SOME NONLINEAR VIBRATION AND RESPONSE PROBLEMS OF CYLINDRICAL PANELS AND SHELLS

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ABSTRACT

Large amplitude vibrations and forced responses of curved panels and shells are studied by the application of the shallow shell equation. The Galerkin procedure is used to reduce the nonlinear partial differential equations to ordinary nonlinear equations. Marked differences are found between the behavior of curved panels and cylindrical shells. Relations for the dependence of the free vibration period on amplitude are given. A two mode analysis of the cylindrical shell problem is included.

The curved panel is found to exhibit a buckling phenomenon for the simple "breathing modes". Shock response methods are used to predict dynamic buckling of the curved panel and the predictions are verified by numerical integration.

The cylindrical shell vibrations and responses are found to be governed by Duffing's equation and certain of the well-known properties of Duffing's equation are applied to the cylindrical shell dynamics.

The two mode analysis of the cylindrical shell is shown to exhibit weak coupling, allowing the separate excitation of the coupled modes.

Some numerical results are given.

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LIST OF SYMBOLS

A, $A(t)$, A_o, A_1, A_ϕ, A_ψ	Amplitude of vibration
$B(t)$	Amplitude of stress function; or amplitude of second mode of vibration
D	Bending stiffness
E	Young's Modulus
$E(\psi, \psi_\tau)$	Energy function for free vibration
F, F_1	Stress functions
H_i	Coefficients of two mode analysis
K, K_{sep}	Energy constants
L	Shell or panel length
M_x, M_y, M_{xy}	Bending Moment resultants
N_x, N_y, N_{xy}, N_{xi}, N_{yi}	Stress resultants
P_o, $P(x, y, t)$, $P_{o\ crt}$, $\mathbb{P}(t)$, $\mathbb{P}_{o\ crt}$	Pressure functions
$Q(\tau)$, Q_o, $Q'(\tau)$, Q_ϕ, Q_ψ	Nondimensional forcing functions

R	Magnitude of a step function in Q; or a ratio of frequencies
$S(t)$, $S(m,n)$, $S(\mu,\nu)$, S_{TOT}	Shell of panel displacement from equilibrium position
T_L, T_{NL}, T_{LM}	Free vibration periods
X_ϕ, X_ψ	Two mode total phase parameters
AR	Aspect Ratio
R_Δ	Shock response ratio
a	Radius of shell or panel
a_H	A parameter of Hill's equation
a_o	A stability determinant
$f(\psi)$	Nonlinearity function
f_i	Stress function coefficients for the two mode analysis
g_ϕ, g_ψ, h_ϕ, h_ψ	Parameter averages for two mode analysis
h	Shell or panel thickness
$\bar{m}$	Mass per unit area
m	Axial mode number
n	Circumferential mode number
q_o, q_{mn}, q_μ	Forcing function constants
q_I, q_{II}	Parameters of Hill's equation
r	R/Ω_L^2
t	Time

u, v, w, w_o	Shell displacements
x, y, z	Cartesian coordinates
Δ	Shock response amplitude parameter
$\Delta_{\phi}, \Delta_{\phi}$	Two mode forcing function amplitudes
Λ, Π, Υ, Φ	Parameters of the two mode analysis
Ω, Ω_L, Ω_*, Ω_{LM}	Nondimensional frequencies
α	Delta function amplitude coefficient
β	Nondimensional frequency
$\beta_{\phi}, \beta_{\psi}$, $\gamma_{\phi}, \gamma_{\psi}$	Two mode analysis parameters
$\bar{\delta}, \bar{\epsilon}$	Mathiev's equation coefficient
ϵ	Nonlinearity parameter
ϵ_x, ϵ_y	Strain components
ξ_I, ξ_{II}, ξ_{ϕ}, ξ_{ψ}	Two mode analysis parameters of phase relations and coupling
ξ	Nondimensional velocity in the numerical integration
η_1, η_2	Two mode analysis parameters
θ_i	Iteration amplitudes
$\varkappa$	Nonlinearity parameter
λ	Nondimensional frequency

μ, ν	Mode numbers for the two mode analysis
ν	Poisson's ratio
ξ_ϕ, ξ_ψ	Two mode phase relation parameters
σ	Nonlinearity parameter for a cylinder
τ, τ_M	Nondimensional time
ϕ_i	Iteration amplitude coefficient
$\psi_0, \psi_1, \psi_2, \psi_3,$	Amplitude coefficients
$\omega, \omega_L, \omega_*$	Circular frequencies

CHAPTER I

INTRODUCTION

The study of cylindrical shell vibrations dates from 1894, the second edition of Rayleigh's famous Theory of Sound (1), where certain displacement modes are assumed in order to compute the associated frequencies of vibration by the application of the Lagrange equations. This approach has developed into the widely used Rayleigh-Ritz method of more recent literature. A review of the developments from the time of Rayleigh until 1957 is given in reference 2. To that time Reissner's paper (3) was the principal shallow shell study of the nonlinear vibrations of cylindrical shells (panels). In 1958 A. S. Vol'mir (4) used the shallow shell equations to study the stability of cylindrical shells (panels) with rapidly applied axial loading, and in 1959 V. L. Agamirov and A. S. Vol'mir (5) again used the shallow shell equations to study both axial and hydrostatic loads which had been applied dynamically. In 1961 Chu's paper (6) appeared with a discussion of the influence of large amplitudes on cylindrical shell vibrations, again utilizing the shallow shell equations.

Even with these several papers using the shallow shell approach to study the large amplitude vibration or response problems of cylindrical shells and panels, there were still a number of questions that remained to be answered by the application of the shallow shell equations. It is the purpose of this thesis to go a little further toward the answering of these questions. The shallow shell equations are used to study both vibration and response problems first for a curved panel, then for a cylindrical shell. A two mode analysis of the cylindrical shell is presented in Chapter VI. In this connection it should be noted that the two mode problem for the cylindrical shell is simpler than the two mode problem for a curved plate.

The final chapter is devoted to numerical results. These are not intended to be comprehensive; rather, representative results are given for several questions of interest.

CHAPTER II

THE CURVED PANEL PROBLEM

2.1 The Equations of Motion

The present analysis will start from the von Karman large deflection plate equation extended to include an initial curvature. The coordinate system and nomenclature are represented in figure 1. A panel (plate) of length L in the x-direction, width $(\pi a/n)$ in the y-direction and thickness h in the z-direction, initially lies in the xy-plane. The xy-coordinate system is located with respect to the panel such that the region of the xy-plane covered by the panel is $-\frac{\pi a}{2n} \le y \le \frac{\pi a}{2n}$, $0 \le x \le L$. The plate is given a small positive vertical displacement (positive displacement is in the upward or z-direction) to form a shallow cylindrical surface

$$w_0(x,y) = \text{CONSTANT} - (y^2/2a). \tag{2.1}$$

The von Karman plate equations are

$$\nabla^4 F_1 = Eh\left[w_{xy}^2 - w_{xx}\, w_{yy}\right] - Eh\left[w_{0xy}^2 - w_{0xx}\, w_{0yy}\right],$$

$$D\nabla^4(w - w_0) = q_1 + F_{1xx}\, w_{yy} + F_{1yy}\, w_{xx} - 2F_{1xy}\, w_{xy}\ , \tag{2.2}$$

where $$D = \frac{Eh^3}{12(1-\nu^2)}.$$

Here the operator ∇^4 is defined by

$$\nabla^4 = \frac{\partial^4}{\partial X^4} + 2\frac{\partial^4}{\partial X^2 \partial Y^2} + \frac{\partial^4}{\partial Y^4}, \tag{2.3}$$

while the stress function F is related to the stress resultants N_X, N_Y and N_{XY} by the relations

$$F_{1_{XX}} = N_Y \; ; \; F_{1_{YY}} = N_X \; ; \; F_{1_{XY}} = -N_{XY} . \tag{2.4}$$

When a cylindrical surface under uniform axial or circumferential stress is to be studied it is convenient to separate the constant initial stress resultants N_{Xi} and N_{Yi} from N_X and N_Y by defining a new stress function

$$F = F_1 - \frac{1}{2}\left(Y^2 N_{Xi} + X^2 N_{Yi}\right) \tag{2.5}$$

so that

$$F_{XX} = N_Y - N_{Yi} \; ; \; F_{YY} = N_X - N_{Xi} \; ; \; F_{XY} = -N_{XY} . \tag{2.6}$$

When equations 2.1 through 2.6 are used together with the simplified notation

$$S = (w - w_0) \tag{2.7}$$

the von Karman plate equations governing an initially curved panel become

$$\begin{aligned} \frac{1}{Eh}\nabla^4 F &= \frac{1}{a} S_{XX} + \left[S_{XY}^2 - S_{XX} S_{YY}\right] \\ D\nabla^4 S &= \left(q_1 - \frac{1}{a} N_{Yi}\right) + N_{Xi} S_{XX} + N_{Yi} S_{YY} - \frac{1}{a} F_{XX} \\ &\quad + \left[F_{XX} S_{YY} + F_{YY} S_{XX} - 2 F_{XY} S_{XY}\right] . \end{aligned} \tag{2.8}$$

Some dynamic problems of very thin walled cylindrical panels will be considered. It is assumed in these problems that of the deformations (w in the z-direction, u in the x-direction, v in the y-direction), the flexural deformation, w, predominates. In reference 3 it has been shown that for the lower (flexural) modes of free vibration of thin cylindrical shells, when n is not too small, say $n > 3$, then the maximum inertial forces due to accelerations tangent to the surface of the shell (i.e., $\ddot{u}\bar{m}$ and $\ddot{v}\bar{m}$) may be neglected in comparison to the maximum transverse inertial force ($\ddot{w}\bar{m}$). This ability to neglect the $\ddot{u}\bar{m}$ and $\ddot{v}\bar{m}$ simplifies the analytical problem tremendously. Hence we shall assume explicitly that the only inertial force which is to be considered is that due to the transverse acceleration $\ddot{w}\bar{m}$. Under this assumption the load normal to surface is considered to consist of

$$q_1 = P_o + P(x,y;t) - \ddot{w}\bar{m} \tag{2.9}$$

where P_o is a uniform constant pressure and $P(x,y;t)$ is a space and time dependent pressure. When equation 2.9 is introduced into equation 2.8, the result is

$$\begin{aligned} D\nabla^4 S &= (P_o - \tfrac{1}{a} N_{yi}) + P(x,y;t) - \ddot{w}\bar{m} \\ &\quad + N_{xi} S_{xx} + N_{yi} S_{yy} - \tfrac{1}{a} F_{xx} \\ &\quad + \left[F_{xx} S_{yy} + F_{yy} S_{xx} - 2F_{xy} S_{xy}\right] , \end{aligned} \tag{2.10}$$

$$\frac{1}{Eh}\nabla^4 F = \frac{1}{a} S_{xx} + \left[S_{xy}^2 - S_{xx} S_{yy}\right] .$$

If the curved panels studied here were a part of a pressurized cylinder without other initial stresses, the relation $N_{yi} = P_o a$ would hold and the term $(P_o - \frac{1}{a} N_{yi})$ would be zero. Otherwise, the equations of motion are equations 2.10.

2.2 Boundary Conditions

Rectangular panels subjected to the so-called "freely supported" boundary conditions will be considered. These conditions are

$$\left.\begin{aligned} S &= S_{xx} = 0 \\ F &= F_{xx} = 0 \end{aligned}\right\} \quad \text{on } x = 0, L \,,$$

$$\left.\begin{aligned} S &= S_{yy} = 0 \\ F &= F_{yy} = 0 \end{aligned}\right\} \quad \text{on } y = \pm \frac{\pi a}{2n} \,. \tag{2.11}$$

The condition $F_{xx} = 0$ on a boundary x = constant requires that N_y be zero on that boundary. The additional condition F = 0 on the boundary x = constant requires that $F_{yy} = 0$ or $N_x = 0$ on that boundary. Since the stress-strain relations may be written

$$\begin{aligned} E h \epsilon_x &= N_x - \nu N_y \,, \\ E h \epsilon_y &= N_y - \nu N_x \,, \\ G h \epsilon_{xy} &= N_{xy} \,, \end{aligned} \tag{2.12}$$

the freely supported boundary conditions require that the normal stress resultants N_x and N_y and the normal strains ϵ_x and ϵ_y vanish on the boundaries while no requirements are imposed on the shear stresses and the shear strains by the boundary conditions.

In a similar manner the boundary condition S = 0 on x = constant requires that $S_{yy} = 0$ on x = constant. The relation between the moment resultants and the displacement, S, are

$$
\begin{aligned}
M_x &= -D\left[S_{xx} + \nu S_{yy}\right], \\
M_y &= -D\left[S_{yy} + \nu S_{xx}\right], \qquad (2.13) \\
M_{xy} &= -(1-\nu) D S_{xy} .
\end{aligned}
$$

Thus the boundary conditions on S provide an additional condition that M_x and M_y be zero on the boundaries but M_{xy} is not prescribed on the boundaries.

2.3 Mode Shapes

Appropriate modes for the displacement function S and the stress function F must be selected in order to apply the Galerkin method. These modes must satisfy the prescribed boundary conditions, equations 2.11. A set of modes which satisfy these boundary conditions (as well as being a solution to the linearized equation obtained when the nonlinear terms are dropped from equations 2.10) are

$$S = (w - w_0) = A(t)\cos\left(\frac{ny}{a}\right)\sin\left(\frac{\pi x}{L}\right),$$

$$F = B(t)\cos\left(\frac{ny}{a}\right)\sin\left(\frac{\pi x}{L}\right) . \qquad (2.14)$$

The choice of these modes carries with it the implicit assumption that any nonlinearity in the problem will influence only the nature of the time dependent amplitudes A(t) and B(t) and will in no way affect the space distributions of stresses and displacements. As an approximation, this assumption can sometimes be justified on an experimental basis.

2.4 Application of the Galerkin Method

The Galerkin procedure requires that the modes (equation 2.14) be substituted for S and F in the equations of motion, 2.10; that each of these equations be weighted with the appropriate modal function (in this case $\cos(\frac{ny}{a})\sin(\frac{\pi x}{L})$) and that the resultant equations be integrated over the domain $0 \le x \le L, -\frac{\pi a}{2n} \le y \le \frac{\pi a}{2n}$. The expressions arising from this procedure are

$$\int_0^L \int_{-\frac{\pi a}{2n}}^{+\frac{\pi a}{2n}} \left\{ \frac{B}{Eh}\left[\left(\frac{\pi}{L}\right)^2+\left(\frac{n}{a}\right)^2\right]^2 \cos\left(\frac{ny}{a}\right)\sin\left(\frac{\pi x}{L}\right) \right.$$

$$+ \frac{1}{a}\left(\frac{\pi}{L}\right)^2 A\cos\left(\frac{ny}{a}\right)\sin\left(\frac{\pi x}{L}\right) \tag{2.15}$$

$$- A^2\left(\frac{n}{a}\right)^2\left(\frac{\pi}{L}\right)^2\sin^2\left(\frac{ny}{a}\right)\cos^2\left(\frac{\pi x}{L}\right)$$

$$\left. + A^2\left(\frac{n}{a}\right)^2\left(\frac{\pi}{L}\right)^2\cos^2\left(\frac{ny}{a}\right)\sin^2\left(\frac{\pi x}{L}\right) \right\} \cos\left(\frac{ny}{a}\right)\sin\left(\frac{\pi x}{L}\right)dx\,dy$$

$$= 0,$$

$$\int_{0}^{L}\int_{-\frac{a\pi}{2n}}^{+\frac{a\pi}{2n}}\Big\{\Big[D\Big[\left(\tfrac{\pi}{L}\right)^2+\left(\tfrac{n}{a}\right)^2\Big]^2 + \bar{m}\,\ddot{A}\Big]\cos\left(\tfrac{nY}{a}\right)\sin\left(\tfrac{\pi X}{L}\right)$$

$$+\Big[A\big[N_{xi}\left(\tfrac{\pi}{L}\right)^2+N_{Yi}\left(\tfrac{n}{a}\right)^2\big]-\tfrac{1}{a}\left(\tfrac{\pi}{L}\right)^2 B\Big]\cos\left(\tfrac{nY}{a}\right)\sin\left(\tfrac{\pi X}{L}\right) \tag{2.16}$$

$$-\Big[2AB\left(\tfrac{n}{a}\right)^2\left(\tfrac{\pi}{L}\right)^2\Big]\Big[\cos^2\left(\tfrac{nY}{a}\right)\sin^2\left(\tfrac{\pi X}{L}\right)-\sin^2\left(\tfrac{nY}{a}\right)\cos^2\left(\tfrac{\pi X}{L}\right)\Big]\cdot$$

$$\cdot\cos\left(\tfrac{nY}{a}\right)\sin\left(\tfrac{\pi X}{L}\right)-\Big[P_o-\tfrac{1}{a}N_{Yi}+P_{(X,Y;t)}\Big]\cos\left(\tfrac{nY}{a}\right)\sin\left(\tfrac{\pi X}{L}\right)\Big\}dx\,dY$$

$$= 0.$$

Evaluation of the integrals of equations 2.15 and 2.16 gives

$$\int_{0}^{L}\int_{-\frac{a\pi}{2n}}^{\frac{a\pi}{2n}}\cos^2\left(\tfrac{nY}{a}\right)\sin^2\left(\tfrac{\pi X}{L}\right)dx\,dY = \left(\tfrac{La\pi}{4n}\right),$$

$$\int_{0}^{L}\int_{-\frac{a\pi}{2n}}^{\frac{a\pi}{2n}}\cos^3\left(\tfrac{nY}{a}\right)\sin^3\left(\tfrac{\pi X}{L}\right)dx\,dY = \left(\tfrac{La}{\pi n}\right)\left(\tfrac{4}{3}\right)^2,$$

$$\int_{0}^{L}\int_{-\frac{a\pi}{2n}}^{\frac{a\pi}{2n}}\sin^2\left(\tfrac{nY}{a}\right)\cos\left(\tfrac{nY}{a}\right)\cos^2\left(\tfrac{\pi X}{L}\right)\sin\left(\tfrac{\pi X}{L}\right)dx\,dY = \left(\tfrac{La}{\pi n}\right)\left(\tfrac{2}{3}\right)^2, \tag{2.17}$$

$$\int_{0}^{L}\int_{-\frac{a\pi}{2n}}^{\frac{a\pi}{2n}}P(X,Y;t)\cos\left(\tfrac{nY}{a}\right)\sin\left(\tfrac{\pi X}{L}\right)dx\,dY = \mathbb{P}(t),$$

$$\int_{0}^{L}\int_{-\frac{a\pi}{2n}}^{\frac{a\pi}{2n}}\left(P_o-\tfrac{1}{a}N_{Yi}\right)\cos\left(\tfrac{nY}{a}\right)\sin\left(\tfrac{\pi X}{L}\right)dx\,dY = -4\left(\tfrac{La}{n\pi}\right)\left(P_o-\tfrac{1}{a}N_{Yi}\right).$$

Then 2.15 becomes

$$B = -Eh\left[\left(\frac{\pi}{L}\right)^2+\left(\frac{n}{a}\right)^2\right]^{-2}\left[\frac{1}{3}A^2\left(\frac{4n}{aL}\right)^2+\frac{A}{a}\left(\frac{\pi}{L}\right)^2\right] \tag{2.18}$$

and equation 2.16 becomes

$$\begin{aligned} \bar{m}\ddot{A} + A\left\{D\left[\left(\frac{\pi}{L}\right)^2+\left(\frac{n}{a}\right)^2\right]^2 + N_{xi}\left(\frac{\pi}{L}\right)^2 + N_{yi}\left(\frac{n}{a}\right)^2\right\} \\ -\frac{1}{a}\left(\frac{\pi}{L}\right)^2 B^2 - \frac{32}{3\pi}\left(\frac{n}{a}\right)^2\left(\frac{\pi}{L}\right)^2 AB \\ = \left(\frac{a}{n}\right)^{-1}\left(\frac{L\pi}{4}\right)^{-1}\mathbb{P}(t) - 16\left(P_o - \frac{1}{a}N_{yi}\right). \end{aligned} \tag{2.19}$$

If equation 2.18 is used to eliminate B from equation 2.19 and if the following notations are used*

$$\begin{aligned} \bar{m}\omega_L^2 &= D\left[\left(\frac{\pi}{L}\right)^2+\left(\frac{n}{a}\right)^2\right]^2 + N_{xi}\left(\frac{\pi}{L}\right)^2 + N_{yi}\left(\frac{n}{a}\right)^2 + \bar{m}\omega_*^2, \\ \bar{m}\omega_*^2 &= \frac{Eh}{a^2}\left(\frac{\pi}{L}\right)^4\left[\left(\frac{\pi}{L}\right)^2+\left(\frac{n}{a}\right)^2\right]^{-2}, \\ \psi &= \frac{16n^2}{\pi^2}\left(\frac{A}{a}\right), \quad Q(\tau) = \left[\pi^3 a^2 L\bar{m}\omega^2\right]^{-1}(4n)^3\,\mathbb{P}(\tau), \\ \omega t &= \tau, \\ \Omega_L^2 &= \omega_L^2/\omega^2, \\ \Omega_*^2 &= \omega_*^2/\omega^2, \\ \epsilon &= \Omega_*^2/\Omega_L^2, \end{aligned} \tag{2.20}$$

* See section 2.5 where Reissner's work is discussed.

the equation

$$\psi_{\tau\tau} + \Omega_L^2 \left[\psi + \epsilon \left(\psi^2 + \frac{2}{9} \psi^3 \right) \right] = Q(\tau) - \frac{(4n)^2 (P_o - \frac{1}{a} N_{yi})}{\pi^2 a \bar{m} \omega^2}, \quad (2.21)$$

is obtained for the nondimensional amplitude coefficient ψ.

2.5 Reissner's Development

In reference 3 Reissner developed, by an entirely different method, an equation identical with equation 2.21 except for the last term. Reissner considers an initially pressurized panel which is represented by the shallow shell equations, 2.8. Hence the condition $N_{yi} = a P_o$ is assumed so that the last term in equation 2.21 vanishes. This agreement between the results obtained by Galerkin's method and Reissner's method provides some confidence in equation 2.21.

Reissner, applying the Lindstedt-Duffing perturbation technique, obtains the following expression relating the free vibration frequency and the amplitude

$$\frac{T_L^2}{T^2} = \frac{\omega^2}{\omega_L^2} = 1 + \frac{1}{6} \epsilon \psi_o^2 (1 - 5\epsilon) \quad (2.22)$$

where ψ_o is the amplitude at $\tau = 0$.

He observes

> ....the remarkable fact that the shell does not spend equal time intervals deflected outwards and deflected inwards. Rather, more than half of the cycle is spent during the inward deflection.

Reissner also shows that for a given maximum outward deflection amplitude there is an associated larger inward deflection amplitude.

2.6 A Curved Panel vs. a Complete Cylindrical Shell

In Chapter III the analysis of a complete cylindrical shell is discussed in detail, thus it is sufficient to make only a brief comment here. In order to extend the application of the shallow shell equations to the complete cylinder, it is necessary to change the limits of the Galerkin integration to include at least one full wave, i.e., $-L \leq x \leq L$; $-\frac{\pi d}{n} \leq Y \leq \frac{\pi d}{n}$. Consequently the results considered in this chapter are not directly applicable to the complete cylindrical shell. It will turn out that for the complete shell, the quadratic nonlinearity vanishes and with it goes many of the interesting features discussed in this chapter.

2.7 Singular Points

For a single panel executing free vibrations in a vacuum, equation 2.21 becomes

$$\psi_{\tau\tau} + \Omega_L^2 \left[\psi + \epsilon\left(\psi^2 + \tfrac{2}{9}\psi^3\right)\right] = 0. \tag{2.23}$$

This equation may be rewritten as

$$\varphi_\psi = -f(\psi)/\varphi, \tag{2.24}$$

$$f(\psi) = \Omega_L^2 \psi\left[1 + \epsilon\left(\psi + \tfrac{2}{9}\psi^2\right)\right]$$

where

$$\varphi = \psi_\tau .$$

Then equation (2.23) is known to have singularities when $f(\psi) = 0$ and $\varphi = 0$, simultaneously. These singularities are located on the ψ axis in the phase plane at the points $(\psi_1, 0), (\psi_2, 0)$, and

$(\psi_3, 0)$, where

$$\psi_1 = 0 ,$$

$$\psi_2 = -\frac{9}{4}\left[1 - \left(1 - \frac{8}{9}\cdot\frac{1}{\epsilon}\right)^{1/2}\right], \tag{2.25}$$

$$\psi_3 = -\frac{9}{4}\left[1 + \left(1 - \frac{8}{9}\cdot\frac{1}{\epsilon}\right)^{1/2}\right].$$

If ψ_2 and ψ_3 are complex, then ψ_1 is a stable center and the only singularity in the plane. If ψ_2 and ψ_3 are real, i.e., if

$$\left(1 - \frac{8}{9}\cdot\frac{1}{\epsilon}\right) \geqslant 0, \tag{2.26}$$

then ψ_1 and ψ_3 are stable centers and ψ_2 is a saddle point (figure 2). The physical interpretation of the instances in which equation 2.26 holds is that stable vibrations may exist about:

1) the undeflected equilibrium point ψ_1 with a limit on the maximum amplitude, $|\psi(\tau)| - |\psi_1|$, at which such a vibration may occur;

2) the "buckled" equilibrium point ψ_3 with a limit on the maximum amplitude, $|\psi(\tau)| - |\psi_3|$, at which such a vibration may occur;

3) the saddle point ψ_2, and encompassing the two equilibrium points ψ_1 and ψ_3 and with a limit on the minimum amplitude $|\psi(\tau) - \psi_2|$ at which such a vibration may occur.

For equation 2.26 to hold, ϵ must satisfy the condition

$$\epsilon \geqslant \frac{8}{9} , \tag{2.27}$$

or if ϵ is written in full

$$\frac{\left(\frac{Eh}{a^2}\right)\left(\frac{\pi}{L}\right)^4\left[\left(\frac{\pi}{L}\right)^2+\left(\frac{n}{a}\right)^2\right]^{-2}}{D\left[\left(\frac{\pi}{L}\right)^2+\left(\frac{n}{a}\right)^2\right]^2 + P_o a\left[\frac{1}{2}\left(\frac{\pi}{L}\right)^2+\left(\frac{n}{a}\right)^2\right]+\left(\frac{Eh}{a^2}\right)\left(\frac{\pi}{L}\right)^4\left[\left(\frac{\pi}{L}\right)^2+\left(\frac{n}{a}\right)^2\right]^{-2}} \geq \frac{8}{9} \ . \tag{2.28}$$

It is clear that for a sufficiently thin shell, the bending stiffness term $D\left[\left(\frac{\pi}{L}\right)^2+\left(\frac{n}{a}\right)^2\right]^2$ will have a small contribution compared to the membrane term $\left(\frac{Eh}{a^2}\right)\left(\frac{\pi}{L}\right)^4\left[\left(\frac{\pi}{L}\right)^2+\left(\frac{n}{a}\right)^2\right]^{-2}$, unless $\left(\frac{n}{a}\right) >> \left(\frac{\pi}{L}\right)$, and then the controlling influence will be that of the initial pressure P_o . And indeed, the initial pressure can be made so large that the inequality 2.28 is never true and then no possibility of a buckled state exists. In that event the vibration is about the single center $\psi = 0$ and is very close to the linear system in its behavior.

If, on the other hand, the pressure is negative (external pressure) it may approach the value

$$\frac{-\left\{\left(\frac{Eh}{a^2}\right)\left(\frac{\pi}{L}\right)^4\left[\left(\frac{\pi}{L}\right)^2+\left(\frac{n}{a}\right)^2\right]^{-2} + D\left[\left(\frac{\pi}{L}\right)^2+\left(\frac{n}{a}\right)^2\right]^2\right\}}{\left(\frac{1}{a}\right)\left[\frac{1}{2}\left(\frac{\pi}{L}\right)^2+\left(\frac{n}{a}\right)^2\right]} = P_{o_{CRT}} \tag{2.29}$$

which corresponds to buckling of the cylindrical panel under external hydrostatic pressure. From this ϵ may be expressed as

$$\epsilon = \frac{\omega_*^2}{(P_o - P_{o_{CRT}})\, a\left[\frac{1}{2}\left(\frac{\pi}{L}\right)^2+\left(\frac{n}{a}\right)^2\right]} , \tag{2.30}$$

where $-P_{o_{CRT}}$ is a positive number. A plot of ϵ vs P_o would generate a hyperbola with $\epsilon > 0$ for $P_o > P_{o_{CRT}}$, with $\epsilon < 0$ for $P_o < P_{o_{CRT}}$ and with ϵ unbounded for $P_o \equiv P_{o_{CRT}}$. If $\epsilon < 0$ then it follows that $\Omega_L^2 < 0$, i.e., Ω_L becomes an imaginary number. If Ω_L is imaginary, then periodic solutions do not exist, even for the linear system. The non-periodic problem corresponding to $\epsilon < 0$ is not examined here.

2.8 The Energy Integral and Period of Vibration.

Integration of equation 2.24

$$\int \psi_\tau \, d(\psi_\tau) = - \int f(\psi) d\psi , \tag{2.31}$$

yields the energy integral

$$(\psi_\tau)^2 + \Omega_L^2 \left[\psi^2 + \epsilon \left(\tfrac{2}{3} \psi^3 + \tfrac{1}{9} \psi^4 \right) \right] = 2K , \tag{2.32}$$

whence the nondimensional velocity is

$$\psi_\tau = \pm \left[2K - \Omega_L^2 \left\{ \psi^2 + \epsilon \left(\tfrac{2}{3} \psi^3 + \tfrac{1}{9} \psi^4 \right) \right\} \right]^{1/2} , \tag{2.33}$$

Now if the definition of τ is restricted to the special case,

$$\tau = \omega_L t , \tag{2.34}$$

and the frequency ω_L is replaced by its associated period

$$T_L = \frac{2\pi}{\omega_L} , \tag{2.35}$$

so that

$$\tau = \frac{2\pi t}{T_L} , \tag{2.36a}$$

and

$$\Omega_L^2 = 1 , \tag{2.36b}$$

then the result of separation of variables and integration is

$$\frac{t(\psi_1, \psi_2)}{T_L} = \frac{1}{2\pi} \int_{\psi_1}^{\psi_2} \frac{d\psi}{\{2K - [\psi^2 + \epsilon(\frac{2}{3}\psi^3 + \frac{1}{9}\psi^4)]\}^{1/2}} . \qquad (2.37)$$

Since the system described by equation 2.23 is conservative, the half period of the system is represented by equation 2.37 if ψ_1 and ψ_2 are the extremes of amplitude of the system represented by equation 2.23 for a particular value of K . And so the ratio of the period of one complete cycle of the non-linear oscillation to the period of the equivalent linear system is

$$\frac{T_{NL}}{T_L} = \frac{1}{\pi} \int_{\psi_1}^{\psi_2} \frac{d\psi}{\{2K - [\psi^2 + \epsilon(\frac{2}{3}\psi^3 + \frac{1}{9}\psi^4)]\}^{1/2}} . \qquad (2.38)$$

Equation 2.38 is an elliptic integral and its evaluation is discussed in Appendix I and the results are presented in Chapter V. Equation 2.32 can also be used to describe the phase plane ($\psi_\tau - \psi$-plane) trajectories and, with ψ_τ set identically zero, to describe the maximum displacements for a given energy level. In fact, if equation 2.32 is written in the form

$$E(\psi, \psi_\tau) = 2K - \psi_\tau^2 - \Omega_L^2[\psi^2 + \epsilon(\tfrac{2}{3}\psi^3 + \tfrac{1}{9}\psi^4)], \qquad (2.39)$$

where ψ_τ is to be taken as zero, then the curve $E(\psi, 0)$ is a vertical cross section of the energy surface. It also provides a visualization of how buckling depends on the energy level, on the initial pressure and on the other nonlinear contributions (see, for instance, figures 3, 4, 5 and 6).

2.9 Phase Plane Trajectories

The general character of the phase plane trajectories, determined from equation 2.32, is controlled by the two parameters ϵ and K which represent the degree of nonlinearity and the energy level, respectively. If equation 2.27 holds, there are two stable centers. The curves which enclose either of these centers are in turn enclosed by the separatrix*. If the motion is at an energy level less than that of the separatrix, then initial conditions will determine the center with which the motion is associated. If the energy level is greater than that for the separatrix, then the motion will encompass the separatrix and, hence, encompass both stable centers, independent of initial conditions.

The slope of these trajectories is

$$\left(\psi_\tau\right)_\psi = \frac{\pm\ \Omega_L^2\, \psi\left(1+\epsilon\left[\psi + \frac{2}{9}\psi^2\right]\right)}{\left\{2K - \Omega_L^2\, \psi^2\left[1 + \epsilon\left(\frac{2}{3}\psi + \frac{1}{9}\psi^2\right)\right]\right\}^{1/2}}, \qquad (2.40)$$

where the upper sign corresponds to the upper half plane. If the value of ϵ is less than 8/9, the slope of these trajectories is zero only at $\psi = 0$. But the maximum inward deflection, ψ in max, has a greater absolute value than does ψ out max. Now the time of transit from ψ in max to $\psi = 0$ and the time of transit from $\psi = 0$ to ψ out max are determined by

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* The separatrix is the integral curve passing through the saddle point.

$$t_{IN} = \int_{\psi_{IN\,MAX}}^{0} \frac{d\psi}{\psi_\tau}\,, \qquad \text{and} \qquad t_{OUT} = \int_{0}^{\psi_{OUT\,MAX}} \frac{d\psi}{\psi_\tau}\,, \tag{2.41}$$

in the upper half plane. For each value of ψ_τ in the upper half plane, there corresponds one value of ψ in each of the inward and outward deflection states. Since the distance $|\psi \text{ in max}|$ covered by the integral t_{IN} is greater than the distance $|\psi \text{ out max}|$ then $t_{IN} > t_{OUT}$. A similar argument may be made for other trajectories when $\epsilon > 8/9$. This result has also been verified by direct numerical integration of the equations of motion. The types of trajectories discussed are shown in figure 2.

CHAPTER III

THE RESPONSE OF A CURVED PANEL

3.1 The Response Problem

A few of the many methods of analyzing the response of a nonlinear system will now be applied. The differential equation analyzed is

$$\psi_{\tau\tau} + \Omega_L^2 \left[\psi + \epsilon \left(\psi^2 + \frac{2}{9} \psi^3 \right) \right] = Q(\tau) \quad . \tag{2.23a}$$

Since the discussion of the last section has dealt with the phase plane, a response problem which may be handled with the aid of phase plane trajectories will be considered first. If $Q(\tau)$ is a delta function and if the initial conditions associated with the problem are $\psi(0) = 0, \psi_\tau(0) = 0$, a new problem which is equivalent to this may be formulated. In the new problem $Q(\tau) = 0$ for all values of τ and the new initial conditions are $\psi(0) = 0$, $\psi_\tau(0) = \nu$. The relation between α, ν and a pressure pulse of magnitude P_0 is

$$\alpha = \nu = \left(\frac{4n}{\pi} \right)^3 \frac{\mathbb{P}_0}{a^2 \, \bar{m} \, \omega^2 L} \tag{3.1}$$

where $\mathbb{P}_0$ is obtained from the fourth of equations 2.17, by setting $P(x, y; t) = P_0$. Now the phase plane trajectories discussed in 2.9 can be used to describe the response of the system to a delta function of magnitude α by simply finding the point in the $\psi - \psi_\tau$ plane which corresponds to $\psi(0) = 0$, $\psi_\tau(0) = \alpha$. The response will simply follow the trajectory passing through that point. A special problem of considerable interest is to find the magnitude of P_o for which the trajectory will enclose the buckled equilibrium point ψ_3, $(\epsilon \geq \frac{8}{9})$. This value of P_o must correspond to the velocity intercept of the separatrix. Thus the critical value of P_o will be determined from setting

$$E(0, \alpha_{CRT}) = E(\psi_2, 0) . \tag{3.2}$$

The equivalent statement is

$$\left(\frac{\alpha_{CRT}}{\Omega_L}\right)^2 = \psi_2^2 + \epsilon\left(\frac{2}{3}\psi_2^3 + \frac{1}{9}\psi_2^4\right). \tag{3.3}$$

In terms of the physical parameters

$$\mathbb{P}_{o_{CRT}} = \alpha_{CRT}\left(\frac{\pi}{4n}\right)^3 a^2 \bar{m}\, \omega_L^2 , \tag{3.4}$$

or

$$P_{o_{CRT}}^{(s)} = \frac{\alpha_{CRT}\left(\frac{\pi}{4n}\right)^3 a^2 \bar{m}\, \omega_L^2}{\int_0^L \int_{-\frac{\pi a}{2n}}^{\frac{\pi a}{2n}} \cos\left(\frac{nY}{a}\right) \sin\left(\frac{\pi X}{L}\right) dX\, dY} , \tag{3.5}$$

from which

$$P_{o_{CRT}}^{(s)} \leq -\alpha_{CRT}\left(\frac{\pi}{4}\right)^4 \left(\frac{a\bar{m}\,\omega_L^2}{n^2}\right) . \tag{3.6}$$

The explanation of the sign in equation 3.6 emerges from a consideration of the effects of damping. In the absence of damping, the phase plane trajectories are symmetric with respect to velocity, thus an internally applied pressure pulse would buckle the system as easily as would an external pressure pulse. This is not physically reasonable. A more careful examination is in order. Recent experimental work by George Watts* at C.I.T. has indicated that the damping of the vibrations of thin circular cylindrical shells is small-equivalent to fifty or one-hundred cycles to damp to half amplitude. Damping of such small magnitude is satisfactorily treated as viscous damping. If a viscous damping term is introduced into equation 2.23, the phase plane trajectories are no longer like figure 2 but are now like figures 7 and 8. From figure 7 it is clear that for a trajectory to enter the buckled region, the equivalent initial velocity must be negative, that is, inward. Hence, the pressure pulse must be external. This requires the sign in equation 3.6 to be negative.

3.2 Response to an Impulsive Loading

It is worth noting that the phase plane trajectories provide a determination of the maximum amplitude response, of this shell system, to an applied impulsive pressure loading. The initial velocity, corresponding to a pressure pulse, identifies a given phase plane trajectory. The maximum amplitude associated with that trajectory is the maximum elastic deformation that the shell can experience.

An iteration scheme may be applied to the response problem for a delta function loading. A description of the largest nonlinear effect that may be treated with an equivalent linear system

* Thesis to be published.

is obtained. Consider the iteration scheme

$$\psi_{(n)} = \sum_{j=0}^{n} \phi_{(j)} \quad , \quad \phi_{(K)} << \phi_{(K-1)} \; . \tag{3.7}$$

Introduce equation 3.7 into equation 2.23 with the initial conditions $\psi(0) = 0$, $\psi_\tau(0) = \nu$. And further assume that $\phi(0) = \phi_{(0)}$ is a constant. Then $\phi_{(0)}$ is determined by

$$\phi_{(0)} + \epsilon\left(\phi_{(0)}^2 + \tfrac{2}{9}\phi_{(0)}^3\right) = 0, \tag{3.8}$$

where the roots are the same as those found in equation 2.25. When $\psi_1 = \phi_0 + \phi_1$ is introduced into equation 2.23, there follows

$$\phi_{(1)\tau\tau} + \Omega_L^2 \phi_{(1)}\left[1 + 2\epsilon\phi_{(0)}\left(1 + \tfrac{1}{3}\phi_{(0)}\right)\right] = 0. \tag{3.9}$$

The solution to equation 3.9 is

$$\phi_{(1)} = C_1 \sin\beta\tau + C_2 \cos\beta\tau \tag{3.10}$$

where

$$(\beta/\Omega_L)^2 = 1 + 2\epsilon\,\phi_{(0)}\left(1 + \tfrac{1}{3}\phi_{(0)}\right) . \tag{3.11}$$

After application of the initial conditions to $\psi_{(1)}$, the expression obtained is

$$\psi_{(1)}(\tau) = \phi_{(0)}\left[1 - \cos\beta\tau\right] + \frac{\nu}{\beta}\sin\beta\tau . \tag{3.12}$$

This solution can have meaning only if β is real. If β is not real, the solution is not periodic. If $\phi_{(0)}$ is taken to be zero, $\beta = \Omega_L$, and nothing has been achieved since a linear system has been matched to itself. If $\phi_{(0)}$ is taken to be the buckled equilibrium position (for $\epsilon > 8/9$),

$$\phi_{(0)} = -\frac{9}{4}\left[1 + \left(1 - \frac{8}{9}\frac{1}{\epsilon}\right)^{1/2}\right], \tag{3.13}$$

then the condition,

$$(\beta/\Omega_L)^2 = 1 + 2\epsilon\phi_{(0)}\left(1 + \tfrac{1}{3}\phi_{(0)}\right) \geq 0 \tag{3.14}$$

determines the values of $\phi_{(0)}$ and ϵ which are an acceptable solution of equation 2.23 corresponding to equation 3.12. Figure 9 shows several curves representing $(\beta/\Omega_L)^2$ as a function of $\phi_{(0)}$ for several values of ϵ. It is observed that as ϵ increases above 8/9 that a larger and larger part of the curve in the region $-3 < \phi_{(0)} < 0$ corresponds to unacceptable values for $(\beta/\Omega_L)^2$. But if $\phi_{(0)} < -3$ there will always be an acceptable solution of the form of equation 3.12. For instance, as ϵ becomes unboundedly large, the value of $\phi_{(0)}$ tends to -4.5 and for $\epsilon = 1$, $\phi_{(0)}$ is -3. The region $\infty > \epsilon > 1$ can have solutions of the form given in equation 3.12 for vibrations about the buckled equilibrium point given by equation 2.53.

3.3 Response to a Step Function Loading

The next problem considered is the response (equation 2.23a) to a step function with zero initial conditions imposed, i.e., $\psi(0) = \psi_\tau(0) = 0$, $Q(\tau) = R$ for $\tau \geq 0$, where R is a constant. The system considered is

$$\psi_{\tau\tau} + \Omega_L^2\left[\psi + \epsilon\left(\psi^2 + \tfrac{2}{9}\psi^3\right)\right] = R\,1(t) . \tag{3.15}$$

Again the iteration technique is applied, seeking a linearized solution approximating the action of the nonlinear system. The form of the iteration is

$$\psi_{(n)} = \sum_{j=0}^{n} \theta_{(j)} \qquad \text{and} \qquad \theta_{(K)} << \theta_{(K-1)} \tag{3.16}$$

where $\theta_{(0)}$ is a constant. The $\theta_{(0)}$ must satisfy

$$\theta_{(0)} \left[1 + \epsilon \left(\theta_{(0)} + \tfrac{2}{9} \theta_{(0)}^2 \right) \right] = r \tag{3.17}$$

where $r = (R/\Omega_L^2)$. The next approximation requires

$$\theta_{(1)_{\tau\tau}} + \Omega_L^2 \left[\theta_{(1)} + \epsilon \left(2\theta_{(0)} + \tfrac{2}{3} \theta_{(0)}^2 \right) \theta_{(1)} \right] = 0. \tag{3.18}$$

Application of the initial conditions $\psi_1(0) = \psi_{1\tau}(0) = 0$ yields

$$\psi_{(1)}(\tau) = \theta_{(0)} (1 - \cos \beta\tau) \tag{3.19}$$

with β defined by equation 3.11. To determine the range of validity of equation 3.19, it is assumed the $\psi_{(1)}(\tau)$ is very nearly the correct solution, then the true solution, $\psi_{(T)}$, becomes

$$\psi_{(T)} = \psi_{(1)}(\tau) + \psi_{(2)}(\tau) \tag{3.20}$$

where $\psi_{(2)}(\tau)$ contains all the deviations of the true solution from $\psi_{(1)}(\tau)$.

The differential equation governing $\psi_{(2)}(\tau)$ is obtained by putting equations 3.20 and 3.19 into equation 3.15. As a consequence

$$\psi_{(2)\tau\tau} + \Omega_L^2 \left[1 + \epsilon \left(2\theta_{(0)} + \theta_{(0)}^2 \right) - \epsilon \theta_{(0)} \left(2 + \tfrac{4}{3} \theta_{(0)} \right) \cos \beta\tau \right.$$
$$\left. + \tfrac{1}{3} \epsilon \theta_{(0)}^2 \cos 2\beta\tau \right] \psi_{(2)} = 0. \tag{3.21}$$

with initial conditions

$$\psi_{(2)}(0) = \psi_{(2)\tau}(0) = 0 . \tag{3.22}$$

Equation 3.21 is a form of Hill's equation. To study this equation it is rewritten in the form

$$\psi_{(2)\tau\tau} + \psi_{(2)} [a - q_{I} \cos\beta\tau + q_{II} \cos 2\beta\tau] = 0. \tag{3.23}$$

One method of solution for equation 3.23 is to assume a solution of the form (ref. 7)

$$\psi_{(2)} = e^{i\lambda\tau} \sum_{-\infty}^{+\infty} b_m e^{im\beta\tau} \tag{3.24}$$

which, when introduced into equation 3.23, yields

$$\begin{aligned} &-\sum_{-\infty}^{+\infty} (\lambda + m\beta)^2 b_m e^{i(\lambda + m\beta)\tau} + e^{i\lambda\tau} \sum_{-\infty}^{+\infty} (a\, b_m \\ &- \tfrac{1}{2} q_{I} [b_{m+1} + b_{m-1}] + \tfrac{1}{2} q_{II} [b_{m+2} + b_{m-2}]) e^{im\beta\tau} = 0. \end{aligned} \tag{3.25}$$

From this

$$\begin{aligned} &a\, b_m - \tfrac{1}{2} q_{I} (b_{m+1} + b_{m-1}) + \tfrac{1}{2} q_{II} (b_{m+2} + b_{m-2}) - (\lambda + m\beta)^2 b_m = 0, \\ &[a - (\lambda + m\beta)^2] b_m - \tfrac{1}{2} q_{I} (b_{m+1} + b_{m-1}) + \tfrac{1}{2} q_{II} (b_{m+2} + b_{m-2}) = 0, \end{aligned} \tag{3.26}$$

is obtained. Associated with these equations there is an infinite determinant with the central seven by seven determinant. The condition that this determinant vanish (or for that matter, any finite central determinant taken from the infinite determinant) is an approximate condition to determine the values of λ and hence the stability of the solution assumed in equation 3.24. If this solution is unstable, then $\psi_{(1)}$ is not a good approximation to the true solution of equation 3.15. It is certainly that the vanishing of λ is the boundary between stable and unstable values of $\psi_{(2)}$. Making use of this fact the first order stability criterion for $\psi_{(2)}$ is

$$\frac{1}{4} q_I^2 q_{II}^2 - \frac{1}{4} q_{II}^2 a - \frac{1}{2} q_I^2 (a - \beta^2) = 0, \tag{3.27}$$

where

$$a = \Omega_L^2 [1 + \epsilon \theta_{(0)} (2 + \theta_{(0)})],$$

$$\beta^2 = \Omega_L^2 [1 + 2 \theta_{(0)} (1 + \tfrac{1}{3} \theta_{(0)})],$$

$$q_I = \Omega_L^2 [2 \epsilon \theta_{(0)} (1 + \tfrac{2}{3} \theta_{(0)})],$$

$$q_{II} = \Omega_L^2 [\tfrac{1}{2} \epsilon \theta_{(0)}].$$

When the necessary operations have been carried out in equation 3.27 the conditions for stability of $\psi_{(2)}$ and hence the limiting conditions under which $\psi_{(1)}$ can represent the response of the system is found to be

$$\epsilon = \left\{ \theta_{(0)} \left[10 + 9 \theta_{(0)} - \tfrac{8}{3} \theta_{(0)}^2 (1 + \theta_{(0)}) \right] \right\}^{-1}, \tag{3.28}$$

where $\theta_{(0)}$ is defined by equation 3.17. Equations 3.17 and 3.27 have been used to construct a stability nomograph which is presented in figure 10.

A consideration of the energy of the problem of a step function loading as represented in equation 3.15 yields

$$\psi_\tau^2 + \Omega_L^2 [\psi^2 + \epsilon (\tfrac{2}{3} \psi^3 + \tfrac{1}{9} \psi^4)] = 2R\psi + 2K. \tag{3.29}$$

A comparison with equation 2.39 shows that the effect of a pressure step function is to add a term, linear in ψ, to the energy equation previously obtained for the free vibration. These energy curves are shown in figure 5. As expected, an internal pressure step stabilizes the cylinder and an external pressure step destabilizes the cylinder. In addition, there are other, less expected results. With an external pressure step, the saddle point moves closer to the origin, $\psi = 0$, and to a lower energy level. The buckled equilibrium point moves to a lower energy level and a larger inward displacement. The static equilibrium point moves to a higher energy level and an inward displacement. Internal pressure has precisely the opposite effect. Figure 11 shows how the phase plane trajectories are modified. Figure 11a depicts the minimum energy buckling trajectory (separatrix) for a given system. Figure 11b shows how this trajectory would be changed by an external pressure step while figure 11c shows the influence of an internal pressure step.

3.4 Buckling Due to a Step Function Loading

An important and interesting question is raised by this influence of the step function loading. It is clear that the larger the external step function, the lower will be the energy level required to cause buckling of the shell, until the step pressure is equal to the static buckling pressure in which case the shell will buckle without dynamic effects. The question that arises is this: is there a condition such that an applied external pressure step, of magnitude less than the static buckling pressure, will buckle the shell? To answer this question it is necessary to know the maximum value that ψ will attain for any external pressure step and to compare this value with the location of the saddle point. If this maximum response is greater than the amplitude which corresponds with the saddle point, buckling is possible.

Application of the work of Fung and Barton (8) for the shock response of nonlinear systems leads to a solution of the problem. In their work, Fung and Barton present the concept of a shock response ratio which is the ratio of the maximum response of a nonlinear system to the maximum response of the equivalent linear system $(\epsilon = 0)$ when each has been subjected to the same loading. This ratio depends on the nature of the nonlinearity of the restoring force in the system considered and on the type of loading to be applied to these two systems. For a step load, they have shown that the response ratio $\mathcal{R}_\Delta$ has the following form

$$\mathcal{R}_\Delta = \left[1 + 2\epsilon \int_0^{-1} g(\psi^*)\, d\psi^* \right]^{-1}, \tag{3.30}$$

where $g(\psi^*)$ represents the nonlinearity function of the restoring force. The limit (-1) occurs here since buckling occurs in the negative displacement domain. For the problem examined here

$$g(\psi^*) \equiv (\psi^*)^2 + \tfrac{2}{9}(\psi^*)^3, \tag{3.31}$$

where $\psi^* = \psi/\Delta$

and Δ is the maximum "allowable" non-dimensional deflection of the nonlinear system. The choice for Δ is that deflection which is the least that will allow buckling to occur, i.e., the saddle point. The critical loading ratio obtained in this manner is

$$\mathcal{R}_\Delta = \left(1 - \frac{5\epsilon}{18}\right)^{-1}. \tag{3.32}$$

The associated linear system is

$$\begin{gathered} \psi_{\tau\tau} + \Omega_L^2 \psi = r\Omega_L^2\, 1(\tau), \\ r\Omega_L^2 = R, \end{gathered} \tag{3.33}$$

with initial conditions

$$\psi(0) = \psi_\tau(0) = 0. \tag{3.34}$$

The maximum response of this linear system is $|2r| = \text{MAX}\,|\psi_{LIN}|$.
But recalling that

$$\mathcal{R}_\Delta = \frac{\text{MAX}\,|\psi_{NONLIN.}|}{\text{MAX}\,|\psi_{LIN}|} = \frac{\Delta}{\text{MAX}\,|\psi_{LIN}|}, \tag{3.35}$$

the maximum nonlinear response to a step input of magnitude $r\Omega_L^2$ is found to be

$$\Delta = |2r|\left(1 - \frac{5\epsilon}{18}\right)^{-1}. \tag{3.36}$$

The saddle point is located by differentiation of equation 3.29 with respect to ψ and setting the derivative equal to zero, with the result

$$\epsilon\left[\frac{4}{9}\psi^3 + 2\psi^2\right] + 2\psi - 2r = 0. \tag{3.37}$$

The three roots of this equation will be ordered in the fashion $\psi_{o(3)} < \psi_{o(2)} < \psi_{o(1)}$. The magnitude of $\psi_{o(2)}$ must correspond to equation 3.39 in order to get the condition on $|r|$ for buckling, i.e.,

$$|\psi_{o(2)}| \leq \frac{2|r|}{1-\frac{5\epsilon}{18}} . \qquad (3.38)$$

It is not possible that an internal pressure loading will generate a dynamic response great enough to buckle the shell, thus an external pressure step may be taken as the physically important condition. The external loading situation corresponds to $r < 0$. If equation 3.38 is introduced into equation 3.37 after dropping the absolute value signs, the equation for the value of r which will just allow buckling is

$$\epsilon\left[\frac{4}{9}\left(\frac{2}{1-\frac{5\epsilon}{18}}\right)^3 r^2 + 2\left(\frac{2}{1-\frac{5\epsilon}{18}}\right)^2 r\right] + 2\left[\left(\frac{2}{1-\frac{5\epsilon}{18}}\right) - 1\right] = 0 . \qquad (3.39)$$

The roots of this equation are ordered as $r_{(2)} < r_{(1)}$. The value of $r_{(1)}$ will provide the correct magnitude (and sign) for the pressure step required to buckle the cylinder, provided only that $\epsilon > \frac{8}{9}$. Figure 17 shows the variation of $r_{(1)}$ as a function of ϵ.

The magnitude of R (and hence r) can be related to a physical pressure P_o. The relation is

$$P_{o_{CRT}}^{(ST)} \leq R_{CRT}\left(\frac{\pi}{4}\right)^4\left(\frac{a\bar{m}\omega^2}{n^2}\right) \qquad (3.40)$$

where $R_{CRT} < 0$. The corresponding equation in terms of r_{CRT} is

$$P_{o_{CRT}}^{(ST)} \leq r_{CRT} \left(\frac{\pi}{4}\right)^4 \left(\frac{a\bar{m}\omega_L^2}{n^2}\right). \tag{3.41}$$

Care must be taken at this point since ω_L^2 depends on an initial pressure. The external pressure is not a part of the initial pressure which determined ω_L^2 . The value of $P_{o_{CRT}}^{(ST)}$ may be compared with $P_{o_{CRT}}$ from equation 2.29 to determine which is larger at each particular condition.

CHAPTER IV

A SECOND DEVELOPMENT OF THE CURVED PANEL PROBLEM

4.1 Equations of Motion

In reference 3 Reissner assumed a stress function that satisfied the boundary conditions but which did not satisfy the compatibility equation exactly. In section 2.3 et seq. the same procedure is followed. The equations of motion will be obtained using an assumed displacement mode shape and the corresponding exact stress function (which satisfies the compatibility equation in a manner similar to the work of Chu, (6)). If the displacement $S(x, y; t)$ as given in equation 2.14 is used and substituted into the compatibility equation (the first of equations 2.8) the stress function may be determined by integration of that equation. The integration gives

$$F = EAh\left\{-\frac{1}{a}\left(\frac{\pi}{L}\right)^2\left[\left(\frac{\pi}{L}\right)^2+\left(\frac{n}{a}\right)^2\right]^{-2}\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{ny}{a}\right) + \frac{A}{32}\left[\left(\frac{n}{a}\right)^2\left(\frac{\pi}{L}\right)^{-2}\cos\left(\frac{2\pi x}{L}\right) - \left(\frac{\pi}{L}\right)^2\left(\frac{n}{a}\right)^{-2}\cos\left(\frac{2ny}{a}\right)\right]\right\}. \tag{4.1}$$

This stress function satisfies the compatibility conditions exactly but the boundary conditions on F and its second derivatives are not satisfied exactly, but are satisfied only on the average. None the less, the results obtained are expected to be improved if equation 4.1 is used with the Galerkin averaging technique to satisfy the equilibrium equation (first of equations 2.10) under the restrictions of section 2.6.

When equations 2.14 and 4.1 are introduced into the first of equations 2.10, the following equation is obtained:

$$\Big\{ \bar{m}\ddot{A} + A\Big[D\big[(\tfrac{\pi}{L})^2 + (\tfrac{n}{a})^2\big]^2 + \frac{Eh}{a^2}(\tfrac{\pi}{L})^4\big[(\tfrac{\pi}{L})^2 + (\tfrac{n}{a})^2\big]^{-2}$$

$$+ P_o a\big[\tfrac{1}{2}(\tfrac{\pi}{L})^2 + (\tfrac{n}{a})^2\big]\Big]\Big\} \sin(\tfrac{\pi x}{L})\cos(\tfrac{n y}{a})$$

$$= EhA^2\Big\{\frac{1}{8a}(\tfrac{n}{a})^2\cos(\tfrac{2\pi x}{L}) + \frac{1}{a}(\tfrac{\pi}{L})^4(\tfrac{n}{a})^2\big[(\tfrac{\pi}{L})^2 + (\tfrac{n}{a})^2\big]^{-2}\cdot \tag{4.2}$$

$$\cdot\cos^2(\tfrac{\pi x}{L})\sin^2(\tfrac{n y}{a}) - \frac{2}{a}(\tfrac{\pi}{L})^4(\tfrac{n}{a})^2\big[(\tfrac{\pi}{L})^2 + (\tfrac{n}{a})^2\big]^{-2}\sin^2(\tfrac{\pi x}{L})\cdot$$

$$\cdot\cos^2(\tfrac{n y}{a})\Big\} + EhA^3\Big\{\frac{1}{8}\Big[(\tfrac{n}{a})^4\cos(\tfrac{2\pi x}{L}) - (\tfrac{\pi}{L})^4\cos(\tfrac{2 n y}{a})\Big]\sin(\tfrac{\pi x}{L})\cos(\tfrac{n y}{a})\Big\}.$$

Now if the Galerkin averaging is applied to equation 4.2, using S(x,y;t) as in the weighting function, the integrals of equations 2.17 are used along with the integrals

$$\int_0^L \int_{-\frac{\pi a}{2n}}^{\frac{\pi a}{2n}} \cos(\tfrac{2\pi x}{L})\sin^2(\tfrac{\pi x}{L})\cos^2(\tfrac{n y}{a})\,dx\,dy = (\tfrac{a\pi}{2n}),$$

$$\int_0^L \int_{-\frac{\pi a}{2n}}^{\frac{\pi a}{2n}} \cos(\tfrac{2 n y}{a})\cos^2(\tfrac{n y}{a})\sin^2(\tfrac{\pi x}{L})\,dx\,dy = (\tfrac{La\pi}{2n}). \tag{4.3}$$

When this is done it is possible to find that

$$\bar{m}\ddot{A} + A\left\{D\left[\left(\tfrac{\pi}{L}\right)^2+\left(\tfrac{n}{a}\right)^2\right]^2 + \frac{Eh}{a^2}\left(\tfrac{\pi}{L}\right)^4\left[\left(\tfrac{\pi}{L}\right)^2+\left(\tfrac{n}{a}\right)^2\right]^{-2} + P_o a\left[\tfrac{1}{2}\left(\tfrac{\pi}{L}\right)^2+\left(\tfrac{n}{a}\right)^2\right]\right\}$$

$$+ A^2\left\{\left(\frac{Eh}{a}\right)\left(\frac{\pi}{L}\right)^4\left(\frac{n}{a}\right)^2\left[\left(\frac{\pi}{L}\right)^2+\left(\frac{n}{a}\right)^2\right]^{-2}\left(\frac{112}{9\pi^2}\right)\right\}$$

(4.4)

$$+ \quad A^3\left\{\left(\frac{Eh}{4}\right)\left[\left(\frac{\pi}{L}\right)^4-\left(\frac{n}{a}\right)^4\right]\right\} = \mathbb{P}(t)\left(\frac{aL\pi}{4n}\right)^{-1}.$$

If a pseudo aspect ratio AR is defined by

$$AR = \frac{nL}{\pi a}, \tag{4.5}$$

and if the notations of equations 2.20 are used, the nondimensional equation obtained from equation 4.4 is

$$\psi_{\tau\tau} + \Omega_L^2\left\{\psi + \left(\tfrac{7}{9}\right)\epsilon\left[\psi^2 + \left(\tfrac{9}{7}\right)\left(\tfrac{1}{4}\right)\left(\tfrac{\pi}{4}\right)^4 AR^{-4}(1+AR^2)^2(1-AR^4)\psi^3\right]\right\} = Q(\tau) \tag{4.6}$$

But if the coefficient of the cubic term is

$$\chi(AR) = \left(\tfrac{9}{7}\right)\left(\tfrac{1}{4}\right)\left(\tfrac{\pi}{4}\right)^4 AR^{-4}(1+AR^2)^2(1-AR^4), \tag{4.7}$$

then equation 4.6 may be written in the form

$$\psi_{\tau\tau} + \Omega_L^2\left[\psi + \tfrac{7}{9}\epsilon\left(\psi^2+\chi(AR)\psi^3\right)\right] = Q(\tau). \tag{4.8}$$

4.2 <u>The Influence of Aspect Ratio</u>

The nonlinearity parameter ϵ, expressed as a function of aspect ratio, is

$$\epsilon =$$

$$\left\{1+(1+AR^2)^2\left[\frac{n^4}{12(1-\nu^2)}(h/a)^2 AR^{-4} + \frac{n^2 P_o}{E}(h/a)^{-1}AR^{-2}\left(\tfrac{1}{2}+AR^2\right)^2\right]\right\}^{-1}. \tag{4.9}$$

As AR tends to zero, ϵ tends to zero but χ tends to plus infinity and the product $\frac{7}{9}\epsilon(AR)\,\chi(AR)$ tends to $3(1-\nu^2)(\pi/4)^4 n^{-4}(h/a)^{-2}$ and for small AR $(AR \ll 1)$ equation 4.8 is

$$\psi_{\tau\tau} + \Omega_L^2\left[\psi + \frac{3(1-\nu^2)\,(\pi/4)^4}{n^4\,(h/a)^2}\,\psi^3\right] = Q(\tau). \tag{4.10}$$

This is the Duffing equation with a hard spring and hence when the aspect ratio of the curved panel is small compared to unity, equation 2.23a may be replaced by equation 4.10.

For AR tending to infinity ϵ tends to zero as AR^{-6} while χ tends to negative infinity as AR^4 (see figure 12). The product $(\frac{7}{9})\,\epsilon(AR)\,\chi(AR)$ tends to zero as AR^{-2} for P_o different from . But if P_o is only slightly different from the buckling condition $P_{o_{CRT}}$ the product $(\frac{7}{9})\,\epsilon\,\chi$ can become finite and the equation of motion reduces to

$$\psi_{\tau\tau} + \Omega_L^2\left[\psi - \tfrac{1}{4}\epsilon\, AR^4\,\psi^3\right] = Q(\tau), \tag{4.11}$$

which is Duffing's equation for a soft spring. Since equation 4.11 requires a very restrictive set of conditions on P_o and AR, it is of limited usefulness. In most instances it will be more appropriate to use equation 4.8.

The comparison of equation 4.8 with equation 2.23a indicates that the two will give similar results for AR in the neighborhood of 0.9. To carry out the studies of sections 2.7 through 2.10 for equation 4.8 in all ranges of the aspect ratio AR is not practical for the present work. But since equation 2.23a has most of the features of equation 4.8, sections 2.7 through 2.10 may be regarded as depicting many of the features of equation 4.8.

A brief examination of the effect of $\mathcal{R}$ on the singularities of equation 4.8 (with $Q(\tau) = 0$) in the phase plane will point out how this equation differs from that studied in sections 2.7 through 2.10. The phase plane singularities are the roots of the equation

$$f(\psi) = \Omega_L^2 \psi \left[1 + \tfrac{7}{9}\epsilon\left(\psi + \chi(\mathcal{R})\psi^2\right)\right] = 0 \tag{4.12}$$

which are

$$\begin{aligned} \psi_1 &= 0 \,, \\ \psi_2 &= -\frac{1}{2\chi}\left[1 - \left(1 - \frac{36}{7}\left(\frac{\chi}{\epsilon}\right)\right)^{1/2}\right], \\ \psi_3 &= -\frac{1}{2\chi}\left[1 + \left(1 - \frac{36}{7}\left(\frac{\chi}{\epsilon}\right)\right)^{1/2}\right]. \end{aligned} \tag{4.13}$$

Again if

$$\left[1 - \frac{36}{7}\left(\frac{\chi}{\epsilon}\right)\right] \geq 0 \tag{4.14}$$

ψ_1 and ψ_3 are stable centers while ψ_2 is a saddle point. If the opposite inequality

$$\left[1 - \frac{36}{7}\left(\frac{\chi}{\epsilon}\right)\right] < 0 \tag{4.15}$$

is valid, ψ_2 and ψ_3 are complex and ψ_1 is a stable center. Now note that for $\mathcal{R}$ tending toward zero the latter inequality holds and there is no possibility of a solution of the "buckled" type. If the aspect ratio is large, equation 4.12 becomes

$$f(\psi) = \psi\left[1 - \tfrac{1}{4}\epsilon\, A\!R^4 \psi^2\right] \tag{4.16}$$

with roots

$$\begin{aligned} \psi_1 &= 2\,\epsilon^{-1/2}\, A\!R^{-2}, \\ \psi_2 &= 0, \\ \psi_3 &= -2\,\epsilon^{-1/2}\, A\!R^{-2}. \end{aligned} \tag{4.17}$$

From these roots ψ_2 is found to be a stable center while ψ_1 and ψ_3 are saddle points. An important feature of the large aspect ratio problem is that both $\epsilon^{-1/2}(A\!R \rightarrow \infty)$ and $A\!R^{-2}$ are small numbers so that as $A\!R$ tends to infinity the amplitude of the stable oscillations about the point ψ_2 diminish to zero. Thus the oscillations for a very long narrow panel become less stable as the length of the panel increases. Figure 13 shows the variations in the type of phase plane trajectories that may be expected to accompany changing aspect ratios in equation 4.8.

4.3 A Second Treatment of the Step Function Response Problem

The inhomogeneous differential equation with homogeneous initial conditions considered in section 3.3 can be treated exactly by a simple transformation. The differential equation then becomes homogeneous while the initial conditions become inhomogeneous. Consider a step function applied to equation 4.8

$$\psi_{\tau\tau} + \Omega_L^2\, g(\psi) = R\, 1(\tau), \tag{4.18}$$

where

$$g(\psi) = \psi + \tfrac{7}{9}\epsilon\left[\psi^2 + \alpha(R)\psi^3\right].$$

And let the initial conditions be

$$\psi(0) = \psi_\tau(0) = 0. \tag{4.19}$$

Now let

$$\psi = \delta + \theta(\tau) \tag{4.20}$$

where δ is a constant such that

$$g(\delta) = R/\Omega_L^2 = r. \tag{4.21}$$

Then

$$\theta_{\tau\tau} + \Omega_L^2\left[g'(\delta)\theta + \tfrac{1}{2}g''(\delta)\theta^2 + \tfrac{1}{6}g'''(\delta)\theta^3\right] = 0, \tag{4.22}$$

where primes indicate differentiation with respect to the argument. The initial conditions become

$$\theta(0) = -\delta, \quad \theta_\tau(0) = 0. \tag{4.23}$$

Phase plane curves are given by

$$\theta_\tau^2 + \Omega_L^2\left[g'(\delta)\theta^2 + \tfrac{1}{3}g''(\delta)\theta^3 + \tfrac{1}{12}g'''(\delta)\theta^4\right] = 2K. \tag{4.24}$$

Maximum amplitudes may be determined from the phase plane trajectories as a function of δ . A given value of R will fix a value of δ and hence a given value of K, from

$$2K = \Omega_L^2 \left[\xi'(\delta(R))\,\delta^2(R) + \tfrac{1}{3}\xi''(\delta(R))\,\delta^3(R) + \tfrac{1}{12}\xi'''(\delta(R))\,\delta^4(R) \right. . \qquad (4.25)$$

The period associated with equation 4.22 is determined from

$$\frac{T_{NL}}{T_L} = \frac{1}{\pi}\int_{\theta_1}^{\theta_2} \frac{d\theta}{\left[2K-\left(\xi'(\delta)\theta^2 + \tfrac{1}{3}\xi''(\delta)\,\theta^3 + \tfrac{1}{12}\xi'''(\delta)\,\theta^4\right)\right]^{1/2}} \qquad (4.26)$$

where $\tau = \omega_L t$ and hence $\Omega_L^2 = 1$ and θ_1, θ_2 are the maximum and minimum points on the phase plane trajectory.

CHAPTER V

THE CYLINDRICAL SHELL PROBLEM

5.1 Equations of Motion

It has been pointed out in section 2.6 that the manner in which the Galerkin averaging process is applied makes a fundamental difference in the nature of the physical system that corresponds to the resulting equation. Indeed, it appears that if the Galerkin integration of the equations of motion is taken over only one half wave in both axial and circumferential directions, the resultant equations cannot be said to apply to a breathing mode vibration of a circular cylindrical shell. However, if the Galerkin integration which led to equation 2.21 had been carried over a full wave in both axial and circumferential directions, an equation quite different from 2.21 would have been found. In fact, the quadratic nonlinearity would have been gone, leaving a form of Duffing's equation. And the objectionable moment along the boundaries Y = constant would have departed with the quadratic nonlinearity. But, the interesting "buckling" phenomena and the difference of time during inward deflection from time during outward deflection will also be missing from the analysis.

If the analysis was to start with equation 4.2 the only variation being introduced by integrating over the domain $-L \leq x \leq L$, $-\frac{\pi a}{n} \leq Y \leq \frac{\pi a}{n}$, the resulting equations still would be limited to a "shallow shell mode", i.e., there would still be the restriction on the value of n. The analysis would apply only if $n > 3$. In order to relieve this restriction use will be made of a nonlinear form of the recently developed Morley equations. In his paper (ref. 9) Morley has shown that a slight modification of the Donnell equations leads to a new and simple equation which shows very good agreement with the Flügge equations even for small values of n in static problems. This conclusion has been substantiated in a paper by Houghton and Johns (10) where Morley's equation is compared not only with Donnell's equation but also with the equations bearing the names of Biezeno and Grammel, Vlassov, Timoshenko, Bijlaard, Naghdi and Berry, and Kennard.

It does not appear that there is a mathematically consistent way of arriving at Morley's equations currently available. The justification for its use lies in the remarkable agreement between numerical results obtained from it and Flügge's equation. With this brief note of caution, the Morley equation will be applied. If a severe doubt about the Morley equations is held by the reader, he may obtain the Donnell equation results by dropping one term in the final equation.*

It can be shown that the compatibility and equilibrium equations which correspond to the Morley equation, with nonlinear terms included, are

$$\frac{1}{Eh}\nabla^4 F = \frac{1}{a} S_{xx} + \left[S_{xy}^2 - S_{xx} S_{yy}\right],$$

$$D\left(\nabla^2 + \frac{1}{a^2}\right)^2 S = P(x,y;t) - \bar{m}\ddot{S} + N_{xi} S_{xx} + N_{yi} S_{yy} \qquad (5.1)$$

$$-\frac{1}{a} F_{xx} + \left[F_{xx} S_{yy} + F_{yy} S_{xx} - 2 F_{yx} S_{yx}\right].$$

*In the final equation the Donnell equation results are obtained by replacing $D\left[\left(\frac{\pi}{L}\right)^2 + \left(\frac{n}{a}\right)^2 + \left(\frac{1}{a}\right)^2\right]^2$ with $D\left[\left(\frac{\pi}{L}\right)^2 + \left(\frac{n}{a}\right)^2\right]^2$.

5.2 Boundary Conditions

The boundary conditions are

$$\left.\begin{aligned} & S = S_{xx} = 0 \\ & \int_{-\left(\frac{\pi a}{n}+\frac{j\pi a}{n}\right)}^{\left(\frac{\pi a}{n}+\frac{j\pi a}{n}\right)} F\,dy = \int_{-\left(\frac{\pi a}{n}+\frac{j\pi a}{n}\right)}^{\left(\frac{\pi a}{n}+\frac{j\pi a}{n}\right)} F_{xx}\,dy = 0 \end{aligned}\right\} \text{ ON } x = \pm jL \quad (j = 0, 1, 2 \ldots),$$

$$\left.\begin{aligned} & S = S_{yy} = 0 \\ & \int_{-L}^{+L} F\,dx = \int_{-L}^{+L} F_{yy}\,dx = 0 \end{aligned}\right\} \text{ ON } y = \pm\left[\frac{\pi a}{n} + \frac{j\pi a}{n}\right] \quad (j = 0, 1, 2 \ldots).$$

The meaning of the boundary conditions on S are the same as in section 2.2. The meaning of the boundary conditions on F are similar to that of section 2.2, but here they are true on the average over the entire boundary where, before they were true on a point by point basis throughout the boundary.

5.3 Galerkin Integration

Now, since a full cylinder is considered, $N_{yi} = a P_o$ and $N_{xi} = \frac{1}{2} a P_o$ for initial stresses caused by pressurization. The displacement mode will be taken to be

$$S = A(t) \cos\left(\frac{ny}{a}\right) \sin\left(\frac{\pi x}{L}\right), \tag{5.2}$$

and the associated stress function, which satisfies compatibility, is given by equation 4.1. Substitution of equations 5.2 and 4.1 into the equilibrium equation from equations 5.1 produces

$$\begin{aligned}
&\Big\{\bar{m}\ddot{A} + A\Big[D\Big[\left(\tfrac{\pi}{L}\right)^2+\left(\tfrac{n}{a}\right)^2+\left(\tfrac{1}{a}\right)^2\Big]^2 + \tfrac{Eh}{a^2}\left(\tfrac{\pi}{L}\right)^4\Big[\left(\tfrac{\pi}{L}\right)^2+\left(\tfrac{n}{a}\right)^2\Big]^{-2} \\
&+ P_o a\Big[\tfrac{1}{2}\left(\tfrac{\pi}{L}\right)^2+\left(\tfrac{n}{a}\right)^2\Big]\Big]\Big\}\sin\left(\tfrac{\pi x}{L}\right)\cos\left(\tfrac{ny}{a}\right) = E\,hA^2\Big\{\tfrac{1}{8a}\left(\tfrac{n}{a}\right)^2\cos\left(\tfrac{2\pi x}{L}\right) \\
&+ \tfrac{1}{a}\left(\tfrac{\pi}{L}\right)^4\left(\tfrac{n}{a}\right)^2\Big[\left(\tfrac{\pi}{L}\right)^2+\left(\tfrac{n}{a}\right)^2\Big]^{-2}\cos^2\left(\tfrac{\pi x}{L}\right)\sin^2\left(\tfrac{ny}{a}\right) \\
&- \tfrac{2}{a}\left(\tfrac{\pi}{L}\right)^4\left(\tfrac{n}{a}\right)^2\Big[\left(\tfrac{\pi}{L}\right)^2+\left(\tfrac{n}{a}\right)^2\Big]^{-2}\sin^2\left(\tfrac{\pi x}{L}\right)\cos^2\left(\tfrac{ny}{a}\right)\Big\} \\
&+ E\,h\,A^3\Big\{\tfrac{1}{8}\Big[\left(\tfrac{n}{a}\right)^4\cos\left(\tfrac{2\pi x}{L}\right) - \left(\tfrac{\pi}{L}\right)^4\cos\left(\tfrac{2ny}{a}\right)\Big]\Big\}\sin\left(\tfrac{\pi x}{L}\right)\cos\left(\tfrac{ny}{a}\right) + P(x,y;t).
\end{aligned} \tag{5.3}$$

Equation 5.3 is to be weighted with equation 5.2 and the Galerkin averaging integral is to be carried out over the domain $-L \leq x \leq L$; $-\frac{\pi a}{n} \leq y \leq \frac{\pi a}{n}$. The following integrals are used

$$\begin{aligned}
&\int_{-L}^{L}\int_{-\frac{\pi a}{n}}^{\frac{\pi a}{n}} \sin^2\left(\tfrac{\pi x}{L}\right)\cos^2\left(\tfrac{ny}{a}\right)dx\,dy = \left(\tfrac{La\pi}{n}\right), \\
&\int_{-L}^{L}\int_{-\frac{\pi a}{n}}^{\frac{\pi a}{n}} \cos\left(\tfrac{2\pi x}{L}\right)\sin^2\left(\tfrac{\pi x}{L}\right)\cos^2\left(\tfrac{ny}{a}\right)dx\,dy = -\left(\tfrac{La\pi}{n}\right), \\
&\int_{-L}^{L}\int_{-\frac{\pi a}{n}}^{\frac{\pi a}{n}} \cos\left(\tfrac{2ny}{a}\right)\cos^2\left(\tfrac{ny}{a}\right)\sin^2\left(\tfrac{\pi x}{L}\right)dx\,dy = \left(\tfrac{La\pi}{n}\right), \\
&\int_{-L}^{L}\int_{-\frac{\pi a}{n}}^{\frac{\pi a}{n}} \cos\left(\tfrac{ny}{a}\right)\sin\left(\tfrac{\pi x}{L}\right)\,P(x,y;t)\,dx\,dy = \mathbb{P}'(t),
\end{aligned} \tag{5.4}$$

where all other integrals in this Galerkin average vanish.

The Galerkin integration of the equilibrium equation then produces

$$\begin{aligned}\ddot{A} &+ \frac{A}{\bar{m}}\left\{D\left[\left(\frac{\pi}{L}\right)^2+\left(\frac{n}{a}\right)^2+\left(\frac{1}{a}\right)^2\right]^2 + \frac{Eh}{a^2}\left(\frac{\pi}{L}\right)^4\left[\left(\frac{\pi}{L}\right)^2+\left(\frac{n}{a}\right)^2\right]^{-2}\right. \\ &\left. + aP_0\left[\frac{1}{2}\left(\frac{\pi}{L}\right)^2+\left(\frac{n}{a}\right)^2\right]\right\} + \frac{Eh}{\bar{m}}\left(\frac{1}{8}\right)\left[\left(\frac{\pi}{L}\right)^4+\left(\frac{n}{a}\right)^4\right]A^3 \\ &= \frac{\mathbb{P}(t)}{\bar{m}\left(\frac{La\pi}{n}\right)} \quad .\end{aligned} \tag{5.5}$$

Once again the notation of equation 2.20 is used with the exception that $\bar{m}\,\omega_L^2$ is now replaced by*

$$\bar{m}\,\omega_{LM}^2 = D\left[\left(\frac{\pi}{L}\right)^2+\left(\frac{n}{a}\right)^2+\left(\frac{1}{a}\right)^2\right]^2 + aP_0\left[\frac{1}{2}\left(\frac{\pi}{L}\right)^2+\left(\frac{n}{a}\right)^2\right] + \bar{m}\,\omega_*^2 \ . \tag{5.6}$$

When these expressions are used to simplify equation 5.5, equation 4.5 is used to introduce the aspect ratio, the nondimensional amplitude ψ is governed by the equation

$$\psi_{\tau\tau} + \Omega_{LM}^2\,\psi + \frac{1}{8}\left(\frac{\pi}{4}\right)^4\left(\frac{Eh}{a^2}\right)\left(\bar{m}\,\omega^2\right)^{-1}\mathcal{R}^{-4}\left(1+\mathcal{R}^4\right)\psi^3 = Q'(\tau), \tag{5.7}$$

where
$$Q'(\tau) = \frac{1}{4}\left[\frac{(4n)^3\,\mathbb{P}'(\tau)}{\pi^3 a^2\,L\,\omega^2\,\bar{m}}\right] \quad .$$

In a recent paper (ref. 6), Chu has obtained a similar equation by a different process. He collects terms which contain first harmonics of the space variables from an equation equivalent to equation 5.3 and neglects the higher harmonic space terms. By this procedure he obtains an equation which differs from equation 5.7

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* The subscript M is to denote that the Morley equation has been used.

by a factor of two in the cubic term. However, since the intent of his paper was to show the trends of frequency dependence on the amplitude of vibration, the loss of this factor of two should not invalidate his general conclusions. Of course it must be noted that Chu's development does not start from the Morley equation and so has Ω_L^2 instead of Ω_{LM}^2 in his development of the equation.

Because equation 5.7 is a form of Duffing's equation, a great deal of information about it can be obtained from the literature of nonlinear mechanics. Since this information is generally available, it is unnecessary to repeat the derivations here*. It will be the work of the next several sections to make use of the already known data for Duffing's equation in order to discuss the shell vibration problem.

5.4 Singularities

Equation 5.7 will be written in the simplified form

$$\psi_{\tau\tau} + \Omega_{LM}^2 \left[\psi + \sigma \psi^3 \right] = Q'(\tau) \qquad (5.8)$$

where

$$\sigma = \left(\frac{1}{\Omega_{LM}^2}\right) \left[\frac{1}{8}\left(\frac{\pi}{4}\right)^4 \left(\frac{Eh}{a^2}\right) (\bar{m}\,\omega^2)^{-1} R^{-4} (1 + R^4) \right].$$

Singularities of equation 5.8 will be at the roots of

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* See, for instance, ref. 11.

$$\psi\left[1+\sigma\psi^{2}\right]=0 \tag{5.9}$$

But since $\sigma \geq 0$ the only singularity is a stable center at $\psi = 0$.

5.5 Energy Equation and Period

The energy integral for free vibration $(Q'(\tau) = 0)$ is obtained, as before, from

$$(\psi_{\tau})_{\psi} = -f(\psi)/\psi_{\tau} \tag{5.10}$$

where $f(\psi) = \Omega_{LM}^{2}\psi\left[1+\sigma\psi^{2}\right]$.

Integration yields

$$(\psi_{\tau})^{2}+\Omega_{LM}^{2}\left[\psi^{2}+\tfrac{1}{2}\sigma\psi^{4}\right]=2K. \tag{5.11}$$

Then the period may be found as before in the form

$$\left.\frac{T_{NL}}{T_{L}}\right|_{M}=\frac{1}{\pi}\int_{\psi_1}^{\psi_2}\frac{d\psi}{\left[2K-(\psi^{2}+\frac{1}{2}\sigma\psi^{4})\right]^{1/2}} \tag{5.12}$$

where the subscript "M" on the period symbol is to designate that the Morley equation has been used. The limits ψ_1 and ψ_2 are the real roots of

$$2K-\psi^{2}-\tfrac{1}{2}\sigma\psi^{4}=0, \tag{5.13}$$

and correspond to maximum values of displacement. Hence the limits on the integral of equation 5.13 are determined from

$$\psi_{1,2}^{2}=\frac{2}{\sigma}\left[(1+4K\sigma)^{1/2}-1\right] \tag{5.14}$$

where the sign has been chosen so that ψ_1 and ψ_2 are real. The remaining roots of the quartic 5.13 are obtained from

$$\psi_{3,4}^{2} = -\frac{2}{\sigma}\left[\left(1+4K\sigma\right)^{1/2}+1\right]. \qquad (5.15)$$

With the aid of equations 5.14 and 5.15, equation 5.12 may be evaluated following the method presented in Appendix I.

5.6 Harmonic Response

The response of the Duffing equation to harmonic excitation has received considerable attention. The following procedure is to be found in ref. 11 and is repeated here with only sufficient detail to make the translation to the notation of the present problem understandable.

The frequency used to nondimensionalize the time will be chosen to be ω_{LM} so that $\tau_M = \omega_{LM} t$ and $\Omega^2_{LM} = 1$ and in addition the frequency in the denomination of σ is also ω_{LM}. Consider the case where the coefficient σ is a quantity small relative to unity.

Let the forcing function be

$$Q'(\tau) = \sigma q_0 \cos(\lambda \tau_M). \qquad (5.16)$$

The forcing function is small of order σ and the parameter λ is the ratio of the driving frequency to the natural frequency of the linearized equation associated with this system. Now equation 5.8 may be written as

$$\psi_{\tau\tau} + \lambda^2 \psi = (\lambda^2 - 1)\psi - \sigma \psi^3 + \sigma q_0 \cos \lambda \tau_M \qquad (5.17)$$

where $\lambda^2 \psi$ has been added to each side of equation 5.8.

If the investigation is in the vicinity of the natural frequency of the linear system, i.e., if the investigation is to seek the conditions near resonance, then λ is close to unity and $(\lambda^2 - 1)$ can be small of order σ. In that case all quantities on the right hand side of equation 5.17 are small of order σ. Under these conditions, equation 5.17 is an instrument for obtaining an approximate solution by iteration. Let the iterative solution be

$$\psi_n = \sum_{j=0}^{n} \phi_j \qquad (5.18)$$

where $\phi_K << \phi_{K-1}$. First introduce $\psi_0 = \phi_0$ into equation 3.17. Then drop terms of order σ with the result

$$\phi_{0\,\tau\tau} + \lambda^2 \phi_0 = 0. \qquad (5.19)$$

The solution of equation 5.19 is

$$\phi_0 = C_1 \cos \lambda \tau_M + C_2 \sin \lambda \tau_M \qquad (5.20)$$

According to Stoker* it can be shown that only $\cos \lambda \tau_M$ (where λ is an odd integer) will occur in the solution to the system of equation 5.17. So equation 5.20 may be written

$$\psi_0 = \phi_0 = A_0 \cos \lambda \tau_M \qquad (5.21)$$

* Ref. 11, page 86.

where A_0 is a constant. Introducing $\psi_1 = \phi_0 + \phi_1$ into equation 5.17 gives

$$\psi_{1\tau\tau} + \lambda^2 \psi_1 = (\lambda^2 - 1)\phi_0 - \sigma \phi_0^3 + \sigma q_0 \cos \lambda \tau_M$$
$$+ \left[(\lambda^2-1)\phi_1 - \sigma(\phi_1^3 + 3\phi_1^2 \phi_0 + 3\phi_1 \phi_0^2)\right]. \tag{5.22}$$

The terms in square brackets in equation 5.22 are second order terms and as such may be neglected. Then substituting equation 5.21 into equation 5.22 and expanding the cosine cubed term

$$\psi_{1\tau\tau} + \lambda^2 \psi_1 = (\lambda^2 - 1) A_0 \cos \lambda \tau_M + \sigma q_0 \cos \lambda \tau_M$$
$$- A_0^3 \sigma \frac{3}{4} \cos \lambda \tau_M - \sigma \frac{A_0^3}{4} \cos 3\lambda \tau_M . \tag{5.23}$$

If the coefficient of $\cos \lambda \tau_M$ is non-zero then the solution to equation 5.23 will contain a secular term which is of the form $\tau_M \cos \lambda \tau_M$. Since this would be a non-periodic solution it is not acceptable and is prohibited by setting

$$\left[(\lambda^2 - 1) A_0 + \sigma q_0 - \frac{3}{4} \sigma A_0^3 \right] = 0. \tag{5.24}$$

Solving for λ^2

$$\lambda^2 = 1 + \frac{3}{4} \sigma A_0^2 - \frac{\sigma q_0}{A_0} . \tag{5.25}$$

The solution of equation 5.23 is

$$\psi_1(\tau_M) = A_1 \cos \lambda \tau_M + \frac{\sigma A_0^3}{32 \lambda^2} \cos 3\lambda \tau_M . \tag{5.26}$$

According to Stoker* the choice of setting $A_1 \equiv A_0$ is "decisive" at this point and the iteration may proceed always applying this principle. If $A_1 \equiv A_0$ at this point, then

$$\psi_1(\tau_M) = A_0 \cos\lambda\tau_M + \left[\frac{\sigma A_0^3}{1+\frac{3}{4}\sigma A_0^2 - \sigma q_0/A_0}\right]\cos 3\lambda\tau_M. \quad (5.27)$$

Curves of $|A_0|$ as a function of λ, for $\sigma = 0.01$, are shown in figure 14. Note that the central curve marked $q_0 = 0$ in figure 14 depicts the free vibration dependence of λ on A_0 as determined by the "Duffing" iteration scheme. To test the range of validity of equation 5.25 (with $q_0 = 0$) it should be compared with the results of equation 5.12 using $\psi_{MAX} = A_0$.

When the response curve has multiple values of amplitude which correspond to a single value of the driving frequency, or in the notation used here, to a single value of λ, the possibility of a jump phenomena may exist. The jump causes the amplitude of vibration to change dramatically without a change in driving frequency. But it is not an instability in the sense of unbounded amplitude; indeed, it reduces the amplitude of vibration. T. K. Caughey has shown** that a necessary and sufficient condition for the existence of the jump phenomena is that the amplitude frequency response curves possess a vertical tangent. In addition he has shown that the "jump instability" cannot occur if the state of response corresponds to a point lying outside the region bounded by the loci

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* Ref. 11, page 86.
** Ref. 12.

of vertical tangents. The loci of vertical tangency in the limiting case of vanishingly small damping are determined by the relations

$$\begin{aligned} \lambda^2 - \tfrac{3}{4}\,\sigma A_o^2 &= 1\,, \\ \lambda^2 - \tfrac{9}{4}\,\sigma A_o^2 &= 1\,, \end{aligned} \qquad (5.28)$$

which both come to the value $\lambda = 1$ for $A_o = 0$.

5.7 Stability of Harmonic Solutions

In the solution of nonlinear differential equations the criterion of stability is the following: let $\psi(\tau)$ be a solution of a nonlinear differential equation; let $\psi'(\tau) = \psi(\tau) + \delta\psi(\tau)$ be another solution of the same equation; and let $\delta\psi(\tau)$ be arbitrarily small at some time τ_o. If $\delta\psi(\tau)$ continues to be arbitrarily small for all other values of τ, then the solution $\psi(\tau)$ is said to be stable, otherwise it is unstable.

If this criterion is applied to equation 5.17 with $\psi(\tau) = A_o \cos(\lambda\tau)$, the equation governing $\delta\psi(\tau)$ is the Mathieu equation,

$$\delta\psi_{\tau\tau} + \left[\left(1 + \frac{3\sigma A_o^2}{2}\right) + \left(\frac{3\sigma A_o^2}{2}\right)\cos 2\lambda\tau_M\right]\delta\psi = 0\,. \qquad (5.29)$$

The harmonic solution $\psi(\tau)$ was obtained under the assumption that σ was small, therefore this assumption must be carried into equation 5.29. In addition λ must satisfy equation 5.25. Combinations of λ and A_o for which the solution to equation 5.29 is bounded for all values of τ_M determine the stability regions for the solution $\psi(\lambda, A_o)$. For convenience of notation the following transformation is made

$$
\begin{aligned}
z &= 2\lambda \tau_M , \\
4\lambda^2 \bar{\delta} &= 1 + \frac{3\sigma A_0^2}{2} , \\
4\lambda^2 \bar{\epsilon} &= \frac{3\sigma A_0^2}{2} ,
\end{aligned}
\tag{5.30}
$$

then equation 5.29 becomes

$$
\delta\psi_{zz} + (\bar{\delta} + \bar{\epsilon}\cos z)\,\delta\psi = 0. \tag{5.31}
$$

If equation 5.25 is used with equation 5.30 to eliminate λ in the expressions for $\bar{\delta}$ and $\bar{\epsilon}$ then

$$
\begin{aligned}
\bar{\delta} &= \frac{1 + \frac{3\sigma A_0^2}{2}}{4\left[1 + \frac{3}{4}\sigma A_0^2 - \frac{\sigma q_0}{A_0}\right]} , \\
\bar{\epsilon} &= \frac{\frac{3\sigma A_0^2}{2}}{4\left[1 + \frac{3}{4}\sigma A_0^2 - \frac{\sigma q_0}{A_0}\right]} .
\end{aligned}
\tag{5.32}
$$

Since σ is a small quantity, equations 5.32 may be put in the form

$$
\begin{aligned}
\bar{\delta} &= \tfrac{1}{4}\left[1 + \left(\tfrac{3}{4}A_0^2 + \tfrac{q_0}{A_0}\right)\sigma\right], \\
\bar{\epsilon} &= \tfrac{1}{4}\left[\tfrac{3}{2}A_0^2\sigma\right].
\end{aligned}
\tag{5.33}
$$

Due to the smallness of σ and hence of $\bar{\epsilon}$, an approximate representation of the stability boundaries may be obtained, as shown in figure 15.

5.8 Subharmonic Response

Since the subharmonic response of the Duffing equation is well documented for a subharmonic of order 1/3, the details of the derivation will not be given here. Consider equation 3.8 written in the form

$$\psi_{\tau\tau} + \Omega_{LM}^2 \left[\psi + \sigma\psi^3\right] = Q_0 \cos\tau \tag{5.34}$$

where $\tau = \omega t$ and $\Omega_{LM}^2 = \omega_{LM}^2/\omega^2$. It is desired to find the condition under which a solution of the form

$$\psi = \psi_{1/3}\cos\left(\tfrac{1}{3}\tau\right) + \psi_1 \cos\tau \tag{5.35}$$

can exist. Upon introducing equation 5.35 into equation 5.34 and dropping of higher order terms (which include higher harmonics) the following conditions are found to be necessary if a harmonic of 1/3 order is to be obtained

$$\begin{aligned}(\omega_{LM}^2 - \omega^2)\psi_{1/3} + \tfrac{3}{4}\omega_{LM}^2\sigma\left(\psi_{1/3}^3 + \psi_{1/3}^2\psi_1 + 2\psi_{1/3}\psi_1^2\right) &= 0 \\ (\omega_{LM}^2 - \omega^2)\psi_1 + \tfrac{1}{4}\omega_{LM}^2\sigma\left(\psi_{1/3}^3 + 6\psi_{1/3}^2\psi_1 + 3\psi_1^3\right) &= \omega^2 Q_0 .\end{aligned} \tag{5.36}$$

These may be rewritten in the form

$$\begin{aligned}\omega^2 &= 9\omega_{LM}^2 + \tfrac{27}{4}\sigma\omega_{LM}^2\left(\psi_{1/3}^2 + \psi_{1/3}\psi_1 + 2\psi_1^2\right) \\ -8\psi_1 &= \omega^2 Q_0 + (\omega^2 - 9\omega_{LM}^2)\psi_1 - \tfrac{1}{4}\sigma\omega_{LM}^2\left(\psi_{1/3}^3 + 6\psi_{1/3}^2\psi_1 + 3\psi_1^3\right)\end{aligned} \tag{5.37}$$

Now if ω^2 is eliminated and the iteration of these equations is started with

$$\omega = (3)^{1/2}\,\omega_{LM} \quad \text{and} \quad \psi_1 = -\frac{\omega^2 Q_0}{8} = \Psi$$

the condition for the existence of a subharmonic of order 1/3 is

$$\omega > 3\omega_{LM}\left(1 + \tfrac{21}{16}\sigma\Psi\right)^{1/2}. \tag{5.38}$$

This subharmonic vibration develops through the branching of the harmonic vibration when

$$\psi_1 = \Psi - \tfrac{51}{32}\left(\omega_{LM}^2\sigma\right)\Psi^3. \tag{5.39}$$

5.9 Ultraharmonic of Order Two

The surprising observation that an ultraharmonic of order 2 can exist in a nonlinear system of the Duffing type has been verified by T. K. Caughey. In order to study the conditions under which such a solution might exist in the response of a cylindrical shell, consider the equation

$$\psi_{\tau\tau} + \psi + \chi(R)\,\psi^3 = q_0 \cos\tfrac{1}{2}\tau_M\,, \tag{5.40}$$

where $\tau_M = \omega_{LM} t$ and $\chi(R)$ is considered to be small in comparison to unity. To find the ultraharmonic of order 2 in the system represented by equation 5.40, it is necessary to look for a motion of the form

$$\psi = \psi_0\left[\alpha\cos\tau_M + \beta\cos\tfrac{1}{2}\tau_M \pm 1\right]. \tag{5.41}$$

If equation 5.41 is introduced into equation 5.40 and higher harmonics neglected, the following result is obtained

$$\left\{\chi \psi_0^3\left[\frac{3}{4}\alpha^3 + \frac{3\alpha\beta^2}{2} + 3\alpha + \frac{3\beta^2}{2}\right]\right\}\cos\tau_M$$

$$+\left\{\frac{3}{4}\beta\psi_0 + \chi\psi_0^3\left[\frac{3}{4}\beta^3 + \frac{3\alpha\beta}{2} + 3\beta\right]\right\}\cos\tfrac{1}{2}\tau_M \tag{5.42}$$

$$+\psi_0 + \chi\psi_0^3\left\{1 + \frac{3\alpha\beta}{4} + \frac{3\beta^2}{2}\right\} = q_0\cos\tfrac{1}{2}\tau_M .$$

For a nontrivial solution, the following conditions are obtained from equation 5.42

$$1 + \chi\left[1 + \frac{3}{4}\alpha\beta^2 + \frac{3}{2}\beta^2\right] = 0, \tag{5.43a}$$

$$\frac{3}{4}\beta\psi_0 + \chi\left[\frac{3}{4}\beta^3 + \frac{3}{2}\alpha^2\beta + 3\beta\right]\psi_0^3 = q_0, \tag{5.43b}$$

$$\frac{3}{4}\alpha^3 + \frac{3\alpha\beta^2}{2} + 3\alpha + \frac{3}{2}\beta^2 = 0. \tag{5.43c}$$

Equation 5.43a yields

$$\alpha = -2\left[1 + \frac{2}{3\beta^2}\left(\frac{1+\chi}{\chi}\right)\right] \tag{5.44}$$

and with this, equation 5.43c becomes

$$\left(\frac{\chi}{1+\chi}\right)^3(\beta^2)^4 + \left[8\left(\frac{\chi}{1+\chi}\right)^3 + \frac{1}{2}\left(\frac{\chi}{1+\chi}\right)^2\right](\beta^2)^3$$

$$+\frac{56}{3}\left(\frac{\chi}{1+\chi}\right)^2(\beta^2)^2 - (\beta^2)16\left(\frac{\chi}{1+\chi}\right) - \frac{32}{27} = 0 . \tag{5.45}$$

Use of equation 5.44 to eliminate α from equation 5.43b gives

$$\frac{3}{4}\beta\phi_0 + \chi\left\{\frac{3}{4}\beta^3 + \frac{3}{2}\beta^{-3}\left[2\beta^2 + \frac{4}{3}\left(\frac{1+\chi}{\chi}\right)\right]^2 + 3\beta\right\}\phi_0^3 = q_0 . \tag{5.46}$$

Now since $\chi \ll 1$ has already been assumed it may be used to obtain an approximate solution. Thus equation 5.45 becomes

$$\beta^2 = \frac{2}{27}\left(\frac{1}{\chi}\right) . \tag{5.47}$$

Using this in equation 5.46 and using $\chi \ll 1$ produces

$$\phi_0 = \left(\frac{27}{2}\right)^{1/6} \chi^{1/2} q_0^{1/3} . \tag{5.48}$$

Again using equation 5.47 with equation 5.44 α is found, for small χ, to be

$$\alpha = -20. \tag{5.49}$$

So if $\chi \ll 1$, then

$$\phi = \left(\frac{27}{2}\right)^{1/6} \chi^{1/2} q_0^{1/3}\left[1 - 20\cos\tau_M + \left(\frac{2}{27}\right)^{1/2}\chi^{-1/2}\cos\tfrac{1}{2}\tau_M\right]. \tag{5.50}$$

In terms of the deformation response problem of a shell, equation 5.50 indicates that when a circular cylindrical shell is subjected to a forcing function at one-half the natural frequency of the equivalent linear system, it is possible to get a response that includes a pure dilatation as well as the fundamental and a large response of the ultrahamonic of order 2. The fundamental and the ultraharmonic are out of phase. The fundamental ($\cos \frac{1}{2} \tau_{N}$) is of the order of unity while the ultraharmonic and pure dilatation are of order $\varkappa^{1/2}$.

CHAPTER VI

TWO MODE CYLINDER ANALYSIS

6.1 Equations of Motion

The equations of motion are not determined by the number of modes used. Therefore, the equations of motion are the same here as used in Chapter IV, that is, the Morley equations.

$$\left(\frac{1}{Eh}\right) \nabla^4 F = \frac{1}{a} S_{xx} + \left[S_{xy}^2 - S_{xx} S_{yy} \right],$$

$$D\left(\nabla^2 + \frac{1}{a^2}\right)^2 S = P(x,y;t) - \bar{m}\ddot{S} + N_{xi} S_{xx} + N_{yi} S_{yy} \tag{6.1}$$

$$- \frac{1}{a} F_{xx} + \left[F_{xx} S_{yy} + F_{yy} S_{xx} - 2 F_{xy} S_{xy} \right].$$

6.2 Boundary Conditions of the Two Mode Analysis

The boundary conditions of section 5.2 are applied separately to each of the two mode functions $S(m, n)$ and $S(\mu, \nu)$ and to the composite stress function F which is associated with them.

6.3 Two Mode Displacements

The modes to be used are

$$S(m,n) = A(t) \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{ny}{a}\right),$$

$$S(\mu, \nu) = B(t) \sin\left(\frac{\mu\pi x}{L}\right) \cos\left(\frac{\nu y}{a}\right). \tag{6.2}$$

The choice of the mode parameters (m, μ, n, ν) is arbitrary and will be considered after the final equations governing A and B have been obtained.

6.4 Two Mode Stress Function

If the total displacement mode is defined as S_{TOT} = $S(m,n) + S(\mu,\nu)$ and S_{TOT} is used with the compatibility equation, the associated stress function F is found in the same manner as previously. The stress function found is

$$
\begin{aligned}
F = &- A(t) f_1 \sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{ny}{a}\right) - B(t) f_2 \sin\left(\frac{\mu\pi x}{L}\right)\cos\left(\frac{\nu y}{a}\right) \\
&+ A^2(t) f_3 \cos\left(\frac{2m\pi x}{L}\right) - A^2(t) f_4 \cos\left(\frac{2ny}{a}\right) \\
&+ B^2(t) f_5 \cos\left(\frac{2\mu\pi x}{L}\right) - B^2(t) f_6 \cos\left(\frac{2\nu y}{a}\right) \\
&+ A(t) B(t) f_7 \cos\frac{(m+\mu)\pi x}{L} \cos\frac{(\nu-n)y}{a} \\
&- A(t) B(t) f_8 \cos\frac{(m-\mu)\pi x}{L} \cos\frac{(\nu+n)y}{a} \\
&+ A(t) B(t) f_9 \cos\frac{(m-\mu)\pi x}{L} \cos\frac{(\nu-n)y}{a} \\
&- A(t) B(t) f_{10} \cos\frac{(m+\mu)\pi x}{L} \cos\frac{(\nu+n)y}{a} ,
\end{aligned}
\tag{6.3}
$$

where

$$
\begin{aligned}
f_1 &= \left(\frac{Eh}{a}\right)\left(\frac{m\pi}{L}\right)^2\left[\left(\frac{m\pi}{L}\right)^2+\left(\frac{n}{a}\right)^2\right]^{-2}, \\
f_2 &= \left(\frac{Eh}{a}\right)\left(\frac{\mu\pi}{L}\right)^2\left[\left(\frac{\mu\pi}{L}\right)^2+\left(\frac{\nu}{a}\right)^2\right]^{-2}, \\
f_3 &= \left(\frac{1}{2}\right)^5\left(\frac{m\pi}{L}\right)^{-2}\left(\frac{n}{a}\right)^2 Eh , \\
f_4 &= \left(\frac{1}{2}\right)^5\left(\frac{m\pi}{L}\right)^2\left(\frac{n}{a}\right)^{-2} Eh,
\end{aligned}
\tag{6.3a}
$$

$$f_5 = \left(\tfrac{1}{2}\right)^5 \left(\tfrac{\nu}{a}\right)^2 \left(\tfrac{\mu\pi}{L}\right)^{-2} E h,$$

$$f_6 = \left(\tfrac{1}{2}\right)^5 \left(\tfrac{\nu}{a}\right)^{-2} \left(\tfrac{\mu\pi}{L}\right)^2 E h,$$

$$f_7 = \left[\frac{(m+\mu)^2\pi^2}{L^2} + \frac{(\nu-n)^2}{a^2} \right]^{-2} E h g_1,$$

$$f_8 = \left[\frac{(m-\mu)^2\pi^2}{L^2} + \frac{(\nu+n)^2}{a^2} \right]^{-2} E h g_1,$$

$$f_9 = \left[\frac{(m-\mu)^2\pi^2}{L^2} + \frac{(\nu-n)^2}{a^2} \right]^{-2} E h g_2, \qquad (6.3a) \text{ (Cont'd)}$$

$$f_{10} = \left[\frac{(m+\mu)^2\pi}{L^2} + \frac{(\nu+n)^2}{a^2} \right]^{-2} E h g_2,$$

$$g_1 = \tfrac{1}{2}\left(\tfrac{m\pi}{L}\right)\left(\tfrac{\mu\pi}{L}\right)\left(\tfrac{n}{a}\right)\left(\tfrac{\nu}{a}\right) + \tfrac{1}{4}\left[\left(\tfrac{m\pi}{L}\right)^2\left(\tfrac{\nu}{a}\right)^2 + \left(\tfrac{\mu\pi}{L}\right)^2\left(\tfrac{n}{a}\right)^2\right],$$

$$g_2 = \tfrac{1}{2}\left(\tfrac{m\pi}{L}\right)\left(\tfrac{\mu\pi}{L}\right)\left(\tfrac{n}{a}\right)\left(\tfrac{\nu}{a}\right) - \tfrac{1}{4}\left[\left(\tfrac{m\pi}{L}\right)^2\left(\tfrac{\nu}{a}\right) + \left(\tfrac{\mu\pi}{L}\right)^2\left(\tfrac{n}{a}\right)^2\right].$$

6.5 Galerkin Integration.

If equation 4.3 is substituted into the equilibrium equation the resulting equation takes the form

$$\bar{m}\ddot{A}H_1 + \bar{m}\ddot{B}H_2 + AD\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{a}\right)^2 + \left(\frac{1}{a}\right)^2\right]^2 H_1 + BD\left[\left(\frac{\mu\pi}{L}\right)^2\right.$$

$$\left. + \left(\frac{\nu}{a}\right)^2 + \left(\frac{1}{a}\right)^2\right]^2 + N_{xi}\left(\frac{m\pi}{L}\right)^2 H_1 A + N_{xi}\left(\frac{\mu\pi}{L}\right)^2 H_2 B + H_i N_{yi}\left(\frac{n}{a}\right)^2 A$$

$$+ N_{yi}\left(\frac{\nu}{a}\right)^2 H_2 A + \frac{1}{a}\left[Af_1\left(\frac{m\pi}{L}\right)^2 H_1 + Bf_2\left(\frac{\mu\pi}{L}\right)^2 H_2 - A^2 f_3\left(\frac{2m\pi}{L}\right)^2 H_3 - B^2 f_5\left(\frac{2\mu\pi}{L}\right)^2 H_4\right.$$

$$- ABf_7\frac{(m+\mu)^2\pi^2}{L^2}H_5 + ABf_8\frac{(m-\mu)^2\pi^2}{L^2}H_6 - ABf_9\frac{(m-\mu)^2\pi^2}{L^2}H_7$$

$$\left. + ABf_{10}\frac{(m+\mu)^2\pi^2}{L^2}H_8\right] + \left(\frac{m\pi}{L}\right)^2 A\left[Af_1\left(\frac{n}{a}\right)^2 H_9 + Bf_2\left(\frac{\nu}{a}\right)^2 H_{10}\right.$$

$$+ \left(\frac{2n}{a}\right)^2 A^2 f_4 H_{11} + \left(\frac{2\nu}{a}\right)^2 B^2 f_6 H_{12} - ABf_7\frac{(\nu-n)^2}{a^2}H_{13}$$

$$\left. + \frac{(\nu+n)^2}{a^2}ABf_8 H_{14} - ABf_9\frac{(\nu-n)^2}{a^2}H_{15} + ABf_{10}\frac{(\nu+n)^2}{a^2}H_{16}\right] \qquad (6.4)$$

$$+ \left(\frac{\mu\pi}{L}\right)^2 B\left[Af_1\left(\frac{n}{a}\right)^2 H_{10} + Bf_2\left(\frac{\nu}{a}\right)^2 H_{17} + A^2 f_4\left(\frac{2n}{a}\right)^2 H_{18}\right.$$

$$+ B^2 f_6\left(\frac{2\nu}{a}\right)^2 H_{19} - ABf_7\frac{(\nu-n)^2}{a^2}H_{20} + ABf_8\frac{(\nu+n)^2}{a^2}H_{21}$$

$$\left. - ABf_9\frac{(\nu-n)^2}{a^2}H_{22} + ABf_{10}\frac{(\nu+n)^2}{a^2}H_{23}\right]$$

$$+ \left(\frac{n}{a}\right)^2 A\left[Af_1\left(\frac{m\pi}{L}\right)^2 H_9 + Bf_2\left(\frac{\mu\pi}{L}\right)^2 H_{10} - A^2 f_3\left(\frac{2m\pi}{L}\right)^2 H_{24}\right.$$

$$- B^2 f_5\left(\frac{2\mu\pi}{L}\right)^2 H_{25} - ABf_7\frac{(m+\mu)^2\pi^2}{L^2}H_{13} + ABf_8\frac{(m-\mu)^2\pi^2}{L^2}H_{14}$$

$$\left. - ABf_9\frac{(m+\mu)^2\pi^2}{L^2}H_{15} + ABf_{10}\frac{(m+\mu)^2\pi^2}{L^2}H_{16}\right] + \left(\frac{\nu}{a}\right)^2 B\left[Af_1\left(\frac{m\pi}{L}\right)^2 H_{10}\right.$$

$$+ B f_2 \left(\frac{\mu\pi}{L}\right)^2 H_{17} - A^2 f_3 \left(\frac{2m\pi}{L}\right)^2 H_{18} - B^2 f_5 \left(\frac{2\mu\pi}{L}\right)^2 H_{26}$$

$$- AB f_7 \frac{(m+\mu)^2\pi}{L^2} H_{20} + AB f_8 \frac{(m-\mu)^2\pi^2}{L^2} H_{21}$$

$$- AB f_9 \frac{(m-\mu)^2\pi^2}{L^2} H_{22} + AB f_{10} \frac{(m+\mu)^2\pi^2}{L^2} H_{23} \Big]$$

$$- 2 \left(\frac{m\pi}{L}\right)\left(\frac{n}{a}\right) A \Big[-\left(\frac{m\pi}{L}\right)\left(\frac{n}{a}\right) f_1 A H_{27} + B f_1 \left(\frac{\mu\pi}{L}\right)\left(\frac{\nu}{a}\right) H_{28}$$

$$+ AB f_7 \frac{(m+\mu)\pi}{L} \cdot \frac{(\nu-n)}{a} H_{29} - AB f_8 \frac{(m-\mu)\pi}{L} \cdot \frac{(\nu+n)}{a} H_{20} \qquad \text{(6.4) (Cont'd)}$$

$$+ AB f_9 \frac{(m-\mu)\pi}{L} \frac{(\nu-n)}{a} H_{31} - AB f_{10} \frac{(m+\mu)\pi}{L} \frac{(\nu+n)}{a} H_{32} \Big]$$

$$- \left(\frac{\mu\pi}{L}\right)\left(\frac{\nu}{a}\right) B \Big[-\left(\frac{m\pi}{L}\right)\left(\frac{n}{a}\right) f_1 A H_{29} + \left(\frac{\mu\pi}{L}\right)\left(\frac{\nu}{a}\right) B f_1 H_{33} + AB f_7 \frac{(m+\mu)\pi}{L} \frac{(\nu-n)}{a} H_{34}$$

$$- AB f_8 \frac{(m-\mu)\pi}{L} \frac{(\nu+n)}{a} H_{35} + AB f_9 \frac{(m-\mu)\pi}{L} \frac{(\nu-n)}{a} H_{36} - AB f_{10} \frac{(m+\mu)\pi}{L} \frac{(\nu+n)}{a} H_{37} \Big] = P.$$

In this the quantities H_i are

$$H_1 = \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n y}{a}\right),$$

$$H_2 = \sin\left(\frac{\mu\pi x}{L}\right) \cos\left(\frac{\nu y}{a}\right),$$

$$H_3 = \cos\left(\frac{2m\pi x}{L}\right), \qquad \text{(6.5)}$$

$$H_4 = \cos\left(\frac{2\mu\pi x}{L}\right),$$

$$H_5 = \cos \frac{(m+\mu)\pi x}{L} \cos \frac{(\nu-n)y}{a},$$

$$H_6 = \cos \frac{(m-\mu)\pi x}{L} \cos \frac{(\nu+n)y}{a},$$

$$H_7 = \cos \frac{(m-\mu)\pi x}{L} \cos \frac{(\nu+n)y}{a},$$

$$H_8 = \cos\frac{(m+u)\pi x}{L}\cos\frac{(v+n)y}{a},$$

$$H_9 = \sin^2\left(\frac{m\pi x}{L}\right)\cos^2\left(\frac{ny}{a}\right),$$

$$H_{10} = \sin\left(\frac{u\pi x}{L}\right)\cos\left(\frac{vy}{a}\right)\sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{ny}{a}\right),$$

$$H_{11} = \cos\left(\frac{2ny}{a}\right)\sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{ny}{a}\right),$$

$$H_{12} = \cos\left(\frac{2vy}{a}\right)\sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{ny}{a}\right),$$

$$H_{13} = \cos\frac{(m+u)\pi x}{L}\cos\frac{(v-n)y}{a}\sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{ny}{a}\right),$$

$$H_{14} = \cos\frac{(m-u)\pi x}{L}\cos\frac{(v+n)y}{a}\sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{ny}{a}\right),$$

$$H_{15} = \cos\frac{(m-u)\pi x}{L}\cos\frac{(v-n)y}{a}\sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{ny}{a}\right),$$

$$H_{16} = \cos\frac{(m+u)\pi x}{L}\cos\frac{(v+n)y}{a}\sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{ny}{a}\right), \qquad \text{(6.5) (Cont'd)}$$

$$H_{17} = \sin^2\left(\frac{u\pi x}{L}\right)\cos^2\left(\frac{vy}{a}\right),$$

$$H_{18} = \cos\left(\frac{2m\pi x}{L}\right)\sin\left(\frac{u\pi x}{L}\right)\cos\left(\frac{vy}{a}\right),$$

$$H_{19} = \cos\left(\frac{2vy}{a}\right)\sin\left(\frac{u\pi x}{L}\right)\cos\left(\frac{vy}{a}\right),$$

$$H_{20} = \cos\frac{(m+u)\pi x}{L}\cos\frac{(v-n)y}{a}\sin\left(\frac{u\pi x}{L}\right)\cos\left(\frac{vy}{a}\right),$$

$$H_{21} = \cos\frac{(m-u)\pi x}{L}\cos\frac{(v+n)y}{a}\sin\left(\frac{u\pi x}{L}\right)\cos\left(\frac{vy}{a}\right),$$

$$H_{22} = \cos\frac{(m-u)\pi x}{L}\cos\frac{(v-n)y}{a}\cos\left(\frac{u\pi x}{L}\right)\cos\left(\frac{vy}{a}\right),$$

$$H_{23} = \cos\frac{(m+u)\pi x}{L}\cos\frac{(v+n)y}{a}\sin\left(\frac{u\pi x}{L}\right)\cos\left(\frac{vy}{a}\right),$$

$$H_{24} = \cos\left(\frac{2m\pi X}{L}\right)\sin\left(\frac{m\pi X}{L}\right)\cos\left(\frac{nY}{a}\right),$$

$$H_{25} = \cos\left(\frac{2\mu\pi X}{L}\right)\sin\left(\frac{m\pi X}{L}\right)\cos\left(\frac{nY}{a}\right),$$

$$H_{26} = \cos\left(\frac{2\mu\pi X}{L}\right)\sin\left(\frac{\mu\pi X}{L}\right)\cos\left(\frac{\nu Y}{a}\right),$$

$$H_{27} = \cos^2\left(\frac{m\pi X}{L}\right)\sin^2\left(\frac{nY}{a}\right),$$

$$H_{28} = \cos^2\left(\frac{\mu\pi X}{L}\right)\sin^2\left(\frac{\nu Y}{a}\right),$$

$$H_{29} = \sin\frac{(m+\mu)\pi X}{L}\sin\frac{(\nu-n)Y}{a}\cos\left(\frac{m\pi X}{L}\right)\sin\left(\frac{nY}{a}\right),$$

$$H_{30} = \sin\frac{(m-\mu)\pi X}{L}\sin\frac{(\nu+n)Y}{a}\cos\left(\frac{m\pi X}{L}\right)\sin\left(\frac{nY}{a}\right), \qquad \begin{matrix}(6.5)\\ \text{(Cont'd)}\end{matrix}$$

$$H_{31} = \sin\frac{(m-\mu)\pi X}{L}\sin\frac{(\nu-n)Y}{a}\cos\left(\frac{m\pi X}{L}\right)\sin\left(\frac{nY}{a}\right),$$

$$H_{32} = \sin\frac{(m+\mu)\pi X}{L}\sin\frac{(\nu+n)Y}{a}\cos\left(\frac{m\pi X}{L}\right)\sin\left(\frac{nY}{a}\right),$$

$$H_{33} = \cos\left(\frac{\mu\pi X}{L}\right)\cos\left(\frac{m\pi X}{L}\right)\sin\left(\frac{\nu Y}{a}\right)\sin\left(\frac{nY}{a}\right),$$

$$H_{34} = \sin\frac{(m+\mu)\pi X}{L}\sin\frac{(\nu-n)Y}{a}\cos\left(\frac{\mu\pi X}{L}\right)\sin\left(\frac{\nu Y}{a}\right),$$

$$H_{35} = \sin\frac{(m-\mu)\pi X}{L}\sin\frac{(\nu+n)Y}{a}\cos\left(\frac{\mu\pi X}{L}\right)\sin\left(\frac{\nu Y}{a}\right),$$

$$H_{36} = \sin\frac{(m-\mu)\pi X}{L}\sin\frac{(\nu-n)Y}{a}\cos\left(\frac{\mu\pi X}{L}\right)\sin\left(\frac{\nu Y}{a}\right),$$

$$H_{37} = \sin\frac{(m+\mu)\pi X}{L}\sin\frac{(\nu+n)Y}{a}\cos\left(\frac{\mu\pi X}{L}\right)\sin\left(\frac{\nu Y}{a}\right).$$

When equation 6.4 is weighted with $\sin\left(\frac{m\pi X}{L}\right)\cos\left(\frac{nY}{a}\right)$ and integrated over the interval $0 \le X \le L$, $-\pi a \le Y \le \pi a$ the integrated contributions of the H_i are used to obtain the first Galerkin equation in the form

$$\bar{m}\,\ddot{A} + \bar{m}\,\omega_{LM}^2(m,n)\,A + \left(\frac{Eh}{4}\right)\left[\left(\frac{m\pi}{L}\right)^4 - \left(\frac{n}{a}\right)^4\right]A^3$$
$$+ \left(\frac{1}{2}\right)^4 Eh\left(\frac{m\pi}{L}\right)^4\left(\frac{n}{a}\right)^4 \zeta_I\, AB^2 = \left(\mathbb{P}(t)\big/(La\pi/2)\right), \tag{6.6}$$

where

$$\zeta_I = \left(\frac{\mu}{m}\right)^2\Bigg[\left(1-\frac{\nu}{n}\right)^2\Bigg\{\frac{2\left(\frac{\mu}{m}\right)\left(\frac{\nu}{n}\right)+\left(\frac{\mu}{m}\right)^2+\left(\frac{\nu}{n}\right)^2}{\left[\left(\frac{m\pi}{L}\right)^2\left(1+\frac{\mu}{m}\right)^2+\left(\frac{n}{a}\right)^2\left(1-\frac{\nu}{n}\right)^2\right]^2}+\frac{2\left(\frac{\mu}{m}\right)\left(\frac{\nu}{n}\right)-\left(\frac{\mu}{m}\right)^2-\left(\frac{\nu}{n}\right)^2}{\left[\left(\frac{m\pi}{L}\right)^2\left(1-\frac{\mu}{m}\right)^2+\left(\frac{n}{a}\right)^2\left(1-\frac{\nu}{n}\right)^2\right]^2}\Bigg\}$$

$$-\left(1+\frac{\nu}{n}\right)^2\Bigg\{\frac{2\left(\frac{\mu}{m}\right)\left(\frac{\nu}{n}\right)+\left(\frac{\mu}{m}\right)^2+\left(\frac{\nu}{n}\right)^2}{\left[\left(\frac{m\pi}{L}\right)^2\left(1-\frac{\mu}{m}\right)^2+\left(\frac{n}{a}\right)^2\left(1+\frac{\nu}{n}\right)^2\right]^2}+\frac{2\left(\frac{\mu}{m}\right)\left(\frac{\nu}{n}\right)-\left(\frac{\mu}{m}\right)^2-\left(\frac{\nu}{n}\right)^2}{\left[\left(\frac{m\pi}{L}\right)^2\left(1+\frac{\mu}{m}\right)^2+\left(\frac{n}{a}\right)^2\left(1+\frac{\nu}{n}\right)^2\right]^2}\Bigg\}\Bigg]$$

$$+\left(\frac{\nu}{n}\right)^2\Bigg[\left(1-\frac{\mu}{m}\right)^2\Bigg\{\frac{2\left(\frac{\mu}{m}\right)\left(\frac{\nu}{n}\right)+\left(\frac{\mu}{m}\right)^2+\left(\frac{\nu}{n}\right)^2}{\left[\left(\frac{m\pi}{L}\right)^2\left(1-\frac{\mu}{m}\right)^2+\left(\frac{n}{a}\right)^2\left(1+\frac{\nu}{n}\right)^2\right]^2}-\frac{2\left(\frac{\mu}{m}\right)\left(\frac{\nu}{n}\right)-\left(\frac{\mu}{m}\right)^2-\left(\frac{\nu}{n}\right)^2}{\left[\left(\frac{m\pi}{L}\right)^2\left(1-\frac{\mu}{m}\right)^2+\left(\frac{n}{a}\right)^2\left(1-\frac{\nu}{n}\right)^2\right]^2}\Bigg\}$$

$$+\left(1+\frac{\mu}{m}\right)^2\Bigg\{\frac{2\left(\frac{\mu}{m}\right)\left(\frac{\nu}{n}\right)-\left(\frac{\mu}{m}\right)^2-\left(\frac{\nu}{n}\right)^2}{\left[\left(\frac{m\pi}{L}\right)^2\left(1+\frac{\mu}{m}\right)^2+\left(\frac{n}{a}\right)^2\left(1+\frac{\nu}{n}\right)^2\right]^2}-\frac{2\left(\frac{\mu}{m}\right)\left(\frac{\nu}{n}\right)+\left(\frac{\mu}{m}\right)^2+\left(\frac{\nu}{n}\right)^2}{\left[\left(\frac{m\pi}{L}\right)^2\left(1+\frac{\mu}{m}\right)^2+\left(\frac{n}{a}\right)^2\left(1-\frac{\nu}{n}\right)^2\right]^2}\Bigg\}\Bigg]$$

$$+2\left(\frac{\mu}{m}\right)\Bigg[\left(1-\frac{\mu}{m}\right)\Bigg\{\left(1+\frac{\nu}{n}\right)\frac{2\left(\frac{\mu}{m}\right)\left(\frac{\nu}{n}\right)-\left(\frac{\mu}{m}\right)^2-\left(\frac{\nu}{n}\right)^2}{\left[\left(\frac{m\pi}{L}\right)^2\left(1+\frac{\mu}{m}\right)^2+\left(\frac{n}{a}\right)^2\left(1+\frac{\nu}{n}\right)^2\right]^2}+\left(1-\frac{\nu}{n}\right)\frac{2\left(\frac{\mu}{m}\right)\left(\frac{\nu}{n}\right)+\left(\frac{\mu}{m}\right)^2+\left(\frac{\nu}{n}\right)^2}{\left[\left(\frac{m\pi}{L}\right)^2\left(1-\frac{\mu}{m}\right)^2+\left(\frac{n}{a}\right)^2\left(1-\frac{\nu}{n}\right)^2\right]^2}\Bigg\}$$

$$+\left(1+\frac{\mu}{m}\right)\Bigg\{\left(1+\frac{\nu}{n}\right)\frac{2\left(\frac{\mu}{m}\right)\left(\frac{\nu}{n}\right)-\left(\frac{\mu}{m}\right)^2-\left(\frac{\nu}{n}\right)^2}{\left[\left(\frac{m\pi}{L}\right)^2\left(1+\frac{\mu}{m}\right)^2+\left(\frac{n}{a}\right)^2\left(1+\frac{\nu}{n}\right)^2\right]^2}+\left(1-\frac{\nu}{n}\right)\frac{2\left(\frac{\mu}{m}\right)\left(\frac{\nu}{n}\right)+\left(\frac{\mu}{m}\right)^2+\left(\frac{\nu}{n}\right)^2}{\left[\left(\frac{m\pi}{L}\right)^2\left(1+\frac{\mu}{m}\right)^2+\left(\frac{n}{a}\right)^2\left(1-\frac{\nu}{n}\right)^2\right]^2}\Bigg\}\Bigg].$$

Now if the notations

$$\psi = \left(\frac{16\,n^2}{\pi^2 a}\right)A,$$

$$\Phi = \left(\frac{16\,v^2}{\pi^2 a}\right)B,$$

$$\tau = \omega t,$$

$$\bar{m}\,\omega_{LM}^2 = D\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{a}\right)^2 + \left(\frac{1}{a}\right)^2\right]^2 + N_{xi}\left(\frac{m\pi}{L}\right)^2 + N_{yi}\left(\frac{n}{a}\right)^2$$

$$+\bar{m}\,\omega_*^2, \tag{6.7}$$

$$\bar{m}\,\omega_*^2 = \frac{Eh}{a^2}\left(\frac{m\pi}{L}\right)^4\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{a}\right)^2\right]^{-2},$$

$$\Omega_{LM}^2 = \omega_{LM}^2/\omega^2,$$

$$\Omega_*^2 = \omega_*^2/\omega^2,$$

$$Q_\psi(\tau) = \mathbb{P}(t)\left[\left(\frac{La\pi}{2}\right)(\bar{m}\,\omega^2)(\pi^2 a/16n^2)\right]^{-1},$$

are used the equation obtained from equation 6.6 is

$$\psi_{\tau\tau} + \Omega_{LM}^2(m,n)\,\psi + \left(\frac{Eh}{a^2}\right)\left(\frac{\pi^2 a^2}{16n^2}\right)^2\left(\frac{1}{\bar{m}\omega^2}\right)\frac{1}{4}\left[\left(\frac{m\pi}{L}\right)^4 - \left(\frac{n}{a}\right)^4\right]\psi^3$$

$$+\left(\frac{1}{2}\right)^4\left(\frac{1}{\bar{m}\omega^2}\right)\left(\frac{Eh}{a^2}\right)\left(\frac{m\pi}{L}\right)^4\left(\frac{\pi^2 a^2}{16v^2}\right)^2 \zeta_I\,\psi\,\Phi^2 = Q_\psi(\tau). \tag{6.8}$$

The second equation may be obtained by interchange of the mode number variables (m,n) with (μ,v) and vice versa. If this is done and a more simplified notation is used then the two mode equations are

$$\psi_{\tau\tau} + \Omega_{LM}^2(m,n)\,\psi + \beta_\psi \psi^3 + \gamma_\psi \psi \Phi^2 = Q_\psi(\tau),$$

$$\Phi_{\tau\tau} + \Omega_{LM}^2(\mu,n)\,\Phi + \beta_\phi \Phi^3 + \gamma_\phi \Phi \psi^2 = Q_\phi(\tau), \qquad (6.9)$$

where

$$\beta_\psi = \left(\frac{Eh}{a^2}\right)\left(\frac{\pi^2 a^2}{16 n^2}\right)^2\left(\frac{1}{\overline{m}\,\omega^2}\right)\left(\frac{1}{4}\right)\left[\left(\frac{m\pi}{L}\right)^4 - \left(\frac{n}{a}\right)^4\right],$$

$$\beta_\phi = \left(\frac{Eh}{a^2}\right)\left(\frac{\pi^2 a^2}{16 \nu^2}\right)^2\left(\frac{1}{\overline{m}\,\omega^2}\right)\left(\frac{1}{4}\right)\left[\left(\frac{\mu\pi}{L}\right)^4 - \left(\frac{\nu}{a}\right)^4\right],$$

$$\gamma_\psi = \left(\frac{1}{2}\right)^4\left(\frac{1}{\overline{m}\,\omega^2}\right)\left(\frac{Eh}{a^2}\right)\left(\frac{m\pi}{L}\right)^4\left(\frac{n}{a}\right)^4\left(\frac{\pi^2 a^2}{16 \nu^2}\right)^2 \zeta_I \qquad (6.10)$$

$$\gamma_\phi = \left(\frac{1}{2}\right)^4\left(\frac{1}{\overline{m}\,\omega^2}\right)\left(\frac{Eh}{a^2}\right)\left(\frac{\mu\pi}{L}\right)^4\left(\frac{\nu}{a}\right)^4\left(\frac{\pi^2 a^2}{16 n^2}\right)^2 \zeta_{II},$$

and ζ_{II} is obtained from equation 6.6a by replacing ν by n, μ by m, and vice versa.

6.6 Weak Coupling

If the natural frequencies $\omega_{LM}(m,n)$ and $\omega_{LM}(\mu,\nu)$ are well separated the coupling of equations 6.9 is very weak unless $\gamma_\psi >> \beta_\psi$ and $\gamma_\phi >> \beta_\phi$. If these coefficients are of the same order of magnitude and if the excitations are periodic with a frequency near $\omega_{LM}(m,n)$, for instance, then equations 6.9 will take on the form

$$\psi_{\tau\tau} + \Omega_{LM}^2(m,n)\,\psi + \beta_\psi \psi^3 = Q_\psi(\omega_{LM}),$$

$$\Phi_{\tau\tau} + \Omega_{LM}^2(\mu,\nu)\,\Phi + \beta_\phi \Phi^3 = Q_\phi(\omega_{LM}), \qquad (6.11)$$

since $\psi^2 >> \Phi^2$ under these conditions (see figure 16). The main response is in the ψ mode which is uncoupled from the influence of the Φ mode. At the same time the response of the Φ mode is excited in part by the ψ mode, but its amplitude is still small compared to its companion mode. If the space distributed of the forcing function $P(x,y;t)$ is such that $Q_\psi >> Q_\Phi$ then equations 6.11 further simplify to

$$\psi_{\tau\tau} + \Omega^2_{LM}(m,n)\psi + \beta_\psi \psi^3 = Q_\psi(\omega_{LM}(m,n)),$$

$$\Phi_{\tau\tau} + \Omega^2_{LM}(\mu,\nu)\Phi + \beta_\phi \Phi^3 = 0 \qquad (6.12)$$

But in this instance the Φ equation admits a solution

$$\Phi \equiv 0. \qquad (6.13)$$

Thus a possible solution is complete decoupling!

Since such a simple solution can exist for modes whose frequencies are well separated, it is fortunate that such problems are of considerable interest. When a mode shape of n-half waves around the circumference is studied, there frequently is associated with this mode one with 2n, 3n or jn (j = 4, 5, . . .) half waves in the circumferential direction. A similar association occurs with the number of waves in the axial direction. Unfortunately this does not assure the separation of the two frequencies under consideration. In figure 17, a set of frequency curves determined for small amplitude, single mode vibrations according to the method of

Arnold and Warburton (13) is presented for one thickness ratio $(h/a) = .004$. This plot shows quite definitely that for $n = 2$, $\left(\frac{m\pi a}{L}\right) = \left(\frac{\mu\pi a}{L}\right) = 2$ and $\nu = 20$, the two modal frequencies are not separated. But for $n = 2$, $\left(\frac{m\pi a}{L}\right) = \left(\frac{\mu\pi a}{L}\right) = 5$, $\nu = 20$ the frequencies are separated by a factor of almost 2. Accordingly, for certain shells it will be possible to achieve a representation similar to equations 6.12. Then the study reduces to that for an uncoupled system.

The coupling may also be weak if γ_ψ and γ_ϕ are small compared to β_ψ and β_ϕ respectively. However, an examination of these four parameters indicates that in general such a relationship is hard to achieve.

6.7 Harmonic Motion

In order to study steady state harmonic response to a harmonic forcing function, equations 6.9 will be rewritten in the following manner

$$\begin{aligned} &\psi_{\tau\tau} + \psi + \beta_\psi\left[\psi^3 + \Upsilon\psi\Phi^2\right] = \beta_\psi \Delta_\psi q_{mn}(\tau), \\ &\phi_{\tau\tau} + \left(\frac{\omega_{LM}^2(\mu,\nu)}{\omega_{LM}^2(m,n)}\right)\Phi + \beta_\psi\left[\Pi\Phi^3 + \Lambda\Phi\psi^2\right] = \beta_\psi \Delta_\phi q_{\mu\nu}(\tau), \end{aligned} \tag{6.13}$$

with the definition

$$\tau \equiv \omega_{LM}(m,n)t\,, \qquad Q_\psi(\tau) \equiv \beta_\psi \Delta_\psi q_{mn}(\tau),$$

$$(\beta_\phi/\beta_\psi) \equiv \Pi\,, \qquad Q_\phi(\tau) \equiv \beta_\psi \Delta_\phi q_{\mu n}(\tau),$$

$$(\gamma_\psi/\beta_\psi) \equiv \Upsilon\,, \qquad q_{mn}(\tau) \equiv \cos(\lambda\tau + \xi_\psi), \tag{6.13a}$$

$$(\gamma_\phi/\beta_\psi) \equiv \Lambda\,, \qquad q_{\mu\nu}(\tau) \equiv \cos(\lambda\tau + \xi_\phi),$$

in which $\beta_\psi << 1$ and Π, Λ, Υ, Δ_ψ, Δ_ϕ are not large compared to unity.

The study of the two modes system will follow the work of T. K. Caughey (12) and so no extensive repetition of his development will be included. The general method is an extension of the Kryloff and Bogliuboff method to the forced response of a two mode nonlinear system. A solution to equations 6.13 is looked for in the form

$$\psi = A_\psi \cos X_\psi\,, \qquad \Phi = A_\phi \cos X_\phi\,,$$

$$\psi_\tau = -\lambda A_\psi \sin X_\psi\,, \qquad \Phi_\tau = -\lambda A_\phi \sin X_\phi\,, \tag{6.14}$$

$$X_\psi = \lambda\tau + \xi_\psi\,, \qquad X_\phi = \lambda\tau + \xi_\phi\,.$$

In the Kryloff-Bogliuboff analysis these solutions are subjected to the conditions

$$(A_\psi)_\tau \cos X_\psi - (\xi_\psi)_\tau A_\psi \sin X_\psi = 0,$$

$$(A_\phi)_\tau \cos X_\phi - (\xi_\phi)_\tau A_\phi \sin X_\phi = 0, \qquad (6.15)$$

as a consequence of the requirement that the solution be periodic. In addition A_ψ, A_ϕ, ξ_ψ and ξ_ϕ are viewed as slowly varying functions of τ. Because of this, A_ψ, A_ϕ, ξ_ψ, ξ_ϕ and their first derivatives may be replaced by their average value over one cycle in computing the response of the system. The average values are denoted by the superscript bars. Thus these averages are

$$2\lambda \bar{A}_\psi \overline{(\xi_\psi)_\tau} = h_\psi,$$

$$2\lambda \overline{(A_\psi)_\tau} = g_\psi,$$

$$2\lambda \bar{A}_\phi \overline{(\xi_\phi)_\tau} = h_\phi, \qquad (6.16)$$

$$2\lambda \overline{(A_\phi)_\tau} = g_\phi,$$

with

$$h_\psi = \frac{\beta_\psi}{\pi} \int_0^{2\pi} \left\{ (\bar{A}_\psi \cos X_\psi)^3 + \Upsilon \bar{A}_\psi \cos X_\psi \bar{A}_\phi^2 \cos^2 X_\phi \right\} \cos X_\psi \, dX_\psi, \qquad (6.17)$$

$$\xi_\psi = \frac{\beta_\psi}{\pi}\int_0^{2\pi}\left\{(\bar{A}_\psi\cos X_\psi)^3 + \Upsilon\bar{A}_\psi\cos X_\psi\bar{A}_\phi^2\cos^2 X_\phi\right\}\sin X_\psi\, dX_\psi$$

$$-\beta_\psi\Delta_\psi\sin(\bar{\xi}_\psi - \xi_\phi),$$

$$h_\phi = \frac{\beta_\psi}{\pi}\int_0^{2\pi}\left\{\Pi\bar{A}_\phi^3\cos^3 X_\phi + \Lambda\bar{A}_\phi\bar{A}_\psi^2\cos X_\phi\cos^2 X_\psi\right\}\cos X_\phi\, dX_\phi$$

$$-\left(\lambda^2 - \frac{\omega_{LM}^2(m,n)}{\omega_{LM}^2(\mu,\nu)}\right)\bar{A}_\phi - \beta_\psi\Delta_\phi\cos(\bar{\xi}_\phi - \xi_\phi), \qquad \begin{array}{r}(6.17)\\ \text{(Cont'd)}\end{array}$$

$$\xi_\phi = \frac{\beta_\psi}{\pi}\int_0^{2\pi}\left\{\Pi\bar{A}_\phi^3\cos^3 X_\phi + \Lambda\bar{A}_\phi\bar{A}_\psi^2\cos X_\phi\cos^2 X_\psi\right\}\sin X_\phi\, dX_\phi$$

$$-\beta_\psi\Delta_\phi\sin(\bar{\xi}_\phi - \xi_\phi).$$

These equations are evaluated to obtain

$$h_\psi = \frac{\beta_\psi}{\pi}\left\{\bar{A}_\psi^3\left(\frac{3\pi}{4}\right) + \Upsilon\bar{A}_\psi\bar{A}_\phi^2\left[\frac{\pi}{2} + \frac{1}{8}\sin 2(\bar{\xi}_\phi - \bar{\xi}_\psi)\right.\right.$$

$$-\frac{1}{8}\sin 2\bar{\xi}_\phi + \frac{1}{32}\sin(2\bar{\xi}_\phi + \bar{\xi}_\psi) - \frac{1}{32}\sin 2(\bar{\xi}_\phi + \bar{\xi}_\psi)$$

$$\left.\left.+\frac{\pi}{2}\cos 2(\bar{\xi}_\psi - \bar{\xi}_\phi)\right]\right\} - (\lambda^2 - 1)\bar{A}_\psi \qquad (6.18)$$

$$-\beta_\psi\Delta_\psi\cos(\bar{\xi}_\psi - \xi_\psi),$$

$$h_\phi = \frac{\beta_\psi}{\pi}\Big\{\pi \bar{A}_\phi^3 \left(\tfrac{3\pi}{4}\right) + \Lambda \bar{A}_\phi \bar{A}_\psi^2 \Big[\tfrac{\pi}{2} + \tfrac{1}{8}\sin 2(\bar{\xi}_\psi - \bar{\xi}_\phi)$$

$$- \tfrac{1}{8}\sin(2\bar{\xi}_\psi) + \tfrac{1}{32}\sin 2(\bar{\xi}_\psi + \bar{\xi}_\phi)$$

$$- \tfrac{1}{32}\sin 2(\bar{\xi}_\phi + \bar{\xi}_\psi) + \tfrac{\pi}{2}\cos 2(\bar{\xi}_\phi - \bar{\xi}_\psi)\Big]\Big\},$$

$$\xi'_\psi = \frac{\beta_\psi}{\pi} \Upsilon \bar{A}_\psi \bar{A}_\phi^2 \Big\{\tfrac{\pi}{4}\sin 2(\bar{\xi}_\psi - \bar{\xi}_\phi) + \tfrac{1}{32}\cos 2(\bar{\xi}_\psi - \bar{\xi}_\phi)$$

$$- \tfrac{1}{32}\cos 2(\bar{\xi}_\psi - \bar{\xi}_\phi)\Big\} - \beta_\psi \Delta_\psi \sin(\bar{\xi}_\psi - \xi_\psi), \qquad \begin{array}{c}(6.18)\\ \text{(Cont'd)}\end{array}$$

$$\xi'_\phi = \frac{\beta_\psi}{\pi}\Lambda \bar{A}_\phi \bar{A}_\psi^2 \Big\{\tfrac{1}{32}\cos 2(\bar{\xi}_\phi - \bar{\xi}_\psi) - \tfrac{1}{32}\cos 2(\bar{\xi}_\phi - \bar{\xi}_\psi)$$

$$+ \tfrac{\pi}{2}\sin 2(\bar{\xi}_\phi - \bar{\xi}_\psi)\Big\} - \beta_\psi \Delta_\phi \sin(\bar{\xi}_\phi - \xi_\phi).$$

The steady state condition is

$$h_\psi = h_\phi = \xi'_\psi = \xi'_\phi = 0. \qquad (6.19)$$

The parameters ξ_ψ and ξ_ϕ may immediately be set equal to zero and all the phase effect thrown into $\bar{\xi}_\psi$ and $\bar{\xi}_\phi$. This is done since the excitation of the two modes will seldom be from different or non-synchronous sources. The parameters $\omega^2_{LM}(m,n)/\omega^2_{LM}(\mu,\nu)$ and β_ψ, Υ, Π, Λ are properties of the system and fixed, $\Delta\psi$ and $\Delta\phi$ are prescribed as a part of the forcing functions (or vibration inducing environment) and there remain four unknowns, $\overline{A_\psi}$, $\overline{A_\phi}$, $\overline{\xi_\psi}$ and $\overline{\xi_\phi}$ to be determined by the four equations 6.19.

6.8 Stability of Harmonic Motion

The criterion for stable motion is $a_0 < 0$ with the boundary at $a_0 = 0$ where a_0 is defined by

$$a_0 = \mathrm{DET} \begin{vmatrix} \frac{\partial \xi_\psi}{\partial \bar{A}_\psi} & \frac{\partial \xi_\psi}{\partial \bar{A}_\phi} & \frac{\partial \xi_\psi}{\partial \bar{\xi}_\psi} & \frac{\partial \xi_\psi}{\partial \bar{\xi}_\phi} \\ \frac{\partial \xi_\phi}{\partial \bar{A}_\psi} & \frac{\partial \xi_\phi}{\partial \bar{A}_\phi} & \frac{\partial \xi_\phi}{\partial \bar{\xi}_\psi} & \frac{\partial \xi_\phi}{\partial \bar{\xi}_\phi} \\ \frac{\partial h_\psi}{\partial \bar{A}_\psi} & \frac{\partial h_\psi}{\partial \bar{A}_\phi} & \frac{\partial h_\psi}{\partial \bar{\xi}_\psi} & \frac{\partial h_\psi}{\partial \bar{\xi}_\phi} \\ \frac{\partial h_\phi}{\partial \bar{A}_\psi} & \frac{\partial h_\phi}{\partial \bar{A}_\phi} & \frac{\partial h_\phi}{\partial \bar{\xi}_\psi} & \frac{\partial h_\phi}{\partial \bar{\xi}_\phi} \end{vmatrix} . \qquad (6.20)$$

6.9 Second Application of the Galerkin Integration

The Galerkin averaging technique has been used so far to achieve a separation of variables and hence reduce a partial differential equation in the three independent variables x, y, t, to an ordinary differential equation in the single variable t. In view of this first use of the Galerkin averaging technique it is homobusic to use the same technique to reduce the ordinary differential equation to an algebraic equation.

Consider the system represented by

$$\psi_{\tau\tau} + \psi + \beta_\psi \psi^3 + \gamma_\psi \psi \Phi^2 = q_{mn} \cos \lambda \tau_M,$$

$$\Phi_{\tau\tau} + \Phi R + \beta_\phi \Phi^3 + \gamma_\phi \Phi \psi^2 = q_{\mu\nu} \cos \lambda \tau_M, \qquad (6.21)$$

where

$$R = \frac{\omega^2_{LM}(m,n)}{\omega^2_{LM}(\mu,\nu)},$$

$$\tau_M = \omega_{LM}(m,n)\, t.$$

And suppose that it is desired to find the conditions under which a harmonic solution of the form

$$\psi = \psi_o \cos \lambda \tau_M,$$

$$\Phi = \Phi_o \cos \lambda \tau_M, \qquad (6.22)$$

can exist. Then application of the Galerkin averaging technique will introduce equations 6.22 into equations 6.21, weight the result with cos $\lambda \tau_M$ and integrate over $0 \le \tau_M \le \frac{2\pi}{\lambda}$. This gives

$$(1-\lambda^2) - (q_{mn}/\psi_o) + \frac{3}{4}\beta_\psi \psi_o^2 + \frac{3}{4}\gamma_\psi \Phi_o^2 = 0,$$

$$(R-\lambda^2) - (q_{\mu\nu}/\Phi_o) + \frac{3}{4}\beta_\phi \Phi_o^2 + \frac{3}{4}\gamma_\phi \psi_o^2 = 0. \qquad (6.23)$$

When λ^2 has been eliminated and the ratios

$$\eta_1 = \Phi_0 / \psi_0 ,$$
$$\eta_2 = q_{mn} / q_{\mu\nu} , \tag{6.24}$$

are used, equations 4.23 reduce to

$$\frac{3}{4}\left[(\beta_\psi - \gamma_\phi) - \eta_1^2(\beta_\phi - \gamma_\psi)\right]\psi_0^3 + (1-R)\psi_0 = q_{mn}\left(1 - \frac{\eta_2}{\eta_1}\right). \tag{6.25}$$

In equation 6.25 the quantities $\beta_\psi, \beta_\phi, \gamma_\psi, \gamma_\phi$ and R are physical parameters which depend on the nature of the shell considered, q_{mn} and η_2 depend on the nature of the vibration inducing environment in which the shell is studied and η_1 depends on the nature of solution desired. Thus the only truly unknown quantity in equation 6.25 is ψ_0, the amplitude of the response of the (n, m)-mode, from it Φ_0, the amplitude of response of the (ν, μ)-mode is determined and hence λ is found from equations 6.23.

CHAPTER VII

CONCLUSIONS AND NUMERICAL RESULTS

7.1 Curved Panel Energy Curves

When ψ_c is set equal to zero in equation 2.39, the energy function $E(\psi, 0)$ is obtained in the form shown in figures 3, 4 and 5. These figures, as plotted, represent a total energy level of $K = 0.5$. However, an upward shift of the axis, $E = 0$, by an amount equal to 2K in figures 3, 4, and 5 yields the correct energy curve for any value of K. These figures may be used to estimate the maximum response amplitude when the applied loading can be interpreted as a set of initial conditions (which then allows the identification of the corresponding energy level).

The increase of the nonlinearity parameter, ϵ, provides a reduction of the outward deflection amplitude while the inward deflection amplitude increases. In addition, when $\epsilon > 8/9$ the inward deflection can include a low energy buckled vibration as indicated by the curve marked $\epsilon = 9.1$ in figure 4. Figure 6 is a schematic representing the connection between $E(\psi, 0)$ and the phase plane trajectories.

7.2 Curved Panel Vibration Period.

Equation 2.38 describes the dependence of the period on amplitude in the form

$$\frac{T_{NL}}{T_L} = \frac{1}{\pi} \int_{\psi_1}^{\psi_2} \frac{d\psi}{\left[2K - \left(\psi^2 + \epsilon \left\{ \frac{2}{3}\psi^3 + \frac{1}{9}\psi^4 \right\} \right) \right]^{1/2}} \qquad (2.38)$$

The evaluation of this integral requires the extraction of the roots of a quadric equation as described in Appendix I. These roots are seldom easy to obtain and will usually be obtained with the aid of a digital computor. After the roots are obtained there remains a significant difficulty in the interpolation of the elliptic integrals from most existing tables. On the other hand, the obvious simplicity of Reissner's formula, valid for small ϵ,

$$\frac{T_{NL}}{T_L} = \left[1 + \frac{\epsilon}{6}\left(1 - 5\epsilon\right)\psi_0^2\right]^{-1/2}, \tag{7.1}$$

makes a comparison of these results of equation 2.38 and 7.1 desirable. This comparison, with $\epsilon = 1$, is presented in figure 18. At a value of $\psi_0 = 4$ these two curves differ by approximately three per cent. In this case equation 7.1 is preferable because of the great difference in difficulty and small difference in accuracy. Unfortunately equation 7.1 is not valid for values of ϵ that are much larger than one-tenth. In fact, for $\epsilon \geq 0.2$ and $\epsilon\psi_0^2 \leq 6$, equation 7.1 yields $T_{NL}/T_L > 1$ which is not possible for the system considered here. When $\epsilon = 0.2$ and $\psi_0 = \sqrt{30}$ equation 7.1 predicts $T_{NL}/T_L = 0$. This definitely shows the limitation of equation 7.1. Consequently, there is a large range of ϵ and ψ_0 for which equation 2.38 must be used to determine T_{NL}/T_L.

7.3 Curved Panel Response by Runge-Kutta Integration

The Runge-Kutta method is applied to the equation

$$\psi_{\tau\tau} + \Omega_L^2 \left[\psi + \epsilon \left(\psi^2 + \tfrac{2}{9} \psi^3 \right) \right] = \Omega_L^2 r, \tag{7.2}$$

with homogeneous boundary conditions. This second order differential equation is reduced to two first order differential equations

$$\begin{aligned} \psi_\tau &= \xi \;, \\ \xi_\tau &= \Omega_L^2 \left[r - \psi - \epsilon \left(\psi^2 + \tfrac{2}{9} \psi^3 \right) \right]. \end{aligned} \tag{7.3}$$

The recurrence equations for integration of equation 7.3 by the Runge-Kutta method are

$$\begin{aligned}
\psi_{K+1} &= \psi_K + \tfrac{1}{6}\left(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 \right), \\
\xi_{K+1} &= \xi_K + \tfrac{1}{6}\left(\beta_1 + 2\beta_2 + 2\beta_3 + \beta_4 \right), \\
\alpha_1 &= \xi(\tau_K, \psi_K)\Delta\tau \;, \\
\alpha_2 &= \xi\left(\tau_K + \tfrac{\Delta\tau}{2}, \psi_K + \tfrac{\alpha_1}{2}\right)\Delta\tau \;, \\
\alpha_3 &= \xi\left(\tau_K + \tfrac{\Delta\tau}{2}, \psi_K + \tfrac{\alpha_2}{2}\right)\Delta\tau, \\
\alpha_4 &= \xi\left(\tau_K + \Delta\tau, \psi_K + \alpha_3\right)\Delta\tau, \\
\beta_1 &= \Omega_L^2\left[r - \psi_K - \epsilon \left(\psi_K^2 + \tfrac{2}{9}\psi_K^3 \right) \right]\Delta\tau, \\
\beta_2 &= \Omega_L^2\left[r - \left(\psi_K + \tfrac{\beta_1}{2}\right) - \epsilon\left(\psi_K + \tfrac{\beta_1}{2}\right)^2 - \tfrac{2}{9}\epsilon\left(\psi_K + \tfrac{\beta_1}{2}\right)^3 \right]\Delta\tau, \\
\beta_3 &= \Omega_L^2\left[r - \left(\psi_K + \tfrac{\beta_2}{2}\right) - \epsilon\left(\psi_K + \tfrac{\beta_2}{2}\right)^2 - \tfrac{2}{9}\epsilon\left(\psi_K + \tfrac{\beta_2}{2}\right)^3 \right]\Delta\tau, \\
\beta_4 &= \Omega_L^2\left[r - \left(\psi_K + \beta_3\right) - \epsilon\left(\psi_K + \beta_3\right)^2 - \tfrac{2}{9}\epsilon\left(\psi_K + \beta_3\right)^3 \right]\Delta\tau,
\end{aligned} \tag{7.4}$$

with $\tau_0 = 0$, $\psi_0 = 0$, $\xi_0 = 0$ and $\Delta\tau = 0.005$.

Since the truncation error is proportional to $(\Delta \tau)^2$ for the Runge-Kutta method, the truncation error in this computation is of the order 10^{-5}. The actual computation was made on an IBM 704 digital computor with print-out for ψ and ψ_τ at intervals of $\tau = 0.04$. These data, for $\epsilon = 1$, have been presented in various ways in figures 19, 20, 21, 22, 24 and 25.

Figure 19 contains partial time histories for various amplitudes of the step function, r. The response is always the same sign as the applied step function. The time to maximum amplitude, for a given magnitude of step, is greater for external pressure $(r < 0)$ than for internal pressure $(r > 0)$ although the ratio

$$\frac{\text{maximum } |\text{ amplitude }|}{\text{time to maximum amplitude}} \tag{7.5}$$

is greater for $r > 0$ than for $r < 0$, at given $|r|$.

Figure 20 compares the phase plane trajectories of the linearized system ($\epsilon = 0$) to a step of $r = \pm 5.0$ with the response of the nonlinear system ($\epsilon = 1$) subjected to steps of $r = + 5.0$ and $r = - 5.0$. It is clear from this figure that the amplitudes of response are strongly limited by the nonlinear system, even when buckling occurs.

Figure 21 is the phase plane trajectory for $r = - 0.1$. This trajectory does not exhibit the buckling phenomena. The maximum values of velocity are limited for this kind of trajectory. This is seen by comparison of $2|\psi_\tau \max|$ with $|\psi_{\max.}|$. In figure 20, for instance, the linear system trajectory has $2|\psi_{\tau\, max}| = |\psi_{max}|$, while for the nonlinear system response curves $2|\psi_{\tau\, max}| > |\psi_{max}|$. For the trajectory of figure 21, $2|\psi_{\tau\, max}| < |\psi_{max}|$. The different character of the trajectory of figure 21 is related to the smallness of the step function, rather than the lack of buckling, since both the buckled and unbuckled trajectories (figure 20) differ from figure 21 but not from each other in shape.

Figure 22 is a family of phase plane response curves for external pressure step loads which exceed the minimum required for dynamic buckling (at $\epsilon = 1$). The rate of increase of amplitude of response with increase of step magnitude is clearly shown in this figure to be one which diminishes sharply with increase of loading magnitude.

The shock response concept was used in equation 3.39 to predict the value of r at which buckling will occur dynamically. Figure 23 is a plot of this relation. The prediction has no meaning for $0 \le \epsilon \le 8/9$ since for this range of ϵ there is no buckling. The relation of r to the physical parameters of the system is

$$r = R/\Omega_i^2 = \left(\frac{P_0}{P_{0_{CRT}}}\right)\left(\frac{4}{\pi}\right)^4 AR^2\left(AR^2 + \frac{1}{2}\right)^{-1}. \tag{7.6}$$

For instance, if $AR = 1$, $\epsilon = 1$, then $r = -0.286$ and $P_0 = 0.163\ P_{0_{CRT}}$. This surprising result has been verified by the numerical integration as shown in figures 24 and 25 where the maximum amplitude and velocity of the step function response, respectively, have been presented as a function of r (at $\epsilon = 1$).

7.4 Vibration Period of the Cylinder

The dependence of the period of vibration of the cylinder is given by equation 5.12. However, the computation is simplified by a different approach. Consider the equation

$$\psi_{\tau\tau} + \Omega_{LM}^2 (\psi + \sigma\psi^3) = 0. \tag{7.3}$$

If equation 7.3 is integrated once, then

$$\psi_\tau = -\Omega_{LM} (\psi_0^2 - \psi^2)^{1/2} \left[1 + \frac{\sigma}{2} (\psi_0^2 - \psi^2)\right]^{1/2}. \tag{7.4}$$

Separating variables and integrating again

$$\int d\tau = -\frac{1}{\Omega_{LM}} \int_{\psi_1}^{\psi_2} \frac{dw}{(\psi_0^2 - w^2)^{1/2} \left[1 + \frac{\sigma}{2} (\psi_0^2 - w^2)\right]^{1/2}} . \tag{7.5}$$

Now making the substitutions

$$\tau = \omega_{LM} t,$$

$$\Omega_{LM} = 1,$$

$$\lambda^2 = \frac{\sigma \psi_0^2}{2(1+\sigma\psi_0^2)} ,$$

$$F(\lambda,\phi) = \int_0^{\phi} \frac{d\theta}{(1-\lambda^2 \sin^2\theta)^{1/2}} , \tag{7.6}$$

equation 7.5 becomes

$$\frac{t}{T_{LM}} = \left(\frac{1}{2\pi}\right) \frac{F(\lambda,\phi)}{(1+\sigma\psi_0^2)} . \tag{7.7}$$

When the integration is carried out for a full cycle, equation 7.7 becomes

$$\frac{T_{NL}}{T_{LM}} = \left(\frac{2}{\pi}\right) \frac{F_1(\lambda)}{(1+\sigma\psi_0^2)^{1/2}} , \tag{7.8}$$

where $F_1(\lambda)$ is the complete elliptic integral of the first kind. Equation 7.8 is presented for two different ranges of in figures 26 and 27. In order to relate these periods of the physical parameters of the problem, $\sigma\psi_0^2$ is

$$\sigma \psi_0'^2/(A_0/h)^2 =$$

$$\frac{(1+AR^4)\ AR^{-4}\ 6434\ (n/\pi)^8}{\left[\frac{n^4}{12(1-\nu^2)}\right]\left[1+\left(1+\frac{1}{n^2}\right)AR^{-2}\right]^2+\left[1+AR^{-2}\right]^{-2}+\left(\frac{n^2 P_0}{E}\right)AR^{-2}\,(a/h)^3} \quad (7.9)$$

Figures 28, 29 and 30 present equation 7.9 as a function of n for three values of AR and three values of $\left(\frac{n^2 P_0}{E}\right)AR^{-2}(a/h)^3$. Thus figures 26 through 30 provide a means of determining the influence of response amplitude on period for certain values of AR and $\left(\frac{n^2 P_0}{E}\right)AR^{-2}\ (a/h)^3$.

7.5 Conclusions

The difference in large amplitude behavior of complete cylindrical shells and of cylindrical panels was studied by the application of the shallow shell equations.

The curved panel exhibits a buckling behavior associated with a very simple mode shape while the cylinder shows no similar phenomena. The response of the curved panel is periodic for a full cycle but is not periodic for a half cycle, spending more time deflected inward than outward. This verifies results previously obtained by Reissner though a different method. On the other hand, cylindrical shells behave like a Duffing system having a hard spring.

In the curved panel, the initial stresses are the main influence in determining the magnitude of the nonlinear effects. As P_0 approaches the buckling pressure, the nonlinear effect becomes very large. The complete cylinder exhibits no similar effect. When either the curved panel or cylindrical shell has a large internal pressure, their behavior resembles the linear system. For curved panels, the effect of an external pressure step is very important. It results in dynamic buckling at a pressure much less than the static buckling load.

The aspect ratio of the curved panel strongly influences its vibrations and forced response characteristics. The cylindrical shell is similarly influenced by the aspect ratio of its mode shape. In both cases, the phase plane trajectories may be used to determine amplitudes of response when the loading can be interpreted as a set of initial conditions (i. e., an energy level). In both problems, perturbation methods provide considerable information with minimum effort.

Reissner's formula for the dependence of the period of vibration on the amplitude of vibration for a curved panel gives good results with minimum effort for a limited range of ϵ and ψ_0. For larger values of ϵ and ψ_0 an exact solution for the dependence of period on amplitude is obtained in terms of complete elliptic integrals. The period of vibration of the cylindrical shell is also represented by complete elliptic integrals.

With curved panels, the effect of damping rules out dynamic buckling caused by an internal pressure pulse. Shock response methods were used to accurately predict the pressure at which the curved panel will buckle dynamically. It is expected that similar methods will yield good results when applied to other transient response problems of the same system.

If, in the two mode analysis, the spacial distribution of the forcing function is such that the resulting generalized force is restricted to a single mode, then the modes are weakly coupled. Under these conditions a single mode may be excited independently of the other modes. This indicates that a careful experimental investigation of large amplitude cylindrical shell vibrations will require precise control of the spacial distribution of the forcing function.

The Morley equations were used to include $n = 0, 1, 2$ in the cylindrical shell study. The result indicates that the Morley correction to the shallow shell equations is important only when bending and initial pressure, or when bending and membrane forces, are of equal importance. That is, the Morley correction is important for shells that do not have extremely large radius-to-thickness ratios.

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APPENDIX I

Evaluation of the elliptic integral

$$\frac{T_{NL}}{T_L} = \frac{1}{\pi} \int_{\psi(\tau_1)}^{\psi(\tau_2)} \frac{dw}{\sqrt{a_0}\,\sqrt{G(w)}} \tag{AI.1}$$

where $\quad a_0 = (\epsilon/9)$

and $\quad G(w) = -(w^4 + 6w^3 + \frac{9}{\epsilon} w^2) + \frac{18}{\epsilon} K,$

$$G(w) = (w-\alpha_1)(w-\alpha_2)[w-(b_1+ic_1)][w-(b_1-ic_1)]$$

$$\alpha_1 > \alpha_2$$

is accomplished by transformation to the form

$$\mu \int_0^{\phi} \frac{d\gamma}{\sqrt{1-k^2 \sin^2\gamma}} \quad . \tag{AI.2}$$

The necessary transformation is given in Table AI.1, where the following quantities are used

$$\alpha_{ik} = \alpha_k - \alpha_i \;, \quad (i, k = 1, 2, 3, 4),$$

$$(\alpha, \beta, \gamma, \delta) = \frac{(\alpha-\gamma)(\beta-\delta)}{(\alpha-\delta)(\beta-\gamma)} \;,$$

$$\tan\theta_1 = \frac{\alpha_1 - b_1}{c_1} \;, \tag{AI.3}$$

$$\tan\theta_2 = \frac{\alpha_2 - b_1}{c_1} \;,$$

$$\nu = \left\{\tan[(\theta_2-\theta_1)/2]\right\}\left\{\tan[(\theta_1+\theta_2)/2]\right\}.$$

Two cases are of interest:

(1) when $G(w)$ has four real roots;

(2) when $G(w)$ has two real roots and a complex conjugate pair of roots.

Case (1) can occur only if $\epsilon > 8/9$ and if $K_1 \leq K_{SEP}$. When Case (1) occurs there are two types of vibration, one about the undeformed equilibrium point and one about the buckled equilibrium point, thus there are two integrals to evaluate. They do not necessarily have the same value. Case (2) covers all other possibilities.

A sample calculation follows for $K_1 = 0.1$, $\epsilon = 0.1$. This is a Case (2) example, no example is given for Case (1).

$$\alpha_1 = 0.4403\,, \quad \alpha_2 = -0.4535, \quad b_1 = -2.993, \; c_1 = 9.009$$

$$\tan\theta_1 = 0.381\,, \quad \tan\theta_2 = 0.282$$

$$\cos\theta_1 = 0.935\,, \quad \cos\theta_2 = 0.963,$$

$$\cos\left(\frac{\theta_1 - \theta_2}{2}\right) = 0.044$$

$$\int_0^{\pi} \frac{d\gamma}{\sqrt{1 - k^2 \sin^2\gamma}} = 3.144$$

$$\mu = 0.105$$

$$\frac{T_{NL}}{T_L} \cong \frac{0.105}{\pi} \sqrt{\frac{9}{\epsilon}}\; 3.144 = 0.992.$$

TABLE AI-1*

Case	Interval	Transformation	Auxiliary Quantities	ψ ..... Corresponding Values	ϕ	K^2	μ
Four Real Roots	$\alpha_4 \le \psi \le \alpha_3$	$\psi = \frac{\alpha_4 \alpha_{31} + \alpha_1 \alpha_{43} \sin^2\phi}{\alpha_{31} + \alpha_{43} \sin^2\phi}$	$\sin^2\phi = \left(\frac{\alpha_{31}}{\alpha_{43}}\right)\left(\frac{\psi - \alpha_4}{\alpha_1 - \psi}\right)$	α_4 α_3	0 $\pi/2$	$(\alpha_3, \alpha_2, \alpha_4, \alpha_1)$	$\frac{2}{(\alpha_{31}\alpha_{42})^{1/2}}$
Negative 1st Term	$\alpha_2 \le \psi \le \alpha_1$	$\psi = \frac{\alpha_2 \alpha_{31} - \alpha_3 \alpha_{21} \sin^2\phi}{\alpha_{31} - \alpha_{21} \sin^2\phi}$	$\sin^2\phi = \left(\frac{\alpha_{31}}{\alpha_{21}}\right)\left(\frac{\psi - \alpha_2}{\psi - \alpha_3}\right)$	α_2 α_1	0 $\pi/2$		
Two Real, Two Complex Roots 1st Term Negative	$\alpha_2 \le \psi \le \alpha_1$	$\psi = \left(\frac{\alpha_1 + \alpha_2}{2}\right) - \left(\frac{\alpha_1 - \alpha_2}{2}\right)\left(\frac{\nu - \cos\phi}{1 - \nu\cos\phi}\right)$; $\tan^2\left(\frac{\phi}{2}\right) = \frac{\cos\theta_1}{\cos\theta_2}\left(\frac{\alpha_1 - \psi}{\psi - \alpha_2}\right)$	$\theta_1 \le \pi/2$ $\theta_2 \le \pi/2$	α_1 α_2	0 π	$\sin^2\left(\frac{\theta_1 - \theta_2}{2}\right)$	$\frac{(\cos\theta_1 \cos\theta_2)^{1/2}}{C_1}$

* After Erdelyi, et al., Higher Transcendental Functions, Vol. 2, McGraw-Hill, New York, 1953.

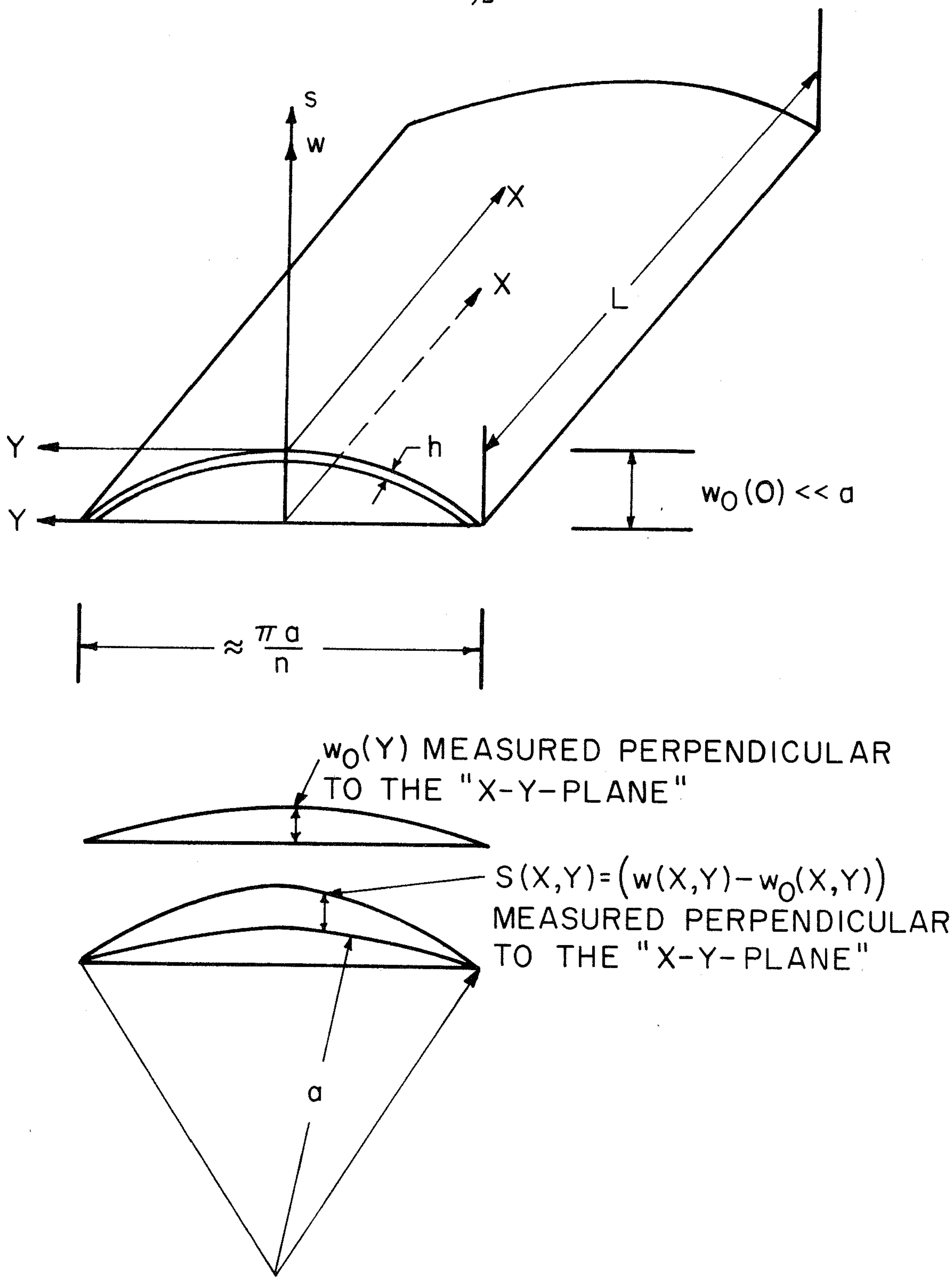
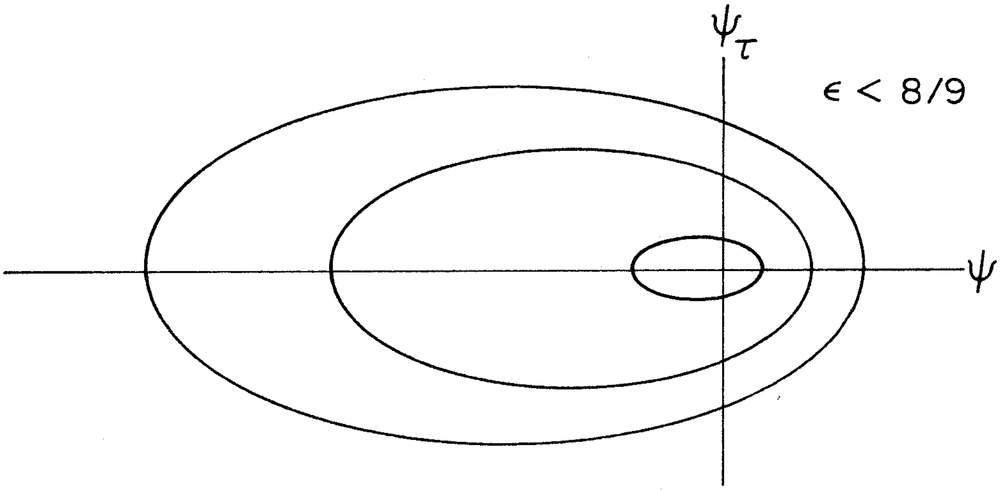

s
w
X
X
L
Y
Y
h
w0(0) << a
≈ πa/n
w0(Y) MEASURED PERPENDICULAR TO THE "X-Y-PLANE"
S(X,Y)=(w(X,Y)-w0(X,Y)) MEASURED PERPENDICULAR TO THE "X-Y-PLANE"
a

Figure 1



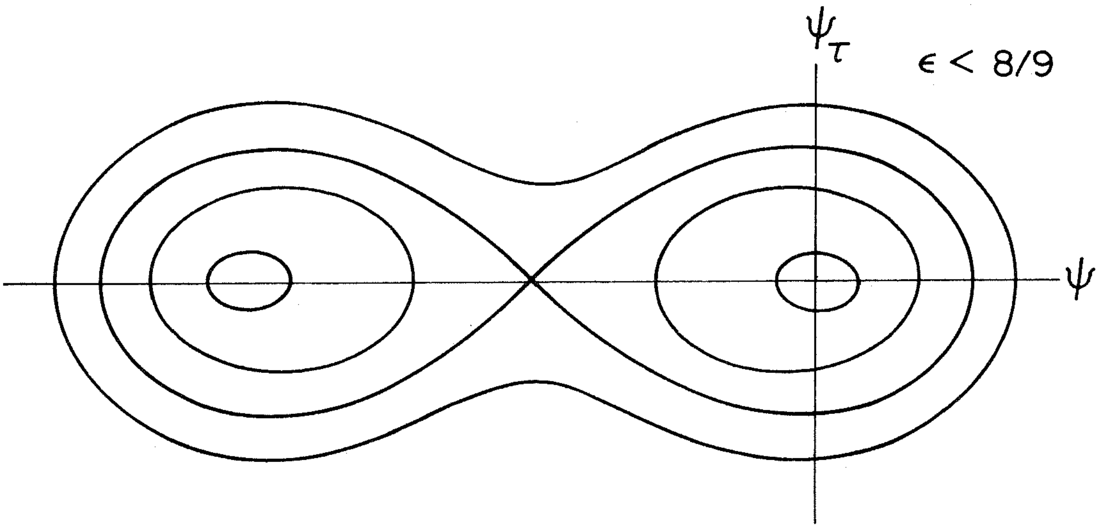


Figure 2

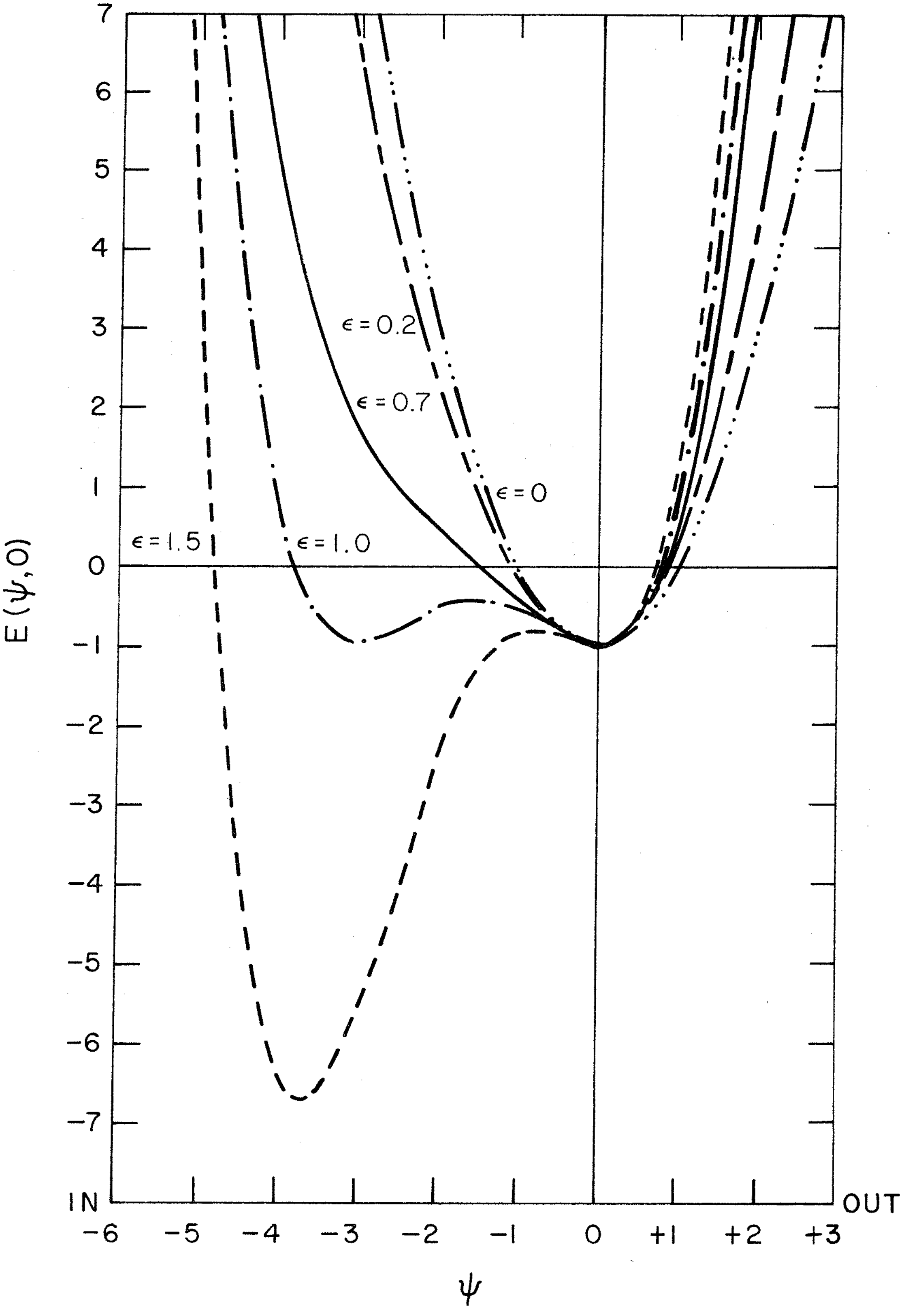
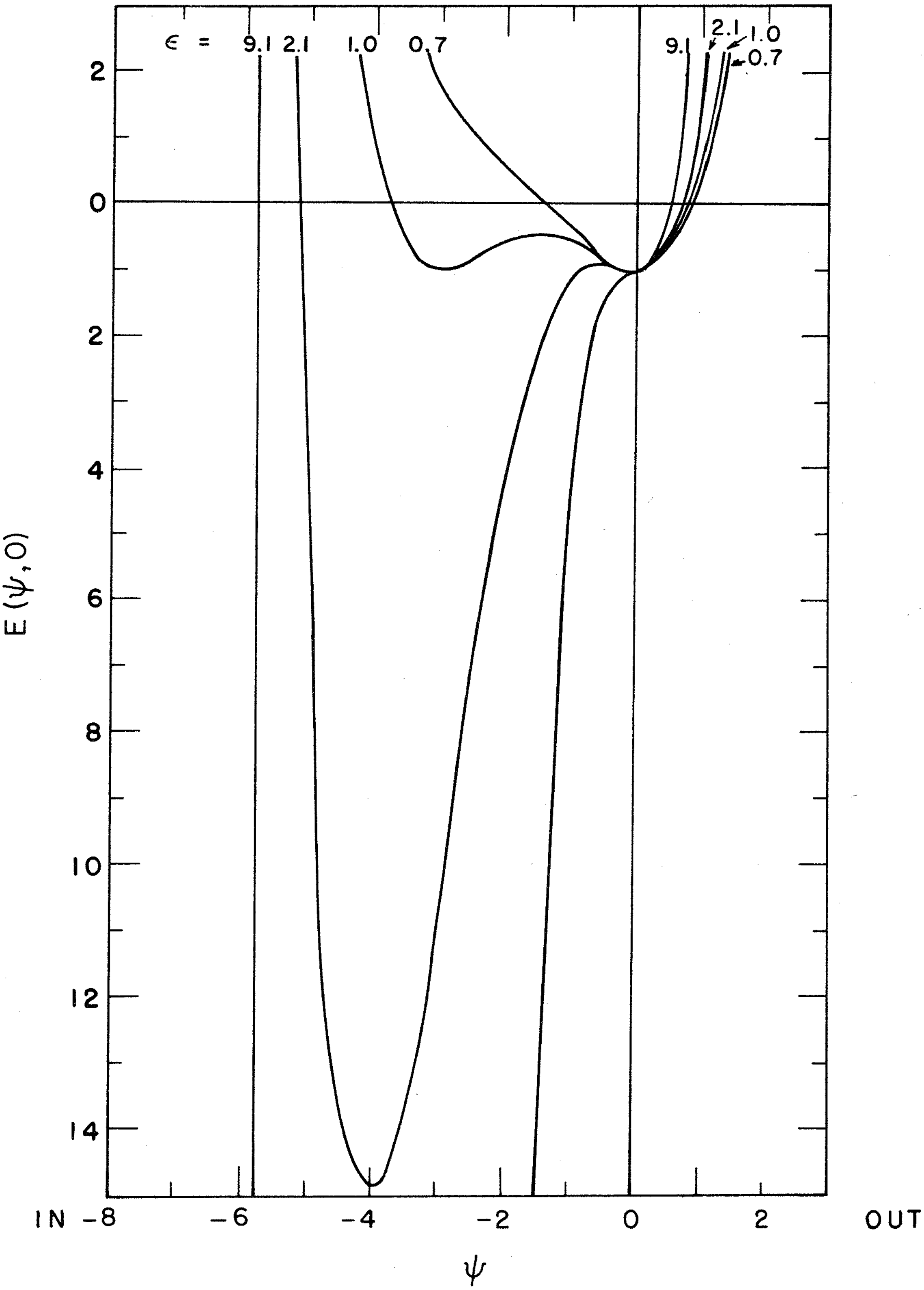


Figure 3



ϵ = 9.1 2.1 1.0 0.7
9.1
2.1
1.0
0.7
2
0
2
4
6
8
10
12
14
E (ψ, 0)
IN -8
-6
-4
-2
0
2
OUT
ψ

Figure 4

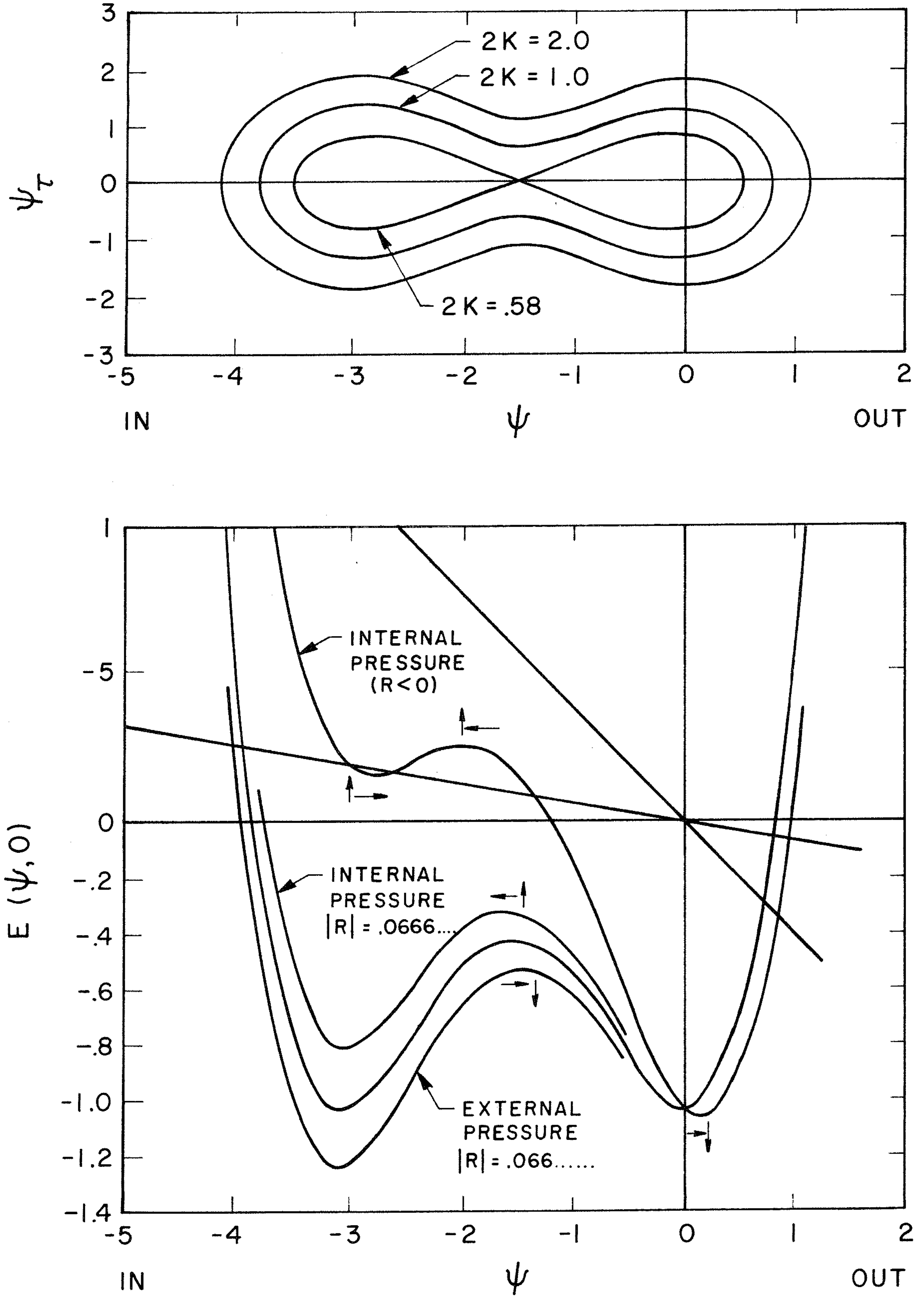

2K = 2.0
2K = 1.0
2K = .58
ψ_τ
ψ
IN
OUT
INTERNAL PRESSURE (R<0)
INTERNAL PRESSURE |R| = .0666....
EXTERNAL PRESSURE |R| = .066......
E (ψ,0)

Figure 5

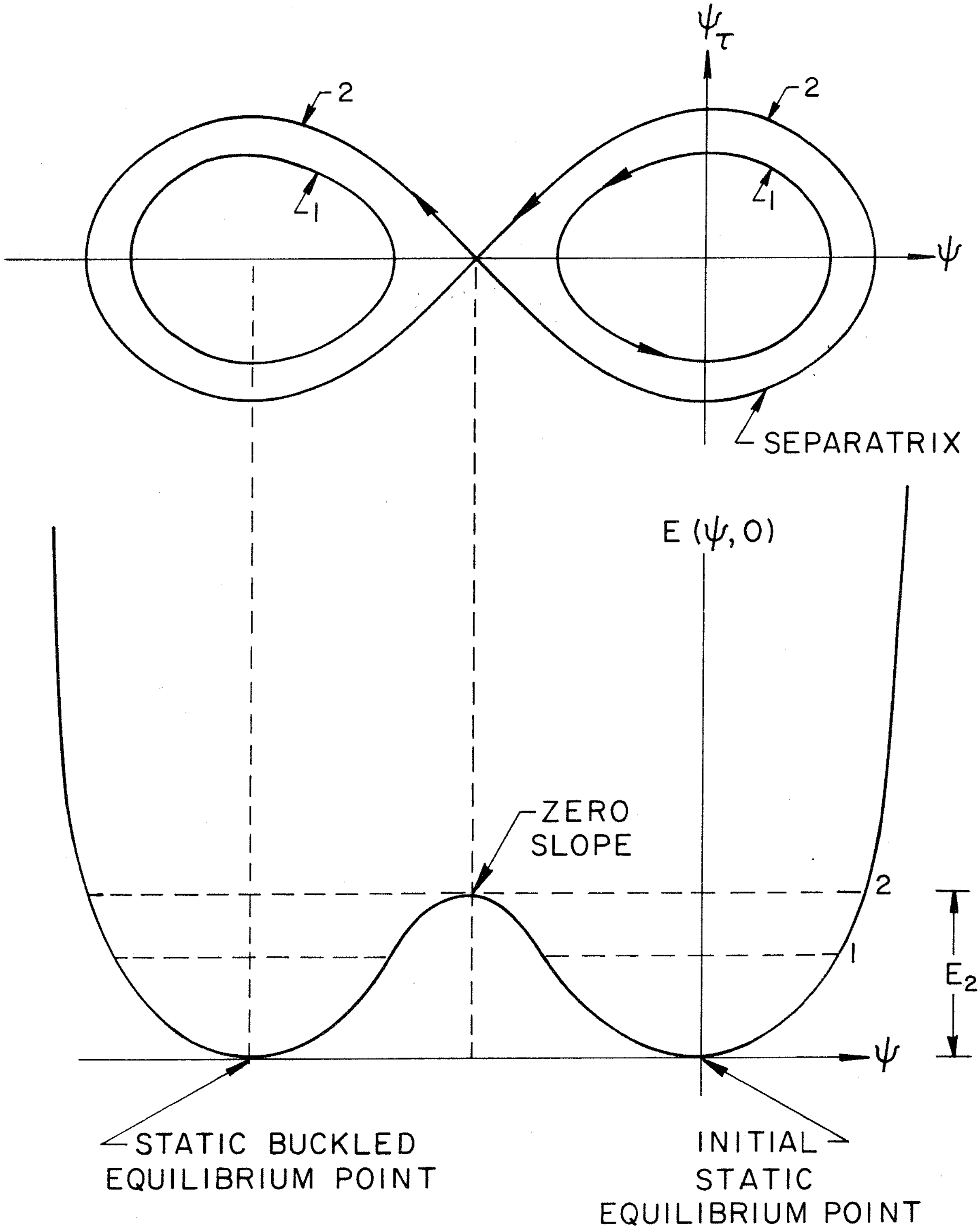

ψ_τ
2
2
1
1
ψ
SEPARATRIX
$E(\psi, 0)$
ZERO SLOPE
2
1
E_2
ψ
STATIC BUCKLED EQUILIBRIUM POINT
INITIAL STATIC EQUILIBRIUM POINT

Figure 6

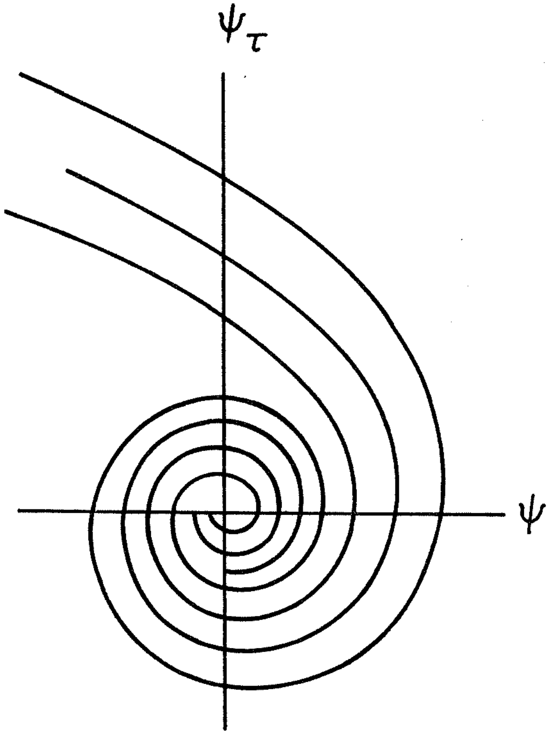


Figure 7

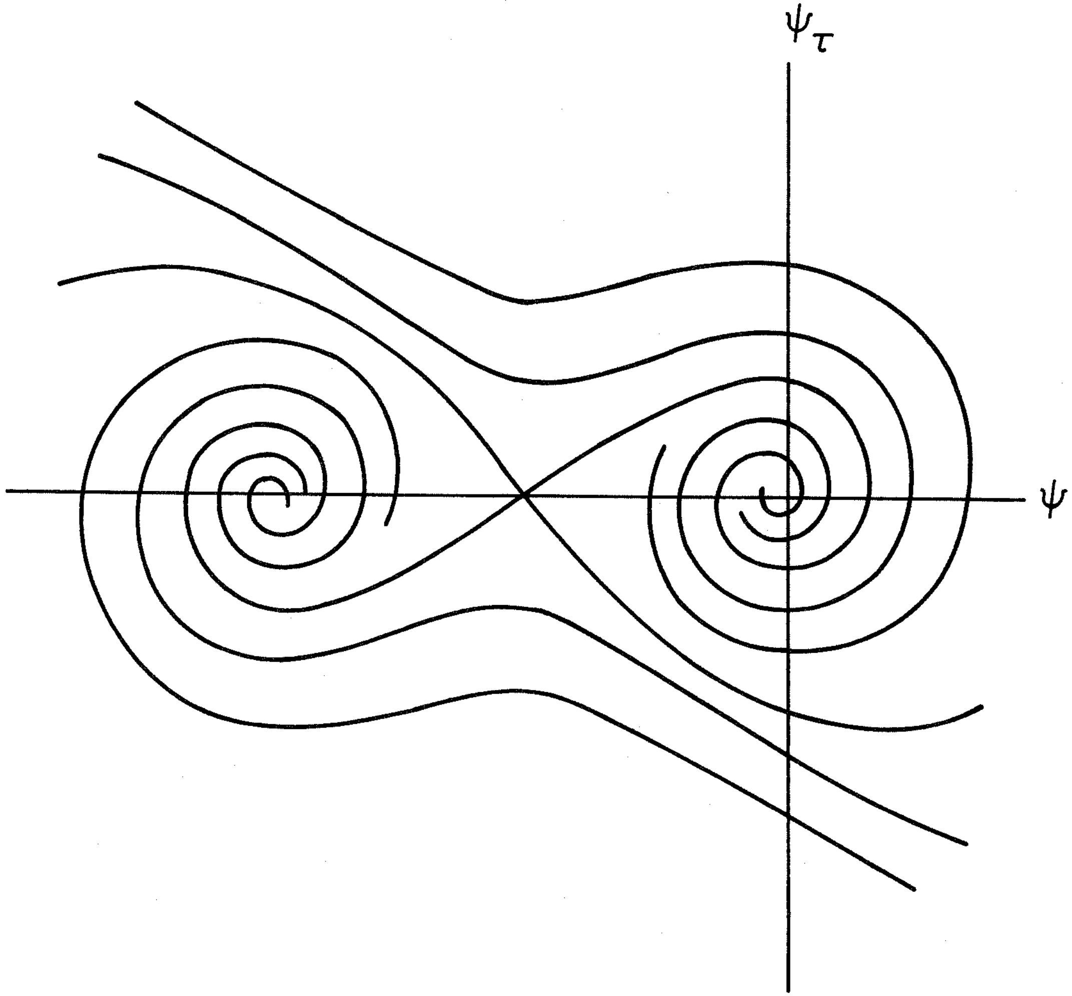
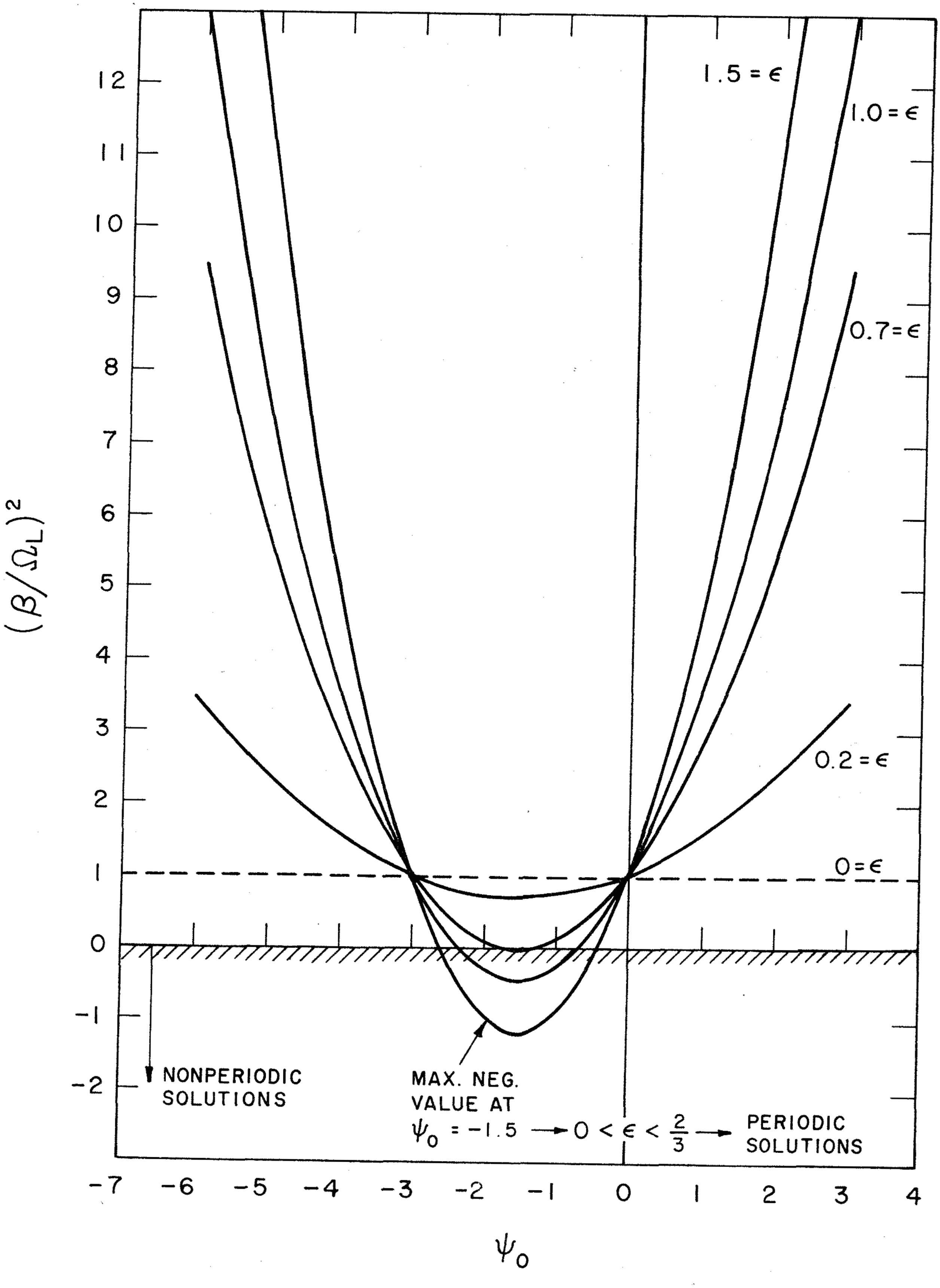
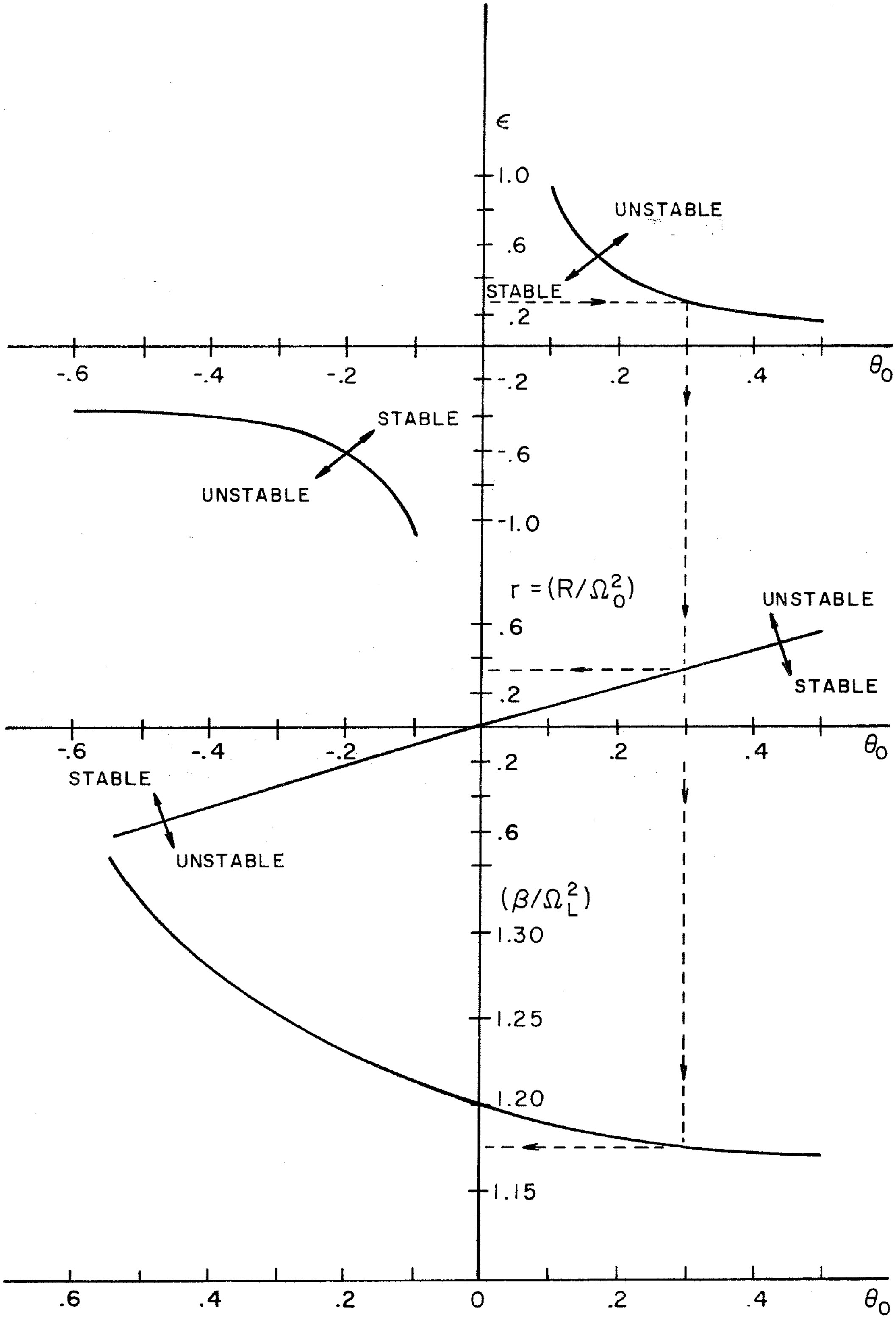


Figure 8



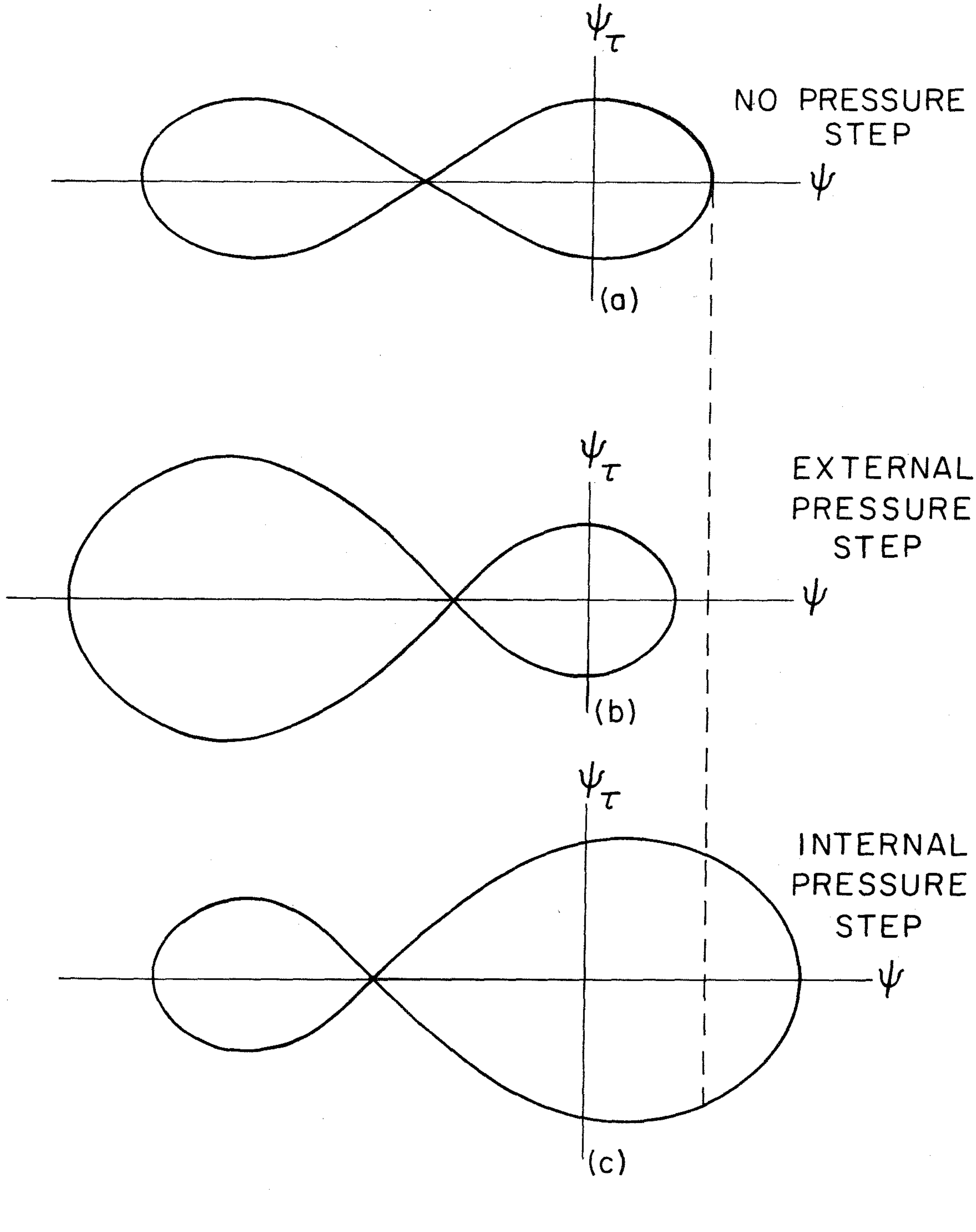
$1.5 = \epsilon$
$1.0 = \epsilon$
$0.7 = \epsilon$
$0.2 = \epsilon$
$0 = \epsilon$
$(\beta/\Omega_L)^2$
ψ_0
NONPERIODIC SOLUTIONS
MAX. NEG. VALUE AT $\psi_0 = -1.5 \rightarrow 0 < \epsilon < \frac{2}{3} \rightarrow$ PERIODIC SOLUTIONS

Figure 9



ϵ
1.0
.6
.2
-.2
-.6
-1.0
UNSTABLE
STABLE
STABLE
UNSTABLE
-.6
-.4
-.2
.2
.4
θ_0
$r = (R/\Omega_0^2)$
.6
.2
UNSTABLE
STABLE
STABLE
UNSTABLE
(β/Ω_L^2)
1.30
1.25
1.20
1.15
.6
.4
.2
0

Figure 10



ψ_τ
NO PRESSURE STEP
ψ
(a)
ψ_τ
EXTERNAL PRESSURE STEP
ψ
(b)
ψ_τ
INTERNAL PRESSURE STEP
ψ
(c)

Figure 11

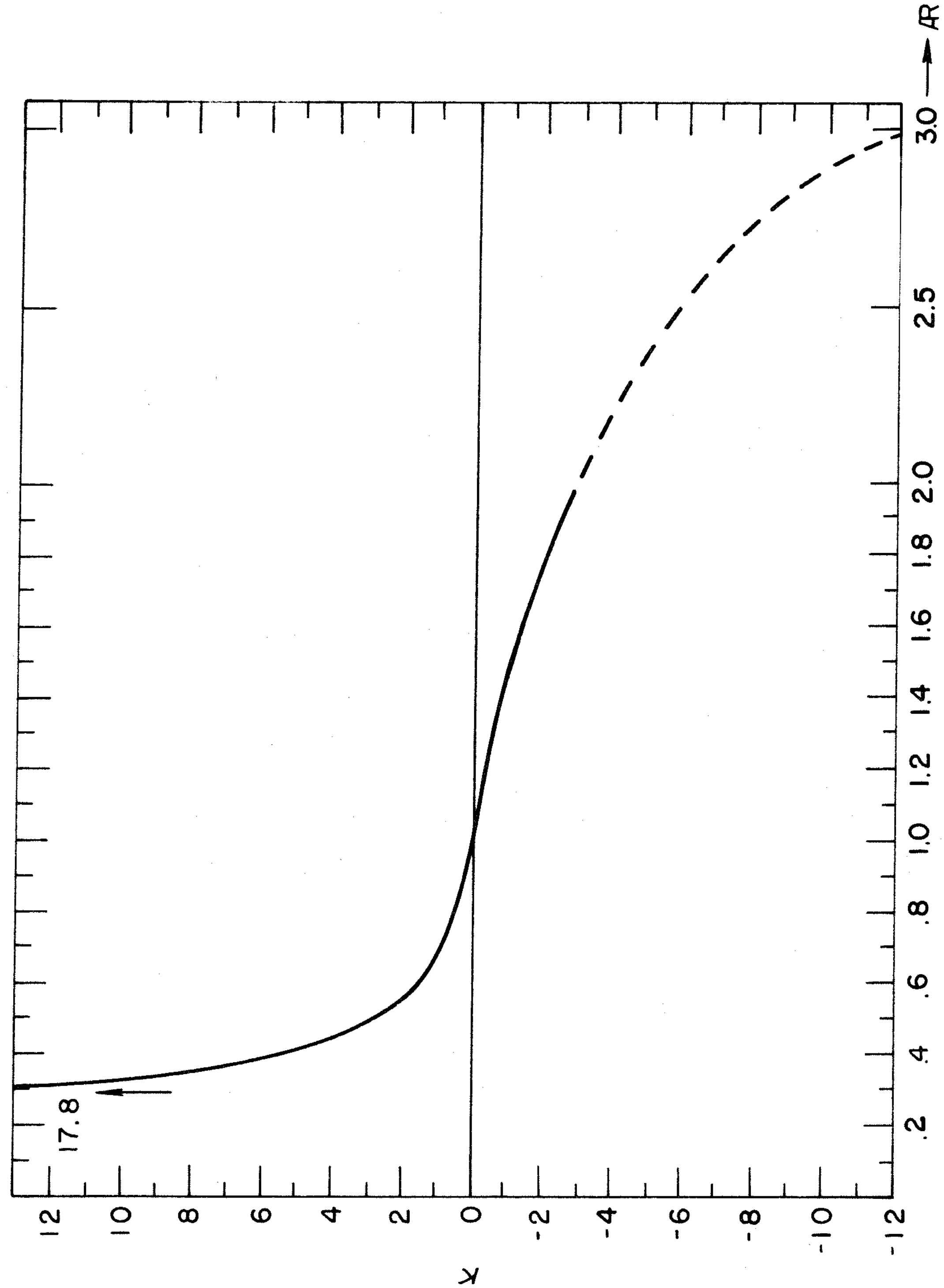


Figure 12

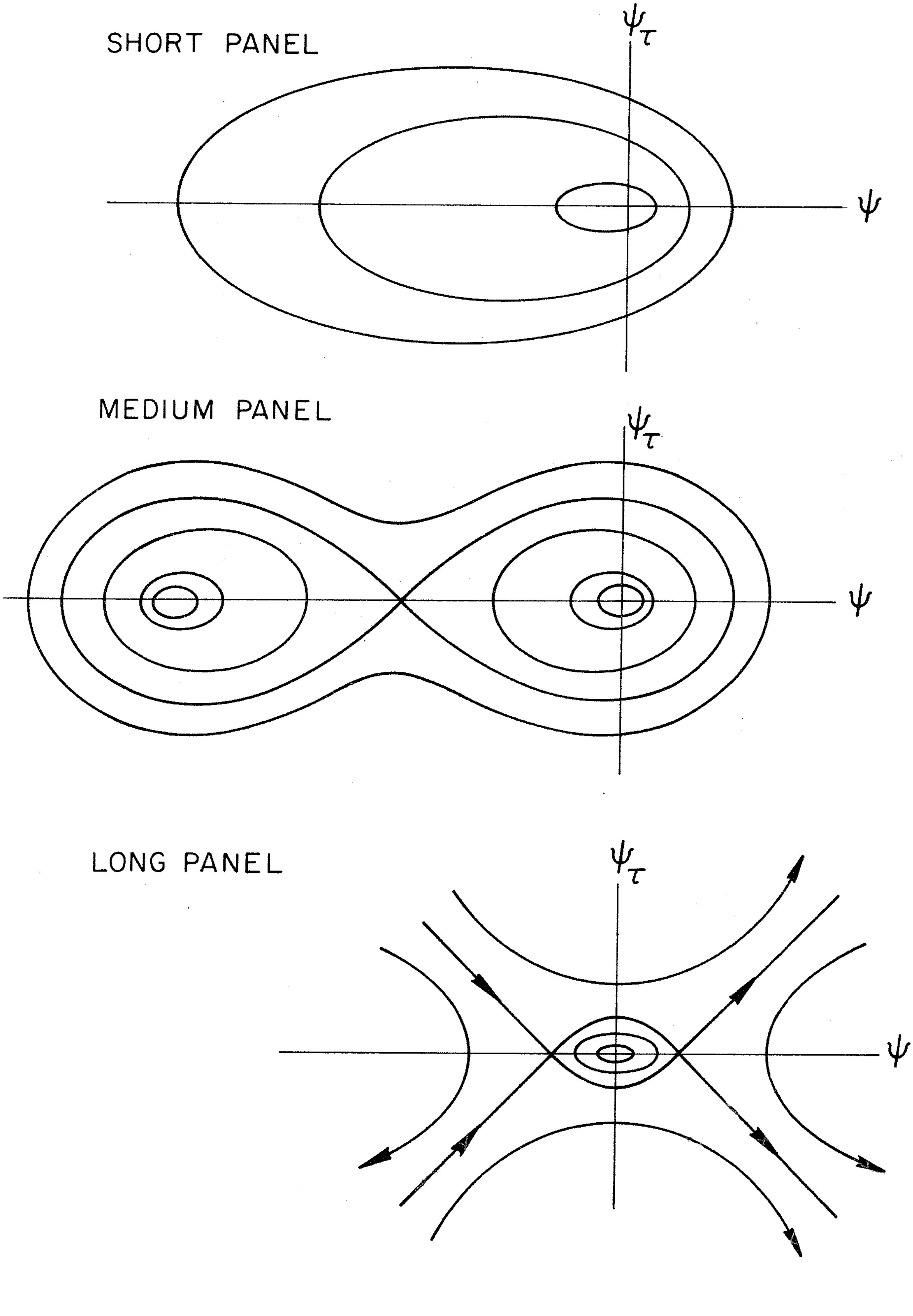

SHORT PANEL
ψ_τ
ψ
MEDIUM PANEL
ψ_τ
ψ
LONG PANEL
ψ_τ
ψ

Figure 13

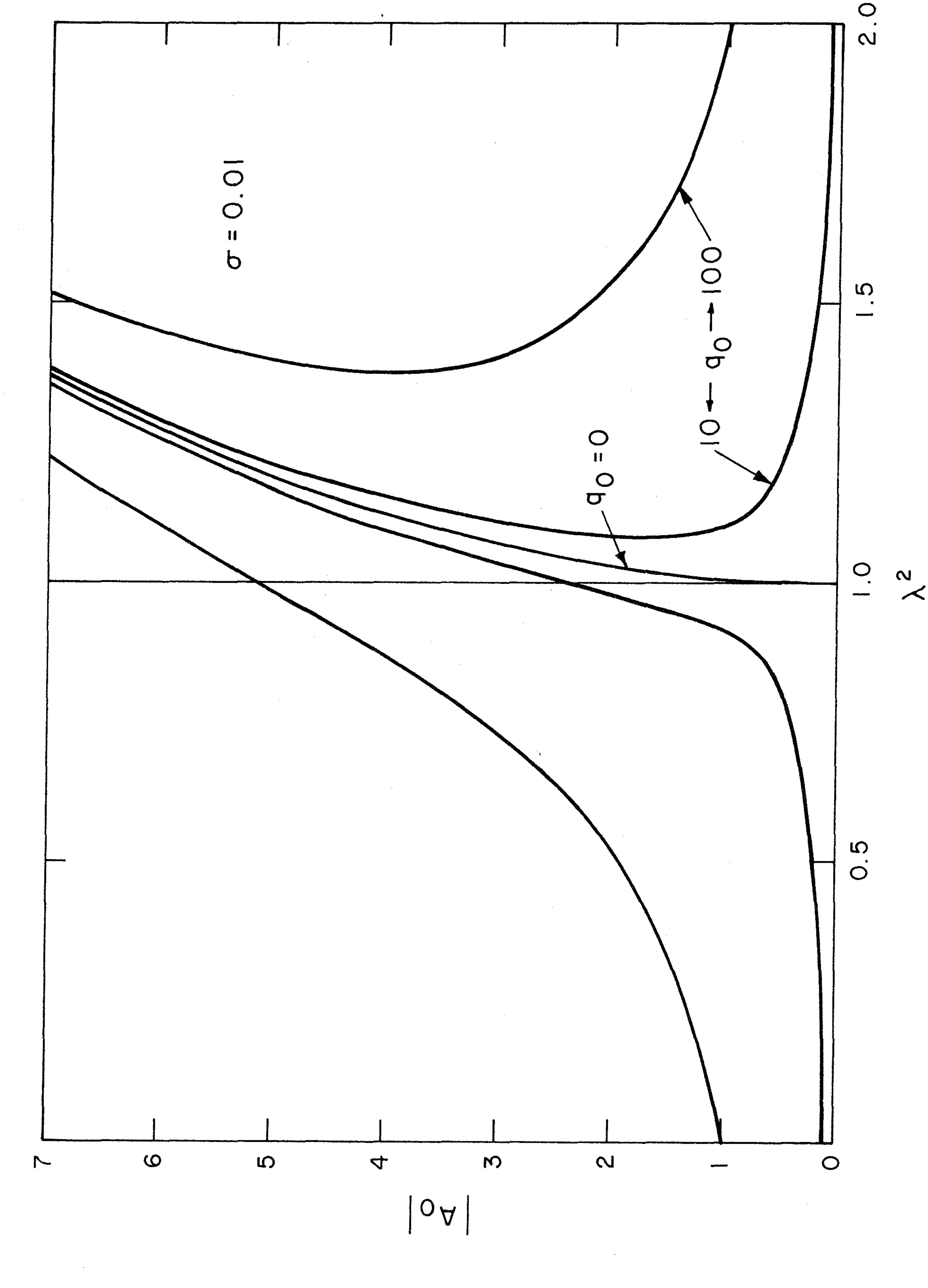
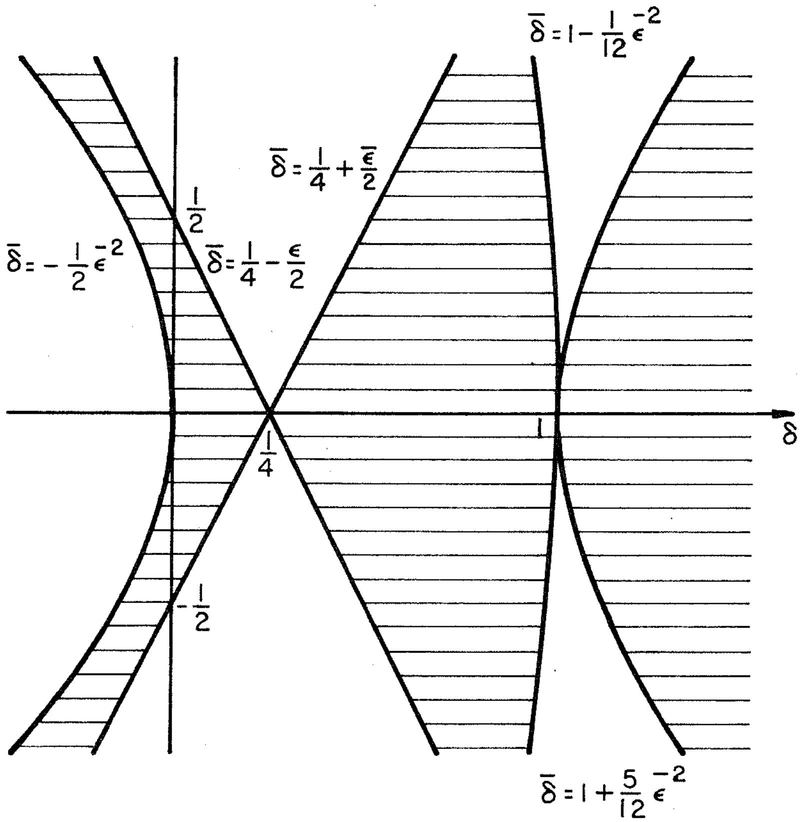
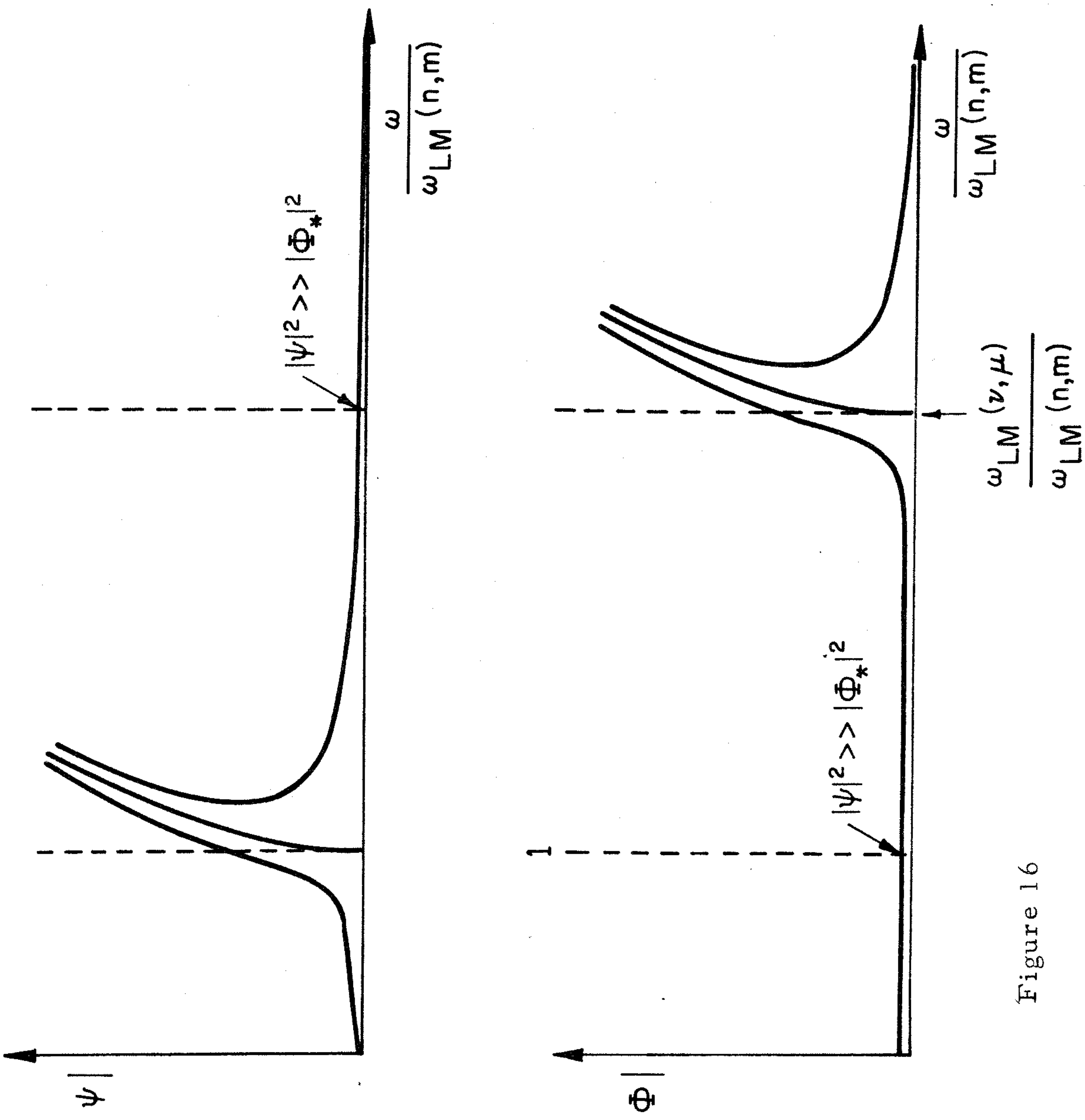


Figure 14



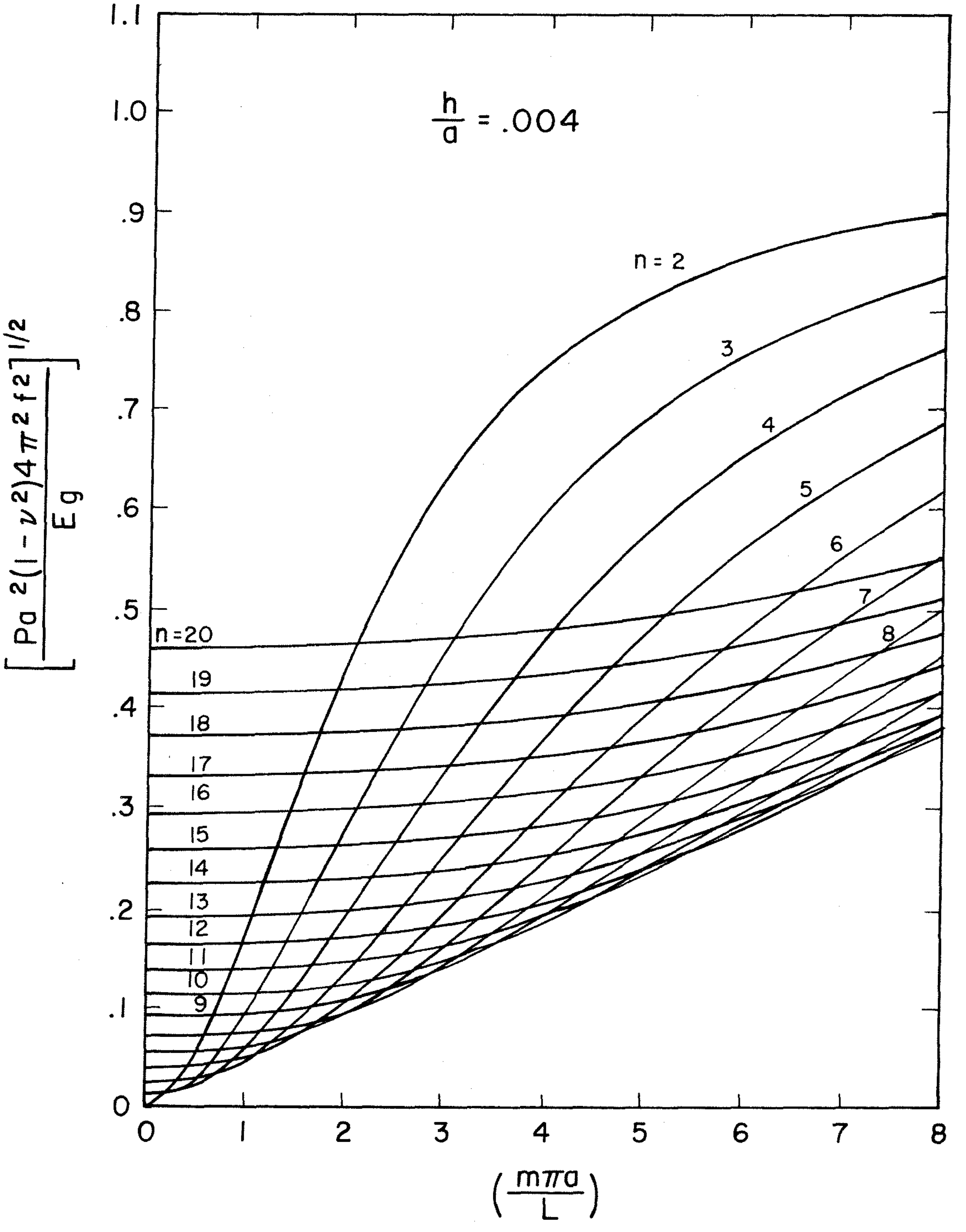
$\bar{\delta} = 1 - \frac{1}{12}\epsilon^{-2}$
$\bar{\delta} = \frac{1}{4} + \frac{\epsilon}{2}$
$\frac{1}{2}$
$\bar{\delta} = -\frac{1}{2}\epsilon^{-2}$
$\bar{\delta} = \frac{1}{4} - \frac{\epsilon}{2}$
1
δ
$\frac{1}{4}$
$-\frac{1}{2}$
$\bar{\delta} = 1 + \frac{5}{12}\epsilon^{-2}$

Figure 15



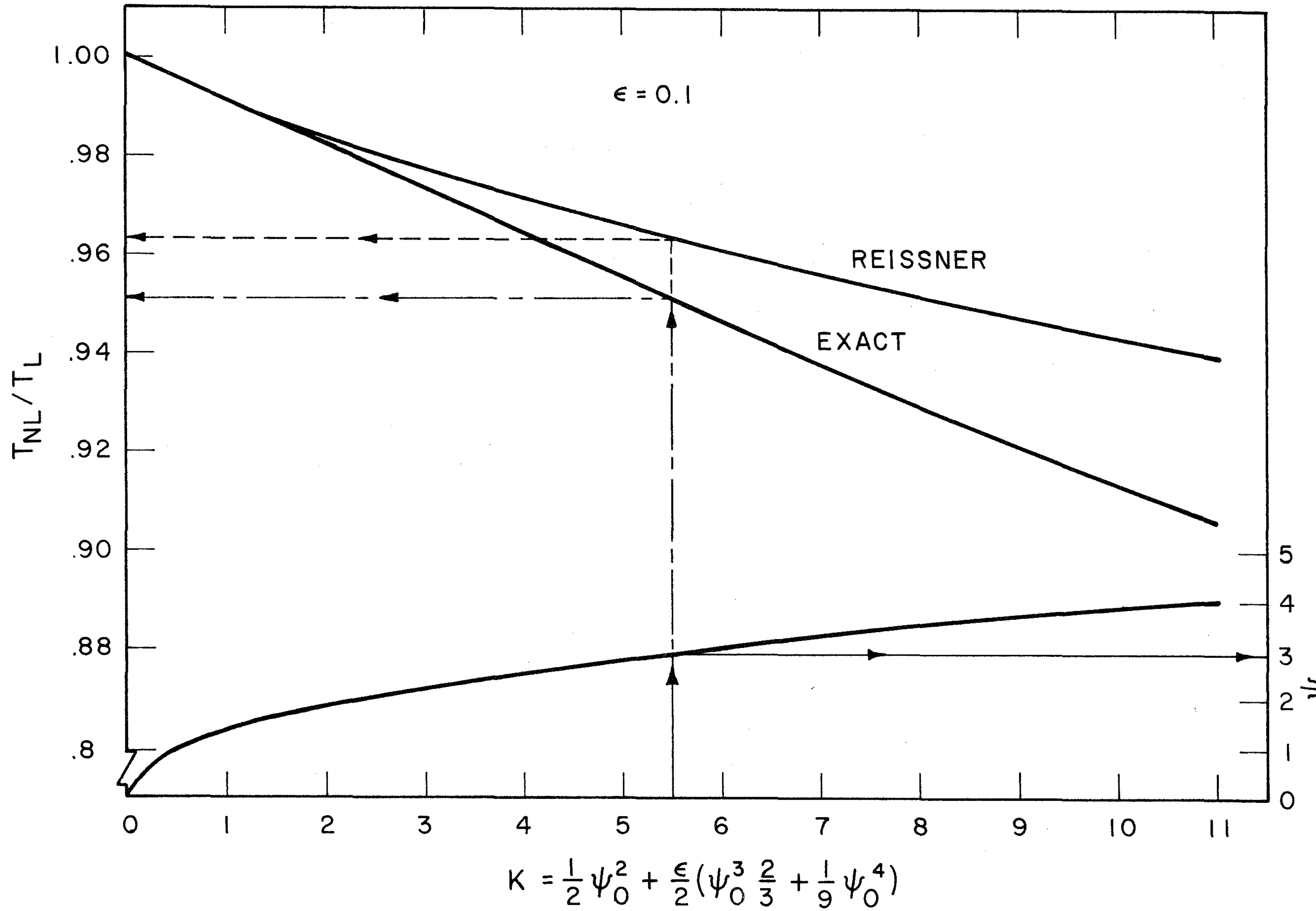
$|\psi|$
$|\psi|^2 >> |\Phi_*|^2$
$\frac{\omega}{\omega_{LM}(n,m)}$
$|\Phi|$
1
$|\psi|^2 >> |\Phi_*|^2$
$\frac{\omega_{LM}(\nu,\mu)}{\omega_{LM}(n,m)}$
$\frac{\omega}{\omega_{LM}(n,m)}$

Figure 16



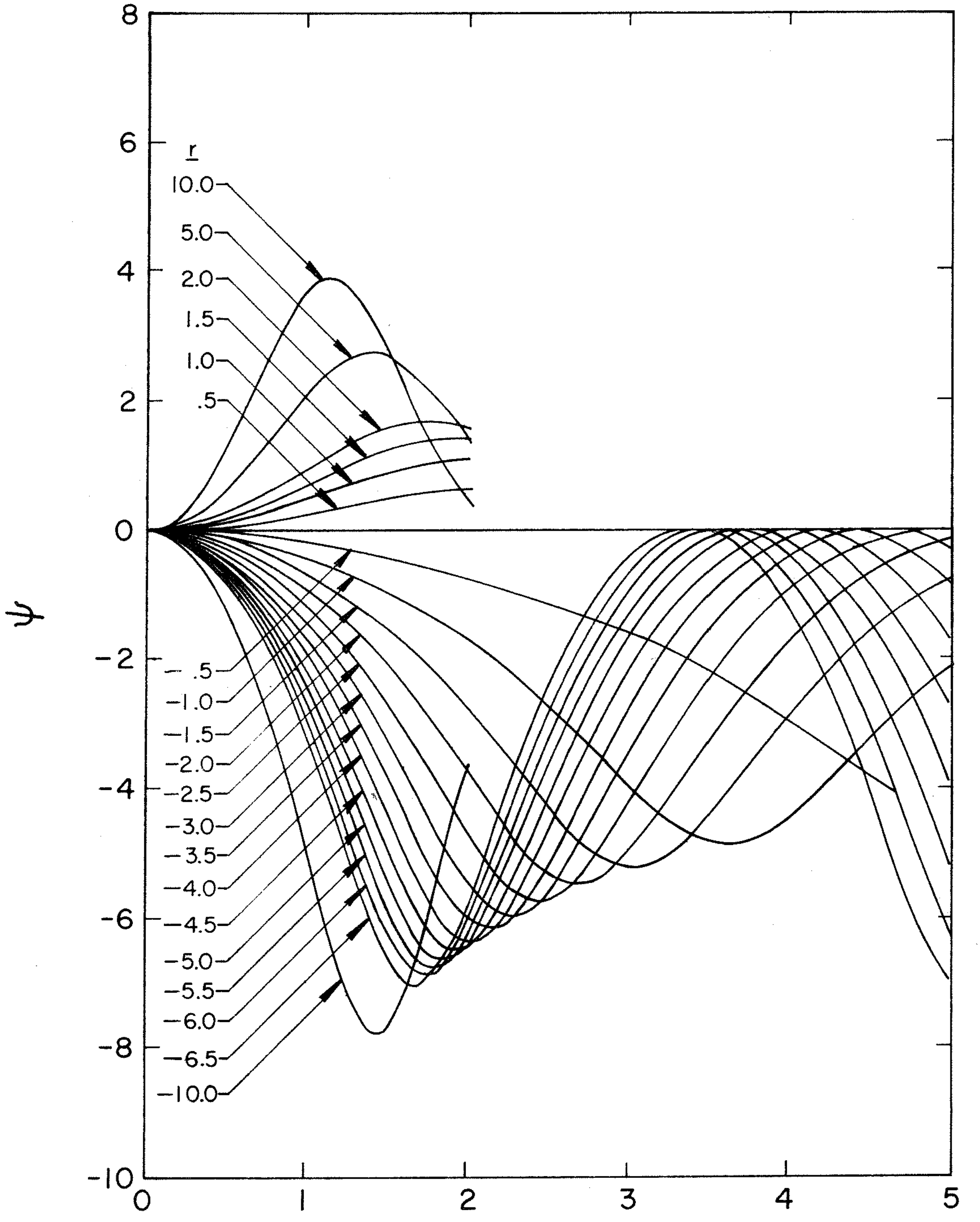
$\frac{h}{a} = .004$
n = 2
3
4
5
6
7
8
n=20
19
18
17
16
15
14
13
12
11
10
9
$\left[\frac{Pa^2(1-\nu^2)4\pi^2 f^2}{Eg}\right]^{1/2}$
$\left(\frac{m\pi a}{L}\right)$
1.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
0
1
2
3
4
5
6
7
8

Figure 17



ε = 0.1
REISSNER
EXACT
T_{NL}/T_L
1.00
.98
.96
.94
.92
.90
.88
.8
0
1
2
3
4
5
6
7
8
9
10
11
ψ_0
$K = \frac{1}{2}\psi_0^2 + \frac{\epsilon}{2}(\psi_0^3 \frac{2}{3} + \frac{1}{9}\psi_0^4)$

Figure 18



8
6
4
2
0
-2
-4
-6
-8
-10
ψ
r
10.0
5.0
2.0
1.5
1.0
.5
−.5
−1.0
−1.5
−2.0
−2.5
−3.0
−3.5
−4.0
−4.5
−5.0
−5.5
−6.0
−6.5
−10.0
0
1
2
3
4
5
TIME (ϵ=1)

Figure 19

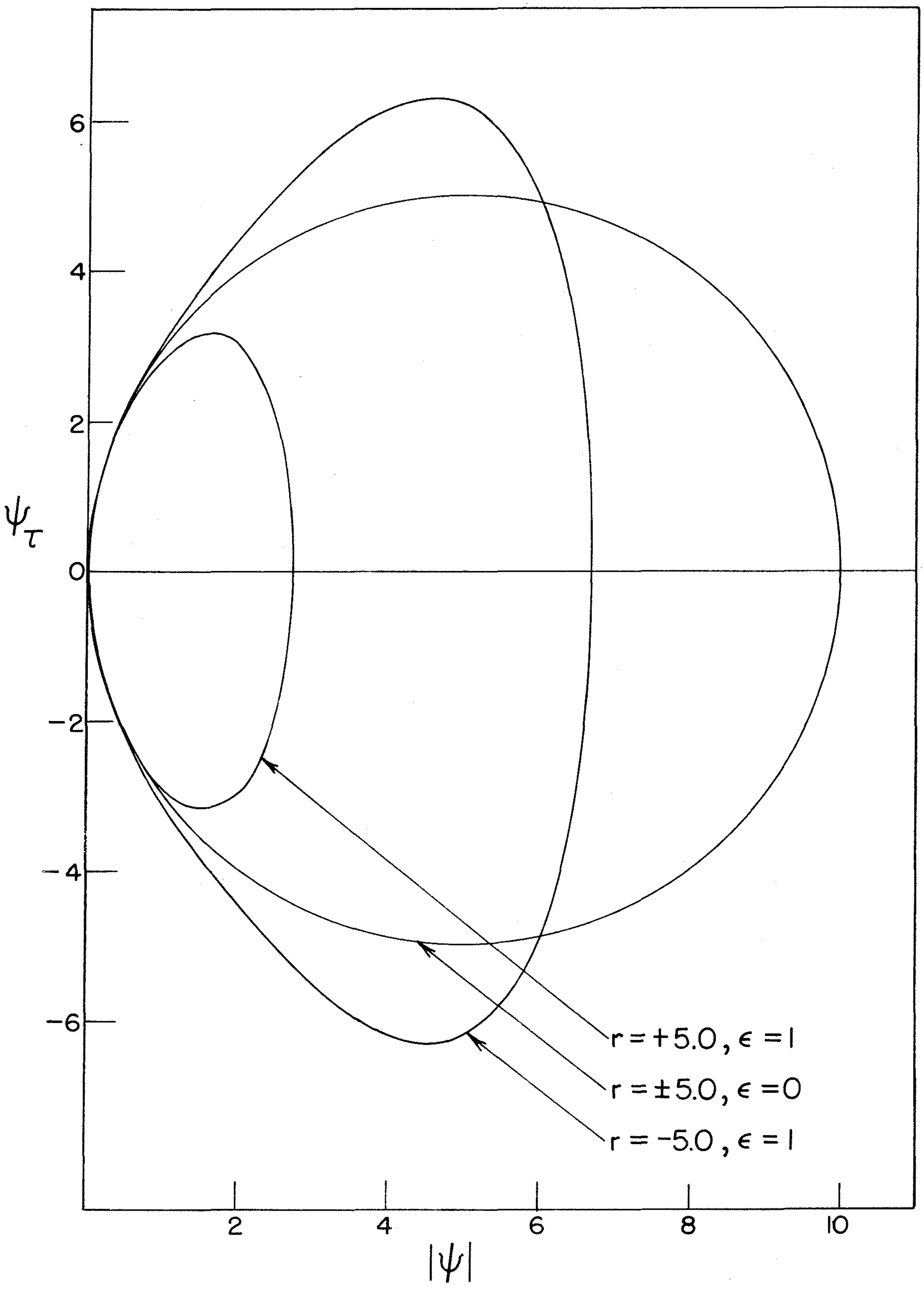

6
4
2
0
-2
-4
-6
ψ_τ
$r = +5.0, \epsilon = 1$
$r = \pm 5.0, \epsilon = 0$
$r = -5.0, \epsilon = 1$
2
4
6
8
10
$|\psi|$

Figure 20

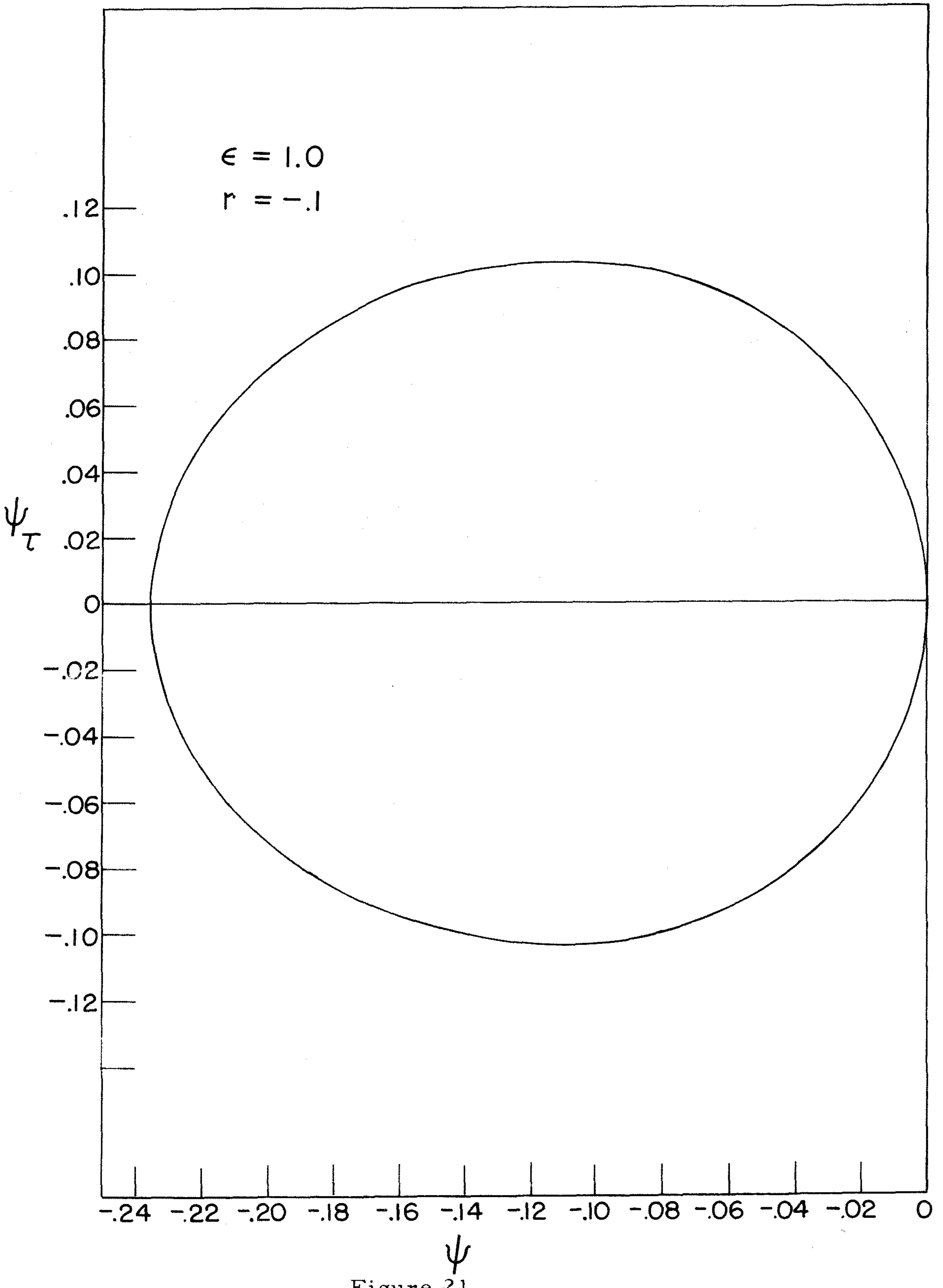
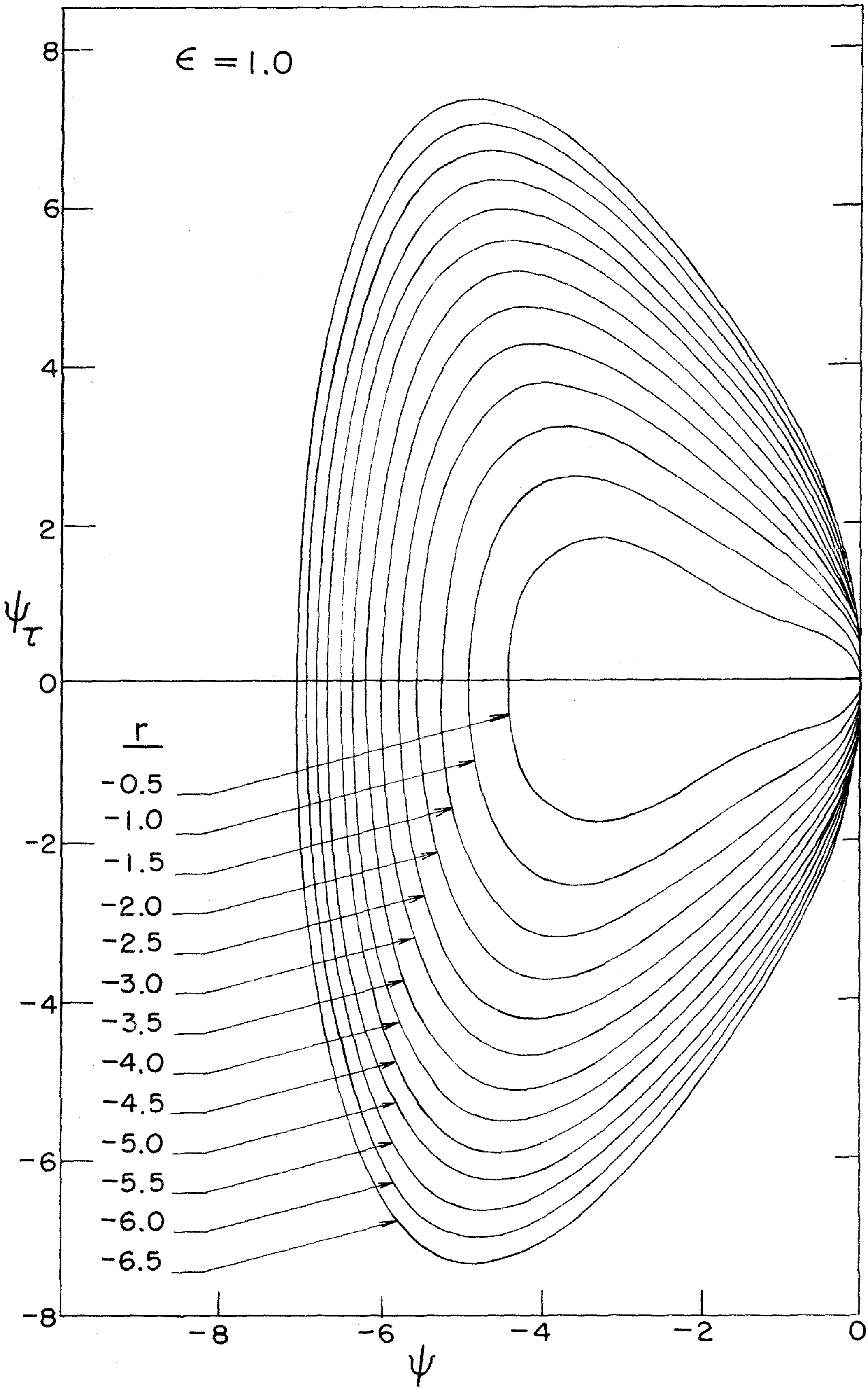

$\epsilon = 1.0$
$r = -.1$
.12
.10
.08
.06
.04
.02
0
-.02
-.04
-.06
-.08
-.10
-.12
ψ_τ
-.24
-.22
-.20
-.18
-.16
-.14
-.12
-.10
-.08
-.06
-.04
-.02
0
ψ

Figure 21



$\epsilon = 1.0$
8
6
4
2
ψ_τ
0
-2
-4
-6
-8
r
-0.5
-1.0
-1.5
-2.0
-2.5
-3.0
-3.5
-4.0
-4.5
-5.0
-5.5
-6.0
-6.5
-8
-6
-4
-2
0
ψ

Figure 22

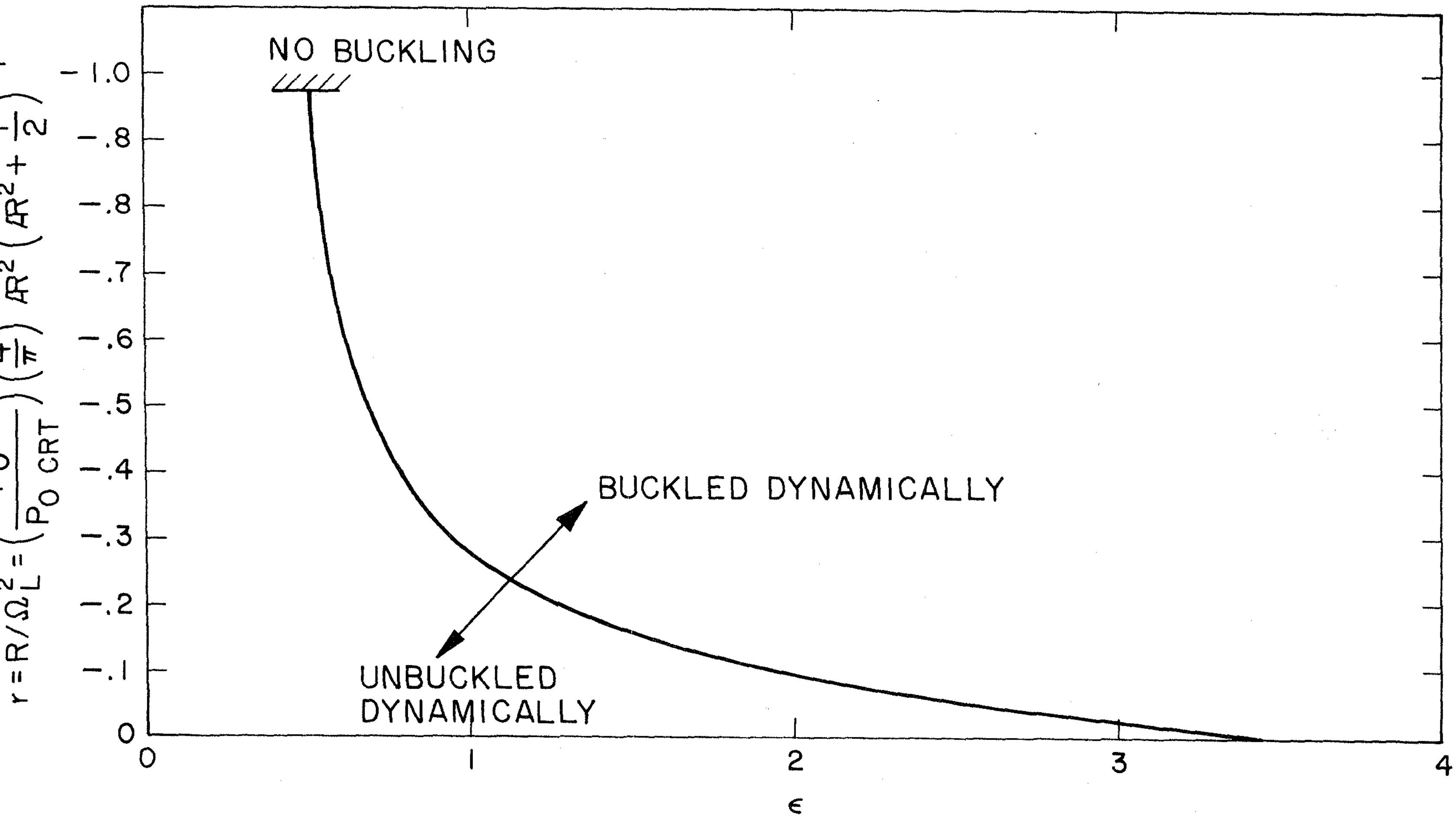

NO BUCKLING
BUCKLED DYNAMICALLY
UNBUCKLED
DYNAMICALLY
$r = R/\Omega_L^2 = \left(\frac{P_O}{P_{O\,CRT}}\right)\left(\frac{4}{\pi}\right)^4 \text{Æ}^2 \left(\text{Æ}^2 + \frac{1}{2}\right)^{-1}$
-1.0
-.8
-.8
-.7
-.6
-.5
-.4
-.3
-.2
-.1
0
0
1
2
3
4
ϵ

Figure 23

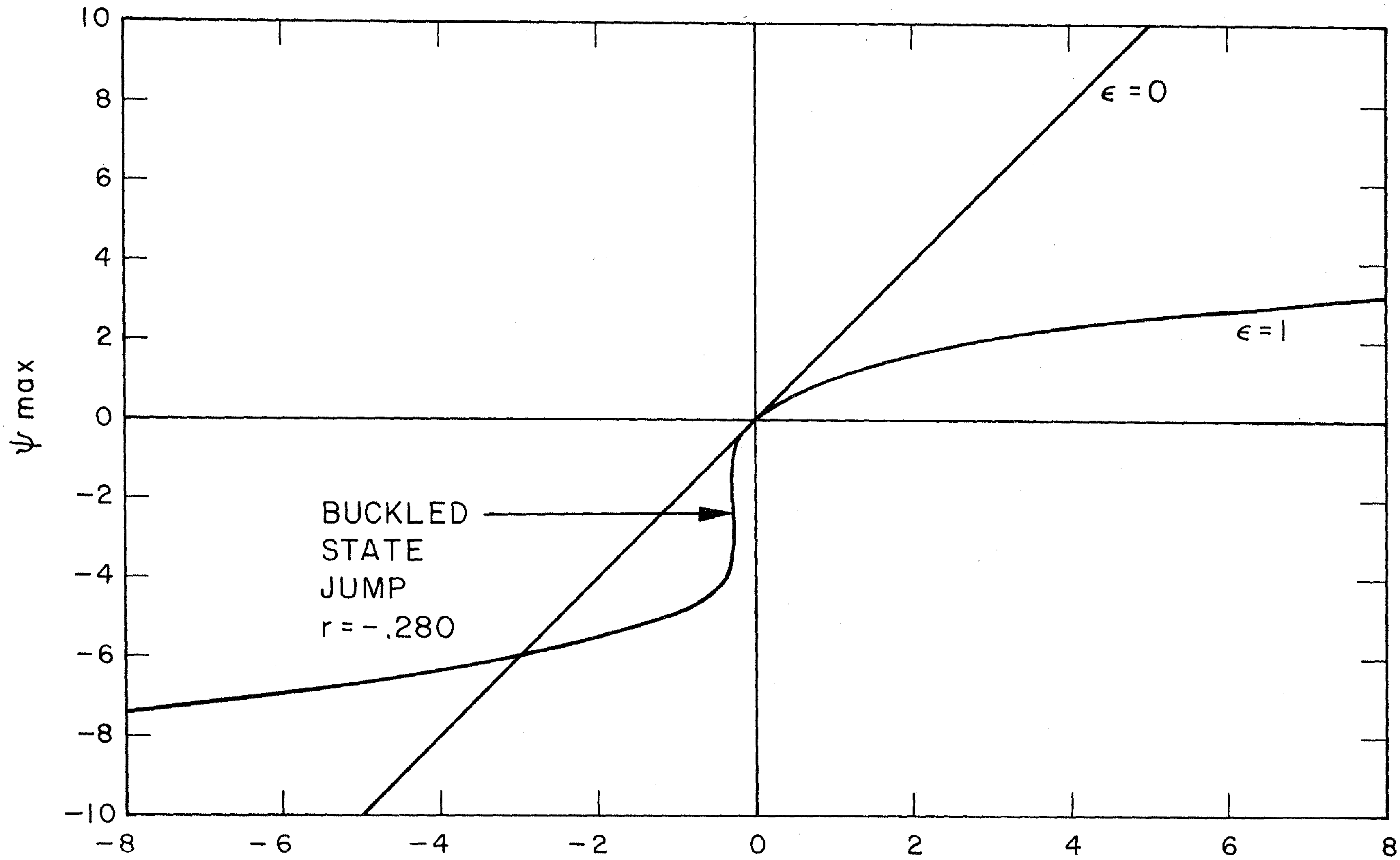

10
8
6
4
2
0
-2
-4
-6
-8
-10
ψ max
-8
-6
-4
-2
0
2
4
6
8
ε = 0
ε = 1
BUCKLED
STATE
JUMP
r = -.280

Figure 24

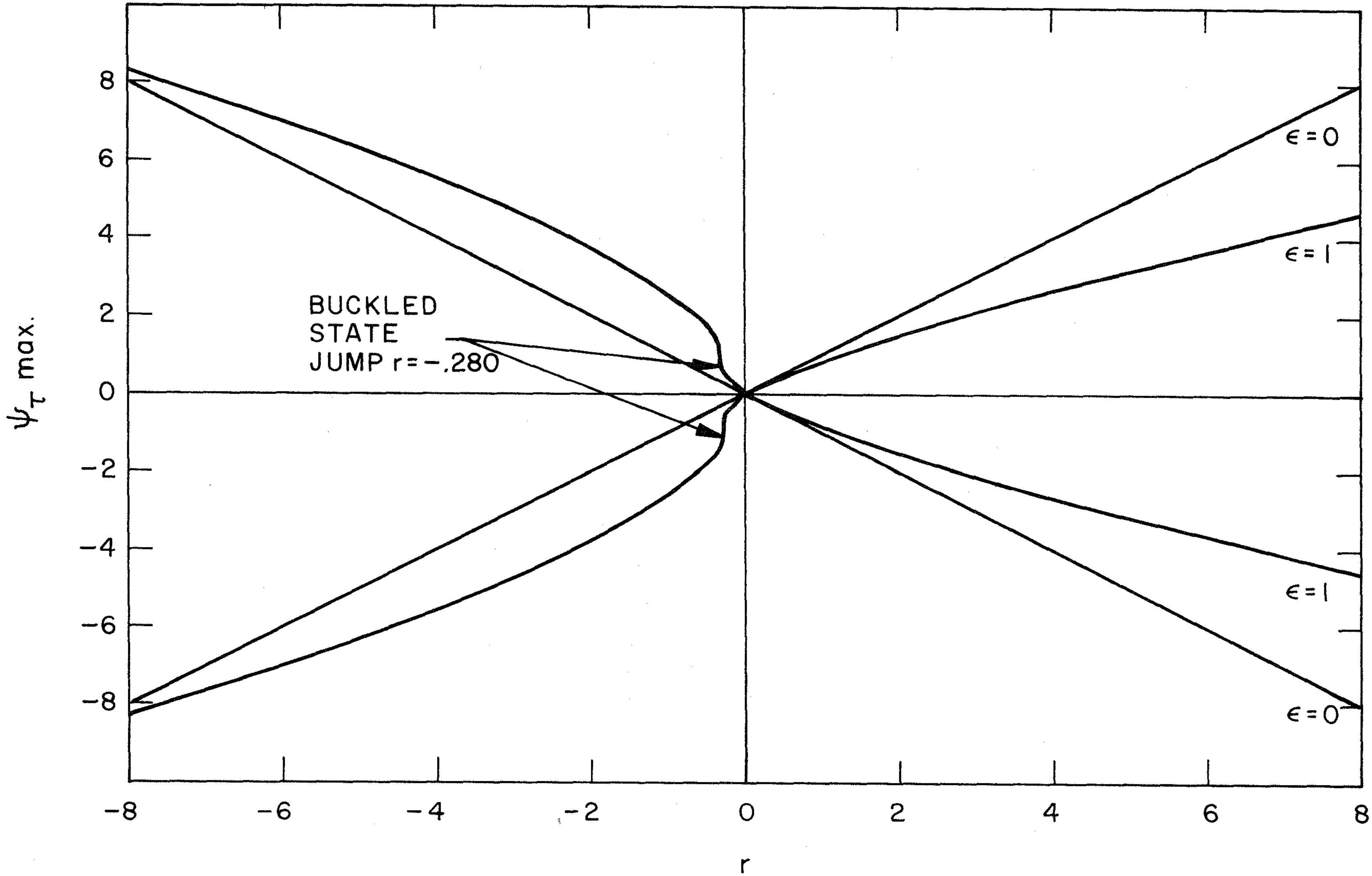

8
6
4
2
0
-2
-4
-6
-8
ψ_τ max.
BUCKLED
STATE
JUMP r = -.280
$\epsilon = 0$
$\epsilon = 1$
$\epsilon = 1$
$\epsilon = 0$
-8
-6
-4
-2
0
2
4
6
8
r

Figure 25

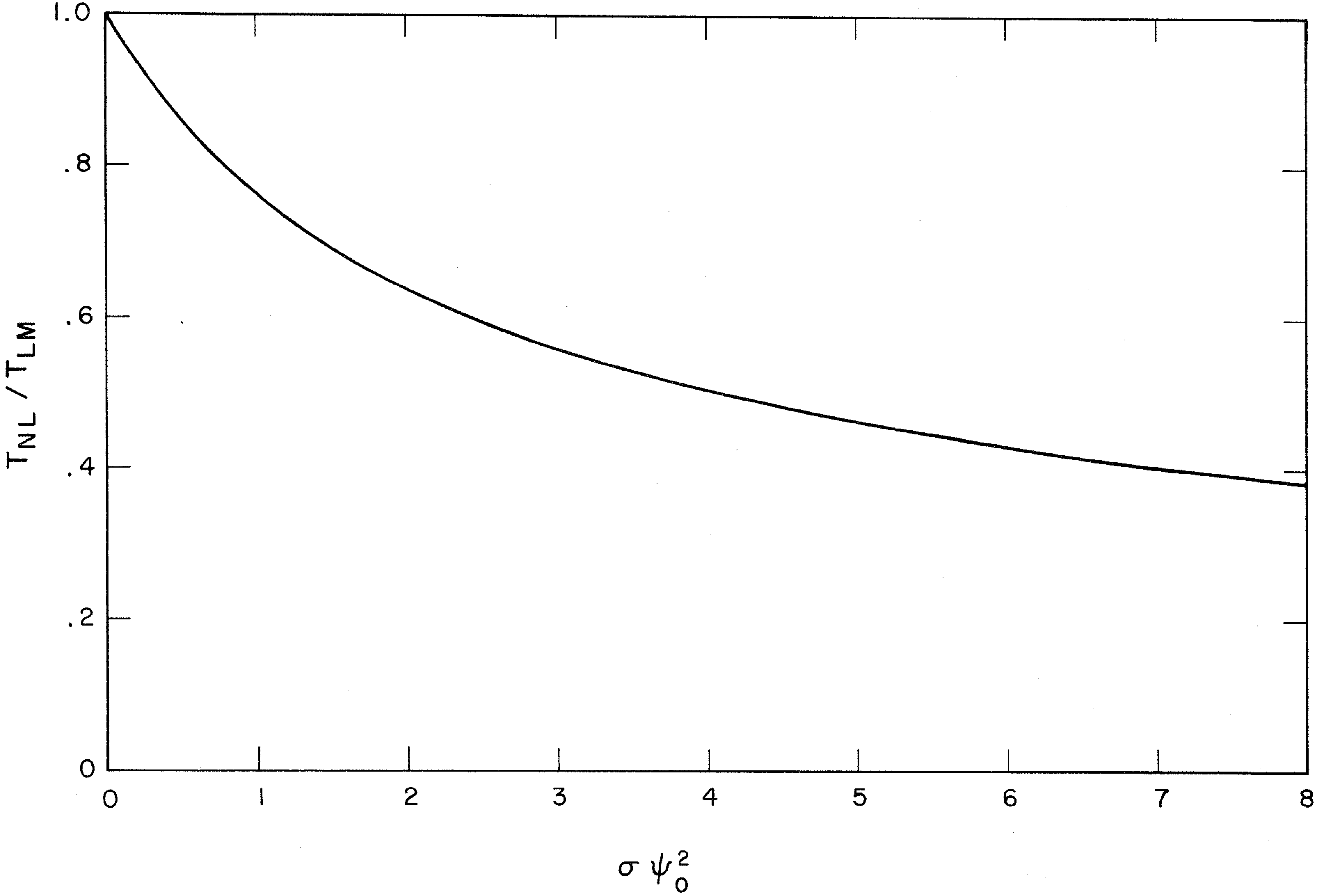


Figure 26

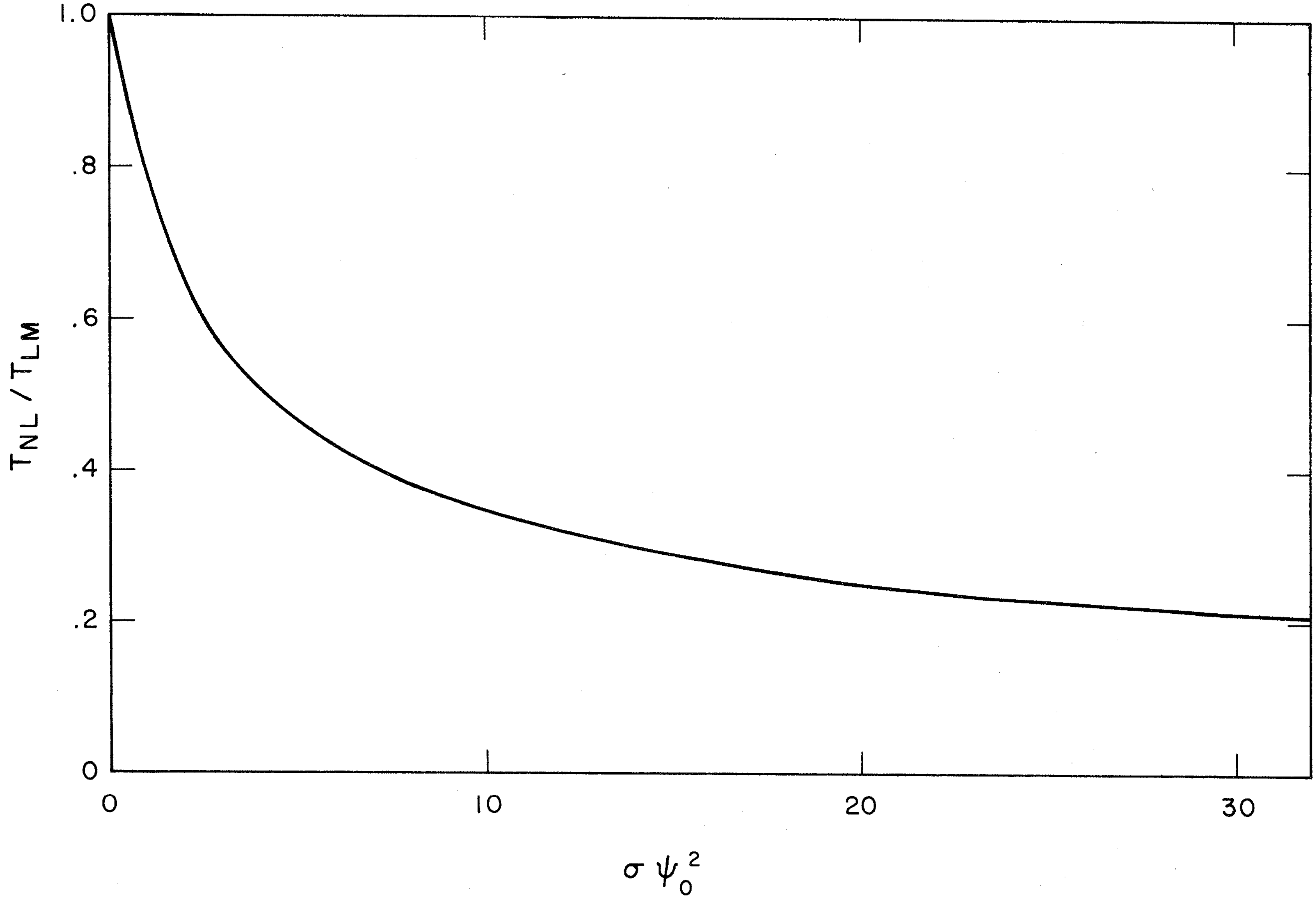


Figure 27

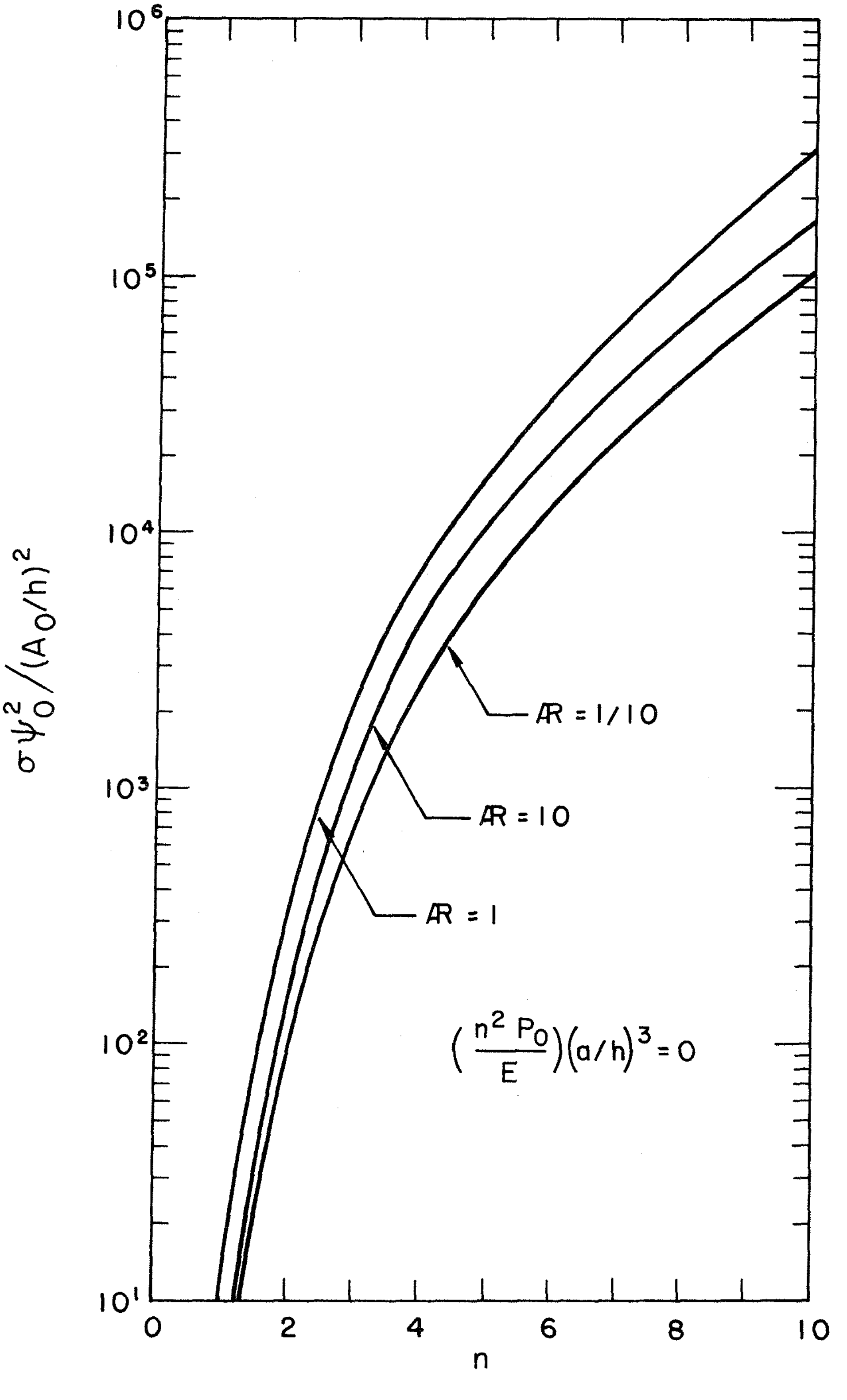

10^6
10^5
10^4
10^3
10^2
10^1
0
2
4
6
8
10
n
$\sigma\psi_0^2/(A_0/h)^2$
AR = 1/10
AR = 10
AR = 1
$\left(\frac{n^2 P_0}{E}\right)(a/h)^3 = 0$

Figure 28

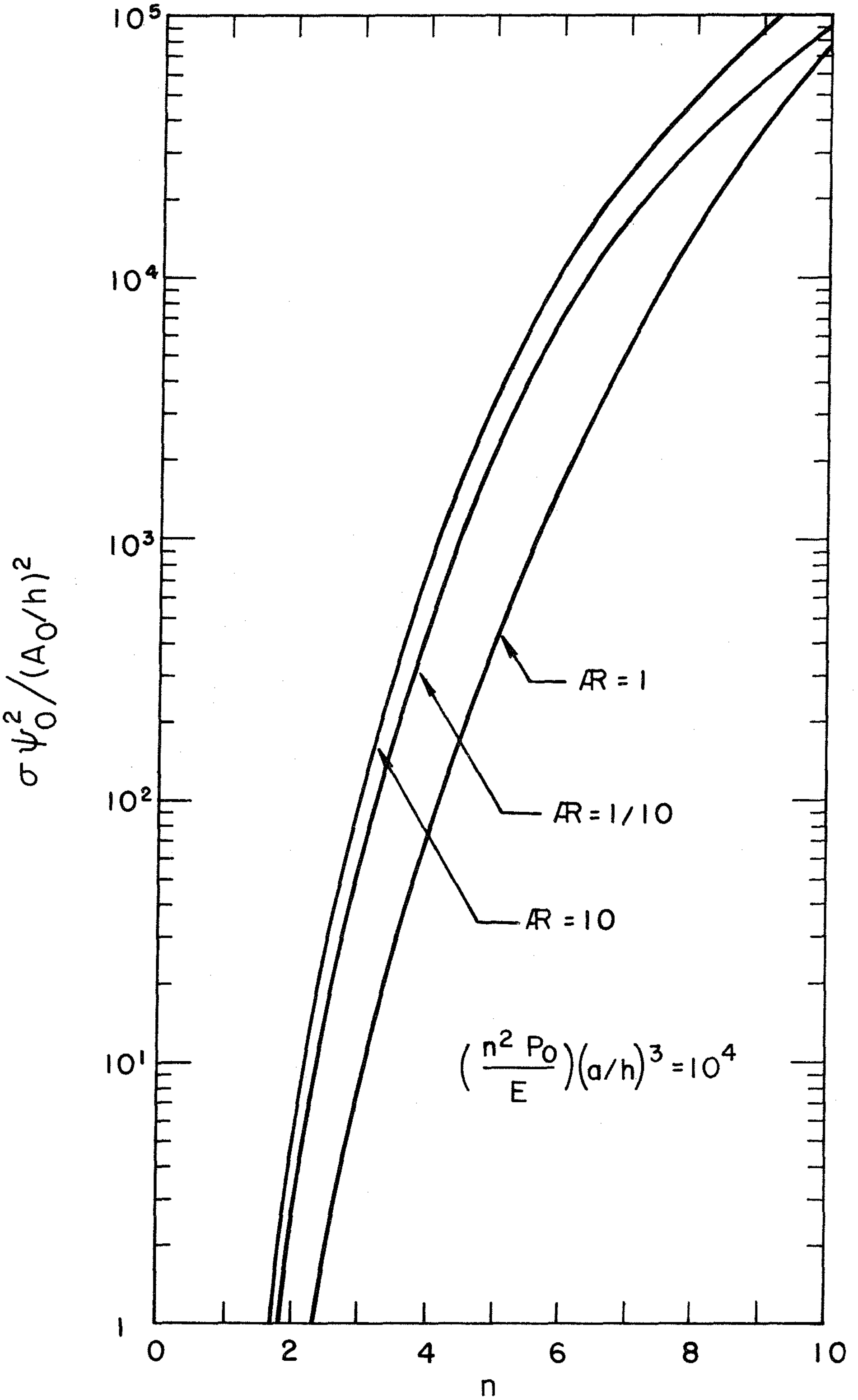

10^5
10^4
10^3
10^2
10^1
1
0
2
4
6
8
10
n
$\sigma \psi_0^2 / (A_0/h)^2$
Æ = 1
Æ = 1/10
Æ = 10
$\left(\frac{n^2 P_0}{E}\right)(a/h)^3 = 10^4$

Figure 29

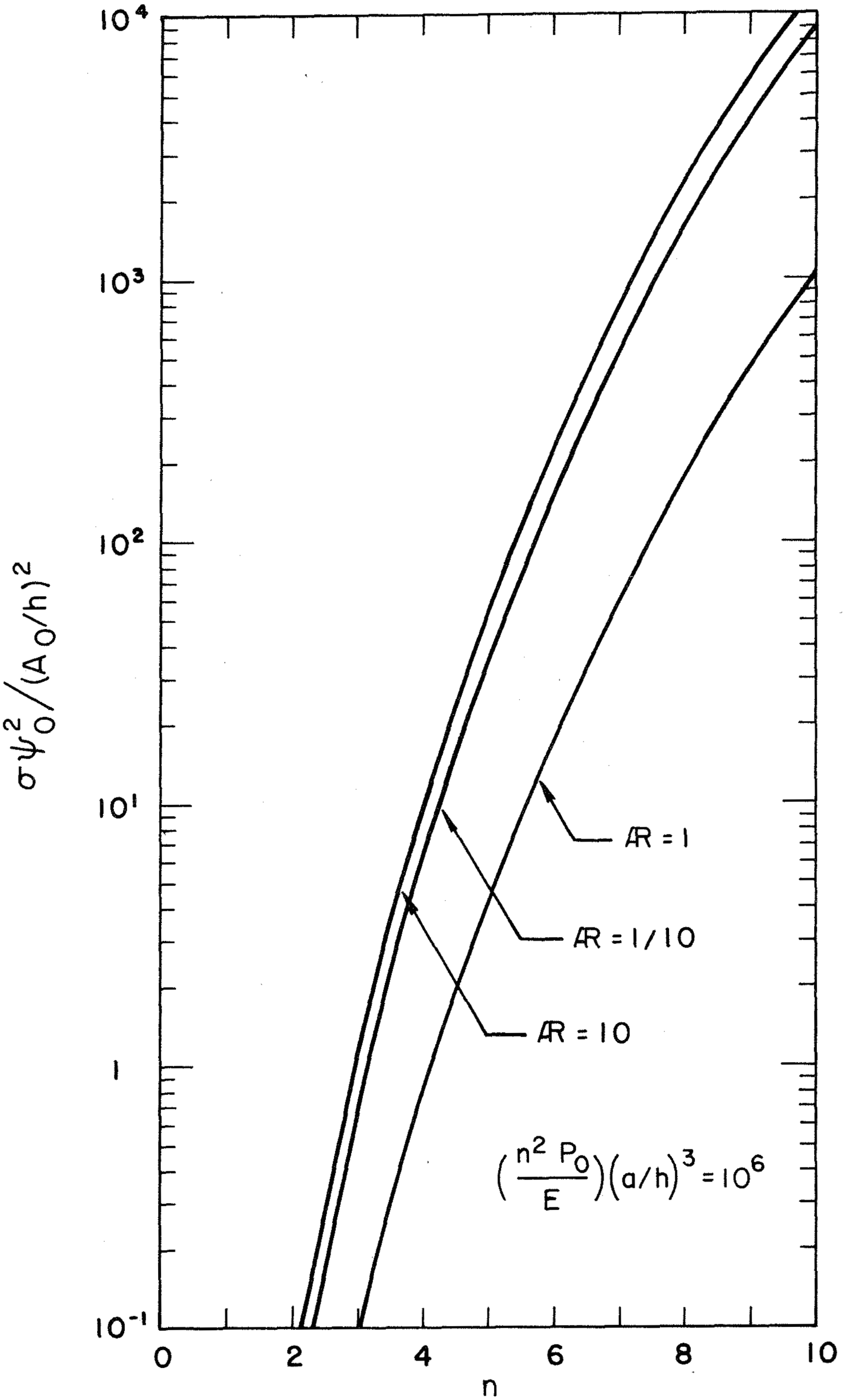

10^4
10^3
10^2
10^1
1
10^{-1}
0
2
4
6
8
10
n
$\sigma\psi_0^2/(A_0/h)^2$
$AR = 1$
$AR = 1/10$
$AR = 10$
$\left(\frac{n^2 P_0}{E}\right)(a/h)^3 = 10^6$

Figure 30