A THEORETICAL INVESTIGATION OF PARTICLE MOTION IN AN OSCILLATING GAS

Thesis by

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In Partial Fulfillment of the Requirements

For the Degree of

Aeronautical Engineer

California Institute of Technology

Pasadena, California

1961

ACKNOWLEDGEMENTS

The author wishes to take this opportunity to thank Dr. Edward E. Zukoski, who gave so freely of his time and advice to aid in the completion of this research. The suggestions and helpful criticism of Dr. Frank E. Marble are also gratefully acknowledged. Thanks are also due to Mrs. JoAnne Rice and Mrs. Elizabeth Fox for typing the manuscript and to Mrs. Betty Wood for preparing the figures.

ABSTRACT

An investigation is made of the drift motion of small particles under the influence of acoustic oscillations. The investigation is made to determine if the motion has a magnitude great enough to produce significant changes on the fuel distribution in the chamber of a liquid propellant rocket motor. The calculations are made for the motion in both a rectangular and a cylindrical chamber.

In the rectangular chamber the gas oscillation is restricted to the fundamental transverse mode and motion in only one dimension is considered. The particle drift velocity, that is the non-oscillating, non-damped term in the expression for the particle velocity is found in the solution of the second order equation.

For particle motion in the cylindrical enclosure, only gas oscillations in the first transverse or sloshing mode is considered and motion is restricted to a transverse plane of the cylinder. The particle drift velocity, again a second order term, is determined.

The magnitude of the drift velocity is calculated using conditions found in a liquid propellant rocket combustion chamber. Distances a typical fuel droplet would move during its average life time are calculated. The distances are small compared to the size of most rocket combustion chambers.

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TABLE OF SYMBOLS

Symbol	Definition	Units
a	Cylinder radius	ft.
c	Local sonic velocity	ft./sec.
J	Bessel function of first kind	
ℓ	Rectangular chamber dimension	ft.
m	Mass of particle	slug
n	Particle density	$1/\text{ft.}^3$
P_o	Amplitude of pressure fluctuations	lb./ft.^2
P_c	Mean chamber pressure	lb./ft.^2
p	Excess pressure in a sound wave	lb./ft.^2
r	Radial dimension	ft.
t	Time	sec.
U	Gas velocity	ft./sec.
v	Particle drift velocity	ft./sec.
y	Rectangular chamber coordinate	ft.
Z	Dimensionless position, r/a, ξ/ℓ	
z	Cylinder axial dimension	ft.
α_{mn}	Constant	
γ	Ratio of specific heats	
μ	Viscosity	slug/sec.ft.
ν	Frequency: $\left(\frac{c}{4\ell}\right)$ for rectangular chamber; $\left(\frac{\alpha_{mn} c}{2a}\right)$ for cylindrical chamber	1/sec.
ξ	Particle position in rectangular chamber	
ρ	Gas density	slug/ft.^3
ρ_p	Particle density	slug/ft.^3

TABLE OF SYMBOLS (Cont'd)

σ	Particle radius	ft.
$\mathcal{T}$	Dimensionless time	
φ, ϕ	Cylindrical angular coordinate	
ω	Angular velocity	1/sec.

Constants in the Solutions to the Equations

A) Rectangular Chamber

D'	$\dfrac{\pi \sin \pi \bar{\bar{Z}}}{4(4\pi^2+K'^2)}$
E'	$\dfrac{K' \sin \pi \bar{\bar{Z}}}{16(4\pi^2+K'^2)}$
F'	$\dfrac{\pi^2 \sin \pi \bar{\bar{Z}}}{-2K'(4\pi^2+K'^2)}$
G'	$\dfrac{K' \sin \pi \bar{\bar{Z}}}{-8(4\pi^2+K'^2)}$
K'	$\dfrac{24\pi\mu\sigma\ell}{mc}$
$\mathcal{E}'$	$\dfrac{-96\pi\mu\sigma P_o \ell}{m\rho c^3}$

B) Cylindrical Chamber

D	$\dfrac{J_1(\pi\alpha_{10}\bar{\bar{Z}}) \sin \bar{\phi}}{\bar{\bar{Z}}^2}$
E	$\dfrac{\pi\alpha_{10}}{2} \cos \bar{\phi} \left[J_2(\pi\alpha_{10}\bar{\bar{Z}}) - J_0(\pi\alpha_{10}\bar{\bar{Z}}) \right]$
F	$\cos \bar{\phi} \left[(\pi\alpha_{10})^2 J_1(\pi\alpha_{10}\bar{\bar{Z}}) - \dfrac{\pi\alpha_{10}}{\bar{\bar{Z}}} J_2(\pi\alpha_{10}\bar{\bar{Z}}) \right]$
G	$\dfrac{\pi\alpha_{10}}{2} \sin \bar{\phi} \left[J_0(\pi\alpha_{10}\bar{\bar{Z}}) - J_2(\pi\alpha_{10}\bar{\bar{Z}}) \right]$
H	$\sin \bar{\phi} \left[\pi\alpha_{10} J_0(\pi\alpha_{10}\bar{\bar{Z}}) - \dfrac{1}{\bar{\bar{Z}}} J_1(\pi\alpha_{10}\bar{\bar{Z}}) \right]$
I	$\cos \bar{\phi}\, J_1(\pi\alpha_{10}\bar{\bar{Z}})$

TABLE OF SYMBOLS (Cont'd)

K	$\frac{6\pi\mu\sigma a}{mc}$
β	$\left[\frac{r}{a} D^2 - (FE + DG)\right]$
δ	$\left[\frac{r}{a} (EH + DI)\right]$
ε	$\frac{3\mu\sigma P_0}{m\rho V c^2}$

I. INTRODUCTION

It is a well known fact that liquid propellant rocket combustion chambers often experience a phenomenon called unstable combustion. The gas properties, such as pressure, density and temperature, fluctuate during periods of unstable combustion. The fluctuations are not random and definite correlation exists between the fluctuations at different positions in space or different instants in time. Characteristically, the fluctuations have a fixed frequency and the amplitude of the variations can grow to values comparable to the steady state value of the property.

The regularity and the large amplitude of the fluctuations distinguish the property variations observed during unstable combustion from the random variation in gas properties observed during "steady" operation of a combustion chamber.

The observed frequencies of oscillation vary from 10 to as high as 10,000 cycles per second and depend on the driving mechanism or phenomenon and the chamber size. The frequencies can be divided into two general categories: (1) a low frequency or chugging instability with frequencies in the neighborhood of 100 cycles per second and (2) a high frequency or screeching instability with frequencies in the range of 1000 to 10,000 cycles per second.

The low frequency phenomena are often caused by an interaction between the rates of propellant injection, combustion and rate of ejection of material from the chamber.

In the high frequency case the variations of gas properties, during unstable combustion, are accompanied by motion of the gas in the chamber. In general, the motion of the gas and variations of properties are similar to the conditions which would be generated by a strong acoustic oscillation. This observation suggests that the large amplitude variations observed during high frequency unstable combustion are a result of a resonance which occurs when a driving mechanism, associated with the combustion phenomenon, is in phase with the natural acoustic modes of the combustion chamber. For example, in cylindrical chambers, the oscillations occur in the radial, tangential and longitudinal modes plus their various combinations. The first transverse or sloshing mode is the mode of oscillation most frequently found experimentally[(1),(2)] in combustion chambers with high frequency instability.

The present investigation will be concerned with the high frequency instability. A complete and satisfactory answer to the question as to why this instability occurs has, as yet, not been found. In the attempt to answer the above question, many mechanisms have been proposed that are based on the interaction of the pressure and temperature oscillations with the combustion process. The interaction depends on the existence of a time delay which is essentially the delay time in the response of the burning rate to a change in temperature or pressure.

Since the delay times in combustion systems are very sensitive functions of the fuel-oxidizer mixture ratios, a systematic variation of the mixture ratio in the chamber may set up conditions that are favorable for unstable burning. The suggestion has been made that variations in the fuel-oxidizer ratio can be produced by the relative mean motion of droplets of fuel and oxidizer under the influence of acoustic oscillations in the combustion chamber gas.

This investigation is concerned with the initial requirements for the production of the mean particle motion mentioned above. The requirements are first, that a particle drift motion does occur in a chamber with acoustic oscillations and, second, that the magnitude be large enough to result in the appreciable particle motion which would be necessary to produce significant changes in fuel composition in the chamber.

The following two idealized problems will be solved. First, as a simple example, particle motion in a rectangular chamber with acoustic oscillations in the fundamental transverse mode is considered. In the second, more complex problem the calculation is carried out assuming a cylindrical enclosure with acoustic oscillations. The small particle drift velocity induced by the first transverse mode of oscillation is determined.

To calculate the particle drift velocity the equation for the velocity of the gas undergoing acoustic oscillations in the chamber is combined with Stokes drag law to calculate the forces acting on small particles. The resulting equation is linearized by introducing a perturbation expansion in powers of a term proportional to the amplitude of the drag force acting on the particle as a result of the gas motion. It is necessary to go to second order terms in the solution for the particle position in order to obtain a mean drift velocity.

Representative values of the drift velocities are calculated using conditions found in a liquid propellant rocket combustion chamber. Directions and distances the droplets would move, in their average life time, are determined. Finally, an upper limit is obtained for a maximum variation in particle density as a result of the drift velocity.

II. RECTANGULAR CHAMBER

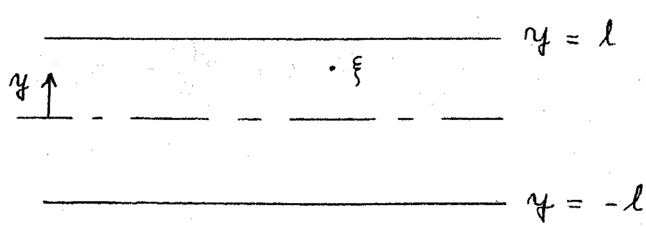
Development of the Equations. Before proceeding to the more complex and more realistic problem of the investigation of particle motion in a cylindrical enclosure with acoustic oscillations it will be illuminating to look at particle motion in a rectangular enclosure containing acoustic waves. The rectangular case will illustrate the important features of the problem in a straightforward manner since motion in just one dimension need be considered.

The method of solution for the particle drift velocity will be as outlined in the introduction.

In solving the problem for the case of the rectangular chamber the following idealized assumptions will be made:

1. The fluid medium is considered to be at rest and in a uniform state except for weak isoenergetic fluctuations in velocity and state properties.
2. The small particles are considered to be spherical and to experience Stokes law forces.
3. The particles have negligible effect upon the gas motion in the chamber.

Consider a rectangular chamber, as shown below, with acoustic oscillations in the fundamental transverse mode.



Gas motion is restricted to oscillations in the y direction. The linearized partial differential equation for the gas pressure disturbance in the chamber, under the conditions considered here, is a simple wave equation of the form:

$$\frac{\partial^2 p}{\partial y^2} = \frac{1}{C^2}\frac{\partial^2 p}{\partial t^2} . \tag{2.1}$$

The well-known solution to equation 2.1, restricting the solution to real values, is:

$$p = P_0 \sin\frac{\omega y}{C} \cos \omega t . \tag{2.2}$$

Here, P_0 is the amplitude of pressure fluctuations of the gas in the chamber, C is the local velocity of sound, ω is 2π times the frequency, ν , and t is the time. ω must be $\frac{m\pi C}{2\ell}$ to satisfy the boundary condition that the gas velocity and, hence, the pressure gradient vanish at the chamber walls ($y = \pm \ell$). For the first transverse mode considered here m is 1. The velocity and pressure fluctuations of the gas are related by the following equation:

$$\rho \frac{\partial U}{\partial t} = -\nabla p = -\frac{\partial p}{\partial y} . \tag{2.3}$$

The velocity of the gas in the chamber is given by integration of 2.3 as:

$$U = -\frac{P_0}{\rho C} \cos\frac{\omega y}{C} \sin \omega t . \tag{2.4}$$

Equation 2.4 shows that the gas velocity is zero at the chamber walls and a maximum at the center of the chamber. The amplitude of the gas velocity oscillations is proportional to $\cos \frac{\pi y}{2\ell}$.

If the location of the particle is given by ξ, and if assumption 2 holds, then the drag force on the particle is given by Stokes law and is:

$$6\pi\mu\sigma\left(U - \frac{d\xi}{dt}\right) . \tag{2.5}$$

The equation of motion for the particle is obtained by equating the inertial forces and drag force, and is:

$$m\frac{d^2\xi}{dt^2} = 6\pi\mu\sigma\left(U - \frac{d\xi}{dt}\right) . \tag{2.6}$$

It is important to note that the gas velocity must be evaluated at the position of the particle, that is $U = U(\xi, t)$. If assumption 3 holds, the gas speed, U, can be eliminated by substituting equation 2.4 into 2.6; this gives:

$$m\frac{d^2\xi}{dt^2} = -\frac{6\pi\mu\sigma P_o}{\varrho c}\cos\frac{\pi\xi}{2\ell}\sin\omega t - 6\pi\mu\sigma\frac{d\xi}{dt} . \tag{2.7}$$

This result can be simplified by the use of the following dimensionless variables:

$$Z = \xi/\ell \quad \text{and} \quad \mathcal{T} = \frac{t}{4\ell/c} = \frac{\omega t}{2\pi} . \tag{2.8}$$

Introducing the dimensionless variables in equation 2.7 gives:

$$\frac{d^2 Z}{d\mathcal{T}^2} + K'\frac{dZ}{d\mathcal{T}} = \varepsilon'\cos\frac{\pi}{2}Z\,\sin 2\pi\mathcal{T} . \tag{2.9}$$

Here, the constants are defined as:

$$K' \equiv \frac{24\pi\mu\sigma l}{mc} \quad \text{and} \quad \mathcal{E}' \equiv \frac{-96\pi\mu\sigma P_0 l}{m\rho c^3} \quad . \tag{2.10}$$

The solution to equation 2.9 will give the position of the small particles, Z, at any time, $\mathcal{T}$, while they are under the influence of the oscillating gas in the chamber.

Note that equation 2.9 is non-linear due to the appearance of Z in the trigonometric function on the right hand side of the equation. Because of this nonlinearity, an approximate method of solution of the equation must be used. The approximation used depends on the fact that the term $\mathcal{E}'$ can be held to small values by limiting the magnitude of the viscosity, or other flow properties appearing in $\mathcal{E}'$. Note that this restriction on the calculation is independent of that required to obtain a linear acoustic equation (2.1) in the first place. Thus, when the expansion in terms of $\mathcal{E}'$ is continued to terms of higher order than the first, there is no obligation to extend the approximation in the acoustic equation to higher powers of pressure or velocity amplitude. By restricting the analysis to arbitrarily small values of $\mathcal{E}'$, it is possible to linearize equation 2.9 by introducing a perturbation expansion of the form:

$$Z(\mathcal{T}) = Z_0(\mathcal{T}) + \mathcal{E}' Z_1(\mathcal{T}) + \mathcal{E}'^2 Z_2(\mathcal{T}) + \cdots \quad . \tag{2.11}$$

Substituting the expansion into equation 2.9 and separating the equation into equations for the various orders of ε' gives:

$$\frac{d^2 Z_0}{d\mathcal{T}^2} + K' \frac{dZ_0}{d\mathcal{T}} = 0 \tag{2.12}$$

$$\frac{d^2 Z_1}{d\mathcal{T}^2} + K' \frac{dZ_1}{d\mathcal{T}} = \cos \tfrac{\pi}{2} Z_0 \sin 2\pi \mathcal{T} \tag{2.13}$$

$$\frac{d^2 Z_2}{d\mathcal{T}^2} + K' \frac{dZ_2}{d\mathcal{T}} = -\tfrac{\pi}{2} Z_1 \sin \tfrac{\pi}{2} Z_0 \sin 2\pi \mathcal{T} \, . \tag{2.14}$$

In making the above substitution, $\cos \frac{\pi}{2} Z$ is expanded as

$$\cos \tfrac{\pi}{2} Z = \left(\cos \tfrac{\pi}{2} Z_0\right)\left(1 - \tfrac{\varepsilon'^2}{2}\left[Z_1 \tfrac{\pi}{2} + \varepsilon' \tfrac{\pi}{2} Z_2\right]^2 + \cdots\right) - \left(\sin \tfrac{\pi}{2} Z_0\right)\left(\varepsilon' \tfrac{\pi}{2} Z_1 + \varepsilon'^2 \tfrac{\pi}{2} Z_2 \cdots\right).$$

For a particle at rest at $\mathcal{T} = 0$ and at the point $\bar{Z}$, the appropriate initial conditions are:

$$\left.\begin{aligned} Z_0(0) &= \bar{Z}\,, \quad \frac{dZ_0}{d\mathcal{T}}(0) = 0 \\ Z_1(0) &= \frac{dZ_1}{d\mathcal{T}}(0) = 0 \\ Z_2(0) &= \frac{dZ_2}{d\mathcal{T}}(0) = 0\,. \end{aligned}\right\} \tag{2.15}$$

Solutions to equations 2.12, 2.13, and 2.14 contain terms which are constant, oscillatory, exponential and linear functions of the time. The purely oscillatory solutions to the above equations are not of interest since no net transport of particles is involved and the solutions involving decaying exponentials have a negligible effect on the drift velocity after a few oscillations. Hence, the only parts of the solution that are of interest and that result in a particle drift velocity are those solutions which are proportional to time.

Looking at equations 2.12 and 2.13, it is seen that the solutions to these equations are of the form of constants, decaying exponentials, and sinusoidal terms. However, equation 2.14 has Z_1, multiplying the right hand side so that the product of the trigonometric solutions of Z_1 with $[\sin 2\pi\mathcal{T}]$ will produce a constant term. The solution for Z_2 will then contain the first and presumably most important term proportional to time.

In the following paragraphs the solutions for Z_0, Z_1 and Z_2 will be found. The chief aim of the analysis is to obtain the single term which is proportional to time so that the drift velocity, as defined here, can be evaluated.

<u>Solutions to the Equations.</u> By inspection it is obvious that the solution to equation 2.12, with the given boundary conditions, is $Z_0 = \overline{Z} =$ constant.

Substituting $Z_0 = \overline{Z}$ in equation 2.13 leads to:

$$\frac{d^2 Z_1}{d\mathcal{T}^2} + K' \frac{dZ_1}{d\mathcal{T}} = \cos\frac{\pi}{2}\,\overline{Z}\,\sin 2\pi\mathcal{T} \quad . \tag{2.16}$$

The solution to the homogeneous equation is:

$$Z_1 = C_1 + C_2 e^{-K'\mathcal{T}} \quad . \tag{2.17}$$

The trial particular solution is:

$$Z_1 = C_3 \sin 2\pi\mathcal{T} + C_4 \cos 2\pi\mathcal{T} \quad . \tag{2.18}$$

Substituting equation 2.18 into 2.16 gives:

$$-4\pi^2 C_3 \sin 2\pi\mathcal{T} - 4\pi^2 C_4 \cos 2\pi\mathcal{T} + K' 2\pi C_3 \cos 2\pi\mathcal{T}$$
$$-K' 2\pi C_4 \sin 2\pi\mathcal{T} = \left[\cos\frac{\pi}{2}\bar{Z} \sin 2\pi\mathcal{T}\right]. \tag{2.19}$$

Equating the coefficients of like terms gives the following values for the constants:

$$C_3 = \frac{\cos\frac{\pi}{2}\bar{Z}}{-4\pi^2 - K'^2} \quad , \quad C_4 = \frac{K'\cos\frac{\pi}{2}\bar{Z}}{-K'^2 2\pi - 8\pi^3}. \tag{2.20}$$

The solution for Z_1 can now be written as:

$$Z_1 = \left[C_1 + C_2 e^{-K'\mathcal{T}} + \frac{\cos\frac{\pi}{2}\bar{Z}}{-4\pi^2 - K'^2} \sin 2\pi\mathcal{T} + \frac{K'\cos\frac{\pi}{2}\bar{Z}}{-K'^2 2\pi - 8\pi^3} \cos 2\pi\mathcal{T}\right]. \tag{2.21}$$

The initial conditions that $Z_1(0) = \frac{dZ_1}{d\mathcal{T}}(0) = 0$ are used to evaluate the constants C_1 and C_2. The following expression is obtained for the complete solution to equation 2.13:

$$Z_1 = \left[\frac{2\pi\cos\frac{\pi}{2}\bar{Z}}{4\pi^2 K' + K'^3} + \frac{K'\cos\frac{\pi}{2}\bar{Z}}{K'^2 2\pi + 8\pi^3} + \frac{2\pi\cos\frac{\pi}{2}\bar{Z}}{-4\pi^2 K' - K'^3} e^{-K'\mathcal{T}}\right.$$
$$\left. + \frac{\cos\frac{\pi}{2}\bar{Z}}{-4\pi^2 - K'^2} \sin 2\pi\mathcal{T} + \frac{K'\cos\frac{\pi}{2}\bar{Z}}{-K'^2 2\pi - 8\pi^3} \cos 2\pi\mathcal{T}\right]. \tag{2.22}$$

The differential equation for Z_2 can be found by substituting 2.22 and $Z_0 = \bar{Z}$ into equation 2.14:

$$\frac{d^2 Z_2}{d\mathcal{T}^2} + K'\frac{dZ_2}{d\mathcal{T}} = \left[-\frac{\pi}{2}\sin\frac{\pi}{2}\bar{Z}\left(\frac{2\pi\cos\frac{\pi}{2}\bar{Z}}{4\pi^2 K' + K'^3} + \frac{K'\cos\frac{\pi}{2}\bar{Z}}{K'^2 2\pi + 8\pi^3}\right)\sin 2\pi\mathcal{T}\right.$$
$$-\frac{\pi}{2}\sin\frac{\pi}{2}\bar{Z}\left(\frac{2\pi\cos\frac{\pi}{2}\bar{Z}}{-4\pi^2 K' - K'^3}\right)e^{-K'\mathcal{T}}\sin 2\pi\mathcal{T}$$
$$-\frac{\pi}{2}\sin\frac{\pi}{2}\bar{Z}\left(\frac{\cos\frac{\pi}{2}\bar{Z}}{-4\pi^2 - K'^2}\right)(\sin 2\pi\mathcal{T})^2$$
$$\left.-\frac{\pi}{2}\sin\frac{\pi}{2}\bar{Z}\left(\frac{K'\cos\frac{\pi}{2}\bar{Z}}{-K'^2 2\pi - 8\pi^3}\right)\sin 2\pi\mathcal{T}\cos 2\pi\mathcal{T}\right]. \tag{2.23}$$

The above equation may be simplified slightly by using the following trigonometric identities:

$$\left.\begin{aligned} \sin\frac{\pi}{2}\bar{z}\cos\frac{\pi}{2}\bar{z} &= \frac{1}{2}\sin\pi\bar{z} \\ \sin 2\pi\mathcal{T}\cos 2\pi\mathcal{T} &= \frac{1}{2}\sin 4\pi\mathcal{T} \\ (\sin 2\pi\mathcal{T})^2 &= \frac{1}{2} - \frac{1}{2}\cos 4\pi\mathcal{T} \end{aligned}\right\} \quad . \tag{2.24}$$

Equation 2.23 reduces to:

$$\frac{d^2 Z_2}{d\mathcal{T}^2} + K'\frac{dZ_2}{d\mathcal{T}} = \Big[(F'+G')\sin 2\pi\mathcal{T} - F'e^{-K'\mathcal{T}}\sin 2\pi\mathcal{T}$$

$$+ \frac{D'}{2} - \frac{D'}{2}\cos 4\pi\mathcal{T} + E'\sin 4\pi\mathcal{T}\Big] \tag{2.25}$$

Here,

$$D' = \frac{\pi\sin\pi\bar{z}}{4(4\pi^2+K'^2)} , \qquad E' = \frac{K'\sin\pi\bar{z}}{16(4\pi^2+K'^2)} ,$$

$$F' = \frac{\pi^2\sin\pi\bar{z}}{-2K'(4\pi^2+K'^2)} , \qquad G' = \frac{K'\sin\pi\bar{z}}{-8(4\pi^2+K'^2)} .$$

The solution to the homogeneous equation is again:

$$Z_2 = C_1 + C_2 e^{-K'\mathcal{T}} .$$

The following trial solution will be used to find the particular solution to equation 2.25.

$$Z_2 = \Big[C_3\mathcal{T} + C_4\sin 2\pi\mathcal{T} + C_5\cos 2\pi\mathcal{T}$$

$$+ C_6 e^{-K'\mathcal{T}}\sin 2\pi\mathcal{T} + C_7 e^{-K'\mathcal{T}}\cos 2\pi\mathcal{T} \tag{2.26}$$

$$+ C_8\cos 4\pi\mathcal{T} + C_9\sin 4\pi\mathcal{T}\Big] .$$

Substituting 2.26 into 2.25, equating coefficients of like terms, and solving for the constants gives:

$$Z_2 = \Big[C_1 + C_2 e^{-K'\mathcal{T}} + \left(\frac{\pi \sin \pi \bar{Z}}{8K'(4\pi^2 + K'^2)}\right)\mathcal{T} + \left(\frac{-2\pi D' + K'E'}{-64\pi^3 - K'^2 4\pi}\right)\cos 4\pi\mathcal{T}$$

$$+ \left(\frac{-\frac{K'D'}{2} - 4\pi E'}{K'^2 4\pi + 64\pi^3}\right)\sin 4\pi\mathcal{T} + \left(\frac{2\pi(F' + G')}{-2\pi K'^2 - 8\pi^3}\right)\sin 2\pi\mathcal{T} \tag{2.27}$$

$$+ \left(\frac{K'(F' + G')}{-2\pi K'^2 - 8\pi^3}\right)\cos 2\pi\mathcal{T} - \left(\frac{2\pi F'}{-8\pi^3 - 2\pi K'^2}\right)e^{-K'\mathcal{T}}\sin 2\pi\mathcal{T}$$

$$- \left(\frac{K'F'}{2\pi K'^2 + 8\pi^3}\right)e^{-K'\mathcal{T}}\cos 2\pi\mathcal{T} \Big] .$$

Here it is unnecessary to evaluate the constants C_1 and C_2, since C_1 is a constant and will disappear when the derivative with respect to time is taken to find the particle velocity and C_2 is multiplying a decaying exponential which will not contribute to the steady state particle drift velocity. The non-oscillating, non-decaying term, a second order term, is $Z_2 = \left[\frac{\pi \sin \pi \bar{Z}}{8K'(4\pi^2 + K'^2)}\right]\mathcal{T}$. Using equation 2.11, the non-oscillating and non-decaying part of the expression for the particle position can be written as:

$$Z = \bar{Z} + \varepsilon'^2 \frac{\pi \sin \pi \bar{Z}}{8K'(4\pi^2 + K'^2)}\mathcal{T} + \cdots \tag{2.28}$$

Or, using 2.8, equation 2.28 can be written in terms of the original variables as:

$$\xi = \bar{\xi} + \varepsilon'^2\left(\frac{\pi}{8K'(4\pi^2 + K'^2)}\right)\left(\sin \frac{\pi \xi}{\ell}\right)\left(\frac{ct}{4}\right) + \cdots \tag{2.29}$$

Now taking the derivative of 2.29 with respect to time, the mean particle drift velocity is obtained:

$$\dot{\xi} = v = \frac{\varepsilon'^2 c}{4}\left[\frac{\pi}{8K'(4\pi^2+K'^2)}\right]\left[\sin\frac{\pi\bar{\xi}}{l}\right]. \tag{2.30}$$

Remember here that $\varepsilon' = \dfrac{-96\pi\mu\sigma P_0 l}{m \rho c^3}$ and $K' = \dfrac{24\pi\mu\sigma l}{mc}$.

Equation 2.30 gives the steady state drift velocity of the particle which is at a point ξ in the chamber. The drift term is second order in ε' and depends sinusoidally on position. The greatest drift velocity occurs at an initial particle position of $y = \pm \frac{l}{2}$.

The fact that the drift velocity is second order can be justified by the following argument. The drift velocity exists because of the gradient in the amplitude of the gas oscillations. Consider the forces acting on a particle due to the difference between particle and gas velocity. The gas is oscillating at a fixed frequency but with an amplitude that varies spatially. Thus, during a single oscillation, the particle moves from a region in which the amplitude of gas motion is large to a region in which the amplitude of gas motion is smaller. During that part of an oscillation when the particle is near its maximum excursion in the direction of increasing amplitude of gas motion forces act to accelerate the particle toward the direction of decreasing amplitude. Similarly, when the particle is on the opposite side of its mean position, forces act to accelerate the particle in the direction of increasing amplitude of gas motion. However, since the force acting on the particle is proportional to the difference between the gas speed and the particle speed, the forces acting on the particle are higher in the regions of higher amplitude of gas motion. Hence, over a number of oscillations of the particle, a net force acts to accelerate the particles toward the direction of decreasing amplitude of gas oscillations. The net driving force depends on the difference between the forces acting on the particle on either side of the mean particle position and thus depends on the mean difference in the gas speed on either side of the particle mean position. Hence, the driving force and the resulting drift velocity is proportional to the product of the gradient in amplitude of gas motion and the amplitude of particle motion. Since both of these quantities are first order, the resulting particle drift speed is second order.

The gradient in the amplitude of gas oscillation is proportional to $\sin\frac{\pi}{2}\bar{z}$ and the amplitude of the particle oscillation is proportional to $\cos\frac{\pi}{2}\bar{z}$, e.g. see equation 2.22. Thus, the maximum drift speed would be expected at the value of $\bar{z}$ which maximizes $(\cos\frac{\pi}{2}\bar{z})(\sin\frac{\pi}{2}\bar{z})$ or $\bar{z} = \pm\frac{1}{2}$.

<u>Discussion of Solutions.</u> It is of interest to determine the manner in which this drift depends upon the particle size. To do this, it is convenient to use the fact that $\mathcal{E}'$ can be written as a function of K', $\left(\mathcal{E}' \equiv \frac{-4P_0 K'}{\rho c^2}\right)$. Therefore, $\mathcal{E}'$ can be eliminated from equation 2.30 to give the following expression for the drift velocity:

$$v = \frac{P_o^2}{4\rho^2 c^3}\left[\frac{K'/2\pi}{(K'/2\pi)^2 + 1}\right] \sin\frac{\pi\xi}{l} \quad . \tag{2.31}$$

To make the dependence upon particle radius explicit, write the particle mass as $m = \rho_p \frac{4}{3}\pi\sigma^3$; then $\frac{K'}{2\pi} = \frac{9\mu l}{c\rho_p \pi\sigma^2}$. In these terms the drift velocity is given by:

$$v = \frac{P_o^2}{4\rho^2 c^3}\left[\frac{\frac{9\mu l}{\rho_p c\pi\sigma^2}}{\left(\frac{9\mu l}{\rho_p c\pi\sigma^2}\right)^2 + 1}\right] \sin\frac{\pi\xi}{l} \quad . \tag{2.32}$$

As the particle radius, σ , gets large it is evident from equation 2.32 that the drift velocity approaches zero. If equation 2.32 is rewritten as:

$$v = \frac{P_o^2}{4\rho^2 c^3}\left[\frac{\frac{\rho_p c\pi\sigma^2}{9\mu l}}{1 + \left(\frac{\rho_p c\pi\sigma^2}{9\mu l}\right)^2}\right] \sin\frac{\pi\xi}{l} \quad , \tag{2.33}$$

it is evident that as the particle radius gets small, the drift velocity again approaches zero. Therefore, a maximum drift velocity exists for some finite value of particle radius. This optimum radius, σ^*, resulting in maximum drift velocity, can be found by examining the first and second derivatives with respect to σ. The result of the calculation gives:

$$\sigma^* = \sqrt{\frac{9\mu l}{\rho_p c\pi}} \quad . \tag{2.34}$$

Substituting the optimum particle radius, 2.34, into the drift velocity expression, 2.32, gives the following expression for the maximum drift velocity:

$$v^* = \left(\frac{P_0^2}{8\rho^2 c^3}\right) \sin \frac{\pi \xi}{l} \quad . \tag{2.35}$$

In terms of the chamber pressure, P_c , the maximum drift s:

$$v^* = \frac{c}{8\gamma^2}\left(\frac{P_0}{P_c}\right)^2 \sin \frac{\pi \xi}{l} \quad . \tag{2.36}$$

Note that while the optimum particle radius is a function of the chamber dimension, l , the maximum drift velocity, v^*, is independent of the chamber size. Therefore, the drift motion is less important as the chamber size increases.

Note also that the maximum drift velocity is independent of droplet properties and only depends on the gas properties through the speed of sound c , specific heat ratio γ and pressure ratio $\frac{P_0}{P_c}$.

The ratio of drift velocity for arbitrary particle radius to the drift velocity for optimum particle radius is simply obtained from equations 2.32 and 2.34 and is given by

$$\frac{v}{v^*} = \left[\frac{2\left(\sigma/\sigma^*\right)^2}{1+\left(\sigma/\sigma^*\right)^4}\right] \quad . \tag{2.37}$$

The drift speed drops rapidly for particle radius different from optimum radius. For example, for $\sigma = \frac{1}{2}\sigma^*$ or $2\sigma^*$ the ratio v/v^* is about $\frac{1}{2}$.

III. CYLINDRICAL CHAMBER

Development of the Equations. The techniques used in Section II are now applied to the solution of the problem of particle motion in a cylindrical geometry. The problems are quite similar except that with the cylindrical geometry, (A) motion in two directions must be considered and (B) centrifugal forces may be important.

To investigate the motion of small particles in a cylindrical enclosure under the influence of acoustic oscillations, the same assumptions that were made for the rectangular enclosure will be used; namely:

1. The fluid medium is considered to be at rest and in a uniform state except for weak isoenergetic fluctuations in velocity and state properties.
2. The small particles are spherical and experience Stokes law forces.
3. The particles have negligible effect upon the gas motion in the chamber.

For the conditions given in assumption 1, it can be shown (3) that the gas motion is governed by the wave equation which is written below in cylindrical coordinates for the pressure disturbance:

$$\frac{1}{r^2}\frac{\partial^2 p}{\partial \varphi^2} + \frac{\partial^2 p}{\partial z^2} + \frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} = \frac{1}{C^2}\frac{\partial^2 p}{\partial t^2} \quad . \tag{3.1}$$

Since we are interested in finding a particle drift velocity in a cross sectional plane of the cylinder, in analogy to the simple transverse waves considered in the rectangular chamber, only oscillations independent of z will be considered in this analysis. The solution to equation 3.1 for a circular cylinder with standing acoustic oscillations is given by:

$$P = P_0 \begin{matrix}\cos \\ \sin\end{matrix} (m\varphi) \, J_m \left(\frac{\omega_r r}{c}\right) e^{-2\pi i \nu \tau} \quad . \tag{3.2}$$

Here, as in the previous problem, the frequency, ν, is $\frac{\omega_r}{2\pi}$, and, in this case the angular velocity, ω_r, is $\frac{\pi \alpha_{mn} c}{a}$. The α_{mn} arise from the condition of zero radial velocity at the cylinder walls. The subscripts m and n specify the modes of vibration in the φ and r directions. At the walls α_{mn} is a solution to

$$\frac{d \, J_m (\pi \alpha)}{d r} = 0 \quad . \tag{3.3}$$

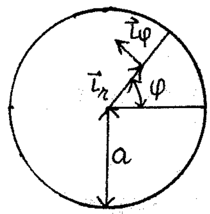
For each value of m there exists an infinite number of solutions to equation 3.3 since J_m is an oscillatory function of its argument. Characteristic values of α_{mn} are listed in reference 3.

For oscillations restricted to a transverse plane of the cylinder the simplest cases are the first radial mode ($m = 0$, $n = 1$, $\alpha_{mn} = 1.22$), and the first transverse or sloshing mode ($m = 1$, $n = 0$, $\alpha_{mn} = .586$). In investigating the possible particle drift velocities induced by the acoustic oscillations only the effect of the sloshing mode will be considered since it is this mode which is most often observed in liquid and solid propellant rocket systems experiencing high frequency instability.

With the restrictions to oscillations independent of z and to real solutions, equation 3.2 can be written:

$$P = P_o \cos\varphi \; J_1\left(\frac{\pi\alpha_{10} r}{a}\right) \cos 2\pi\nu t \tag{3.4}$$

The coordinate system used in this problem is shown in the following diagram:



In this case:

$$\vec{\nabla} P = \vec{i}_\varphi \left\{ \frac{P_o}{r} (-\sin\varphi) \, J_1\left(\frac{\pi\alpha_{10} r}{a}\right) \cos 2\pi\nu t \right\}$$
$$+ \vec{i}_r \left\{ P_o (\cos\varphi)(\cos 2\pi\nu t)\left(\frac{\pi\alpha_{10}}{2a}\right)\left[J_0\left(\frac{\pi\alpha_{10} r}{a}\right) - J_2\left(\frac{\pi\alpha_{10} r}{a}\right)\right]\right\} . \tag{3.5}$$

An expression for the gas velocity can be obtained from equation 3.4 in a manner analogous to that used in the case of the rectangular chamber. The result of the calculation is:

$$\vec{U} = \vec{i}_\varphi \left[\frac{P_o}{2\pi\nu\rho r} \sin\varphi \; J_1\left(\frac{\pi\alpha_{10} r}{a}\right) \sin 2\pi\nu t \right]$$

$$+ \vec{i}_r \left[-\frac{P_o}{4\nu\rho a} \cos\varphi \sin 2\pi\nu t \left\{ J_0\left(\frac{\pi\alpha_{10} r}{a}\right) - J_2\left(\frac{\pi\alpha_{10} r}{a}\right)\right\}\right] . \tag{3.6}$$

Fig. 1 shows the magnitude and direction of the gas velocity, as a function of P_o/P_c, for various points in the cross section of the cylinder. The calculations are made for a time such that

$\sin 2\pi\nu t = 1$. Since the cross section is symmetrical, only the first two quadrants are shown in the plot. It can be seen in Fig. 1 that there is a gas velocity node at φ equals 0° and 180° while the pressure node occurs at $\varphi = 90^{\circ}$ and 270°.

Using assumption 2 the drag force on the particles is given by:

$$6\pi\mu\sigma\left(\vec{U} - \vec{v}\right) \quad . \tag{3.7}$$

Here, as before, $\vec{U}$ is the gas speed and $\vec{v}$ is the particle speed. The particle position is given by $\vec{i}_r\, r$, the particle velocity by $\frac{d\vec{r}}{dt} = \vec{i}_r \frac{dr}{dt} + \vec{i}_\varphi\, r\frac{d\varphi}{dt}$, and the particle acceleration by

$$\frac{d^2\vec{r}}{dt} = \vec{i}_r\left[\frac{d^2r}{dt^2} - r\left(\frac{d\varphi}{dt}\right)^2\right] + \vec{i}_\varphi\left[2\frac{dr}{dt}\frac{d\varphi}{dt} + r\frac{d^2\varphi}{dt^2}\right].$$

The equation of motion for the particle can be obtained by equating the particle inertial forces to the drag force obtained from Stokes law. The result is:

$$\vec{i}_r\, m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\varphi}{dt}\right)^2\right] + \vec{i}_\varphi\, m\left[2\frac{dr}{dt}\frac{d\varphi}{dt} + r\frac{d^2\varphi}{dt^2}\right]$$

$$= 6\pi\mu\sigma\left[\vec{i}_\varphi\left\{\frac{P_0}{2\pi\nu r\rho}\sin\varphi\, J_1\!\left(\frac{\pi\alpha_{10}r}{a}\right)\sin 2\pi\nu t - r\frac{d\varphi}{dt}\right\}\right. \tag{3.8}$$

$$\left. + \vec{i}_r\left\{\frac{-P_0\alpha_{10}}{4\nu\rho a}\cos\varphi\,\sin 2\pi\nu t\left[J_0\!\left(\frac{\pi\alpha_{10}r}{a}\right) - J_2\!\left(\frac{\pi\alpha_{10}r}{a}\right)\right] - \frac{dr}{dt}\right\}\right].$$

In the above equation the same symbols are used for gas position as for particle position since the gas velocity must be evaluated at the position of the particle (by assumption 3). Equation 3.8 can be separated into two more useful equations corresponding to the $\vec{i}_r$ and $\vec{i}_\varphi$ directions:

$\vec{i}_r$ component

$$\frac{d^2 r}{dt^2} + \frac{6\pi\mu\sigma}{m}\frac{dr}{dt} - r\left(\frac{d\varphi}{dt}\right)^2 = -\frac{3\pi\mu\sigma\alpha_{10}P_0}{2m\rho\nu a}\cos\varphi\sin 2\pi\nu t\left[J_0\left(\frac{\pi\alpha_{10}r}{a}\right) - J_2\left(\frac{\pi\alpha_{10}r}{a}\right)\right] \tag{3.9}$$

$\vec{i}_\varphi$ component

$$2\frac{dr}{dt}\frac{d\varphi}{dt} + r\frac{d^2\varphi}{dt^2} + \frac{6\pi\mu\sigma}{m}r\frac{d\varphi}{dt} = \left[\frac{3\mu\sigma P_0}{m\rho\nu r}\sin\varphi\sin 2\pi\nu t\, J_1\left(\frac{\pi\alpha_{10}r}{a}\right)\right]. \tag{3.10}$$

As in the case of the rectangular chamber it will be convenient to nondimensionalize equations 3.9 and 3.10; the following dimensionless variables are used:

$$Z = r/a, \qquad \mathcal{T} = \frac{t}{a/c} = \frac{tc}{a}, \qquad \phi = \varphi. \tag{3.11}$$

Introducing the dimensionless variables into equation 3.9 and 3.10 gives:

$\vec{i}_r$ component

$$\frac{d^2 Z}{d\mathcal{T}^2} + K\frac{dZ}{d\mathcal{T}} - Z\left(\frac{d\phi}{d\mathcal{T}}\right)^2 = -\frac{\varepsilon\pi\alpha_{10}}{2}\cos\phi\sin\frac{2\pi\nu a\mathcal{T}}{c}\left[J_0(\pi\alpha_{10}Z) - J_2(\pi\alpha_{10}Z)\right] \tag{3.12}$$

$\vec{i}_\varphi$ component

$$2Z\frac{dZ}{d\mathcal{T}}\frac{d\phi}{d\mathcal{T}} + Z^2\frac{d^2\phi}{d\mathcal{T}^2} + K Z^2\frac{d\phi}{d\mathcal{T}} = \left[\mathcal{E}\sin\phi\,\sin\frac{2\pi\nu a\mathcal{T}}{c}\,J_1(\pi\alpha_{10}Z)\right]. \tag{3.13}$$

Here, $\mathcal{E} = \frac{3\mu\sigma P_0}{m\rho\nu c^2}$ and $K = \frac{6\pi\mu\sigma a}{mc}$ are parameters of the problem and, except for constants, are the same as the $\mathcal{E}'$ and K' in the rectangular chamber problem.

Equations 3.12 and 3.13 are non-linear due to the appearance of Z in the argument of the Bessel functions and are analogous to equation 2.9 in the case of the rectangular chamber. Since $\mathcal{E}$ is proportional to P_0, μ, and other flow properties, an argument similar to that made about equation 2.9 is valid. The following perturbation expansion in powers of $\mathcal{E}$ is introduced to linearize equations 3.12 and 3.13.

$$Z(\mathcal{T}) = Z_0(\mathcal{T}) + \mathcal{E}Z_1(\mathcal{T}) + \mathcal{E}^2Z_2(\mathcal{T}) + --- \tag{3.14}$$

$$\phi(\mathcal{T}) = \phi_0(\mathcal{T}) + \mathcal{E}\phi_1(\mathcal{T}) + \mathcal{E}^2\phi_2(\mathcal{T}) + --- \tag{3.15}$$

In substituting the above expansions in equations 3.12 and 3.13, $\sin\phi$ is expanded as $\sin\phi = (\sin\phi_0)\left(1 - \frac{\mathcal{E}^2}{2}[\phi_1 + \mathcal{E}\phi_2]^2 + ---\right) + (\cos\phi_0)(\mathcal{E}\phi_1 + \mathcal{E}^2\phi_2 - ---)$ and $\cos\phi$ is expanded as $\cos\phi = (\cos\phi_0)\left(1 - \frac{\mathcal{E}^2}{2}[\phi_1 + \mathcal{E}\phi_2]^2 + ---\right) - (\sin\phi_0)(\mathcal{E}\phi_1 + \mathcal{E}^2\phi_2 - ---)$.

The Bessel functions are expanded in a Taylor series about the point Z_0 to get:

$$J_0(\pi\alpha_{10}Z) = \Big[J_0(\pi\alpha_{10}Z_0) - \varepsilon Z_1 \pi\alpha_{10} J_1(\pi\alpha_{10}Z_0) - \varepsilon^2 Z_2 \pi\alpha_{10} J_1(\pi\alpha_{10}Z_0)$$

$$- \frac{\varepsilon^2 Z_1^2}{2}(\pi\alpha_{10})^2 J_0(\pi\alpha_{10}Z_0) + \frac{\varepsilon^2 Z_1^2 \pi\alpha}{2 Z_0} J_1(\pi\alpha_{10}Z_0)\Big]$$

$$J_1(\pi\alpha_{10}Z) = \Big[J_1(\pi\alpha_{10}Z_0) + \varepsilon\pi\alpha_{10} Z_1 J_0(\pi\alpha_{10}Z_0) - \frac{\varepsilon Z_1}{Z_0} J_1(\pi\alpha_{10}Z_0)$$

$$+ \varepsilon^2 \pi\alpha_{10} Z_2 J_0(\pi\alpha_{10}Z_0) - \frac{\varepsilon^2 Z_2}{Z_0} J_1(\pi\alpha_{10}Z_0) + \frac{\varepsilon^2 Z_1^2}{2}(\pi\alpha_{10})^2\left[\frac{2}{(\pi\alpha_{10}Z_0)^2} - 1\right] J_1(\pi\alpha_{10}Z_0)$$

$$- \frac{\varepsilon^2 Z_1^2 \pi\alpha_{10}}{2 Z_0} J_0(\pi\alpha_{10}Z_0)\Big]$$

$$J_2(\pi\alpha_{10}Z) = \Big[J_2(\pi\alpha_{10}Z_0) + \varepsilon Z_1 \pi\alpha_{10} J_1(\pi\alpha_{10}Z_0) - \frac{2\varepsilon Z_1}{Z_0} J_2(\pi\alpha_{10}Z_0)$$

$$+ \varepsilon^2 Z_2 \pi\alpha_{10} J_1(\pi\alpha_{10}Z_0) - \frac{2\varepsilon^2 Z_2}{Z_0} J_2(\pi\alpha_{10}Z_0) + \frac{\varepsilon^2 Z_1^2}{2}(\pi\alpha_{10})^2 J_0(\pi\alpha_{10}Z_0)$$

$$- \frac{3\varepsilon^2 Z_1^2(\pi\alpha_{10})}{2 Z_0} J_1(\pi\alpha_{10}Z_0) + \frac{3\varepsilon^2 Z_1^2}{Z_0^2} J_2(\pi\alpha_{10}Z_0)\Big].$$

The results of substituting 3.14 and 3.15 into equations 3.12 and 3.13 and separating the results into orders of ε are:

$\vec{i}_r$ component

$$\frac{d^2 Z_0}{d\mathcal{T}^2} + K\frac{dZ_0}{d\mathcal{T}} - Z_0\left(\frac{d\phi_0}{d\mathcal{T}}\right)^2 = 0 \tag{3.16}$$

$$\frac{d^2 Z_1}{d\mathcal{T}} + K\frac{dZ_1}{d\mathcal{T}} - 2Z_0\frac{d\phi_1}{d\mathcal{T}}\frac{d\phi_0}{d\mathcal{T}} - Z_1\left(\frac{d\phi_0}{d\mathcal{T}}\right)^2 =$$
$$\left\{\frac{\pi\alpha_{10}}{2}\sin\frac{2\pi\nu a\mathcal{T}}{c}\cos\phi_0\Big[-J_0(\pi\alpha_{10}Z_0) + J_2(\pi\alpha_{10}Z)\Big]\right\} \tag{3.17}$$

$$\frac{d^2 Z_2}{d\mathcal{T}^2} + K\frac{dZ_2}{d\mathcal{T}} - 2Z_0\frac{d\phi_2}{d\mathcal{T}}\frac{d\phi_0}{d\mathcal{T}} - Z_0\left(\frac{d\phi_1}{d\mathcal{T}}\right)^2 - 2Z_1\frac{d\phi_1}{d\mathcal{T}}\frac{d\phi_0}{d\mathcal{T}} - Z_2\left(\frac{d\phi_0}{d\mathcal{T}}\right)^2 =$$

$$\frac{\pi\alpha_{10}}{2}\sin\frac{2\pi\nu a\mathcal{T}}{c}\Big[2Z_1\pi\alpha_{10}\cos\phi_0\, J_1(\pi\alpha_{10}Z_0)$$

$$- \frac{2Z_1}{Z_0}\cos\phi_0\, J_2(\pi\alpha_{10}Z_0) + \phi_1\sin\phi_0\left\{J_0(\pi\alpha_{10}Z_0) - J_2(\pi\alpha_{10}Z_0)\right\}\Big]. \quad (3.18)$$

$\vec{l}_\varphi$ component

$$2Z_0\frac{dZ_0}{d\mathcal{T}}\frac{d\phi_0}{d\mathcal{T}} + Z_0^2\frac{d^2\phi_0}{d\mathcal{T}^2} + Z_0^2 K\frac{d\phi_0}{d\mathcal{T}} = 0 \quad (3.19)$$

$$2Z_0\frac{d\phi_0}{d\mathcal{T}}\frac{dZ_1}{d\mathcal{T}} + 2Z_0\frac{dZ_0}{d\mathcal{T}}\frac{d\phi_1}{d\mathcal{T}} + 2Z_1\frac{dZ_0}{d\mathcal{T}}\frac{d\phi_0}{d\mathcal{T}} + Z_0^2\frac{d^2\phi_1}{d\mathcal{T}^2} + 2Z_0Z_1\frac{d^2\phi_0}{d\mathcal{T}^2}$$

$$+ Z_0^2 K\frac{d\phi_1}{d\mathcal{T}} + 2KZ_0Z_1\frac{d\phi_0}{d\mathcal{T}} = \left[J_1(\pi\alpha_{10}Z_0)\sin\phi_0\sin\frac{2\pi\nu a\mathcal{T}}{c}\right] \quad (3.20)$$

$$2Z_0\frac{d\phi_0}{d\mathcal{T}}\frac{dZ_2}{d\mathcal{T}} + 2Z_0\frac{dZ_1}{d\mathcal{T}}\frac{d\phi_1}{d\mathcal{T}} + 2Z_0\frac{dZ_0}{d\mathcal{T}}\frac{d\phi_2}{d\mathcal{T}} + 2Z_1\frac{d\phi_0}{d\mathcal{T}}\frac{dZ_1}{d\mathcal{T}} + 2Z_1\frac{dZ_0}{d\mathcal{T}}\frac{d\phi_1}{d\mathcal{T}} + 2Z_2\frac{dZ_0}{d\mathcal{T}}\frac{d\phi_0}{d\mathcal{T}}$$

$$+ Z_0^2\frac{d^2\phi_2}{d\mathcal{T}^2} + 2Z_0Z_1\frac{d^2\phi_1}{d\mathcal{T}^2} + 2Z_0Z_2\frac{d^2\phi_0}{d\mathcal{T}^2} + Z_1^2\frac{d^2\phi_0}{d\mathcal{T}^2} + KZ_0^2\frac{d\phi_2}{d\mathcal{T}} + 2KZ_0Z_1\frac{d\phi_1}{d\mathcal{T}}$$

$$+ 2KZ_0Z_2\frac{d\phi_0}{d\mathcal{T}} + KZ_1^2\frac{d\phi_0}{d\mathcal{T}} = \Big\{Z_1\sin\phi_0\left[\pi\alpha_{10}J_0(\pi\alpha_{10}Z_0) - \frac{1}{Z_0}J_1(\pi\alpha_{10}Z_0)\right] \quad (3.21)$$

$$+ \phi_1\cos\phi_0\, J_1(\pi\alpha_{10}Z_0)\Big\}\sin\frac{2\pi\nu a\mathcal{T}}{c}.$$

If a particle is considered to be at rest at $\mathcal{T} = 0$ and at a point $\bar{Z}$, $\bar{\Phi}$, then the appropriate initial conditions are:

$$\left.\begin{aligned}
&Z_0(0) = \bar{Z}, \quad \phi_0(0) = \bar{\Phi}, \quad \frac{dZ_0}{d\mathcal{T}}(0) = \frac{d\phi_0}{d\mathcal{T}}(0) = 0 \\
&Z_1(0) = \frac{dZ_1}{d\mathcal{T}}(0) = \phi_1(0) = \frac{d\phi_1}{d\mathcal{T}}(0) = 0 \\
&Z_2(0) = \frac{dZ_2}{d\mathcal{T}}(0) = \phi_2(0) = \frac{d\phi_2}{d\mathcal{T}}(0) = 0
\end{aligned}\right\} \quad (3.22)$$

It is noted again that purely constant, damped or oscillatory solutions will not be of interest since no net transport of particles is involved.

The form of equations 3.16 to 3.21 can be compared with equations 2.12 to 2.14. Since ϕ_1, and Z_1 multiply the right side of the second order equations it is expected that any terms in the solution which are proportional to time will again appear in the second order equations 3.18 and 3.21.

The expectation that the drift velocity will be second order in ε, specifically for the radial motion of the particles, also follows from the physical arguments given on page 14. However, for the geometry considered here, centrifugal forces also act on the particle. Since the centrifugal forces are proportional to the square of the particle velocity and the particle velocity is first order in ε, the drift which results from centrifugal forces is also a second order term.

<u>Solutions to the Equations.</u> A solution to the zero order equations 3.16 and 3.19 is Z_0 = constant = $\overline{Z}$ and ϕ_0 = constant = $\overline{\phi}$. The constant ε is closely related to a drag force and can be called an interaction parameter so that to zero order in ε there is no drag or interaction between the particles and the gas in the chamber with the result that the particle position is fixed. That is to zero order in ε, the particles are not moved by the gas, but are at a fixed position. Using these simple zero order solutions, the solutions for Z_1, Z_2, ϕ_1, and ϕ_2 can be found by the same

method used in solving equations 2.13 and 2.14 in the rectangular chamber problem. The complete solutions to equations 3.17, 3.18, 3.20 and 3.21 are discussed in more detail in Appendix A.

The first and presumably most important non-oscillatory and non-damped terms appear in the solutions to the second order equations 3.18 and 3.21. These non-oscillatory and non-decaying solutions are:

$$\left.\begin{aligned} z_2 &= \left\{ \frac{\frac{\bar{\bar{z}}}{2} D^2 - \frac{1}{2}(FE + DG)}{K\left[K^2 + \left(\frac{2\pi\nu a}{c}\right)^2\right]} \right\} \mathcal{T} \\ \phi_2 &= \left\{ \frac{-\frac{1}{2}(EH + DI)}{K\left[K^2 + \left(\frac{2\pi\nu a}{c}\right)^2\right]} \right\} \mathcal{T} \end{aligned}\right\} \tag{3.23}$$

Here,

$$D = \frac{J_1(\pi\alpha_{10}\bar{\bar{z}})}{\bar{\bar{z}}^2} \sin\bar{\phi}$$

$$E = \frac{\pi\alpha_{10}}{2} \cos\bar{\phi} \left[J_2(\pi\alpha_{10}\bar{\bar{z}}) - J_0(\pi\alpha_{10}\bar{\bar{z}}) \right]$$

$$F = \cos\bar{\phi} \left[(\pi\alpha_{10})^2 J_1(\pi\alpha_{10}\bar{\bar{z}}) - \frac{\pi\alpha_{10}}{\bar{\bar{z}}} J_2(\pi\alpha_{10}\bar{\bar{z}}) \right]$$

$$G = \frac{\pi\alpha_{10}}{2} \sin\bar{\phi} \left[J_0(\pi\alpha_{10}\bar{\bar{z}}) - J_2(\pi\alpha_{10}\bar{\bar{z}}) \right]$$

$$H = \sin\bar{\phi} \left[\pi\alpha_{10} J_0(\pi\alpha_{10}\bar{\bar{z}}) - \frac{1}{\bar{\bar{z}}} J_1(\pi\alpha_{10}\bar{\bar{z}}) \right]$$

$$I = \cos\bar{\phi}\, J_1(\pi\alpha_{10}\bar{\bar{z}}) \;.$$

Differentiation of equation 3.23 with respect to time will give the particle drift velocity. The objection might be made that the decaying exponential terms in the solutions to the first order equations may contribute appreciably to the particle motion. A calculation to determine the magnitude of the first order decaying exponential terms, compared with the steady state drift velocity, is shown in Appendix B. In general, these terms appear to be negligible for reasonable values of the parameters.

Moreover, the terms which decay exponentially in time arise from certain assumed initial conditions of the particle position in a certain phase of the motion. Since there is no physical way in which one initial condition or phase may be preferred over another, these can occur at random and hence the decaying terms can not contribute to the mean motion of large masses of particles.

IV. RESULTS AND DISCUSSION

The Drift Velocity. Ignoring the oscillatory terms and decaying exponentials the drift velocity of the particles can now be written down using the solutions listed in 3.23. Remember that the velocity of the particles is $\vec{v} = \vec{i}_r \frac{dr}{dt} + \vec{i}_\varphi r \frac{d\varphi}{dt}$ or in the dimensionless expanded form is:

$$\vec{v} = \vec{i}_r\, c\left[\frac{dZ_0}{d\mathcal{T}} + \varepsilon\frac{dZ_1}{d\mathcal{T}} + \varepsilon^2\frac{dZ_2}{d\mathcal{T}}\right] + \vec{i}_\varphi\, c\left[Z_0\frac{d\phi_0}{d\mathcal{T}}\right.$$

$$\left. + \varepsilon Z_0\frac{d\phi_1}{d\mathcal{T}} + \varepsilon Z_1\frac{d\phi_0}{d\mathcal{T}} + \varepsilon^2 Z_0\frac{d\phi_2}{d\mathcal{T}} + \varepsilon^2 Z_1\frac{d\phi_1}{d\mathcal{T}} + \varepsilon^2 Z_2\frac{d\phi_0}{d\mathcal{T}}\right]. \tag{4.1}$$

The $\vec{i}_r$ component of the drift velocity can be written as:

$$v_r = \frac{\varepsilon^2 c}{2}\left[\frac{\bar{Z}D^2 - (FE + DG)}{K[K^2 + (\frac{2\pi\nu a}{c})^2]}\right] \text{ or } \frac{\varepsilon^2 c}{2}\left[\frac{\frac{\bar{r}}{a}D^2 - (FE + DG)}{K[K^2 + (\frac{2\pi\nu a}{c})^2]}\right]. \tag{4.2}$$

in terms of the original variables. Note that $[\frac{\bar{r}}{a}D^2 - (FE + DG)]$ is a constant for any given $\bar{r}$, $\bar{\varphi}$ and cylinder radius and does not depend on the particle dimensions. Let $[\frac{\bar{r}}{a}D^2 - (FE+DG)] \equiv \beta$, then the r component of drift velocity is more conveniently written as

$$v_r = \frac{\varepsilon^2 c\,\beta}{2K[K^2 + (\frac{2\pi\nu a}{c})^2]}. \tag{4.3}$$

To see how the r component of particle velocity varies with particle size the expressions for ε and K can be substituted into v_r; this substitution gives:

$$v_r = \left[\frac{\frac{3}{16} \frac{\mu P_0^2 \beta \sigma}{\rho^2 \nu^4 \pi^3 a^3 m}}{\left(\frac{3\mu\sigma}{\nu m}\right)^2 + 1} \right] . \tag{4.4}$$

Or, since the mass of the particle equals $\rho_p \frac{4}{3} \pi \sigma^3$, 4.3 can be written as

$$v_r = \left[\frac{\frac{9}{64} \frac{\mu P_0^2 \beta}{\rho_p \rho^2 \nu^4 \pi^4 a^3 \sigma^2}}{\frac{81}{16} \frac{\mu^2}{\rho_p^2 \pi^2 \nu^2 \sigma^4} + 1} \right] . \tag{4.5}$$

As $\sigma \to \infty$ in equation 4.5, $v_r \to 0$ and as $\sigma \to 0$, $v_r \to 0$; therefore, as in the case of the rectangular chamber, there is an optimum particle radius, σ^*, which results in a maximum v_r. This optimum radius is again found by examining the first and second derivatives of the velocity with respect to σ. The result of the calculation gives:

$$\sigma_r^* = \sqrt{\frac{9\mu}{4 \rho_p \nu \pi}} . \tag{4.6}$$

The same arguments can also be made for the φ component. Thus, the φ component of particle velocity is

$$v_\varphi = \frac{\varepsilon^2 c}{2} \left[\frac{-\bar{z}(EH + DI)}{k\left[k^2 + \left(\frac{2\pi\nu a}{c}\right)^2\right]} \right] \text{ or } \frac{\varepsilon^2 c}{2} \left[\frac{-\frac{\bar{r}}{a}(EH + DI)}{k\left[k^2 + \left(\frac{2\pi\nu a}{c}\right)^2\right]} \right] \tag{4.7}$$

in terms of the original variables. Again, $\frac{\bar{r}}{a}(EH+DI)$ is constant for a given $\bar{r}$, $\bar{\varphi}$ and cylinder radius. If $\frac{\bar{r}}{a}(EH + DI) \equiv \delta$, then the φ component of drift velocity can be written as

$$v_\varphi = \left[\frac{-\varepsilon^2 c \,\delta}{2k\left[k^2 + \left(\frac{2\pi\nu a}{c}\right)^2\right]} \right] . \tag{4.8}$$

Since the expression for the r component of drift velocity (equation 4.3) differs from the φ component of drift velocity (equation 4.8) only by a multiplying constant, it is evident that the optimum radius for maximum v_φ will again be $\sigma_\varphi^* = \sqrt{\frac{9\mu}{4\rho_p \nu \pi}}$.

Since the optimum particle radius is the same for the r and φ components of drift velocity, a particle of radius σ^* will indeed experience a maximum resultant drift velocity. Note that, since $\nu = \frac{c}{4\ell}$ in the case of the rectangular chamber, the optimum radius given in 4.6 agrees exactly with the optimum radius found in equation 2.34 for particle motion in a rectangular chamber.

Again, in analogy with the rectangular chamber problem, it will be interesting to see whether or not the maximum particle drift velocity is independent of the cylinder radius, a. The particle drift velocity can be written as:

$$\vec{v} = \left\{ \vec{i}_r \frac{\varepsilon^2 c \beta}{2K[K^2 + (\frac{2\pi\nu a}{c})^2]} - \vec{i}_\varphi \frac{\varepsilon^2 c \delta}{2K[K^2 + (\frac{2\pi\nu a}{c})^2]} \right\} . \tag{4.9}$$

Here β and δ contain the cylinder radius in the ratio r/a and for a given r/a they are independent of the radius. It remains to be shown that ε, K and $\left(\frac{2\pi\nu a}{c}\right)$ are independent of the cylinder radius. Since $\nu = \frac{\alpha_{10} c}{2a}$, it is obvious that ν varies as the inverse of the radius, and that $\left(\frac{2\pi\nu a}{c}\right)$ does not vary with the radius. Since the optimum particle radius varies as $a^{1/2}$ and the mass of an optimum sized particle varies as $a^{3/2}$, it is evident that $\varepsilon = \frac{3\mu\sigma P_0}{m \rho \nu c^2}$ and $K = \frac{6\pi\mu\sigma a}{m c}$ are both independent of the cylinder radius. Therefore, while the optimum particle radius is a function of the cylinder size, the magnitude of the maximum particle drift velocity is not. This result is in agreement with the result obtained for the rectangular chamber problem.

Another comparison with the rectangular chamber problem can now be made. If the optimum particle radius, equation 4.6, is substituted into the particle drift velocity expression, equation 4.9, the following is obtained:

$$\vec{v}^{\,*} = \frac{c}{4\gamma^2(\pi\alpha_{10})^3}\left(\frac{P_0}{P_c}\right)^2\left[\vec{i}_r\beta - \vec{i}_\varphi\delta\right] . \qquad (4.10)$$

For $\varphi = 0^\circ$, the constant δ is 0 and the particle motion is entirely in the radial direction. Fig. 1 shows that the gas velocity in the cylinder is also entirely in the radial direction when φ is 0°. Since motion was restricted to one dimension in the rectangular geometry problem, a comparison of the maximum particle drift velocities is most meaningful if compared for a φ of 0° in the cylindrical problem. When $\varphi = 0^\circ$ is substituted into equation 4.10, the equation reduces to:

$$\vec{v}^{\,*} = \vec{i}_r\frac{c}{8\gamma^2}\left(\frac{P_0}{P_c}\right)^2\left\{-\left[J_1(\pi\alpha_{10}\tfrac{r}{a}) - \frac{1}{\pi\alpha_{10}\frac{r}{a}}J_2(\pi\alpha_{10}\tfrac{r}{a})\right]\left[J_2(\pi\alpha_{10}\tfrac{r}{a}) - J_0(\pi\alpha_{10}\tfrac{r}{a})\right]\right\} . \qquad (4.11)$$

For comparison, equation 2.36 is rewritten here:

$$v^* = \frac{c}{8\gamma^2}\left(\frac{P_0}{P_c}\right)^2 \sin\frac{\pi\xi}{\ell} . \qquad (2.36)$$

The expressions are the same except for the terms on the right resulting from the geometry difference in the two problems.

It is also noted that the ratio of particle drift velocity for arbitrary radius to the drift velocity for particles of optimum radius varies in exactly the same manner as given in equation 2.37 for the rectangular chamber problem.

Magnitude of the Drift Velocity and Discussion. Thus far, with the assumptions listed in Section II and the restrictions mentioned for ε, the expressions for particle drift velocity are perfectly general. The expression for particle drift velocity, equation 4.9, will now be used to calculate the drift velocity of small liquid droplets in a liquid propellant rocket combustion chamber. Although the fluid medium in the combustion chamber is not at rest, the Mach number of the gas relative to the particle is usually low near the injector. Stokes law drag will hold if the droplet shape remains nearly spherical and if the Reynolds number of the flow is less than one which, for reasonable values of the parameters, means that the particle radius must be less than ten microns. The results of these drift velocity calculations will not be exact, but will give an indication of the mean motion the droplets experience in a combustion chamber.

The following properties, for the droplets and gas in the chamber will be assumed:

For the particle:

$\rho_p = 2.1$ slug/ft.^{3}

The particle radius will be picked to maximize the drift velocity and for the values listed here:

$\sigma^* = 2.75 \times 10^{-5}$ ft. $= 8.4$ microns

For the gas in the chamber:

$C = 2360$ ft./sec.

$\mu = 1.54 \times 10^{-6}$ slug/sec.ft.

$\rho = 4.16 \times 10^{-4}$ slug/ft.^{3}

$\gamma = 1.3$

$T = 2500^\circ K.$

$\nu = 692 \ 1/\text{sec.}$

For the chamber:

$a = 1$ ft.

In substituting the above values in the equations the form of $\mathcal{E}$ was changed to $\mathcal{E} = \frac{3\mu\sigma P_o}{m\nu\gamma P_c}$ by substituting $\gamma \, P_c/\rho$ for c^2. Here, P_c and ρ are the mean chamber pressure and density.

In calculating the magnitudes of the drift velocities, values of $\bar{r}$ of .1, .5, and .9 ft. were chosen for the initial particle position, and since the drift velocities are completely symmetrical, values of φ in the first two quadrants of the cylinder cross section were considered. In particular, angles of 0°, 22.5°, 45°, 67.5°, 90°, and the corresponding angles in the second quadrant were used. Table I shows the results of the drift velocity calculations. In both Table I and Fig. 2 the magnitude of the particle drift velocity is divided by $(P_o/P_c)^2$. Since the amplitude of pressure fluctuations in an unstable rocket chamber is a parameter which varies quite drastically, it is convenient to multiply the values listed in Table I and Fig. 2 by the square of any desired ratio of pressure amplitude to mean chamber pressure to determine the actual drift velocity.

Fig. 2 shows the directions and magnitudes of the drift velocities in the first two quadrants of the cylinder cross section. It is seen that the drift near the wall of the cylinder is greatest at $\varphi = 90^\circ$ which is to be expected since at this value of φ there is a pressure node and, as can be seen in Fig. 1, the gas velocity is entirely in the φ direction. The drift velocities are generally in the radial direction, due to the centrifugal force exerted on the particles by the oscillating gas. The drift velocities are greatest when the initial particle position is at the mid-radius of the cylinder. Since the drift velocity is a second order phenomenon, it is difficult to find any correlation between the drift direction and magnitude, and the pressure and velocity distribution of the oscillating gas in the chamber except for that mentioned above.

To obtain a range of values for the magnitude of the drift velocity, a value of .316 is chosen for P_o/P_c ; that is, a pressure amplitude amounting to 31.6 per cent of the mean chamber pressure. Then $(P_o/P_c)^2$ is 0.1 and with the values shown in Table I, the particle drift velocity varies from .39 to 4 feet per second.

As mentioned earlier, the magnitude of the first order decaying exponential terms in the solution of the cylindrical chamber problem are investigated in Appendix B. For this numerical example, the results show that after just two oscillations the magnitude of the velocity contribution of the decaying exponential terms is less than one-one hundredth of the magnitude of the second order drift velocity.

It is also interesting to determine how far the particles move in the combustion chamber. In .01 seconds, which is close to an upper limit for liquid droplet life times in a rocket chamber (4), the droplets can move from .05 to .48 inches depending on their initial position in the cylinder. This is a very small distance when compared with the size of most rocket chambers.

In comparison, the drift velocity in the rectangular chamber will reach a maximum of 17.4 feet per second at an initial position in the center of the chamber. In .01 seconds the particles could move 2 inches which is considerably greater than in the cylindrical chamber, but still not significant when compared with chamber size or wave length of the oscillation.

Variation in Particle Density. Although it is not the aim of this present work to calculate in detail the variation in fuel-oxidizer ratio which results from the relative drifts of fuel and oxidizer particles, it is of interest to calculate a few representative values for the change of particle density with time. The variation in particle density, n, can be computed directly from the continuity equation for nonsteady flows. The equation, in vector form, is

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\vec{v}) = 0 \quad . \tag{4.12}$$

$\vec{v}$ is the particle drift velocity and $n = n(t, r, \varphi)$ is the droplet density in number per unit volume. If the problem is restricted to the calculation of variations in density over short periods, then the variation of n with time may be approximated by

$$n = n_0(r, \varphi) + n_1(t, r, \varphi) \quad ; \qquad n_1/n_0 \ll 1 \quad .$$

Also, for short periods, the effects of evaporation of the particles may be neglected and hence the particle radius will be constant. With the above restrictions equation 4.12 reduces to

$$\Delta m / m_0 \approx -(\nabla \cdot \vec{v}) \Delta t \quad . \tag{4.13}$$

Values of $\Delta m / m_0$ are calculated for the chamber conditions given in the previous section. The calculation is carried out for the cylindrical chamber at $r/a = 0.1$ and $\varphi = 0^\circ$ and for the rectangular chamber at $\xi/\ell = 0.1$. A time of about 1/5 of the particle life time of 0.01 seconds is assumed and a radius equal to the optimum value for particle motion is used in the calculation.

A 3 per cent reduction in particle concentration is obtained for the cylindrical chamber and a 10 per cent reduction for the rectangular chamber. These values of density variation are the maximum which can be obtained for the chamber conditions used. For the cylindrical chamber, the changes in particle density appear to be too small to cause a large change in oxidizer-fuel ratio and in fact the variations of the order of 5 per cent are much less than the variations which occur because of the injector design.

V. CONCLUSIONS

It has been demonstrated that small particles in an oscillating gas field experience a steady state drift velocity in both rectangular and cylindrical enclosures. This result is based on the assumptions that (1) the fluid medium is at rest and in a uniform state except for weak isoenergetic fluctuations in velocity and state properties, (2) the particles are spherical and experience Stokes law forces, and (3) the particles have negligible effect upon the fluid motion.

The particle drift velocity is a maximum for certain optimum sized particles. The optimum particle radius is given by $\sigma^* = \sqrt{\frac{9\mu}{4\rho_p \nu \pi}}$ where ν is the frequency of acoustic oscillations in the combustion chamber.

The maximum particle drift velocity is independent of the chamber size although the optimum particle radius is a function of the acoustic frequency and hence of the chamber dimension.

The magnitude of the particle drift velocity in a typical liquid propellant rocket chamber varies from .4 to 4 feet per second and in an average particle or droplet life time the actual movement will amount to just a few tenths of an inch.

It is concluded that, for rocket chambers of reasonable size, the drift movement of particles has a small, if not negligible, effect on the distribution of fuel or oxidizer in rocket combustion chambers.

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TABLE I

PARTICLE DRIFT VELOCITIES IN THE CROSS SECTION OF A CYLINDRICAL ENCLOSURE

$\bar{r}/a$	$\bar{\varphi}$ (degrees)	$v_r/(P_0/P_c)^2$ (ft/sec)	$v_\varphi/(P_0/P_c)^2$ (ft/sec)	$v/(P_0/P_c)^2$ (ft/sec)
.1	0	11.8	0	11.8
.1	22.5	10.6	-.3	10.6
.1	45	7.5	-.4	7.6
.1	67.5	4.5	-.2	4.5
.1	90	3.9	0	3.9
.5	0	37.4	0	37.4
.5	22.5	34.4	-5.3	34.8
.5	45	27.1	-3.7	27.4
.5	67.5	19.7	-2.6	19.9
.5	90	16.7	0	16.7
.9	0	10.9	0	10.9
.9	22.5	12.2	-6.9	14.0
.9	45	15.4	-9.8	18.2
.9	67.5	18.5	-6.9	19.8
.9	90	19.9	0	19.9

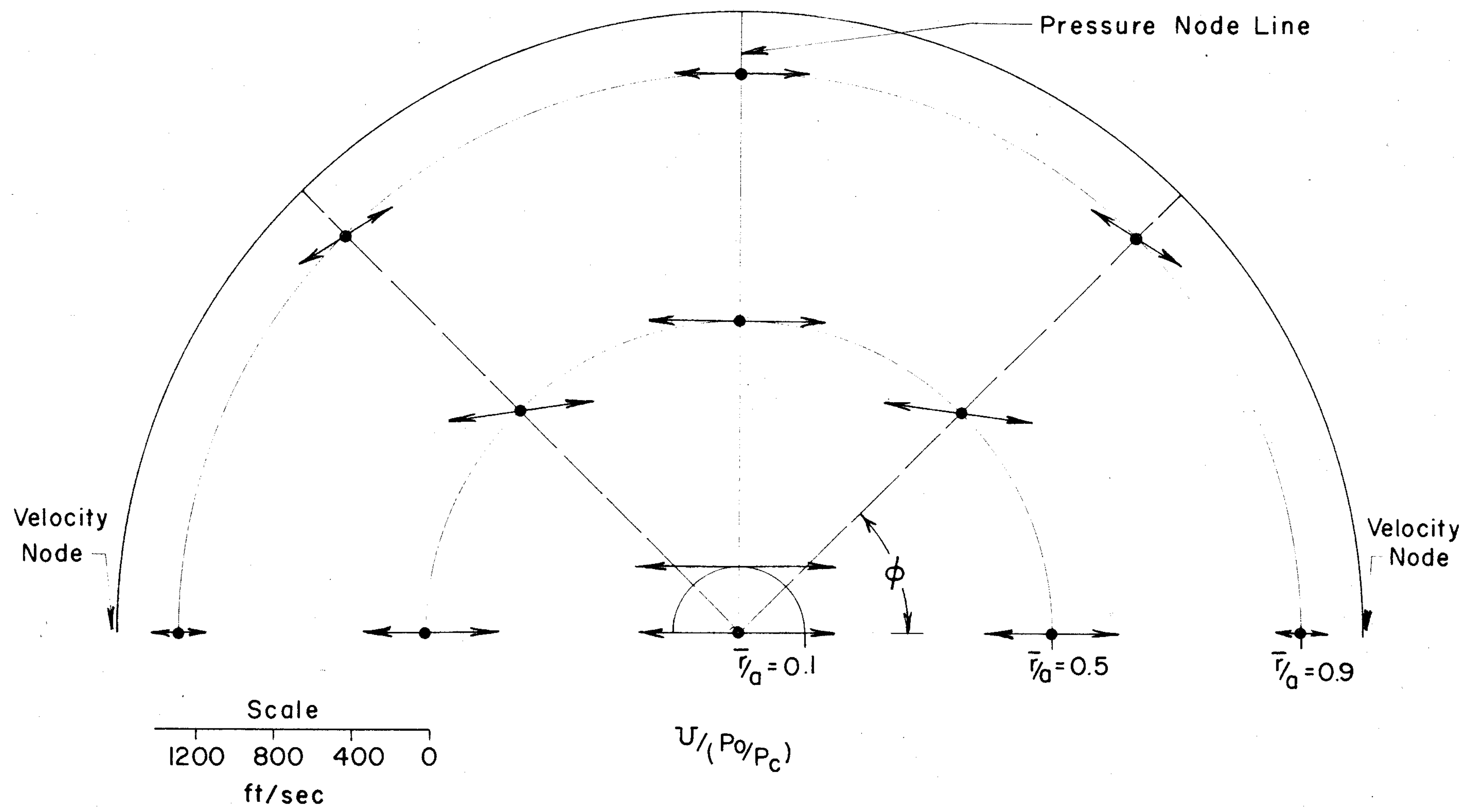

FIG. 1 - GAS VELOCITY MAP OF GAS OSCILLATING IN THE FIRST TRANSVERSE MODE IN A CYLINDRICAL ENCLOSURE

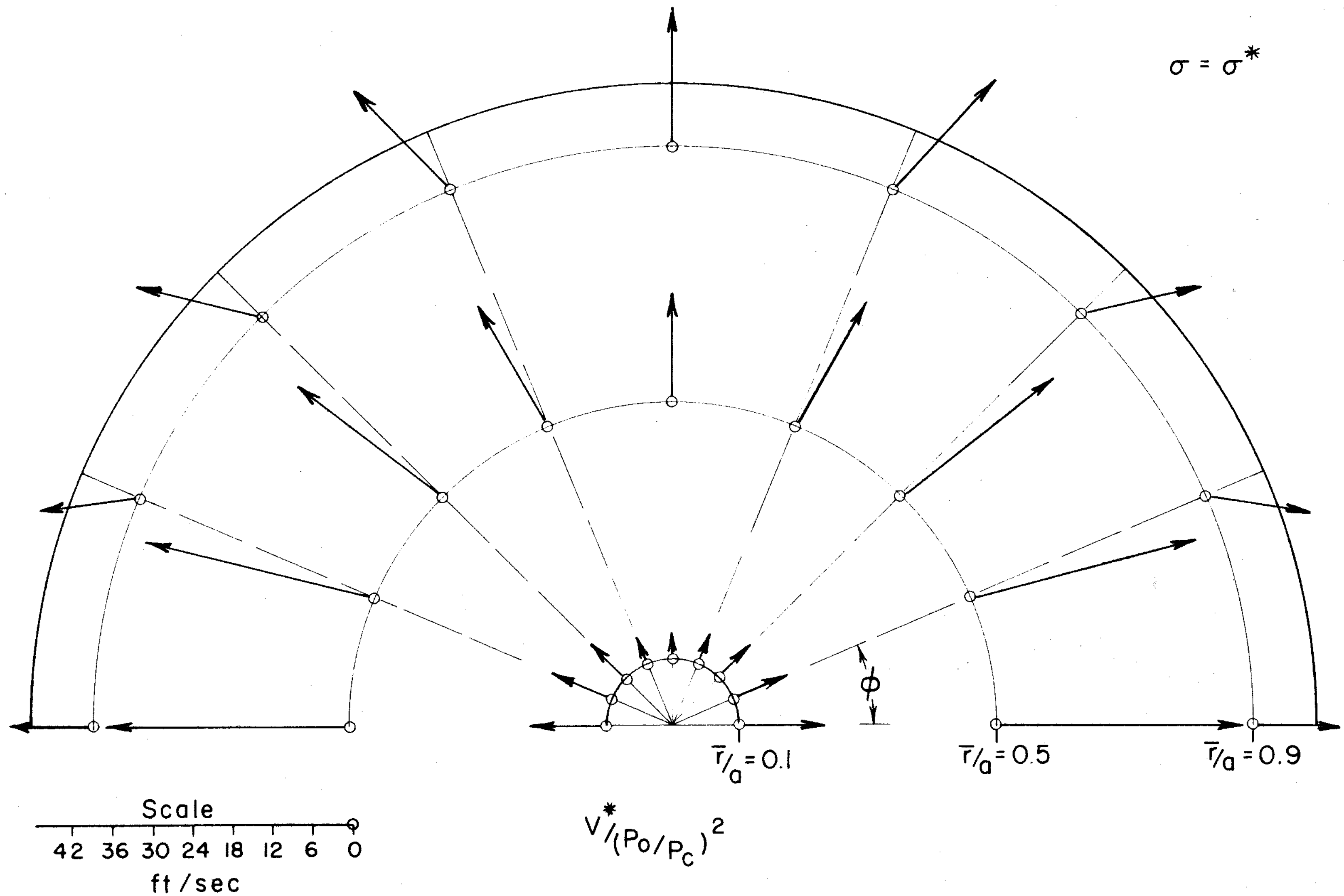

FIG. 2 - DRIFT VELOCITY MAP FOR PARTICLES IN A CYLINDRICAL ENCLOSURE CORRESPONDING TO GAS MOTION OF FIGURE I

APPENDIX A

SOLUTIONS TO EQUATIONS 3.16 TO 3.21

In this appendix solutions are obtained for equations 3.16 to 3.21 which are developed in Chapter III.

If a particle is considered at rest at $\tau = 0$ at a point $\bar{Z}$, $\bar{\Phi}$, then the initial conditions for the equations are:

$$\left.\begin{aligned} & Z_0(0) = \bar{Z}\,,\ \ \phi_0(0) = \bar{\Phi}\,,\ \ \frac{dZ_0}{d\tau}(0) = \frac{d\phi_0}{d\tau}(0) = 0 \\ & Z_1(0) = \frac{dZ_1}{d\tau}(0) = \phi_1(0) = \frac{d\phi_1}{d\tau}(0) = 0 \\ & Z_2(0) = \frac{dZ_2}{d\tau}(0) = \phi_2(0) = \frac{d\phi_2}{d\tau}(0) = 0 \end{aligned}\right\} \qquad (1)$$

<u>Zero Order Equations.</u> The zero order equations, 3.16 and 3.19 are:

$$2Z_0\frac{dZ_0}{d\tau}\frac{d\phi_0}{d\tau} + Z_0^2\frac{d^2\phi_0}{d\tau^2} + Z_0^2 K\frac{d\phi_0}{d\tau} = 0 \qquad (2)$$

$$Z_0\frac{d^2Z_0}{d\tau^2} + K Z_0\frac{dZ_0}{d\tau} - Z_0^2\left(\frac{d\phi_0}{d\tau}\right)^2 = 0 \qquad (3)$$

Z_0 = constant = $\bar{Z}$ and ϕ_0 = constant = $\bar{\phi}$ satisfy equations 2 and 3 and will be used as the solution.

<u>First Order Equations.</u> The first order equations reduce to the following upon substituting $\bar{Z}$ and $\bar{\Phi}$ into 3.17 and 3.20:

<u>$\vec{i}_\varphi$ component:</u>

$$Z^2\frac{d^2\phi_1}{d\tau^2} + \bar{Z}^2 K\frac{d\phi_1}{d\tau} = \left[J_1(\pi\alpha\bar{Z})\sin\bar{\Phi}\,\sin\frac{2\pi\nu a\tau}{c}\right] \qquad (4)$$

$\vec{i}_r$ component:

$$\frac{d^2 Z_1}{d\mathcal{T}^2} + K\frac{dZ_1}{d\mathcal{T}} = \left[\frac{\pi\alpha}{2}\sin\frac{2\pi\nu a\mathcal{T}}{c}\cos\bar{\phi}\left[J_2(\pi\alpha\bar{Z}) - J_0(\pi\alpha\bar{Z})\right]\right] \qquad (5)$$

The solution for ϕ_1 is obtained first and then the solution for Z_1 is found.

If:
$$D \equiv \left[\frac{J_1(\pi\alpha\bar{Z})}{\bar{Z}^2}\sin\bar{\phi}\right] \qquad (6)$$

the $\vec{i}_\varphi$ component equation may be rewritten as

$$\frac{d^2\phi_1}{d\mathcal{T}^2} + K\frac{d\phi_1}{d\mathcal{T}} = \left[D\sin\frac{2\pi\nu a\mathcal{T}}{c}\right]. \qquad (7)$$

The solution to 7 is determined by the usual technique of finding the complementary and a particular solution and adding the two. The constants appearing in these solutions are then eliminated by use of the initial conditions.

The complementary solution to equation 7 is:

$$\phi_{1c} = \left[C_1 + C_2 e^{-K\mathcal{T}}\right] \qquad (8)$$

For the particular solution try:

$$\phi_{1p} = \left[C_3\sin\frac{2\pi\nu a\mathcal{T}}{c} + C_4\cos\frac{2\pi\nu a\mathcal{T}}{c}\right] \qquad (9)$$

In this appendix, the subscript c will pertain to the complementary solution of the equation and the subscript p will pertain to the particular solution.

The first and second derivatives of equation 9 are:

$$\phi'_{1p} = \left[\frac{2\pi\nu a}{c}C_3\cos\frac{2\pi\nu a\mathcal{T}}{c} - \frac{2\pi\nu a}{c}C_4\sin\frac{2\pi\nu a\mathcal{T}}{c}\right] \qquad (10)$$

$$\phi''_{1p} = \left[-\left(\frac{2\pi\nu a}{c}\right)^2 C_3\sin\frac{2\pi\nu a\mathcal{T}}{c} - \left(\frac{2\pi\nu a}{c}\right)^2 C_4\cos\frac{2\pi\nu a\mathcal{T}}{c}\right]. \qquad (11)$$

Substituting 10 and 11 into equation 7 results in:

$$-\left(\frac{2\pi\nu a}{c}\right)^2 C_3 \sin\frac{2\pi\nu a\tau}{c} - \left(\frac{2\pi\nu a}{c}\right)^2 C_4 \cos\frac{2\pi\nu a\tau}{c}$$

$$+K\frac{2\pi\nu a}{c} C_3 \cos\frac{2\pi\nu a\tau}{c} - K\frac{2\pi\nu a}{c} C_4 \sin\frac{2\pi\nu a\tau}{c} = \left[D\sin\frac{2\pi\nu a\tau}{c}\right]. \quad (12)$$

Equating coefficients of like terms gives:

$$-\left(\frac{2\pi\nu a}{c}\right)^2 C_3 - K\frac{2\pi\nu a}{c} C_4 = D, \quad (13)$$

and

$$K\frac{2\pi\nu a}{c} C_3 - \left(\frac{2\pi\nu a}{c}\right)^2 C_4 = 0. \quad (14)$$

The solution of equations 13 and 14 is:

$$C_3 = \frac{-D}{K^2+\left(\frac{2\pi\nu a}{c}\right)^2}, \quad C_4 = \frac{-KD}{\frac{2\pi\nu a}{c}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}. \quad (15)$$

The complete solution for ϕ_1 is then:

$$\phi_1 = \left[C_1 + C_2 e^{-K\tau} + \frac{-D\sin\frac{2\pi\nu a\tau}{c}}{K^2+\left(\frac{2\pi\nu a}{c}\right)^2} - \frac{KD\cos\frac{2\pi\nu a\tau}{c}}{\frac{2\pi\nu a}{c}\left[K^2+\left(\frac{2\pi\nu a}{2}\right)^2\right]}\right] \quad (16)$$

To determine C_1 and C_2 the initial conditions are used.

$$\phi_1(0) = 0 = \left[C_1 + C_2 - \frac{KD}{\frac{2\pi\nu a}{c}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}\right] \quad (17)$$

$$\phi_1'(0) = 0 = \left[-KC_2 - \frac{D\frac{2\pi\nu a}{c}}{K^2+\left(\frac{2\pi\nu a}{c}\right)^2}\right] \quad (18)$$

The solution of 17 and 18 is:

$$\left.\begin{aligned} C_2 &= \frac{-D\,\frac{2\pi\nu a}{c}}{k\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]} \\ C_1 &= \frac{D\,\frac{2\pi\nu a}{c}}{k\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]} + \frac{KD}{\frac{2\pi\nu a}{c}\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]} \end{aligned}\right\} \qquad (19)$$

The complete solution for ϕ_1 is then:

$$\phi_1 = \left[\frac{D\,\frac{2\pi\nu a}{c}}{k\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]} + \frac{KD}{\frac{2\pi\nu a}{c}\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]} - \frac{D\,\frac{2\pi\nu a}{c}\,e^{-K\mathcal{T}}}{K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]} \right.$$

$$\left. - \frac{D\sin\frac{2\pi\nu a\mathcal{T}}{c}}{k^2+\left(\frac{2\pi\nu a}{c}\right)^2} - \frac{KD\cos\frac{2\pi\nu a\mathcal{T}}{c}}{\frac{2\pi\nu a}{c}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}\right] . \qquad (20)$$

A similar process to that carried out above leads to the solution of Z_1 from equation 5.

$$\text{If} \qquad E = \left[\frac{\pi\alpha}{2}\left[J_2(\pi\alpha\bar{Z})\cos\bar{\Phi} - J_0(\pi\alpha\bar{Z})\cos\bar{\Phi}\right]\right] \qquad (21)$$

then equation 5 reduces to:

$$\frac{d^2 Z_1}{d\mathcal{T}^2} + K\frac{dZ_1}{d\mathcal{T}} = \left[E\sin\frac{2\pi\nu a\mathcal{T}}{c}\right] \qquad (22)$$

Since this is exactly the form of equation 7 and since the initial conditions on ϕ_1 and Z_1 are identical, the solution can be written down immediately by analogy and it is given by:

$$Z_1 = \left[\frac{E\frac{2\pi\nu a}{c}}{K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]} + \frac{K E}{\frac{2\pi\nu a}{c}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]} - \frac{E\frac{2\pi\nu a}{c}e^{-K\mathcal{T}}}{K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}\right.$$

$$\left. - \frac{E\sin\frac{2\pi\nu a\mathcal{T}}{c}}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]} - \frac{K E\cos\frac{2\pi\nu a\mathcal{T}}{c}}{\frac{2\pi\nu a}{c}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}\right] \tag{23}$$

Second Order Equations. The second order $\vec{i}_r$ component equation, 3.18, reduces to:

$$\frac{d^2 Z_2}{d\mathcal{T}^2} + K\frac{dZ_2}{d\mathcal{T}} - \overline{Z}\left(\frac{d\phi_1}{d\mathcal{T}}\right)^2 = \left[Z_1\left[\frac{(\pi\alpha)^2}{2}2\,J_1(\pi\alpha\overline{z})\cos\overline{\Phi}\right.\right.$$

$$\left. - \frac{\pi\alpha}{2}J_2(\pi\alpha\overline{z})\cos\overline{\Phi}\right]\sin\frac{2\pi\nu a\mathcal{T}}{c} + \phi_1\left[\frac{\pi\alpha}{2}J_0(\pi\alpha\overline{z})\sin\overline{\Phi}\right.$$

$$\left.\left. - \frac{\pi\alpha}{2}J_2(\pi\alpha\overline{z})\sin\overline{\Phi}\right]\sin\frac{2\pi\nu a\mathcal{T}}{c}\right] \tag{24}$$

when zero order solutions are substituted.

If

$$F \equiv \left[(\pi\alpha)^2 J_1(\pi\alpha\overline{z})\cos\overline{\Phi} - \frac{\pi\alpha}{2}J_2(\pi\alpha\overline{z})\cos\overline{\Phi}\right] \tag{25}$$

and

$$G \equiv \left[\frac{\pi\alpha}{2}J_0(\pi\alpha\overline{z})\sin\overline{\Phi} - \frac{\pi\alpha}{2}J_2(\pi\alpha\overline{z})\sin\overline{\Phi}\right] \tag{26}$$

then equation 24 may be expressed as:

$$\frac{d^2 Z_2}{d\mathcal{T}^2} + K\frac{dZ_2}{d\mathcal{T}} - \overline{Z}\left(\frac{d\phi_1}{d\mathcal{T}}\right)^2 = \left[Z_1 F\sin\frac{2\pi\nu a\mathcal{T}}{c} + \phi_1 G\sin\frac{2\pi\nu a\mathcal{T}}{c}\right]. \tag{27}$$

The values for $\frac{d\phi_1}{d\mathcal{T}}$, Z_1, and ϕ_1 can be obtained from the solutions given in equations 20 and 23. Thus, from equation 20:

$$\frac{d\phi_1}{d\mathcal{T}} = \left[\frac{D\left(\frac{2\pi\nu a}{c}\right)e^{-k\mathcal{T}} - \frac{2\pi\nu a}{c} D \cos\frac{2\pi\nu a\mathcal{T}}{c} + kD \sin\frac{2\pi\nu a\mathcal{T}}{c}}{k^2 + \left(\frac{2\pi\nu a}{c}\right)^2}\right] \tag{28}$$

$$\left(\frac{d\phi_1}{d\mathcal{T}}\right)^2 = \left[\frac{D^2\left(\frac{2\pi\nu a}{c}\right)^2 e^{-2k\mathcal{T}} - 2D^2\left(\frac{2\pi\nu a}{c}\right)^2 \cos\frac{2\pi\nu a\mathcal{T}}{c}\, e^{-k\mathcal{T}}}{\left[k^2 + \left(\frac{2\pi\nu a}{c}\right)^2\right]^2}\right.$$

$$+ \frac{2kD^2\frac{2\pi\nu a}{c} e^{-k\mathcal{T}} \sin\frac{2\pi\nu a\mathcal{T}}{c} + \left(\frac{2\pi\nu a}{c}\right)^2 D^2 \cos^2\frac{2\pi\nu a\mathcal{T}}{c}}{\left[k^2 + \left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$\left. - \frac{2kD^2\frac{2\pi\nu a}{c}\sin\frac{2\pi\nu a\mathcal{T}}{c}\cos\frac{2\pi\nu a\mathcal{T}}{c} + k^2D^2\sin^2\frac{2\pi\nu a\mathcal{T}}{c}}{\left[k^2 + \left(\frac{2\pi\nu a}{c}\right)^2\right]^2}\right] \tag{29}$$

Substituting 29, 20 and 23 into equation 27 gives:

$$\frac{d^2 Z_2}{d\mathcal{T}^2} + K\frac{dZ_2}{d\mathcal{T}} = \left[\frac{\bar{Z} D^2\left(\frac{2\pi\nu a}{c}\right)^2 e^{-2k\mathcal{T}} - 2\bar{Z}D^2\left(\frac{2\pi\nu a}{c}\right)^2 e^{-k\mathcal{T}}\cos\frac{2\pi\nu a\mathcal{T}}{c}}{\left[k^2 + \left(\frac{2\pi\nu a}{c}\right)^2\right]^2}\right.$$

$$+ \frac{2\bar{Z}kD^2\frac{2\pi\nu a}{c} e^{-k\mathcal{T}}\sin\frac{2\pi\nu a\mathcal{T}}{c} + \bar{Z}\left(\frac{2\pi\nu a}{c}\right)^2 D^2\cos^2\frac{2\pi\nu a\mathcal{T}}{c}}{\left[k^2 + \left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$- \frac{2\bar{Z}kD^2\frac{2\pi\nu a}{c}\sin\frac{2\pi\nu a\mathcal{T}}{c}\cos\frac{2\pi\nu a\mathcal{T}}{c} + \bar{Z}k^2D^2\sin^2\frac{2\pi\nu a\mathcal{T}}{c}}{\left[k^2 + \left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$+ \frac{EF\frac{2\pi\nu a}{c}\sin\frac{2\pi\nu a\mathcal{T}}{c}}{k\left[k^2 + \left(\frac{2\pi\nu a}{c}\right)^2\right]} + \frac{kEF\sin\frac{2\pi\nu a\mathcal{T}}{c}}{\frac{2\pi\nu a}{c}\left[k^2 + \left(\frac{2\pi\nu a}{c}\right)^2\right]} - \frac{EF\frac{2\pi\nu a}{c}e^{-k\mathcal{T}}\sin\frac{2\pi\nu a\mathcal{T}}{c}}{k\left[k^2 + \left(\frac{2\pi\nu a}{c}\right)^2\right]}$$

$$- \frac{EF\sin^2\frac{2\pi\nu a\mathcal{T}}{c}}{\left[k^2 + \left(\frac{2\pi\nu a}{c}\right)^2\right]} - \frac{DG\frac{2\pi\nu a}{c}e^{-k\mathcal{T}}\sin\frac{2\pi\nu a\mathcal{T}}{c}}{k\left[k^2 + \left(\frac{2\pi\nu a}{c}\right)^2\right]} + \frac{DG\frac{2\pi\nu a}{c}\sin\frac{2\pi\nu a\mathcal{T}}{c}}{k\left[k^2 + \left(\frac{2\pi\nu a}{c}\right)^2\right]}$$

$$- \frac{KEF\cos\frac{2\pi\nu a\mathcal{T}}{c}\sin\frac{2\pi\nu a\mathcal{T}}{c}}{\frac{2\pi\nu a}{c}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]} + \frac{KDG\sin\frac{2\pi\nu a\mathcal{T}}{c}}{\frac{2\pi\nu a}{c}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]} - \frac{DG\sin^2\frac{2\pi\nu a\mathcal{T}}{c}}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}$$

$$\left. - \frac{KDG\cos\frac{2\pi\nu a\mathcal{T}}{c}\sin\frac{2\pi\nu a\mathcal{T}}{c}}{\frac{2\pi\nu a}{c}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}\right] . \qquad (30)$$

Equation 30 can be simplified by using the following trigonometric identities:

$$\left.\begin{aligned} \sin\frac{2\pi\nu a\mathcal{T}}{c}\cos\frac{2\pi\nu a\mathcal{T}}{c} &\equiv \tfrac{1}{2}\sin\frac{4\pi\nu a\mathcal{T}}{c} \\ \sin^2\frac{2\pi\nu a\mathcal{T}}{c} &\equiv \left(\tfrac{1}{2}-\tfrac{1}{2}\cos\frac{4\pi\nu a\mathcal{T}}{c}\right) \\ \cos^2\frac{2\pi\nu a\mathcal{T}}{c} &\equiv \left(\tfrac{1}{2}+\tfrac{1}{2}\cos\frac{4\pi\nu a\mathcal{T}}{c}\right) \end{aligned}\right\} \qquad (31)$$

The results of substituting the above identities into equation 30 is:

$$\frac{d^2z_2}{d\mathcal{T}^2} + k\frac{dz_2}{d\mathcal{T}} = \left[\frac{\bar{z}D^2\left(\frac{2\pi\nu a}{c}\right)^2 e^{-2k\mathcal{T}} - 2\bar{z}D^2\left(\frac{2\pi\nu a}{c}\right)^2 e^{-k\mathcal{T}}\cos\frac{2\pi\nu a\mathcal{T}}{c}}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}\right.$$

$$+ \frac{2\bar{z}KD^2\left(\frac{2\pi\nu a}{c}\right)e^{-k\mathcal{T}}\sin\frac{2\pi\nu a\mathcal{T}}{c} + \frac{\bar{z}}{2}\left(\frac{2\pi\nu a}{c}\right)^2 D^2 + \frac{\bar{z}}{2}\left(\frac{2\pi\nu a}{c}\right)^2 D^2\cos\frac{4\pi\nu a\mathcal{T}}{c}}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$\frac{-\bar{z}KD^2\left(\frac{2\pi\nu a}{c}\right)\sin\frac{4\pi\nu a\mathcal{T}}{c} + \frac{\bar{z}}{2}K^2D^2 - \frac{\bar{z}}{2}K^2D^2\cos\frac{4\pi\nu a\mathcal{T}}{c}}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}$$

$$+ \frac{(FE+DG)\left(\frac{2\pi\nu a}{c}\right)\sin\frac{2\pi\nu a\mathcal{T}}{c}}{K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]} + \frac{(FE+DG)\,K\sin\frac{2\pi\nu a\mathcal{T}}{c}}{\frac{2\pi\nu a}{c}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}$$

$$- \frac{(FE+DG)\left(\frac{2\pi\nu a}{c}\right)e^{-k\mathcal{T}}\sin\frac{2\pi\nu a\mathcal{T}}{c}}{K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]} - \frac{\frac{1}{2}(FE+DG) + \frac{1}{2}(FE+DG)\cos\frac{4\pi\nu a\mathcal{T}}{c}}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}$$

$$- \frac{(FE + DG)\, k/2 \, \sin \frac{4\pi\nu a \tau}{c}}{\frac{2\pi\nu a}{c}\left[k^2 + \left(\frac{2\pi\nu a}{c}\right)^2 \right]} \Bigg] \qquad (32)$$

Equation 32 is the linear second order differential equation which must be solved. The solution is obtained in the same manner as was used in the previous section to obtain values of the first order functions ϕ_1 and Z_1.

The solution of the complementary function is:

$$Z_{2c} = \left[C_1 + C_2 e^{-k\tau} \right] \qquad (33)$$

and the trial particular solution is:

$$Z_{2p} = \Big[C_3 \tau + C_4 e^{-2k\tau} + C_5 e^{-k\tau} \cos \frac{2\pi\nu a \tau}{c} + C_6 e^{-k\tau} \sin \frac{2\pi\nu a\tau}{c}$$
$$+ C_7 \cos \frac{4\pi\nu a \tau}{c} + C_8 \sin \frac{4\pi\nu a\tau}{c} + C_9 \cos \frac{2\pi\nu a\tau}{c} + C_{10} \sin \frac{2\pi\nu a\tau}{c} \Big]. \qquad (34)$$

The first and second derivatives of Z_{2p} are:

$$Z'_{2p} = \Big[C_3 - 2kC_4 e^{-2k\tau} - kC_5 e^{-k\tau} \cos \frac{2\pi\nu a\tau}{c} - \frac{2\pi\nu a}{c} C_5 e^{-k\tau} \sin \frac{2\pi\nu a\tau}{c}$$
$$- kC_6 e^{-k\tau} \sin \frac{2\pi\nu a\tau}{c} + \frac{2\pi\nu a}{c} C_6 e^{-k\tau} \cos \frac{2\pi\nu a\tau}{c} - \frac{4\pi\nu a}{c} C_7 \sin \frac{4\pi\nu a\tau}{c}$$
$$+ \frac{4\pi\nu a}{c} C_8 \cos \frac{4\pi\nu a\tau}{c} - \frac{2\pi\nu a}{c} C_9 \sin \frac{2\pi\nu a\tau}{c} + \frac{2\pi\nu a}{c} C_{10} \cos \frac{2\pi\nu a\tau}{c} \Big] \qquad (35)$$

$$Z''_{2p} = \Big[4k^2 C_4 e^{-2k\tau} + k^2 C_5 e^{-k\tau} \cos \frac{2\pi\nu a\tau}{c} + \frac{2\pi\nu a}{c} k C_5 e^{-k\tau} \sin \frac{2\pi\nu a\tau}{c}$$

$$+ K\frac{2\pi\nu a}{c} C_5 e^{-K\mathcal{J}} \sin\frac{2\pi\nu a\mathcal{J}}{c} - \left(\frac{2\pi\nu a}{c}\right)^2 C_5 e^{-K\mathcal{J}} \cos\frac{2\pi\nu a\mathcal{J}}{c}$$

$$+ K^2 C_6 e^{-K\mathcal{J}} \sin\frac{2\pi\nu a\mathcal{J}}{c} - \frac{2\pi\nu a}{c} K C_6 e^{-K\mathcal{J}} \cos\frac{2\pi\nu a\mathcal{J}}{c} - K\frac{2\pi\nu a}{c} C_6 e^{-K\mathcal{J}} \cos\frac{2\pi\nu a\mathcal{J}}{c}$$

$$-\left(\frac{2\pi\nu a}{c}\right)^2 C_6 e^{-K\mathcal{J}} \sin\frac{2\pi\nu a\mathcal{J}}{c} - \left(\frac{4\pi\nu a}{c}\right)^2 C_7 \cos\frac{4\pi\nu a\mathcal{J}}{c} - \left(\frac{4\pi\nu a}{c}\right)^2 C_8 \sin\frac{4\pi\nu a\mathcal{J}}{c}$$

$$-\left(\frac{2\pi\nu a}{c}\right)^2 C_9 \cos\frac{2\pi\nu a\mathcal{J}}{c} - \left(\frac{2\pi\nu a}{c}\right)^2 C_{10} \sin\frac{2\pi\nu a\mathcal{J}}{c} \Bigg] . \qquad (36)$$

Substituting 35 and 36 into equation 32, and equating coefficients of like term results in the following evaluation of constants:

$$K C_3 = \left[\frac{\frac{\bar{Z}}{2} D^2 - \frac{1}{2}(FE + DG)}{\left[K^2 + \left(\frac{2\pi\nu a}{c}\right)^2\right]} \right] \qquad (37)$$

$$2K^2 C_4 = \left[\frac{\bar{Z} D^2 \left(\frac{2\pi\nu a}{c}\right)^2}{\left[K^2 + \left(\frac{2\pi\nu a}{c}\right)^2\right]^2} \right] \qquad (38)$$

$$K\left(\frac{2\pi\nu a}{c}\right) C_5 - \left(\frac{2\pi\nu a}{c}\right)^2 C_6 = \left[\frac{2\bar{Z} K D^2 \left(\frac{2\pi\nu a}{c}\right)}{\left[K^2 + \left(\frac{2\pi\nu a}{c}\right)^2\right]^2} - \frac{(FE + DG)\frac{2\pi\nu a}{c}}{K\left[K^2 + \left(\frac{2\pi\nu a}{c}\right)^2\right]} \right] \qquad (39)$$

$$-\left(\frac{2\pi\nu a}{c}\right)^2 C_5 - K\left(\frac{2\pi\nu a}{c}\right) C_6 = \left[\frac{-2\bar{Z} D^2 \left(\frac{2\pi\nu a}{c}\right)^2}{\left[K^2 + \left(\frac{2\pi\nu a}{c}\right)^2\right]^2} \right] \qquad (40)$$

$$-\left(\frac{4\pi\nu a}{c}\right)^2 C_7 + \frac{4\pi\nu a}{c} K C_8 = \left[\frac{\frac{\bar{Z}}{2}\left(\frac{2\pi\nu a}{c}\right)^2 D^2 - \frac{\bar{Z}}{2} K^2 D^2}{\left[K^2 + \left(\frac{2\pi\nu a}{c}\right)^2\right]^2} + \frac{\frac{1}{2}(FE + DG)}{\left[K^2 + \left(\frac{2\pi\nu a}{c}\right)^2\right]} \right] \qquad (41)$$

$$-\frac{4\pi\nu a}{c} K C_7 - \left(\frac{4\pi\nu a}{c}\right)^2 C_8 = \left[\frac{-\bar{Z} K D^2 \left(\frac{2\pi\nu a}{c}\right)}{\left[K^2 + \left(\frac{2\pi\nu a}{c}\right)^2\right]^2} - \frac{(FE + DG)\,K/2}{\frac{2\pi\nu a}{c}\left[K^2 + \left(\frac{2\pi\nu a}{c}\right)^2\right]} \right] \qquad (42)$$

$$-\left(\frac{2\pi\nu a}{c}\right)^2 C_9 + \frac{2\pi\nu a}{c} K C_{10} = 0 \qquad (43)$$

$$-\frac{2\pi\nu a}{c} K C_9 - \left(\frac{2\pi\nu a}{c}\right)^2 C_{10} = \left[\frac{(FE+DG)\frac{2\pi\nu a}{c}}{k\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]} + \frac{(FE+DG)\,k}{\frac{2\pi\nu a}{c}\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}\right]. \tag{44}$$

Solving equations 37 to 44 for the constants gives:

$$C_3 = \left[\frac{\frac{Z}{2}D^2 - \frac{1}{2}(FE+DG)}{k\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}\right] \qquad C_4 = \left[\frac{Z D^2\left(\frac{2\pi\nu a}{c}\right)^2}{2k^2\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}\right] \tag{45}$$

$$C_5 = \left[\frac{2 Z D^2 - (FE+DG)}{\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}\right] \qquad C_6 = \left[\frac{(FE+DG)\left(\frac{2\pi\nu a}{c}\right)}{k\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}\right] \tag{46}$$

$$C_7 = \left[\frac{-\frac{Z}{2}\left(\frac{2\pi\nu a}{c}\right)^2 D^2 \frac{4\pi\nu a}{c} + 2 Z k^2 D^2 \frac{2\pi\nu a}{c}}{\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[\frac{4\pi\nu a}{c}\left(k^2+\left(\frac{4\pi\nu a}{c}\right)^2\right)\right]} - \frac{\frac{2\pi\nu a}{c}(FE+DG)}{\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\left[\frac{4\pi\nu a}{c}\left(k^2+\left(\frac{4\pi\nu a}{c}\right)^2\right)\right]}\right.$$

$$\left. + \frac{\frac{k^2}{2}(FE+DG)}{\frac{2\pi\nu a}{c}\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\left[\frac{4\pi\nu a}{c}\left(k^2+\left(\frac{4\pi\nu a}{c}\right)^2\right)\right]}\right] \tag{47}$$

$$C_8 = \left[\frac{\frac{5}{2} Z\left(\frac{2\pi\nu a}{c}\right)^2 k D^2 - \frac{Z}{2} k^3 D^2}{\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2 \frac{4\pi\nu a}{c}\left[k^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]} + \frac{\frac{3}{2} k\,(FE+DG)}{\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\frac{4\pi\nu a}{c}\left[k^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}\right] \tag{48}$$

$$C_9 = \left[\frac{-(FE+DG)}{\left(\frac{2\pi\nu a}{c}\right)^2\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}\right] \qquad C_{10} = \left[\frac{-(FE+DG)}{k\left(\frac{2\pi\nu a}{c}\right)\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}\right]. \tag{49}$$

The complete solution for Z_2 can now be written as:

$$Z_2 = \left[C_1 + C_2 e^{-k\mathcal{J}} + \frac{\frac{Z}{2}D^2\mathcal{J} - \frac{1}{2}(FE+DG)\mathcal{J}}{k\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]} + \frac{Z D^2\left(\frac{2\pi\nu a}{c}\right)^2 e^{-2k\mathcal{J}}}{2k^2\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}\right.$$

$$+ \frac{\left[2 Z D^2 - (FE+DG)\right] e^{-k\mathcal{J}}\cos\frac{2\pi\nu a\mathcal{J}}{c}}{\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2} + \frac{(FE+DG)\left(\frac{2\pi\nu a}{c}\right)e^{-k\mathcal{J}}\sin\frac{2\pi\nu a\mathcal{J}}{c}}{k\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$+ \frac{\left[-\frac{Z}{2}\left(\frac{2\pi\nu a}{c}\right)^2 D^2 \frac{4\pi\nu a}{c} + 2 Z k^2 D^2 \frac{2\pi\nu a}{c}\right]\cos\frac{4\pi\nu a\mathcal{J}}{c}}{\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2 \frac{4\pi\nu a}{c}\left[k^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]} - \frac{\frac{2\pi\nu a}{c}(FE+DG)\cos\frac{4\pi\nu a\mathcal{J}}{c}}{\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\frac{4\pi\nu a}{c}\left[k^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}$$

$$+\frac{\frac{K^2}{2}(FE+DG)\cos\frac{4\pi\nu a\sigma}{c}}{\frac{2\pi\nu a}{c}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\frac{4\pi\nu a}{c}\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}+\frac{\left[\frac{5}{2}\bar{Z}\left(\frac{2\pi\nu a}{c}\right)^2 K D^2-\frac{\bar{Z}}{2}K^3D^2\right]\sin\frac{4\pi\nu a\sigma}{c}}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left(\frac{4\pi\nu a}{c}\right)\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}$$

$$+\frac{\frac{3}{2}K(FE+DG)\sin\frac{4\pi\nu a\sigma}{c}}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\frac{4\pi\nu a}{c}\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}-\frac{(FE+DG)\cos\frac{2\pi\nu a\sigma}{c}}{\left(\frac{2\pi\nu a}{c}\right)^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}-\frac{(FE+DG)\sin\frac{2\pi\nu a\sigma}{c}}{K\left(\frac{2\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}\Bigg]. \quad (50)$$

To determine C_1 and C_2 the initial conditions are substituted into 50:

$$Z_2(0)=0=\Bigg[C_1+C_2+\frac{\bar{Z}D^2\left(\frac{2\pi\nu a}{c}\right)^2}{2K^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}+\frac{2\bar{Z}D^2-(FE+DG)}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$+\frac{\left[-\frac{\bar{Z}}{2}\left(\frac{2\pi\nu a}{c}\right)^2D^2\frac{4\pi\nu a}{c}+2\bar{Z}K^2D^2\frac{2\pi\nu a}{c}\right]}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left(\frac{4\pi\nu a}{c}\right)\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}-\frac{\frac{2\pi\nu a}{c}(FE+DG)}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\frac{4\pi\nu a}{c}\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}$$

$$+\frac{\frac{K^2}{2}(FE+DG)}{\frac{2\pi\nu a}{c}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\frac{4\pi\nu a}{c}\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}-\frac{(FE+DG)}{\left(\frac{2\pi\nu a}{c}\right)^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}\Bigg] \quad (51)$$

$$Z_2'(0)=0=\Bigg[-KC_2+\frac{\frac{\bar{Z}}{2}D^2-\frac{1}{2}(FE+DG)}{K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}-\frac{\bar{Z}D^2\left(\frac{2\pi\nu a}{c}\right)^2}{K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$-\frac{K\left[2\bar{Z}D^2-(FE+DG)\right]}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}+\frac{(FE+DG)\left(\frac{2\pi\nu a}{c}\right)^2}{K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}+\frac{\left[\frac{5}{2}\bar{Z}\left(\frac{2\pi\nu a}{c}\right)^2KD^2-\frac{\bar{Z}}{2}K^3D^2\right]}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}$$

$$+\frac{\frac{3}{2}K(FE+DG)}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}-\frac{(FE+DG)}{K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}\Bigg]. \quad (52)$$

Solving for C_2 and C_1, results in:

$$C_2=\Bigg[\frac{\frac{\bar{Z}}{2}D^2-\frac{1}{2}(FE+DG)}{K^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}-\frac{\bar{Z}D^2\left(\frac{2\pi\nu a}{c}\right)^2}{K^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}-\frac{\left[2\bar{Z}D^2-(FE+DG)\right]}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$+\frac{(FE+DG)\left(\frac{2\pi\nu a}{c}\right)^2}{K^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}+\frac{\left[\frac{5}{2}\bar{Z}\left(\frac{2\pi\nu a}{c}\right)^2D^2-\frac{\bar{Z}}{2}K^2D^2\right]}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}+\frac{\frac{3}{2}(FE+DG)}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}$$

$$\left.-\frac{(FE+DG)}{K^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}\right] \tag{53}$$

$$C_1=\left[\frac{-\left[\frac{\bar{Z}}{2}D^2-\frac{1}{2}(FE+DG)\right]}{K^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}+\frac{\bar{Z}D^2\left(\frac{2\pi\nu a}{c}\right)^2}{K^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}+\frac{\left[2\bar{Z}D^2-(FE+DG)\right]}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}\right.$$

$$-\frac{(FE+DG)\left(\frac{2\pi\nu a}{c}\right)^2}{K^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}-\frac{\left[\frac{5}{2}\bar{Z}\left(\frac{2\pi\nu a}{c}\right)^2D^2-\frac{\bar{Z}}{2}K^2D^2\right]}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}-\frac{\frac{3}{2}(FE+DG)}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}$$

$$+\frac{(FE+DG)}{K^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}-\frac{\bar{Z}D^2\left(\frac{2\pi\nu a}{c}\right)^2}{2K^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}-\frac{\left[2\bar{Z}D^2-(FE+DG)\right]}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}+\frac{(FE+DG)}{\left(\frac{2\pi\nu a}{c}\right)^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}$$

$$-\frac{\left[-\frac{\bar{Z}}{2}\left(\frac{2\pi\nu a}{c}\right)^2D^2\,\frac{4\pi\nu a}{c}+2\bar{Z}K^2D^2\,\frac{2\pi\nu a}{c}\right]}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left(\frac{4\pi\nu a}{c}\right)\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}+\frac{\frac{2\pi\nu a}{c}(FE+DG)}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\frac{4\pi\nu a}{c}\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}$$

$$\left.-\frac{\frac{K^2}{2}(FE+DG)}{\frac{2\pi\nu a}{c}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\frac{4\pi\nu a}{c}\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}\right]\;. \tag{54}$$

The complete solution for Z_2 is obtained by substitution of equations 53 and 54 into 50 and is:

$$Z_2=\left[\frac{-\left[\frac{\bar{Z}}{2}D^2-\frac{1}{2}(FE+DG)\right]}{K^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}+\frac{\bar{Z}D^2\left(\frac{2\pi\nu a}{c}\right)^2}{K^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}+\frac{\left[2\bar{Z}D^2-(FE+DG)\right]}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}\right.$$

$$-\frac{(FE+DG)\left(\frac{2\pi\nu a}{c}\right)^2}{K^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}-\frac{\left[\frac{5}{2}\bar{Z}\left(\frac{2\pi\nu a}{c}\right)^2D^2-\frac{\bar{Z}}{2}K^2D^2\right]}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}-\frac{\frac{3}{2}(FE+DG)}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}$$

$$+\frac{(FE+DG)}{K^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}-\frac{\bar{Z}D^2\left(\frac{2\pi\nu a}{c}\right)^2}{2K^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}-\frac{\left[2\bar{Z}D^2-(FE+DG)\right]}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$-\frac{\left[-\frac{\bar{Z}}{2}\left(\frac{2\pi\nu a}{c}\right)^2D^2\frac{4\pi\nu a}{c}+2\bar{Z}K^2D^2\frac{2\pi\nu a}{c}\right]}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left(\frac{4\pi\nu a}{c}\right)\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}+\frac{\frac{2\pi\nu a}{c}(FE+DG)}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\frac{4\pi\nu a}{c}\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}$$

$$-\frac{\frac{K^2}{2}(FE+DG)}{\frac{2\pi\nu a}{c}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\frac{4\pi\nu a}{c}\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}+\frac{(FE+DG)}{\left(\frac{2\pi\nu a}{c}\right)^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}+\frac{\left[\frac{\bar{Z}}{2}D^2-\frac{1}{2}(FE+DG)\right]e^{-KJ}}{K^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}$$

$$-\frac{\bar{Z}D^2\left(\frac{2\pi\nu a}{c}\right)^2e^{-KJ}}{K^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}-\frac{\left[2\bar{Z}D^2-(FE+DG)\right]e^{-KJ}}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}+\frac{(FE+DG)\left(\frac{2\pi\nu a}{c}\right)^2e^{-KJ}}{K^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$+\frac{\left[\frac{5}{2}\bar{Z}\left(\frac{2\pi\nu a}{c}\right)^2D^2-\frac{\bar{Z}}{2}K^2D^2\right]e^{-KJ}}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}+\frac{\frac{3}{2}(FE+DG)e^{-KJ}}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}-\frac{(FE+DG)e^{-KJ}}{K^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}$$

$$+\boxed{\frac{\frac{\bar{Z}}{2}D^2J-\frac{1}{2}(FE+DG)J}{K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}}+\frac{\bar{Z}D^2\left(\frac{2\pi\nu a}{c}\right)^2e^{-2KJ}}{2K^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$+\frac{\left[2\bar{Z}D^2-(FE+DG)\right]e^{-KJ}\cos\frac{2\pi\nu aJ}{c}}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}+\frac{(FE+DG)\left(\frac{2\pi\nu a}{c}\right)e^{-KJ}\sin\frac{2\pi\nu aJ}{c}}{K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$+\frac{\left[-\frac{\bar{Z}}{2}\left(\frac{2\pi\nu a}{c}\right)^2D^2\frac{4\pi\nu a}{c}+2\bar{Z}K^2D^2\frac{2\pi\nu a}{c}\right]\cos\frac{4\pi\nu aJ}{c}}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left(\frac{4\pi\nu a}{c}\right)\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}-\frac{\frac{2\pi\nu a}{c}(FE+DG)\cos\frac{4\pi\nu aJ}{c}}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\frac{4\pi\nu a}{c}\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}$$

$$+\frac{\frac{K^2}{2}(FE+DG)\cos\frac{4\pi\nu aJ}{c}}{\frac{2\pi\nu a}{c}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\frac{4\pi\nu a}{c}\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}+\frac{\left[\frac{5}{2}\bar{Z}\left(\frac{2\pi\nu a}{c}\right)^2KD^2-\frac{\bar{Z}}{2}K^3D^2\right]\sin\frac{4\pi\nu aJ}{c}}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left(\frac{4\pi\nu a}{c}\right)\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}$$

$$\left.+\frac{\frac{3}{2}K(FE+DG)\sin\frac{4\pi\nu aJ}{c}}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\frac{4\pi\nu a}{c}\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}-\frac{(FE+DG)\cos\frac{2\pi\nu aJ}{c}}{\left(\frac{2\pi\nu a}{c}\right)^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}-\frac{(FE+DG)\sin\frac{2\pi\nu aJ}{c}}{K\left(\frac{2\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}\right] \quad (55)$$

The second order $\vec{i}_\varphi$ component equation can be treated in a similar manner. Thus, substitution of zero order solutions in 3.21 leads to:

$$2\bar{Z}\frac{dZ_1}{d\mathcal{T}}\frac{d\Phi_1}{d\mathcal{T}} + \bar{Z}^2\frac{d^2\Phi_2}{d\mathcal{T}^2} + 2\bar{Z}Z_1\frac{d^2\Phi_1}{d\mathcal{T}} + K\bar{Z}^2\frac{d\Phi_2}{d\mathcal{T}} + 2K\bar{Z}Z_1\frac{d\Phi_1}{d\mathcal{T}} =$$

$$\left[Z_1\pi\alpha\sin\bar{\Phi}\left[J_0(\pi\alpha\bar{Z}) - \frac{1}{\pi\alpha\bar{Z}}J_1(\pi\alpha\bar{Z})\right]\sin\frac{2\pi\nu a\mathcal{T}}{c} + \Phi_1 J_1(\pi\alpha\bar{Z})\cos\bar{\Phi}\sin\frac{2\pi\nu a\mathcal{T}}{c}\right]. \quad (56)$$

If

$$H \equiv \sin\bar{\Phi}\left[\pi\alpha J_0(\pi\alpha\bar{Z}) - \frac{1}{\bar{Z}}J_1(\pi\alpha\bar{Z})\right] \quad (57)$$

and

$$I \equiv \left[J_1(\pi\alpha\bar{Z})\cos\bar{\Phi}\right] \quad (58)$$

then equation 56 can be expressed as:

$$2\bar{Z}\frac{dZ_1}{d\mathcal{T}}\frac{d\Phi_1}{d\mathcal{T}} + \bar{Z}^2\frac{d^2\Phi_2}{d\mathcal{T}^2} + 2\bar{Z}Z_1\frac{d^2\Phi_1}{d\mathcal{T}^2} + K\bar{Z}^2\frac{d\Phi_2}{d\mathcal{T}} + 2K\bar{Z}Z_1\frac{d\Phi_1}{d\mathcal{T}} =$$

$$\left[Z_1 H\sin\frac{2\pi\nu a\mathcal{T}}{c} + \Phi_1 I\sin\frac{2\pi\nu a\mathcal{T}}{c}\right]. \quad (59)$$

Terms involving Z_1 and Φ_1 are evaluated by use of equations 20 and 23 and are given by:

$$\frac{dZ_1}{d\mathcal{T}} = \left[\frac{E\left(\frac{2\pi\nu a}{c}\right)e^{-K\mathcal{T}} - \frac{2\pi\nu a}{c}E\cos\frac{2\pi\nu a\mathcal{T}}{c} + KE\sin\frac{2\pi\nu a\mathcal{T}}{c}}{\left[K^2 + \left(\frac{2\pi\nu a}{c}\right)^2\right]}\right] \quad (60)$$

$$\frac{d\Phi_1}{d\mathcal{T}} = \left[\frac{D\left(\frac{2\pi\nu a}{c}\right)e^{-K\mathcal{T}} - \frac{2\pi\nu a}{c}D\cos\frac{2\pi\nu a\mathcal{T}}{c} + KD\sin\frac{2\pi\nu a\mathcal{T}}{c}}{\left[K^2 + \left(\frac{2\pi\nu a}{c}\right)^2\right]}\right] \quad (61)$$

$$\frac{dZ_1}{dJ}\frac{d\phi_1}{dJ} = \Bigg[\frac{DE\left(\frac{2\pi\nu a}{c}\right)e^{-2kJ} - 2D\left(\frac{2\pi\nu a}{c}\right)^2 E\, e^{-kJ}\cos\frac{2\pi\nu a J}{c}}{\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$+\frac{2D\left(\frac{2\pi\nu a}{c}\right)kE\, e^{-kJ}\sin\frac{2\pi\nu a J}{c} + \left(\frac{2\pi\nu a}{c}\right)^2 DE\cos^2\frac{2\pi\nu a J}{c}}{\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$-\frac{2\left(\frac{2\pi\nu a}{c}\right)DkE\sin\frac{2\pi\nu a J}{c}\cos\frac{2\pi\nu a J}{c} + k^2 DE\sin^2\frac{2\pi\nu a J}{c}}{\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}\Bigg] \qquad (62)$$

$$\frac{d^2\phi_1}{dJ^2} = \left[\frac{-kD\left(\frac{2\pi\nu a}{c}\right)e^{-kJ} + \left(\frac{2\pi\nu a}{c}\right)^2 D\sin\frac{2\pi\nu a J}{c} + \frac{2\pi\nu a}{c}kD\cos\frac{2\pi\nu a J}{c}}{\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}\right] \qquad (63)$$

$$Z_1\frac{d^2\phi_1}{dJ^2} = \Bigg[\frac{-kED\left(\frac{2\pi\nu a}{c}\right)^2 e^{-kJ} + E\left(\frac{2\pi\nu a}{c}\right)^3 D\sin\frac{2\pi\nu a J}{c} + \left(\frac{2\pi\nu a}{c}\right)^2 kED\cos\frac{2\pi\nu a J}{c}}{k\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$\frac{-k^2 ED\left(\frac{2\pi\nu a}{c}\right)e^{-kJ} + \left(\frac{2\pi\nu a}{c}\right)^2 kED\sin\frac{2\pi\nu a J}{c} + \frac{2\pi\nu a}{c}k^2 ED\cos\frac{2\pi\nu a J}{c}}{\frac{2\pi\nu a}{c}\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$+\frac{kED\left(\frac{2\pi\nu a}{c}\right)^2 e^{-2kJ} - \left(\frac{2\pi\nu a}{c}\right)^3 DE\, e^{-kJ}\sin\frac{2\pi\nu a J}{c} - \left(\frac{2\pi\nu a}{c}\right)^2 kEDe^{-kJ}\cos\frac{2\pi\nu a J}{c}}{k\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$+\frac{kED\left(\frac{2\pi\nu a}{c}\right)e^{-kJ}\sin\frac{2\pi\nu a J}{c} - \left(\frac{2\pi\nu a}{c}\right)^2 ED\sin^2\frac{2\pi\nu a J}{c}}{\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$-\frac{\left(\frac{2\pi\nu a}{c}\right)kED\sin\frac{2\pi\nu a J}{c}\cos\frac{2\pi\nu a J}{c}}{\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2} + \frac{k^2 ED\left(\frac{2\pi\nu a}{c}\right)e^{-kJ}\cos\frac{2\pi\nu a J}{c}}{\frac{2\pi\nu a}{c}\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$-\frac{\left(\frac{2\pi\nu a}{c}\right)^2 kED\sin\frac{2\pi\nu a J}{c}\cos\frac{2\pi\nu a J}{c} - \left(\frac{2\pi\nu a}{c}\right)k^2 ED\cos^2\frac{2\pi\nu a J}{c}}{\frac{2\pi\nu a}{c}\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}\Bigg] \qquad (64)$$

$$Z_1 \frac{d\phi_1}{d\mathcal{T}} = \left[\frac{ED\left(\frac{2\pi\nu a}{c}\right)^2 e^{-k\mathcal{T}} - \left(\frac{2\pi\nu a}{c}\right)^2 ED\cos\frac{2\pi\nu a\mathcal{T}}{c} + KED\frac{2\pi\nu a}{c}\sin\frac{2\pi\nu a\mathcal{T}}{c}}{k\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2} \right.$$

$$+ \frac{kED\left(\frac{2\pi\nu a}{c}\right)e^{-k\mathcal{T}} - \left(\frac{2\pi\nu a}{c}\right)KED\cos\frac{2\pi\nu a\mathcal{T}}{c} + k^2ED\sin\frac{2\pi\nu a\mathcal{T}}{c}}{\frac{2\pi\nu a}{c}\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$\frac{-ED\left(\frac{2\pi\nu a}{c}\right)^2 e^{-2k\mathcal{T}} + \left(\frac{2\pi\nu a}{c}\right)^2 EDe^{-k\mathcal{T}}\cos\frac{2\pi\nu a\mathcal{T}}{c} - KED\frac{2\pi\nu a}{c}e^{-k\mathcal{T}}\sin\frac{2\pi\nu a\mathcal{T}}{c}}{k\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$\frac{-ED\left(\frac{2\pi\nu a}{c}\right)e^{-k\mathcal{T}}\sin\frac{2\pi\nu a\mathcal{T}}{c} + \frac{2\pi\nu a}{c}DE\cos\frac{2\pi\nu a\mathcal{T}}{c}\sin\frac{2\pi\nu a\mathcal{T}}{c}}{\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$\frac{-KED\sin^2\frac{2\pi\nu a\mathcal{T}}{c}}{\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2} \quad \frac{-KED\left(\frac{2\pi\nu a}{c}\right)e^{-k\mathcal{T}}\cos\frac{2\pi\nu a\mathcal{T}}{c}}{\frac{2\pi\nu a}{c}\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$\left. + \frac{\frac{2\pi\nu a}{c}KED\cos^2\frac{2\pi\nu a\mathcal{T}}{c} - K^2ED\sin\frac{2\pi\nu a\mathcal{T}}{c}\cos\frac{2\pi\nu a\mathcal{T}}{c}}{\frac{2\pi\nu a}{c}\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2} \right] \tag{65}$$

Substituting 62, 64 and 65 into equation 59 gives:

$$\frac{d^2\phi_2}{d\mathcal{T}^2} + k\frac{d\phi_2}{d\mathcal{T}} = \left[\frac{-2DE\left(\frac{2\pi\nu a}{c}\right)^2 e^{-2k\mathcal{T}} + 4D\left(\frac{2\pi\nu a}{c}\right)^2 Ee^{-k\mathcal{T}}\cos\frac{2\pi\nu a\mathcal{T}}{c}}{\overline{Z}\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2} \right.$$

$$\frac{-4D\left(\frac{2\pi\nu a}{c}\right)KEe^{-k\mathcal{T}}\sin\frac{2\pi\nu a\mathcal{T}}{c} - 2\left(\frac{2\pi\nu a}{c}\right)^2 DE\cos^2\frac{2\pi\nu a\mathcal{T}}{c}}{\overline{Z}\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$+\frac{4\left(\frac{2\pi\nu a}{c}\right)DKE\sin\frac{2\pi\nu a\mathcal{T}}{c}\cos\frac{2\pi\nu a\mathcal{T}}{c} - 2k^2DE\sin^2\frac{2\pi\nu a\mathcal{T}}{c}}{\overline{Z}\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$+\frac{2KED\left(\frac{2\pi\nu a}{c}\right)^2 e^{-K\tau} - 2E\left(\frac{2\pi\nu a}{c}\right)^3 D\sin\frac{2\pi\nu a\tau}{c} - 2\left(\frac{2\pi\nu a}{c}\right)^2 KED\cos\frac{2\pi\nu a\tau}{c}}{\bar{\bar{Z}}\,K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$+\frac{2K^2ED\left(\frac{2\pi\nu a}{c}\right) e^{-K\tau} - 2\left(\frac{2\pi\nu a}{c}\right)^2 KED\sin\frac{2\pi\nu a\tau}{c} - 2\left(\frac{2\pi\nu a}{c}\right) K^2ED\cos\frac{2\pi\nu a\tau}{c}}{\bar{\bar{Z}}\,\frac{2\pi\nu a}{c}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$-\frac{2KED\left(\frac{2\pi\nu a}{c}\right)^2 e^{-2K\tau} + 2\left(\frac{2\pi\nu a}{c}\right)^3 DE e^{-K\tau}\sin\frac{2\pi\nu a\tau}{c} + 2\left(\frac{2\pi\nu a}{c}\right)^2 KEDe^{-K\tau}\cos\frac{2\pi\nu a\tau}{c}}{\bar{\bar{Z}}\,K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$-\frac{2KED\left(\frac{2\pi\nu a}{c}\right) e^{-K\tau}\sin\frac{2\pi\nu a\tau}{c} + 2\left(\frac{2\pi\nu a}{c}\right)^2 ED\sin^2\frac{2\pi\nu a\tau}{c}}{\bar{\bar{Z}}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$+\frac{2\left(\frac{2\pi\nu a}{c}\right) KED\sin\frac{2\pi\nu a\tau}{c}\cos\frac{2\pi\nu a\tau}{c}}{\bar{\bar{Z}}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2} - \frac{2K^2ED\left(\frac{2\pi\nu a}{c}\right) e^{-K\tau}\cos\frac{2\pi\nu a\tau}{c}}{\bar{\bar{Z}}\left(\frac{2\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$+\frac{2\left(\frac{2\pi\nu a}{c}\right)^2 KED\sin\frac{2\pi\nu a\tau}{c}\cos\frac{2\pi\nu a\tau}{c} + 2\left(\frac{2\pi\nu a}{c}\right) K^2ED\cos^2\frac{2\pi\nu a\tau}{c}}{\bar{\bar{Z}}\left(\frac{2\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$-\frac{2KED\left(\frac{2\pi\nu a}{c}\right)^2 e^{-K\tau} + 2K\left(\frac{2\pi\nu a}{c}\right)^2 ED\cos\frac{2\pi\nu a\tau}{c} - 2K^2ED\left(\frac{2\pi\nu a}{c}\right)\sin\frac{2\pi\nu a\tau}{c}}{\bar{\bar{Z}}\,K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$-\frac{2K^2ED\left(\frac{2\pi\nu a}{c}\right) e^{-K\tau} + 2\left(\frac{2\pi\nu a}{c}\right) K^2ED\cos\frac{2\pi\nu a\tau}{c} - 2K^3ED\sin\frac{2\pi\nu a\tau}{c}}{\bar{\bar{Z}}\left(\frac{2\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$+\frac{2KED\left(\frac{2\pi\nu a}{c}\right)^2 e^{-2K\tau} - 2\left(\frac{2\pi\nu a}{c}\right)^2 KEDe^{-K\tau}\cos\frac{2\pi\nu a\tau}{c}}{\bar{\bar{Z}}\,K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$+\frac{2K^2ED\left(\frac{2\pi\nu a}{c}\right)e^{-K\mathcal{T}}\sin\frac{2\pi\nu a\mathcal{T}}{c}}{\bar{Z}K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}+\frac{2KED\left(\frac{2\pi\nu a}{c}\right)e^{-K\mathcal{T}}\sin\frac{2\pi\nu a\mathcal{T}}{c}}{\bar{Z}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$\frac{-2\left(\frac{2\pi\nu a}{c}\right)KED\cos\frac{2\pi\nu a\mathcal{T}}{c}\sin\frac{2\pi\nu a\mathcal{T}}{c}+2K^2ED\sin^2\frac{2\pi\nu a\mathcal{T}}{c}}{\bar{Z}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$\frac{+2K^2ED\left(\frac{2\pi\nu a}{c}\right)e^{-K\mathcal{T}}\cos\frac{2\pi\nu a\mathcal{T}}{c}-2\left(\frac{2\pi\nu a}{c}\right)K^2ED\cos^2\frac{2\pi\nu a\mathcal{T}}{c}}{\bar{Z}\left(\frac{2\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$+\frac{2K^3ED\sin\frac{2\pi\nu a\mathcal{T}}{c}\cos\frac{2\pi\nu a\mathcal{T}}{c}}{\bar{Z}\left(\frac{2\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}+\frac{(EH+DI)\frac{2\pi\nu a}{c}\sin\frac{2\pi\nu a\mathcal{T}}{c}}{K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}$$

$$+\frac{(EH+DI)K\sin\frac{2\pi\nu a\mathcal{T}}{c}}{\frac{2\pi\nu a}{c}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}-\frac{(EH+DI)\frac{2\pi\nu a}{c}e^{-K\mathcal{T}}\sin\frac{2\pi\nu a\mathcal{T}}{c}}{K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}$$

$$\left.-\frac{(EH+DI)\sin^2\frac{2\pi\nu a\mathcal{T}}{c}}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}-\frac{(EH+DI)K\cos\frac{2\pi\nu a\mathcal{T}}{c}\sin\frac{2\pi\nu a\mathcal{T}}{c}}{\frac{2\pi\nu a}{c}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}\right]. \qquad (66)$$

Using trigonometric identities, equation 66 is simplified to the following:

$$\frac{d^2\phi_2}{d\mathcal{T}^2}+K\frac{d\phi_2}{d\mathcal{T}}=\left[\frac{-2DE\left(\frac{2\pi\nu a}{c}\right)^2e^{-2K\mathcal{T}}+4\left(\frac{2\pi\nu a}{c}\right)^2EDe^{-K\mathcal{T}}\cos\frac{2\pi\nu a\mathcal{T}}{c}}{\bar{Z}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}\right.$$

$$\frac{-2\left(\frac{2\pi\nu a}{c}\right)KED\,e^{-K\mathcal{T}}\sin\frac{2\pi\nu a\mathcal{T}}{c}-2\left(\frac{2\pi\nu a}{c}\right)^2DE\cos\frac{4\pi\nu a\mathcal{T}}{c}}{\bar{Z}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$+\frac{3\left(\frac{2\pi\nu a}{c}\right)KED\sin\frac{4\pi\nu a\mathcal{T}}{c}}{\bar{Z}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}-\frac{2\left(\frac{2\pi\nu a}{c}\right)^3ED\sin\frac{2\pi\nu a\mathcal{T}}{c}}{\bar{Z}K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$-\frac{4\left(\frac{2\pi\nu a}{c}\right)KED\sin\frac{2\pi\nu a\mathcal{T}}{c}}{\bar{Z}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}+\frac{2\left(\frac{2\pi\nu a}{c}\right)^3ED\,e^{-K\mathcal{T}}\sin\frac{2\pi\nu a\mathcal{T}}{c}}{\bar{Z}K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$\frac{-2K^3ED\sin\frac{2\pi\nu a\mathcal{T}}{c}+K^3ED\sin\frac{4\pi\nu a\mathcal{T}}{c}}{\bar{Z}\left(\frac{2\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}+\frac{(EH+DI)\left(\frac{2\pi\nu a}{c}\right)\sin\frac{2\pi\nu a\mathcal{T}}{c}}{K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}$$

$$+\frac{(EH+DI)\,K\sin\frac{2\pi\nu a\mathcal{T}}{c}}{\frac{2\pi\nu a}{c}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}-\frac{(EH+DI)\left(\frac{2\pi\nu a}{c}\right)e^{-K\mathcal{T}}\sin\frac{2\pi\nu a\mathcal{T}}{c}}{K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}$$

$$\left.\frac{-\frac{1}{2}(EH+DI)+\frac{1}{2}(EH+DI)\cos\frac{4\pi\nu a\mathcal{T}}{c}}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}-\frac{\frac{K}{2}(EH+DI)\sin\frac{4\pi\nu a\mathcal{T}}{c}}{\frac{2\pi\nu a}{c}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}\right] \tag{67}$$

The complementary solution is again:

$$\phi_{2c}=\left[C_1+C_2e^{-K\mathcal{T}}\right]. \tag{68}$$

The trial particular solution is:

$$\phi_{2p}=\left[C_3\mathcal{T}+C_4e^{-2K\mathcal{T}}+C_5e^{-K\mathcal{T}}\cos\frac{2\pi\nu a\mathcal{T}}{c}+C_6e^{-K\mathcal{T}}\sin\frac{2\pi\nu a\mathcal{T}}{c}\right.$$

$$\left.+C_7\cos\frac{4\pi\nu a\mathcal{T}}{c}+C_8\sin\frac{4\pi\nu a\mathcal{T}}{c}+C_9\cos\frac{2\pi\nu a\mathcal{T}}{c}+C_{10}\sin\frac{2\pi\nu a\mathcal{T}}{c}\right]. \tag{69}$$

The derivatives of ϕ_{2p} are:

$$\phi_{2p}'=\left[C_3-2KC_4e^{-2K\mathcal{T}}-KC_5e^{-K\mathcal{T}}\cos\frac{2\pi\nu a\mathcal{T}}{c}-\frac{2\pi\nu a}{c}C_5e^{-K\mathcal{T}}\sin\frac{2\pi\nu a\mathcal{T}}{c}\right.$$

$$-KC_6 e^{-K\tau}\sin\frac{2\pi\nu a\tau}{c} + \frac{2\pi\nu a}{c} C_6 e^{-K\tau}\cos\frac{2\pi\nu a\tau}{c} - \frac{4\pi\nu a}{c} C_7 \sin\frac{4\pi\nu a\tau}{c}$$

$$+\frac{4\pi\nu a}{c} C_8 \cos\frac{4\pi\nu a\tau}{c} - \frac{2\pi\nu a}{c} C_9 \sin\frac{2\pi\nu a\tau}{c} + \frac{2\pi\nu a}{c} C_{10}\cos\frac{2\pi\nu a\tau}{c}\Bigg] \quad (70)$$

$$\phi''_{2p} = \Bigg[4K^2 C_4 e^{-2K\tau} + K^2 C_5 e^{-K\tau}\cos\frac{2\pi\nu a\tau}{c} + \frac{2\pi\nu a}{c} K C_5 e^{-K\tau}\sin\frac{2\pi\nu a\tau}{c}$$

$$+K\frac{2\pi\nu a}{c} C_5 e^{-K\tau}\sin\frac{2\pi\nu a\tau}{c} - \left(\frac{2\pi\nu a}{c}\right)^2 C_5 e^{-K\tau}\cos\frac{2\pi\nu a\tau}{c}$$

$$+K^2 C_6 e^{-K\tau}\sin\frac{2\pi\nu a\tau}{c} - \frac{2\pi\nu a}{c} K C_6 e^{-K\tau}\cos\frac{2\pi\nu a\tau}{c} - K\frac{2\pi\nu a}{c} C_6 e^{-K\tau}\cos\frac{2\pi\nu a\tau}{c}$$

$$-\left(\frac{2\pi\nu a}{c}\right)^2 C_6 e^{-K\tau}\sin\frac{2\pi\nu a\tau}{c} - \left(\frac{4\pi\nu a}{c}\right)^2 C_7\cos\frac{4\pi\nu a\tau}{c} - \left(\frac{4\pi\nu a}{c}\right)^2 C_8 \sin\frac{4\pi\nu a\tau}{c}$$

$$-\left(\frac{2\pi\nu a}{c}\right)^2 C_9 \cos\frac{2\pi\nu a\tau}{c} - \left(\frac{2\pi\nu a}{c}\right)^2 C_{10}\sin\frac{2\pi\nu a\tau}{c}\Bigg]. \quad (71)$$

Substituting the above results into equation 67 and equating coefficients of like terms we get:

$$KC_3 = \left[\frac{-\frac{1}{2}(EH+DI)}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}\right] \quad (72)$$

$$2K^2C_4 = \left[\frac{-2DE\left(\frac{2\pi\nu a}{c}\right)^2}{\overline{Z}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}\right] \quad (73)$$

$$K\left(\frac{2\pi\nu a}{c}\right)C_5 - \left(\frac{2\pi\nu a}{c}\right)^2 C_6 = \left[\frac{-2KED\left(\frac{2\pi\nu a}{c}\right)}{\overline{Z}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2} + \frac{2\left(\frac{2\pi\nu a}{c}\right)^3 DE}{\overline{Z}K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2} - \frac{\left(\frac{2\pi\nu a}{c}\right)(EH+DI)}{K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}\right] \quad (74)$$

$$-\left(\frac{2\pi\nu a}{c}\right)^2 C_5 \quad -K\left(\frac{2\pi\nu a}{c}\right) C_6 = \left[\frac{4DE\left(\frac{2\pi\nu a}{c}\right)^2}{\bar{\bar{Z}}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2} \right] \tag{75}$$

$$-\left(\frac{4\pi\nu a}{c}\right)^2 C_7 + K\left(\frac{4\pi\nu a}{c}\right) C_8 = \left[\frac{-2\left(\frac{2\pi\nu a}{c}\right)^2 DE}{\bar{\bar{Z}}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2} + \frac{\frac{1}{2}(EH+DI)}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]} \right] \tag{76}$$

$$-K\left(\frac{4\pi\nu a}{c}\right) C_7 - \left(\frac{4\pi\nu a}{c}\right)^2 C_8 = \left[\frac{3\left(\frac{2\pi\nu a}{c}\right) KED}{\bar{\bar{Z}}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2} - \frac{K/2\,(EH+DI)}{\frac{2\pi\nu a}{c}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]} \right.$$

$$\left. + \frac{K^3 ED}{\bar{\bar{Z}}\left(\frac{2\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2} \right] \tag{77}$$

$$-\left(\frac{2\pi\nu a}{c}\right)^2 C_9 \quad + K\left(\frac{2\pi\nu a}{c}\right) C_{10} = \left[\, 0 \,\right] \tag{78}$$

$$-K\left(\frac{2\pi\nu a}{c}\right) C_9 - \left(\frac{2\pi\nu a}{c}\right)^2 C_{10} = \left[\frac{-2\left(\frac{2\pi\nu a}{c}\right)^3 ED}{\bar{\bar{Z}}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2 K} - \frac{4\left(\frac{2\pi\nu a}{c}\right) KED}{\bar{\bar{Z}}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2} \right.$$

$$\left. + \frac{(EH+DI)\left(\frac{2\pi\nu a}{c}\right)}{K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]} - \frac{2K^3 ED}{\bar{\bar{Z}}\left(\frac{2\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2} + \frac{K(EH+DI)}{\frac{2\pi\nu a}{c}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]} \right] \tag{79}$$

Solving equations 72 to 79 gives:

$$C_3 = \left[\frac{-\frac{1}{2}(EH+DI)}{K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]} \right] \qquad C_4 = \left[\frac{-DE\left(\frac{2\pi\nu a}{c}\right)^2}{\bar{\bar{Z}}K^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2} \right] \tag{80}$$

$$C_5 = \left[\frac{-2DE\left(\frac{2\pi\nu a}{c}\right)^2}{\bar{\bar{Z}}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3} - \frac{(EH+DI)}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2} - \frac{2K^2 ED}{\bar{\bar{Z}}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3} \right] \tag{81}$$

$$C_6 = \left[\frac{-2\left(\frac{2\pi\nu a}{c}\right)^3 DE}{\bar{\bar{Z}} K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3} + \frac{(EH+DI)\left(\frac{2\pi\nu a}{c}\right)}{K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2} - \frac{2KED\left(\frac{2\pi\nu a}{c}\right)}{\bar{\bar{Z}}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3} \right] \tag{82}$$

$$C_7 = \left[\frac{2\left(\frac{2\pi\nu a}{c}\right)^2 ED}{\bar{\bar{Z}}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]} - \frac{\frac{1}{2}(EH+DI)}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]} - \frac{\frac{3}{2}K^2 ED}{\bar{\bar{Z}}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]} \right.$$

$$\left. + \frac{\frac{K^2}{2}(EH+DI)}{\left(\frac{2\pi\nu a}{c}\right)\left(\frac{4\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]} - \frac{K^4 ED}{\bar{\bar{Z}}\left(\frac{2\pi\nu a}{c}\right)\left(\frac{4\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]} \right] \tag{83}$$

$$C_8 = \left[\frac{-2KED\left(\frac{2\pi\nu a}{c}\right)^2}{\bar{\bar{Z}}\left(\frac{4\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]} + \frac{\frac{K}{2}(EH+DI)}{\left(\frac{4\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]} - \frac{3\left(\frac{2\pi\nu a}{c}\right)KED}{\bar{\bar{Z}}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]} \right.$$

$$\left. - \frac{K^3 ED}{\bar{\bar{Z}}\left(\frac{2\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]} + \frac{\frac{K}{2}(EH+DI)}{\left(\frac{2\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]} \right] \tag{84}$$

$$C_9 = \left[\frac{2KED\left(\frac{2\pi\nu a}{c}\right)^2}{\bar{\bar{Z}} K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3} + \frac{4K^2 ED}{\bar{\bar{Z}}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3} + \frac{2K^4 ED}{\bar{\bar{Z}}\left(\frac{2\pi\nu a}{c}\right)^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3} \right.$$

$$\left. - \frac{(EH+DI)}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2} - \frac{K^2(EH+DI)}{\left(\frac{2\pi\nu a}{c}\right)^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2} \right] \tag{85}$$

$$C_{10} = \left[\frac{2\left(\frac{2\pi\nu a}{c}\right)^3 ED}{\bar{\bar{Z}} K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3} + \frac{4\left(\frac{2\pi\nu a}{c}\right)KED}{\bar{\bar{Z}}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3} + \frac{2K^3 ED}{\bar{\bar{Z}}\left(\frac{2\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3} \right.$$

$$\left. - \frac{(EH+DI)\left(\frac{2\pi\nu a}{c}\right)}{K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2} - \frac{(EH+DI)K}{\left(\frac{2\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2} \right] \tag{86}$$

The complete solution for ϕ_2 is obtained by substituting the above constant into equation 69 and adding the complementary solution.

$$\phi_2 = \Bigg[C_1 + C_2 e^{-KJ} - \frac{\frac{1}{2}(EH+DI)J}{K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]} - \frac{DE\left(\frac{2\pi\nu a}{c}\right)^2 e^{-2KJ}}{\bar{Z}K^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$- \frac{2DE\left(\frac{2\pi\nu a}{c}\right)^2 e^{-KJ}\cos\frac{2\pi\nu a J}{c} - 2K^2 ED e^{-KJ}\cos\frac{2\pi\nu a J}{c}}{\bar{Z}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3} - \frac{(EH+DI)e^{-KJ}\cos\frac{2\pi\nu a J}{c}}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$- \frac{2\left(\frac{2\pi\nu a}{c}\right)^3 DE e^{-KJ}\sin\frac{2\pi\nu a J}{c}}{\bar{Z}K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3} + \frac{(EH+DI)\left(\frac{2\pi\nu a}{c}\right)e^{-KJ}\sin\frac{2\pi\nu a J}{c}}{K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$- \frac{2KED\left(\frac{2\pi\nu a}{c}\right)e^{-KJ}\sin\frac{2\pi\nu a J}{c}}{\bar{Z}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3} + \frac{2\left(\frac{2\pi\nu a}{c}\right)^2 ED\cos\frac{4\pi\nu a J}{c}}{\bar{Z}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}$$

$$- \frac{\frac{1}{2}(EH+DI)\cos\frac{4\pi\nu a J}{c}}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]} - \frac{\frac{3}{2}K^2 ED\cos\frac{4\pi\nu a J}{c}}{\bar{Z}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]} + \frac{\frac{K^2}{2}(EH+DI)\cos\frac{4\pi\nu a J}{c}}{\left(\frac{2\pi\nu a}{c}\right)\left(\frac{4\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}$$

$$- \frac{K^4 ED\cos\frac{4\pi\nu a J}{c}}{\bar{Z}\left(\frac{2\pi\nu a}{c}\right)\left(\frac{4\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]} - \frac{2KED\left(\frac{2\pi\nu a}{c}\right)^2\sin\frac{4\pi\nu a J}{c}}{\bar{Z}\left(\frac{4\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]} + \frac{\frac{K}{2}(EH+DI)\sin\frac{4\pi\nu a J}{c}}{\left(\frac{4\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}$$

$$- \frac{3\left(\frac{2\pi\nu a}{c}\right)KED\sin\frac{4\pi\nu a J}{c}}{\bar{Z}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]} - \frac{K^3 ED\sin\frac{4\pi\nu a J}{c}}{\bar{Z}\left(\frac{2\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]} + \frac{\frac{K}{2}(EH+DI)\sin\frac{4\pi\nu a J}{c}}{\left(\frac{2\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}$$

$$+ \frac{2KED\left(\frac{2\pi\nu a}{c}\right)^2\cos\frac{2\pi\nu a J}{c}}{\bar{Z}K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3} + \frac{4K^2 ED\cos\frac{2\pi\nu a J}{c}}{\bar{Z}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3} + \frac{2K^4 ED\cos\frac{2\pi\nu a J}{c}}{\bar{Z}\left(\frac{2\pi\nu a}{c}\right)^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3}$$

$$- \frac{(EH+DI)\cos\frac{2\pi\nu a J}{c}}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2} - \frac{K^2(EH+DI)\cos\frac{2\pi\nu a J}{c}}{\left(\frac{2\pi\nu a}{c}\right)^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2} + \frac{2\left(\frac{2\pi\nu a}{c}\right)^3 ED\sin\frac{2\pi\nu a J}{c}}{\bar{Z}K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3}$$

$$+ \frac{4\left(\frac{2\pi\nu a}{c}\right)KED\sin\frac{2\pi\nu a J}{c}}{\bar{Z}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3} + \frac{2K^3 ED\sin\frac{2\pi\nu a J}{c}}{\bar{Z}\left(\frac{2\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3} - \frac{(EH+DI)\left(\frac{2\pi\nu a}{c}\right)\sin\frac{2\pi\nu a J}{c}}{K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$- \frac{(EH + DI) K \sin \frac{2\pi\nu a J}{c}}{\left(\frac{2\pi\nu a}{c}\right)\left[K^2 + \left(\frac{2\pi\nu a}{c}\right)^2\right]^2} \Bigg] . \tag{87}$$

To determine C_1 and C_2 the initial conditions are used:

$$\Phi_2(0) = 0 = \Bigg[C_1 + C_2 - \frac{DE\left(\frac{2\pi\nu a}{c}\right)^2}{\bar{\bar{Z}} K^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2} - \frac{2DE\left(\frac{2\pi\nu a}{c}\right)^2}{\bar{\bar{Z}}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3} - \frac{2(EH+DI)}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}$$

$$+ \frac{2\left(\frac{2\pi\nu a}{c}\right)^2 ED}{\bar{\bar{Z}}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]} - \frac{\frac{1}{2}(EH+DI)}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]} - \frac{\frac{3}{2} K^2 ED}{\bar{\bar{Z}}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}$$

$$+ \frac{\frac{K^3}{2}(EH+DI)}{\left(\frac{2\pi\nu a}{c}\right)\left(\frac{4\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]} - \frac{K^4 ED}{\bar{\bar{Z}}\left(\frac{2\pi\nu a}{c}\right)\left(\frac{4\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]} + \frac{2ED\left(\frac{2\pi\nu a}{c}\right)^2}{\bar{\bar{Z}}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3}$$

$$+ \frac{2K^2 ED}{\bar{\bar{Z}}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3} + \frac{2K^4 ED}{\bar{\bar{Z}}\left(\frac{2\pi\nu a}{c}\right)^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3} - \frac{K^2(EH+DI)}{\left(\frac{2\pi\nu a}{c}\right)^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2} \Bigg] \tag{88}$$

$$\Phi_2'(0) = 0 = \Bigg[-K C_2 - \frac{\frac{1}{2}(EH+DI)}{K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]} + \frac{2DE\left(\frac{2\pi\nu a}{c}\right)^2}{\bar{\bar{Z}} K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2} + \frac{4K^3 ED}{\bar{\bar{Z}}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3}$$

$$- \frac{2KED\left(\frac{2\pi\nu a}{c}\right)^2}{\bar{\bar{Z}}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]} + \frac{\frac{K}{2}(EH+DI)}{\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]} - \frac{3\left(\frac{2\pi\nu a}{c}\right)\left(\frac{4\pi\nu a}{c}\right) KED}{\bar{\bar{Z}}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}$$

$$- \frac{\left(\frac{4\pi\nu a}{c}\right) K^3 ED}{\bar{\bar{Z}}\left(\frac{2\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]} + \frac{\frac{K}{2}\left(\frac{4\pi\nu a}{c}\right)(EH+DI)}{\left(\frac{2\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\left[K^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]} + \frac{4\left(\frac{2\pi\nu a}{c}\right)^2 KED}{\bar{\bar{Z}}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3} \Bigg] . \tag{89}$$

The constants C_2 and C_1 are found from 88 and 89 and are:

$$C_2 = \Bigg[- \frac{\frac{1}{2}(EH+DI)}{K^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]} + \frac{2DE\left(\frac{2\pi\nu a}{c}\right)^2}{\bar{\bar{Z}} K^2\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2} + \frac{4K^2 ED}{\bar{\bar{Z}}\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3}$$

$$-\frac{2ED\left(\frac{2\pi\nu a}{c}\right)^2}{\bar{\bar{Z}}\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[k^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}+\frac{\frac{1}{2}(EH+DI)}{\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\left[k^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}-\frac{3\left(\frac{2\pi\nu a}{c}\right)\left(\frac{4\pi\nu a}{c}\right)ED}{\bar{\bar{Z}}\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[k^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}$$

$$-\frac{\left(\frac{4\pi\nu a}{c}\right)k^2ED}{\bar{\bar{Z}}\left(\frac{2\pi\nu a}{c}\right)\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[k^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}+\frac{(EH+DI)}{\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\left[k^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}+\frac{4\left(\frac{2\pi\nu a}{c}\right)^2ED}{\bar{\bar{Z}}\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3}\Bigg] \qquad (90)$$

$$C_1=\Bigg[\frac{\frac{1}{2}(EH+DI)}{k^2\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]}-\frac{2DE\left(\frac{2\pi\nu a}{c}\right)^2}{\bar{\bar{Z}}k^2\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}-\frac{4k^2ED}{\bar{\bar{Z}}\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3}$$

$$+\frac{2ED\left(\frac{2\pi\nu a}{c}\right)^2}{\bar{\bar{Z}}\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[k^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}-\frac{\frac{1}{2}(EH+DI)}{\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\left[k^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}+\frac{3\left(\frac{2\pi\nu a}{c}\right)\left(\frac{4\pi\nu a}{c}\right)ED}{\bar{\bar{Z}}\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[k^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}$$

$$+\frac{\left(\frac{4\pi\nu a}{c}\right)k^2ED}{\bar{\bar{Z}}\left(\frac{2\pi\nu a}{c}\right)\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[k^2+\left(\frac{4\pi\nu a}{c}\right)\right]}-\frac{(EH+DI)}{\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\left[k^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}-\frac{4\left(\frac{2\pi\nu a}{c}\right)^2ED}{\bar{\bar{Z}}\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3}$$

$$+\frac{DE\left(\frac{2\pi\nu a}{c}\right)^2}{\bar{\bar{Z}}k^2\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}+\frac{2DE\left(\frac{2\pi\nu a}{c}\right)^2}{\bar{\bar{Z}}\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3}+\frac{2(EH+DI)}{\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}-\frac{2\left(\frac{2\pi\nu a}{c}\right)^2ED}{\bar{\bar{Z}}\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[k^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}$$

$$+\frac{\frac{1}{2}(EH+DI)}{\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\left[k^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}+\frac{\frac{3}{2}k^2ED}{\bar{\bar{Z}}\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[k^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}-\frac{\frac{k^2}{2}(EH+DI)}{\left(\frac{2\pi\nu a}{c}\right)\left(\frac{4\pi\nu a}{c}\right)\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]\left[k^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}$$

$$+\frac{k^4ED}{\bar{\bar{Z}}\left(\frac{2\pi\nu a}{c}\right)\left(\frac{4\pi\nu a}{c}\right)\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2\left[k^2+\left(\frac{4\pi\nu a}{c}\right)^2\right]}-\frac{2ED\left(\frac{2\pi\nu a}{c}\right)^2}{\bar{\bar{Z}}\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3}-\frac{2k^2ED}{\bar{\bar{Z}}\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3}$$

$$-\frac{2k^4ED}{\bar{\bar{Z}}\left(\frac{2\pi\nu a}{c}\right)^2\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^3}+\frac{k^2(EH+DI)}{\left(\frac{2\pi\nu a}{c}\right)^2\left[k^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2}\Bigg]. \qquad (91)$$

The solution for ϕ_2 is then obtained from 87, 90 and 91 and is:

$$\phi_2 = \Bigg[\frac{\frac{1}{2}(EH+DI)}{K^2[K^2+(\frac{2\pi\nu a}{c})^2]} - \frac{2DE(\frac{2\pi\nu a}{c})^2}{\bar{Z}K^2[K^2+(\frac{2\pi\nu a}{c})^2]^2} - \frac{4K^2ED}{\bar{Z}[K^2+(\frac{2\pi\nu a}{c})^2]^3} + \frac{2ED(\frac{2\pi\nu a}{c})^2}{\bar{Z}[K^2+(\frac{2\pi\nu a}{c})^2]^2[K^2+(\frac{4\pi\nu a}{c})^2]}$$

$$+ \frac{3(\frac{2\pi\nu a}{c})(\frac{4\pi\nu a}{c})ED}{\bar{Z}[K^2+(\frac{2\pi\nu a}{c})^2]^2[K^2+(\frac{4\pi\nu a}{c})^2]} - \frac{\frac{1}{2}(EH+DI)}{[K^2+(\frac{2\pi\nu a}{c})^2][K^2+(\frac{4\pi\nu a}{c})^2]} + \frac{(\frac{4\pi\nu a}{c})K^2ED}{\bar{Z}(\frac{2\pi\nu a}{c})[K^2+(\frac{2\pi\nu a}{c})^2]^2[K^2+(\frac{4\pi\nu a}{c})^2]}$$

$$- \frac{(EH+DI)}{[K^2+(\frac{2\pi\nu a}{c})^2][K^2+(\frac{4\pi\nu a}{c})^2]} - \frac{4(\frac{2\pi\nu a}{c})^2ED}{\bar{Z}[K^2+(\frac{2\pi\nu a}{c})^2]^3} + \frac{DE(\frac{2\pi\nu a}{c})^2}{\bar{Z}K^2[K^2+(\frac{2\pi\nu a}{c})^2]^2} + \frac{2DE(\frac{2\pi\nu a}{c})^2}{\bar{Z}[K^2+(\frac{2\pi\nu a}{c})^2]^3}$$

$$+ \frac{2(EH+DI)}{[K^2+(\frac{2\pi\nu a}{c})^2]^2} - \frac{2(\frac{2\pi\nu a}{c})^2ED}{\bar{Z}[K^2+(\frac{2\pi\nu a}{c})^2]^2[K^2+(\frac{4\pi\nu a}{c})^2]} + \frac{\frac{1}{2}(EH+DI)}{[K^2+(\frac{2\pi\nu a}{c})^2][K^2+(\frac{4\pi\nu a}{c})^2]}$$

$$+ \frac{\frac{3}{2}K^2ED}{\bar{Z}[K^2+(\frac{2\pi\nu a}{c})^2]^2[K^2+(\frac{4\pi\nu a}{c})^2]} - \frac{\frac{K^2}{2}(EH+DI)}{(\frac{2\pi\nu a}{c})(\frac{4\pi\nu a}{c})[K^2+(\frac{2\pi\nu a}{c})^2][K^2+(\frac{4\pi\nu a}{c})^2]} + \frac{K^4ED}{\bar{Z}(\frac{2\pi\nu a}{c})(\frac{4\pi\nu a}{c})[K^2+(\frac{2\pi\nu a}{c})^2]^2[K^2+(\frac{4\pi\nu a}{c})^2]}$$

$$- \frac{2ED(\frac{2\pi\nu a}{c})^2}{\bar{Z}[K^2+(\frac{2\pi\nu a}{c})^2]^3} - \frac{2K^2ED}{\bar{Z}[K^2+(\frac{2\pi\nu a}{c})^2]^3} - \frac{2K^4ED}{\bar{Z}(\frac{2\pi\nu a}{c})^2[K^2+(\frac{2\pi\nu a}{c})^2]^3} + \frac{K^2(EH+DI)}{(\frac{2\pi\nu a}{c})^2[K^2+(\frac{2\pi\nu a}{c})^2]^2}$$

$$- \frac{\frac{1}{2}(EH+DI)e^{-K\sigma}}{K^2[K^2+(\frac{2\pi\nu a}{c})^2]} + \frac{2DE(\frac{2\pi\nu a}{c})^2e^{-K\sigma}}{\bar{Z}K^2[K^2+(\frac{2\pi\nu a}{c})^2]^2} + \frac{4K^2EDe^{-K\sigma}}{\bar{Z}[K^2+(\frac{2\pi\nu a}{c})^2]^3}$$

$$- \frac{2ED(\frac{2\pi\nu a}{c})^2e^{-K\sigma}}{\bar{Z}[K^2+(\frac{2\pi\nu a}{c})^2]^2[K^2+(\frac{4\pi\nu a}{c})^2]} + \frac{\frac{1}{2}(EH+DI)e^{-K\sigma}}{[K^2+(\frac{2\pi\nu a}{c})^2][K^2+(\frac{4\pi\nu a}{c})^2]} - \frac{3(\frac{2\pi\nu a}{c})(\frac{4\pi\nu a}{c})EDe^{-K\sigma}}{\bar{Z}[K^2+(\frac{2\pi\nu a}{c})^2]^2[K^2+(\frac{4\pi\nu a}{c})^2]}$$

$$- \frac{(\frac{4\pi\nu a}{c})K^2EDe^{-K\sigma}}{\bar{Z}(\frac{2\pi\nu a}{c})[K^2+(\frac{2\pi\nu a}{c})^2]^2[K^2+(\frac{4\pi\nu a}{c})^2]} + \frac{(EH+DI)e^{-K\sigma}}{[K^2+(\frac{2\pi\nu a}{c})^2][K^2+(\frac{4\pi\nu a}{c})^2]} + \frac{4(\frac{2\pi\nu a}{c})^2EDe^{-K\sigma}}{\bar{Z}[K^2+(\frac{2\pi\nu a}{c})^2]^3}$$

$$\boxed{- \frac{\frac{1}{2}(EH+DI)\sigma}{K[K^2+(\frac{2\pi\nu a}{c})^2]}} - \frac{DE(\frac{2\pi\nu a}{c})^2e^{-2K\sigma}}{\bar{Z}K^2[K^2+(\frac{2\pi\nu a}{c})^2]^2} - \frac{2DE(\frac{2\pi\nu a}{c})^2e^{-K\sigma}\cos\frac{2\pi\nu a\sigma}{c}}{\bar{Z}[K^2+(\frac{2\pi\nu a}{c})^2]^3}$$

$$-\frac{2K^2ED\,e^{-KJ}\cos\frac{2\pi\nu aJ}{c}}{\bar{Z}\,[K^2+(\frac{2\pi\nu a}{c})^2]^3}-\frac{(EH+DI)e^{-KJ}\cos\frac{2\pi\nu aJ}{c}}{[K^2+(\frac{2\pi\nu a}{c})^2]^2}$$

$$-\frac{2(\frac{2\pi\nu a}{c})^3DE\,e^{-KJ}\sin\frac{2\pi\nu aJ}{c}}{\bar{Z}K\,[K^2+(\frac{2\pi\nu a}{c})^2]^3}+\frac{(EH+DI)(\frac{2\pi\nu a}{c})e^{-KJ}\sin\frac{2\pi\nu aJ}{c}}{K\,[K^2+(\frac{2\pi\nu a}{c})^2]^2}$$

$$-\frac{2KED(\frac{2\pi\nu a}{c})e^{-KJ}\sin\frac{2\pi\nu aJ}{c}}{\bar{Z}\,[K^2+(\frac{2\pi\nu a}{c})^2]^3}+\frac{2(\frac{2\pi\nu a}{c})^2ED\cos\frac{4\pi\nu aJ}{c}}{\bar{Z}\,[K^2+(\frac{2\pi\nu a}{c})^2]^2[K^2+(\frac{4\pi\nu a}{c})^2]}$$

$$-\frac{\frac{1}{2}(EH+DI)\cos\frac{4\pi\nu aJ}{c}}{[K^2+(\frac{2\pi\nu a}{c})^2][K^2+(\frac{4\pi\nu a}{c})^2]}-\frac{\frac{3}{2}K^2ED\cos\frac{4\pi\nu aJ}{c}}{\bar{Z}[K^2+(\frac{2\pi\nu a}{c})^2]^2[K^2+(\frac{4\pi\nu a}{c})^2]}+\frac{\frac{K^3}{2}(EH+DI)\cos\frac{4\pi\nu aJ}{c}}{(\frac{2\pi\nu a}{c})(\frac{4\pi\nu a}{c})[K^2+(\frac{2\pi\nu a}{c})^2][K^2+(\frac{4\pi\nu a}{c})^2]}$$

$$-\frac{K^4ED\cos\frac{4\pi\nu aJ}{c}}{\bar{Z}(\frac{2\pi\nu a}{c})(\frac{4\pi\nu a}{c})[K^2+(\frac{2\pi\nu a}{c})^2]^2[K^2+(\frac{4\pi\nu a}{c})^2]}-\frac{2KED(\frac{2\pi\nu a}{c})^2\sin\frac{4\pi\nu aJ}{c}}{\bar{Z}(\frac{4\pi\nu a}{c})[K^2+(\frac{2\pi\nu a}{c})^2]^2[K^2+(\frac{4\pi\nu a}{c})^2]}$$

$$+\frac{\frac{K}{2}(EH+DI)\sin\frac{4\pi\nu aJ}{c}}{(\frac{4\pi\nu a}{c})[K^2+(\frac{2\pi\nu a}{c})^2][K^2+(\frac{4\pi\nu a}{c})^2]}-\frac{3(\frac{2\pi\nu a}{c})KED\sin\frac{4\pi\nu aJ}{c}}{\bar{Z}\,[K^2+(\frac{2\pi\nu a}{c})^2]^2[K^2+(\frac{4\pi\nu a}{c})^2]}$$

$$-\frac{K^3ED\sin\frac{4\pi\nu aJ}{c}}{\bar{Z}(\frac{2\pi\nu a}{c})[K^2+(\frac{2\pi\nu a}{c})^2]^2[K^2+(\frac{4\pi\nu a}{c})^2]}+\frac{\frac{K}{2}(EH+DI)\sin\frac{4\pi\nu aJ}{c}}{(\frac{2\pi\nu a}{c})[K^2+(\frac{2\pi\nu a}{c})^2][K^2+(\frac{4\pi\nu a}{c})^2]}$$

$$+\frac{2KED(\frac{2\pi\nu a}{c})^2\cos\frac{2\pi\nu aJ}{c}}{\bar{Z}K\,[K^2+(\frac{2\pi\nu a}{c})^2]^3}+\frac{4K^2ED\cos\frac{2\pi\nu aJ}{c}}{\bar{Z}\,[K^2+(\frac{2\pi\nu a}{c})^2]^3}+\frac{2K^4ED\cos\frac{2\pi\nu aJ}{c}}{\bar{Z}(\frac{2\pi\nu a}{c})^2[K^2+(\frac{2\pi\nu a}{c})^2]^3}$$

$$-\frac{(EH+DI)\cos\frac{2\pi\nu aJ}{c}}{[K^2+(\frac{2\pi\nu a}{c})^2]^2}-\frac{K^2(EH+DI)\cos\frac{2\pi\nu aJ}{c}}{(\frac{2\pi\nu a}{c})^2[K^2+(\frac{2\pi\nu a}{c})^2]^2}+\frac{2(\frac{2\pi\nu a}{c})^3ED\sin\frac{2\pi\nu aJ}{c}}{\bar{Z}K\,[K^2+(\frac{2\pi\nu a}{c})^2]^3}$$

$$+\frac{4(\frac{2\pi\nu a}{c})KED\sin\frac{2\pi\nu aJ}{c}}{\bar{Z}\,[K^2+(\frac{2\pi\nu a}{c})^2]^3}+\frac{2K^3ED\sin\frac{2\pi\nu aJ}{c}}{\bar{Z}(\frac{2\pi\nu a}{c})[K^2+(\frac{2\pi\nu a}{c})^2]^3}-\frac{(EH+DI)(\frac{2\pi\nu a}{c})\sin\frac{2\pi\nu aJ}{c}}{K\,[K^2+(\frac{2\pi\nu a}{c})^2]^2}$$

$$- \frac{(EH+DI)K \sin \frac{2\pi\nu a J}{c}}{\left(\frac{2\pi\nu a}{c}\right)\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]^2} \Bigg] \tag{92}$$

The terms proportional to time in equations 55 and 92 are circled and are the terms used to find the particle drift velocity.

APPENDIX B

A DETERMINATION OF THE MAGNITUDE OF THE FIRST ORDER DECAYING EXPONENTIAL TERMS OF THE PARTICLE VELOCITY

The first order decaying exponential terms in the solution for the particle position in the cylindrical enclosure are:

$$\left.\begin{aligned} z_1 &= \frac{-E\left(\frac{2\pi\nu a}{c}\right)e^{-K\tau}}{K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]} \\ \phi_1 &= \frac{-D\left(\frac{2\pi\nu a}{c}\right)e^{-K\tau}}{K\left[K^2+\left(\frac{2\pi\nu a}{c}\right)^2\right]} \end{aligned}\right\} \quad (1)$$

Here,

$$D = \frac{J_1(\pi\alpha_{10}\bar{z})}{\bar{z}^2}\sin\bar{\phi}$$

$$E = \frac{\pi\alpha_{10}}{2}\cos\bar{\phi}\left[J_2(\pi\alpha_{10}\bar{z}) - J_0(\pi\alpha_{10}\bar{z})\right]$$

$$K = \frac{6\pi\mu\sigma a}{mc} .$$

From equation 4.1 the expression for the particle velocity, containing just the first order decaying terms, can be written as:

$$\vec{v} = \left[\vec{i}_r\, c\varepsilon\frac{dz_1}{d\tau} + \vec{i}_\varphi\, c\varepsilon\,\bar{z}\,\frac{d\phi_1}{d\tau}\right] . \quad (2)$$

Substituting the expressions for z_1 and ϕ_1 into equation 2 the following is obtained:

$$\vec{v} = \left[\vec{i}_r\,\frac{\varepsilon c E\left(\frac{2\pi\nu a}{c}\right)e^{-K\tau}}{K^2+\left(\frac{2\pi\nu a}{c}\right)^2} + \vec{i}_\varphi\,\frac{\varepsilon c\bar{z}D\left(\frac{2\pi\nu a}{c}\right)e^{-K\tau}}{K^2+\left(\frac{2\pi\nu a}{c}\right)^2}\right] . \quad (3)$$

For an initial position of the particle of $\bar{z} = \frac{\bar{r}}{a} = .5$, ϕ = 45°, for P_0/P_c = .316 and for gas and particle properties evaluated for the conditions used in the calculations for Table I, the following expression is obtained for the particle velocity due to the damped terms:

$$\vec{v} = \left[\vec{i}_r \left(71.1\, e^{-4350t} \right) + \vec{i}_\varphi \left(91.3\, e^{-4350t} \right) \right]. \qquad (4)$$

For $t \approx .0023$ seconds, which is time for approximately two oscillations in the chamber, $e^{-4350t} \doteq e^{-10}$, and the velocity amounts to:

$$\vec{v} = \vec{i}_r (-.003) + \vec{i}_\varphi (.004) \quad \text{ft./sec.} \qquad (5)$$

For these same conditions the steady state drift velocity as calculated in Table I is:

$$\vec{v} = \vec{i}_r (2.7) + \vec{i}_\varphi (-.37) \quad \text{ft./sec.} \qquad (6)$$

Comparing equations 5 and 6, it is seen that after just two oscillations the velocity contribution of the decaying term is negligible compared to the drift velocity.