

S Characteristic area, ft^2

t time, seconds

U_o, W_o Components of velocity along x_o and z_o axes respectively, ft/sec

V Forward velocity, ft/sec

u, v, w Perturbation velocities along x, y, z axes respectively, ft/sec

$\dot{u} = \frac{du}{dt}, \dot{v} = \frac{dv}{dt}, \dot{w} = \frac{dw}{dt}$, ft/sec^2

W Weight, lbs

x, y, z Axes fixed in body, x, z plane is plane of symmetry

X_I, Y_I, Z_I Axes fixed in space, Z_I is vertical

X_o, Y_o, Z_o Axes fixed in space which coincides with x, y, z during steady state flights

x_B Distance from center of gravity to center of buoyancy along x axis + for c.g. aft of c.b., ft

z_B Distance from center of gravity to center of buoyancy along z axis. + for c.g. above c.b., ft.

X_H, Y_H, Z_H Components of hydrodynamic force along x, y, z axes respectively, lbs

$X_u = \frac{\partial X}{\partial u}$, $X_w = \frac{\partial X}{\partial w}$, etc.

X_T, Y_T, Z_T Component of thrust along x, y, z axes respectively, lbs

α Pertubation in angle of attack, radians

ω Angular velocity; frequency

ζ_o Initial angle of attack, radians

ζ $\zeta_o + \alpha$ Angle of attack, radians

ψ Pertubation in yaw angle, radians

θ Pertubation in pitch angle, radians

θ_o Initial pitch angle, radians

ϕ Pertubation in roll angle, radians

ρ Density, slug/ft^3

μ $\frac{m}{Sc}$

<u>Dimensionless Coefficients</u>

$$C_{D_o} = D_o \Big/ \frac{\rho}{2} SV^2$$

$$C_D = \frac{\partial D}{\partial \alpha} \Big/ \frac{\rho}{2} SV^2$$

$$C_{D_{\dot{\alpha}}} = \frac{\partial D}{\partial \dot{\alpha}} \Big/ \frac{\rho}{2} SVc$$

$$C_{D_q} = \frac{\partial D}{\partial q} \Big/ \frac{\rho}{2} SVc$$

$$C_{D_{\dot{q}}} = \frac{\partial D}{\partial q} \Big/ \frac{\rho}{2} Sc^2$$

$$C_{D_\delta} = \frac{\partial D}{\partial \delta} \Big/ \frac{\rho}{2} SV^2$$

$$C_{D_{\dot{u}}} = C_{D_{\dot{V}}} = \frac{\partial D}{\partial \dot{u}} \Big/ \frac{\rho}{2} Sc$$

$$C_{L_o} = L_o \Big/ \frac{\rho}{2} SV^2$$

$$C_{L_\alpha} = \frac{\partial L}{\partial \alpha} \Big/ \frac{\rho}{2} SV^2$$

$$C_{L_{\dot{\alpha}}} = \frac{\partial L}{\partial \dot{\alpha}} \Big/ \frac{\rho}{2} SVc$$

$$C_{L_q} = \frac{\partial L}{\partial q} \Big/ \frac{\rho}{2} SVc$$

$$C_{L_{\dot{q}}} = \frac{\partial L}{\partial \dot{q}} \Big/ \frac{\rho}{2} Sc^2$$

$$C_{L_\delta} = \frac{\partial L}{\partial \delta} \Big/ \frac{\rho}{2} SV^2$$

$$C_{L_{\dot{u}}} = C_{M_{\dot{V}}} = \frac{\partial L}{\partial u} \Big/ \frac{\rho}{2} Sc$$

$$C_{M_o} = M_o \Big/ \frac{\rho}{2} SV^2c$$

$$C_{M_\alpha} = \frac{\partial M}{\partial \alpha} \Big/ \frac{\rho}{2} SV^2c$$

$$C_{M_{\dot{\alpha}}} = \frac{\partial M}{\partial \dot{\alpha}} \Big/ \frac{\rho}{2} SVc^2$$

$$C_{M_q} = \frac{\partial M}{\partial q} \Big/ \frac{\rho}{2} \; SVc^2$$

$$C_{M_{\dot{q}}} = \frac{\partial M}{\partial \dot{q}} \Big/ \frac{\rho}{2} \; Sc^3$$

$$C_{M_\delta} = \frac{\partial M}{\partial \delta} \Big/ \frac{\rho}{2} \; SV^2c$$

$$C_{M_{\dot{u}}} = C_{M_{\dot{V}}} = \frac{\partial M}{\partial u} \Big/ \frac{\rho}{2} \; Sc^2$$

V. DISCUSSION OF TRANSIENT FLIGHT TESTING

The transient flight test techniques consist of applying a sudden disturbance to a test vehicle and recording the subsequent transient motion. Such a disturbance may be due to a control surface displacement or a non-hydrodynamic force such as that produced by rocket fired from the test vehicle.

These tests are of very short duration. The time for an entire test is of the order of two or three seconds for a vehicle such as an airplane or torpedo. Thus, many test points may be made on one single flight test.

The equations of motion assume small disturbances from steady flight. By selecting the proper type of forcing function the motions can be made small enough so that this condition is satisfied.

The flight test measurements required are the transient values of the variables in terms of which the equations of motion are written and the initial value of the forward velocity. It is known that the stability derivatives are functions of angle of attack. Hence, it is desirable to know the average value of the angle of attack at which the test is made.

The equations of motion may be written in terms of pitching velocity, normal acceleration, control surface deflection and time; these are the only transient measurements required in flight. Angle of attack, however, [redacted] may be substituted for normal acceleration.

The output of the instruments used to measure the transient motions depends upon the frequency spectrum of the motion. Corrections for the dynamic characteristics of the measuring instruments can be made by the method outlined in section 10. Thus, instruments with high natural frequencies are not necessarily required.

VI. THE LONGITUDINAL STABILITY DERIVATIVES IN TERMS OF WIND AXES

6.1 Hydrodynamic Forces and Moments

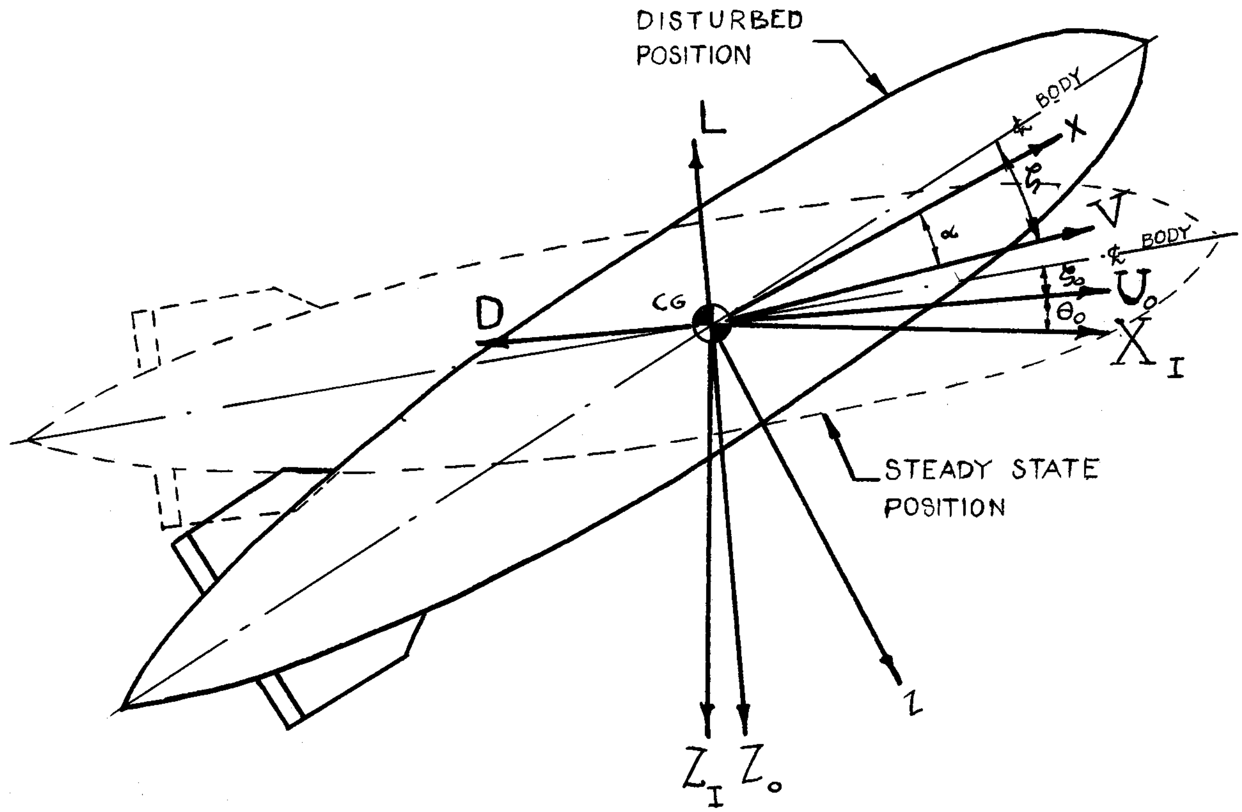

Fig. C-1

Referring to Figure C-1

$V =$ forward velocity

Let $\zeta =$ angle of attack $= \zeta_o + \alpha$

$\zeta_o =$ steady state angle of attack

$\alpha =$ perturbation in angle of attack

$$X = L \sin \alpha - D \cos \alpha \tag{6.1}$$

$$Z = -L \cos \alpha - D \sin \alpha \tag{6.2}$$

Considering α to be a small angle and $\sin \alpha = \alpha$, $\cos \alpha \simeq 1$

$$X_H = L\alpha - D \tag{6.3}$$

$$Z_H = -L - D\alpha \tag{6.4}$$

D and L are assumed to depend upon α, $\dot{\alpha}$, V, $\dot{V}$, q, $\dot{q}$, δ_e

$$L = C_{L_o} \frac{\rho}{2} SV^2 + C_{L_\alpha} \alpha \frac{\rho}{2} SV^2 + C_{L_{\dot{\alpha}}} \dot{\alpha} \frac{\rho}{2} SV_c$$

$$+ C_{L_q} q \frac{\rho}{2} SV_c + C_{L_{\dot{q}}} \dot{q} \frac{\rho}{2} Sc^2 \tag{6.5}$$

$$+ C_{L_{\dot{V}}} \dot{V} \frac{\rho}{2} Sc + C_{L_\delta} \frac{\rho}{2} SV^2$$

$$D = C_{D_o} \frac{\rho}{2} SV^2 + C_{D_\alpha} \alpha \frac{\rho}{2} SV^2 + C_{D_{\dot{\alpha}}} \dot{\alpha} \frac{\rho}{2} SV_c$$

$$+ C_{D_q} q \frac{\rho}{2} SV_c + C_{D_{\dot{q}}} \dot{q} \frac{\rho}{2} Sc^2 + C_{D_{\dot{V}}} \dot{V} \frac{\rho}{2} Sc + C_{D_\delta} \delta \frac{\rho}{2} SV^2 \tag{6.6}$$

Substituting the expressions for L and D in equations 6.3 and 6.4 and retaining only first order terms, the equations for X and Z become

$$Z_H = - C_{L_o} \frac{\rho}{2} SV^2 - C_{L_\alpha} \alpha \frac{\rho}{2} SV^2 - C_{L_{\dot{\alpha}}} \dot{\alpha} \frac{\rho}{2} SV_c$$

$$- C_{L_q} q \frac{\rho}{2} SV_c - C_{L_{\dot{q}}} \dot{q} \frac{\rho}{2} Sc^2 + C_{L_{\dot{V}}} \dot{V} \frac{\rho}{2} Sc \qquad (6.7)$$

$$+ C_{L_\delta} \delta \frac{\rho}{2} SV^2 - C_{D_o} \frac{\rho}{2} SV^2 \alpha$$

$$X_H = (C_{L_o} - C_{D_o}) \frac{\rho}{2} SV^2 - C_{D_\alpha} \alpha \frac{\rho}{2} SV^2 - C_{D_{\dot{V}}} \dot{V} \frac{\rho}{2} Sc$$

$$- C_{D_\delta} \delta \frac{\rho}{2} SV^2 - C_{D_q} q \frac{\rho}{2} SV_c - C_{D_{\dot{q}}} \dot{q} \frac{\rho}{2} Sc^2 \qquad (6.8)$$

The hydrodynamic moment (M_H) is assumed to depend upon V, $\dot{V}$, α, $\dot{\alpha}$, q, $\dot{q}$, δ.

$$M_H = C_{M_o} \frac{\rho}{2} SV^2 c \quad C_{M_\alpha} \alpha \frac{\rho}{2} SV^2 \quad C_{M_{\dot{\alpha}}} \dot{\alpha} \frac{\rho}{2} SVc^2$$

$$+ C_{M_q} q \frac{\rho}{2} SVc^2 \quad C_{M_{\dot{q}}} \dot{q} \frac{\rho}{2} Sc^2 \quad C_{M_{\dot{V}}} \dot{V} \frac{\rho}{2} Sc^2$$

$$+ C_{M_\delta} \delta \frac{\rho}{2} SV^2 c \qquad (6.9)$$

6.2 Relations Between u, w, and V

$$U_o + u = V \cos \alpha \qquad (6.10)$$

$$w = V \sin \alpha \qquad (6.11)$$

$$V^2 = (U_o + u)^2 + w^2 = U_o^2 + 2U_o u + u^2 + w^2 \tag{6.12}$$

$$\frac{\partial V}{\partial u} = \frac{U_o + u}{V} = \cos\alpha \tag{6.13}$$

≈ 1 to 1st order approx.

$$\frac{\partial V}{\partial w} = \frac{w}{V} = \sin\alpha \tag{6.14}$$

to first order approx.

$$\alpha = \tan^{-1} \frac{w}{U_o + u} = \cot^{-1} \frac{U_o + u}{w}$$

$$\frac{\partial \alpha}{\partial u} = \frac{1}{\left[1 + \left(\frac{U_o \quad u}{w}\right)^2\right] w} = \frac{w}{V^2} = \frac{\sin\alpha}{V} \tag{6.15}$$

$= \frac{\alpha}{V}$ to 1st order approx.

$$\frac{\partial \alpha}{\partial w} = \frac{1}{\left[1 + \frac{w}{U_o + w}\right]^2 (U_o + u)} = \frac{\cos\alpha}{V}$$

$\approx \frac{1}{V}$ to 1st order approx. (6.16)

$$\acute{u} = \dot{V}\cos\alpha - V\acute{\alpha}\sin$$

$\approx \dot{V}$ to 1st order approx. (6.17)

$$\dot{w} = \dot{V} \sin\alpha + V \dot{\alpha} \cos\alpha$$
$$\approx V \dot{\alpha} \text{ to 1st order approx.} \qquad (6.18)$$

$$\frac{\partial \dot{\alpha}}{\partial \dot{w}} = \frac{1}{V} \quad \text{to 1st order approx.} \qquad (6.19)$$

Since $\dot{V} \approx \dot{u}$, $\quad C_{D_{\dot{u}}} = C_{D_{\dot{V}}}$ and $C_{L_{\dot{u}}} = C_{L_{\dot{V}}}$

6.3 The Longitudinal Stability Derivatives

The dimensional forms of the stability derivatives to first order approximations are derived below:

$$X_u = \frac{\partial X}{\partial V} \frac{\partial V}{\partial u} + \frac{\partial X}{\partial \alpha} \frac{\partial \alpha}{\partial u} = -2C_{D_o} \frac{\rho}{2} SU_o$$

$$X_w = \frac{\partial X}{\partial V} \frac{\partial V}{\partial w} + \frac{\partial X}{\partial \alpha} \frac{\partial \alpha}{\partial w} = (C_{L_o} - C_{D_\alpha}) \frac{\rho}{2} SU_o$$

$$X_{\dot{u}} = -C_{D_{\dot{u}}} \frac{\rho}{2} Sc \qquad \text{where } C_{D_{\dot{u}}} = C_{D_{\dot{V}}}$$

$$X_\delta = C_{D_\delta} \frac{\rho}{2} SU_o^2$$

$$Z_u = \frac{\partial Z}{\partial V} \frac{\partial V}{\partial u} + \frac{\partial Z}{\partial \alpha} \frac{\partial \alpha}{\partial u} = -2C_{L_o} \frac{\rho}{2} SU_o$$

$$Z_{\dot{u}} = - C_{D_{\dot{u}}} \frac{\rho}{2} Sc$$

$$Z_w = \frac{\partial Z}{\partial V} \frac{\partial V}{\partial w} + \frac{\partial Z}{\partial \alpha} \frac{\partial \alpha}{\partial w} = - \frac{\rho}{2} SU_o \left[C_{L_\alpha} + C_{D_o} \right]$$

$$Z_{\dot{w}} = - C_{L_{\dot{\alpha}}} \frac{\rho}{2} Sc$$

$$Z_q = - C_{L_q} \frac{\rho}{2} SU_o c$$

$$Z_{\dot{q}} = - C_{L_{\dot{q}}} \frac{\rho}{2} Sc^2$$

$$Z_\delta = - C_{L_\delta} \frac{\rho}{2} SU_o^2$$

$$M_u = \frac{\partial M}{\partial V} \frac{\partial V}{\partial u} + \frac{\partial M}{\partial \alpha} \frac{\partial \alpha}{\partial u} = 2C_{M_o} \frac{\rho}{2} Sc$$

$$M_{\dot{u}} = \frac{\partial M}{\partial \dot{u}} = C_{M_{\dot{u}}} \frac{\rho}{2} Sc^2$$

$$M_w = \frac{\partial M}{\partial V} \frac{\partial V}{\partial w} + \frac{\partial M}{\partial \alpha} \frac{\partial \alpha}{\partial w} = C_{M_\alpha} \frac{\rho}{2} SU_o c$$

$$M_{\dot{w}} = \frac{\partial M}{\partial \alpha} \quad \frac{\partial \alpha}{\partial \dot{w}} \quad C_{M_{\dot{\alpha}}} \quad \frac{\rho}{2} \quad Sc^2$$

$$M_q = \frac{\partial M}{\partial q} = C_{M_q} \quad \frac{\rho}{2} \quad SU_o c^2$$

$$M_{\dot{q}} = \frac{\partial M}{\partial \dot{q}} \quad C_{M_{\dot{q}}} \quad \frac{\rho}{2} \quad Sc^3$$

$$M_{\delta} = \frac{\partial M}{\partial \delta} \quad C_{M_{\delta}} \quad \frac{\rho}{2} \quad SU_o{}^2 c$$

VII. THE LONGITUDINAL EQUATIONS OF MOTION

7.1 The Linearized Equations of Motion

The linearized equations of motion for a body moving through a fluid such as air or water are derived in Appendix B. These equations are written for small disturbances from steady symmetric flight. Assuming constant forward velocity the longitudinal set of equations listed on page reduce to the following two equations:

$$\left[Z_w + (Z_{\dot{w}} - m)\,D\right] w + \left[Z_{\dot{q}}D^2 + (Z_q + mU_o)\,D - G\sin\theta_o\right]\theta + Z_\delta\,\delta = 0 \tag{7.1}$$

$$(M_w + M_{\dot{w}}D)\,\dot{w} + \left[(M_{\dot{q}} - I_{yy})\,D^2 + M_qD + Bz_B\cos\theta_o - Bx_B\sin\theta_o\right]\theta + M_\delta\,\delta = 0 \tag{7.2}$$

where $Z_{\dot{w}} - m$ = apparent mass, $M_{\dot{q}} - I_{yy}$ = apparent inertia.

The further assumption of nearly level flight ($-15^o < \theta_o < 15^o$) is made and the terms $G\sin\theta_o$, $Bx_B\sin\theta_o$ considered negligible. In addition, the distance of the center of gravity from the center of buoyancy z_B is assumed small enough so that the term $Bz_B\cos\theta_o$ can be neglected. With these assumptions, the equations of motion reduce to 7.3 and 7.4 and these equations are considered in the subsequent analysis.

$$\left[Z_w + (Z_{\dot{w}} - m)\,D\right] w + \left[(Z_q + mU_o) + Z_{\dot{q}}D\right] q = -Z_\delta\,\delta \tag{7.3}$$

$$\left[M_w + M_{\dot{w}}D\right] w + \left[M_q + (M_{\dot{q}} - I_{yy})\,D\right] q = -M_\delta\,\delta \tag{7.4}$$

7.2 The Operational Form of the Longitudinal Equations of Motion in Terms of $\bar{q}$, $\bar{w}$, $\bar{\delta}$.

Define

$$\bar{w}(s) = \int_0^\infty e^{-st}\, w(t)\, dt$$

$\bar{w}(s)$ is the Laplace transform of $w(t)$ and is a function of the operator s. The bar is used over the variable to indicate the Laplace transform.

The initial values of w and q are assumed to be zero in the derivation of equations 7.3 and 7.4. With these assumptions, these equations may be written in operational form by replacing $D = \frac{d}{dt}$ by s and $w(t)$, $q(t)$, and $\delta(t)$ by $\bar{w}(s)$, $\bar{q}(s)$ and $\bar{\delta}(s)$, respectively.

The operational form of the equations become

$$\left[Z_w + (Z_{\dot{w}} - m)\, s\right] \bar{w} + \left[(Z_q + mU_o) + Z_{\dot{q}} s\right] q = -Z_\delta \delta \qquad (7.5)$$

$$\left[M_w + M_{\dot{w}} s\right] \bar{w} + \left[M_q + (M_{\dot{q}} - I_{yy})\, s\right] q = -M_\delta \delta \qquad (7.6)$$

7.3 The Operational Form of the Equations of Motion in Terms of $\bar{q}, \bar{\alpha}, \bar{\delta}$.

From Section VI page 15 it is shown that to the first order approximation

$$\alpha = \frac{w}{U_o}$$

$$\dot{\alpha} = \frac{\dot{w}}{U_o} \qquad (7.7)$$

Define

$$\mu = \frac{m}{\rho S c}$$

$$k_{yy} = \frac{I_{yy}}{\rho/2\, S c^3} \qquad (7.8)$$

Dividing 7.5 by $\rho/2\, SU_o^{\,2}$ and 7.6 by $\rho/2\, SU_o^{\,2} c$ and using 7.7 and 7.8 and the expressions for the stability derivatives listed on page 16, the operational form of equations of motion expressed in dimensionless form become

$$\left[(C_{L_\alpha} + C_{D_o}) + (C_{L_{\dot\alpha}} + 2\mu) \frac{c}{U_o} s\right]\bar\alpha + \left[(C_{L_q} - 2\mu) \frac{c}{U_o} + C_{L_{\dot q}} \frac{c^2}{U_o^{\,2}} s\right] \bar q$$

$$= - C_{L_\delta}\delta \qquad (7.9)$$

$$\left[C_{M_\alpha} + C_{M_{\dot\alpha}} \frac{c}{U_o} s\right]\bar\alpha + \left[C_{M_q} \frac{c}{U_o} + \left(C_{M_{\dot q}} - k_{yy}\right) \frac{c^2}{U_o^2} s\right] \bar q$$

$$= -C_{M_\delta}\delta \qquad (7.10)$$

7.4 The Operational Form of the Equations of Motion in Terms of $\bar a_z$, $\bar q$, $\bar\delta$

An accelerometer placed at the center of gravity with its axis parallel to the z axis fixed in the body will read R_z

$$R_z = a_z + g \cos z \wedge Z_I \qquad (7.11)$$

where

Z_I is the true vertical

a_z = the component of acceleration along the z axis

= normal acceleration

$$a_z = \frac{dw}{dt} - U_o q = U_o \dot{\alpha} - U_o q \qquad (7.12)$$

$$\dot{\alpha} = \frac{a_z}{U_o} + q$$

or written in operational form (7.13)

$$\bar{\alpha} = \frac{1}{s}\left[\frac{\bar{a}_z}{U_o} + \bar{q}\right]$$

The equations of motion 7.9 and 7.10 expressed in terms of pitching velocity and normal acceleration become

$$\left[\frac{1}{U_o s}\left(C_{L_\alpha} + C_{D_o}\right) + \left(C_{L_{\dot{\alpha}}} + 2\mu\right)\frac{c}{U_o^2}\right]\bar{a}_z$$

$$+\left[\frac{1}{s}\left(C_{L_\alpha} + C_{D_o}\right) + \left(C_{L_{\dot{\alpha}}} + C_{L_q}\right)\frac{c}{U_o} + C_{L_{\dot{q}}}\frac{c^2}{U_o^2}s\right]\bar{q} = -C_{L_\delta}\bar{\delta} \qquad (7.14)$$

$$\left[\frac{1}{U_o s}C_{M_\alpha} + C_{M_{\dot{\alpha}}}\frac{c}{U_o^2}\right]\bar{a}_z$$

$$+\left[\frac{C_{M_\alpha}}{s}\left(C_{M_{\dot{\alpha}}} + C_{M_q}\right)\frac{c}{U_o} + \left(C_{M_{\dot{q}}} - k_{yy}\right)\frac{c^2}{U_o^2}s\right]\bar{q} = -C_{M_\delta}\delta \qquad (7.15)$$

VIII. TRANSFER FUNCTIONS

8.1 Definition

The transfer function is defined in servomechanism theory as the ratio of the Laplace transform of the output of a system to the Laplace transform of its input. The transfer function is defined here as the ratio of the Laplace transform of the response to a disturbance to the Laplace transform of the disturbance. For example, the transfer function $\frac{\bar{q}(a)}{\bar{\delta}(s)}$ is the ratio of the Laplace transform of the pitching velocity, q, to the Laplace transform of the elevator displacement, δ .

The transfer functions are determined from the operational form of the equations of motion by solving for the desired ratios.

8.2 The Transfer Function $\frac{\bar{q}(s)}{\bar{\delta}(s)}$ and $\frac{\bar{a}_z(s)}{\bar{\delta}(s)}$

Solving equations 7.14 and 7.15 for $\frac{\bar{q}(s)}{\bar{\delta}(s)}$ and $\frac{\bar{a}_z(s)}{\bar{\delta}\, s}$ give 8.1 and 8.6

$$\frac{\bar{q}(s)}{\bar{\delta}(s)} = \frac{A_1 s + A_2}{s^2 + A_3 s + A_4} \qquad (8.1)$$

where

$$A_1 = \frac{U_o^2}{\lambda c^2} = \frac{-\left(C_{L_{\dot{\alpha}}} + 2\mu\right) C_{M_\delta} + C_{M_{\dot{\alpha}}} C_{L_\delta}}{\lambda} \qquad (8.2)$$

where $$\lambda = \left(C_{L_{\dot{\alpha}}} + 2\mu\right)\left(C_{M_{\dot{q}}} - k_{yy}\right) - C_{M_{\dot{\alpha}}} C_{L_{\dot{q}}}$$

$$A_2 = \frac{U_o^3}{\lambda c^3}\left[-\left(C_{L_\alpha} + C_{D_o}\right)C_{M_\delta} + C_{M_\alpha} C_{L_\delta}\right] \tag{8.3}$$

$$A_3 = \frac{U_o}{\lambda c}\left[\left(C_{L_{\dot{\alpha}}} + 2\mu\right)C_{M_q} + \left(C_{L_\alpha} + C_{D_o}\right)\left(C_{M_{\dot{q}}} - k_{yy}\right) - C_{L_{\dot{q}}} C_{M_\alpha} - \left(C_{L_q} - 2\mu\right)C_{M_{\dot{\alpha}}}\right] \tag{8.4}$$

$$A_4 = \frac{U_o^2}{\lambda c^2}\left[\left(C_{L_\alpha} + C_{D_o}\right)\left(C_{M_q}\right) - \left(C_{L_q} - 2\mu\right)C_{M_\alpha}\right] \tag{8.5}$$

$$\frac{\bar{a}_z(s)}{\bar{\delta}(s)} = \frac{A_5 s^2 + A_6 s + A_7}{s^2 + A_3 s + A_4} \tag{8.6}$$

where

$$A_5 = \frac{U_o^2}{\lambda c}\left[-\left(C_{M_{\dot{q}}} - k_{yy}\right)C_{L_\delta} + C_{L_{\dot{q}}} C_{M_\delta}\right] \tag{8.7}$$

$$A_6 = \frac{U_o^3}{\lambda c^2}\left[-\left(C_{M_\alpha} + C_{M_q}\right)C_{L_\delta} + \left(C_{L_{\dot{\alpha}}} + C_{L_q}\right)C_{M_\delta}\right] \tag{8.8}$$

$$A_7 = -U_o A_2 \tag{8.9}$$

8.3 The Transfer Functions $\frac{\bar{q}(s)}{\bar{F}(s)}$ and $\frac{\bar{a}_z(s)}{\bar{F}(s)}$

If a non-hydrodynamic force F acting in the z direction is used to disturb the vehicle in place of a control surface motion, δ , the term $C_{L_\delta}\delta$ in equation 7.14 is replaced by $\frac{F}{\rho/2\ SU_o^2}$ and the term $C_{M_\delta}\delta$ by $\frac{F\ell}{\rho/2\ SU_o^2c}$ where ℓ is the distance of the point of application of the force F behind the center of gravity. Making these substitutions 8.1 and 8.6 become

$$\frac{\bar{q}(s)}{\bar{F}(s)} = \frac{A_8 s + A_9}{s^2 + A_3 s + A_4} \tag{8.10}$$

$$\frac{\bar{a}_z(s)}{\bar{F}(s)} = \frac{A_{10}s^2 + A_{11}s + A_{12}}{s^2 + A_3 s + A_4} \tag{8.11}$$

where

$$A_8 = \frac{1}{\lambda\ \rho/2\ Sc^2}\left[-\left(C_{L_{\dot{\alpha}}} + 2\mu\right)\frac{\ell}{c} - C_{M_{\dot{\alpha}}}\right] \tag{8.12}$$

$$A_9 = \frac{U_o}{\lambda\ \rho/2\ Sc^3}\left[-\left(C_{L_\alpha} + C_{D_o}\right)\frac{\ell}{c} - C_{M_\alpha}\right] \tag{8.13}$$

$$A_{10} = \frac{U_o}{\lambda\ \rho/2\ Sc}\left[\left(C_{M_{\dot{q}}} - k_{yy}\right) + C_{L_{\dot{q}}}\frac{\ell}{c}\right] \tag{8.14}$$

$$A_{11} = \frac{U_o}{\lambda \rho/2\ Sc^2}\left[\left(C_{M_{\dot{\alpha}}} + C_{M_q}\right) + \left(C_{L_{\dot{\alpha}}} + C_{L_q}\right)\frac{\ell}{c}\right] \qquad (8.15)$$

A_3 and A_4 are defined on page 24.

IX. THE LONGITUDINAL STABILITY DERIVATIVES OBTAINABLE FROM DYNAMIC FREE FLIGHT TESTS

9.1 Lift Derivatives

The equations of motion 7.14 and 7.15 are written in terms of normal acceleration, pitching velocity, and control surface motion. Thus, if flight test measurements are made of these three quantities, the transfer function constants A_1 through A_7 appearing in the transfer functions

$\frac{\bar{q}(s)}{\bar{\delta}(s)}$ and $\frac{\bar{a}_z(s)}{\bar{\delta}(s)}$ can be determined from the flight test data by the methods of Section X.

From equations 8.1 and 8.6

$$\bar{q} = \frac{(A_1 s + A_2)\,\bar{\delta}}{s^2 + A_3 s + A_4} \qquad (8.1)$$

$$\bar{a}_z = \frac{(A_5 s^2 + A_6 s + A_7)\,\bar{\delta}}{s^2 + A_3 s + A_4} \qquad (8.6)$$

Substituting these in the lift equation 7.14 and noting $A_7 = - U_o A_2$

$$\begin{aligned}
& s^2\delta \left[A_5 \left(C_{L_{\dot{\alpha}}} + 2\mu \right) \frac{c}{U_o^2} + A_1 \; C_{L_{\dot{q}}} \; \frac{c^2}{U_o^2} \right] \\
& + s\delta \left[\frac{A_5}{U_o} \left(C_{L_\alpha} + C_{D_o} \right) + A_6 \left(C_{L_{\dot{\alpha}}} + 2\mu \right) \frac{c}{U_o^2} + A_1 \left(C_{L_{\dot{\alpha}}} + C_{L_q} \right) \frac{c}{U_o} \right. \\
& \left. + A_2 \; C_{L_{\dot{q}}} \; \frac{c^2}{U_o^2} \right] + \delta \left[\frac{A_6}{U_o} \left(C_{L_\alpha} + C_{D_o} \right) - \frac{A_2}{U_o} \, c \left(C_{L_{\dot{\alpha}}} + 2\mu \right) \right. \\
& \left. + A_1 \left(C_{L_\alpha} + C_{D_o} \right) + A_2 \left(C_{L_{\dot{\alpha}}} + C_{L_q} \right) \frac{c}{U_o} \right] \\
& = - (s^2 + A_3 s + A_4) \, C_{L_\delta} \delta
\end{aligned} \qquad (9.1)$$

Equating coefficients of like powers of the independent variable s, the three equations 9.2, 9.3, 9.4 in the five unknowns $C_{D_o} + C_{L_\alpha}$, $C_{L_{\dot\alpha}}$, C_{L_q}, $C_{L_{\dot q}}$, C_{L_δ} are obtained.

Thus, to determine all of the lift derivatives, two of them must be known from other sources.

If a non-hydrodynamic force is used to disturb the test vehicle in place of a control surface displacement, the terms C_{L_δ}, A_2, A_3, A_5, A_6, A_7 are replaced by $\frac{1}{\rho/2\, S U_o^2}$, A_8, A_9, A_{10}, A_{11}, and A_{12}, respectively in equations 9.2, 9.3, 9.4 giving three equations in five unknowns. Thus, two stability derivatives must be known from other sources. In most practical cases, at least two of these stability derivatives are small enough to be neglected and the remaining derivatives can be determined from the flight test data.

$$A_5 \quad \frac{c}{U_o^2}\left(C_{L_\alpha} + 2\mu\right) \quad A_1 \, C_{L_{\dot q}} \quad \frac{c^2}{U_o^2} + C_{L_\delta} = 0 \qquad (9.2)$$

$$\frac{A_5}{U_o}\left(C_{L_{\dot\alpha}} + C_{D_o}\right)\left(\frac{A_6}{U_o^2} \quad c + A_1 \quad \frac{c}{U_o}\right) C_{L_{\dot\alpha}} + A_1 \quad \frac{c}{U_o} \quad C_{L_q}$$

$$+ \; A_2 \quad C_{L_{\dot q}} \quad \frac{c^2}{U_o^2} + A_3 \quad C_{L_\delta} + 2 \quad \frac{A_6}{U_o} \quad c\mu = 0 \qquad (9.3)$$

$$\left(A_1 + \frac{A_6}{U_o}\right)\left(C_{L_\alpha} + C_{D_o}\right) + \frac{A_2}{U_o} \quad c\left(C_{L_q} - 2\mu\right) + A_4 \quad C_{L_\delta} = 0 \qquad (9.4)$$

9.2 Moment Derivative

Following the procedure of the preceding section, equations 8.1 and 8.6 are substituted in the moment equation 7.15 and like powers of s equated. The three equations, 9.5, 9.6, 9.7, in the five unknowns C_{M_α}, $C_{M_{\dot\alpha}}$, C_{M_q}, $C_{M_{\dot q}}$, C_{M_δ} are obtained. Thus, two of the unknown stability derivatives must be determined from other sources such as wind tunnel tests to evaluate the derivatives from a flight test.

If a non-hydrodynamic force is used to disturb the test vehicle, the term C_{M_δ} is replaced by $\dfrac{\ell}{\rho/2\, U_o{}^2 c}$ and the constants A_2, A_3, A_5, A_6, A_7 by A_8, A_9, A_{10}, A_{11}, and A_{12}, respectively. The number of unknowns is reduced to four and only one stability derivative must be known from other sources. In most practical cases, one of these stability derivatives is small enough to be neglected and the remaining derivatives can be determined from the flight rest data.

$$A_5 \frac{c}{U_o{}^2} C_{M_{\dot\alpha}} + A_1 \frac{c^2}{U_o{}^2}\left(C_{M_{\dot q}} - k_{yy}\right) + C_{M_\delta} = 0 \tag{9.5}$$

$$\frac{A_5}{U_o} C_{M_\alpha} \left(\frac{A_6}{U_o} + A_1\right)\frac{c}{U_o} C_{M_{\dot\alpha}} + \frac{A_1 c}{U_o} C_{M_q} + A_2 \frac{c^2}{U_o{}^2}\left(C_{M_q} - k_{yy}\right) + A_3 C_{M_\delta} = 0 \tag{9.6}$$

$$\left(\frac{A_6}{U_o} + A_1\right) C_{M_\alpha} + A_2 \frac{c}{U_o} C_{M_q} + A_4 C_{M_\delta} = 0 \tag{9.7}$$

9.3 Special Case of the Dirigible or Torpedo

The Stability derivatives $C_{M_{\dot{\alpha}}}$ and $C_{L_{\dot{q}}}$ are assumed negligible for the case. $C_{M_{\dot{\alpha}}}$ depends primarily upon the downwash from a forward lifting surface, such as a wing. $C_{L_{\dot{q}}}$ is determined by the lift due to angular acceleration about the center of gravity. This term would be small compared to the term $C_{L_{\dot{\alpha}}}$ which depends upon the lif due to a linear acceleration normal to the longitudinal axis of the test vehicle.

The lift derivatives obtainable from tests where the forcing function is a control surface displacement are determined from equations 9.2, 9.3, and 9.4. Since $C_{L_{\dot{q}}} = 0$ for this case, there are only three equations in the four unknowns $\left(C_L + C_{D_o}\right)$, C_{L_α}, C_{L_q}, C_{L_δ}. Only the ratios of these derivatives can be determined from the tests.

If a non-aerodynamic force replaces the control surface deflection, the term C_{L_δ} is replaced by $\frac{1}{\rho/2\ SU_o^2}$ and the three equations are sufficient to determine $\left(C_{L_\alpha} + C_{D_o}\right)$, $C_{L_{\dot{\alpha}}}$, C_{L_q}, C_{L_δ}. From A_8 it is seen that if only the normal acceleration and the force are measured, the derivative $C_{L_{\dot{\alpha}}}$ or the apparent mass can be determined explicitly.

The moment derivatives obtainable can be determined by examining equations 9.4, 9.5, 9.6, remembering $C_{M_{\dot{\alpha}}} = 0$. There are only three equations in the four unknowns, C_{M_α}, C_{M_q}, $C_{M_{\dot{q}}}$, C_{M_δ} and only the ratios of these derivatives can be determined from flight tests where a control surface motion is used to disturb the test vehicle.

If a non-aerodynamic force is used to disturb the vehicle, the term C_{M_δ} is replaced by $\frac{\ell}{\rho/2\ SU_o^2 c}$, a known quantity, and the derivatives C_{M_α}, C_{M_q}, $C_{M_{\dot{q}}}$ can be determined from the test. Examination of A_8 shows that if only the pitching velocity and the force are measured, the derivative $C_{M_{\dot{\alpha}}}$ or the apparent mass may be determined explicitly.

9.4 Special Case of the Airplane

The stability derivatives $C_{L_{\dot{q}}}$ and $C_{M_{\dot{q}}}$, which depend upon the apparent inertia of the displaced fluid, are considered to be zero for an airplane. The terms $C_{L_{\dot{\alpha}}}$ and C_{L_q} depend upon the tail lift, while C_{L_α} depends mainly on the lift from the wings. For these reasons, the terms $C_{L_{\dot{\alpha}}} \dot{\alpha} \frac{c}{U_o}$ and $C_{L_q} q \frac{c}{U_o}$ are considered small compared to $C_{L_\alpha} \alpha$ and are neglected.

The lift stability derivatives obtainable from tests where the forcing function is a control surface displacement are determined from equations 9.2, 9.3, and 9.4. Since $C_{L_{\dot{\alpha}}} = C_{L_q} = C_{L_{\dot{q}}} = 0$ for this case, equation 9.2 is identically satisfied, leaving the two equations, 9.3 and 9.4, in the two unknowns $\left(C_{L_\alpha} + C_{D_o}\right)$ and C_{L_δ}. From these two equations the unknown lift derivatives $(C_{L_\alpha} + C_{D_o})$ and C_{L_δ} can be determined.

The moment derivatives obtainable can be determined by examining equations 9.5, 9.6, and 9.7. For this case only $C_{M_{\dot{q}}} = 0$. There are only three equations in the four unknowns C_{L_α}, $C_{L_{\dot{\alpha}}}$, C_{M_q}, C_{M_δ}. Thus, only the ratios of the moment derivatives may be determined.

If a non-aerodynamic force is used to disturb the airplane, the term C_{M_δ} is replaced by a known force and the three equations are sufficient to determine C_{M_α}, $C_{M_{\dot{\alpha}}}$, and $C_{M_{\dot{q}}}$.

If only pitching velocity and the force $\bar{F}$ are measured, examination of the transfer function A_8 shows that $C_{M_{\dot{\alpha}}}$ can be explicitly determined.

X. TRANSIENT ANALYSIS METHOD FOR COMPUTING STABILITY DERIVATIVES FROM FREE FLIGHT TEST DATA

10.1 Analysis Procedure

The flight test data are usually obtained as curves of pitching velocity, normal acceleration and control surface deflection plotted against time or other similar sets of data. This method is applicable when these curves represent the transient response of the test vehicle to a disturbance such as a control surface motion.

The analysis method consists of two steps: (1) determining the transfer function constants from the test data; (2) computing the stability derivatives from these transfer constants. Two analysis methods are considered for computing the transfer function constants from the test data; (1) the Fourier integral method presented in Section 10.2; (2) the Laplace transform method presented in Section 10.3. Both these methods depend upon the assumption of linear equations with constant coefficients. The Fourier integral method can be applied to systems which are stable, that is, systems in which the response to a disturbance reaches a steady state. The Laplace transform method can be used with unstable systems by taking the value of s in the integral

$$x(s) = \int_0^\infty e^{-st}\, x(t)\, dt$$

large enough so that x(s) will be determined by the value of the integral for a small value of t. Thus, the response curve can be analyzed for small values of the response such that the assumptions of small disturbances used in deriving the equations of motion are satisfied.

10.2 The Fourier Integral Method for Computing Transfer Function Constants from Test Data

10.21 Outline of Method

The steps necessary to compute the transfer function constants from the flight test data are listed below. Each step is discussed in detail later in this section.

(1) The flight test data are transformed from time dependent quantities to frequency dependent quantities by the Fourier integral transformation. For example, the pitching velocity $q(t)$, normal acceleration $a_z(t)$, and control surface deflection $\delta(t)$ are transformed to $\bar{q}(\omega)$, $\bar{a}_z(\omega)$, $\bar{\delta}(\omega)$. The value $\bar{x}(\omega)$ of the Fourier transform of $x(t)$ is complex and is represented by $\bar{x}(\omega) = \alpha + i\beta$ where $\alpha =$ real part of $x(\omega)$ and β is the imaginary part of $x(\omega)$.

(2) The transformed data are corrected for errors due to the dynamic characteristics of the measuring instruments.

(3) The experimental values of the transfer functions are computed from the corrected data of step 2. Since the Fourier integral transform $x(\omega)$ is complex, the transfer function will consist of a real part ξ and an imaginary part η. The values of $\xi = \text{Re}\,\dfrac{\bar{q}(\omega)}{\bar{\delta}(\omega)}$ and $\eta = \text{Im}\,\dfrac{\bar{q}(\omega)}{\bar{\delta}(\omega)}$ are plotted as functions of ω.

(4) From the theoretical equations for the transfer function, two equations are written in terms of ξ , η , ω , and the transfer function constants. These two equations are 10.15 and 10.16. The most probable value of the transfer function constants are determined by solving each of these equations using the method of least squares.

10.22 Transformation of Test Data by the Fourier Integral

Given a plot of a variable x(t) as a function of time. The Fourier integral transform is defined as

$$\bar{x}(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-i\omega t}\, dt \qquad (10.1)$$

For cases where x(t) is zero for $t < 0$, the integral

$$\int_{-\infty}^{0} x(t)\, e^{-i\omega t}\, dt$$

vanishes and the unilateral Fourier integral transform is defined as

$$\bar{x}(\omega) = \int_{0}^{\infty} x(t)\, e^{-i\omega t}\, dt \qquad (10.2)$$

where it is understood that $x(t) = 0$ for $t < 0$.

The unilateral Fourier integral transform is just the special case of the Laplace transform

$$x(s) = \int_{0}^{\infty} x(t)\, e^{-st}\, dt$$

with $s = i\omega$

Consider a function, $x(t)$, which approaches a steady state value, x_{ss}, at time T. Substituting this value in 10.2 the resulting integral does not converge (oscillates). Hence to obtain a Fourier Integral representation of such a function it is necessary to consider the function

$$G(t) = x(t)\ e^{-\sigma t} \tag{10.3}$$

where $\sigma > 0$

define $$\bar{x}(\sigma, \omega) = \int_0^\infty G(t)\ e^{-i\omega t}\, dt$$

$$= \int_0^T x(t)\ e^{-(\sigma + i\omega)t}\, dt + x_{ss} \int_T^\infty e^{-(\sigma + i\omega)t}\, dt$$

$$= \int_0^T x(t)\ e^{-(\sigma + i\omega)t}\, dt - \frac{x_{ss}\ e^{-(\sigma + i\omega)t}}{\sigma + i\omega}\Bigg|_T^\infty$$

$$= \int_0^T x(t) e^{-(\sigma + i\omega)t}\, dt + \frac{x_{ss}\ e^{-(\sigma + i\omega)T}}{\sigma + i\omega}$$

now let $\sigma \to 0$ obtaining

$$\bar{x}(\omega) = \lim_{\sigma \to 0} \int_0^T x(t)\ e^{-(\sigma + i\omega)t}\, dt + \lim_{\sigma \to 0} x_{ss} \frac{e^{-(\sigma + i\omega)T}}{\sigma + i\omega}$$

$$\bar{x}(\omega) = \int_0^T x(t)\ e^{-i\omega t} dt + \frac{x_{ss} e^{-i\omega T}}{i\omega} \tag{10.4}$$

writing

$$e^{-i\omega t} = \cos\omega t - i \sin\omega t \tag{10.5}$$

$$\bar{x}(\omega) = \int_0^T x(t) \cos\omega t \, dt - \frac{x_{ss}}{\omega} \sin\omega T$$

$$- i \int_0^T \sin\omega t \, x(t) \, dt + \frac{x_{ss}}{\omega} \cos\omega T \tag{10.6}$$

$\bar{x}(\omega)$ is complex and may be expressed as

$$\bar{x}(\omega) = \alpha + i\beta \tag{10.7}$$

where

$$\alpha = \text{Re } \bar{x}(\omega) = \int_0^T x(t) \cos\omega t \, dt - \frac{x_{ss}}{\omega} \sin\omega T \tag{10.8}$$

$$\beta = \text{Im } \bar{x}(\omega) = -\int_0^T x(t) \sin\omega t \, dt - \frac{x_{ss}}{\omega} \cos\omega T \tag{10.9}$$

10.23 Correction of Test Data for the Dynamic Characteristics of Measuring Instruments

Given an instrument, say a rate gyroscope, which measures pitching velocity. The relation between the output and input of the instrument is expressed by the transfer function, which is a function only of the dynamic characteristics of the instrument. The transfer function $F(\omega)$ may be written as

$$\bar{F}(\omega) = \frac{\bar{I}(\omega)}{\bar{q}(\omega)} \tag{10.10}$$

where

$\bar{I}(\omega)$ is the Fourier integral transform of the output of the instrument.

$\bar{q}(\omega)$ is the Fourier integral transform of the input to the instrument.

$\bar{F}(\omega)$ may be obtained from a dynamic calibration of the instrument by putting a known input function, $q(t)$ on the instrument and measuring its output, $I(t)$. Transforming the input and output of the instrument by the Fourier integral, the values $\bar{q}(\omega)$ and $\bar{I}(\omega)$ are obtained and their ratio $\bar{F}(\omega) = \frac{\bar{I}(\omega)}{\bar{q}(\omega)}$ plotted against ω_o .

The data received from flight test are usually in the form of instrument readings plotted against time. Transforming these data by the Fourier integral transformation $\bar{I}(\omega)$ is obtained. From equation 10.10, the expression for the Fourier integral transform of the input to the instrument becomes

$$\bar{q}(\omega) = \frac{\bar{I}(\omega)}{\bar{F}(\omega)} \tag{10.11}$$

Thus, $\bar{q}(\omega)$ is the value of Fourier integral transform of the test data corrected for the dynamic characteristics of the measuring instrument.

10.24 Computation of Experimental Value of Transfer Functions

The experimental value of the transfer function is computed for a particular value of ω by substituting the values of the transformed data from step 2 in the expression for the transfer function. For example, the value of the transfer function $\frac{\bar{q}(\omega_o)}{\bar{\delta}(\omega_o)}$ is obtained by substituting the value for $\bar{q}(\omega)$ at $\omega = \omega_o$ and $\delta(\omega)$ at $\omega = \omega_o$ in the expression $\frac{\bar{q}(\omega)}{\bar{\delta}(\omega)}$. Since $\bar{q}(\omega)$ and $\bar{\delta}(\omega)$ are complex $\frac{\bar{q}(\omega)}{\bar{\delta}(\omega)}$ is also complex and can be represented by $\frac{\bar{q}(\omega)}{\bar{\delta}(\omega)} = \xi(\omega) + i\,\eta(\omega_o)$

Values of ξ and η are computed for a number of values of ω and are plotted against ω. These curves of ξ and η are used in computing the transfer function constants.

10.25 Computation of Transfer Function Constants from the Experimental Values of the Transfer Function

This procedure is explained by an example of the transfer function for $\frac{\bar{q}(s)}{\bar{\delta}(s)}$ given by equation 8.1.

$$\frac{\bar{q}(s)}{\bar{\delta}(s)} = \frac{A_1 s + A_2}{s^2 + A_3 s + A_4} \qquad (8.1)$$

where $\bar{q}(s)$ is the Laplace transform of q(t) defined in Section VIII,

$$\bar{q}(s) \equiv \int_0^\infty q(t)\, e^{-st}\, dt$$

and s is the complex variable defined by

$$s \equiv \sigma + i\omega$$

Similarly $\bar{\delta}(s)$ is the Laplace transform of $\delta(t)$.

Letting $s = i\omega$, equation 8.1 becomes

$$\frac{\bar{q}(\omega)}{\bar{\delta}(\omega)} \quad \frac{A_1 i\omega + A_2}{-\omega^2 + A_3\omega + A_4}$$

This is the theoretical form for the particular transfer function. Using the definition from step 3

$$\frac{\bar{q}(\omega)}{\bar{\delta}(\omega)} = \xi + i\eta \tag{10.13}$$

$$\xi + i\eta = \frac{i A_1 \omega + A_2}{(-\omega^2 + A_4) + i A_3 \omega} \tag{10.14}$$

$$\left[(\xi + i\eta)(-\omega^2 + A_4) + i A_3 \omega\right] = i A_1 \omega + A_2$$

Equating real and imaginary parts

$$-\xi\omega^2 - \eta A_3 \omega + \xi A_4 - A_2 = 0 \tag{10.15}$$

$$-\eta\omega^2 + (A_3 \xi - A_1)\omega + \eta A_4 = 0 \tag{10.16}$$

For each value of ω, say $\omega = \omega_0$, a value of ξ and η is read from the curves plotted in step 3. These values and the value substituted in 10.15 give one linear equation in A_2, A_3 and A_4. Three such linear equations are sufficient to compute A_2, A_3, and A_4. However, to obtain the most probable value of the A's, many values of ω are chosen and the values of A_2, A_3, and A_4 determined by the method of least squares. This method is outlined in detail in reference 2. Similarly A_1, A_3, and A_4 are determined from 10.16.

10.3 The Laplace Transform Method for Determining the Transfer Function Constants from Test Data

This method is similar to the Fourier integral method discussed previously. The test data are transformed from functions of time to functions of s by the Laplace transformation

$$\bar{x}(s) = \int_0^\infty e^{-st} x(t)\, dt$$

In the general case s is a complex number, $s = \sigma + i\omega$. However, for this method, s is taken to be a real positive constant large enough to insure convergence of the integral.

10.31 Outline of Method

(1) The flight test data are transformed by the Laplace transform.

(2) The transformed data are corrected for errors due to the dynamic characteristics of the measuring instruments.

(3) The experimental values of the transfer functions are computed from the corrected data of step 2 and are plotted as functions of s.

(4) The curve plotted in step 2 is fitted by the theoretical expression for the transfer function using the

method of least squares. The most probable value for the transfer function constants are thus determined.

These steps are discussed in detail below.

10.32 Transformation of Test Data by the Laplace Transform

The Laplace transform x(s) of a variable x(t) is defined by the equation

$$\bar{x}(s) = \int_0^\infty e^{-st}\, x(t)\, dt \qquad (10.17)$$

Given values of the test data x(t). $x(s_o)$ is computed for a particular value s_o by substituting s_o in 10.17 and carrying out the integration by graphical methods, Simpson's rule or any convenient means. This integration gives one particular value $x(s_o)$. This process is repeated for as many values of s as are desired.

10.33 Correction of Test Data for Dynamic Characteristics of Instruments

This step is identical with step 2 of the Fourier Integral method except that the variable s is substituted for $i\omega$.

10.34 Computation of Experimental Values of the Transfer Functions

This is illustrated for the transfer function $\frac{\bar{q}(s)}{\bar{\delta}(s)}$. Values of

$\bar{q}(s_o)$ and $\bar{\delta}(s_o)$ are selected from the corrected test data for a particular value of (s_o). The quotient $\frac{\bar{q}(s_o)}{\bar{\delta}(s_o)}$ is the value of the transfer function corresponding to s_o. Many values of s are chosen and the resulting values of $\frac{\bar{q}(s)}{\bar{\delta}(s)}$ plotted against the independent variable s.

10.35 Computation of Transfer Function Constants from the Experimental Values of the Transfer Functions

The transfer function for $\frac{\bar{q}(s)}{\bar{\delta}(s)}$ is according to equation 8.1

$$\frac{\bar{q}(s)}{\bar{\delta}(s)} = \frac{A_1 s + A_2}{s^2 + A_3 s + A_4} \tag{8.1}$$

$$\frac{\bar{q}(s)}{\bar{\delta}(s)} \left[s^2 + A_3 s + A_4 \right] = \left(A_1 s + A_2 \right)$$

For each value of s, say s_o, the value of $\frac{\bar{q}(s_o)}{\bar{\delta}(s_o)}$ is read from the curves plotted in step 3. Substitution of this value and $s = s_o$ in equations 10.18 gives one linear equation in A_1, A_2, A_3, and A_4. From values of s gives four equations which are sufficient to determine the values of A_1, A_2, A_3, and A_4. For the most probable value for the A's, many values of s are chosen and the A's determined by the method of least squares. The method of least squares is discussed in reference 2.

10.4 Computation of Stability Derivatives from Transfer Function Constants

In Section 9 the three linear equations, 9.2, 9.3, and 9.4, relating the stability derivatives in the lift equations to the transfer function constants were derived. Similarly, the three linear equations, 9.5, 9.6, and 9.7, relate the transfer function constants to the stability derivatives appearing in the moment equation. The simultaneous solution of 9.2, 9.3, and 9.4 yield the combinations of lift stability derivatives obtainable from the test and simultaneous solution of 9.5, 9.6, and 9.7 the moment derivatives obtainable from the test.

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APPENDIX A

THE EQUATIONS OF MOTIONS FOR A RIGID BODY IN TERMS OF MOVING COORDINATES

The equations of motion for a rigid body written in terms of forces, moments, velocities, and accelerations along axes fixed in the body are the well known Euler's equations. Their derivation is brief and is included for the sake of completeness of this work.

The Moment Equations

Let $\bar{i}$, $\bar{j}$, $\bar{k}$ be unit vectors along the axes x, y, z fixed to the moving body.

Let $\bar{\mathfrak{M}}$ be the external moment applied to the rigid body

$$\bar{\mathfrak{M}} = \bar{i}\,L + \bar{j}\,M + \bar{k}\,N \tag{A.1}$$

where L, M, N are components of moment along the moving axes x, y, z.

Let $\bar{H}$ be the angular momentum vector

$$\bar{H} = \underset{\sim}{I} \cdot \bar{\omega} \tag{A.2}$$

where $\bar{\omega}$ is the angular velocity vector

$$\bar{\omega} = \bar{i}\omega_x \quad \bar{j}\omega_y + \bar{k}\omega_z$$

and $\underset{\sim}{I}$ is the dyad of inertia from Reference 3.

$$\underset{\sim}{I} = \begin{bmatrix} \bar{i}\bar{i}\, I_{xx} & \bar{i}\bar{j}\, I_{xy} & \bar{i}\bar{k}\, I_{xz} \\ \bar{j}\bar{i}\, I_{yx} & \bar{j}\bar{j}\, I_{yy} & \bar{j}\bar{k}\, I_{yz} \\ \bar{k}\bar{i}\, I_{zy} & \bar{k}\bar{j}\, I_{zx} & \bar{k}\bar{k}\, I_{zz} \end{bmatrix}$$

$$I_{xx} = \int_{vol} (y^2 + z^2)\, \rho\, d\tau$$

$$I_{yy} = \int_{vol} (x^2 + z^2) \rho d\tau$$

$$I_{zz} = \int_{vol} (x^2 + y^2) \rho d\tau$$

$$I_{xy} = I_{yx} \quad - \int_{vol} xy \rho d\tau$$

$$I_{xz} = I_{zx} = - \int_{vol \; xz} xz \rho d\tau$$

$$I_{yz} = I_{zy} = - \int_{vol} yz \rho d\tau$$

Newton's second law written for inertial space states

$$\bar{\mathcal{M}} = \dot{\bar{H}} \tag{A.4}$$

where $\dot{\bar{H}}$ = time rate of change of $\bar{H}$ in a coordinate system fixed in space

$$\dot{\bar{H}} = \frac{d\bar{H}}{dt} + \bar{\omega} \times \bar{H} \qquad \text{(Ref. 25)}$$

where $\frac{d\bar{H}}{dt} = \bar{i} \frac{dH_x}{dt} + j \frac{dH_y}{dt} + \bar{k} \frac{dH_z}{dt}$

and $\frac{dH_x}{dt}, \frac{dH_y}{dt}, \frac{dH_z}{dt}$ are the time rate of change of the components of the vector H as measured in the moving coordinate system x, y, z.

$$\bar{\omega} \times \bar{H} = \begin{bmatrix} \bar{i} & \bar{j} & \bar{k} \\ \omega_x & \omega_y & \omega_z \\ H_x & H_y & H_z \end{bmatrix}$$

$$= \bar{i} \ (\omega_y \ H_z - \omega_z \ H_y) + \bar{j} \ (\omega_z \ H_x - \omega_x \ H_z) + \bar{k} \ (\omega_x H_y - \omega_y \ H_x)$$

Define

$P \equiv \omega_x$ = component of angular velocity about the x axis

$Q \equiv \omega_y$ = component of angular velocity about the y axis

$R \equiv \omega_z$ = compoent of angular velocity about the z axis

The equations of motion become

$$L = I_{xx} \frac{dP}{dt} + I_{xy}\left(\frac{dQ}{dt} - RP\right) + I_{xz}\left(\frac{dR}{dt} + QP\right)$$

$$+ I_{yz} \ (Q^2 - R^2) + I_{zz} \ (QR) - RQ \ I_{yy} \tag{A.5}$$

$$M = I_{yy} \frac{dQ}{dt} + I_{yz}\left(\frac{dR}{dt} - PQ\right) + I_{yx}\left(\frac{dP}{dt} + RQ\right)$$

$$+ I_{xz} \ (R^2 - P^2) + I_{xx} \ (RP) - PR \ I_{zz} \tag{A.6}$$

$$N = I_{zz} \frac{dR}{dt} + I_{xz}\left(\frac{dP}{dt} - QR\right) + I_{yz}\left(\frac{dQ}{dt} + PR\right)$$

$$+ I_{xy} \ (P^2 - Q^2) + I_{yy} \ (PQ) - I_{yy} \ (QP) \tag{A.7}$$

<u>Force Equations</u>

Let $\bar{F}$ = the external force acting on the body.

$$\bar{F} = \bar{i} \ X + \bar{j} \ Y + \bar{k} \ Z$$

where X, Y, and Z are the components of external force along axes x, y, z fixed in the body.

Newton's second law written for inertial space states,

$$\bar{F} = m\dot{\bar{V}}$$

where $\dot{\bar{V}}$ = time rate of change of the velocity vector V in a coordinate system fixed in inertial space.

$$\dot{\bar{V}} = \frac{d\bar{V}}{dt} + \bar{\omega} \times \bar{V}$$

$$\frac{d\bar{V}}{dt} = \bar{i}\ \frac{dV_x}{dt} + \bar{j}\ \frac{dV_y}{dt} + \frac{\bar{k}dV_z}{dt}$$

where $\frac{dV_x}{dt}$, $\frac{dV_y}{dt}$, $\frac{dV_z}{dt}$ are the time rates of change of the components of $\bar{V}$ along the moving axes x, y, z respectively.

Define

$$U = V_x$$

$$V = V_y$$

$$W = V_z$$

$$\bar{\omega} \times \bar{V} = \bar{i}\ (QW - RV) + \bar{j}\ (RU - WP) + \bar{k}\ (VP - UQ)$$

The force of equations of motions become

$$X = \frac{mdU}{dt} + mQW - mRV \qquad (A.8)$$

$$Y = \frac{mdV}{dt} + mRU - mWP \qquad (A.9)$$

$$Z = \frac{mdW}{dt} + mVP - m\ \ UQ \qquad (A.10)$$

APPENDIX B

THE LINEARIZED EQUATIONS OF MOTION OF A SUBMERGED BODY FOR SMALL DISTURBANCES FROM STEADY SYMMETRIC FLIGHT

Forces Acting on the Body

The external forces on a submerged body consist of the force of gravity (mg), the buoyancy force B, the hydrodynamic forces due to the motion of the body through the fluid, and a thrust T_o from the propulsion system.

Orientation of Axes Fixed in Body with Respect to Axes Fixed in Space

The submerged body is assumed to have one plane of symmetry. Three sets of axes are necessary to develop the equations of motion. Each set of axes has its origin at the body center of gravity and translates with it. The set of axes X_I, Y_I, Z_I are defined such that the Z_I axis is vertical and points toward the center of the earth. The X_I Y_I plane is thus horizontal. These axes translate but do not rotate with the body.

The second set of axes are the X_o, Y_o, Z_o set. The X_o Z_o plane is vertical and coincides with the plane of symmetry of the body. The X_o axis may either point in the direction of the relative wind or along the longitudinal axis of the body when the vehicle is in steady flight. If the X_o axis points in the direction of the relative wind, the X_o, Y_o, Z_o set are called "wind axes" and if the X_o axis points in the direction of the longitudinal axis of the X_o, Y_o, Z_o set are called "body axes". These axes translate but do rotate with the body. The equations which are developed in the following pages may be used for wind or body axes.

The third set of axes are the x, y, z set. These coincide with the X_o, Y_o, Z_o axes during the steady state flight but are fixed to the body and hence move with it during the disturbed motion.

The orientation of the x, y, z axes with the X_o Y_o Z_o set is defined by three angles ψ, θ, ϕ. The angles are found by the following hypothetical rotation of the x, y, z axes from the initial X_o, Y_o, Z_o position to their final position. The x, y, z axes fixed in the body are assumed to be coincident with the X_o, Y_o, Z_o set before the disturbance. The body first rotates about the Z_o axis through the yaw angle ψ and the x, y, z axes in this position define a new set, the X_1, Y_1, Z_1 axes. The body then rotates about the Y_1 axis through the pitch angle θ and the x, y, z axes fixed in the body in this position define a second new X_2 Y_2 Z_2 set of axes. The third rotation is about the X_2 axis through the roll angle ϕ and the x, y, z axes are then coincident with their final position. The order of rotation defines the angles and must be carried out in the order indicated for finite rotations. However, for the case of infinitesimal rotations the order is immaterial.

Table B.1 gives the direction cosines of the x, y, z and the X_o, Y_o, Z_o axes for finite rotations and Table B.2 gives the direction cosines for infinitesimal rotation.

	X_o	Y_o	Z_o
x	$\cos\theta \cos\psi$	$\cos\theta \sin\psi$	$-\sin\theta$
y	$-\cos\phi \sin\psi + \sin\phi \cos\psi \sin\theta$	$\cos\phi \cos\psi + \sin\phi \sin\psi \sin\theta$	$\sin\phi \cos\theta$
z	$\sin\phi \sin\psi + \cos\phi \cos\psi \sin\theta$	$-\sin\phi \cos\psi + \cos\phi \sin\psi \sin\theta$	$\cos\theta \cos\phi$

Table B.1 - Direction Cosines for Finite Rotations

	X_o	Y_o	Z_o
x	1	ψ	$-\theta$
y	$-\psi$	1	ϕ
z	θ	$-\phi$	1

Table B.2 - Direction Cosines for Infinitesimal Rotations

The Hydrodynamic Forces and Moments

The hydrodynamic forces along the x, y, z axes will be designated as X_H, Y_H, Z_H and the hydrodynamic moments along the x, y, z axes by L_H, N_H, M_H. These forces and moments will be assumed to be functions of the velocity components U, V, W, P, Q, R and the acceleration components $\dot{U}$, $\dot{V}$, $\dot{W}$, $\dot{P}$, $\dot{Q}$, $\dot{R}$.

Expanding the hydrodynamic force X_H in a Taylors series gives

$$X_H = X_o + X_u u + X_v v + X_w w + X_{\dot{u}}\dot{u} + X_{\dot{w}}\dot{w} + X_{\dot{v}}\dot{v}$$

$$+ X_p p + X_q q + X_r r + X_{\dot{p}}\dot{p} + X_{\dot{q}}\dot{q} + X_{\dot{r}}\dot{r} \qquad \text{(B.1)}$$

higher order terms

where

$$X_u = \frac{\partial X}{\partial u}, \quad X_w = \frac{\partial X}{\partial w} \qquad \text{etc.}$$

and

$$U = U_o + u, \; V = v, \; W = W_o + w, \quad P = p, \quad Q = q, \quad R = r$$

where u, v, w, p, q, r, $\dot{u}$, $\dot{v}$, $\dot{w}$, $\dot{p}$, $\dot{q}$, $\dot{r}$ are small perturbations. Similar expressions hold for Y_H, Z_H, L_H, M_H, N_H.

The coefficients X_u, $X_{\dot{u}}$ etc., are defined as the stability derivatives. There are 72 stability derivatives. Reference 4 Vol/5 shows by reason of symmetry that X_H, Z_H, M_H are functions only of u, w, q, $\dot{u}$, $\dot{w}$, $\dot{q}$ and Y_H, L_H, N_H functions only of v, p, r, $\dot{v}$, $\dot{p}$, $\dot{r}$. Thus the stability derivatives

$$(X, Z, M)\; v, p, r, \dot{v}, \dot{p}, \dot{r} = 0$$

$$(Y, L, N)\; u, w, q, \dot{u}, \dot{w}, \dot{q} = 0$$

The remaining stability derivatives will be considered in the equations of motion.

Components of Weight and Buoyancy along the x y z Axes

Define $\bar{G} = \bar{W} - \bar{B}$

G is first resolved along the X_o and Z_o axes (B.2)

Component of G along $X_o = - G \sin \theta_o$ (B.3)

Component of G along $Z_o = G \cos \theta_o$ (B.4)

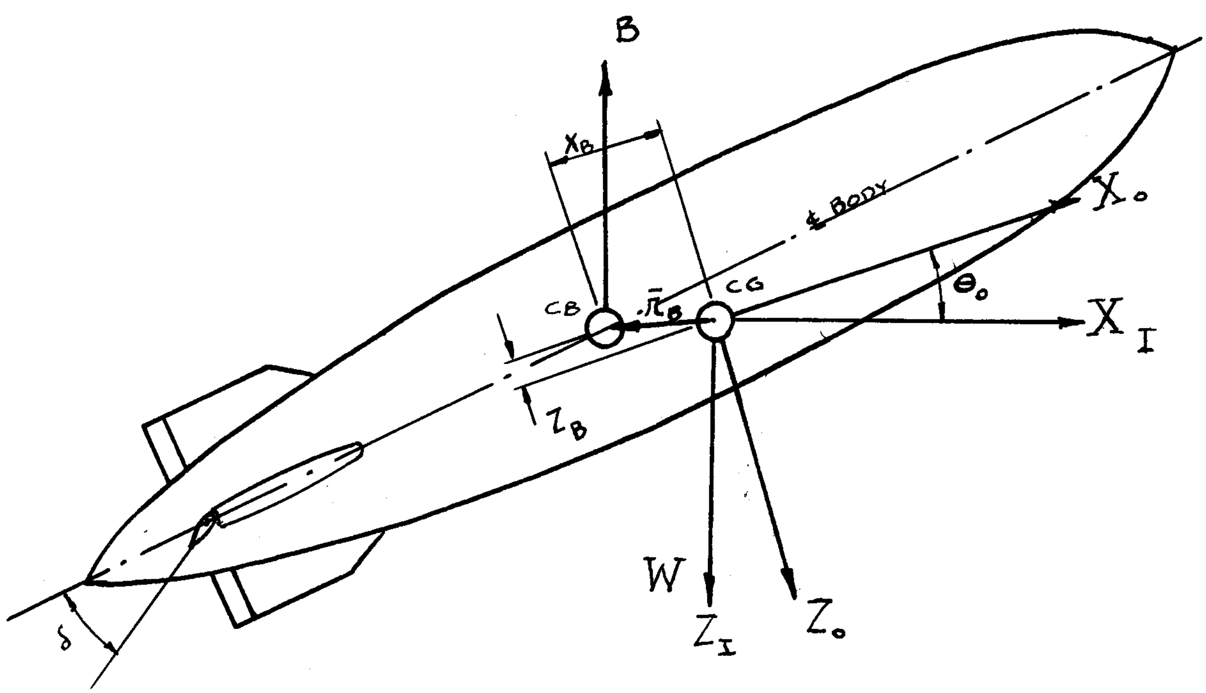
Using the direction cosine table B.2 for infinitesimal rotations, the components of the gravity force along the x y z axes become

G_x component of G along x axis $= - G \sin \theta_o - G \theta \cos \theta_o$ (B.5)

G_y component of G along y axis $= G \psi \sin \theta_o + G \phi \cos \theta_o$ (B.6)

G_z compoent of G along z axis $= - G\theta \sin \theta_o + G \cos \theta_o$ (B.7)

Buoyancy Moment



Referring to Fig. B.1

Let $\bar{r}_B$ = Vector from center of gravity (C.G.) to center of buoyancy (C.B.)

$$\bar{B} = \text{Buoyancy force vector} = -B\bar{e}_{Z_I} \tag{B.8}$$

$$T_B = \text{Buoyancy torque vector} \quad \bar{r}_B \times \bar{B} \tag{B.9}$$

$$T_B = \bar{i}\, L_B + \bar{j}\, M_B + \bar{k}\, N_B \tag{B.10}$$

Components of $\bar{B}$ along X_o, Z_o

along OX_o	$B \sin \theta_o$
along OY_o	0
along OZ_o	$-B \cos \theta_o$

$$\bar{r}_B = \bar{i}\, X_B + \bar{k}\, Z_B$$

<u>Linearized Equations of Motion for Small Disturbance from Steady Symmetric Flight</u>

The external forces acting on the body are

$$X = X_H + G_x + X_T \tag{B.11}$$

$$Y = Y_H + G_y + Y_T \tag{B.12}$$

$$Z = Z_H + G_z + Z_T \tag{B.13}$$

X_T, Y_T, Z_T are the components of the thrust along the x, y, z axes.

The external moments acting on the body are

$$L = L_H + L_B + L_T \tag{B.14}$$

$$M = M_H + M_B + M_T \tag{B.15}$$

$$N = N_H + N_B + N_T \tag{B.16}$$

where L_T, N_T, M_T are the components of the moment along the x, y, z axes respectively due to thrust.

Using relations B.5, B.6, and B.7 in equations A.8, A.9, A.10, relations B.14, B.15, B.16 in equations A.5, A.6, A.7, the following velocity components

$$U = U_o + u \qquad P = p$$

$$V = v \qquad Q = q$$

$$W = W_o + w \qquad R = r$$

and noting that due to symmetry $I_{x_y} = I_{y_z} = 0$.
The equations of motion become

$$\begin{aligned} &X_o + X_u u + X_w w + X_q q + X_{\dot{u}}\dot{u} + X_{\dot{w}}\dot{w} + X_{\dot{q}}\dot{q} + X_T \\ &-G \sin \theta_o - G\theta \cos \theta_o = m\dot{u} + mq\, W_o \end{aligned} \tag{B.17}$$

$$\begin{aligned} &Z_o + Z_u u + Z_w w + Z_q q + Z_{\dot{u}}\dot{u} + Z_{\dot{w}}\dot{w} + Z_{\dot{q}}\dot{q} + Z_T \\ &+ G \cos \theta_o - G\theta \sin \theta_o = m\dot{w} - mU_o q \end{aligned} \tag{B.18}$$

$$M_o + M_u u + M_w w + M_q q + M_{\dot{u}} \dot{u} + M_{\dot{w}} \dot{w} + M_{\dot{q}} \dot{q} + M_T$$

$$+ Bz_B \sin \Theta_o + Bz_B \Theta \cos \Theta_o - Bx_B \Theta \sin \Theta_o + Bx_B \cos \Theta_o = I_{yy} \dot{q} \quad (B.19)$$

$$Y_o + Y_v v + Y_p p + Y_r r + Y_{\dot{v}} \dot{v} + Y_{\dot{p}} \dot{p} + Y_{\dot{r}} \dot{r} + Y_T$$

$$+ G\phi \cos \Theta_o - G \psi \sin \Theta_o = m\dot{v} - mW_o p \quad (B.20)$$

$$L_o + L_v v + L_p p + L_r r + L_{\dot{v}} \dot{v} + L_{\dot{p}} \dot{p} + L_{\dot{r}} \dot{r} + L_T$$

$$+ (Bz_B) \quad (\psi \sin \Theta_o + \phi \cos \Theta_o) = I_{xx} \dot{p} + I_{xz} \dot{r} \quad (B.21)$$

$$N_o + N_v v + N_p p + N_r r + N_{\dot{v}} \dot{v} + N_{\dot{p}} \dot{p} + N_{\dot{r}} \dot{r} + N_T$$

$$- (Bx_B) \quad (\psi \sin \Theta + \phi \cos \Theta) = I_{zz} \dot{r} + I_{xz} \dot{p} \quad (B.22)$$

The steady state equations of motion are

$$X_o + X_T - G \sin \Theta_o = 0 \quad (B.23)$$

$$Z_o + Z_T + G \cos \Theta_o = 0 \quad (B.24)$$

$$M_o + M_T + Bz_B \sin \Theta_o + Bx_B \cos \Theta_o = 0 \quad (B.25)$$

$$Y_o + Y_T = 0 \quad (B.26)$$

$$L_o + L_T = 0 \qquad \text{(B.27)}$$

$$N_o + N_T = 0 \qquad \text{(B.28)}$$

Using the steady state relations in the equations of motion and separating the equations into a longitudinal set in the variables u, w, q, $\dot u$, $\dot w$, $\dot q$, θ and a lateral set in v, p, r, $\dot v$, $\dot p$, $\dot r$, ψ , ϕ the equations become

<u>Longitudinal Equations</u>

$$X_u u + X_w w + X_q q + X_{\dot u}\dot u + X_{\dot w}\dot w + X_{\dot q}\dot q - G\theta \cos \theta_o = m \qquad \text{(B.29)}$$

$$Z_u u + Z_w w + Z_q q + Z_{\dot u}\dot u + Z_{\dot w}\dot w + Z_{\dot q}\dot q - G\theta \sin \theta_o = m\dot w - mU_o q \qquad \text{(B.30)}$$

$$M_u u + M_w w + M_q q + M_{\dot u}\dot u + M_{\dot w}\dot w + M_{\dot q}\dot q + Bz_B\, \theta \cos \theta_o - Bx_B\, \theta \sin \theta_o = I_{yy}\, \dot q \qquad \text{(B.31)}$$

<u>Lateral Equations</u>

$$Y_v v + Y_p p + Y_r r + Y_{\dot v}\dot v + Y_{\dot p}\dot p + Y_{\dot r}\dot r + G\phi \cos \theta_o - G\, \psi \sin \theta_o = m\dot v - mW_o p \qquad \text{(B.32)}$$

$$L_v v + L_p p + L_r r + L_{\dot v}\dot v + L_{\dot p}\dot p + L_{\dot r}\dot r + (Bz_B)\ (\psi \sin \theta_o + \phi \cos \theta_o) = I_{xx}\dot p + I_{xz} r \qquad \text{(B.33)}$$

$$N_v v + N_p p + N_r r + N_{\dot v}\dot v + N_{\dot p}\dot p + N_{\dot r}\dot r - (Bx_B)\ (\psi \sin \theta_o + \phi \cos \theta_o) = I_{zz}\dot r + I_{xz}\dot p \qquad \text{(B.34)}$$