THE UNIFORMLY VALID ASYMPTOTIC APPROXIMATIONS TO THE SOLUTIONS OF CERTAIN NON-LINEAR ORDINARY DIFFERENTIAL EQUATIONS

Thesis by

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ABSTRACT

This work deals with the application of an expansion procedure in terms of two independent time variables for the uniform asymptotic representation of solutions representing certain mechanical systems.

The method is first applied to systems governed by the equation

$$\frac{d^2y}{dt^2} + \epsilon f(y, \frac{dy}{dt}) + y = 0$$

where ϵ is a small parameter, and f has the character of a damping (i. e. y is a bounded function of t for all $t \geqslant 0$).

It is shown that the physical problems which can be brought to the above non-dimensional form possess two characteristic time scales, one associated with the oscillatory behavior of the solution, while the other measures the time interval in which the effects of the non-linear term become apparent.

The dependence of the solution on these time scales is not simple, in the sense that an asymptotic representation of the exact solution which is valid for large times cannot be obtained by a limit process in which a non-dimensional time variable is held fixed. This fact has motivated the introduction of an expansion procedure in functions of two time variables, and it is shown that with the use of certain simple boundedness criteria a uniform asymptotic representation can be derived.

In addition to the above mentioned class of problems a variety of examples possessing certain boundedness properties is studied by this

method, including, for example, the Mathieu equation.

The main emphasis of this paper is on the constructive rather than general approach to the solutions of specific examples. These examples are introduced in turn to illustrate the underlying ideas of the method, whose main advantage is its simplicity especially for computing the higher approximations.

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I. INTRODUCTION

This work deals with the application of an expansion procedure in terms of two independent time variables for the uniform asymptotic representation of the solutions of certain mechanical systems.

We will present a technique which applies to a large number of problems in mechanics. In the first instance we will consider mechanical systems having a predominantly linear restoring force, and a small non-linear damping. This class of problems is characterized by the fact that (i) in the absence of the non-linear perturbing forces, the motion is simple-harmonic, and (ii) the complete motion is a slowly damped oscillation.

In particular, we will first illustrate the method of solution by presenting the detailed calculations for the linear problem, and then we will discuss the following three examples:

(i) The oscillator with a small cubic damping.

(ii) The small amplitude oscillations of a damped pendulum.

(iii) The general solution of Van der Pol's equation.

We will next consider miscellaneous problems which do not belong in the above category, but whose solutions possess certain boundedness criteria.

The following problems will be discussed in detail:

(i) The motion of a charged particle in a slowly varying magnetic field.

(ii) The problem of beats in forced oscillations.

(iii) The solutions of Mathieu's equation in neighborhoods of the transitional curves from stability to instability.

(iv) The drag perturbation of a satellite orbit.

(v) Planar orbits in the vicinity of the small body in the restricted three-body problem.

In the non-dimensional formulation of the differential equations governing both of the above classes of problems, we will find two significant time scales. In all cases, the dependence of the solutions on these two time scales will not be simple, in the sense that expressions valid for large times cannot be derived from the exact solution by a limit process in which only one of the non-dimensional time variables is held fixed. This fact will motivate the quest for an expansion procedure in which both non-dimensional time variables appear. As a consequence, we will transform the ordinary differential equation governing the motion into a partial differential equation with respect to the two time variables. The indeterminancy introduced into the solution by this conversion will be removed by requiring that the asymptotic expansion for the solution be a bounded function of the two time scales. This boundedness criterion, which is a property of the initial value problems we consider, will lead us to uniformly valid asymptotic developments.

The main emphasis in this paper will be on the constructive approach to the solution of specific examples. Although we will derive certain general forms for the solutions of the damped oscillatory motions, we will rely on the numerous examples in order to illustrate the underlying ideas, and draw comparisons between this method and others.

In this connection we should mention that the class of problems having a small damping can be solved by the method of N. Kryloff and N. Bogoliuboff outlined in reference 1. In a recent paper Kuzmak (2) has presented a more general technique applicable to this class of problems. His solution, like ours, is derived from a partial differential equation with respect to two independent time variables. The type of equations Kuzmak considers are more general than ours in the following respect. His method is applicable to systems with a small non-linear damping and a restoring force term which is an arbitrary function of the displacement and time. Thus, the unperturbed motion is not necessarily simple harmonic. His method, like ours, is restricted to systems where the frequency does not depend on the time. The main advantage of our method is its simplicity, especially for computing the higher approximations. Moreover, the basic nature of the underlying principles allows us to tackle a large variety of problems whose only common property is the boundedness of the motion for large times.

II. DAMPED OSCILLATIONS (FORMULATION OF THE PROBLEM)

2.1. General Remarks

We will show in Chapter III that the initial value problems governing the motion of mechanical systems with a small non-linear damping, and a restoring force departing slightly from linearity, can be brought to the non-dimensional form:

$$\frac{d^2y}{dt^{*2}} + \epsilon f(y, \frac{dy}{dt^*}) + y = 0 \qquad (2.1a)$$

$$y(0) = a \qquad (2.1b)$$

$$\frac{dy(0)}{dt^*} = 0 \qquad (2.1c)$$

In the above, ϵ is a small positive non-dimensional parameter proportional to the damping coefficient, and y and t^* are the non-dimensional displacement and time respectively.

In order that equation 2.1a describe damped oscillations, the function f must have certain properties which we can easily deduce by examining the trajectories in the phase plane.

Consider the quantity $R = \sqrt{y^2 + V^2}$ where $V = dy/dt^*$. Clearly $R = \sqrt{2E}$ where E is the total non-dimensional energy of the particle. In the phase plane, the curves $R =$ constant are concentric circles centered at the origin as shown in fig. 2.1.

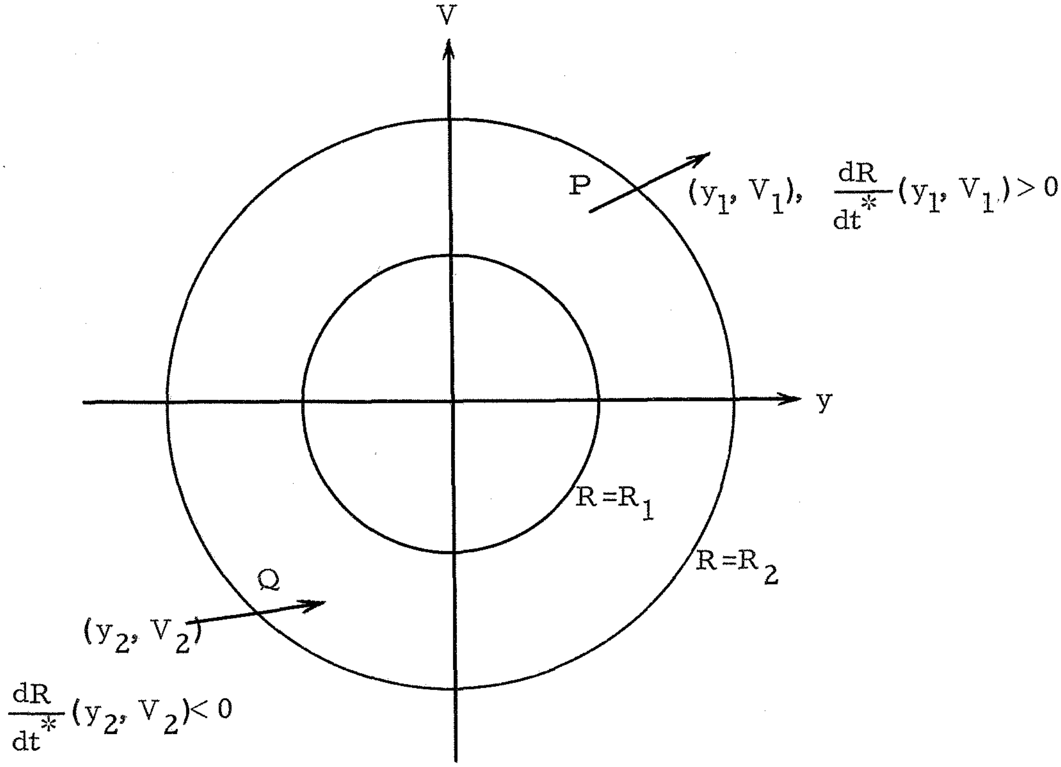


Figure 2.1

Now it is easy to show that R satisfies the equation

$$\frac{dR}{dt^*} = -\frac{\epsilon V f(y, V)}{\sqrt{y^2 + V^2}} \tag{2.2}$$

where y is a solution of equation 2.1a.

Thus for a given function f, equation 2.2 defines dR/dt^* at every point (y, V) in the phase plane, and the sign of dR/dt^* determines whether the trajectory is running into or out of the circle passing through that point. This is illustrated by the trajectories through the points P and Q shown in fig. 2.1.

In this work we will only consider functions f such that dR/dt^* is always negative whenever R is greater than some finite R_L. It is clear that under these conditions the solution of equation 2.1 must be a bounded function of t^* for all $t^* \geq 0$.

There is no loss of generality in assuming, as in equation 2.1c, that the initial velocity is zero, for this merely fixes the origin of the time scale for each value of a.

2.2. Initially valid expansions

Let us now define a limit process which we will use to construct an asymptotic expansion of the solution of equation 2.1.

Let y denote the exact solution of equation 2.1, and let $\{\xi_n(\epsilon)\}$ be an asymptotic sequence as $\epsilon \to 0$ with $\xi_o = 1$. We say that

$$h^{(N)}(t^*, \epsilon) = \sum_{n=0}^{N} h_n(t^*)\xi_n(\epsilon) \tag{2.3}$$

is an "initially valid" asymptotic expansion of y with respect to the sequence $\{\xi_n\}$ if the h_n are derived from y by the successive application of the following limit process for each $N = 0, 1, 2, \ldots$

$$\lim_{\substack{\epsilon \downarrow 0 \\ t^* \text{ fixed}}} \frac{y(t^*, \epsilon) - h^{(N)}(t^*, \epsilon)}{\xi_N(\epsilon)} \tag{2.4}$$

For a given function $y(t^*, \epsilon)$ and a given sequence $\{\xi_n\}$ the h_n can be evaluated uniquely. However, y is known only to the extent that it is the solution of the differential equation 2.1, and the sequence $\{\xi_n\}$ is somewhat arbitrary.

It is plausible (and quite often true) that for physically meaningful differential equations the h_n are the solutions of differential equations obtained by the recursive application of the limit process 2.4 to the original differential equation. For lack of any precise conditions on problems where this interchange of limits is valid, we should regard it as a very plausible assumption.

It is easy to verify that for equation 2.1 only the sequence $\{\epsilon^n\}$ will lead to non-trivial equations for the h_n, and these are

$$\frac{d^2h_o}{dt^{*2}} + h_o = 0 \tag{2.5a}$$

$$\frac{d^2h_1}{dt^{*2}} + h_1 = -f(h_o, \frac{dh_o}{dt^*}) \tag{2.5b}$$

$$\frac{d^2h_2}{dt^{*2}} + h_2 = -h_1 \frac{\partial}{\partial y} f(h_o, \frac{dh_o}{dt^*}) - \frac{dh_1}{dt^*} \frac{\partial}{\partial(\frac{dy}{dt^*})} f(h_o, \frac{dh_o}{dt^*}) \tag{2.5c}$$

$$\frac{d^2h_n}{dt^{*2}} + h_n = H\left[h_o, \ldots, h_{n-1}, \frac{dh_o}{dt^*}, \ldots \frac{dh_{n-1}}{dt^*}, \frac{\partial^{n-1}f}{\partial y^{n-1}}, \ldots \frac{\partial^{n-1}f}{\partial y^{n-m}\partial(\frac{dy}{dt^*})^{m-1}}, \ldots \frac{\partial^{n-1}f}{\partial(\frac{dy}{dt^*})^{n-1}}\right] \tag{2.5d}$$

The general term on the right-hand side of equation 2.5d cannot be given concisely, but can be computed for each value of n by expanding f in its Taylor series in the neighborhood of the arguments h_o, dh_o/dt^*.

Equations 2.1b and c imply that the h_n satisfy the initial conditions

$$h_o(0) = a, \quad h_n(0) = 0 \quad n \neq 0 \tag{2.6a}$$

$$\frac{dh_n(0)}{dt^*} = 0 \tag{2.6b}$$

The solution of equation 2.5a satisfying the conditions 2.6 is $h_o = a \cos t^*$. Since h_o is periodic, we may represent the right-hand side of equation 2.5b by its Fourier series expansion

$$-f(h_o, \frac{dh_o}{dt^*}) = \frac{a_o}{2} + \sum_{n=1}^{\infty} [a_n \cos nt^* + b_n \sin nt^*] \tag{2.7a}$$

where

$$a_n = -\frac{1}{\pi}\int_0^{2\pi} f(a \cos t^*, - a \sin t^*) \cos nt^* \, dt^* \tag{2.7b}$$

$$b_n = -\frac{1}{\pi}\int_0^{2\pi} f(a \cos t^*, - a \sin t^*) \sin nt^* \, dt^* \tag{2.7c}$$

In general, the Fourier series 2.7a will be finite. For example, this is the case if f is a polynomial in y and dy/dt^*.

The solution of equation 2.5b satisfying the appropriate initial condition is given below in terms of the known coefficients a_n and b_n.

$$h_1 = \left[\frac{b_1}{2} - \sum_{n=2}^{\infty} \frac{nb_n}{1-n^2}\right] \sin t^* - \left[\frac{a_o}{2} + \sum_{n=2}^{\infty} \frac{a_n}{1-n^2}\right] \cos t^*$$
$$+ \sum_{n=2}^{\infty} \left[\frac{a_n}{1-n^2} \cos nt^* + \frac{b_n}{1-n^2} \sin nt^*\right] - \frac{t^*}{2}\left[a_1 \sin t^* + b_1 \cos t^*\right] \tag{2.8}$$

In a similar manner all the h_n can be successively computed.

We immediately notice the presence of the "secular" terms $-\frac{a_1}{2} t^* \sin t^*$ and $-\frac{b_1}{2} t^* \cos t^*$ in equation 2.8. In order to show that these terms do not arise due to any particular choice of f, but are merely a consequence of the expansion procedure we have adopted, let us choose $f = 2dy/dt^*$.

For this choice of f the first two terms in the expansion for h are

$$h_o + \epsilon h_1 = a \cos t^* + \epsilon a(\sin t^* - t^* \cos t^*) \qquad (2.9)$$

whereas the exact solution of equation 2.1 is

$$y = ae^{-\epsilon t^*}\left[\cos\sqrt{1-\epsilon^2}\, t^* + \frac{\epsilon}{\sqrt{1-\epsilon^2}} \sin\sqrt{1-\epsilon^2}\, t^*\right] \qquad (2.10)$$

For this example we can directly verify that $h_o + \epsilon h_1$ is the initially valid expansion of equation 2.10 to order ϵ, and that the secular term $-\epsilon at^* \cos t^*$ arises from the non-uniform* representation of the term $ae^{-\epsilon t^*}$. Without computing the explicit formulas for the other h_n, let us note that further secular terms will appear in each of the h_n, and it is easy to verify for this linear equation, that these secular terms are contributed by the non-uniform expansion of the terms $e^{-\epsilon t^*}$, $\sin\sqrt{1-\epsilon^2}\, t^*$, and $\cos\sqrt{1-\epsilon^2}\, t^*$ of the exact solution.

Although the expansion for h is not uniformly valid for large times, it is a valid representation of equation 2.10 (for finite time interval), and is analogous to an inner expansion for a boundary value singular

*That is, for large times.

perturbation problem as defined in reference 3.* To point out this analogy, let us note that for any fixed value of t^*, say T, the difference between the exact solution and the asymptotic expansion tends to zero by the limit process of equation 2.4. Thus in the t^*, ϵ plane the domain of validity for h is the shaded rectangular region sketched in fig. 2.2.

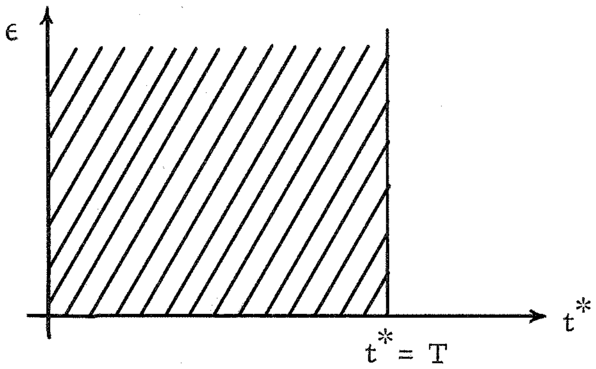


Figure 2.2

If we now transform the independent variable t^* to $\tilde{t} = \epsilon t^*$, this domain maps into the triangular region sketched in fig. 2.3.

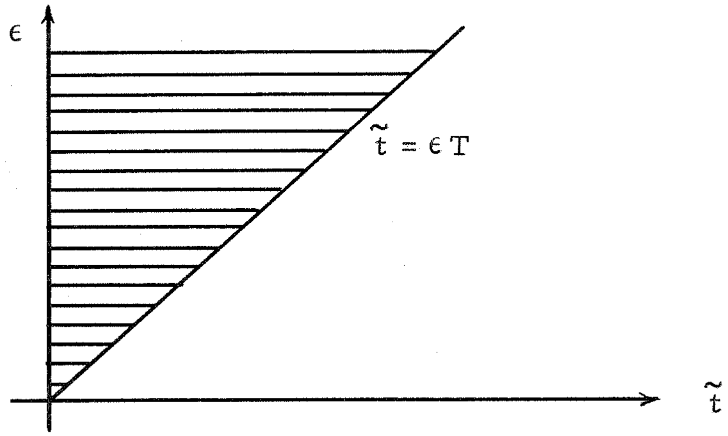


Figure 2.3

*In the ensuing discussion we will assume that the reader is familiar with the fundamental ideas of singular perturbations.

This is in exact analogy with the domain of validity of an inner expansion for a boundary value problem in which $\tilde{t}$ corresponds to a non-dimensional distance (outer variable).

2. 3. An Expansion Valid for Large Times

In view of the analogy between the inner expansion of a singular perturbation problem, and our initially valid expansion, it is natural to ask whether there exists a corresponding strict analogy between an outer expansion as defined in reference 3 and an expansion which will for our case be valid for large times. Unfortunately such an analogy does not exist. Superficially, it is easy to verify that there does not exist a limit process by which a set of meaningful limiting differential equations, valid for large times, can be derived from equation 2. 1. This fact is a reflection of the structure of the exact solutions, as can be illustrated by considering the linear equation with $f = 2dy/dt^*$. We see from the exact solution of equation 2. 10 that for any choice of a modified time variable t_η (where $t_\eta = \eta(\epsilon)t^*$, and $\text{ord}\,\epsilon \leq \text{ord}\,\eta \leq \text{ord}\,1$) there exists no limit for the exact solution whereby t_η is held fixed and $\epsilon \to 0$. Thus the exact solution does not possess a limit process expansion for large times, because the limit $\lim\limits_{\substack{\epsilon \downarrow 0 \\ t_\eta \text{ fixed}}} \begin{matrix}\cos \\ \sin\end{matrix} \sqrt{1-\epsilon^2}\,\frac{t_\eta}{\epsilon}$ does not exist.

We note that the variables t^* and $\tilde{t} = \epsilon t^*$ play distinctly different roles, at least for this linear case. The former only enters in the form $\sqrt{1-\epsilon^2}\,t^*$, in the arguments of the trigonometric terms (and can be thought of as the variable depicting the oscillatory behavior of the solution), while the latter is associated only with the damping term.

The fact that a trigonometric term whose argument is proportional to t^*, cannot have an outer limit, strongly suggests that an expansion which is valid for large times must necessarily involve both variables t^* and $\tilde{t}$. We also note that in order to represent uniformly a term such as $\sin r(\epsilon)t^*$ (where $r(\epsilon) = O(1)$ as $\epsilon \to 0$) we must account for the quantity $r(\epsilon)t^*$ by a new variable of the form $t^+ = (1 + \epsilon^2\omega_2 + \epsilon^3\omega_3 + \ldots)$. In the preceding expression for t^+ we have omitted the term $\epsilon\omega_1 t^*$ because such a term would automatically be classified as $\omega_1\tilde{t}$ and hence appear as such in the solution.

In effect we are proposing to ignore the relationship between t^+ and $\tilde{t}$, and to represent the solution by an expansion in functions where these two variables are treated as independent of one another. Of course such an expansion cannot be a limit process expansion. Nevertheless let us temporarily ignore this difficulty, and attempt to develop y for large times in the form

$$y = F(t^+, \tilde{t}, \epsilon) = \sum_{n=0}^{N} F^{(n)}(t^+, \tilde{t})\epsilon^n \tag{2.11}$$

$$\lim_{\substack{\epsilon \downarrow 0 \\ \tilde{t}\ \text{fixed}}} \frac{y - \sum_{n=0}^{N} F^{(n)}(t^+, \tilde{t})\epsilon^n}{\epsilon^N} \tag{2.12}$$

and keep in mind that as $\epsilon \to 0$ with $\tilde{t}$ fixed $t^* \to \infty$. We will be better able to clarify the ambiguity of this limit process after we have developed explicit expressions for the $F^{(n)}$.

In order that F be a valid representation of the exact solution for large times, we must adduce the following boundedness criteria to equations 2.11 and 2.12.

As a consequence of the restrictions imposed on f in section 2.1, y must be a bounded function of t^+ and hence $\tilde{t}$ for all positive values of t^+ and $\tilde{t}$. We will show that y actually obeys the somewhat stronger boundedness condition outlined below, at least for all the examples we will discuss in this chapter.

There exists a function $\delta(\tilde{t})$, defined for all non-negative values of $\tilde{t}$ such that

a) $\delta(\tilde{t})$ is continuous and possesses continuous first and second derivatives.

b) $\delta(\tilde{t}) \geqslant |F^{(0)}(t^+,\tilde{t})|$ for all non-negative values of t^+ and $\tilde{t}$.

c) y/δ is a bounded function of t^* for all $t^* \geqslant 0$ where y is the exact solution of equation 2.1.

We will show that these boundedness conditions will allow us to define all the ω_n appearing in t^+ completely, and each of the $F^{(n)}$ to within two arbitrary constants which will in turn be evaluated by applying a matching condition we will present later on.

Again, if we assume that the $F^{(n)}$ satisfy equations which are derivable from equation 2.1 by the limit process 2.12, we obtain the following set of partial differential equations for $F^{(0)}$, $F^{(1)}$ and $F^{(2)}$.

$$F_{11}^{(0)} + F^{(0)} = 0 \tag{2.13a}$$

$$F_{11}^{(1)} + F^{(1)} = -2F_{12}^{(0)} - f(F^{(0)}, F_1^{(0)}) \tag{2.13b}$$

$$F_{11}^{(2)} + F^{(2)} = -2F_{12}^{(1)} - F_{22}^{(0)} - 2\omega_2 F_{11}^{(0)} - F^{(1)} \frac{\partial}{\partial y} f(F^{(0)}, F_1^{(0)})$$

$$- (F_1^{(1)} + F_2^{(0)}) \frac{\partial}{\partial \frac{dy}{dt^*}} f(F^{(0)}, F_1^{(0)}) \qquad (2.13c)$$

In the above equations, and in what follows we have used the following notation for partial derivatives

$$F_1^{(n)} = \frac{\partial F^{(n)}}{\partial t^+} \quad , \quad F_2^{(n)} = \frac{\partial F^{(n)}}{\partial \tilde{t}} \qquad (2.14a)$$

$$F_{12}^{(n)} = F_{21}^{(n)} = \frac{\partial^2 F^{(n)}}{\partial t^+ \partial \tilde{t}} \quad , \quad F_{11}^{(n)} = \frac{\partial^2 F^{(n)}}{\partial t^{+2}} \qquad (2.14b)$$

$$F_{22}^{(n)} = \frac{\partial^2 F^{(n)}}{\partial \tilde{t}^2} \qquad (2.14c)$$

We have only given the equations governing $F^{(0)}$, $F^{(1)}$ and $F^{(2)}$. Although there is no concise representation for the right-hand side of the equation for $F^{(n)}$, one can compute the right-hand sides for each value of n in a straightforward manner.

The general solution of equation 2.13a is

$$F^{(0)}(t^+, \tilde{t}) = A^{(0)}(\tilde{t}) \sin t^+ + B^{(0)}(\tilde{t}) \cos t^+ \qquad (2.15)$$

where $A^{(0)}$ and $B^{(0)}$ are as yet undefined functions of $\tilde{t}$.

In order to compute $F^{(1)}$ let us replace the given function $f(F^{(0)}, F_1^{(0)})$ by its Fourier series expansion with respect to t^+:

$$f(F^{(0)}, F_1^{(0)}) = \frac{a_o}{2} + \sum_{n=1}^{\infty} [a_n \cos nt^+ + b_n \sin nt^+] \qquad (2.16)$$

In equation 2.16 the a_n and b_n depend explicitly on $A^{(0)}$ and $B^{(0)}$, and hence on $\tilde{t}$ implicitly.

If we now use the expression for $F^{(0)}$ given in equation 2.15 to compute $F_{12}^{(0)}$ and substitute this result together with the Fourier series expansion for f into the right-hand side of equation 2.13b we obtain

$$F_{11}^{(1)} + F^{(1)} = -(2\frac{dA^{(0)}}{d\tilde{t}} + a_1)\cos t^+ + (2\frac{dB^{(0)}}{d\tilde{t}} - b_1)\sin t^+$$

$$-\frac{a_o}{2} - \sum_{n=2}^{\infty}[a_n \cos nt^+ + b_n \sin nt^+] \qquad (2.17)$$

The condition that F be a bounded function of t^+ requires that the coefficients of the $\sin t^+$ and $\cos t^+$ terms in equation 2.17 must vanish (for otherwise these terms would give rise to secular terms proportional to $t^+\cos t^+$ and $t^+\sin t^+$). This condition has thus provided the following two first order ordinary differential equations governing $A^{(0)}$ and $B^{(0)}$:

$$2\frac{dA^{(0)}}{d\tilde{t}} + a_1(A^{(0)}, B^{(0)}) = 0 \qquad (2.18a)$$

$$2\frac{dB^{(0)}}{d\tilde{t}} - b_1(A^{(0)}, B^{(0)}) = 0 \qquad (2.18b)$$

The general solution of these equations will define $A^{(0)}$ and $B^{(0)}$ up to one arbitrary constant for each.

Let us denote these solutions by:

$$A^{(0)} = A^{(0)}(\tilde{t}, \alpha_o) \;, \qquad B^{(0)} = B^{(0)}(\tilde{t}, \beta_o) \qquad (2.19a, b)$$

With the coefficients of the $\sin t^{+}$ and $\cos t^{+}$ terms equal to zero, the general solution of equation 2.17 is

$$F^{(1)}(t^{+},\tilde{t}) = A^{(1)}(\tilde{t})\sin t^{+} + B^{(1)}(\tilde{t})\cos t^{+} - \frac{a_o}{2}$$

$$-\sum_{n=2}^{\infty}\left[\frac{a_n}{1-n^2}\cos nt^{+} + \frac{b_n}{1-n^2}\sin nt^{+}\right] \qquad (2.20)$$

In order to compute $F^{(2)}$ let us again express the terms $F^{(1)}\frac{\partial}{\partial y}(F^{(0)}, F_1^{(0)})$ and $(F_1^{(1)} + F_2^{(0)})\frac{\partial}{\partial \frac{dy}{dt^*}} f(F^{(0)}, F_1^{(0)})$ by their Fourier series expansions:

$$G = F^{(1)}\frac{\partial}{\partial y} f(F^{(0)}, F_1^{(0)}) = \frac{c_o}{2} + \sum_{n=1}^{\infty}\left[c_n \cos nt^{+} + d_n \sin nt^{+}\right] \qquad (2.21)$$

$$H = (F_1^{(1)} + F_2^{(0)})\frac{\partial}{\partial \frac{dy}{dt^*}}(F^{(0)}, F_1^{(0)}) = \frac{p_o}{2}$$

$$+\sum_{n=1}^{\infty}\left[p_n \cos nt^{+} + q_n \sin nt^{+}\right] \qquad (2.22)$$

For a given function f the coefficients c_n, d_n, p_n and q_n are known functions of $A^{(0)}$, $B^{(0)}$, $A^{(1)}$, and $B^{(1)}$. Hence, by virtue of the relations 2.19a and b these coefficients are known functions of $A^{(1)}$, $B^{(1)}$, and $\tilde{t}$. After substituting for the various terms appearing on the right-hand side of equation 2.13c we obtain the following result:

$$F_{11}^{(2)} + F^{(2)} = -[\,2\frac{dA^{(1)}}{d\tilde{t}} + \frac{db_1}{d\tilde{t}} - 2\omega_2 B^{(0)} + c_1 + p_1]\cos t^+$$

$$+ [\,2\frac{dB^{(1)}}{d\tilde{t}} + \frac{da_1}{d\tilde{t}} + 2\omega_2 A^{(0)} - d_1 - q_1]\,\mathrm{din}\ t^+ - \frac{1}{2}(c_o + p_o)$$

$$-\sum_{n=2}^{\infty}\left[\frac{2n(da_n/d\tilde{t})}{1-n^2} + d_n + q_n\right]\sin nt^+$$

$$+\left[c_n + p_n - \frac{2n(db_n/d\tilde{t})}{1-n^2}\right]\cos nt^+ \qquad (2.13c')$$

Clearly, the boundedness of $F^{(2)}$ requires that $A^{(1)}$ and $B^{(1)}$ satisfy the first order ordinary differential equations

$$2\frac{dA^{(1)}}{d\tilde{t}} + \frac{db_1}{d\tilde{t}} - 2\omega_2 B^{(0)} + c_1 + p_1 = 0 \qquad (2.23a)$$

$$2\frac{dB^{(1)}}{d\tilde{t}} + \frac{da_1}{d\tilde{t}} + 2\omega_2 A^{(0)} - d_1 - q_1 = 0 \qquad (2.23b)$$

The solutions of these equations will define each of the functions $B^{(1)}(\tilde{t})$ and $A^{(1)}(\tilde{t})$ in terms of α_o, β_o, ω_2 and two additional constants α_1 and β_1 in the form:

$$A^{(1)} = A^{(1)}(\tilde{t};\alpha_o, \alpha_1, \beta_o, \beta_1, \omega_2) \qquad (2.24a)$$

$$B^{(1)} = B^{(1)}(\tilde{t};\alpha_o, \alpha_1, \beta_o, \beta_1, \omega_2) \qquad (2.24b)$$

From the form of equations 2.13 it is clear that this process may be continued indefinitely. The requirement that each $F^{(n)}$ be a bounded function of t^+ will provide an ordinary differential equation for $A^{(n-1)}$ and $B^{(n-1)}$, and the solutions of these equations will define $A^{(n-1)}$ and

$B^{(n-1)}$ in the form

$$A^{(n-1)} = A^{(n-1)}(\tilde{t}; \alpha_o, \ldots, \alpha_{n-1}, \beta_o, \ldots, \beta_{n-1}, \omega_2, \ldots, \omega_{n+1}) \tag{2.25a}$$

$$B^{(n-1)} = B^{(n-1)}(\tilde{t}; \alpha_o, \ldots, \alpha_{n-1}, \beta_o, \ldots, \beta_{n-1}, \omega_2, \ldots, \omega_{n+1}) \tag{2.25b}$$

So far we have made no attempt to evaluate the constants α_n, β_n and ω_n. We will show later on that a matching condition for each n will define these constants, so that each $A^{(n)}$ and $B^{(n)}$ can be completely evaluated in succession.

We are now in a better position to clarify the limit process of equation 2.12. Let us note that the condition that F be a bounded function of t^+ restricts the appearance of this variable to only trigonometric terms, in the general expansion for F. Now, <u>if</u> the exact solution of equation 2.1 also possesses this special structure (as it does for the linear case), then the limit process 2.12 is meaningful. For, even though y does not possess a limit process expansion with $\tilde{t}$ fixed and $\epsilon \to 0$, a matching condition such as 2.12 is still valid as long as <u>all</u> the terms in y that do not possess a limit, also appear in F.

In short, the artifice of choosing to treat t^+ and $\tilde{t}$ as separate variables removes the difficulty of the non-existence of a limit process expansion for large times, when we make use of the fact that y must be a bounded function of t^+.

Since it is not possible to make any general statements about the behavior of the functions $A^{(n)}$ and $B^{(n)}$ because of the arbitrary nature of f, let us again use the linear equation in order to illustrate the procedure we shall use to define the ω_n, α_n and β_n.

For $f = 2(dy/dt^*)$ the Fourier series 2.16 only consists of the term $2A^{(0)}\cos t^+ - 2B^{(0)}\sin t^+$, and equations 2.18a and 2.18b become:

$$\frac{dA^{(0)}}{d\tilde{t}} + A^{(0)} = 0 \tag{2.18a'}$$

$$\frac{dB^{(0)}}{d\tilde{t}} + B^{(0)} = 0 \tag{2.18b'}$$

The general solution of this system is

$$A^{(0)} = \alpha_o e^{-\tilde{t}} \tag{2.19a'}$$

$$B^{(0)} = \beta_o e^{-\tilde{t}} \tag{2.19b'}$$

We now propose to evaluate α_o and β_o, by matching $F^{(0)}$ with h_o to order unity in a common overlap domain. We must first show that such an overlap domain exists for the two expansions F and h. We have already discussed the domain of validity for h in section 2.2, so let us examine the domain of validity for F.

Since by construction the difference between the exact solution and F vanishes in the limit as $\epsilon \to 0$ with $\tilde{t}$ fixed (say $\tilde{t} = \tilde{T}$), F is a valid representation of y in the rectangular region to the right of the line $\tilde{t} = \tilde{T}$ in the $\tilde{t}, \epsilon$ plane as shown in fig. 2.4.

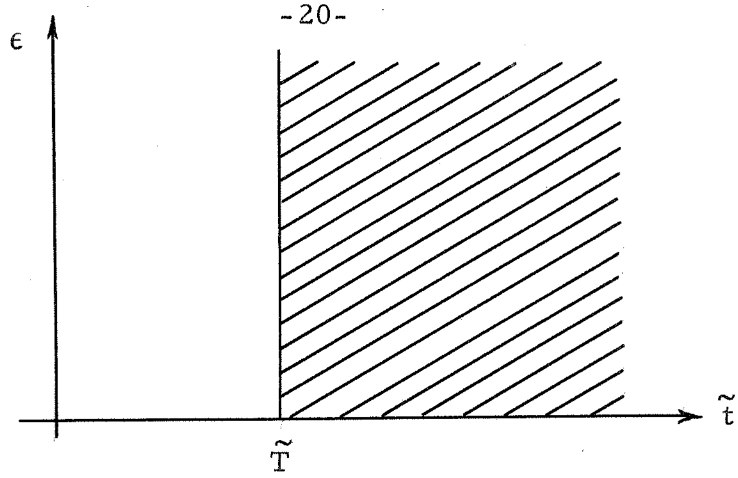


Figure 2.4

Even though F is not a limit process expansion, and equation 2.1 does not possess an intermediate expansion in the strict sense, we note that if F satisfies the initial conditions of the problem, then it contains h (i.e. the initially valid limit of F is h itself). Hence the matching of F with h is automatically effected by imposing the initial condition 2.1b and 2.1c on F.

Now y and dy/dt^* have the following expansions

$$y \sim \sum_{n=0}^{N} F^{(n)}(t^+, \tilde{t})\epsilon^n \tag{2.26}$$

$$\frac{dy}{dt^*} \sim \sum_{n=0}^{N} \epsilon^n [F_1^{(n)} + F_2^{(n-1)} + \sum_{m=2}^{n} \omega_m F_1^{(n-m)}] \tag{2.27}$$

Therefore the conditions $y(0) = a$ and $dy(0)/dt^* = 0$ are satisfied if:

$$F^{(0)}(0, 0) = a \;, \quad F^{(n)}(0, 0) = 0 \qquad n \neq 0 \tag{2.28}$$

and

$$F_1^{(0)}(0,0) = 0, \quad F_1^{(1)}(0,0) = -F_2^{(0)}(0,0)$$

$$F_1^{(n)}(0,0) = -F_2^{(n-1)}(0,0) - \sum_{m=2}^{n} \omega_m F_1^{(n-m)} \tag{2.29}$$

If we use the expression we have derived for $F^{(0)}$ and $F^{(1)}$ in equations 2.15 and 2.20 in conjunction with the above initial conditions we deduce the following conditions for $A^{(0)}$, $A^{(1)}$, $B^{(0)}$ and $B^{(1)}$.

$$A^{(0)}(0) = 0 \tag{2.30a}$$

$$B^{(0)}(0) = a \tag{2.30b}$$

$$A^{(1)}(0) = \sum_{n=2}^{\infty} \frac{nb_n(0)}{1-n^2} - \frac{dB^{(0)}(0)}{d\tilde{t}} \tag{2.31a}$$

$$B^{(1)}(0) = \frac{a_o(0)}{2} + \sum_{n=2}^{\infty} \frac{a_n(0)}{1-n^2} \tag{2.31b}$$

The fact that h is contained in F implies in addition, <u>that F is the uniformly valid expansion of the exact solution.</u>

To illustrate these ideas, we will resume our discussion of the linear example.

The matching of h_o and $F^{(0)}$ is satisfied if we set $\alpha_o = 0$ and $\beta_o = a$. This result could also have been obtained by requiring that F satisfy the initial conditions of the problem. For, the condition $y(0) = a$ implies that $F^{(0)}(0,0) = a$, which in turn implies that $\beta_o = a$. Similarly the condition $dy(0)/dt^* = 0$ provides the relation $\alpha_o = 0$.

Let us now proceed with the higher approximations for this example. If we substitute the known values of $A^{(0)}(\tilde{t})$ and $B^{(0)}(\tilde{t})$ into equations 2. 23a and 2. 23b we obtain

$$2\frac{dA^{(1)}}{d\tilde{t}} + 2A^{(1)} - (2\omega_2 + 1)ae^{-\tilde{t}} = 0 \qquad (2.23a')$$

$$2\frac{dB^{(1)}}{d\tilde{t}} + B^{(1)} = 0 \qquad (2.23b')$$

The general solution of these equations is

$$A^{(1)} = \frac{a}{2}(2\omega_2 + 1)\tilde{t}e^{-\tilde{t}} + \alpha_1 e^{-\tilde{t}} \qquad (2.24a')$$

$$B^{(1)} = \beta_1 e^{-\tilde{t}} \qquad (2.24b)$$

By matching $F^{(0)} + \epsilon F^{(1)}$ with $h_o + \epsilon h_1$ (or by requiring that F satisfy the initial conditions of the problem i. e. $F^{(1)}(0,0) = 0$, $F_1^{(1)}(0,0) = -F_2^{(0)}(0,0)$) we can show that $\alpha_1 = a$ and $\beta_1 = 0$. Thus to order ϵ, F is of the form

$$F = ae^{-\tilde{t}}\cos t^{+} + \epsilon a\left[\frac{(2\omega_2+1)}{2}\tilde{t}e^{-\tilde{t}} + 1\right]e^{-\tilde{t}}\sin t^{+} \qquad (2.2)$$

In order to evaluate ω_2 we will make use of the stronger boundedness condition we discussed earlier. We note that the function $\delta(\tilde{t}) = ae^{-\tilde{t}}$ satisfies all the conditions we have set. For:

a) $ae^{-\tilde{t}}$ is continuous and possess continuous first and second derivatives.

b) $ae^{-\tilde{t}} \geqslant |F^{(0)}|$

and to show that y/δ is a bounded function we note that if y is a solution of equation 2.1 then $\eta = y/\delta$ is a solution of the equation

$$\frac{d^2\eta}{dt^{*2}} + (1 - \epsilon^2)\eta = 0 \qquad (2.2)$$

and hence is bounded.

The important conclusion we draw from the above is that $F/e^{-\tilde{t}}$ must be bounded, which requires that $\omega_2 = -1/2$.

We have now illustrated all the ideas we need to tackle various examples for nonlinear functions f, which we will proceed to do in the next section.

III. DAMPED OSCILLATIONS (EXAMPLES)

3.1. Dimensional Analysis

In this section we will introduce the various problems we propose to solve, in order to convert the equations to the appropriate non-dimensional forms.

First, let us consider the linear system consisting of a mass, spring, and dashpot whose solution we have already discussed in the previous sections. The differential equation for the motion of this system is

$$m\frac{d^2\bar{y}}{dt^2} + 2\beta\frac{d\bar{y}}{dt} + k\bar{y} = 0 \tag{3.1a}$$

and if the motion is started from rest with an initial displacement A from equilibrium, the initial conditions are

$$\bar{y}(0) = A \tag{3.1b}$$

$$\frac{d\bar{y}(0)}{dt} = 0 \tag{3.1c}$$

The only length scale in this problem is the initial displacement, hence it is necessary to non-dimensionalize y by A in the form

$$y = \frac{\bar{y}}{A} \tag{3.2a}$$

The only non-dimensional grouping of the four constants m, k and A which is proportional to the damping β is $\epsilon = \beta/(mk)^{1/2}$. Moreover, we can form the following two independent time scales

$$T_1 = (m/k)^{1/2} \qquad T_2 = m/\beta \tag{3.2b, c}$$

It is clear that T_1 is a measure of the period of the oscillatory behavior of the system, while T_2 measures the period during which the effects of damping are produced. We also note that ϵ is the ratio of these two time scales. Hence, the magnitude of ϵ measures the relative importance of the damping as compared to the oscillatory behavior of the motion. We wish to regard ϵ as a small quantity in the sense that β is small and the limit as $\epsilon \to 0$ to imply that the damping disappears from the problem. This choice leads to the definition of t^* as $t^* = t/T_1$, and the following non-dimensional formulation of equations 3.1.

$$\frac{d^2y}{dt^{*2}} + 2\epsilon \frac{dy}{dt^*} + y = 0 \tag{3.1a'}$$

$$y(0) = 1 \tag{3.1b'}$$

$$\frac{dy(0)}{dt^*} = 0 \tag{3.1c'}$$

Let us now consider a more general system in which the damping is proportional to the cube of the speed, and hence is governed by the equation

$$m \frac{d^2\overline{y}}{dt^2} + \beta\left(\frac{d\overline{y}}{dt}\right)^3 + k\overline{y} = 0 \tag{3.3a}$$

and the initial conditions

$$\overline{y}(0) = A \tag{3.3b}$$

$$\frac{d\overline{y}(0)}{dt} = 0 \tag{3.3c}$$

By a dimensional analysis analogous to that for the linear case, we obtain the following non-dimensional variables

$$y = \frac{\bar{y}}{A} , \quad t^* = \frac{t}{T_1} , \quad \epsilon = \frac{T_1}{T_2}$$

$$T_1 = \left(\frac{m}{k} \right)^{1/2} , \quad T_2 = \frac{m^2}{k\beta A^2} \tag{3.4}$$

and the following non-dimensional differential equation, and initial conditions

$$\frac{dy}{dt^*} + \epsilon \left(\frac{dy}{dt^*} \right)^3 + y = 0 \tag{3.3a'}$$

$$y(0) = 1 \tag{3.3b'}$$

$$\frac{dy(0)}{dt^*} = 0 \tag{3.3c'}$$

Let us next consider a linearly damped system having a slightly non-linear restoring force. The simplest example that comes to mind is the small amplitude motion of a damped pendulum.

The general dimensional equation, and initial conditions for this system are:

$$m\ell \frac{d^2\bar{\theta}}{dt^2} + \beta\ell \frac{d\bar{\theta}}{dt} + mg \sin \bar{\theta} = 0 \tag{3.5a}$$

$$\bar{\theta}(0) = \theta_o \tag{3.5b}$$

$$\frac{d\bar{\theta}(0)}{dt} = 0 \tag{3.5c}$$

In the above, ℓ is the length of the pendulum, m is its mass, β the damping coefficient, and g the acceleration of gravity (a constant).

Since we wish to consider small amplitude oscillations let us replace the term $\sin\overline{\theta}$ by the first two terms of its power series expansion, and write equation 3.5a as

$$m\ell\frac{d^2\overline{\theta}}{dt^2} + \beta\ell\frac{d\overline{\theta}}{dt} + mg\sin\overline{\theta} = 0 \qquad (3.5a')$$

The dimensional analysis of the given constants yields the following variables

$$y = \frac{\overline{\theta}}{\overline{\theta}_o}, \quad t^* = \frac{t}{T_1}, \quad \epsilon = \frac{T_1}{T_2}$$

$$T_1 = \left(\frac{\ell}{g}\right)^{1/2}, \quad T_2 = \frac{m}{\beta} \qquad (3.6)$$

Equation 3.5a' and the initial conditions 3.5b and 3.5c become:

$$\frac{d^2y}{dt^{*2}} + \epsilon\frac{dy}{dt^*} + y - \alpha y^3 = 0 \qquad (3.5a'')$$

$$y(0) = 1 \qquad (3.5b'')$$

$$\frac{dy(0)}{dt^*} = 0 \qquad (3.5c'')$$

where $\alpha = \theta_o^2/6$.

The problem now involves the two non-dimensional parameters α and ϵ. In order to treat equation 3.5a'' within the framework we have established we should regard α to be of the same order as ϵ by setting $\alpha = c\epsilon$, where c is an appropriate constant. This means that the non-linearity of the restoring force is assumed to be of the same order as the damping, which is true for sufficiently small initial displacements.

We can now rewrite equation 3.5a" in a form suitable for solution.

$$\frac{d^2y}{dt^{*2}} + \epsilon\left(\frac{dy}{dt^*} - cy^3\right) + y = 0 \tag{3.5a'''}$$

We will next consider Van der Pol's equation with arbitrary initial conditions. This is a system which for a particular choice of initial values, possesses a periodic solution, and has been studied extensively in this context.

The dimensional initial value problem is:

$$m\frac{d^2\overline{y}}{dt^2} - \left[\alpha\frac{d\overline{y}}{dt} - \frac{\beta}{3}\left(\frac{d\overline{y}}{dt}\right)^3\right] + ky = 0 \tag{3.7a}$$

$$y(0) = A \tag{3.7b}$$

$$\frac{dy(0)}{dt} = 0 \tag{3.7c}$$

We have assigned a mechanical interpretation to this equation even though it has a more natural interpretation for a certain electrical circuit. We should also note that another equation, also denoted as Van der Pol's equation in the literature, is:

$$\frac{d^2\overline{y}}{dt^2} - \epsilon(1 - \overline{y}^2)\frac{d\overline{y}}{dt} + \overline{y} = 0 \tag{3.8}$$

It may be deduced from equation 3.7a by the transformations:

$$t\left(\frac{k}{m}\right)^{1/2} \rightarrow t \tag{3.9a}$$

$$\frac{d\overline{y}}{dt}\left(\frac{\beta k}{m\alpha}\right)^{1/2} \rightarrow \overline{y} \tag{3.9b}$$

As we wish to study equation 3.7a for arbitrary initial displacements, we should choose a length scale which is independent of A. A dimensional analysis of the given constants yields the following characteristic scales:

$$L = \frac{\alpha m}{\beta k}, \quad T_1 = \left(\frac{m}{k}\right)^{1/2}, \quad T_2 = \frac{m}{\alpha} \tag{3.10}$$

Hence the appropriate non-dimensional variables are:

$$y = \frac{\bar{y}}{L}, \quad t^* = \frac{t}{T_1}, \quad \epsilon = \frac{T_1}{T_2} \tag{3.11}$$

and the initial value problem 3.7a becomes:

$$\frac{d^2y}{dt^{*2}} - \epsilon\left[\frac{dy}{dt^*} - \frac{1}{3}\left(\frac{dy}{dt^*}\right)^3\right] + y = 0 \tag{3.7a'}$$

$$y(0) = c = \frac{A\beta k}{\alpha m} \tag{3.7b'}$$

$$\frac{dy(0)}{dt^*} = 0 \tag{3.7c'}$$

The preceding four examples were chosen because they illustrate in a simple manner the various types of problems that can be brought to the form 2.1. In Chapter IV we will consider a variety of examples which are not strictly of the form given in equation 2.1, but which nevertheless can be solved by our method.

3.2. Solution of 3.3'

For this example $f = (dy/dt^*)^3$, $a = 1$ and we have the following expressions for the Fourier coefficients of $f(F^{(0)}, F_1^{(0)})$

$$a_n = 0\,, \qquad n \neq 1, 3 \tag{3.12a}$$

$$a_1 = \frac{3}{4} A^{(0)} B^{(0)^2} + \frac{3}{4} A^{(0)^3} \tag{3.12b}$$

$$a_3 = \frac{A^{(0)^3}}{4} - \frac{3}{4} A^{(0)} B^{(0)^2} \tag{3.12c}$$

$$b_n = 0, \qquad n \neq 1, 3 \tag{3.12d}$$

$$b_1 = -\frac{B^{(0)^3}}{4} + \frac{3}{4} B^{(0)} A^{(0)^2} \tag{3.12e}$$

$$b_3 = -\frac{B^{(0)^3}}{4} + \frac{3}{4} B^{(0)} A^{(0)^2} \tag{3.12f}$$

The solution of equations 2.18 with the initial conditions 2.30 for $A^{(0)}$ and $B^{(0)}$ is:

$$A^{(0)}(\tilde{t}) = 0 \tag{3.13a}$$

$$B^{(0)}(\tilde{t}) = 2(3\tilde{t} + 4)^{-1/2} \tag{3.13b}$$

Therefore the uniformly valid solution to order unity is

$$F^{(0)} = 2(3\tilde{t} + 4)^{-1/2} \cos t^* \tag{3.14}$$

According to equation 2.20 we have

$$F^{(1)} = A^{(1)} \sin t^+ + B^{(1)} \cos t^+ + \frac{B^{(0)^3}}{32} \sin 3t^+ \tag{3.15}$$

If we use the above to evaluate the Fourier coefficients of G and H and hence equations 2.23 we obtain the following two first order ordinary differential equations for $A^{(1)}$ and $B^{(1)}$.

$$2\frac{dA^{(1)}}{d\tilde{t}} + \frac{3}{4} B^{(0)^2} A^{(1)} + \frac{9}{128} B^{(0)^5} - 2\omega_2 B^{(0)} = 0 \tag{3.16a}$$

$$2\frac{dB^{(1)}}{d\tilde{t}} + \frac{9}{4}B^{(0)^2}B^{(1)} = 0 \qquad (3.16b)$$

The initial conditions for $A^{(1)}$ and $B^{(1)}$ according to equations 2.31 are:

$$A^{(1)}(0) = \frac{9}{32} \qquad (3.17a)$$

$$B^{(1)}(0) = 0 \qquad (3.17b)$$

The solution of the system 3.16 satisfying the conditions 3.17 is

$$A^{(1)}(\tilde{t}) = \frac{3}{8}(3\tilde{t} + 4)^{-3/2} + 2\omega_2\tilde{t}(3\tilde{t} + 4)^{-1/2} + \frac{15}{32}(3\tilde{t} + 4)^{-1/2} \qquad (3.18a)$$

$$B^{(0)}(\tilde{t}) = 0 \qquad (3.18b)$$

Although for this example, the stronger boundedness condition also applies with $\delta(\tilde{t}) = B^{(0)}$, we need only require the weaker condition that F itself is a bounded function of $\tilde{t}$ to evaluate ω_2. Clearly, since for large values of $\tilde{t}$, $A^{(1)}$ behaves like $\frac{2}{(3)^{1/2}}\omega_2\tilde{t}^{\,1/2}$ we must set $\omega_2 = 0$. Thus the uniformly valid solution to order ϵ of equation 3.3' is

$$F^{(0)} + \epsilon F^{(1)} = 2(3\tilde{t} + 4)^{-1/2}\cos t^* + \epsilon\left\{\frac{(3\tilde{t} + 4)^{-3/2}}{4}\sin 3t^* + \left[\frac{3}{8}(3\tilde{t} + 4)^{-3/2} + \frac{15}{32}(3\tilde{t} + 4)^{-1/2}\right]\sin t^*\right\} \qquad (3.19)$$

3.3. Solution of equation 3.5'''

For this example $f = dy/dt^* - cy^3$, $a = 1$ and we have the following expressions for the Fourier coefficients of $f(F^{(0)}, F_1^{(0)})$

$$a_n = 0, \quad n \neq 1, 3 \tag{3.20a}$$

$$a_1 = A^{(0)} - \frac{3c}{4}\left(B^{(0)^3} + A^{(0)^2} B^{(0)}\right) \tag{3.20b}$$

$$a_3 = \frac{c}{4}\left(3A^{(0)^2} B^{(0)} - B^{(0)^3}\right) \tag{3.20c}$$

$$b_n = 0, \quad n \neq 1, 3 \tag{3.20d}$$

$$b_1 = -B^{(0)} - \frac{3c}{4}\left(A^{(0)^3} + B^{(0)^2} A^{(0)}\right) \tag{3.20e}$$

$$b_3 = \frac{c}{4}\left(A^{(0)^3} - 3A^{(0)} B^{(0)^2}\right) \tag{3.20f}$$

According to equations 2.18 and 2.30, the equations and initial conditions governing $A^{(0)}$ and $B^{(0)}$ are:

$$2\frac{dA^{(0)}}{d\tilde{t}} + A^{(0)} - \frac{3c}{4}\left(B^{(0)^3} + A^{(0)^2} B^{(0)}\right) = 0 \tag{3.21a}$$

$$2\frac{dB^{(0)}}{d\tilde{t}} + B^{(0)} + \frac{3c}{4}\left(A^{(0)^3} + B^{(0)^2} A^{(0)}\right) = 0 \tag{3.21b}$$

$$A^{(0)}(0) = 0 \tag{3.22a}$$

$$B^{(0)}(0) = 1 \tag{3.22b}$$

We can simplify these equations by introducing the transformations

$$Xe^{-\tilde{t}/2} = B^{(0)} + A^{(0)} \tag{3.23a}$$

$$Ye^{-\tilde{t}/2} = B^{(0)} - A^{(0)} \tag{3.23b}$$

The system 3.21 transforms to

$$8\frac{dY}{d\tilde{t}} + 3ce^{-\tilde{t}}X^3 = 0 \tag{3.24a}$$

$$8\frac{dX}{d\tilde{t}} - 3ce^{-\tilde{t}}Y^3 = 0 \tag{3.24b}$$

with the initial conditions

$$X(0) = Y(0) = 1 \tag{3.25}$$

A first integral, obtained by dividing the first equation in 3.24 by the second is

$$X^4 + Y^4 = 2 \tag{3.26}$$

If we now substitute this result into either one of the two equations 3.24 we obtain an integral representation in the form $\tilde{t} = g(X, c)$ or $\tilde{t} = h(Y, c)$. Unfortunately the functions g and h are not expressible in terms of elementary functions. Nevertheless, we do derive the very useful conclusion from the first integral $X^4 + Y^4 = 2$, that $Y(\tilde{t})$ and $X(\tilde{t})$ are bounded functions of $\tilde{t}$, in fact

$$0 \leq X(\tilde{t}),\ Y(\tilde{t}) \leq 2^{1/4}$$

In terms of these bounded functions $F^{(0)}$ becomes

$$F^{(0)} = e^{-\tilde{t}/2}[P(\tilde{t})\sin t^* + Q(\tilde{t})\cos t^*] \tag{3.27}$$

where

$$P(\tilde{t}) = \frac{1}{2}[X - (2 - X^4)^{1/4}] \tag{3.28}$$

$$Q(\tilde{t}) = \frac{1}{2}[X + (2 - X^4)^{1/4}] \tag{3.29}$$

and P and Q are bounded functions of $\tilde{t}$. The important conclusion regarding this system is the following. In the absence of the nonlinearity in the restoring force the motion would have been exponentially damped. Hence the effect of this nonlinearity is to modify the damping to the extent of multiplying it by the slowly varying functions of $\tilde{t}$, P and Q.

3. 4. Solution of Van der Pol's Equation 3. 7a'

Here $f = - dy/dt^{*} + 1/3(dy/dt^{*})^3$, $a = c$, and the Fourier coefficients a_n and b_n are:

$$a_n = 0, \quad n \neq 1, 3 \tag{3. 30a}$$

$$a_1 = \frac{A^{(0)^3}}{4} + \frac{A^{(0)}B^{(0)^2}}{4} - A^{(0)} \tag{3. 30b}$$

$$a_3 = \frac{A^{(0)^3}}{12} - \frac{A^{(0)}B^{(0)^2}}{4} \tag{3. 30c}$$

$$b_n = 0, \quad n \neq 1, 3 \tag{3. 30d}$$

$$b_1 = - \frac{B^{(0)^3}}{4} - \frac{A^{(0)^2}B^{(0)}}{4} + B^{(0)} \tag{3. 30e}$$

$$b_3 = \frac{B^{(0)^3}}{12} - \frac{B^{(0)}A^{(0)^2}}{4} \tag{3. 30f}$$

According to equations 2. 18 and 2. 30, the functions $A^{(0)}(\tilde{t})$ and $B^{(0)}(\tilde{t})$ are governed by the equations

$$2\frac{dA^{(0)}}{d\tilde{t}} + \frac{A^{(0)^3}}{4} + \frac{A^{(0)}B^{(0)^2}}{4} - A^{(0)} = 0 \tag{3. 31a}$$

$$2\frac{dB^{(0)}}{d\tilde{t}} + \frac{B^{(0)^3}}{4} + \frac{A^{(0)^2}B^{(0)}}{4} - B^{(0)} = 0 \tag{3. 31b}$$

and the initial conditions

$$A^{(0)}(0) = 0 \tag{3.32a}$$

$$B^{(0)}(0) = c \tag{3.32b}$$

The solution of the above system is

$$A^{(0)}(\tilde{t}) = 0 \tag{3.33a}$$

$$B^{(0)}(\tilde{t}) = 2c\left[c^2 - (c^2 - 4)e^{-\tilde{t}}\right]^{-1/2} \tag{3.33b}$$

hence $F^{(0)}$ is given by

$$F^{(0)} = 2c\left[c^2 - (c^2 - 4)e^{-\tilde{t}}\right]^{-1/2}\cos t^+$$

We note that if $c = 2$, $B^{(0)} = 2$, and $F^{(0)}$ reduces to the periodic solution of equation 3.7. We also note that this periodic solution is stable in the very strong sense that for <u>any</u> value of c the solution will tend to the "limit cycle" as $\tilde{t} \to \infty$.

Let us now proceed to calculate the next approximation for F. We have the following expression for $F^{(1)}$ from equation 2.20

$$F^{(1)} = A^{(1)}(\tilde{t})\sin t^+ + B^{(1)}(\tilde{t})\cos t^+ + \frac{B^{(0)^3}}{96}\sin 3t^+ \tag{3.33c}$$

The equations governing $A^{(1)}$ and $B^{(1)}$ are, according to equation 2.23

$$2\frac{dA^{(1)}}{d\tilde{t}} - \frac{B^{(0)}}{4} - \frac{B^{(0)^5}}{128} + A^{(1)} - \frac{A^{(1)}B^{(0)^2}}{4} + 2\omega_2 B^{(0)} = 0 \tag{3.34a}$$

$$2\frac{dB^{(1)}}{d\tilde{t}} - B^{(1)} + \frac{3}{4}B^{(0)^2}B^{(1)} = 0 \tag{3.34b}$$

The initial conditions 2.31 for this case, reduce to

$$A^{(1)}(0) = \frac{c(3c^2 - 16)}{32} \tag{3.35a}$$

$$B^{(1)}(0) = 0 \tag{3.35b}$$

The solution of the system 3.34 satisfying the above initial conditions is:

$$A^{(1)}(\tilde{t}) = \frac{B^{(0)}}{8}\log\left(\frac{B^{(0)}}{c}\right) + \frac{B^{(0)}}{64}\left(B^{(0)^2} + 5c^2 - 32\right) + \frac{B^{(0)}\tilde{t}}{2}\left(\frac{1}{8} + 2\omega_2\right) \tag{3.36a}$$

$$B^{(1)}(\tilde{t}) = 0 \tag{3.36b}$$

Here again we may impose the stronger boundedness condition on $F^{(1)}$ by choosing $\delta_1(\tilde{t}) = B^{(0)}(\tilde{t})$. In order to evaluate ω_2, it is sufficient to require that F itself be bounded. For, since the term $\frac{B^{(0)}\tilde{t}}{2}(\frac{1}{8} + 2\omega_2)$ approaches $\tilde{t}(\frac{1}{8} + 2\omega_2)$ for large values of $\tilde{t}$ we must set $\omega_2 = -\frac{1}{16}$.

The uniformly valid solution of Van der Pol's equation to order ϵ is thus:

$$F^{(0)} + \epsilon F^{(1)} = B^{(0)}\cos\left(1 - \frac{\epsilon^2}{16}\right)t^* + \epsilon\left[\frac{B^{(0)}}{8}\log\left(\frac{B^{(0)}}{c}\right) + \frac{B^{(0)}}{64}\left(B^{(0)^2} + 5c^2 - 32\right)\right]\sin\left(1 - \frac{\epsilon^2}{16}\right)t^* + \frac{\epsilon B^{(0)^3}}{96}\sin 3\left(1 - \frac{\epsilon^2}{16}\right)t^* \tag{3.37}$$

Again we note that for $c = 2$ this reduces to the correct periodic solution, and that for $\tilde{t} \to \infty$ and c arbitrary, the general solution approaches the limit cycle.

The leading terms of the solutions we have derived in the preceeding three examples have been checked by the method of Kryloff and Bogoliuboff of reference 1, and the agreement is exact.

IV. BOUNDED OSCILLATIONS (MISCELLANEOUS EXAMPLES)

In this chapter we will apply the method we have developed for solving equation 2. la to a variety of problems in which the function f has a more general form.

For the sake of simplicity we will not parallel here the general discussion we gave in section 2. 1. We will rather solve each problem in detail separately, and will again make use of the basic boundedness, and matching criteria which will, of course, still hold.

4. 1. The Motion of a Charged Particle in a Slowly Varying Magnetic Field.

In reference 4, Broer and Wijngaarden show that the equations of motion for a charged particle in a slowly varying magnetic field can be transformed to the linear second order differential equation

$$\frac{d^2\varphi}{dt^2} + \frac{\omega^2(t)}{4}\varphi = 0 \tag{4. 1a}$$

where φ is the complex variable related to the Cartesian coordinates x, y by

$$u = x + iy = \varphi e^{-\frac{i}{2}\int \omega\, dt} \tag{4. 2}$$

and ω is the cyclotron frequency $\frac{eB(t)}{m}$.

In the above m is the mass of the particle, e its charge, and B(t) the slowly varying magnetic field strength.

Let us denote ω by 2μ and $\varphi = \xi + i\eta$, and consider the equation for ξ

$$\frac{d^2\xi}{dt^2} + \mu^2(t)\xi = 0 \tag{4.1b}$$

We wish to study equation 4.1b for functions μ which vary slowly with time. In particular, we wish to consider functions μ which are of the form

$$T_1\mu = \mu(\mu_o t) \ , \quad \mu_o T_1 \ll 1 \tag{4.3}$$

where μ_o is a characteristic frequency and T_1 a characteristic time. We should mention that not all slowly varying functions necessarily belong to this class, and this particular choice was made in order to derive some general results for which an explicit knowledge of the function μ is not needed.

Let us choose the initial conditions

$$\xi(0) = A \tag{4.4a}$$

$$\frac{d\xi(0)}{dt} = 0 \tag{4.4b}$$

The appropriate non-dimensional variables are

$$\xi^* = \frac{\xi}{A} \ , \quad t^* = \frac{t}{T_1} \ , \quad \epsilon = \mu_o T_1, \quad \tilde{t} = \epsilon t^* \tag{4.5}$$

Equation 4.3, and the initial conditions 4.5 become

$$\frac{d^2\xi^*}{dt^{*2}} + \mu^2(\tilde{t})\xi^* = 0 \tag{4.6a}$$

$$\xi^*(0) = 1 \tag{4.6b}$$

$$\frac{d\xi(0)}{dt^*} = 0 \tag{4.6c}$$

It is incorrect to assume that the two-variable expansion procedure we developed earlier applies directly to equation 4.6 for the following reason: With such an expansion we would be tacitly assuming that $F^{(0)}$, the leading term of the asymptotic expansion was of the form

$$F^{(0)}(t^*, \tilde{t}) = A^o(\tilde{t})\sin \mu(\tilde{t})t^* + B^{(0)}(\tilde{t})\cos \mu(\tilde{t})t^* \qquad (4.7)$$

and in fact that the homogeneous parts of all the $F^{(n)}$ have the "frequency" $\mu(\tilde{t})$.

We can show that this is incorrect (unless μ = constant) by actually carrying out the expansion for ξ^* in the form

$$\xi^* = F(t^*, \tilde{t}) = \sum_{n=0}^{N} \epsilon^n F^{(n)}(t^*, \tilde{t}) \qquad (4.8)$$

Then equation 4.6a implies that $F^{(0)}$ is as in equation 4.7, and that $F^{(1)}$ is the solution of

$$F^{(1)}_{11} + \mu^2(\tilde{t})F^{(1)} = -2F^{(0)}_{12} \qquad (4.9)$$

When we substitute the result obtained for $-2F^{(0)}_{12}$ from equation 4.7 into the right-hand side of equation 4.9 we have:

$$\begin{aligned} F^{(1)}_{11} + \mu^2(\tilde{t})F^{(1)} = &-2\left[\mu \frac{dA^{(0)}}{d\tilde{t}} + A^{(0)} \frac{d\mu}{d\tilde{t}}\right] \cos \mu t^* \\ &- 2\left[\mu \frac{dB^{(0)}}{d\tilde{t}} + B^{(0)} \frac{d\mu}{d\tilde{t}}\right] \sin \mu t^* \\ &- 2t^* \mu \frac{d\mu}{d\tilde{t}} \left[A^{(0)} \sin \mu t^* + B^{(0)} \cos \mu t^*\right] \end{aligned} \qquad (4.9a)$$

Thus the dependence of the frequency of $F^{(0)}$ on $\tilde{t}$ has introduced the bothersome terms $-2t^* \mu \frac{d\mu}{d\tilde{t}} [A^{(0)} \sin \mu t^* + B^{(0)} \cos \mu t^*]$

which can only be eliminated by setting $A^{(0)} = B^{(0)} = 0$.

This is an indirect verification of the fact that the solution of equation 4.6 cannot have the fundamental frequency $\mu(\tilde{t})$. We can show this more directly by transforming the variable t^* to another appropriate variable in terms of which equation 4.6 takes on the more familiar form where the restoring force term has a constant coefficient.

If we denote this new variable by $\tau = \tau(t^*)$ we obtain the following transformed equation from equation 4.6

$$\frac{d^2\xi^*}{d\tau^2} + \left[\frac{1}{(\frac{d\tau}{dt^*})} \frac{d^2\tau}{dt^{*2}}\right]\frac{d\xi^*}{d\tau} + \frac{\mu^2}{(\frac{d\tau}{dt^*})^2}\xi^* = 0 \tag{4.10}$$

Thus if $\mu = c\frac{d\tau}{dt^*}$ (where c is a constant, say $c = 1$), the solution of equation 4.10 will have no $\tilde{t}$ dependent frequency. If we make the substitution $\mu = \frac{d\tau}{dt^*}$ and observe that μ being a function of $\tilde{t}$ implies that

$$\frac{1}{\frac{d\tau}{dt^*}}\left(\frac{d^2\tau}{dt^{*2}}\right) = \frac{1}{\mu}\frac{d\mu}{d\tau} = \frac{\epsilon}{\mu^2}\frac{d\mu}{d\tilde{t}} = \epsilon f(\tilde{t}), \quad f(\tilde{t}) = \mu^{-2}\frac{d\mu}{d\tilde{t}} \tag{4.11}$$

we obtain a more familiar transformed equation:

$$\frac{d^2\xi^*}{d\tau^2} + \epsilon f(\tilde{t})\frac{d\xi^*}{d\tau} + \xi^* = 0 \tag{4.12a}$$

The initial conditions 4.6b and 4.6c become

$$\xi^*(0) = 1 \tag{4.12b}$$

$$\frac{d\xi^*(0)}{d\tau} = 0 \quad \text{if} \quad \mu(0) \neq 0 \tag{4.12c}$$

In these variables, the function f represents a slowly varying damping coefficient, and is positive if μ is an increasing function of $\tilde{t}$. Therefore, at least for this case, ξ^* must be a bounded function of τ. However, even if f is negative, the solution should only grow with $\tilde{t}$, and must be a bounded function of τ if we hold $\tilde{t}$ fixed.

Let us carry out the solution of the system 4.12 by letting

$$\xi^* = F(\tau, \tilde{t}) = \sum_{n=0}^{N} F^{(n)}(\tau, \tilde{t})\epsilon^n \tag{4.13}$$

The equations governing $F^{(0)}$ and $F^{(1)}$ are:

$$F_{11}^{(0)} + F^{(0)} = 0 \tag{4.14a}$$

$$F_{11}^{(1)} + F^{(1)} = -f(\tilde{t})F_1^{(0)} - \frac{2F_{12}^{(0)}}{\mu} \tag{4.14b}$$

If the solution of equation 4.14a

$$F^{(0)} = A^{(0)}(\tilde{t})\sin\tau + B^{(0)}(\tilde{t})\cos\tau \tag{4.15}$$

is used to compute the right-hand side of equation 4.14b we obtain

$$F_{11}^{(1)} + F^{(1)} = -[fA^{(0)} + \frac{2}{\mu}\frac{dA^{(0)}}{d\tilde{t}}]\sin\tau + [fB^{(0)} + \frac{2}{\mu}\frac{dB^{(0)}}{d\tilde{t}}]\cos\tau \tag{4.16}$$

Clearly, both bracketed terms in equation 4.16 must vanish if F is to be a bounded function of τ.

The solution of the two first order ordinary differential equations thus obtained is

$$A^{(0)}(\tilde{t}) = A^{(0)}(0)\left(\frac{\mu(0)}{\mu}\right)^{1/2}, \quad B^{(0)}(\tilde{t}) = B^{(0)}(0)\left(\frac{\mu(0)}{\mu}\right)^{1/2} \tag{4.17}$$

The initial conditions imply that $A^{(0)}(0) = 0$, $B^{(0)}(0) = 1$, hence $F^{(0)}$ is given by

$$F^{(0)}(\tau, \tilde{t}) = \left(\frac{\mu(0)}{\mu(\tilde{t})}\right)^{1/2} \cos \tau \tag{4.18}$$

This result agrees with that obtained by the WKB method directly to equation 4.6a. The higher approximations can be found in a straightforward manner, and since no new ideas are involved we will not carry this analysis further.

The successful application of the two-variable expansion procedure to this example has thus extended the validity of this method to include functions $\epsilon f(y, \frac{dy}{dt^*}, \tilde{t})$. In the next section we will apply the two-variable expansion to the linear system having a periodic forcing function whose frequency is close to the natural frequency of the system.

4.2. Beats

Consider the following linear mechanical system

$$m \frac{d^2\bar{y}}{dt^2} + k\bar{y} = F_o \cos \omega t \tag{4.19a}$$

$$\bar{y}(0) = \frac{d\bar{y}(0)}{dt} = 0 \tag{4.19b}$$

where m, k, F_o and ω are constants.

We wish to study the solution of 4.19 for the case where ω is close to the natural frequency of the system $(k/m)^{1/2}$.

Clearly the appropriate non-dimensional variables are:

$$y = \frac{\bar{y}}{F_o/k}, \quad t^* = \frac{t}{(m/k)^{1/2}}, \quad \epsilon = \frac{(k/m)^{1/2} - \omega}{(k/m)^{1/2}}, \quad \tilde{t} = \epsilon t^* \tag{4.20}$$

The non-dimensional form of the initial value problem 4.19 is

$$\frac{d^2y}{dt^{*2}} + y = \cos(t^* - \tilde{t}) \tag{4.21a}$$

$$y(0) = \frac{dy(0)}{dt^*} = 0 \tag{4.21b}$$

We can of course compute the following exact solution of the above system:

$$y = \frac{1}{\epsilon(2-\epsilon)}[\cos t^*(\cos \epsilon t^* - 1) + \sin t^* \sin \epsilon t^*] \tag{4.22}$$

We note that $\lim_{\epsilon \downarrow 0} y(t^*, \epsilon) = \frac{t^*}{2} \sin t^*$, namely resonant oscillations. However, for any positive value of ϵ, the motion is bounded and represents a long period oscillation modulating a short period oscillation.

Let us now construct the asymptotic representation of this solution. We should note that since for $\tilde{t}$ fixed the leading term behaves like $1/\epsilon$ the correct expansion for F must have the form:

$$y = F(t^*, \tilde{t}) = \frac{1}{\epsilon} F^{(0)}(t^*, \tilde{t}) + F^{(1)}(t^*, \tilde{t}) + \epsilon F^{(2)}(t^*, \tilde{t}) \tag{4.23}$$

The equations governing $F^{(0)}$ and $F^{(1)}$ are

$$F^{(0)}_{11} + F^{(0)} = 0 \tag{4.24a}$$

$$F^{(1)}_{11} + F^{(1)} = -2F^{(0)}_{12} + \cos t^* \cos \tilde{t} + \sin t^* \sin \tilde{t} \tag{4.24b}$$

where the $F^{(n)}$ satisfy the following initial conditions

$$F^{(0)}(0, 0) = 0, \quad F^{(1)}(0, 0) = 0, \quad F^{(n)}(0, 0) = 0 \tag{4.25a}$$

$$F^{(0)}_1(0, 0) = 0, \quad F^{(1)}_1(0, 0) = -F^{(0)}_2(0, 0), \quad F^{(n)}_1(0, 0) = -F^{(n-1)}_2(0, 0) \tag{4.25b}$$

The solution of equation 4. 24a is

$$F^{(0)} = A^{(0)}(\tilde{t})\sin t^* + B^{(0)}(\tilde{t})\cos t^* \tag{4. 26}$$

where $A^{(0)}(0) = B^{(0)}(0) = 0$ by equation 4. 25.

If we use the above to compute $-2F^{(0)}_{12}$, equation 4. 24b becomes

$$F^{(1)}_{11} + F^{(1)} = (\cos \tilde{t} - \frac{2dA^{(0)}}{d\tilde{t}})\cos t^* + (\sin \tilde{t} + \frac{2dB^{(0)}}{d\tilde{t}})\sin t^* \tag{4. 27}$$

In order that $F^{(1)}$ be bounded we should require that the coefficients of the $\sin t^*$ and $\cos t^*$ terms in equation 4. 27 vanish for all values of $\tilde{t}$. The solution of the resulting equations for $A^{(0)}$ and $B^{(0)}$, satisfying the appropriate initial conditions are

$$A^{(0)}(\tilde{t}) = \frac{1}{2}\sin \tilde{t} \tag{4. 28a}$$

$$B^{(0)}(\tilde{t}) = \frac{1}{2}(\cos \tilde{t} - 1) \tag{4. 28b}$$

Thus $F^{(0)}$ is given by

$$F^{(0)}(t^*, \tilde{t}) = \frac{1}{2}\sin \tilde{t}\sin t^* + \frac{1}{2}(\cos \tilde{t} - 1)\cos t^* \tag{4. 29}$$

which is the appropriate leading term in the expansion for y, as can be verified from equation 4. 22.

The only essential difference between this example and the previous ones, was the fact that the leading term in the expansion for F was of order ϵ^{-1}.

4.3. Mathieu's Equation

The non-dimensional form of Mathieu's equation is

$$\frac{d^2y}{dt^{*2}} + (\delta + \epsilon \cos t^*)y = 0 \tag{4.30}$$

In reference 5, Stoker studies the stability of the solution of equation 4.30 in the δ, ϵ plane. He shows that corresponding to transitional values of δ and ϵ from stability to instability, there exists a periodic solution of equation 4.30 with period 2π or 4π, and that the general solutions of equation 4.30 for these transitional values of δ and ϵ are unstable.*

Thus by finding all the functions $\delta(\epsilon)$ for which equation 4.30 admits a periodic solution one can define the boundaries of the regions of stability and instability in the δ, ϵ plane.

It is especially easy to compute the periodic solutions and their corresponding transitional curves $\delta(\epsilon)$ if ϵ is small. To summarize the results given by Stoker and McLachlan in reference 6, we note the following.

The transitional curves intersect the $\epsilon = 0$ axis at the critical points $\delta_c = n^2/4$, $n = 0, 1, 2, \ldots$. Through all these points pass two transitional curves $\delta^{(n)}$ and ${}^{(n)}\delta$, except at the origin which admits only one such curve ${}^{(0)}\delta$. The region to the left of the curve ${}^{(0)}\delta$ corresponds to unstable solutions, and as each of the transitional curves are crossed the stability of the solution of equation 4.30 changes.

The asymptotic representation of the first four $\delta^{(n)}$ and ${}^{(n)}\delta$,

*By stability we mean boundedness for all non-negative values of t^*.

correct to $O(\epsilon^4)$ is given below. For the expansions of all the $\delta^{(n)}$ and ${}^{(n)}\delta$ correct to $O(\epsilon^6)$, the reader is referred to page 17 of reference 6.

$$ {}^{(0)}\delta = -\frac{\epsilon^2}{2} + \frac{7}{32}\epsilon^4 \tag{4.31a}$$

$$ \delta^{(1)} = \frac{1}{4} + \frac{\epsilon}{2} - \frac{\epsilon^2}{8} - \frac{\epsilon^3}{32} - \frac{1}{768}\epsilon^4 \tag{4.31b}$$

$$ {}^{(1)}\delta = \frac{1}{4} - \frac{\epsilon}{2} - \frac{\epsilon^2}{8} + \frac{\epsilon^3}{32} - \frac{1}{768}\epsilon^4 \tag{4.31c}$$

$$ \delta^{(2)} = 1 - \frac{\epsilon^2}{12} + \frac{5}{6912}\epsilon^4 \tag{4.31d}$$

$$ {}^{(2)}\delta = 1 + \frac{5\epsilon^2}{12} - \frac{763}{6912}\epsilon^4 \tag{4.31e}$$

$$ \delta^{(3)} = \frac{9}{4} + \frac{\epsilon^2}{16} + \frac{\epsilon^3}{32} + \frac{13}{10,240}\epsilon^4 \tag{4.31f}$$

$$ {}^{(3)}\delta = \frac{9}{4} + \frac{\epsilon^2}{16} - \frac{\epsilon^3}{32} + \frac{13}{10,240}\epsilon^4 \tag{4.31g}$$

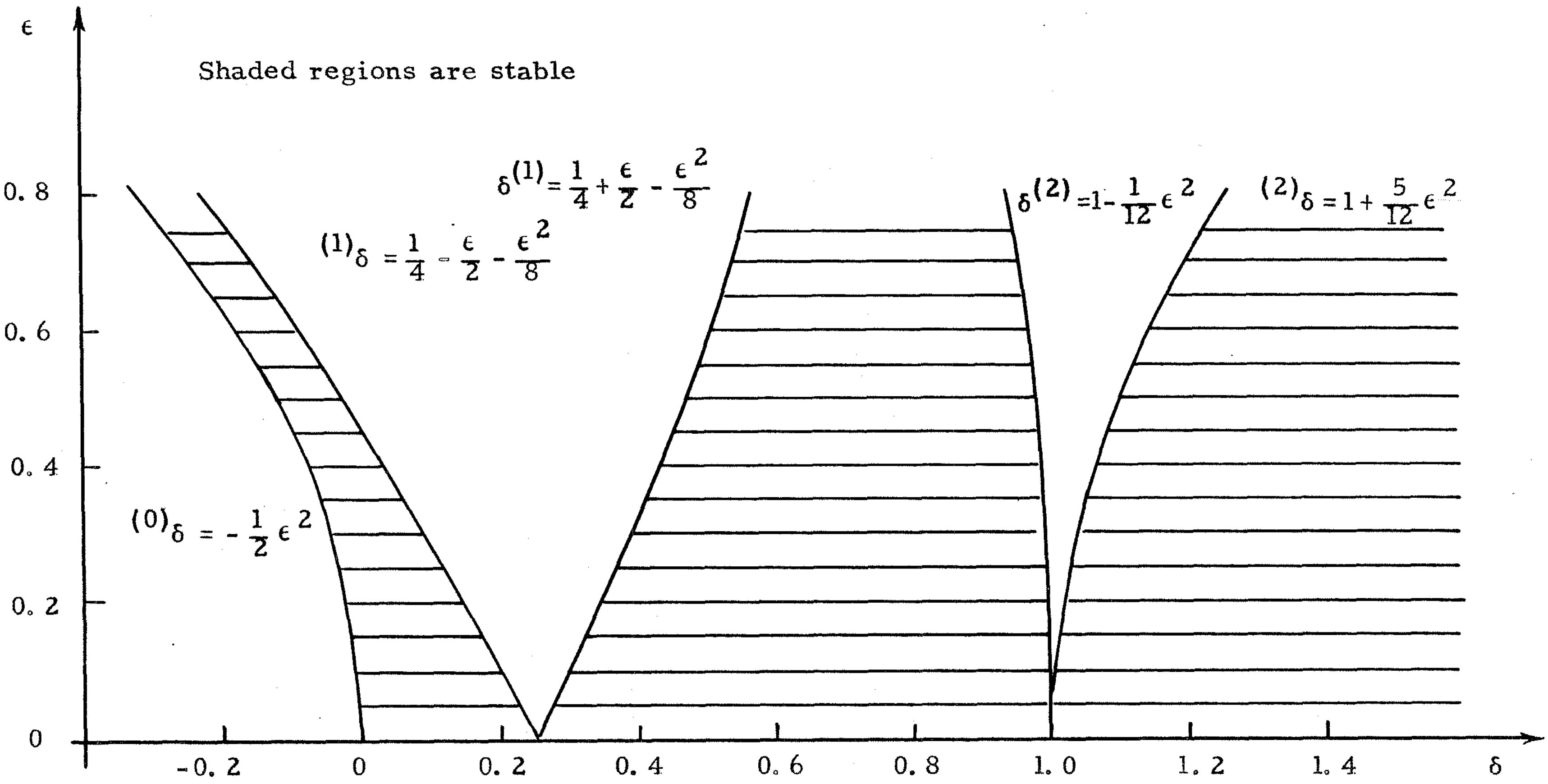
The regions of stability and instability bounded by the first five of the above curves is illustrated in fig. 4.1.

In this section we will investigate the behavior of the solutions of equation 4.30 in neighborhoods of the transitional curves emanating from the first three critical points $\delta_c = 0, \frac{1}{4}, 1$. More precisely, we will solve Mathieu's equation with δ set equal to $\delta_c + \sum_{i=1}^{n} \delta_i \epsilon^i$ with δ_c fixed, and δ_i arbitrary. In the δ, ϵ plane the curves $\delta = \delta_c + \sum_{i=1}^{n} \delta_i \epsilon^i$ form an n-parameter family of curves passing through δ_c for each value of δ_c.

In order to apply the two-variable expansion procedure to both the stable and unstable solutions we will make the following assumption regarding the structure of the solutions of equation 4.30.

Figure 4.1

REGIONS OF STABILITY FOR MATHIEU'S EQUATION FOR SMALL VALUES OF ϵ



On the n-parameter family of curves $\delta = \delta_c + \sum_{i=1}^{n} \delta_i \epsilon^i$ the solutions of Mathieu's equation can be uniformly represented by an asymptotic expansion in terms of the two variables t^*, and $t_\eta = \eta(\epsilon)t^*$ for some appropriate $\eta(\epsilon) = o(1)$ as $\epsilon \to 0$. The particular η depends on the critical point δ_c.

It can be shown that for each critical point δ_c there is only one order class of functions $\eta_c(\epsilon)$ for which the expansion for F has the following properties, and we will determine η by demonstrating these properties.

i) For values of δ_i associated with a stable region F will be a bounded function of the two variables t^* and t_η.

ii) For values of δ_i associated with an unstable region F will be a bounded function of t^*, but an unbounded function of t_η.

The above is our assumption regarding the structure of the solutions of equation 4.30, and cannot be proved for this case since no t_η appears in the differential equation. (Contrast this with the example in section 4.1.)

Assuming the validity of this assumption, the expansion for F must conform with the results of the general theory in the following respects.

(i) On the transition curves F must admit a periodic solution for the appropriate initial conditions, and the general initial value problem must be unstable.

(ii) Since F is presumed to represent y uniformly, the basic stability, and instability of the solution must be exhibited by the leading term in the expansion for F.

Let us adopt the initial conditions

$$y(0) = b \tag{4.32a}$$

$$\frac{dy(0)}{dt^*} = a \tag{4.32b}$$

and in the first instance study the solutions of equation 4.30 in the neighborhood of the origin of the δ, ϵ plane.

For this purpose let us choose $\delta = O(\epsilon)$ as $\epsilon \to 0$ in the form

$$\delta = \epsilon \delta_1 + \epsilon^2 \delta_2 + O(\epsilon^3) \tag{4.33}$$

Equation 4.30 becomes

$$\frac{d^2 y}{dt^{*2}} + \epsilon[(\delta_1 + \cos t^*) + \epsilon \delta_2 + O(\epsilon^2)]\, y = 0 \tag{4.34}$$

We will show by actually carrying out the expansion, that the appropriate $\eta(\epsilon)$ is in this case ϵ. Let us now expand y in terms of t^* and $\tilde{t} = \epsilon t^*$ thus:

$$y = F(t^*, \tilde{t}) = \frac{1}{\epsilon} F^{(0)}(t^*, \tilde{t}) + F^{(1)}(t^*, \tilde{t}) + \epsilon F^{(2)}(t^*, \tilde{t}) + \dots \tag{4.35}$$

The leading term in this expansion is of order ϵ^{-1} in order to accomodate the non-zero initial condition $\frac{dy(0)}{dt^*} = a$.

The equations and initial conditions obtained by substituting the expansion 4.33 into 4.34 and 4.32 are:

$$F_{11}^{(0)} = 0 \tag{4.36a}$$

$$F_{11}^{(1)} = -2F_{12}^{(0)} - (\delta_1 + \cos t^*)F^{(0)} \tag{4.36b}$$

$$F_{11}^{(2)} = -2F_{12}^{(1)} - (\delta_1 + \cos t^*)F^{(1)} - \delta_2 F^{(0)} - F_{22}^{(0)} \tag{4.36c}$$

$$F_{11}^{(n)} = -2F_{12}^{(n-1)} - (\delta_1 + \cos t^*)F^{n-1} - F_{22}^{(n-2)} - \delta_2 F^{(n-2)} - \dots - \delta_n F^{(0)} \tag{4.36n}$$

$$F^{(1)}(0,0) = b, \quad F^{(n)}(0,0) = 0, \quad n \neq 0 \tag{4.37a}$$

$$F_1^{(1)}(0,0) = a - F_2^{(0)}(0,0), \quad F_1^{(n)}(0,0) = -F_2^{(n-1)}(0,0) \quad n \neq 1 \tag{4.37b}$$

We note that the homogeneous solutions of all the $F^{(n)}$ are

$$F^{(n)}(t^*, \tilde{t}) = A^{(n)}(\tilde{t})t^* + B^{(n)}(\tilde{t}) \tag{4.38}$$

In view of our assumption regarding the boundedness of the $F^{(n)}$ for large t^* we must set all the $A^{(n)}(\tilde{t}) = 0$.

The solution for $F^{(0)}$ is therefore

$$F^{(0)} = B^{(0)}(\tilde{t}) \tag{4.39}$$

The initial condition 4.37 for $F^{(0)}$ requires that

$$B^{(0)}(0) = 0 \tag{4.40}$$

If we use the fact that $F^{(0)}$ is a function of $\tilde{t}$ only in equation 4.36b we obtain

$$F_{11}^{(1)} = -\delta_1 B^{(0)}(\tilde{t}) - \cos t^* \, B^{(0)}(\tilde{t}) \tag{4.41}$$

Again in order that $F^{(1)}$ be a bounded function of t^*, δ_1 must vanish, and this produces the correct stability criterion in the neighborhood of the origin. For to the first order in ϵ the transitional curve is indeed $\delta = 0$.

If we now solve for $F^{(1)}$ we obtain

$$F^{(1)} = B^{(1)}(\tilde{t}) + B^{(0)}(\tilde{t}) \cos t^* \tag{4.42}$$

The initial conditions for $F^{(1)}$ imply that

$$B^{(1)}(0) = b, \quad \frac{dB^{(0)}(0)}{d\tilde{t}} = a \tag{4.43}$$

If we now use the preceding expressions for $F^{(0)}$ and $F^{(1)}$ to evaluate the right-hand side of equation 4.36c we obtain

$$F_{11}^{(2)} = \frac{2dB^{(0)}}{d\tilde{t}} \sin t^* - \left[\frac{d^2B^{(0)}}{d\tilde{t}^2} + (\delta_2 + \tfrac{1}{2})B^{(0)} \right]$$

$$- B^{(1)}(\tilde{t}) \cos t^* - (B^{(0)}/2) \cos 2t^* \tag{4.44}$$

The bracketed term in the right-hand side of equation 4.44 is a function of $\tilde{t}$ only, and hence must vanish in order that $F^{(2)}$ be bounded.

The solution of the equation obtained by setting this bracketed term equal to zero is:

$$B^{(0)}(\tilde{t}) = a(\delta_2 + \tfrac{1}{2})^{-1/2} \sin (\delta_2 + \tfrac{1}{2})^{1/2} \tilde{t} \quad \text{if } \delta_2 \geq -\tfrac{1}{2} \tag{4.45a}$$

$$B^{(0)}(\tilde{t}) = \frac{a}{2}(-\delta_2 - \tfrac{1}{2})^{-1/2}\left[e^{(-\delta_2-\frac{1}{2})^{1/2}\tilde{t}} - e^{-(-\delta_2-\frac{1}{2})^{1/2}\tilde{t}} \right] \text{if } \delta_2 \leq -\tfrac{1}{2}$$

(4.45b)

Thus we have established the correct stability criterion for δ_2, and if we proceed with the higher order terms we will successively be able to compute more accurate expressions for the transitional curve and the solution on either side of this curve. To illustrate the situation it will suffice to evaluate $B^{(1)}$ by considering the next step. Necessarily our calculation must fall into two categories from now on depending on whether $\delta_2 > -\frac{1}{2}$ or $\delta_2 < -\frac{1}{2}$.

Let us first consider the stable case $\delta_2 > -\frac{1}{2}$. The solution of equation 4. 44 for $F^{(2)}$ gives

$$F^{(2)} = -2a \sin p\tilde{t} \sin t^* + B^{(1)}(\tilde{t})\cos t^* + \frac{a}{8p} \sin p\tilde{t} \cos 2t^* + B^{(2)}(\tilde{t}) \qquad (4.46)$$

where $p = (\delta_2 + \frac{1}{2})^{1/2}$. By applying the initial conditions on $F^{(2)}$ and $F_1^{(2)}$ we obtain

$$B^{(2)}(0) = -b, \qquad \frac{dB^{(1)}(0)}{d\tilde{t}} = -a \qquad (4.47)$$

If we use the preceding expressions for $F^{(0)}$, $F^{(1)}$ and $F^{(2)}$ to compute the right-hand side of equation 4. 36 we obtain:

$$F_{22}^{(3)} = -\left[\frac{d^2B^{(1)}}{d\tilde{t}^2} + pB^{(1)} + \frac{\delta_3 a}{p} \sin p\tilde{t} \right] + \left[3ap \cos p\tilde{t} - \frac{a}{p}\left(\frac{1}{16} + \delta_2\right)\sin p\tilde{t} - B^{(2)}(\tilde{t}) \right] \cos t^* + \frac{2dB^{(1)}}{d\tilde{t}} \sin t^* + \frac{1}{2}(a \cos p\tilde{t} - B^{(1)})\cos 2t^* + a \sin p\tilde{t} \sin 2t^* - \frac{a}{16p} \sin p\tilde{t} \cos 3t^* \qquad (4.48)$$

In order that $F^{(3)}$ be a bounded function of t^* the first bracketed term on the right-hand side of equation 4.48 must vanish. Now in order that the ensuing expression for $B^{(1)}$ be a bounded function of $\tilde{t}$ we must further require that $\delta_3 = 0$, and this is in agreement with the results given in equation 4.31a. With $\delta_3 = 0$ the solution of the bracketed term in question is

$$B^{(1)}(\tilde{t}) = -\frac{a}{p}\sin p\tilde{t} + b\cos p\tilde{t} \tag{4.49}$$

We may summarize the results as follows. The general representation of the stable solution corresponding to $\delta_2 > -\frac{1}{2}$, correct to $O(1)$ is

$$y = \frac{1}{\epsilon}\,\frac{a}{p}\sin p\tilde{t} + \frac{a}{p}\sin p\tilde{t}(\cos t^* - 1) + b\cos p\tilde{t} + O(\epsilon) \tag{4.50a}$$

The leading term of the unstable solution on the transitional curve is

$$y = \frac{a\tilde{t}}{\epsilon} + b + O(\epsilon) \tag{4.50b}$$

The unstable solution for $\delta_2 < -\frac{1}{2}$, correct to $O(\frac{1}{\epsilon})$ is

$$y = \frac{a}{2\epsilon(-\frac{1}{2} - \delta_2)^{1/2}}\left[e^{(-\frac{1}{2} - \delta_2)^{1/2}\tilde{t}} - e^{-(-\frac{1}{2} - \delta_2)^{1/2}\tilde{t}}\right] \tag{4.51}$$

These solutions lie on the family of parabolas $\delta = \delta_2\epsilon + O(\epsilon^4)$ in the δ, ϵ plane. On the transition curve we have a periodic solution if $a = 0$, but the general solution with $a \neq 0$ is unstable as postulated by the general theory.

Let us now return to the question of the choice of $\eta(\epsilon)$. If we had chosen any $\eta(\epsilon)$ other than $\eta(\epsilon) = O(\epsilon)$ as $\epsilon \to 0$, either one of the following two eventualities would have arisen.

(i) For $\eta(\epsilon) = o(\epsilon)$ as $\epsilon \to 0$, it would have been impossible to make F a bounded function of t_η for values of δ_i consistent with the transitional curve. This can be seen easily if we consider a typical term in the expansion such as $\sin\,(\delta_2 + \frac{1}{2})^{1/2}\tilde{t} = \sin(\delta_2 + \frac{1}{2})^{1/2}\epsilon t^*$. If, for example, we had chosen $\eta(\epsilon) = \epsilon^{1/2}$ (i. e. $(t_\eta = \epsilon^{1/2} t^*)$) the term in question would have appeared in the solution in terms of its non-uniform expansion in powers of $t_\eta \epsilon^{1/2}$. This would have led to a term proportional to $\epsilon^{1/2}(\delta_2 + \frac{1}{2})^{1/2} t_\eta$ in the solution, and to the erroneous conclusion that $\delta_2 = -\frac{1}{2}$ was the <u>only</u> possible value for stability.

(ii) For $\epsilon = o(\eta(\epsilon))$ as $\epsilon \to 0$, it would have been impossible to make F a bounded function of t^* for the following reason. If the two variables we are expanding in are t^*, and $t_\eta = \epsilon^2 t^*$, the above term would appear in the solution in terms of its non-uniform expansion in powers of ϵt^* thus: $(\delta_2 + \frac{1}{2})^{1/2}\epsilon t^* + \dots$.

We conclude from the above that if we restrict ourselves to those solutions which lie on the family of curves to which the transitional curve belongs, (i. e. the m-parameter family obtained by varying the <u>non-zero</u> coefficients $\delta_1, \dots, \delta_m$ appearing in the transitional curve) then we have a unique way of determining the second time variable in our expansion.

In the preceding example, and in what follows the choice of the t_η variable we have used was verified by actually carrying out the expansions for two other $\eta(\epsilon)$ in adjacent order classes. In all cases the results were as outlined above.

Next, let us consider the family of solutions in the neighborhood of $\delta_c = \frac{1}{4}$. For this purpose we let $\delta = \frac{1}{4} + \epsilon\delta_1 + \ldots$, and equation 4.30 becomes

$$\frac{d^2y}{dt^{*2}} + \left[\frac{1}{4} + \epsilon(\delta_1 + \cos t^*) + O(\epsilon^2)\right] y = 0 \qquad (4.52)$$

Here again the appropriate function $\eta(\epsilon)$ is ϵ, hence $t_\eta = \tilde{t} = \epsilon t^*$. We will expand y in terms of t^* and $\tilde{t}$ in the form

$$y = F(t^*, \tilde{t}) = F^{(0)}(t^*, \tilde{t}) + \epsilon F^{(1)}(t^*, \tilde{t}) + \ldots \qquad (4.53)$$

The equations, and initial conditions governing $F^{(0)}$ and $F^{(1)}$ are

$$F_{11}^{(0)} + \frac{1}{4} F^{(0)} = 0 \qquad (4.54a)$$

$$F_{11}^{(1)} + \frac{1}{4} F^{(1)} = -2F_{12}^{(0)} - \delta_1 F^{(0)} - \cos t^* F^{(0)} \qquad (4.54b)$$

$$F^{(0)}(0, 0) = b, \quad F^{(1)}(0, 0) = 0 \qquad (4.55a)$$

$$F_1^{(0)}(0, 0) = a, \quad F_1^{(1)}(0, 0) = -F_2^{(0)}(0, 0) \qquad (4.55b)$$

The solution of equation 4.54a for $F^{(0)}$ gives

$$F^{(0)} = A^{(0)}(\tilde{t}) \sin \frac{t^*}{2} + B^{(0)}(\tilde{t}) \cos \frac{t^*}{2} \qquad (4.56)$$

The initial conditions imply that

$$B^{(0)}(0) = b, \quad A^{(0)}(0) = 2a \qquad (4.57)$$

When this expression for $F^{(0)}$ is substituted into equation 4.54b we obtain

$$F_{11}^{(1)} + \frac{1}{4} F^{(1)} = \left[\frac{dB^{(0)}}{d\tilde{t}} - \delta_1 A^{(0)} + \frac{A^{(0)}}{2} \right] \sin \frac{t^*}{2}$$

$$- \left[\frac{dA^{(0)}}{d\tilde{t}} + (\delta_1 + \frac{1}{2}) B^{(0)} \right] \cos \frac{t^*}{2} - \frac{A^{(0)}}{2} \sin \frac{3t^*}{2}$$

$$- \frac{B^{(0)}}{2} \cos \frac{3t^*}{2} \qquad (4.58)$$

Thus the boundedness of $F^{(1)}$ requires that $A^{(0)}$ and $B^{(0)}$ satisfy the equations

$$\frac{dA^{(0)}}{d\tilde{t}} + (\delta_1 + \frac{1}{2}) B^{(0)} = 0 \qquad (4.59a)$$

$$\frac{dB^{(0)}}{d\tilde{t}} - (\delta_1 - \frac{1}{2}) A^{(0)} = 0 \qquad (4.59b)$$

The solution of the above equations falls into two categories. First if $\delta_1 > \frac{1}{2}$ or $\delta_1 < -\frac{1}{2}$ the solutions are stable, and we have

$$A^{(0)} = b\left(\frac{2\delta_1 + 1}{2\delta_1 - 1}\right)^{1/2} \sin (\delta_1^2 - \frac{1}{4})^{1/2} \tilde{t} + 2a \cos (\delta_1^2 - \frac{1}{4})^{1/2} \tilde{t} \qquad (4.60a)$$

$$B^{(0)} = b \cos (\delta_1^2 - \frac{1}{4})^{1/2} \tilde{t} - 2a\left(\frac{2\delta_1 - 1}{2\delta_1 + 1}\right)^{-1/2} \sin (\delta_1^2 - \frac{1}{4})^{1/2} \tilde{t} \qquad (4.60b)$$

Conversely if $-\frac{1}{2} < \delta_1 < \frac{1}{2}$, the solutions are unstable, and we have

$$A^{(0)}\tilde{t} = \frac{[b\sqrt{r} + 2a\sqrt{s}]^2 e^{-2\sqrt{rs}\,\tilde{t}} + (4sa^2 - rb^2)}{2\sqrt{s}\,(b\sqrt{r} + 2a\sqrt{s}\,) e^{-\sqrt{rs}\,\tilde{t}}} \qquad (4.61a)$$

$$B^{(0)}\tilde{t} = \frac{[b\sqrt{r} + 2a\sqrt{s}]^2 e^{-2\sqrt{rs}\,\tilde{t}} - (4sa^2 - rb^2)}{2\sqrt{r}\,(b\sqrt{s} + 2a\sqrt{s}) e^{-\sqrt{rs}\,\tilde{t}}} \qquad (4.61b)$$

where $r = (\delta_1 + \frac{1}{2})$, $s = (\frac{1}{2} - \delta_1)$.

When $\delta_1 = -\frac{1}{2}$ we have

$$A^{(0)} = 2a \ , \qquad B^{(0)} = b - 2a\tilde{t} \tag{4.62a}$$

and for $\delta_1 = \frac{1}{2}$ we obtain

$$A^{(0)} = 2a + b\tilde{t} \ , \quad B^{(0)} = b \tag{4.62b}$$

Thus the choice of $t_\eta = \epsilon t^*$ has led to the correct stability requirements (and this choice is unique). Furthermore, it can be verified that the expression we have derived for $F^{(0)}$ possesses the appropriate properties on, and in the neighborhoods of the two transitional curves through $\delta_c = \frac{1}{4}$.

By carrying out the next step in the expansion we will obtain the more accurate criteria for $\delta^{(1)}$ and ${}^{(1)}\delta$ as well as the next term in the expansion for F.

The analysis for the third critical point $\delta_c = 1$ is quite similar to the preceding with the exception that here we must set $t_\eta = \epsilon^2 t^* = \tilde{\tilde{t}}$.

We will not carry out the details here, but will simply present the final results for $F^{(0)}$ the leading term in the expansion for F.

We have for $\delta = 1 + \epsilon\delta_1 + \epsilon^2\delta_2$

$$y = F(t^*, \tilde{\tilde{t}}) = F^{(0)}(t^*, \tilde{\tilde{t}}) + \epsilon F^{(1)}(t^*, \tilde{\tilde{t}}) + \ldots \tag{4.63}$$

$$F^{(0)}(t^*, \tilde{\tilde{t}}) = A^{(0)}(\tilde{\tilde{t}})\sin t^* + B^{(0)}(\tilde{\tilde{t}})\cos t^* \tag{4.64}$$

For the boundedness with respect to t^* we must have $\delta_1 = 0$. The solutions are stable if $\delta_2 > \frac{5}{12}$ or $\delta_2 < -\frac{1}{12}$ and

$$A^{(0)}(\tilde{\tilde{t}}) = b\left(\frac{12\delta_2-5}{12\delta_2+1}\right)^{1/2} \sin\left[(\delta_2-\frac{5}{12})(\delta_2+\frac{1}{12})\right]^{1/2}\tilde{\tilde{t}}$$

$$+ a\cos\left[(\delta_2-\frac{5}{12})(\delta_2+\frac{1}{12})\right]^{1/2}\tilde{\tilde{t}} \qquad (4.65a)$$

$$B^{(0)}(\tilde{\tilde{t}}) = b\cos\left[(\delta_2-\frac{5}{12})(\delta_2+\frac{1}{12})\right]^{1/2}\tilde{\tilde{t}}$$

$$- a\left(\frac{12\delta_2+1}{12\delta_2-5}\right)^{1/2} \sin\left[(\delta_2-\frac{5}{12})(\delta_2+\frac{1}{12})\right]^{1/2}\tilde{\tilde{t}} \qquad (4.65b)$$

The solutions are unstable if $-\frac{1}{12} < \delta_2 < \frac{5}{12}$ and

$$A^{(0)}(\tilde{\tilde{t}}) = \frac{[b\sqrt{r} + a\sqrt{s}]^2 e^{2\sqrt{rs}\,\tilde{\tilde{t}}} + (sa^2 - rb^2)}{2\sqrt{s}\,(b\sqrt{r} + a\sqrt{s}\,)e^{\sqrt{rs}\,\tilde{\tilde{t}}}} \qquad (4.66a)$$

$$B^{(0)}(\tilde{\tilde{t}}) = \frac{[b\sqrt{r} + a\sqrt{s}\,]^2 e^{2\sqrt{sr}\,\tilde{\tilde{t}}} - (sa^2 - rb^2)}{2\sqrt{r}\,(b\sqrt{r} + a\sqrt{s}\,)e^{\sqrt{rs}\,\tilde{\tilde{t}}}} \qquad (4.66b)$$

where $r = \frac{5}{12} - \delta_2$, and $s = \delta_2 + \frac{1}{12}$.

On the transitional curve $\delta^{(2)} = 1 + \frac{5\epsilon^2}{12}$, $A^{(0)}$ and $B^{(0)}$ take the following values

$$A^{(0)}(\tilde{\tilde{t}}) = a, \quad B^{(0)}(\tilde{\tilde{t}}) = b - \frac{a\tilde{\tilde{t}}}{2} \qquad (4.67a)$$

On ${}^{(2)}\delta = 1 - \frac{\epsilon^2}{12}$ we have

$$A^{(0)}(\tilde{\tilde{t}}) = a + \frac{b\tilde{\tilde{t}}}{2}, \quad B^{(0)}(\tilde{\tilde{t}}) = b \qquad (4.67b)$$

Again, it can be verified that the expression we have derived for $F^{(0)}$ possesses the appropriate properties on, and in the neighborhood of the transitional curves through the critical point $\delta_c = 1$.

As mentioned earlier, this solution (and similar solutions for the remainder of the critical points) are valid only in limited regions surrounding the transitional curves. It is not clear whether the two-variable expansion procedure would apply in regions not described by the families of curves $\delta = \delta_c + \sum_{i=1}^{n} \delta_i \epsilon^i$. For example, if we wish to solve equation 4.30 with $\delta = \frac{1}{2} + \sum_{i=1}^{n} \delta_i \epsilon^i$ (where $\frac{1}{2} \neq \delta_c$), the choice of the second time variable is not clear, nor is it clear in what sense such a solution should match with the solutions through either of the two neighboring points $\delta_c = \frac{1}{4}$ and $\delta_c = 1$, at corresponding points in the δ, ϵ plane.

4.4. The Motion of a Satellite in a Central Gravitational Field and a Thin Constant Density Atmosphere.

The purpose of this example is to illustrate the application of the two variable expansion procedure to a wide variety of problems in celestial mechanics, for which the motion in the absence of small perturbing forces, is Keplerian.

A number of such problems may be solved by Poincare's method. These problems are characterized by the fact that the motion can be described by periodic terms, either in an inertial or in a uniformly precessing coordinate system.

If, however, the motion does not fall in this category Poincare's method fails. We will study such an example in this section.

Consider the motion of a spherical satellite in the gravitational field of a homogeneous, spherical, and non-rotating earth. Let us further assume that the drag coefficient of the satellite is constant, and that the earth is surrounded by a thin constant density atmosphere. Although this simple model of an earth is not physically realistic, it retains the essential features which serve to illustrate the effect of drag perturbation in the two body problem.

Since the earth is stationary, there will be no Coriolis force, the orbit will remain planar, and we can write the equations of motion in polar coordinates in the plane of the orbit, and centered at the origin of attraction:

$$m\frac{d^2r}{dt^2} - mr\left(\frac{d\varphi}{dt}\right)^2 = -m\frac{GM}{r^2} - \frac{C_D\rho S}{2}\frac{dr}{dt}\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\varphi}{dt}\right)^2\right]^{1/2} \tag{4.68a}$$

$$mr\frac{d^2\varphi}{dt^2} + 2m\frac{dr}{dt}\frac{d\varphi}{dt} = -\frac{C_D\rho S}{2}r\frac{d\varphi}{dt}\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\varphi}{dt}\right)^2\right]^{1/2} \tag{4.68b}$$

In the above, m is the mass of the satellite, M the mass of the earth, G the universal gravitational constant, C_D the drag coefficient, ρ the density of the atmosphere and S the cross sectional area of the satellite.

We can make some further simplifications in the choice of the following initial conditions:

$$r(0) = R \tag{4.69a}$$

$$\frac{dr(0)}{dt} = 0 \tag{4.69b}$$

$$\varphi(0) = 0 \tag{4.70a}$$

$$\frac{d\varphi(0)}{dt} = (GM/R^3)^{1/2} \tag{4.70b}$$

In the absence of an atmosphere, the above initial conditions correspond to a circular orbit of radius R.

We have the following characteristic dimensions in the problem

$$L_1 = R \tag{4.71a}$$

$$L_2 = C_D \rho S/2m \tag{4.71b}$$

$$T_1 = (R^3/GM)^{1/2} \tag{4.72a}$$

$$T_2 = \frac{C_D \rho SR}{2m} (R^3/GM)^{1/2} \tag{4.72b}$$

We wish to study the motion in the large, hence the appropriate length and time scales are L_1 and T_1. The small parameter in this problem is $\epsilon = C_D \rho RS/2m$, the ratio of drag to centrifugal force at radius R.

Let us introduce the non-dimensional variables

$$r^* = \frac{r}{L_1} , \quad t^* = \frac{t}{T_1} , \quad \tilde{t} = \frac{t}{T_2} = \epsilon t^* \tag{4.73}$$

Equations 4.68, 4.69, and 4.70 take on the non-dimensional form

$$\frac{d^2 r^*}{dt^{*2}} - r^* \left(\frac{d\varphi}{dt^*}\right)^2 = -\frac{1}{r^{*2}} - \epsilon \frac{dr^*}{dt^*} \left[\left(\frac{dr^*}{dt^*}\right)^2 + r^{*2} \left(\frac{d\varphi}{dt^*}\right)^2 \right]^{1/2} \tag{4.74a}$$

$$r^* \frac{d^2\varphi}{dt^{*2}} + \frac{2\, dr^*}{dt^*} \frac{d\varphi}{dt^*} = -\epsilon r^* \left[\left(\frac{dr^*}{dt^*}\right)^2 + r^{*2} \left(\frac{d\varphi}{dt^*}\right)^2 \right]^{1/2} \frac{d\varphi}{dt^*} \tag{4.74b}$$

$$r^*(0) = 1 \tag{4.75a}$$

$$\frac{dr^*(0)}{dt^*} = 0 \tag{4.75b}$$

$$\varphi(0) = 0 \tag{4.76a}$$

$$\frac{d\varphi(0)}{dt^*} = 1 \tag{4.76b}$$

The two variable expansion procedure, when applied directly to these equations, will fail for the following reason.

Let us denote the term

$$r^{*2}\frac{d\varphi}{dt^*}\left[\left(\frac{dr^*}{dt^*}\right)^2 + r^{*2}\left(\frac{d\varphi}{dt^*}\right)^2\right]^{1/2} = \frac{d}{dt^*}f(t^*, \epsilon)$$

allowing us to integrate equation 4.74b to

$$r^{*2}\frac{d\varphi}{dt^*} = a + \epsilon f(t^*, \epsilon) \tag{4.77}$$

where a is a constant. If we now solve for $d\varphi/dt^*$ from equation 4.77 and substitute the result into equation 4.74a we obtain

$$\frac{d^2r^*}{dt^{*2}} + \epsilon\frac{dr^*}{dt^*}\left[\left(\frac{dr^*}{dt^*}\right)^2 + \frac{1}{r^{*2}}(a+\epsilon f)^2\right]^{1/2} + \left(\frac{1}{r^{*2}} - \frac{a}{r^*}\right)$$

$$= \epsilon\left[\frac{2af}{r^{*3}} + \frac{\epsilon f^2}{r^{*3}}\right] \tag{4.78}$$

Even though for $\epsilon = 0$ the motion is oscillatory, we note that because of the nature of the restoring force term $1/r^{*2} - a/r^*$, the unperturbed motion will not be simple-harmonic. This in turn will introduce $\tilde{t}$ dependent terms in the frequency for the representation of r^*. As we noted earlier in this chapter, the method cannot apply under these circumstances.

Fortunately from our knowledge of the unperturbed motion we can deduce the appropriate variables for this problem. We know that a Kepler ellipse has the following form in polar coordinates

$$r^* = \frac{a(1-e^2)}{1 - e\cos(\varphi - \varphi_o)} \tag{4.79}$$

where a is the semi-major axis, e the eccentricity and φ_o the argument of apogee.

Thus $1/r^* = u$ must be the solution of a differential equation of the form

$$\frac{d^2u}{d\varphi^2} + u = \text{constant} \tag{4.80}$$

This immediately suggests that we should let φ be the independent, time-like variable, and let the dependent variables be t^*, and u. If we perform the above transformations, equations 4.74 and the initial conditions 4.75 and 4.76 become

$$\frac{d^2u}{d\varphi^2} + u = u^4\left(\frac{dt^*}{d\varphi}\right)^2 \tag{4.81a}$$

$$u^2\frac{d^2t^*}{d\varphi^2} + 2u\frac{du}{d\varphi}\frac{dt^*}{d\varphi} = \epsilon\frac{dt^*}{d\varphi}\left[u^2 + \left(\frac{du}{d\varphi}\right)^2\right]^{1/2} \tag{4.81b}$$

$$u(0) = 1 \tag{4.82a}$$

$$\frac{du(0)}{d\varphi} = 0 \tag{4.82b}$$

$$t^*(0) = 0 \tag{4.83a}$$

$$\frac{dt^*(0)}{d\varphi} = 1 \tag{4.83b}$$

It is easy to verify that for $\epsilon = 0$ the term $u^4(dt^*/d\varphi)^2$ is a constant of the motion, equal to the reciprocal of the square of the angular momentum. In fact for our initial conditions we have

$$u^4\left(\frac{dt^*}{d\varphi}\right)^2 = 1 + O(\epsilon) \quad \text{as } \epsilon \to 0 \tag{4.84}$$

Therefore u is governed by an equation equivalent to that of a linear oscillator with an almost constant forcing function, and an expansion procedure in terms of the two variables φ and $\tilde{\varphi} = \epsilon\varphi$ will apply if we can show that u must be a bounded function of φ. We cannot prove this latter assumption but will use a physical argument to justify it.

For $\epsilon = 0$, u is a bounded function of φ (for our special initial conditions u is constant). For $\epsilon \neq 0$ we expect r to approach zero slowly as t^* approaches infinity. In fact since drag has the same effect as a damping (c. f. equation 4. 78) one would expect the decay of r to depend on $\tilde{t} = \epsilon t^*$. Since the mean value of φ is proportional to t^*, the assumption that the decay of r depends on $\tilde{t}$ alone implies that u should be a bounded function of φ, and an unbounded function of $\tilde{\varphi} = \epsilon\varphi$.

We will now expand u, and t^* in the form:

$$u(\varphi, \epsilon) = V(\varphi, \tilde{\varphi}; \epsilon) = V^{(0)}(\varphi, \tilde{\varphi}) + \epsilon V^{(1)}(\varphi, \tilde{\varphi}) + \ldots \tag{4.85a}$$

$$t^*(\varphi, \epsilon) = T(\varphi, \tilde{\varphi}; \epsilon) = T^{(0)}(\varphi, \tilde{\varphi}) + \epsilon T^{(1)}(\varphi, \tilde{\varphi}) + \ldots \tag{4.85b}$$

We wish only to solve for $V^{(0)}$ and $T^{(0)}$, hence will only carry out the expansions to the first power in ϵ. Equations 4. 81 require that $V^{(0)}$, $V^{(1)}$, $T^{(0)}$ and $T^{(1)}$ satisfy the following equations:

$$V_{11}^{(0)} + V^{(0)} = {V^{(0)}}^4 {T_1^{(0)}}^2 \tag{4.86a}$$

$${V^{(0)}}^2 T_{11}^{(0)} + 2V^{(0)} V_1^{(0)} T_1^{(0)} = 0 \tag{4.86b}$$

$$\begin{aligned} V_{11}^{(1)} + 2T_1^{(0)} V_{12}^{(0)} + T_{11}^{(0)} V_2^{(0)} + V^{(1)} - 2{V^{(0)}}^4 T_1^{(0)} T_1^{(1)} \\ - 2{V^{(0)}}^4 {T_1^{(0)}}^2 T_2^{(0)} - 4{V^{(0)}}^3 {T_1^{(0)}}^2 V^{(1)} = 0 \end{aligned} \tag{4.87a}$$

$$V^{(0)^2}T_{11}^{(1)} + 2V^{(0)^2}T_1^{(0)}T_{12}^{(0)} + V^{(0)^2}T_{11}^{(0)}T_2^{(0)} + V^{(0)}V^{(1)}T_{11}^{(0)}$$

$$+ 2V^{(0)}V_1^{(0)}T_1^{(1)} + 2V^{(0)}V_1^{(0)}T_1^{(0)}T_2^{(0)} + 2T_1^{(0)}V_1^{(1)}V^{(0)}$$

$$+ 2V^{(0)}T_1^{(0)^2}V_2^{(0)} - T_1^{(0)}[V^{(0)^2} + V_1^{(0)^2}]^{1/2} = 0 \qquad (4.87b)$$

The initial conditions are satisfied if we set

$$V^{(0)}(0, 0) = 1 \qquad (4.88a)$$

$$V^{(1)}(0, 0) = 0 \qquad (4.88b)$$

$$V_1^{(0)}(0, 0) = 0 \qquad (4.89a)$$

$$V_1^{(1)}(0, 0) = -V_2^{(0)}(0, 0) \qquad (4.89b)$$

$$T^{(0)}(0, 0) = 0, \quad T^{(1)}(0, 0) = 0 \qquad (4.90a)$$

$$T_1^{(0)}(0, 0) = 1, \quad T_1^{(1)}(0, 0) = -T_2^{(0)}(0, 0) \qquad (4.90b)$$

Before solving equations 4.86, let us note that the two conservation laws of angular momentum, and energy now become statements that certain groups of variables depend only on $\tilde{\varphi}$. Thus by integrating equation 4.86b with respect to φ we have

$$V^{(0)^2}T_1^{(0)} = f(\tilde{\varphi}) = \frac{1}{[a^2(\tilde{\varphi})(1-e^2(\tilde{\varphi}))]^{1/2}} \qquad (4.91a)$$

We have introduced the functions $a(\tilde{\varphi})$ and $e(\tilde{\varphi})$ in order to establish the correspondence between the reciprocal of the angular momentum and the orbital elements (a = semi-major axis, e = eccentricity) in the unperturbed problem. For our initial value problem we have

$$f(0) = 1 \qquad (4.92)$$

The "conservation" of energy can be derived by substituting the result for $T_1^{(0)}$ from equation 4.91a into equation 4.86a and integrating with respect to φ to obtain

$$V_1^{(0)^2} + V^{(0)^2} - \frac{2V^{(0)}}{f^2} = g(\tilde{\varphi}) = a^{-2} \tag{4.91b}$$

The general solution of equations 4.86 for arbitrary initial conditions can be conveniently expressed in the form:

$$V^{(0)}(\varphi, \tilde{\varphi}) = \frac{1}{a(1-e^2)} [1 - e \cos(\varphi - B)] \tag{4.93}$$

$$T^{(0)}(\varphi, \tilde{\varphi}) = \tau + a^{3/2}\left[\frac{e(1-e^2)\sin(\varphi-B)}{1-e\cos(\varphi-B)} + \cos^{-1}\frac{\cos(\varphi-B)-e}{1-e\cos(\varphi-B)}\right] \tag{4.94}$$

In the above a, e, B, τ are the four arbitrary functions of $\tilde{\varphi}$ introduced by the four integrations of equations 4.86 with respect to φ. The function B is the argument of apogee, (i.e. the angle from the axis to that point in the orbit which is the farthest from the center of attraction). In order to interpret the function τ it is more convenient to invert equations 4.93 and 4.94 to the more conventional form

$$V^{(0)}(t^*, \tilde{\varphi}) = F(a, e; \xi) \tag{4.95}$$

$$\varphi^{(0)}(t^*, \tilde{\varphi}) - B = \xi t^* + G(a, e; \xi) \tag{4.96}$$

where $\xi = a^{-3/2}(t^* - \tau)$. The functions F and G are periodic in the argument ξ. Thus a slow change in the function τ represents a slow change in the period of the orbit. Conversely, a slow change in B corresponds to a slow precession of the apse.

For our initial conditions we have

$$a(0) = 1 \tag{4.97a}$$

$$e(0) = 0 \tag{4.97b}$$

$$B(0) = 0 \tag{4.97c}$$

$$\tau(0) = 0 \tag{4.97d}$$

The problem now is to define the four functions a, e, B and τ by requiring that $V^{(1)}$ be a bounded function of φ.

We proceed to solve for $V^{(1)}$ and $T^{(1)}$ by first substituting the expression for $T_1^{(0)}$ obtained from equation 4.91a into equation 4.87a. This gives

$$V^{(0)^2}T_{11}^{(1)} + V^{(0)^2}T_{12}^{(0)} - 2fV^{(1)}V_1^{(0)}V^{(0)^{-2}} + 2V^{(0)}V_1^{(0)}T_1^{(1)}$$
$$+ 2V^{(0)}V_1^{(0)}T_2^{(0)} + 2fV_1^{(1)}V^{(0)^{-1}} + \frac{df}{d\tilde{\varphi}}$$
$$- fV^{(0)^{-2}}[V^{(0)^2} + V_1^{(0)^2}]^{1/2} \tag{4.98}$$

If we introduce the notation

$$\frac{\partial Q}{\partial \varphi} = -fV^{(0)^{-2}}[V^{(0)^2} + V_1^{(0)^2}]^{1/2} \tag{4.99}$$

we can integrate equation 4.98 with respect to φ, and express the results in terms of the known function Q as follows:

$$V^{(0)^2}T_1^{(1)} + V^{(0)^2}T_2^{(0)} + 2fV^{(0)^{-1}}V^{(1)} + \frac{\varphi\, df}{d\tilde{\varphi}} + Q(\varphi, \tilde{\varphi}) = c(\tilde{\varphi}) \tag{4.100}$$

In the above $c(\tilde{\varphi})$ is a function of $\tilde{\varphi}$ introduced by the integration, and is to be evaluated by making $V^{(2)}$ a bounded function of φ.

If we use the expression given by equation 4.100 for $T_1^{(1)}$ in equation 4.87a we obtain the simple result

$$V_{11}^{(1)} + V^{(1)} = -2V_{12}^{(0)} - 2f\frac{df}{d\tilde{\varphi}}\varphi - 2fQ + 2fc \qquad (4.101)$$

The first three terms on the right-hand side of equation 4.101 contain critical terms of the form $g\varphi$, $h \sin\varphi$, and $\ell \cos\varphi$, where the coefficients g, h, ℓ are functions of a, e, B and their first derivatives. (Note: τ is not involved in these coefficients.)

In order to evaluate g, h, and ℓ we note that $V_{12}^{(0)}$ and $\frac{\partial Q}{\partial \varphi}$ are periodic functions of φ. Thus in order to evaluate g, h, and ℓ it is only necessary to compute the Fourier coefficients of the terms $\sin\varphi$, $\cos\varphi$ and the "constant" term in $\frac{\partial Q}{\partial \varphi}$.

We will not give the expressions for g, h, and ℓ (which are quite lengthy), since for our special initial conditions the solution of the system of three first order equations obtained by setting g, h, and ℓ equal to zero is simply

$$e(\tilde{\varphi}) = 0 \qquad (4.102a)$$

$$B(\tilde{\varphi}) = 0 \qquad (4.102b)$$

$$a(\tilde{\varphi}) = \frac{1}{2\tilde{\varphi} + 1} \qquad (4.102c)$$

Hence

$$V^{(0)} = a^{-1} = 2\tilde{\varphi} + 1 \qquad (4.103a)$$

$$T^{(0)} = \tau(\tilde{\varphi}) + a^{3/2}\varphi$$

This result could have been anticipated by noting that since e was initially zero, there is no mechanism in the problem by which e

could have changed. In fact, the effect of drag is to equalize any irregularities in the initial orbit, and make it tend towards a circular shape. Since the unperturbed orbit had no point of maximum separation, and e was found to remain unchanged, it follows that B must also be zero.

According to equations 4.103 the orbit, to the first approximation, is a spiral slowly approaching the center of attraction.

So far, we have not defined the function $\tau(\tilde{\varphi})$. In order to do so, let us compute $V^{(1)}$ and $T^{(1)}$. With the critical terms set equal to zero, $V^{(1)}$ is simply:

$$V^{(1)}(\varphi, \tilde{\varphi}) = A^{(1)}(\tilde{\varphi})\sin\varphi + B^{(1)}(\tilde{\varphi})\cos\varphi + 2c(2\tilde{\varphi} + 1) \qquad (4.104)$$

If this result is substituted into equation 4.100, and the ensuing expression for $T_1^{(1)}$ is integrated, we obtain

$$T^{(1)} = -\left[\frac{d\tau}{d\tilde{\varphi}} + \frac{4c}{2\tilde{\varphi}+1} - \frac{c}{(2\tilde{\varphi}+1)^2}\right]\varphi + \left[\frac{3}{(2\tilde{\varphi}+1)^{3/2}} + \frac{1}{(2\tilde{\varphi}+1)^2}\right]\frac{\varphi^2}{2}$$

$$- \frac{2}{(2\tilde{\varphi}+1)^2}\left[A^{(1)}\sin\varphi + B^{(1)}\cos\varphi\right] + d(\tilde{\varphi}) \qquad (4.105)$$

We will set the coefficient of φ, given by the first bracketed term on the right-hand side of equation 4.105, equal to zero, not because $T^{(1)}$ is to be a bounded function of φ (which cannot be since T is the time) but simply for consistency in notation. We are denoting the variable $\epsilon\varphi$ by $\tilde{\varphi}$ hence by setting the bracketed term in question equal to zero, and solving the resulting first order equation for τ, we will relegate this term to the first order.

The above also has a definite physical interpretation. As we mentioned earlier, a variation in τ denotes a slow change in the period of the orbit. Clearly the drag perturbation should introduce such a change. Unfortunately, since $c(\tilde{\varphi})$ appears in the equation for τ this quantity can only be evaluated after c has been determined.

The purpose of presenting this example here was not to give a detailed solution for the drag-perturbed orbit. Rather, we wish to point out the applicability of the two-variable expansion procedure to all those problems in celestial mechanics, for which in the absence of small perturbations, the motion is Keplerian. We will formulate another such problem in the next section.

4.5. The Planar Motion of a Satellite in the Vicinity of the Smaller Body in the Restricted Three-Body Problem*

Consider the motion of a body of negligible mass in the gravitational field of two other bodies, one of which (the sun) is much larger than the other (the planet). It is further assumed that the sun and planet move in circular orbits about their common center of mass. This is the statement of the restricted three-body problem (c. f. reference 7, which contains a systematic introduction to the problem).

The dimensional equations of motion with respect to a Cartesian frame (ξ, η, ζ) centered at the center of mass of the sun and planet, and rotating with the planet are: (see figure 4. 2)

* The solution of this problem was sponsored by the Douglas Aircraft Company. A detailed solution for the corresponding three-dimensional problem will be published in the "Astronomical Journal."

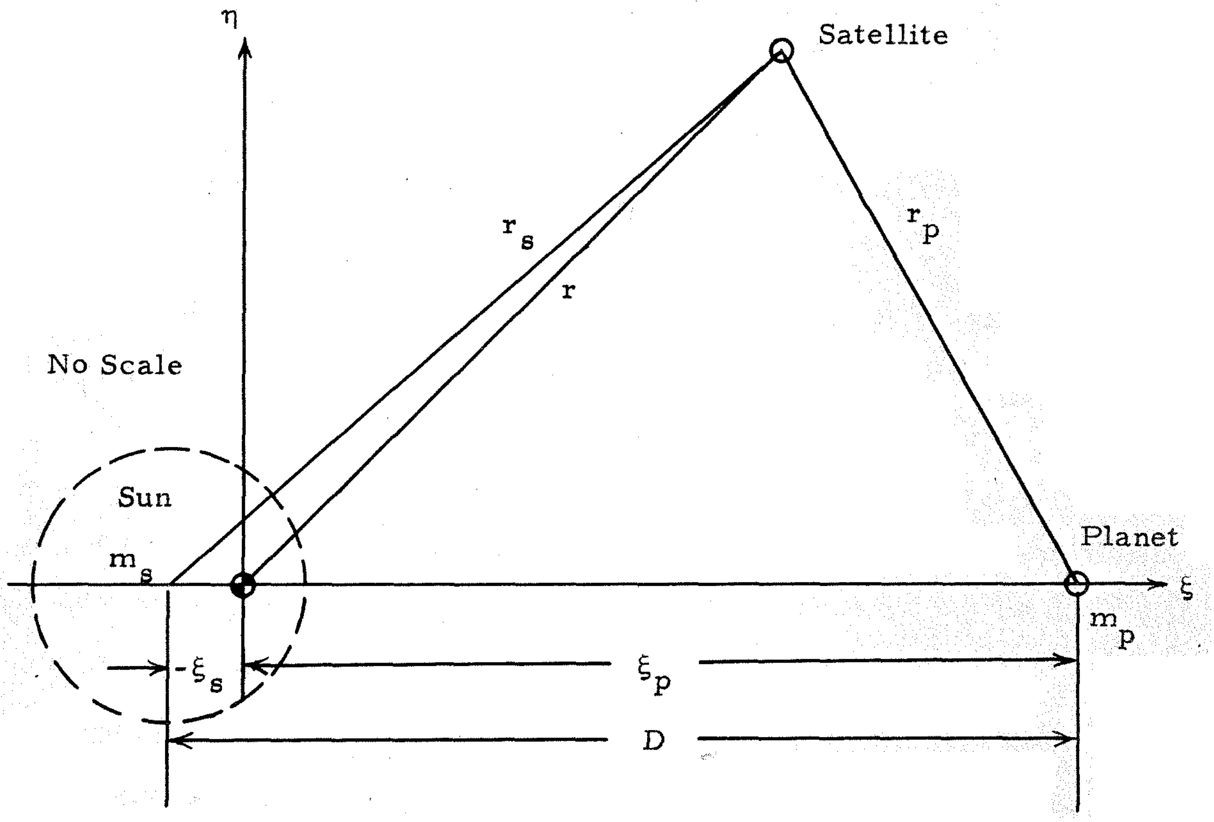


Figure 4.2

DIMENSIONAL COORDINATE SYSTEM

$$\frac{d^2\xi}{dt^2} = 2\omega\frac{d\eta}{dt} + \omega^2\xi - \frac{Gm_s}{r_s^2}\left(\frac{\xi-\xi_s}{r_s}\right) - \frac{Gm_p}{r_p^2}\left(\frac{\xi-\xi_p}{r_p}\right) \tag{4.106a}$$

$$\frac{d^2\eta}{dt^2} = -2\omega\frac{d\xi}{dt} + \omega^2\eta - \frac{Gm_s}{r_s^2}\frac{\eta}{r_s} - \frac{Gm_p}{r_p^2}\frac{\eta}{r_p} \tag{4.106b}$$

$$\frac{d^2\zeta}{dt^2} = -\frac{Gm_s}{r_s^2}\frac{\zeta}{r_s} - \frac{Gm_p}{r_p^2}\frac{\zeta}{r_p} \tag{4.106c}$$

In the above ξ_p and ξ_s are the distances of the planet and sun respectively from their common center of mass. The quantities r_p and r_s are the distances to the satellite from the centers of the planet and sun respectively. Thus

$$r_s^2 = (\xi - \xi_s)^2 + \eta^2 + \zeta^2, \qquad r_p^2 = (\xi - \xi_p)^2 + \eta^2 + \zeta^2 \tag{4.107}$$

The quantity ω is the angular velocity of the planet about the center of mass, and since this motion is circular ω is simply $[GMD^{-3}]^{1/2}$. Here G is the universal gravitational constant, D is the distance between the centers of the sun and planet, and M is the sum of the masses of the sun and planet, m_s and m_p respectively.

The only non-dimensional parameter in equation 4.106 is the ratio $\mu = m_p/M$. Clearly for motion in the field of both sun and planet the characteristic length and time scales are D and $1/\omega$ respectively. This suggests the following non-dimensional variables

$$\xi^* = \frac{\xi}{D}, \quad \eta^* = \frac{\eta}{D}, \quad t^* = \omega t, \quad \zeta^* = \frac{\zeta}{D} \tag{4.108}$$

In order to keep the non-dimensional distance between the sun and planet independent of μ, we should introduce the variables:

$$\xi^{**} = \xi^{*} + \mu, \qquad \eta^{**} = \eta^{*}, \qquad \zeta^{**} = \zeta^{*} \tag{4.109}$$

In terms of these variables equations 4.106 take the form:

$$\frac{d^2\xi^{**}}{dt^{*2}} = \frac{2d\eta^{**}}{dt^{*}} + \xi^{**} - \mu - (1-\mu)\frac{\xi^{**}}{r_s^{**3}} - \frac{\mu(\xi^{**} - 1)}{r_p^{**3}} \tag{4.110a}$$

$$\frac{d^2\eta^{**}}{dt^{*2}} = -\frac{2d\xi^{**}}{dt^{*}} + \eta^{**} - (1-\mu)\frac{\eta^{**}}{r_s^{**3}} - \frac{\mu\eta^{**}}{r_p^{**3}} \tag{4.110b}$$

$$\frac{d^2\zeta^{**}}{dt^{*2}} = -(1-\mu)\frac{\zeta^{**}}{r_s^{**3}} - \frac{\mu\zeta^{**}}{r_p^{**3}} \tag{4.110c}$$

where

$$r_s^{**2} = \xi^{**2} + \eta^{**2} + \zeta^{**2} \tag{4.111a}$$

$$r_p^{**2} = (\xi^{**} - 1)^2 + \eta^{**2} + \zeta^{**2} \tag{4.111b}$$

When the motion of the third body is not restricted to the vicinity of either the sun or the planet, but can range over the entire space, these are the appropriate differential equations governing the motion. Since we are interested in planetary satellites, we should not only translate the origin to the center of the planet but also introduce new "planetary" variables in terms of which the effect of the sun becomes secondary. Thus, we translate the origin by:

$$x^{**} = \xi^{**} - 1, \quad y^{**} = \eta^{**}, \quad z^{**} = \zeta^{**} \tag{4.112}$$

and introduce the planetary variables:

$$x_a = \frac{x^{**}}{\mu^a}, \quad y_a = \frac{y^{**}}{\mu^a}, \quad z_a = \frac{z^{**}}{\mu^a}, \quad t_\beta = \frac{t^*}{\mu^\beta} \tag{4.113}$$

We will show how an order of magnitude analysis of the various terms in the equations will define a and β. If we anticipate the fact that $a \geqslant 0$ and $\beta \geqslant 0$, the significance of the $x_a, \ldots, t_\beta$ variables is clear. We will be taking the limits as $\mu \to 0$ with $x_a, \ldots, t_\beta$ fixed. This will imply that the motion is characterized by being close to the planet and having a time scale which is small compared to the planetary year.

If we rewrite equation 4.110a in the new variables (and there is no need at this stage to consider all three equations), we have:

$$\frac{\mu^{a-2\beta} d^2 x_a}{dt_\beta^2} = \frac{2\mu^{a-\beta}\, dy_a}{dt_\beta} + \mu^a x_a + 1 - \mu - \frac{(1-\mu)(\mu^a x_a + 1)}{[1+2\mu^a x_a + \mu^{2a}(x_a^2 + y_a^2)]^{3/2}} - \frac{\mu^{1-2a} x_a}{[x_a^2 + y_a^2]^{3/2}} \tag{4.114}$$

We have at this point set $z_a = 0$ and will for the remainder of this paper consider only planar motion.

If the planetary gravitation is of the same order as the centrifugal acceleration due to motion of the planet around the sun, the exponents of μ for the above two terms must be equal and this gives $a = 1 - 2a$.

If further the Coriolis acceleration and the centrifugal acceleration due to the planet's motion are comparable, then the exponent of the Coriolis term $a - \beta$ must be equal to a. The above two conditions uniquely specify the values of $a = 1/3$ and $\beta = 0$. These are the variables that lead to Hill's equation (c. f. reference 8) and in fact Hill's equations are obtained by holding the variables $\tilde{x} = x_{1/3}$, $\tilde{y} = y_{1/3}$, $\tilde{t} = t_o$ fixed and letting $\mu \to 0$. The reader can easily verify that the limiting equations are

$$\frac{d^2\tilde{x}}{d\tilde{t}^2} = \frac{2\,d\tilde{y}}{d\tilde{t}} - \frac{\tilde{x}}{[\tilde{x}^2 + \tilde{y}^2]^{3/2}} + 3\tilde{x} \tag{4.115a}$$

$$\frac{d^2\tilde{y}}{d\tilde{t}^2} = -\frac{2\,d\tilde{x}}{d\tilde{t}} - \frac{\tilde{y}}{[\tilde{x}^2 + \tilde{y}^2]^{3/2}} \tag{4.115b}$$

In the above, the term $3\tilde{x}$ is contributed by both the gravitational attraction of the sun and the centrifugal acceleration.

We will discuss equations 4.115 further in the next section. Let us now return to equation 4.114. If we assume that the planetary gravitation is the dominant term, we obtain $1 - 2\alpha = 0$ and $\alpha - 2\beta = 0$ which imply that $\alpha = 1/2$, $\beta = 1/4$. The corresponding planetary variables are $x = x_{1/2}$, $y = y_{1/2}$, $t = t_{1/4}$. Clearly, x, y, t are more restricted variables than $\tilde{x}, \tilde{y}, \tilde{t}$ in the sense that the limiting equations obtained by holding x, y, t fixed and letting $\mu \to 0$ are the two-body equations; thus, the effect of the sun is ignored as a first approximation. This "restrictedness" is further demonstrated by the fact that as $\mu \to 0$ with x fixed, $x^{**} \to 0$ faster than it does in the limit $\tilde{x}$ fixed and $\mu \to 0$ (i.e., x is more "planetary" than $\tilde{x}$).

The equations of motion in x, y, t variables are to $O(\mu^{1/2})$ given below:

$$\frac{d^2x}{dt^2} = -\frac{x}{[x^2+y^2]^{1/2}} + \frac{2\mu^{1/2}dy}{dt} + 3\mu^{1/2} + O(\mu^{3/4}) \tag{4.116a}$$

$$\frac{d^2y}{dt^2} = -\frac{y}{[x^2+y^2]^{3/2}} - \frac{2\mu^{1/4}dx}{dt} + O(\mu^{3/4}) \qquad (4.116b)$$

For orbits which are quite close to the planet, it is not necessary to solve Hill's equations which are much too general. We will show that the more restricted equations 4.116 are appropriate, and lead to meaningful results.

Periodic Solutions of Hill's Equations

The brief exposition in this section is taken from the results given in reference 7, to which the reader is referred for a thorough discussion on the periodic solutions of equations 4.115.

Before proceding with a discussion of the solution of 4.115, we note that this system possesses a Jacobi integral, which is simply

$$\left(\frac{d\tilde{x}}{d\tilde{t}}\right)^2 + \left(\frac{d\tilde{y}}{d\tilde{t}}\right)^2 - \frac{2}{[\tilde{x}^2+\tilde{y}^2]^{1/2}} - 3\tilde{x}^2 = -c \qquad (4.117)$$

As equation 4.117 represents the dominant part of Jacobi's integral for the complete equations 4.110, all comments pertinent to Jacobi's integral apply here, and need not be repeated. The symmetry of the integral curves with respect to $\tilde{y}$ is due to the fact that in this limit the sun has approached $-\infty$ like $\mu^{-1/3}$ when $\mu \rightarrow 0$.

The approximate solution of equations 4.115 depends upon a small parameter $m = 2\pi\tau$, where τ is the period of the orbit. Furthermore, the periodic solutions are completely characterized by this parameter, and are of the form

$$\tilde{x}(\tilde{t}) = \sum_{k=-\infty}^{\infty} a_k \cos (2k + 1)\frac{\tilde{t}}{m} \tag{4.118a}$$

$$\tilde{y}(\tilde{t}) = \sum_{k=-\infty}^{\infty} a_k \sin (2k + 1)\frac{\tilde{t}}{m} \tag{4.118b}$$

where the coefficients, a_k are power series in m and are given below to $O(m^3)$:

$$\frac{a_1}{a_o} = \frac{3}{16} m^2 + O(m^3) \tag{4.119a}$$

$$\frac{a_{-1}}{a_o} = - \frac{19}{16} m^2 + O(m^3) \tag{4.119b}$$

$$a_j = o(m^3) \quad \text{if} \quad |j| \geqslant 2 \tag{4.119c}$$

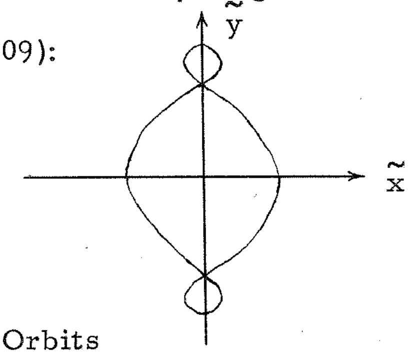
$$a_o = m^{2/3}[1 - \frac{2m}{3} + \frac{7}{18} m^2 + O(m^3)] \tag{4.119d}$$

Moreover, m is related to C by:

$$C = m^{-2/3}[1 + \frac{8m}{3} + \frac{7m^2}{18} + O(m^3)]$$

Thus, <u>if</u> for a given set of initial conditions, $\tilde{x}(0)$, $\tilde{y}(0)$, $\frac{d\tilde{x}(0)}{d\tilde{t}}$, $\frac{d\tilde{y}(0)}{d\tilde{t}}$ a periodic solution exists, the period of the orbit is uniquely defined by C from equation 4.120.

A graph of the solution would show that the effect of the solar perturbation is to elongate the unperturbed orbit in the direction along the tangent to the planet's path. Although for larger values of m no analytic representation of the solution is available, numerical integrations have shown that increasing m will tend to elongate the orbit further until two symmetrically located cusps are formed on the $\tilde{y}$ axis. For larger values of m, these cusps develop into small loops such that a periodic orbit for very large m looks like this (c. f. reference 9, pp. 104-109):



General Non-periodic Orbits

Let us now return to equations 4.116. Since the unperturbed motion is Keplerian, it is advantageous to introduce the polar coordinates:

$$x = r \cos \varphi, \qquad y = r \sin \varphi \tag{4.121}$$

in terms of which 4.116 takes the form

$$\frac{d^2r}{dt^2} - r\left(\frac{d\varphi}{dt}\right)^2 + \frac{1}{r^2} = \frac{2\mu^{1/4} r\, d\varphi}{dt} + 3\mu^{1/2} r \cos^2\varphi + O(\mu^{3/4}) \tag{4.122a}$$

$$r\frac{d^2\varphi}{dt^2} + 2\frac{dr}{dt}\frac{d\varphi}{dt} = \frac{-2\mu^{1/4} dr}{dt} - 3\mu^{1/2} r \sin\varphi \cos\varphi + O(\mu^{3/4}) \tag{4.122b}$$

The perturbations introduced to $O(\mu^{1/4})$ due to the Coriolis acceleration terms can be most conveniently accounted for by introducing the precessing frame of reference defined by

$$\overline{\varphi} = \varphi + \mu^{1/4} t \tag{4.123}$$

In terms of $\overline{\varphi}$ equations 4.122 take the form

$$\frac{d^2r}{dt^2} - r\left(\frac{d\overline{\varphi}}{dt}\right)^2 + \frac{1}{r^2} = 3\epsilon^2 r\cos^2(\overline{\varphi} - \epsilon t) + \epsilon^2 r + O(\epsilon^3) \qquad (4.124a)$$

$$r\frac{d^2\overline{\varphi}}{dt^2} + \frac{2r\,d\overline{\varphi}}{dt}\frac{dr}{dt} = -3\epsilon^2 r\sin(\overline{\varphi} - \epsilon t)\cos(\overline{\varphi} - \epsilon t) + O(\epsilon^3) \qquad (4.124b)$$

where $\epsilon = \mu^{1/4}$.

The limiting equations obtained by holding $r, \overline{\varphi}, t$ fixed and letting $\epsilon \to 0$ correspond to the familiar two-body equations in the $r, \overline{\varphi}$ frame, which by definition preserves its orientation in inertial space as the planet moves around the sun. The fact that equations 4.124 are free of terms of order ϵ is a verification of the well-known fact that orbits will retain their orientation to the first approximation.

Before proceding with the solution, let us consider the initial conditions to be imposed. Since we are interested in bounded orbits, we will have two slowly varying values of φ at which $dr/dt = 0$. For convenience let us start to measure time when the satellite passes through applanet, and with the knowledge that the motion has an inherent precession let us allot an appropriate part of the given initial angular velocity to this precession. Thus, if we are given the following initial values:

$$r(0) = R_o \qquad (4.125a)$$

$$\varphi(0) = \varphi_o \qquad (4.125b)$$

$$\frac{dr(0)}{dt} = 0 \qquad (4.125c)$$

$$\frac{d\varphi(0)}{dt} = \Omega_o \qquad (4.125d)$$

where R_o, φ_o and Ω_o have been suitably non-dimensionalized we may set $\Omega_o = \overline{\Omega}_o + \epsilon$ such that the initial conditions pertaining to equation 4.124 will be

$$r(0) = R_o \tag{4.126a}$$

$$\overline{\varphi}(0) = \varphi_o \tag{4.126b}$$

$$\frac{dr(0)}{dt} = 0 \tag{4.126c}$$

$$\frac{d\overline{\varphi}(0)}{dt} = \overline{\Omega}_o \tag{4.126d}$$

As in section 4.4, we will now transform equations 4.124 and 4.126 to the variables $1/r = u = u(\overline{\varphi})$ and $t = t(\overline{\varphi})$ to obtain

$$\frac{d^2u}{d\overline{\varphi}^2} + u - u^4\left(\frac{dt}{d\overline{\varphi}}\right)^2 = -3\epsilon^2\left[-\frac{du}{d\overline{\varphi}}\left(\frac{dt}{d\overline{\varphi}}\right)^2 \sin(\overline{\varphi} - \epsilon t)\cos(\overline{\varphi} - \epsilon t) + u\left(\frac{dt}{d\overline{\varphi}}\right)^2 \cos^2(\overline{\varphi} - \epsilon t)\right] + \epsilon^2 u\left(\frac{dt}{d\overline{\varphi}}\right)^2 + O(\epsilon^2) \tag{4.127a}$$

$$u\frac{d^2t}{d\overline{\varphi}^2} + 2\frac{du}{d\overline{\varphi}}\frac{dt}{d\overline{\varphi}} = 3\epsilon^2 u\left(\frac{dt}{d\overline{\varphi}}\right)^2 \sin(\overline{\varphi} - \epsilon t)\cos(\overline{\varphi} - \epsilon t) \tag{4.127b}$$

$$u(\varphi_o) = \frac{1}{R_o} \tag{4.128a}$$

$$t(\varphi_o) = 0 \tag{4.128b}$$

$$\frac{du(\varphi_o)}{d\overline{\varphi}} = 0 \tag{4.128c}$$

$$\frac{dt(\varphi_o)}{d\overline{\varphi}} = \frac{1}{\overline{\Omega}_o} \tag{4.128d}$$

For the sake of convenience we will assume that $\varphi_o = 0$, and will transform R_o and $\overline{\Omega}_o$ to the more familiar expressions in terms of the orbital elements a and e, to obtain

$$u(0) = \frac{1}{a(1+e)} \tag{4.129a}$$

$$t(0) = 0 \tag{4.129b}$$

$$\frac{du(0)}{d\overline{\varphi}} = 0 \tag{4.129c}$$

$$\frac{dt(0)}{d\overline{\varphi}} = \frac{a^{3/2}(1+e^2)}{(1-e^2)^{1/2}} \tag{4.129d}$$

Having derived the above formulation of the problem, we had anticipated a solution by Poincare's method, and indeed this is possible for this problem.

It is natural to ask whether there exists a precessing coordinate system, and a modified time variable such that the initially valid expansions of equations 4.127 and 4.129 written in terms of these variables are uniformly valid. For this problem this is true. Without giving the laborious calculations we will state the following results.

We can find quantities ω_2 and ν_2 such that $u(\psi, \epsilon)$ is a bounded function of ψ where

$$\overline{\overline{\varphi}} = \overline{\varphi} + \epsilon^2 \nu_2 t$$

$$\psi = (1 + \epsilon^2 \omega_2)\overline{\overline{\varphi}}$$

$$\omega_2 = a^3\left[-\frac{7}{12} + 2e + O(e^2)\right] \quad , \quad \nu_2 = a^{3/2}\left[-\frac{1}{6} - 2e + O(e^2)\right]$$

$$u(\psi, \epsilon) = u_o(\psi) + O(\epsilon^2) \qquad \text{uniformly in } 0 \leq \psi \leq \infty$$

$$t(\psi, \epsilon) = t_o(\psi) + O(\epsilon^2) \qquad \text{uniformly in } 0 \leq \psi \leq \infty$$

and u_o and t_o are the familiar expressions for a Kepler ellipse in the u, ψ frame (c. f. equations 4.93 and 4.94, with $B = 0$ and φ replaced by ψ).

In order to point out the significance of ω_2 and ν_2 let us only consider u_o and t_o, which represent the solution uniformly in ψ.

We have that

$$\psi = (1 + \epsilon^2\omega_2)(\overline{\varphi} + \epsilon^2\nu_2 t) \tag{a}$$

From equation 4.94 it is easy to verify that

$$t_o(\psi) = a^{3/2}\psi + eF(\psi, e) \tag{b}$$

where F is a periodic function of ψ, and e, the eccentricity, is a small number.

Thus

$$\psi = \overline{\varphi} + \epsilon^2\nu_2 a^{3/2}\psi + \epsilon^2\omega_2\overline{\varphi} + \epsilon^2\nu_2(eF + O(\epsilon^2)) \tag{c}$$

If we only consider the mean values of the various functions in (c) (and denote these mean values by the subscript m), we obtain

$$\psi_m = [1 - \epsilon^2 a^3 \frac{(3 + O(e^2))}{4}]\,\overline{\varphi} \tag{d}$$

If we substitute the expression for ψ_m given in (d) into the equation for $u_o(\psi)$ we see that the quantity $3\epsilon^2 a^3/4$ represents a slow counterclockwise advance for the mean motion of the apse. Superimposed upon this mean advance, the apse undergoes oscillations which average out after the satellite makes a complete revolution in its orbit. The resulting orbit in the inertially oriented $r, \overline{\varphi}$ frame would appear as in fig. 4. 3 for an exaggerated value of $3\epsilon^2 a^3/4$.

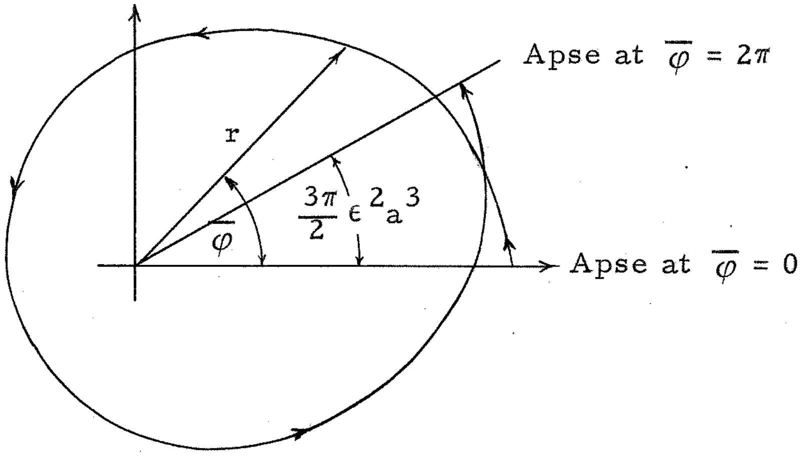


Figure 4. 3

The more systematic approach to this problem is provided by the two-variable expansion procedure.

As we have shown in the preceding section, the appropriate variables are $\overline{\varphi}$ and $\tilde{\overline{\varphi}} = \epsilon\overline{\varphi}$. It is well worth pointing out that if we apply the two-variable expansion procedure to the equations in the $u(\varphi)$ and $t(\varphi)$ variables we will be able to derive the result that the orbit preserves its orientation to the first order.

It is more convenient to start with equations 4. 127, and by a very simple calculation the results given by Poincare's method can be immediately derived.

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