STABILITY OF LAMINAR WAKES

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ABSTRACT

This investigation deals with the effects of compressibility on the hydrodynamic stability of wake flows. It is found that the effect of temperature is two-fold: (1), as the wake core temperature increases, the range of Mach numbers over which neutral and self-excited subsonic disturbances can exist also increases; (2) as long as the relative Mach number is below the critical Mach number the neutral inviscid wave number will decrease with increasing core temperature, implying that a hot wake will be more stable than a cool one.

The analysis of Batchelor and Gill for the inviscid stability of axi-symmetric incompressible jets has been extended to the more general problem of compressible wakes and jets. It is shown that the results are directly analogous to those obtained for the two-dimensional problem. The sinuous (n = 1) mode is the most unstable allowable mode. This unstable mode is observed in a hypersonic wake.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

The symbols used in the present report are in general those commonly used in the literature on hydrodynamic stability. In some regretable instances a symbol will represent more than one item. To minimize the confusion, the different definitions of the same symbol will have listed the section of the report in which they appear.

	Dimensional Quantities	Dimensionless Quantities	Reference Quantities
Two-Dimensional Flows:			
Positional Coordinates:			
longitudinal	x*	x	L*
normal	y*	y	L*
Velocity Components:			
longitudinal	$U^* = \overline{W}^* + U^{*'}$	$\omega(y) + f(y)e^{i\alpha(x-ct)}$	V^*
	$W^* = \overline{U}^* - \overline{U}_e^*$		
normal	$V^* = \overline{V}^* + V^{*'}$	$v(y) + \alpha\phi(y)e^{i\alpha(x-ct)}$	V^*
Density	$\rho^* = \overline{\rho}^* + \rho^{*'}$	$\rho(y) + s(y)e^{i\alpha(x-ct)}$	ρ_e^*
Pressure	$p^* = \overline{p}^* + p^{*'}$	$p(y) + \pi(y)e^{i\alpha(x-ct)}$	P_e^*
Temperature	$T^* = \overline{T}^* + T^{*'}$	$T(y) + \theta(y)e^{i\alpha(x-ct)}$	T_e^*
Axi-Symmetric Flows:			
Positional Coordinates:			
axial	x*	x	L*
radial	r*	r	L*
angular		ϕ	

Velocity Components:			
axial	$q_x^* = \bar{\omega}^* + q_x^{*\prime}$ $\bar{\omega}^* = \bar{q}_x^* - U_e^*$	$\omega(r) + q_x(r)\, e^{i\alpha(x-ct)+in\phi}$	V^*
radial	$q_r^* = \bar{q}_r^* + q_r^{*\prime}$	$\overline{q_r(r)} + q_r(r)\, e^{i\alpha(x-ct)+in\phi}$	V^*
angular	$q_\phi^* = \bar{q}_\phi^* + q_\phi^{*\prime}$	$\overline{q_\phi(r)} + q_\phi(r)\, e^{i\alpha(x-ct)+in\phi}$	V^*
Angular Wave Number:		n	
Density	$\rho^* = \bar{\rho}^* + \rho^{*\prime}$	$\rho(r) + s(r)\, e^{i\alpha(x-ct)+in\phi}$	ρ_e^*
Pressure	$p^* = \bar{p}^* + p^{*\prime}$	$p(r) + \pi(r)\, e^{i\alpha(x-ct)+in\phi}$	p_e^*
Temperature	$T^* = \bar{T}^* + T^{*\prime}$	$T(r) + \theta(r)\, e^{i\alpha(x-ct)+in\phi}$	T_e^*
Time	t^*	t	L^*/V^*
Wave Length	Λ^*	Λ	L^*
Wave Number	$\alpha^* = 2\pi/\Lambda^*$	$\alpha = 2\pi/\Lambda$	$1/L^*$
Disturbance Propagation Velocity	c^*	$\bar{c}$	U_e^*
	$c^* - U_e^*$	c	V^*

a	speed of sound
b	unknown constant (Section IV. 2)
C_D	drag coefficient
c_g^*	group velocity of disturbance
$\bar{c}_g$	c_g^*/U_e^* dimensionless group velocity
c_p^*	local relative propagation velocity (Eq. (5. 20))
D	(d/dy) (Section III. 1, Appendix C)
d*	characteristic body dimension
E	exp $(i\alpha(x-ct) + in\phi)$ [Eq. (5. 49)]

G	$\pi'/a^2 T^2 \pi$ [Eq. (4.18)]
$\underset{\sim}{G}$	defined by Eq. (5.52)
H	ηG (Appendix D)
H^*	stagnation enthalpy
$H^{(1),(2)}_{1/3}$	Hankel functions of order 1/3, first and second kind
h	dimensionless static enthalpy (Appendix B)
$h_{1,2}(z)$	$\left[\frac{2}{3}(iz)^{3/2}\right] H^{(1),(2)}_{1/3}\left(\frac{2}{3}(iz)^{3/2}\right)$
$I_1(a\sqrt{\Omega}\, r)$	modified Bessel function of the second kind of order one
i	$\sqrt{-1}$
$K_1(a\sqrt{\Omega}\, r)$	modified Bessel function of the second kind of order one
k	gradient of density-vorticity product (Section V)
L^*	characteristic length
$L(z)$	Lommel function (Appendix G)
M	V^*/a_e^* relative Mach number
M_e	local Mach number outside the mean wake
M_L	local Mach number of disturbance [Eq. (5.21)]
M_∞	free stream Mach number
m	$\sqrt{a^2 + (n^2/r^2)}$ total wave number
N^2	Eq. (5.68)
p	Eq. (5.10)
$Q^*(\vec{r}^*, t^*)$	quantity of total flow
$\bar{Q}^*(\vec{r}^*)$	mean or steady component of flow quantity
$Q^{*\prime}(\vec{r}^*, t^*)$	fluctuating component of flow quantity
$\tilde{q}^{*\prime}(\vec{r}^*)$	fluctuation amplitude

$q(y)$, $q(r)$	fluctuation amplitude for nearly parallel flow
q_1	$\frac{n}{mr} q_\phi + \frac{\alpha q_x}{m}$
q_2	q_r
q_3	$\frac{\alpha q_\phi}{m} - \frac{n}{mr} q_x$
R	$V^* L^* / \nu_e^*$ wake Reynolds number
R^*	gas constant
$R_{\delta *}$	Reynolds number based on displacement thickness
$R_{e_{d*}}$	local external Reynolds number based on d*
Re_{x*}	local external Reynolds number based on x*
$Re_{\theta *}$	local external Reynolds number based on θ*
$R_{\infty_{d*}}$	$\rho_\infty^* U_\infty^* d^* / \mu_\infty^*$
$\vec{r}^*$	position vector
r	$\int (dr/\xi_R)$ (Appendix F)
ΔT	temperature excess at centerline of wake
U	transformed mean velocity component (Appendix B)
U^*	mean velocity component in x* direction
ΔU	velocity defect at center of wake
V	transformed mean velocity component (Appendix B)
V^*	characteristic velocity
W	1 - U (Appendix B)
$\underline{W}$	Imag $(\xi \hat{\psi} \psi')$ (Section V)
X	transformed x coordinate (Appendix B)

X	$(1/\xi rT)$	(Appendix E)
X_c	$(1/\xi_c r_c T_c)$	
Y	transformed y coordinate	(Appendix B)
Y	$(1/\xi)(\xi\omega')'$	(Appendix E)
z	$(\omega_c')^{1/3}\eta = (\omega_c' \alpha R)^{1/3} (r-r_c)$	
β	$\sqrt{\alpha^2 - i\alpha R c}$	
Γ	disturbance vorticity	
$\bar{\Gamma}$	mean vorticity	
γ	ratio of specific heats	
Δ	dimensionless heat transfer coefficient	(Appendix B)
δ	$\tan^{-1}(\alpha r/n)$	(Section V)
δ^*	net heat transfer	(Appendix B)
ϵ	$1/(\alpha R)^{1/3}$	(Appendix G)
η	Dorodnitsyn-Howarth variable	[Eqs. (2.47) and (2.56)]
η	$r - r_c/\epsilon$	(Appendix G)
Θ	dimensionless momentum thickness	(Appendix B)
θ^*	momentum thickness	
μ	viscosity	
ν	kinematic viscosity	
ξ	Eq. (5.10)	
σ	$\beta - \alpha = \alpha\sigma$	(Section III and Appendix C)
τ	$-\rho\overline{q_x' q_r'}$	Reynolds shear stress
χ	$r - r_c$	
ψ	rq_r	
Ω	$1 - M^2 c^2$	

Subscripts

c	critical point
e	local condition outside mean wake
I	imaginary part of quantity
i	point at which disturbances begin to amplify
o	initial values (Appendix B)
o	values at axis (Appendix D)
R	real part of quantity
s	neutral, inviscid values
∞	free stream conditions
1, 2...	first, second solution, etc.

Superscripts

$\hat{}$	complex conjugate
(0), (1), (2)	zero, first, second order quantities, etc.

A bar over a quantity indicates mean value.

Primes generally denote differentiation with respect to y or r. The few instances where primes denote a fluctuating quantity should not cause any confusion.

I. INTRODUCTION

Experimental[2-4] studies have shown that transition in incompressible free boundary layers is preceded by a linear and non-linear wave-type instability. The linear instability can be described by the small disturbance theory of hydrodynamic stability[5-31]. The non-linear instability is a very complex phenomena and is not yet understood. Recently, the problem of laminar-turbulent transition in hypersonic wakes has also been of considerable interest.[1] The purpose of this present investigation is to study the effects of compressibility on the hydrodynamic stability of wake-type flows.

The stability characteristics of wake flows are relatively insensitive to Reynolds number, for sufficiently high Reynolds numbers, because of the occurrence of a point of inflection in the density vorticity product[22, 32]. Therefore, interesting and important results can be obtained by considering the "inviscid limit" of the small disturbance equations, in which the viscosity and conductivity of the fluid can be neglected to a certain order.

This study will be restricted to subsonic disturbances, i. e., disturbances whose propagation velocity is subsonic with respect to the free stream velocity. These disturbances have amplitudes that die out exponentially far from the wake axis. If such disturbances exist then the mean flow is unstable to small disturbances. The question of the stability of a flow to supersonic disturbances has not been resolved[32-35].

The inviscid stability characteristics of two-dimensional incompressible and compressible wake flows are studied using Gaussian distributions for velocity and temperature in the Dorodnitsyn-Howarth variable. The incompressible wake was also studied at low Reynolds numbers and wave numbers. The results of Batchelor and Gill[36] and Gill[37] for axi-symmetric incompressible wake-type flows have been extended to include the effects of compressibility.

II. FORMULATION OF THE PROBLEM

An infinitesimally small disturbance is imposed upon a mean or a steady flow and the behavior of the amplitude of the disturbance is examined as time progresses. If, for large values of time, the disturbance is damped out, the motion is said to be stable; if not, the motion is said to be unstable with respect to infinitesimally small disturbances. It is much easier to prove that a motion is unstable than stable. If the flow is unstable to disturbances of any kind, even the simplest kind, it is always unstable, but the flow may be stable with respect to one type of disturbance and not another. In hydrodynamic stability theory, the disturbance is assumed to have a wave-like nature. The problem is to find certain combinations of the wave number and wave speed of the disturbance and the Reynolds number of the mean flow for which the fluid motion is unstable, or neutrally stable.

II. 1. Outline of the Stability Problem

The total flow consists of a time-independent or mean component, $\overline{Q}^*$, and an infinitesimally small component, $Q^{*\prime}$, which is both space and time dependent:

$$Q^* (\vec{r}^*, t^*) = \overline{Q}^* (\vec{r}^*) + Q^{*\prime} (\vec{r}^*, t^*) \tag{2.1}$$

where

$$|Q^*| \,/\, |\overline{Q}^*| << 1 \quad .$$

The total flow, Q^*, satisfies the conservation equations of mass, momentum and energy and an equation of state. The mean flow, $\overline{Q}^*$, satisfies the steady flow equations or some approximation to them, for example, the boundary layer equations. The conservation equations for the disturbance are obtained by substituting expressions of the form, Eq. (2.1), for the flow variables into the total flow equations and subtracting out the mean flow equations. The hydrodynamic stability equations are obtained from this set by neglecting quadratic and higher order terms of the disturbance quantities.

The coefficients of the resulting linear partial differential equations depend upon the mean flow quantities. Time appears only as the derivative $(\partial/\partial t^*)$ and hence solutions containing an exponential time factor

$$Q^{*\prime}(\vec{r}^*, t^*) = \tilde{q}^{*\prime}(\vec{r}^*)\, e^{-i\alpha^* c^* t^*} \tag{2.2}$$

may be assumed. The resulting differential equations will contain the space coordinates as the only independent variables.

This study will be limited to parallel or quasi-parallel flows, i.e., motion in which the mean normal velocity is zero or very small compared to the main velocity component. The following order of magnitude relations from boundary layer theory apply to the mean flow quantities:

$$\begin{aligned} &\text{(a)} \quad \bar{V}^*/\bar{U}^* \sim (1/R_{\delta^*}) << 1 \\ &\text{(b)} \quad (\partial\bar{Q}^*/\partial x^*) \;/\; (\partial\bar{Q}^*/\partial y^*) \sim (1/R_{\delta^*}) << 1 \quad , \end{aligned} \tag{2.3}$$

where

x^* longitudinal coordinate in two-dimensional flow or axial coordinate in axi-symmetric flow

y^* normal coordinate in two-dimensional flow or radial coordinate, r^*, in axi-symmetric flow

$\bar{U}^*$ mean velocity component in x^* direction

$\bar{V}^*$ mean velocity component in y^* direction

R_{δ^*} Reynolds number based upon displacement thickness.

The mean flow quantity, $\bar{Q}^*$ is a function of the position coordinates, x^* and y^*. Expand $\bar{Q}^*$ about the point $x^* = x^*_p$:

$$\bar{Q}^*(x^*, y^*) = \bar{Q}^*(x^*_p, y^*) + (\partial\bar{Q}^*/\partial x^*)_{x^* = x^*_p} (x^* - x^*_p) + . \tag{2.4a}$$

The region adjacent to the point under consideration is taken to be of the order of a few wavelengths of the disturbance in the x^* direction

$$x^* - x^*_p \sim \lambda^* \sim (1/\alpha^*) \quad . \tag{2.4b}$$

Then from Eqs. (2.3b) and (2.4b)

$$\begin{aligned}(x^*-x^*_p)\,(\partial\bar{Q}^*/\partial x^*)_{x^*=x^*_p} &\sim (1/\alpha^* R_{\delta^*})\,(\partial\bar{Q}^*/\partial y^*)_{x^*=x^*_p} \\ &\sim \frac{\bar{Q}^*(x^*_p, y^*)}{\alpha\, R_{\delta^*}}\end{aligned} \tag{2.4c}$$

and for large values of αR_{δ^*}, Eq. (2.4a) becomes

$$\bar{Q}^*(x^*, y^*) = \bar{Q}^*(x_p^*, y^*)\left[1 + O(1/\alpha R_{\delta^*})\right] \quad . \tag{2.4d}$$

Therefore all mean quantities can be considered to be independent of the normal (or radial) space variable to order $(1/\alpha R_{\delta^*})$.

By considering disturbances that are spatially periodic, both in the direction of flow and in the direction perpendicular to the plane of symmetry of the mean motion, Squire[38] has shown for incompressible flow that two dimensional disturbances are less stable than three-dimensional disturbances. For compressible flow, Dunn and Lin[39] have shown, by neglecting dissipation terms and some terms involving the fluctuating viscosity and thermal conductivity (which are valid at moderate Mach numbers), that an equivalent two-dimensional disturbance is not possible, but that the transformed three-dimensional disturbance equations are of the same form as those for two-dimensional disturbances. In particular, by neglecting viscosity and thermal conductivity, the three-dimensional disturbance equations are exactly of the same form as the two-dimensional ones. Therefore, important features of the stability problem can be obtained by considering two-dimensional disturbances alone. For parallel and quasi-parallel flow, the coefficients of the equations are independent of x* and consequently solutions of the form

$$\tilde{q}^{*\prime}(x^*, y^*) = q^{*\prime}(y^*)\, e^{i\alpha^* x^*} \tag{2.5}$$

might be expected. The exponent is purely imaginary since the disturbance must be bounded for x* at both $+\infty$ and $-\infty$. For two-dimensional flows a disturbance of the form

$$Q^{*\prime}(x^*, y^*, t^*) = q^{*\prime}(y^*)\, e^{i\alpha^*(x^*-c^*t^*)} \tag{2.6}$$

will reduce the set of linear partial differential equations to a set of ordinary differential equations in y*.

There is no direct analogue of Squire's result for axi-symmetric parallel and quasi-parallel flows. However, Lessen, et al[21] and Pai[27] indicate that for <u>rotationally symmetric</u> disturbances, the incompressible disturbance equations, except for the obvious coordinate scale factors, are similar to those for two dimensional disturbances. Batchelor and

Gill[36] show, by a suitable velocity transformation, that for disturbances with an angular dependence, the incompressible equations are exactly analogous to the two-dimensional ones, again, except for the obvious coordinate scale factors. In Section V. 1 this latter result is extended to the compressible case. The mean flow quantities for parallel and quasi-parallel flows do not depend on the angular coordinate, ϕ, and depend only on the coordinate normal to the direction of the mean motion, r^*, to order $(1/\alpha R_{\delta^*})$. Since the amplitude of the disturbance must be single-valued with respect to the angular coordinate, ϕ, a disturbance of the form

$$Q^{*\prime}(r^*, \phi, x^*, t^*) = q^{*\prime}(r^*)\, e^{i\alpha^*(x^* - c^* t) + in\phi} \tag{2.7}$$

where n is an integer, may be assumed. The resultant set of ordinary differential equations will have r^* as the only independent variable.

In Eq. (2. 6) [or Eq. (2. 7)] the disturbance amplitude $q^{*\prime}(y^*)$ [or $q^{*\prime}(r^*)$] and the wave velocity c^* are taken to be complex. The main flow is stable, neutrally stable, or unstable to these waves according to whether the imaginary part of c^* is negative, zero, or positive, respectively. The quantity α^* is the wave number of the disturbance and is taken to be real and positive. The real part of c^* is the phase or propagation velocity of the wavy disturbance.

The assumption that the disturbance has the form $e^{-i\alpha^* c^* t^*}$ [Eq. (2. 2)] is known as the normal mode approach to hydrodynamic stability. If there are some values of $\alpha^* c^*$, with $c_I^* > 0$, such that a non-trivial solution satisfying the disturbance equations and boundary conditions exists, then the flow is said to be unstable; if not, it is stable. Recently, the initial value problem has been emphasized by Case[40-44] and Lin[45]. An arbitrary small perturbation is introduced into the flow at time, $t^* = 0$, and its subsequent motion is followed by means of a normal mode expansion. If there is a single mode with $c_I^* > 0$ the perturbation grows exponentially with time and is said to be unstable. The normal mode approach should be equivalent to the initial value method although there seems to be some inconsistencies between the two results as the Reynolds number becomes infinite[40-42]. Lin[45] has pointed out that these can be resolved by considering the limit of a normal mode in the viscous theory. This limit is <u>not</u> the normal mode in the inviscid theory, and vice versa. The normal mode approach is applicable to differential operators having discrete eigenvalues while the initial value method should be used with singular operators and/or continuous eigenvalues. However, it appears that the modes leading to instability are associated with the discrete eigenvalues. It is for this reason that the normal mode approach will be used in this text.

The amplification rate of the disturbance is defined as follows:

$$(1/Q^{*\prime})\,(\partial Q^{*\prime}/\partial t^*) \;=\; \alpha^*\, c_I^* \tag{2.8a}$$

and

$$Q^{*\prime} \;=\; Q_i^{*\prime}\, \exp \int_{t_i^*}^{t^*} \alpha^*\, c_I^*\, dt^* \quad . \tag{2.8b}$$

In the laboratory, the experimenter has a quasi-stationary problem. As the disturbance propagates downstream of its origin, its amplitude changes in both space and time. One measures the spatial amplification rate, or the rate at which a disturbance will amplify with distance in the mean flow direction. The wave speed is a function of both the wave number of the disturbance and the Reynolds number of the mean flow. If the parallel and quasi-parallel assumptions are made, then the group velocity of the disturbance, i.e., the velocity at which the disturbance energy must propagate[46], is

$$
\begin{aligned}
(dx^*/dt^*) &= c_g^* = (d/d\alpha^*)\left[\alpha^* c^*\right] \\
&= c^* + (\alpha^* dc^*/d\alpha^*) \quad .
\end{aligned}
\tag{2.9}
$$

The spatial amplification rate is

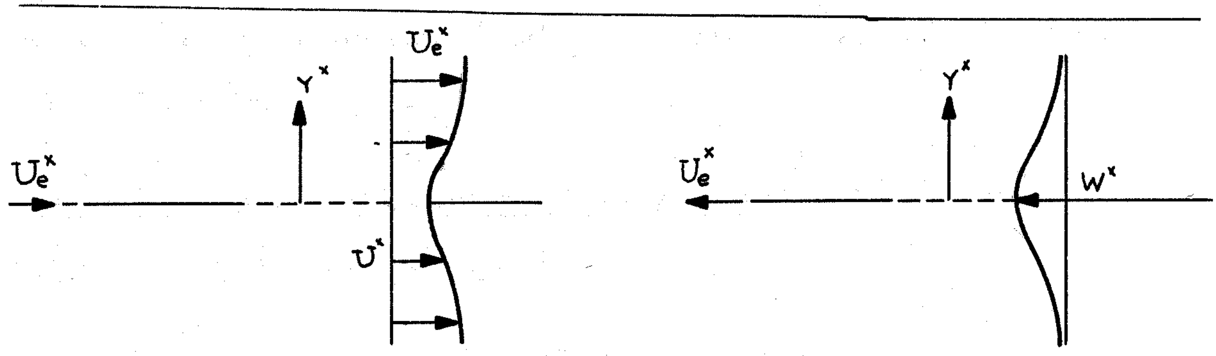
$$(1/Q^{*\prime})(\partial Q^{*\prime}/\partial x^*) = (\alpha^* c_I^*/c_g^*) \tag{2.10a}$$

and

$$Q^{*\prime} = Q_i^{*\prime} \exp \int_{x_i^*}^{x^*} (\alpha^* c_I^*/c_g^*)\, dx^* \quad . \tag{2.10b}$$

The spatial amplification rate is constant for parallel flows. For quasi-parallel flows, the spatial amplification rate is computed at each streamwise station by using a mean velocity profile that is assumed to be independent of x at that station. The total amplification (or decay) of the disturbance as it moves downstream is found by the piecewise integration of the local amplification (or decay) rates.

For wake-type flows, it is expected that the spatial amplification rates depend upon the decay of the mean velocity profile. Therefore, it is convenient to transform from a coordinate system in which the observer is fixed in the body to one in which the observer is fixed in a fluid at rest. In this latter system, the observer sees the "velocity defect" of the wake. [See Sketch 2.1]



$$w^*(y^*) \;=\; U^*(y^*) - U_e^*$$

Body-Centered Coordinates — Coordinates Fixed in Fluid at Rest

Sketch 2.1

In this coordinate system the wave has a propagation velocity $(c^* - U_e^*)$. Let the mean flow be dimensionally represented by a characteristic length, L^*, and a characteristic velocity, V^*, and the temperature, density, pressure and viscosity by their external values, so that [See List of Symbols.] :

$$w(y) = \frac{w^*(y^*)}{V^*} \quad , \quad c = \frac{c^* - U_e^*}{V^*} \quad , \quad \alpha = \alpha^* L^*$$
$$R = \frac{V^* L^*}{\nu_e^*} \quad , \quad M = \frac{V^*}{\sqrt{\gamma R^* T_e^*}} \quad . \tag{2.11}$$

V^* and L^* will be taken to be the velocity defect at the centerline $[w(0) = 1]$ and the half-width of the wake, respectively [Section II.4] .

The disturbance equations and boundary conditions will be derived for two-dimensional [Section II.2] and axi-symmetric [Section II.3] flows. The outer boundary conditions for the compressible problem will be discussed in detail in Section IV. 1 (two-dimensional case) and Section V. 2 (axi-symmetric case). The quantities are defined in the List of Symbols.

II. 2. Two-Dimensional Free Shear Flows. Small Disturbance Equations and Boundary Conditions

II. 2. a. Viscous, Incompressible Problem

The incompressible, small disturbance equations are a limiting case of the complete compressible equations. This case is obtained by neglecting the viscous dissipation and heat conduction terms in the conservation equations and assuming that the mean temperature, pressure, density, viscosity and thermal conductivity are constants. In addition, if the gradient of the temperature fluctuation vanishes at the axis and the outer edge of the mean flow, the temperature fluctuation and hence the density fluctuation can be set equal to zero[32]. The dimensionless small disturbance equations are[32*]

Continuity

$$\phi' + i\,f = 0 \tag{2.12}$$

x- Momentum

$$i(w-c)f + w'\phi = -i\pi + (1/\alpha R)\left[f'' + \alpha^2\,(i\phi' - 2f)\right] \tag{2.13}$$

y- Momentum

$$i\alpha^2(w-c)\phi = -\,\pi' + (\alpha/R)\left[2\phi'' + if' - \alpha^2\phi\right] \quad . \tag{2.14}$$

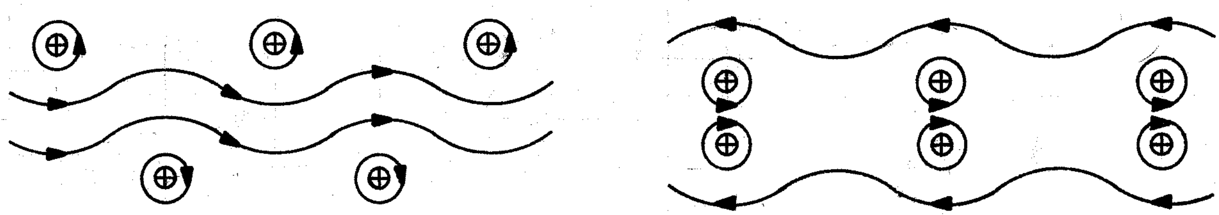
---

* Primes indicate differentiation with respect to y for the two-dimensional case and differentiation with respect to r for the axi-symmetric case.

This system consists of three linear disturbance equations in the three dependent perturbation amplitudes f, ϕ and π, where the mean velocity w is determined from the mean or steady-state equations. The system is of the fourth order in the dependent variables.

If the mean velocity profile is symmetrical $[w'(0) = 0]$, then the disturbance amplitudes can be decomposed into even and odd parts, each part satisfying Eqs. (2.12) - (2.14). The even part of the longitudinal velocity disturbance, f, corresponds to anti-symmetrical (or sinuous) oscillations and the odd part to symmetrical (or varicose) oscillations.

The anti-symmetrical oscillations are analogous to two parallel rows of equally spaced vortices in alternate positions (Sketch 2.2.a) (Kármán vortex street) and the symmetrical oscillations to symmetrically placed vortices (Sketch 2.2.b). Kármán[47] has shown that the symmetrical



(a) Flow in Alternate Vortex Street (b) Flow in Symmetrical Vortex Street

Sketch 2.2

vortex street is unstable for all values of the spacing ratio (ratio of transverse to longitudinal dimension) and will tend to rearrange itself into the alternate vortex street. The alternate position is stable for only one spacing ratio and unstable for all others. Physically, the anti-symmetrical disturbances are observed more often than the symmetrical ones; for example, in the wake behind a circular cylinder at very low Reynolds numbers. This fact suggests that the anti-symmetrical disturbance is more unstable than the symmetrical one and the minimum critical Reynolds number, below which all disturbances are damped, will be lower for the former.

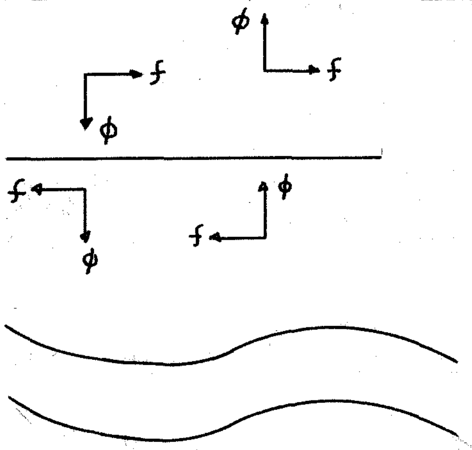
Using Eqs. (2.12) and (2.13) the boundary conditions at the axis are:

Anti-symmetrical oscillations (Sketch 2.3.a)

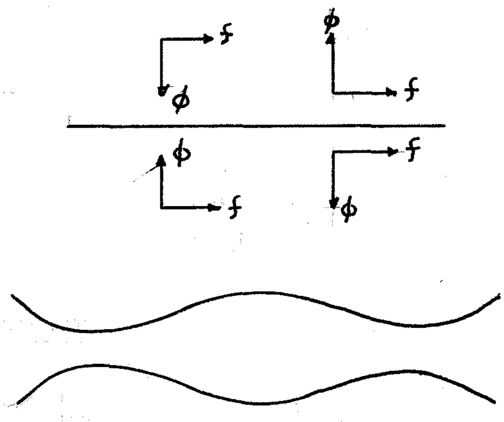
$$\begin{aligned} &f(0) = 0 \quad , \quad \pi(0) = 0 \quad , \quad \phi'(0) = 0 \\ &f''(0) = 0 \quad , \quad \pi''(0) = 0 \quad , \quad \phi'''(0) = 0 \end{aligned} \tag{2.15}$$

Symmetrical oscillations (Sketch 2.3.b)

$$\begin{aligned} &f'(0) = 0 \quad , \quad \pi'(0) = 0 \quad , \quad \phi(0) = 0 \\ &f'''(0) = 0 \quad , \quad \pi'''(0) = 0 \quad , \quad \phi''(0) = 0 \quad . \end{aligned} \tag{2.16}$$



(a) Anti-symmetric oscillations



(b) Symmetrical oscillations

Sketch 2.3

The quantities f and π can be eliminated from Eqs. (2.12) - (2.14) and a fourth order equation in ϕ can be found

$$(w-c)(\phi''-\alpha^2\phi) - w''\phi = (1/i\alpha R)\left[\phi^{IV} - 2\alpha^2\phi'' + \alpha^4\phi\right] \quad . \tag{2.17}$$

The boundary conditions for large values of y are obtained from Eq. (2.17). As $y \longrightarrow \infty$, $w \longrightarrow 0$ exponentially and Eq. (2.17) becomes

$$\phi^{IV} - \left|\alpha^2 + \beta^2\right| \phi'' + \alpha^2 \beta^2 \phi = 0 \tag{2.18}$$

where

$$\beta^2 = \alpha^2 - i\alpha Rc \quad .$$

The solutions of Eq. (2.18) are:

$$\phi \sim e^{\pm\alpha y} \quad , \quad e^{\pm\beta y} \quad . \tag{2.19}$$

Now ϕ must be bounded for large values of y $(y > 0)$. If the real part of β is positive then the solutions with the positive exponent must be rejected and the outer boundary conditions for Eq. (2.17) are

$$\phi \sim e^{-\alpha y} \quad , \quad e^{-\beta y} \tag{2.20}$$

and from Eqs. (2.12) and (2.13):

$$f, \pi \sim e^{-\alpha y} \quad , \quad e^{-\beta y} \quad . \tag{2.21}$$

II. 2. b. Inviscid, Compressible Problem

If the solution of the disturbance equations is assumed to be of the form

$$q(y) = q^{(0)}(y) + (1/\alpha R)\; q^{(1)}(y) + \cdots \tag{2.22}$$

and the limit $\alpha R \longrightarrow \infty$ is taken, the resulting equations for the zeroth approximation, $q^{(0)}$, are called the inviscid small disturbance equations. They are identical with the equations obtained by ignoring viscosity and thermal conductivity. The dimensionless inviscid equations are[32]:

Continuity

$$\phi' + i f = (T'/T)\,\phi - i\,(w-c)\,(s/\rho) \tag{2.23}$$

x- Momentum

$$\rho\left[i\,(w-c)\,f + w'\,\phi\right] = -\,(i\pi/\gamma M^2) \tag{2.24}$$

y- Momentum

$$\alpha^2\left[\rho\, i\,(w-c)\,\phi\right] = -\,(\pi'/\gamma M^2) \tag{2.25}$$

Energy

$$\rho\left[i\,(w-c)\,\theta + T'\,\phi\right] = -\,(\gamma - 1)\,(\phi + i f) \tag{2.26}$$

State

$$(s/\rho) = \pi - (\theta/T) \qquad \rho T = 1 \quad . \tag{2.27}$$

This is a system of five equations in the five disturbance variables f, ϕ, π, s, and θ, where the mean flow quantities ρ, T, and w are determined from the mean equations of motion. Upon eliminating four out of the five dependent variables, the system is seen to be of the second order.

Again, as in the incompressible case, if the mean velocity and temperature profiles are symmetrical, then the disturbance amplitudes can be decomposed into even and odd parts, each part satisfying Eqs. (2.23) - (2.27). The boundary conditions at the axis are:

Anti-symmetrical oscillations

$$f(0) = 0 \quad , \quad \pi(0) = 0 \quad , \quad \phi'(0) = 0$$
$$s(0) = 0 \quad , \quad \theta(0) = 0 \tag{2.28}$$

Symmetrical oscillations

$$f'(0) = 0 \quad , \quad \pi'(0) = 0 \quad , \quad \phi(0) = 0$$
$$s'(0) = 0 \quad , \quad \theta'(0) = 0 \quad . \tag{2.29}$$

The outer boundary conditions will be derived in Section IV. 1 but will be included here for completeness. It is

$$\pi' + \alpha \sqrt{\Omega}\, \pi = 0 \qquad y \longrightarrow \infty \tag{2.30}$$

where

$$\Omega = 1 - M^2 c^2 \qquad - \pi < \arg \Omega < \pi \quad . \tag{2.31*}$$

II. 3. Axi-Symmetric Free Shear Flows. Small Disturbance Equations and Boundary Conditions

The same assumptions regarding the derivation of the axi-symmetric small disturbance equations apply as for the two-dimensional case and will not be repeated here.

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* In this equation, π equals 3.141.... .

II. 3. a. Viscous, Incompressible Problem

The dimensionless small disturbance equations are[36]:

Continuity

$$(1/r)\left[r\, q_r\right]' + i\, q_x + (in/\alpha)\,(q_\phi/r) = 0 \tag{2.32}$$

x- Momentum

$$i(w-c)q_x + w'q_r = -i\pi + (1/\alpha R)\left[(1/r)(rq_x')' - (\alpha^2 + \frac{n^2}{r^2})\, q_x\right] \tag{2.33}$$

r- Momentum

$$i\alpha^2(w-c)q_r = -\pi' + (\alpha/R)\left[(1/r)(rq_r')' - (\alpha^2 + \frac{n^2+1}{r^2})q_r - \frac{2in}{r^2}\,\frac{q_\phi}{\alpha}\right] \tag{2.34}$$

ϕ- Momentum

$$i(w-c)q_\phi = -(in/\alpha)(\pi/r) + (1/\alpha R)\left[(1/r)(rq_\phi')' - (\alpha^2 + \frac{n^2+1}{r^2})q_\phi + \frac{2in}{r^2}\alpha q_r\right] . \tag{2.35}$$

This is a system of four linear equations in four dependent variables q_r, q_ϕ, q_x and π. This system is of the sixth order in the dependent variables since q_ϕ and q_ϕ' can be eliminated algebraically from Eqs. (2.32), (2.34) and (2.35).

For the axi-symmetric wake, the boundary conditions on the axis are kinematic in nature (do not depend upon viscosity) and can be derived from the inviscid equations (Appendix A). All the disturbance amplitudes and the vorticity disturbance must be finite on the axis. The boundary conditions are:

$$\begin{array}{lll} n = 0 & q_r = 0 \quad , & q_\phi = 0 \\ & q_x, \pi \text{ are arbitrary} & \\ n \neq 0 & q_x = 0 \quad , & \pi = 0 \\ \quad n = 1 & q_\phi = -\alpha q_r & q_r \text{ is arbitrary} \\ \quad n > 1 & q_\phi = 0 \quad , & q_r = 0 \quad . \end{array} \tag{2.36}$$

The n = 0 and n = 1 modes are shown in Figure 1.

Far from the axis, the boundary conditions should be the same as for the two-dimensional case (Section V. 2). These conditions can also be derived by taking the limit of Eqs. (2. 32) - (2. 35) as $r \longrightarrow \infty$. The outer boundary conditions are:

$$\begin{aligned} \pi, \; q_x, \; q_r &\sim (1/\sqrt{r})\, e^{-\alpha r} \quad , \quad (1/\sqrt{r})\, e^{-\beta r} \\ q_\phi &\sim \sqrt{r} \quad e^{-\alpha r} \quad , \quad \sqrt{r} \quad e^{-\beta r} \end{aligned} \tag{2.37}$$

where the real part of β is positive and

$$\beta^2 = \alpha^2 - i\alpha Rc \quad .$$

III. 3. b. Inviscid, Compressible Problem

The dimensionless inviscid small disturbance equations are:

Continuity

$$(1/r)\left[rq_r\right]' + iq_x + (in/\alpha)(q_\phi/r) + i(w-c)(s/\rho) - q_r(T'/T) = 0 \tag{2.38}$$

x- Momentum

$$\rho\left[\, i(w-c)\, q_x + w'\, q_r\right] = -\,(i\pi/\gamma M^2) \tag{2.39}$$

r- Momentum

$$i\,\alpha^2\,\rho\,(w-c)\,q_r = -\,(\pi'/\gamma M^2) \tag{2.40}$$

ϕ- Momentum

$$i\,\rho\,(w-c)\,q_\phi = -\,(in\pi/\alpha r\gamma M^2) \tag{2.41}$$

Energy

$$\rho\left[i(w-c)\theta + T'q_r\right] = -(\gamma-1)\left[(1/r)(rq_r)' + iq_x + (in/\alpha)(q_\phi/r)\right] \tag{2.42}$$

State

$$(s/\rho) = \pi - (\theta/T) \qquad \rho T = 1 \tag{2.43}$$

This is a system of six linear equations in six dependent variables q_r, q_x, q_ϕ, s, θ, and π. As in the two-dimensional inviscid, compressible case, the system is of the second order.

The boundary conditions on the axis do not depend on the viscosity or compressibility of the fluid and are [Appendix A] :

$$\begin{array}{lll} n = 0 & & q_r = 0 \quad q_\phi = 0 \\ & & q_x,\ \pi,\ \theta,\ s \ \text{ are arbitrary} \\ n \neq 0 & & q_x = 0 \quad , \quad \pi = 0 \\ & & \theta = 0 \quad , \quad s = 0 \\ & n = 1 & q_\phi = -\alpha\, q_r \quad q_r \text{ is arbitrary} \\ & n > 1 & q_\phi = 0 \quad q_r = 0 \quad . \end{array} \tag{2.44}$$

The outer boundary condition is the same as in the two-dimensional inviscid, compressible case and will be derived in

Section V. 2. It is:

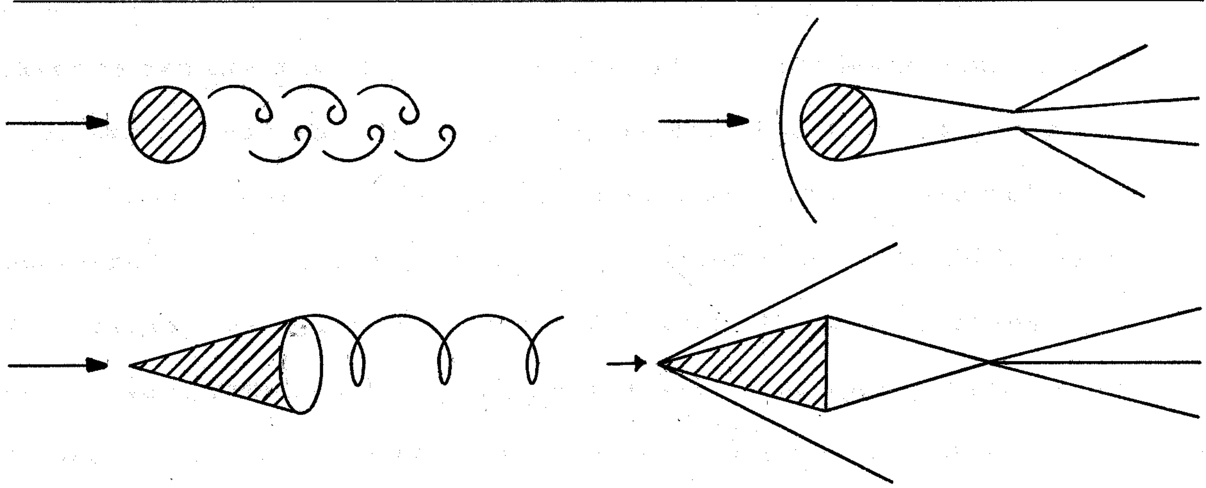
$$\frac{d(r\,q_r)}{dr} + a\sqrt{\Omega}\,(r\,q_r) = 0 \qquad r \longrightarrow \infty \tag{2.45}$$

where

$$\Omega = 1 - M^2 c^2 \qquad -\pi < \arg \Omega < \pi \quad .$$

II. 4. Mean Flow Model

The wake in back of a blunt or slender body can be divided into two regions of interest; the "near" wake and the "far" wake. The "near" wake refers to the region near the body and the "far" wake to the region far downstream of the body. At low Mach numbers ($M_\infty << 1$), the near wake is characterized by the formation of vortices and unsteady phenomena over a wide range of Reynolds numbers[48] [Sketch 2. 4. a].



(a) $M_\infty << 1$ (b) $M_\infty >> 1$

Sketch 2. 4

For $M_\infty << 1$ the boundary layer assumptions do not give an accurate description at the near wake because $R_{e_{x*}}$ is low and the gradient in the streamwise direction ($\partial/\partial x$) is not small compared to the normal gradient ($\partial/\partial y$). The flow is unstable at low Reynolds numbers and becomes turbulent. Thus, there is no region of laminar flow when the Reynolds number becomes large. For flat plates at low Mach numbers, the vortices and unsteadiness "disappear" and the wake becomes laminar at low Reynolds numbers[49]. The boundary layer assumptions apply in this case and an analytical solution can be found for the far wake[50].

For $M_\infty > 1$, or more specifically, $M_\infty >> 1$, the near wake is characterized by two free shear layers (or an annulus) shed from the body surface, that converge into a "neck"[51]. For blunt bodies the Mach number external to the shear layer is "frozen" at about three, while for slender bodies it is of the order of the free stream Mach number. Theoretical[23] and experimental[52, 53] studies show that a laminar shear layer is remarkably stable for supersonic external Mach numbers. This same result applies in the neck region. Therefore, all or part of the "inner" wake will be laminar over a wide range of Reynolds numbers[1, 54]. The boundary layer approximations are valid in the inner wake region, except very near the neck where the mean profiles change very rapidly. In the far field, Oseen-type approximations can be used to linearize the equations and relatively simple analytical expressions can be obtained. Since the major trends of the stability problem are of interest here, these analytical expressions will be used in the stability analysis.

Kubota's[55] solution for the two-dimensional compressible wake with zero pressure gradient will be used (Appendix B):

$$w = -e^{-\eta^2} \qquad T = 1 + \Delta T\, e^{-\eta^2} \tag{2.46}$$

where

$$\eta = \int_0^y (dy/T) \qquad \Delta T = \frac{T^*(0) - T_e^*}{T_e^*} \sim \frac{1}{\sqrt{(x^*/d^*)}} \tag{2.47}$$

In Eq. (2.11) the characteristic length

$$L^* = \frac{2\, d^*}{\sqrt{R_{\infty_{d^*}}}} \; \frac{U_\infty^*}{U_e^*} \sqrt{\frac{U_e^* \rho_e^* \mu_e^*}{U_\infty^* \rho_\infty^* \mu_\infty^*}} \sqrt{\frac{x^*}{d^*}} \tag{2.48}$$

is related to the half-width of the wake, and the characteristic velocity

$$V^* = \Delta U\, U_e^* = \left[U_e^* - U^*(0)\right] \sim 1/\sqrt{(x^*/d^*)} \tag{2.49}$$

is the maximum velocity defect in the wake. The quantity d^* is a characteristic body dimension. The Reynolds number of the wake

$$R = \frac{L^* V^*}{\nu_e^*} = (2/\sqrt{\pi})\, R_{e_{\theta^*}} = \frac{R_{e_{d^*}}}{2\sqrt{\pi}} \quad C_D \tag{2.50}$$

is constant. The y coordinate in the stability equations must be "stretched" by the temperature (Eq. (2.47)). In the derivation of Eq. (2.46), it is assumed that ΔT and M are very small (ΔT, $M << 1$). However, since the relative effects (and not the absolute effects) of temperature and Mach number are desired, values of ΔT and M greater than unity will be used in the numerical calculations.

For the two-dimensional incompressible wake

$$w = -e^{-y^2} \qquad T = 1 \qquad L^* = (2d^*/\sqrt{R_{e_{d^*}}}\,)\sqrt{(x^*/d^*)} \tag{2.51}$$

A transformation analogous to Kubota's does not exist for the axi-symmetric compressible wake[56]. The viscous stress term in the momentum equation does not transform to the equivalent incompressible form and therefore the momentum and energy equations must be integrated simultaneously. For the axi-symmetric incompressible wake[50]:

$$U^* = U_e^*\left[1 - \Delta U\, e^{-r^2}\right], \qquad L^* = (2d^*/\sqrt{R_{e_{d^*}}})\sqrt{(x^*/d^*)} \quad . \qquad (2.52)$$

The momentum thickness [or drag coefficient ,

$$C_D = \frac{\text{Drag}}{\frac{1}{2}\rho_e^* U_e^{*2}\,\pi d^{*2}}\Bigg] \qquad \text{is given by}$$

$$\theta^{*2} = C_D\,(\pi d^{*2}/2) = 2\pi\int_0^\infty \left[1 - (U^*/U_e^*)\right] r^*\, dr^*$$

and

$$\Delta U = (\theta^*/d^*)^2\,(R_{e_{d^*}}/4\pi)\,\frac{1}{(x^*/d^*)} = (R_{e_{d^*}}/8)\,\frac{C_D}{(x^*/d^*)} \qquad (2.53)$$

$$V^* = U_e^*\ \Delta U \sim 1/x^*/d^* .$$

The Reynolds number for axi-symmetric flow is

$$R = \frac{L^*V^*}{\nu_e^*} = \frac{R_{e_{d^*}}^{\ 2}\ C_D}{4\sqrt{R_{e_{x^*}}}}\qquad\frac{1}{\sqrt{(x^*/d^*)}} \qquad (2.54)$$

and varies as the reciprocal of the square root of the distance downstream of the origin.

The mean profiles for the axi-symmetric compressible case are assumed to be Gaussian in the Dorodnitsyn-Howarth variable, η,

$$w = -e^{-\eta} \qquad T = 1 + \Delta T\, e^{-\eta} \tag{2.55}$$

where

$$\eta = \int_0^r (r dr/T) \qquad \Delta T,\ \Delta U \sim \frac{1}{(x^*/d^*)}\,. \tag{2.56}$$

These mean profiles, although not strictly valid (as mentioned previously), will be used to illustrate the stability characteristics of slowly varying axi-symmetric velocity and temperature mean profiles.

III. STABILITY OF TWO-DIMENSIONAL INCOMPRESSIBLE WAKE FLOWS

Wakes belong to the class of two-dimensional quasi-parallel flows known as "free boundary layers", i. e., flow fields in which solid boundaries are not present. Usually flows which belong to this class of quasi-parallel flows have one or more points of inflection in the velocity profile. The presence of an inflection point indicates that the flow is dynamically unstable in the limiting case of vanishing viscosity, and that it would become unstable at relatively low Reynolds numbers. Hence the classical methods of solution for large Reynolds number, or more precisely large αR, cannot be used to determine the minimum critical Reynolds number. The quantity, $1/(\alpha R)$, is a measure of the diffusion distance for vorticity during one period. New methods of solution for the Orr-Sommerfeld equation have to be found for small values of αR. In addition, the asymptotic methods developed by Tollmien[57], Heisenberg[58] and Lin[22] have to be modified for large values of αR.

Another problem arises in that the quasi-parallel flow assumptions leading to the Orr-Sommerfeld equation are not valid throughout the entire flow field, since the transverse mean velocity component is of the same order as the longitudinal mean velocity in certain regions of the field for the small values of Reynolds number of interest in this problem. Near the trailing edge of a flat plate, for instance, the transverse mean velocity must be taken into account if a "precise" prediction of the flow stability is desired. However, general quantitative results are of interest here, and the quasi-parallel assumptions will be retained for all values of αR.

The subtleties of incompressible wake-type flows will be discussed in this section. Section III. 1 will deal with the solutions of the Orr-Sommerfeld equation for small values of αR. A minimum critical Reynolds number of 39, based upon the length of a flat plate, is found for anti-symmetrical disturbances. The stability characteristics of a smoothly varying profile at long wave lengths can be found by using discontinuous velocity profiles [Section III. 2] In Section III. 3 the inviscid stability of an incompressible Gaussian flat plate wake is determined by numerical methods. These theoretical results agree very well with the experimental results of Sato and Kuriki[2]. The effect of viscosity on the eigen-value equation for large, but finite, αR flows comes in through the "inviscid solutions" and does not depend on the "viscous solutions". However, the calculation of the disturbance amplitudes must include the "viscous solutions", since the "inviscid solutions" are singular at w = c. [Section III. 4] .

The Orr-Sommerfeld equation can be derived by eliminating the pressure and longitudinal velocity perturbation amplitudes from Eqs. (2. 12) - (2. 14) (or by considering the disturbance vorticity equation),

$$(w-c)\left[\frac{d^2\phi}{dy^2}-\alpha^2\phi\right]-\frac{d^2w}{dy^2}\,\phi=\frac{1}{i\alpha R}\left[\phi^{iv}-2\alpha^2\phi''+\alpha^4\phi\right] \qquad (3.1)$$

or

$$\phi^{iv}-\phi''\left[2\alpha^2-i\alpha R(w-c)\right]+\phi\left[\alpha^4+i\alpha R(w-c)\alpha^2+w''i\alpha R\right] \qquad (3.2)$$

The boundary conditions at the axis are [Eq. (2. 29) and Eq. (2. 30)]:

$$\phi'(0) = \phi'''(0) = 0 \text{ anti-symmetric oscillations} \qquad (3.3)$$
$$\phi(0) = \phi''(0) = 0 \text{ symmetric oscillations} \quad .$$

The boundary condition for large values of y is obtained from Eq. (3. 2). As $y \longrightarrow \infty$, $w \longrightarrow 0$ exponentially, and Eq. (3. 2) becomes

$$\phi^{IV} - \left[2\alpha^2 - i\alpha Rc\right]\frac{d^2\phi}{dy^2} + \alpha^2\left[\alpha^2 - i\alpha Rc\right]\phi = 0 \qquad (3.4)$$

The solutions of Eq. (3. 4) are

$$\phi \sim e^{\pm\sqrt{\alpha^2 - i\alpha Rc}\, y} , \; e^{\pm \alpha y} \qquad (3.5)$$

Now ϕ must be bounded for large values of y ($y > 0$); if for definiteness, we take

$$-\pi < \arg\,(\alpha^2 - i\,\alpha\,R\,c\,) < \pi \quad , \qquad (3.6)$$

solutions with the positive exponent must be rejected, and

$$\phi \sim e^{-\sqrt{\alpha^2 - i\alpha Rc}\, y} , \; e^{-\alpha y}$$

or

$$(d\phi/dy) + \alpha\,\phi = 0$$
$$(d\phi/dy) + \sqrt{\alpha^2 - i\,\alpha\,R\,c}\;\phi = 0 \quad . \qquad y \longrightarrow \infty \qquad (3.7)$$

Eq. (3. 1) together with the boundary conditions Eq. (3. 3) and Eq. (3. 7) constitute a characteristic-value problem. The characteristic-values (or eigen-values) are determined by the usual secular determinant, leading to a relation of the form

$$E\,(\alpha\,,\;c\,,\;R\,) = 0 \qquad (3.8)$$

where E is some general function of the arguments.

If the imaginary part of c is positive, the disturbances will amplify with time and the motion is said to be unstable. If it is negative, the disturbances will eventually be damped out. If c_I is zero, the disturbances are considered to be neutral. At each Reynolds number, the spatial amplification rates can be computed. A typical neutral stability curve for wake type flows is sketched in Figure 2. The history of a disturbance as it progresses downstream is indicated in this figure. For Gaussian wake profiles, the Reynolds number is constant [Eq. (2.50)] and the wave number is proportional to $\sqrt{x}$ [Eq. (2.50)] . The disturbance will be amplified within the neutral stability curve and damped outside of it.

III. 1. Solutions for Small α R

Wake-type flows are very unstable because of the occurrence of a point of inflection in the mean velocity profile, i. e., the minimum critical Reynolds number, below which all disturbances are stable, is relatively low. The effect of viscosity is not confined to a thin layer, as in the boundary layer case, but is felt throughout the entire flow field. More precisely, the relative distance that the disturbance is diffused in one period is proportional to the reciprocal of some power of α R, and for small values of α R, this distance is of the order of one, or the full extent of the wake.

An energy balance shows that the rate of increase of the kinetic energy of the disturbance is equal to the conversion of energy from the basic flow into the disturbance by the Reynolds shear stress, minus the viscous dissipation[22]. Viscous dissipation is always a stabilizing factor (always leads to a decrease in energy) and for small Reynolds numbers is very large. Disturbances will be damped out very rapidly in this region.

Tatsumi and Kakutani[31], anticipating a small value of the minimum critical Reynolds number, have expanded the solution in powers of αR as follows:

$$\phi(y) = \sum_{n=0}^{\infty} (i\alpha R)^n \, \phi^{(n)}(y; \alpha, \beta) \tag{3.9}$$

where

$$\beta^2 = \alpha^2 - i\alpha R c.$$

Substituting Eq. (3.9) into the Orr-Sommerfeld equation, Eq. (3.2), and matching powers of $i\alpha R$, the following equations relating the $\phi^{(n)}$'s are obtained:

$$(D^2 - \alpha^2)(D^2 - \beta^2)\,\phi^{(0)} = 0 \tag{3.10a}$$

$$(D^2 - \alpha^2)(D^2 - \beta^2)\,\phi^{(n)} = W\left[D^2\phi^{(n-1)} - \alpha^2\phi^{n-1}\right] - D^2W\,\phi^{(n-1)} \tag{3.10b}$$

where

$$D = \frac{d}{dy}.$$

The solutions of Eq. (3.10a) are:

$$\phi_1^{(0)} = e^{-\alpha y}, \quad \phi_2^{(0)} = e^{\alpha y}, \quad \phi_3^{(0)} = e^{-\beta y}, \quad \phi_4^{(0)} = e^{\beta y} \tag{3.11}$$

The solutions of Eq. (3.10b) can be found by the method of variation of parameters [Appendix C.]* The general solution of Eq. (3.2) is

$$\phi = C_1 \phi_1 + C_2 \phi_2 + C_3 \phi_3 + C_4 \phi_4 \qquad (3.12)$$

where C_1, C_2, C_3, and C_4 are arbitrary constants.

From the outer boundary conditions, Eq. (3.7), $C_2 = C_4 = 0$, and for a non-trivial solution, ϕ_1 and ϕ_3, must satisfy the following characteristic equations [Eq. (3.3)]:

Anti-symmetric Oscillations

$$\begin{vmatrix} \phi_1'(0) & \phi_3'(0) \\ \phi_1'''(0) & \phi_3'''(0) \end{vmatrix} = 0 \qquad (3.13)$$

Symmetric Oscillations

$$\begin{vmatrix} \phi_1(0) & \phi_3(0) \\ \phi_1''(0) & \phi_3''(0) \end{vmatrix} = 0 \qquad (3.14)$$

Eq. (3.9) is substituted into Eqs. (3.13) and (3.14) and a complex eigenvalue equation is obtained as a power series in $(i\alpha R)$.**

---

* The solution Eq. (3.9) converges uniformly for the Gaussian wake profile, $w = -e^{-y^2}$, when $\alpha < 1$ and $R/\beta < 1$ [Reference 31, page 270,] where $\left|\frac{U(\xi)}{\xi}\right| <$ constant $|U'(\xi)| <$ constant .

** Tatsumi and Kakutani[31] considered only anti-symmetrical oscillations. Their method is easily extended to the case of symmetrical oscillations (Appendix C).

Terms of the fourth order and higher were neglected, because it was found that these terms did not affect the first term of the asymptotic behavior of the lower branch of the neutral stability curve. [Appendix C, Eq. (C. 18) and Eq. (C. 19)] . The eigenvalue equation was further simplified by neglecting terms of order σ^3 and higher where

$$\beta - a = a\sigma \quad , \tag{3.15}$$

and

$$\begin{aligned} c_R &= -\frac{2\alpha\sigma_I}{R}\left[1 + \sigma_R\right] \\ \sigma_R &= -1 + \sqrt{1 + \sigma_I^2 + \frac{R c_I}{\alpha}} \end{aligned} \tag{3.16}$$

Since the anti-symmetric oscillations are more unstable than the symmetric ones, the minimum critical Reynolds number was determined only for the former case. Eq. (C. 18) [Appendix C] was solved graphically for the neutral curve $[c_I = 0]$. The results are indicated in the following table for a Gaussian profile:

σ_I	R	a	R	c_R
+0. 40	. 077	. 120	4. 8	-. 022
+0. 45	. 095	. 16	4. 7	-. 034
+0. 50	. 125	. 19	4. 7	-. 046

Table 3. 1

The minimum critical Reynolds number is R = 4. 7 at a = 0. 17. At this point, a R = 0. 8 which is exactly the same result found by Tatsumi and Kakutani[31] for a jet.

For an incompressible flat plate (length d) Gaussian wake [Eq. (2.43)] :

$$R = \frac{R_{e_{d^*}}}{2\sqrt{\pi}} C_D = \frac{R_{e_{d^*}}}{2\sqrt{\pi}} \left(\frac{2.656}{\sqrt{R_{e_{d^*}}}} \right) = .75\sqrt{R_{e_{d^*}}} \tag{3.17}$$

Therefore the minimum critical Reynolds number based upon the length of the plate is $Re_{d*} = 39$. This value is considerably below the experimental value of about 600 measured by Hollingdale[5], and about 700 measured by Taneda[49], for which oscillations were observed. At low Reynolds numbers the amplification rate is a strong function of Reynolds number, i.e., viscous dissipation tends to damp out the disturbances. Therefore, oscillations will begin to occur far downstream of the plate and the Reynolds number at which they are first observed will be considerably higher than the minimum critical Reynolds number determined by stability.

The asymptotic behavior of the lower branch was found by taking the limit of Eq. (C.18) as $\sigma_I \longrightarrow 0$, and Eq. (C.19) as $\sigma_I \longrightarrow -\infty$.

Any other limit did not produce any meaningful results. The first approximation consisted of solving the eigenvalue equations using only terms up to order αR; the second approximation, using only terms up to order $(\alpha R)^2$; etc. The results are tabulated below:

<u>Anti-symmetric Disturbance</u>

Approximation	αR^2		c_R/α^2	
1st	3.00		-1.50	
2nd	1.48	36.40	-1.64	1.29
3rd	1.51	22.96	-1.57	1.22

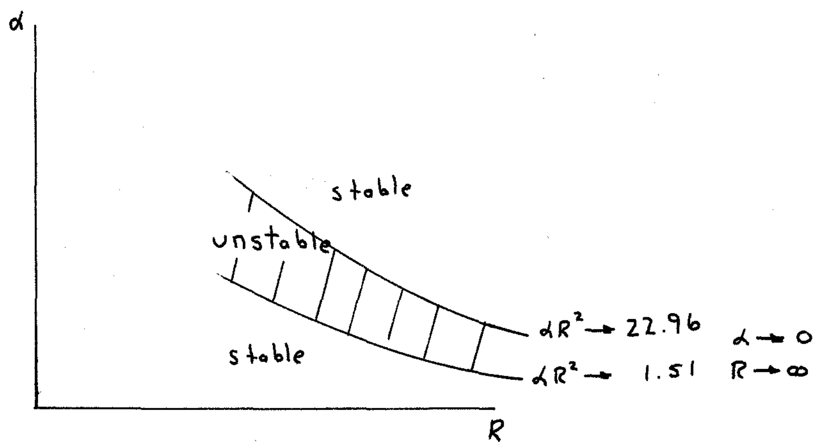
Symmetric Disturbances

Approximation	αR	c_R
1st	0.67	-1.05
2nd	0.47	-1.26

Table 3.2

For the anti-symmetric disturbances, the fourth approximation involves terms of order $(\alpha R)^4$, which do not modify the third order approximation. For the symmetric disturbances, the real part of the eigenvalue equation has terms of order $\alpha^2 R^2$ to the approximations made (even though the imaginary part has terms of order $(\alpha R)^3$), and it is not obvious that the 3rd approximation will not effect the 2nd approximation.

In the limit $\alpha \longrightarrow 0$, $R \longrightarrow \infty$, for anti-symmetric disturbances, the product αR^2 approaches a constant along two branches. The lower



Sketch 3.1

of the two branches, in a reference system fixed in a fluid at rest, corresponds to a wave travelling in the same sense as the centerline velocity while the upper branch corresponds to a wave travelling in the sense opposite to this velocity. The flow is unstable if $-1.57\,\alpha^2 < c_R < 1.22\,\alpha^2$ and stable outside this region.

Along these two branches $\alpha R \sim \sqrt{\alpha}$ as $\alpha \longrightarrow 0$, $R \longrightarrow \infty$, thus confirming the validity of the expansion [Eq. (3.9)] . At the critical point $\alpha R = 0.8$; this value is higher so that more terms in the eigenvalue equation should probably be retained in order to find a more precise value of the minimum critical Reynolds number. However, the purpose of the computation was to find an approximate value of the minimum critical Reynolds number, and it was felt that any additional calculations were not commensurate with the aims of the investigation. Moreover, at these low Reynolds numbers, the boundary layer equations themselves are not accurate.

In reviewing the paper of Tatsumi and Kakutani[31], this author found that terms of order $(\alpha R)^3$, which were neglected in Eq. (6.4) of their paper, should have been retained. In the limiting case $\alpha \longrightarrow 0$, $R \longrightarrow \infty$, they are of the same order as the terms that were retained. In addition, the second branch was not recognized by these authors. Drazin[14] reported these additional results in 1961.

For the symmetric oscillations, αR and c_R approach a constant as $\alpha \longrightarrow 0$, $R \longrightarrow \infty$. The value of $|c_R|$, however, is larger than the maximum value of $|w|$. This behavior of the lower branch was first suggested by Curle[12] and verified numerically by Clenshaw and Elliot[7] for the Bickley jet, $w = \mathrm{sech}^2\, y$.

The results obtained by the method of Tatsumi and Kakutani[31] suggest another type of expansion for small values of αR. Along the lower branch, R varies from its minimum value to infinity while α is always very small ($\alpha < .2$). Another type of expansion, then, would be to hold R fixed and expand c and ϕ as a power series in α. Howard[18] used this method for the Bickley jet and obtained results that are identical to those of Tatsumi and Kakutani [if the additional terms are added to Eq. (6.4), Reference 31]. This author has used this method for the Gaussian flat plate wake and also obtained the same minimum critical Reynolds number, and the same asymptotic behavior of the two lower branches (anti-symmetric oscillations) as by the method of Tatsumi and Kakutani.

III. 2. Idealized Profiles at Long Wave Lengths

In both the methods of Tatsumi and Kakutani[31] and Howard[18], the mean velocity profile appears only in integrals, indicating that the precise shape of the velocity profile is unimportant for small wave-numbers or large wave-lengths of the disturbances. Drazin and Howard[15] and Drazin[14] have used discontinuous velocity profiles to find the stability characteristics of flows at low wave-numbers. Some of their ideas will be presented here as applied to the wake-type flow stability problem.

From Eq. (2.11),

$$y = (y^*/L^*) \quad , \qquad w(y) = \frac{W^*(Y^*)}{L^*}$$
$$R = (v^* L^*/\nu^*) \qquad \alpha = \alpha^* L^* \quad , \tag{3.18}$$

where the starred quantities represent dimensional quantities, and

$w(y) \longrightarrow 0$ as $y \longrightarrow \pm\infty$. The eigenvalue equation, Eq. (3.8), leads to a relation between c, α^2, and αR or (R/α), i.e.,

$$c = c\,(\alpha,\ R/\alpha\ ) \quad . \tag{3.19}$$

For a fixed α^*, $\alpha \longrightarrow 0$, $(R/\alpha) = V^*/(\alpha^* \nu^*)$ = constant as $L^* \longrightarrow 0$. Therefore, $c \longrightarrow c\,(0,\ R/\alpha\ ) = f\,(V^*/\alpha^* \nu^*)$ and

$$c^* \rightarrow V^*\, f\left(\frac{V^*}{\alpha^* \nu^*}\right) \tag{3.20}$$

As $L^* \longrightarrow 0$ for a fixed dimensionless velocity profile, $w(y)$, then $w^*(y^*) \longrightarrow 0$ since $y = (y^*/L^*) \longrightarrow \infty$ ($y^* \gtrless 0$).

Therefore the two limits, $\alpha^* \longrightarrow 0$, L^* and $(R/\alpha) = (V^*/\alpha^* \nu^*)$ fixed, and $L^* \longrightarrow 0$, α^* and (V^*/ν^*) fixed, give the same result $\alpha \longrightarrow 0$, R/α fixed. In other words, for $w(y)$ fixed, each limit gives the same limiting form of the eigenvalue relation, Eq. (3.20), and the stability characteristics of the flow are the same for both the limiting profile $w^*(y^*)$ as $\alpha^* \longrightarrow 0$. In other words, by using the limiting profiles $w^*(y^*)$ as $L^* \longrightarrow 0$, which may be discontinuous, the stability characteristics of smoothly varying profiles [actual $w^*(y^*)$] as $\alpha^* \longrightarrow 0$ can be determined.

Drazin[14] derives jump conditions [for α, αR bounded] at the points where w and/or (dw/dy) are discontinuous and applies them to the case of a broken line jet.

$$w = \begin{cases} 0 & |y| > 1 \\ 1 & |y| < 1 \end{cases} \quad .$$

He finds that three neutral "branches" exist for anti-symmetric disturbances

$$
\begin{array}{lll}
\alpha R \longrightarrow \infty, \ \alpha \text{ fixed} & (a) \ \alpha \longrightarrow \tanh^{-1}(7 - 4\sqrt{3}\ ) & \\
\alpha R^2 \text{ — constant } \begin{array}{l}\alpha \longrightarrow 0 \\ R \longrightarrow \infty\end{array} & (b) \ \alpha R^2 \longrightarrow 1.34 \quad c_R \longrightarrow -1.54\, \alpha^2 & (3.21)^* \\
 & (c) \ \alpha R^2 \longrightarrow 32.4 \quad c_R \longrightarrow 1.21\, \alpha^2 &
\end{array}
$$

and for symmetric disturbances

$$
\alpha R \text{ fixed}, \ \alpha \longrightarrow 0, \quad c \longrightarrow -\left[1 + 2n^2 (\pi^2/\alpha R)\right]
$$

$$
\alpha R = \frac{2 n^2 \pi^2}{\left[1 - \frac{3}{2} \cosh^2 n\pi \left\{1 - \sqrt{1 - \frac{8}{9} \operatorname{sech}^2 n\pi}\right\}\right]} \qquad (3.22)^*
$$

$$
n = 1, 2, 3 \quad .
$$

The branches (b) and (c) of Eq. (3.21) correspond to those found in Section III. 1 and have approximately the same limiting values. The wave speeds agree to within 2 %, while the product αR^2 agrees to within 10 % on the lower branch (b) and to within 50 % on the upper branch (c). Thus the use of discontinuous velocity profiles at low wave numbers is justified at least for qualitative purposes.

The first branch (a) is not meaningful in that the assumption αR bounded is not met. This branch was not found in Section III. 1. However, this calculation indicates that the flow is unstable between the branches (b) and (c) and above branch (a). In Section III. 3, another limit is found, namely the inviscid limit, on which $\alpha R \longrightarrow \infty$, $\alpha \longrightarrow \alpha_s$. Below this branch, the flow is unstable, which leads to the hypothesis that there might be an "island of stability" within the unstable region[14]. This situation is illustrated in the following sketch. It was suggested

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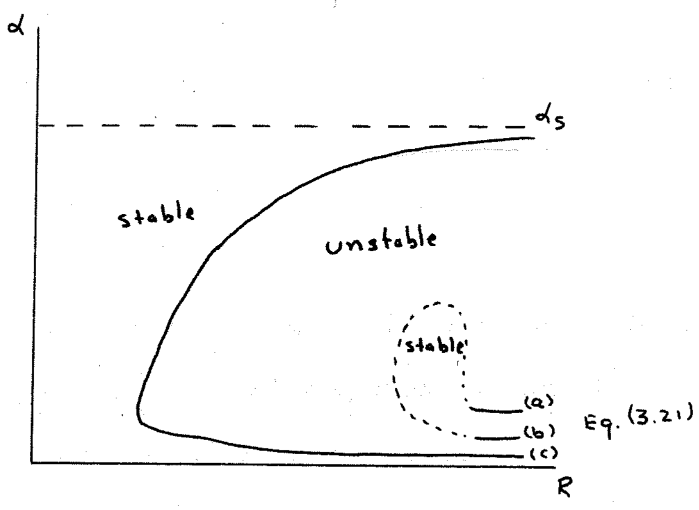
* In this text, c_R is defined as the relative wave velocity and is the negative of the values found in Reference 14.



Sketch 3. 2

by Stuart[14] that branch (a) corresponds to the limit $\alpha R^P \longrightarrow$ constant, $R \longrightarrow \infty$, $0 < p < 1$.

For symmetric disturbances, $\alpha R \longrightarrow$ constant, $c_R \longrightarrow$ constant as $R \longrightarrow \infty$, $\alpha \longrightarrow 0$[14]. This behavior is exactly that found in Section III. 1 for the smoothly varying profile. For $n = 1$, $c_R \longrightarrow -1.33$ and $\alpha R \longrightarrow 59$.

Therefore, for low wave numbers or long wave lengths, the stability characteristics of smoothly varying velocity profiles can be found by using suitable discontinuous velocity profiles.

III. 3. Inviscid Limit

A point of inflection in the mean velocity profile indicates that the flow is unstable in the limiting case of vanishing viscosity, and that the main features of the instability mechanism can be obtained by neglecting the viscous forces[22]. The effect of viscosity is a stabilizing one and can be taken into account once the inviscid instability mechanism is understood.

The inviscid limit is formally obtained by expanding ϕ in a power series in $(1/\alpha R)$,

$$\phi = \phi^{(0)} + (1/\alpha R)\,\phi^{(1)} + \ldots \quad ,$$

substituting this expansion into Eq. (3.1), and taking the limit $\alpha R \to \infty$. The equation for the zeroeth-order approximation, $\phi^{(0)}$, is known as the inviscid Orr-Sommerfeld equation and is

$$(w-c)(\phi'' - \alpha^2 \phi) - w'' \phi = 0 \quad . \tag{3.23*}$$

For neutral disturbances ($c_I = 0$) the wave speed, c_R, is equal to the mean velocity, w, at the point where w'' vanishes. The vanishing of w'' is a necessary and sufficient condition for the existence of a neutral disturbance[22], and the disturbance amplitudes are finite everywhere in the flow region if this condition holds. The inviscid neutral wave speed and wave number are commonly denoted by c_s and α_s.

For a Gaussian flat plate wake, $w = -e^{-y^2}$, the inviscid neutral wave speed is $c_s = -e^{-\frac{1}{2}} = -0.606$. McKoen[24] approximated the velocity profile in three different regions and found that $\alpha_s = 2.0$. In the present investigation, α_s was determined by numerically integrating Eq. (3.23) on an IBM 7090 electronic computer. This method is described in Section IV. 2; α_s was found to be equal to 1.90.

The neutral inviscid wave number is an indication of the extent of the region of instability (Figure 2) and is of an order of magnitude

---

* *It is understood that the ϕ occurring in this equation is the zeroth-order approximation, ϕ^{o}.*

greater than the wave numbers encountered in boundary layer stability theory. A very important parameter in stability theory is the spatial amplification rate, [Eq. (2.10a)] which indicates how fast a disturbance will amplify in a wavelength. The dimensionless amplification rate [Eq. 4.28] was determined by the method described in Appendix D. The results are indicated in Table 2 and Figures 3 and 4. The complex wave velocity determined in this investigation versus the wave number is compared with the theoretical results of Sato and Kuriki[3]. The values of the imaginary part of the complex wave velocity at low wave numbers obtained by them are considerably higher than those obtained in the present investigation. For low values of the wave number c_I should decrease, a trend which is not indicated by their results. Therefore it is felt that the results of the present investigation are more accurate than those obtained by Sato and Kuriki .

The group velocity of the disturbance must be calculated in a frame of reference fixed in the body. The amplification rates were calculated using both the group velocity and the phase velocity of the disturbance; the maximum value of the former was found to be 0.33 and of the latter, 0.26 (Table 2). The spatial amplification rate (using the phase velocity) is compared with the theoretical and experimental values of Sato and Kuriki in Figure 4. The notation of Sato and Kuriki is used in this figure.* The preferred frequency for natural oscillations

---

* The wave speed and wave number used by Sato and Kuriki[3] are defined differently in this text. The conversions are

$$c_R \text{ (Sato and Kuriki)} = 1 + .692\, c_R \text{ (Gold)}$$

$$\alpha \text{ (Sato and Kuriki)} = 0.833\, \alpha \text{ (Gold)} \quad .$$

corresponds to the frequency at which the theoretical spatial amplification is a maximum. This behavior is similar to that found in free shear layer and jet flows[2, 4] and seems to be characteristic of unbounded flows. The stability characteristics of such flows are relatively insensitive to the effects of viscosity over a wide range of (large) Reynolds numbers.

III. 4. Large, Finite Reynolds Number Stability

It is very desirable, although extremely tedious, to determine the entire neutral stability curve in the α-R plane. The solution for small values of the wave number and Reynolds number was discussed in Sections III. 1 and III. 2, and the inviscid limit (infinite Reynolds number) in Section III. 3. A description of the problems arising for large but finite Reynolds numbers will be given in this section.

For channel flows, Heisenberg[58] has shown that of the four independent solutions of the Orr-Sommerfeld equation, two of the solutions are slowly varying (inviscid solutions) and satisfy the inviscid equation throughout the channel (except at the singular point), and the other two are rapidly varying (viscous solutions) and very sensitive to the effects of viscosity.

For wake type flows, the absence of any solid boundaries implies that the viscous solutions will not play an important role in the stability of these flows and can probably be neglected in the first approximation, and that the effect of viscosity must be found from the higher order terms of the inviscid solutions. Foote and Lin[17] show that the effect of the viscous solutions does not enter into the eigenvalue problem for both free shear layers and wakes of large αR. For free shear layers, the viscous solutions must be rejected because the disturbance amplitudes must damp out at infinity. For wake flows (symmetrical velocity profiles), one viscous solution must be rejected because it becomes infinite for $y \longrightarrow \infty$. However, the other viscous solution is shown to be of higher order, in the eigenvalue problem, and can be neglected. However this does not imply that the viscous solutions are entirely neglected. The inviscid solutions have a logarithmic singularity at the point $w = c$, which must be "smoothed" out by the action of viscosity. If ϕ_1 and ϕ_2 are the inviscid solutions, and ϕ_3 and ϕ_4 are the viscous solutions [ϕ_3 and ϕ_4 become infinite exponentially as $y \longrightarrow -\infty$ and $+\infty$, respectively] then

$$\phi = C_1 \phi_1 + C_2 \phi_2 + C_3 \phi_3 + C_4 \phi_4 \quad . \tag{3.24}$$

For symmetric or anti-symmetric disturbances, $C_4 = 0$, and Eq. (3.24) becomes

$$\phi = C_1 \phi_1 + C_2 \phi_2 + C_3 \phi_3 \quad . \tag{3.25}$$

The coefficients in Eq. (3.25) must be chosen in such a way that in the vicinity of the critical point, $w = c$, the discontinuities in the inviscid solutions ϕ_1 and ϕ_2 must be smoothed out by the action of viscosity due to the viscous solution ϕ_3. In other words, even though the viscous solutions can be <u>neglected</u> in the <u>eigenvalue</u> equation, they must be <u>retained</u> in determining the <u>distribution</u> of the eigen-functions.

Tatsumi and Kakutani[31] have used this approach to find the upper branch of the neutral stability curve, and have shown that the stability characteristics of the Bickely jet are relatively insensitive to the effects of viscosity over a wide range of very large Reynolds numbers.

McKoen[24] neglected the fourth-order terms ϕ'''' in Eq. (3.1) and perturbed the solution about the inviscid solution. This procedure enabled him to find a simple expression for the neutral stability curve for large values of R. This assumption cannot be justified although his results look reasonable. However, this method did not predict a minimum critical Reynolds number. Curle[13], using an extension of McKoen's method, approximated the solution by a linear combination of two inviscid solutions. Again, this additional assumption cannot be justified, although it predicts a minimum critical Reynolds number which is very close to that found by Tatsumi and Kakutani[31] and Howard[18].

IV. STABILITY OF TWO-DIMENSIONAL COMPRESSIBLE WAKE FLOWS

Wake-type flows are dynamically unstable because of the occurrence of a point of inflection in the density-vorticity product[22, 32]. Therefore the stability characteristics of such flows are relatively insensitive to Reynolds number, for sufficiently high Reynolds numbers, and interesting and important results can be obtained by considering the "inviscid limit" of the small disturbance equations, in which the viscosity and conductivity of the fluid can be neglected to a certain order.

For an incompressible wake, Sato and Kuriki[3] show that inviscid small disturbance theory compares very favorably with experimental results, and that natural oscillations occur at a frequency at which the theoretical spatial amplification rate is a maximum. The present investigation was motivated by these ideas and was extended to find the effect of compressibility on the stability characteristics of wake-type flows.

Lees and Lin[32] considered the inviscid stability of laminar compressible fluid flow and applied their results to the flat plate boundary layer. The reader is referred to their paper for a complete description of the problem. The only points that will be discussed in this section are those relating to the compressible wake problem.

The nature of the disturbances far from the axis of the wake was investigated. It is found that the disturbances can be classified as subsonic, sonic and supersonic according to whether the wave velocity of the disturbance (in the direction of the free stream velocity), relative to the free stream velocity is less than, equal to or greater than the external velocity of sound. Neutral and self-excited subsonic disturbances are possible only when the gradient of the density-vorticity product $(\rho w')'$ vanishes for some $-w < (1/M)$ in a coordinate system fixed in the fluid at rest. Thus, when M is sufficiently high, no subsonic disturbances occur. If one assumes that only subsonic disturbances are important for stability, then many of the transition phenomena occurring in the hypersonic wake can be explained on this basis.

A numerical method of solving the inviscid compressible small disturbance equations is presented in Section IV. 2 and Appendix D for both neutral and amplified disturbances. Numerical results are presented in Section IV. 3 for a compressible wake using the mean flow model of Kubota[55]. The amplification rates at four stations of a hypersonic wake were calculated and the results indicated that the maximum spatial amplification rate is constant in the streamwise direction and occurs at one preferred frequency. This amplification rate is approximately half of that calculated for an incompressible wake. However, for hot wakes, the range of relative Mach numbers over which subsonic disturbances can exist increases. Therefore, as long as the relative Mach number is below the critical Mach number a hot wake will be more stable than a cool one. Finally, the hypersonic wake stability problem is discussed in Section IV. 4 using the results obtained in the previous sections.

IV. 1. Inviscid Disturbance Equation and Outer Boundary Condition

The following self-adjoint equation for the pressure perturbation can be obtained from Eqs. (2. 23) - (2. 27), (Lees and Lin[32]):

$$\pi'' - \left[\frac{2w'}{w-c} - \frac{T'}{T}\right] \pi' - a^2 \left[1 - \frac{M^2(w-c)^2}{T}\right] \pi = 0 \tag{4.1}$$

The boundary condition on the axis, for anti-symmetrical disturbances is Eq. [(2.28)] :

$$\pi = 0 \qquad y = 0 \tag{4.2}$$

The boundary condition for large values of y must be determined from the self-adjoint equation. When $y \longrightarrow \infty$, $w \longrightarrow 0$, $T \longrightarrow 1$, and Eq. (3.1) takes the limiting form

$$\pi'' - a^2 \Omega \pi = 0 \tag{4.3}$$

where

$$\Omega = 1 - M^2 c^2 \tag{4.4}$$

The solution of Eq. (4.3) is

$$\pi = C_1 e^{a\sqrt{\Omega} y} + C_2 e^{-a\sqrt{\Omega} y} \tag{4.5}$$

Following the suggestion of Lees and Lin[32], introduce a "cut" along the negative real axis of the complex Ω - plane so that the real part of $a\sqrt{\Omega}$ will always be positive as long as $-\pi < \arg \Omega < \pi$. From physical considerations, π must be bounded for large y and must behave like $e^{-a\sqrt{\Omega}y}$ as $y \longrightarrow \infty$. Therefore the boundary condition as $y \longrightarrow \infty$ is

$$\pi' + a\sqrt{\Omega}\,\pi = 0 \tag{4.6}$$

when $-\pi < \arg \Omega < \pi$. The self-adjoint equation Eq. [(4.1)] and the boundary conditions [Eq. (4.2) and Eq. (4.6)] constitute a

Sturm-Liouville system with discrete characteristic values.

The product of the asymptotic solution $e^{-\alpha\sqrt{\Omega} y}$ and the y independent part of the disturbance, $\exp i \alpha (x - ct)$, represents progressive waves with the direction of propagation of the wave dependent upon the frame of reference of the observer. If a wave propagates outward and in the negative x direction with respect to an observer fixed in a fluid at rest, it will propagate inward and in the positive x direction to an observer fixed in the body, and vice-versa. It is to be emphasized that the component of the propagation velocity of the wave front in the x direction, in a reference system in which the coordinates are fixed in a fluid at rest, is $c_R^* - U_e^*$; the component is c_R^* in a body - centered frame of reference [Figure 5]. The asymptotic solution for the complete disturbance is

$$\exp\left[i\left(\alpha x - \alpha\sqrt{\Omega}_I\, y - \alpha c_R t\right)\right] \exp\left[c_I t - \alpha\sqrt{\Omega}_R\, y\right] \tag{4.7}$$

to an observer fixed in a fluid at rest ($c_R < 0$). The quantities $\alpha\sqrt{\Omega}_I$ and Ω_I take the sign of c_I. For amplified disturbances, $c_I > 0$ and the disturbance is an incoming wave with an exponentially damped amplitude as $y \longrightarrow \infty$. If $c_I = 0$ (neutral disturbances), then $\Omega_I = 0$ and from Eq. (2.11)

$$\begin{aligned} \Omega = \Omega_R &= 1 - M^2\, c_R^{\,2} \\ &= 1 - \frac{(c^* - U_e^*)^2}{a_e^{*2}} \end{aligned} \tag{4.8}$$

The disturbance can be classified according to whether $\Omega_R \gtreqless 0$; corresponding to subsonic ($U_e^* - a_e^* < c_R^* < U$), sonic ($U_e^* - a_e^* = c_R^*$)

and supersonic ($c_R^* < U_e^* - a_e^*$) disturbances, respectively. The neutral subsonic disturbance is propagated parallel to the x axis and is exponentially damped in y as $y \longrightarrow \infty$.

If $c_I = 0$ and $\Omega_R < 0$ (neutral supersonic disturbances), the pressure disturbance is composed of both incoming and outgoing waves, in general, of unequal amplitudes. Therefore, unless the Sommerfeld radiation condition (pure oncoming or outgoing waves) is imposed as the boundary condition for $y \longrightarrow \infty$, the characteristic values will not be discrete. This problem and the problem of neutral sonic disturbances are discussed in Lees and Lin[32] and will not be treated here.

This condition must somehow be related to the conditions on the axis since the proper frame of reference for stability considerations is one in which the observer is fixed in the fluid at rest and only sees the velocity defect of the wake. Lees and Lin[32] show that the necessary and sufficient condition for the existence of a neutral subsonic disturbance is that the wave speed must equal the mean velocity at the point where the gradient of the mean density-vorticity product vanishes, namely,

$$c_R = w \text{ at } (w'/T)' = 0 \tag{4.9}$$

and must lie between the maximum and the minimum of the mean velocity in the interval $0 < y < \infty$

$$-1 < c_R < 0 \quad . \tag{4.10}$$

The wave speed is a function of both the mean velocity and temperature profiles, and, as will be shown in Section IV. 3, only depends on the temperature excess ΔT, since the velocity is normalized to minus one on the axis and zero at infinity. The dimensional wave speed in a coordinate system fixed in the body is

$$c_R^* = U_e^* \left[1 + c_R \ \Delta U \right] \tag{4.11}$$

and depends on c_R and the velocity defect of the wake.

This result becomes more evident in a coordinate system fixed in the fluid at rest. In this system,

$$c_R \ \substack{<\\=\\>} \ (1/M) \tag{4.12}$$

corresponding to subsonic, sonic and supersonic disturbances, respectively. The relative wave speed, c_R , is a slowly varying function of the mean profiles and temperature excess ΔT, while the reciprocal of the relative Mach number is a rapidly varying function of the velocity defect. This result is very important for the hypersonic wake problem and will be discussed in Section IV. 4.

For shear layer type profiles, Lin[23] has shown that instability might occur when the wave speed is subsonic relative to <u>both</u> external streams. This results in the condition that the difference in the external velocities is less than the sum of the external speeds of sound, i. e.,

$\longrightarrow U_1^*$ $\qquad$ $U_1^* > U_2^*$

$\longrightarrow U_2^*$

$$U_1^* - U_2^* < a_1^* + a_2^* \tag{4.13}$$

IV. 2. Solution of the Inviscid Equation

Since the mean velocity and temperature profiles are functions of the Dorodnitsyn-Howarth variable, η , it is more convenient to transform the inviscid pressure disturbance equation, Eq. (4. 1) , and the boundary conditions Eq. (4.2) and Eq. (4.6) to the following form, using η as the independent variable:

$$\frac{d^2\pi}{d\eta^2} - \frac{2\frac{dw}{d\eta}}{w-c}\frac{d\pi}{d\eta} - \alpha^2 T^2\left[1 - (M^2/T)(w-c)^2\right]\pi = 0$$

$$\pi = 0 \qquad \eta = 0 \quad ; \quad (d\pi/d\eta) + \alpha\sqrt{\Omega}\,\pi = 0 \qquad \eta \longrightarrow \infty \tag{4.14}$$

where

$$d\eta = (dy/T) \quad .$$

The system (4.14) was solved for both neutral and amplified subsonic disturbances using two different methods of solution. The method of solution for the neutral subsonic disturbance will be described below, while the method of solution for the amplified disturbance will be described in Appendix D.

Lees and Lin[32] show that a necessary and sufficient condition for the existence of a neutral inviscid disturbance is that the quantity

$$(w'/T)' = (1/T)(d/d\eta)\left[(1/T^2)(dw/d\eta)\right] = 0 \tag{4.15}$$

must vanish at some value of $w = c_R$. The solution for π is regular at this point if this condition holds. In addition the imaginary part of π is zero everywhere. For numerical purposes it was found convenient to divide the region of integration into two parts: (1), an "inner region" between the axis and the critical point, η_c (where $w = c$); (2), an "outer region" between the critical point and infinity.

In the "inner region", Eq. (4.14) is used. The solution for π in the neighborhood of the critical point is obtained by a series expansion, the details of which are given by Reshotko[59]*:

$$\pi = 1 - \frac{\alpha^2 T_c^2}{2}(\eta - \eta_c)^2 + \frac{\alpha^2 T_c^2}{4}\Big[\frac{T_c''}{T_c} - \frac{2}{3}\frac{W_c'''}{W_c'} + 3\frac{T_c'^2}{T_c^2} - \frac{M^2 W_c'^2}{T_c} - \frac{\alpha^2 T_c^2}{2}\Big](\eta - \eta_c)^4 + \frac{\alpha^2 T_c^2}{3} b\Big[1 + \frac{3}{4}\frac{W_c''}{W_c'}(\eta - \eta_c) + \cdots\Big](\eta - \eta_c)^3 \qquad (4.16)^{**}$$

where b is an unknown constant.

Following the suggestion of Reshotko[59], Eq. (4.14) can be transformed in a Ricatti-type first order non-linear differential equation

$$G' = \Big[1 - \frac{M^2 (W - C)^2}{T}\Big] + \Big[\frac{2W'}{W-C} \quad \frac{2T'}{T}\Big]G - \alpha^2 T^2 G^2 \qquad (4.17)$$

where

$$G = (\pi'/\alpha^2 T^2 \pi) \quad . \qquad (4.18)$$

The boundary condition at infinity then becomes

$$G = -\sqrt{\frac{1 - M^2 c^2}{\alpha}} \qquad \eta \to \infty \quad . \qquad (4.19)$$

Around the critical point

---

* Reshotko found the expansion around the singular point using Eq. (4.1); therefore, Eq. (4.16) is slightly different from that obtained by Reshotko [Reference 59, Eq. (A-8)].

** Primes (') indicate differentiation with respect to η .

$$G = -[\eta - \eta_c] + \left[b + 2\frac{T_c'}{T_c}\right][\eta - \eta_c]^2$$
$$-\left[\frac{2}{3}\frac{W_c'''}{W_c'} - 2\frac{T_c''}{T_c} + M^2\frac{W_c'^2}{T_c} + \alpha^2 T_c^2\right][\eta - \eta_c]^3 + \cdots \tag{4.20a}$$

or inverting Eq. (4.20a)

$$b = \frac{1}{[\eta - \eta_c]^2}\left[G + (\eta - \eta_c) - 2\frac{T_c'}{T_c}(\eta - \eta_c)^2\right.$$
$$\left.\left(\frac{2}{3}\frac{W_c'''}{W_c'} - 2\frac{T_c''}{T_c} + M^2\frac{W_c'^2}{T_c} + \alpha^2 T_c^2\right)(\eta - \eta_c)^3 + \cdots\right] \tag{4.20b}$$

Eq. (4.17) has a singular point at infinity. This can be seen in the following way. The second term on the right hand side tends to zero exponentially as $y \longrightarrow \infty$. Therefore, for large values of η, Eq. (4.14) reduces to

$$G' = (1 - M^2 c^2) - a^2 G^2 \tag{4.21}$$

whose solution is

$$G = \frac{\sqrt{1 - M^2c^2}}{\alpha}\left[\frac{C_1 e^{2\alpha\sqrt{1-M^2c^2}\,\eta} - 1}{C_1 e^{2\alpha\sqrt{1-M^2c^2}\,\eta} + 1}\right]$$

Therefore as $\eta \longrightarrow \infty$

$$G \longrightarrow \frac{\sqrt{1 - M^2 c^2}}{a} \tag{4.22}$$

unless $C_1 \equiv 0$; in this case

$$G \longrightarrow -\frac{\sqrt{1 - M^2 c^2}}{a}$$

If the integration of Eq. (4.17) is started at the critical point, for any arbitrary value of b (unknown constant), all solutions will tend to

Eq. (4.22) at infinity. If b is chosen exactly right the solution will tend to the correct boundary condition (4.19). The correct method of integration then is to start from infinity and integrate in towards the critical point.

The calculation procedure used to obtain the neutral inviscid solution for the given profiles $w(\eta)$ and $T(\eta)$ and the relative Mach number, M , is as follows:

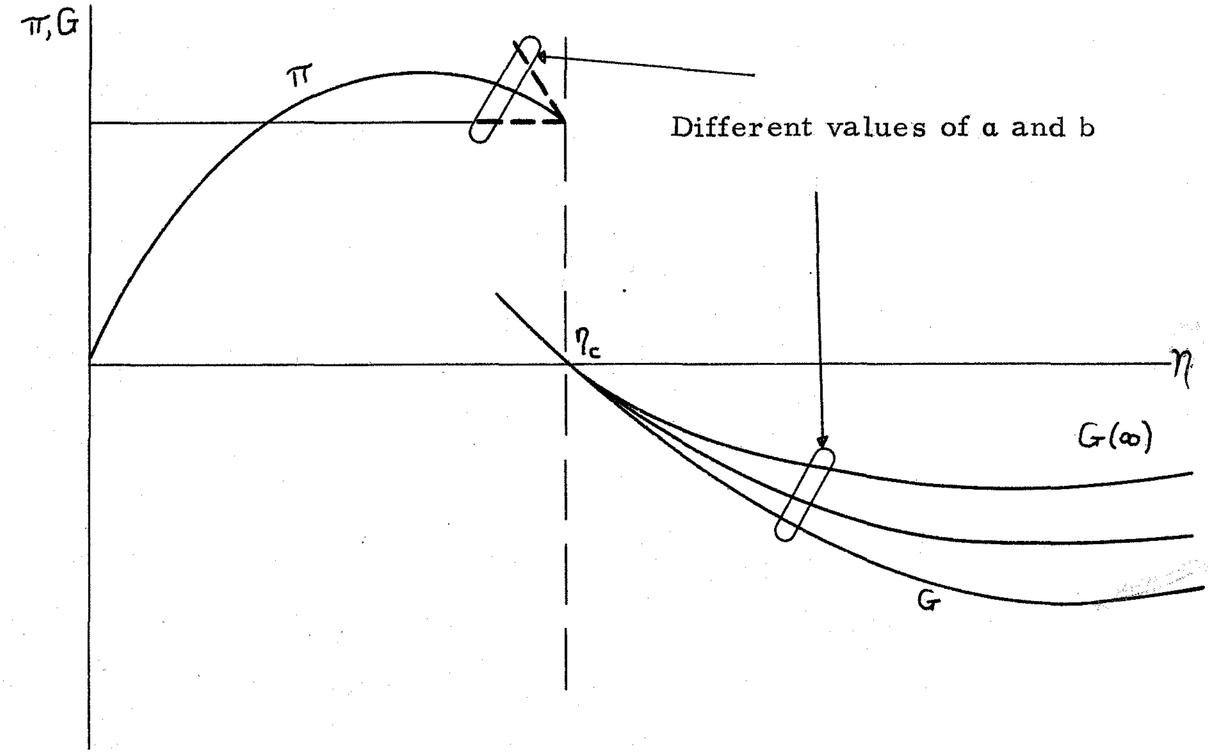
Integration from Infinity to Critical Point

1. Evaluate c and η_c from Eq. (4.15).
2. Assume a value of α and evaluate $G(\infty)$ $\left[\text{Eq. (4.19)}\right]$.
3. Continue the calculation of G by integration of Eq. (4.17) to some small positive value of $(\eta - \eta_c)$.
4. Evaluate the unknown constant, b , at $(\eta - \eta_c)$ from Eq. (4.10b) since G is known from step 3 at this point.

Integration from Critical Point to Wall

5. Using the value of the constant, b , from step (4) evaluate π from Eq. (4.16).
6. Continue the calculation of π by integration of Eq. (4.14) to the wake axis.
7. Repeat steps (2) to (6) until the boundary condition, $\pi = 0$, is satisfied.

The nature of the integral curves for π and G are shown in the following sketch.



Sketch 4. 1

IV. 3. Numerical Results

In order to find the effect of relative Mach number, M, and temperature excess, ΔT, on the stability of a hypersonic wake, the wave numbers, wave speeds and amplification rates of a typical blunt body wake were obtained using the methods described in Section IV. 2 and Appendix D. The inviscid equations were solved numerically on the IBM 7090 of the California Institute of Technology Computing Center by the Runge-Kutta-Gill method.

The mean flow model of Kubota (Section II. 4)

$$w = - e^{-\eta^2} \qquad T = 1 + \Delta T \, e^{-\eta^2} \tag{4.23}$$

was used in the numerical calculations. For neutral subsonic disturbances

$$(w'/T^2)' = 0 \tag{4.24}$$

and, using Eq. (4.23)

$$(1 - 2\eta_c^{\,2}) + \Delta T \; e^{-\eta_c^{\,2}} \left[1 + 2\,\eta_c^{\,2} \right] = 0 \tag{4.25}$$

$$c = w(\eta_c) = -\, e^{-\eta_c^{\,2}} \quad . \tag{4.26}$$

Eq. (4.25) is a transcendental equation for η_c as a function of the temperature excess, ΔT. The neutral inviscid wave speed is determined once η_c is known and is a function only of ΔT (independent of M). As T increases, η_c moves out, towards the outer edge, i.e., the density-vorticity product spreads out and shifts to higher values of , and c increases towards zero; in other words $c_R{}^*$ approaches the free stream velocity $U_e{}^*$. These results are shown in Figure 6 and are listed in Table I.

As mentioned in Section IV.1, the disturbances can be classified according to whether

$$- c_R \lesseqgtr (1/M) \tag{4.27}$$

corresponding to subsonic, sonic and supersonic disturbances, respectively. For neutral subsonic disturbances the mean flow is unstable with respect to small disturbances provided a value $- c_R < (1/M)$ exists for which $(w'/T^2)'$ vanishes. As c_R increases, the critical relative Mach number, $M_{cr} = \left[U_e{}^* - U^*(0) \right] / a_e{}^* = - (1/c_R)$ increases very rapidly. This result is shown in Figure 7. As ΔT increases, the wave speed c_R increases (critical Mach number also increases).

This result is also shown in Figure 7.

The neutral inviscid wave number, α_s, was then determined for various values of ΔT and M. For a fixed value of M, the inviscid wave number decreases with increasing ΔT, implying that a hot wake will be more stable than a cool one. These results are listed in Table 1 and are shown in Figure 8.

For a fixed value of ΔT, the inviscid wave number decreases with increasing M, verifying the results of Lin[23]. The value of α_s seems to be linearly dependent on M^2, the slope being a function of ΔT. Corresponding to each value of ΔT, there is a critical Mach number above which subsonic disturbances are impossible. Therefore as M increases (for a fixed ΔT), a critical wave number is reached below which subsonic disturbances are impossible. This critical wave number decreases with increasing ΔT (Figure 8).

Thus, we see that the effect of temperature is two-fold. As ΔT increases, the critical Mach number increases, and the range of relative Mach numbers over which subsonic disturbances can exist also increases (Figure 7). However, as long as the relative Mach number is below the critical Mach number the neutral inviscid wave number, α_s, will decrease with increasing ΔT (Figure 8), implying greater stability of the wake flow. In order to make this statement more definite, the amplification rates must be compared at these various conditions. The amplification rate depends upon the velocity defect of the wake, the temperature excess and relative Mach number. It was decided to take a typical hypersonic wake and compute the dimensionless maximum amplification rate at each station. These results are compared with the results of Sato and Kuriki[3] for a flat plate incompressible wake.

A hypersonic cylinder wake was considered under the following conditions:

Free stream Mach number - 5.8

Diameter of cylinder - 0.100"

Free stream Reynolds number based on diameter - 8,280.

Under these conditions, McCarthy[54] found that transition from laminar to turbulent flow occurred 47 diameters downstream of the neck. He also was able to compute the temperature excess, ΔT, velocity defect, ΔU and the relative Mach number M, as a function of the downstream coordinate, (x^*/d^*). At four typical stations

x^*/d^*	ΔU	ΔT	M
5	.285	.500	.65
10	.160	.380	.41
20	.083	.300	.26
40	.049	.200	.16

Table 3.1

At each station, the Gaussian distributions of Kubota were fitted to the temperature excess and the velocity defect, and the stability characteristics determined by the method of Section IV.2 and Appendix D. The results are tabulated in Table 2.

For the range of temperatures and Mach numbers considered, the wave number is almost a universal function of the complex wave velocity for $\alpha < 0.8$. In fact it compares reasonably well with the stability characteristics of the incompressible wake. For $\alpha > 0.8$ the real part of the complex wave velocity is also roughly independent of the temperature, but the imaginary part is a strong function of temperature; hence the spatial amplification rate is also a very strong function of temperature. These results can be explained by the following argument: at low wave numbers, or large wave lengths the mean profiles become unimportant since the length scale of the disturbance is larger than the length scale of the mean flow; for large wave numbers, or small wave lengths, the length scale of the disturbance is of the order of the scale of the mean flow and the mean profile then becomes important.

For the four cases considered, the propagation velocity of the wave was practically equal to the group velocity (within 10 %) and for calculation purposes, the amplification rate was determined by using the former. These results are listed in Table 2. The dimensionless spatial amplification rate is shown in Figure 9 as a function of the dimensionless frequency. The maximum rate of amplification occurs at a frequency of between 0.7 and 0.8. This result is remarkably similar to that obtained by Sato and Kuriki[3]* for the incompressible flat plate wake. The natural oscillations seem to occur at a preferred frequency, and they amplify experimentally with a constant spatial amplification rate in the streamwise direction (Figure 8). The maximum dimensionless spatial amplification rate was found to be 0.165

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* The wave speed and wave number used by Sato and Kuriki[3] are defined differently than in the text. The conversions are

$$C(\text{Gold}) = \left[c\ (\text{Sato and Kuriki}) - 1 \right] \times \quad (1.45)$$

$$\alpha(\text{Gold}) = \left[\alpha\ (\text{Sato and Kuriki}) \right] \quad (1.20)$$

for the compressible wake under consideration and 0.330 for the incompressible flat plate wake. The effect of temperature seems to be a stabilizing one if the dimensionless amplification rate is a measure of the relative stability between two flows. The ratio of the disturbance amplitudes (in the linear regime) at two different stations, is given by Eq. (2.10b).

$$Q^{x'}/Q_i^{x'} = \exp \int_{x_i^*/d^*}^{x^*/d^*} d^* \frac{\alpha^* C_I^*}{C_g^*} \, d\, x^*/d^*$$

Since

$$\alpha^* = \alpha / L^* \qquad c_I^* = c_I V^* = c_I \, \Delta U \; U_e^*$$

$$c_R^* = \bar{c}_R U_e^* \qquad c_g^* = (d/d\alpha^*)(\alpha^* c_R^*) = \bar{c}_g U_e^*$$

$$\bar{c}_R = 1 + c_R \Delta U \qquad \bar{c}_g = \bar{c}_R + \alpha (dc_R/d\alpha) \, \Delta U$$

where L* and ΔU are defined by Eqs. (2.48) and (2.49), then Eq. (2.10b) becomes

$$Q^{x'}/Q_i^{x'} = \exp \int_{x_i^*/d^*}^{x^*/d^*} \frac{\alpha C_I}{\bar{C}_g} \frac{\rho_\infty^*}{\rho_e^*} C_D \frac{R_{ed^*}}{8\sqrt{\pi}} \, d \ln x^*/d^* \tag{4.28}$$

The appropriate drag coefficient is not the total drag of the body but the value of C_D in the inner laminar wake, which swallows momentum defect in the outer flow very slowly. Therefore C_D is approximately equal to the initial drag coefficient at the neck and therefore[1]

$$(\rho_\infty^*/\rho_e^*) \; C_D \simeq \left(3 / \sqrt{R_{e_{d^*}}} \right) \; . \tag{4.29}$$

Eq. (4.28) then becomes

$$Q^{*'}/Q_i^{*'} = \exp \int_{x_i^*/d^*}^{x^*/d^*} \frac{\alpha C_I}{\bar{C}_g} \frac{3}{8\sqrt{\pi}} \sqrt{R_{e_{d^*}}} \, d \ln x^*/d^* \tag{4.30}$$

If the dimensionless amplification rate is independent of (x^*/d^*) as the numerical results seem to indicate then Eq. (4.28) can be integrated, to give the following result

$$Q^{*'}/Q_i^{*'} = \left[\frac{x^*/d^*}{x_i^*/d^*}\right]^{\frac{3}{8\sqrt{\pi}}\sqrt{R_{e_{d^*}}}\frac{\alpha C_I}{\bar{C}_g}} \quad . \tag{4.31}$$

If the disturbance originates 5 diameters downstream of the neck, and the linear region (region in which the theory of small disturbances applies) is assumed to extend to the transition point, then for the case just considered

$$(Q^{*'}/Q_i^{*'}) = 13 \qquad \left(R_{e_{d^*}} \approx \frac{R_{\infty_{d^*}}}{8}\right) \quad . \tag{4.32}$$

This result is of the same order as the experimental results of Sato and Kuriki for a flat plate incompressible wake in the linear region.

Extreme care must be taken in applying these numerical results to the formulation of a theory of transition for hypersonic wakes, although these results do indicate some trends (concerning transition) that are observed in wind tunnel and ballistic range experiments (Section IV. 4). The mechanism of transition is a very complex one and cannot be explained fully by a linear theory.

IV. 4. The Hypersonic Wake Problem

In this section some of the laminar-turbulent transition phenomena observed in the hypersonic wake of blunt and slender bodies will be explained by the small disturbance theory of laminar stability using the results obtained in Sections IV. 1 to IV. 3.

If the free stream Reynolds number based upon a characteristic body dimension is very low the wake will be completely laminar. As the free stream Reynolds is increased, transition occurs in the wake far down stream of the body, and begins to move upstream as the Reynolds number is further increased. Eventually, transition "sticks" in the region of the neck, i. e., the transition point approaches a fixed value over a wide range of Reynolds numbers, and the wake downstream of this transition point is completely turbulent.

The fact that the wake is laminar below a certain critical Reynolds number can be explained by recognizing that there is a minimum critical Reynolds number below which the turbulence in the wake cannot maintain itself against the action of viscous dissipation. One way that this Reynolds number can be found is by assuming that the effective turbulent diffusivity is equal to the laminar diffusivity[1]. If the Reynolds number is below this value the wake will be always laminar; if it is slightly above turbulent flow is possible. (There is also a minimum critical Reynolds number in laminar stability theory below which all small disturbances are stable, according to the solution of the full viscous small disturbance equations.)

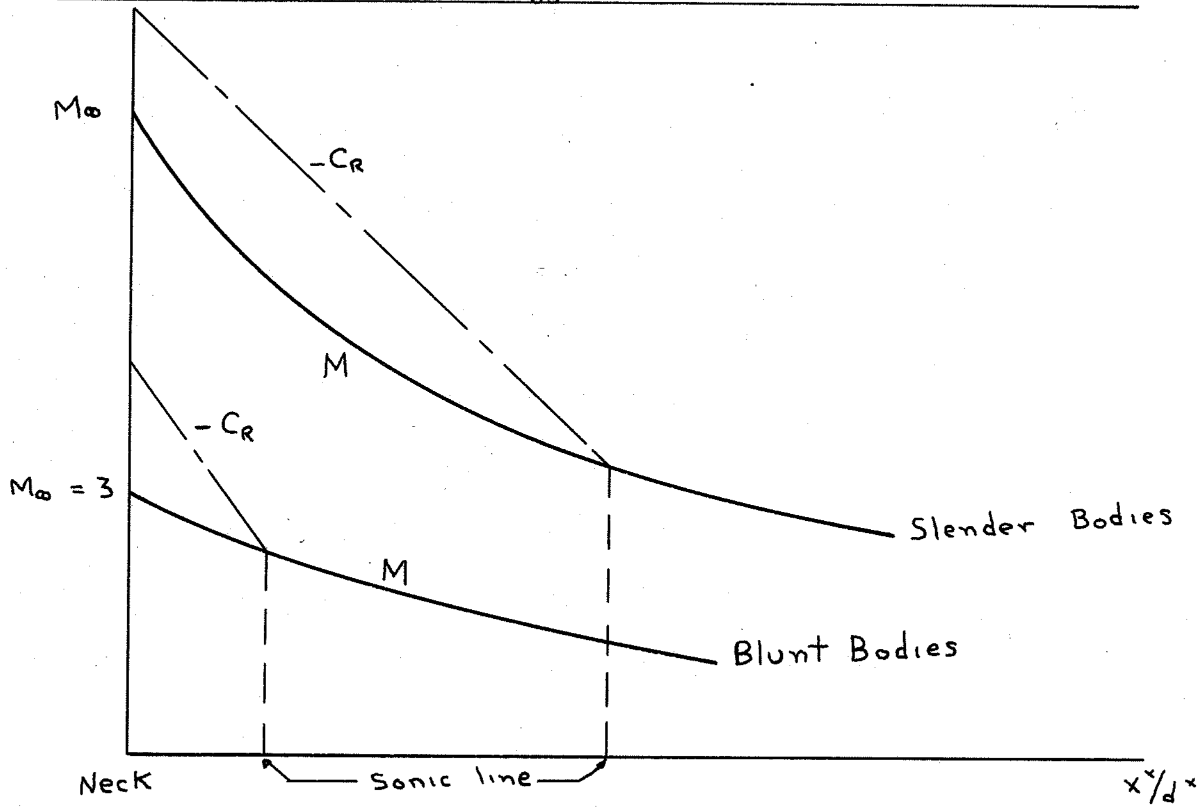
The upstream motion of the transition point as the Reynolds number is increased is not yet understood. It seems probable, however, that transition is preceded by linear and non-linear regions similar to those found by Sato and Kuriki for an incompressible flat plate wake. Considerations based on Eq. (4. 31) may indeed furnish an explanation of this phenomenon.

The "sticking" phenomena is caused by the fact that subsonic disturbances are impossible in the wake neck region and in the free shear layer because the relative Mach number is so high there. In the neck region, the relative Mach number is practically equal to the local "external" Mach number, since the centerline velocity is very small. For blunt bodies the external Mach number is "frozen" at about three, while for slender bodies the external Mach number is approximately equal to the free stream Mach number. The velocity defect and relative Mach number decrease very rapidly downstream of the neck, while the wave speed decreases to roughly one-half of the velocity defect. At some point downstream of the neck, subsonic disturbances will,occur implying instability. The length of the stable region is determined mainly by the external Mach number and the rate of decay of the velocity defect. The stable region will be longer for slender bodies than for blunt bodies because the external Mach number is much larger in the former case. [Sketch 4. 2] This prediction is verified experimentally by the results of Slattery and Clay[60] for spheres and cones.

The experimental studies of Chapman, et al[52] and Larson[53] show that a laminar free shear layer is very stable for high external Mach numbers. Lin[23] [Section IV. 1] indicates that instability occurs if the wave speed is subsonic relative to both streams and from Eq. (4. 13)

$$U_1^* - U_2^* < a_1^* + a_2^* \quad .$$

For the free shear layer behind a body, $U_2^* \cong 0$ and



Sketch 4. 2

$$M_1 < 1 + (a_1^*/a_2^*) \quad .$$

If the body is adiabatic then the condition for the existence of subsonic disturbances is that $M_1 < 2.5$. Again the local external Mach number for a blunt body is frozen at about three while for slender bodies it is of the order of the free stream Mach number. Therefore, subsonic disturbances will not exist for most cases of practical interest and the shear layers will be very stable. The free stream Reynolds number must be raised one to two orders of magnitude in order that transition jump from the wake to the body boundary layer[1].

V. STABILITY OF AXI-SYMMETRIC COMPRESSIBLE WAKE FLOWS

The stability of inviscid axi-symmetric incompressible jets has recently been studied by Batchelor and Gill[36].* The more general problem of the stability of inviscid, compressible fluids will be discussed in this section with special emphasis on wake flows. This study will give valuable insight into the stability problem of a real fluid at very large Reynolds numbers ($\alpha R >> 1$). It is known from experience that wake-type flows are very unstable. This suggests that the dynamical properties of the flow are very important in the stability problem.

The eigenvalue equation and boundary conditions for the inviscid radial velocity disturbance amplitude constitute a Sturm-Liouville system analogous to that treated by Lees and Lin[32] for two-dimensional disturbances. By transforming the velocity components to a new orthogonal set, the similarity is even more startling. This fact suggests that results similar to those for the two dimensional case can be obtained for the axi-symmetric problem. These results are derived in Sections V.1 - V.4.

Again, as in the case of two-dimensional flows, it is found convenient to classify the disturbances as "subsonic", "sonic" and "supersonic", according to whether the phase velocity of the disturbance relative to the free-stream velocity is less than, equal to, or greater than the mean speed of sound in the free stream.

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* This work was carried on at the same time as the present investigations, unbeknown to this author.

It was also found that neutral and self-excited subsonic disturbances are possible only when the gradient of a density-vorticity product vanishes for some $-w < (1/M)$(in a coordinate system fixed in a fluid at rest), which is exactly analogous to the two-dimensional case.

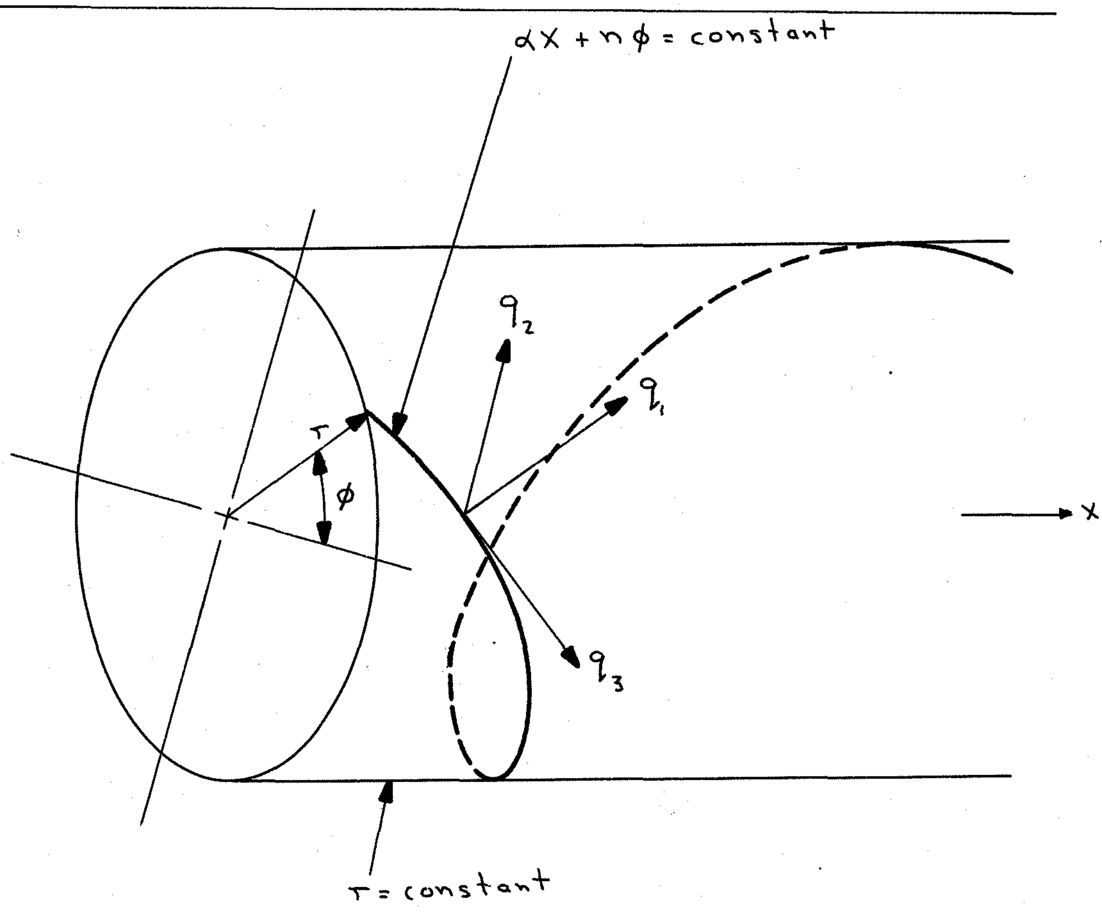
The energy transfer mechanism between the mean flow and the disturbance flow is studied in the inviscid limit. It is found that the Reynolds shear stress is composed of two terms: one associated with a density-vorticity product, which produces a discontinuity in the shear stress at the critical point; the other associated with a singularity in the radial disturbance vorticity, which produces a delta function behavior near the critical point. The latter contribution is a destabilizing influence.

The special case of an axi-symmetric wake is worked out. For incompressible flow only the $n = 1, 2$ modes are unstable. If the temperature profile is of a "top-hat" nature then only the $n = 0, 1$ modes are unstable; if it is "slowly varying" then the $n = 1, 2$ modes are unstable. The $n = 1$ mode seems to be the most unstable mode because the radial velocity component is free to "flop" around on the axis, giving the motion an extra "degree-of-freedom". These oscillations are analogous to the anti-symmetric oscillations in two-dimensional flow which are known to be more unstable than the symmetric ones. However, it is necessary to calculate the amplification rates of different modes before a definite statement can be made.

V. 1. Similarity Between the Small Disturbance Equations and Boundary Conditions for Axi-symmetric and Two-Dimensional Flows

Batchelor and Gill[36] show that by a suitable transformation of velocity components, the incompressible small disturbance equations for axi-symmetric flow become similar to the two-dimensional small disturbance equations. These results are now extended to the case of compressible, axi-symmetric flows.

Following Batchelor and Gill, the lines of intersection of the



Sketch 5. 1

family of surfaces

$$r = \text{constant}, \ \alpha x + n\phi = \text{constant}$$

are circular helices on which the phase of the disturbance wave is constant [see Sketch 5.1]. The disturbance amplitudes depend only on the variables r and $\alpha x + n\phi$ and are constant on a helix of this family.

It is convenient to define new orthogonal velocity coordinates as follows [See Sketch 5.1] :

$$\begin{aligned} q_1 &= \frac{n}{mr} q_\phi + \frac{\alpha}{m} q_x \\ q_2 &= q_r \\ q_3 &= \frac{\alpha}{m} q_\phi - \frac{n}{mr} q_x \end{aligned} \tag{5.1}$$

where q_1 is the velocity component perpendicular to both the radial line and the helix of constant phase [r = constant, $\alpha x + n\phi$ = constant], q_3 is the velocity component parallel to the tangent to the helix of constant phase, and $m = \sqrt{\alpha^2 + (n^2/r^2)}$ is the magnitude of the total wave number. The tangent to the helix of constant phase makes an angle, $\tan^{-1}(\alpha r/n)$ with the axis of the cylinder.

In this coordinate system, Eqs. (2.38) - (2.43) become:

Continuity

$$i(W-C)s + q_2 \rho' + \rho\left[\frac{1}{r}(rq_2)' + iq_1 \frac{m}{\alpha}\right] \tag{5.2}$$

1- Momentum

$$\rho\left[i(W-C)q_1 + \frac{\alpha}{m} q_2 W'\right] = -\frac{i\pi}{\gamma M^2}\frac{m}{\alpha} \tag{5.3}$$

2- Momentum

$$i\alpha^2 \rho (W-c)\, q_2 = -\frac{\pi'}{\gamma M^2} \tag{5.4}$$

3- Momentum

$$\rho\left[i(W-c)\, q_3 - \frac{n}{mr}\, q_2 W'\right] = 0 \tag{5.5}$$

Energy

$$\rho\left[i(W-c)\theta + T' q_2\right] = -\left[\gamma - 1\right]\left[\frac{1}{r}(r q_2)' + i q_1 \frac{m}{\alpha}\right] \tag{5.6}$$

State

$$\frac{s}{\rho} = \pi - \frac{\theta}{T} \qquad\qquad \rho T = 1 \tag{5.7}$$

These equations are exactly the same as those for two-dimensional inviscid flow, Eqs. (2.23) - (2.27), except for the obvious coordinate scale factors. q_1 and q_2 correspond, respectively, to the longitudinal and normal velocity disturbances in two dimensional flow. The velocity component q_3 appears only in Eq. (5.5) and is determined once q_2 is known. It plays the same role as the sweep velocity in boundary layer theory.

The two-dimensional small-disturbance equations can be reduced to a single second order equation in π or ϕ [Section IV. 1], while the disturbance equations for axi-symmetrical flow can be reduced to a similar single second order equation in π or q_r [Section V. 2]. The boundary conditions on the axis for these perturbation amplitudes are:

Axi-symmetric [Eq. (2.44)]

$n = 0$	$\pi(0)$ arbitrary	$q_r(0) = 0$
$n = 1$	$\pi(0) = 0$	q_r arbitrary
$n > 1$	$\pi(0) = 0$	$q_r(0) = 0$

Two-Dimensional [Eqs. (2.28) and (2.29)]

Anti-symmetrical oscillations	$\pi(0) = 0$
	$\phi(0)$ arbitrary
Symmetrical oscillations	$\pi(0)$ arbitrary
	$\phi(0) = 0$

Far away from the axis, the boundary conditions are exactly the same [Eqs. (2.30) and (2.45)]. Therefore, the $n = 0$ mode corresponds to symmetrical oscillations and the $n = 1$ mode corresponds to the anti-symmetrical ones. There is no direct comparison between the $n > 1$ modes and the two-dimensional oscillations. It is expected, therefore, that the $n = 1$ mode will be the most unstable mode [Section V.6].

Since the small disturbance axi-symmetric flow equations and boundary conditions are analogous to those for two-dimensional flow, it is expected that many of the basic results will be the same in both cases.

V.2. Inviscid Disturbance Equation and Outer Boundary Condition

A single second order equation in either q_r or π can be obtained from the system of Eqs. (5.2) - (5.5). Since the inviscid disturbances are particle-isentropic

$$\frac{1}{p^*}\frac{dp^*}{dt^*} = \frac{\gamma}{\rho^*}\frac{d\rho^*}{dt^*} = -\operatorname{div}\vec{W}$$

or

$$i(w-c)\,\pi = -\gamma\left[\frac{1}{r}(rq_r)' + iq_1\frac{m}{\alpha}\right] \tag{5.8}$$

The variables q_1 and π can then be eliminated from Eqs. (5.3), (5.4), and (5.8), resulting in the following self-adjoint equation

$$(\xi\psi')' - \left(p + \frac{\alpha^2}{Tr}\right)\psi = 0 \tag{5.9}$$

where

$$\begin{aligned}
\psi &= rq_r \\
\xi &= \frac{1}{r\left[\frac{m^2}{\alpha^2}T - M^2(w-c)^2\right]} \\
p &= \frac{1}{w-c}\left[\xi w'\right]'
\end{aligned} \tag{5.10}$$

The other disturbance amplitudes can be found in terms of ψ as follows:

$$\begin{aligned}
\frac{i\pi}{\gamma M^2} &= (w-c)^2\xi\left(\frac{\psi}{w-c}\right)' \\
iq_1 &= -\xi T(w-c)\frac{m}{\alpha}\left(\frac{\psi}{w-c}\right)' - \frac{\alpha}{m}\frac{\psi}{w-c}\frac{w'}{r} \\
iq_3 &= \frac{n}{mr^2}w'\frac{\psi}{w-c} \\
is &= \frac{M^2(w-c)^2}{T}\xi\left(\frac{\psi}{w-c}\right)' + \frac{\psi}{w-c}\frac{T'}{rT^2} \\
i\theta &= (\gamma-1)M^2T(w-c)^2\xi\left(\frac{\psi}{w-c}\right)' - \frac{\psi}{w-c}\frac{T'}{r}
\end{aligned} \tag{5.11}$$

A self adjoint equation can also be written for the pressure disturbance

$$\left[\frac{rT}{(W-C)^2}\pi'\right]' - \left[\frac{\alpha^2}{(W-C)^2\xi}\right]\pi = 0 \qquad (5.12)$$

The boundary condition on the axis is

$$\psi = 0 \qquad r = 0 \quad . \qquad (5.13)$$

The boundary condition for large values of r must be determined from the self-adjoint equation. When $r \longrightarrow \infty$, $w \longrightarrow 0$, $T \longrightarrow 1$, $p \longrightarrow 0$ $\xi \longrightarrow (1/r)\left[1 - M^2 c^2\right]^{-1}$. Then Eq. (5.9) takes the limiting form

$$\psi'' - (\psi'/r) - a^2\Omega\,\psi = 0 \qquad (5.14)$$

where

$$\Omega = 1 - M^2 c^2 \quad . \qquad (5.14a)$$

The solution of Eq. (5.14) is

$$\psi = r\left[A\, I_1(a\sqrt{\Omega}\, r) + B\, K_1(a\sqrt{\Omega}\, r)\right] \quad . \qquad (5.15)$$

Following the suggestion of Lees and Lin[32], introduce a "cut" along the negative real axis of the complex Ω - plane so that the real part of $\sqrt{\Omega}$ will always be positive as long as $-\pi < \arg \Omega < \pi$. The asymptotic form of the solution (5.15) is

$$\begin{aligned} I_1(\alpha\sqrt{\Omega}\, r) &\longrightarrow \sqrt{\frac{\pi}{2r\alpha\sqrt{\Omega}}}\; e^{\alpha\sqrt{\Omega}\, r} \\ K_1(\alpha\sqrt{\Omega}\, r) &\longrightarrow \sqrt{\frac{\pi}{2r\alpha\sqrt{\Omega}}}\; e^{-\alpha\sqrt{\Omega}\, r} \\ R1\sqrt{\Omega} > 0 &\qquad \alpha\sqrt{\Omega}\, r \longrightarrow \infty \end{aligned} \qquad (5.16)$$

From physical considerations q_r must be bounded for large r; therefore Ψ must behave like $\sqrt{r}\quad e^{-\alpha\sqrt{\Omega}\, r}$ as $r \longrightarrow \infty$ or

$$\Psi' + \alpha\sqrt{\Omega}\;\Psi = 0 \qquad -\pi < \arg\Omega < \pi \quad . \tag{5.17}$$

This is the same boundary condition as for the two-dimensional case, [Eq. (4.6)] . A simple interpretation of this result will be given below.

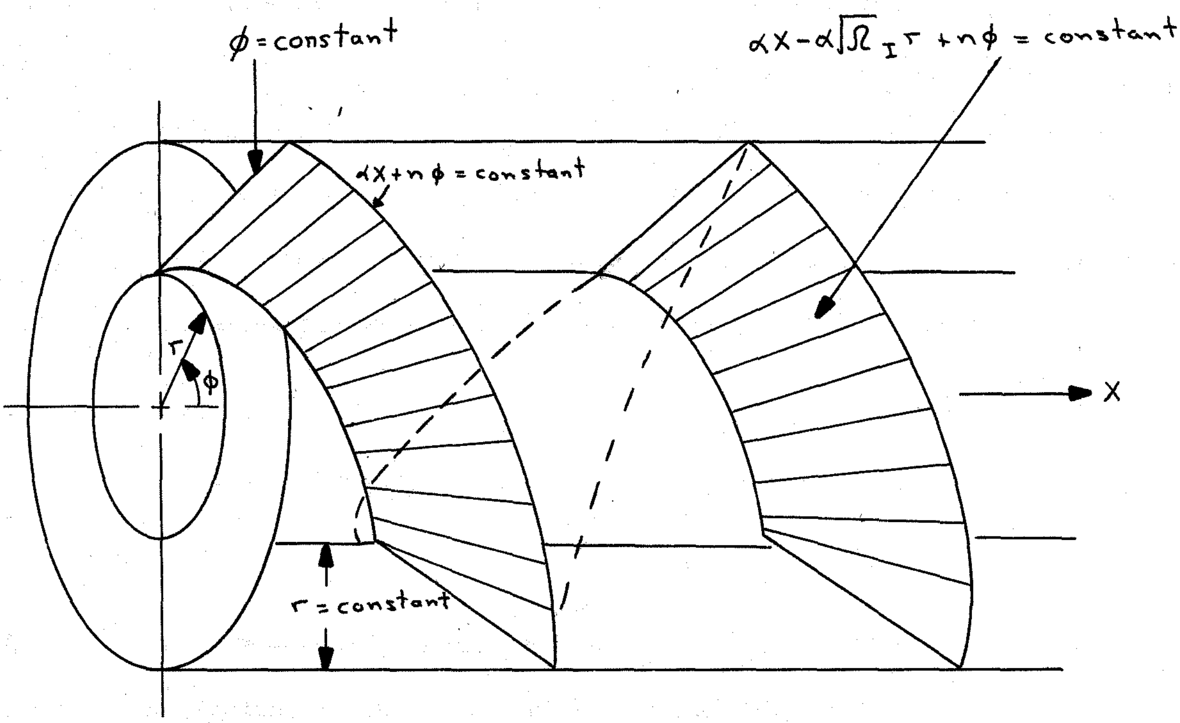
The product of the asymptotic solution, $e^{-\alpha\sqrt{\Omega}\, r}$, and the r independent part of the disturbance, $\exp\left[i\alpha\,(x - ct) + in\phi\right]$, represents progressive waves with the direction of propagation of the wave dependent upon the frame of reference of the observer. If a wave propagates outward and in the negative x direction with respect to an observer fixed in a fluid at rest, it will propagate inward and in the positive x direction to an observer fixed in the body and vice versa [See Section IV. 1.] . The asymptotic form of the solution (large values of r) is

$$\sqrt{r}\;\exp\left[i\left(\alpha x - \alpha\sqrt{\Omega}_I\, r + n\phi - \alpha c_R t\right)\right]\exp\left[c_I t - \alpha\sqrt{\Omega}_R\, r\right] \tag{5.18}$$

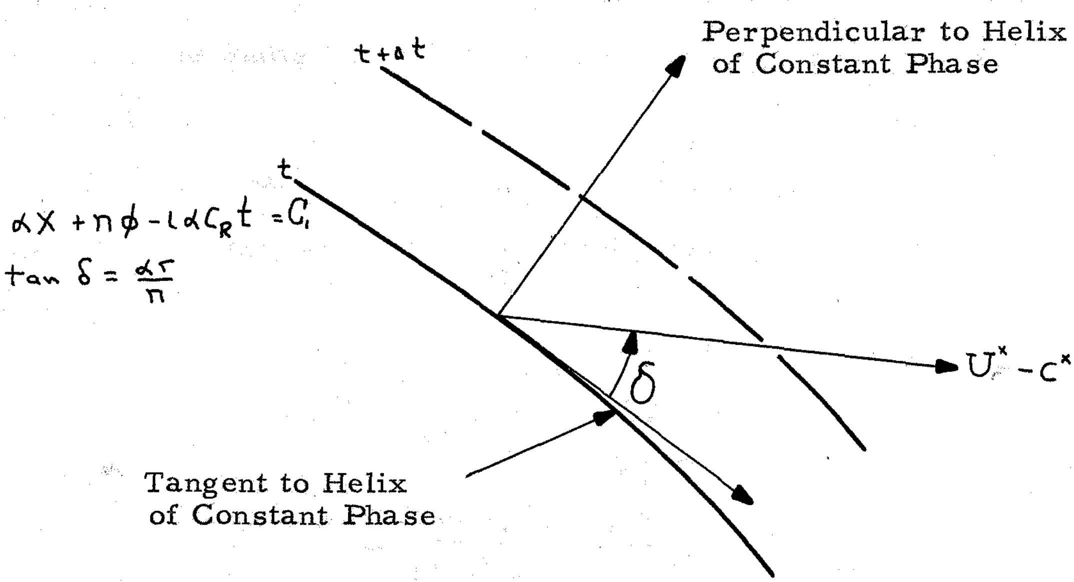
to an observer fixed in a fluid at rest ($c_R < 0$). The surface

$\alpha x - \alpha\sqrt{\Omega}_I\, r + n\,\phi =$ constant is generated by the circular helix $\alpha x + n\phi =$ constant, for various values of r [See Sketch 5. 2.] . The quantities Ω_I and $\sqrt{\Omega}_I$ take the sign of c_I. For amplified disturbances, $c_I > 0$, and the disturbance is an incoming wave (for each ϕ) with an exponentially damped amplitude as $r \longrightarrow \infty$. If $c_I = 0$, then $\Omega_I = 0$ and from Eq. (2.11)

$$\Omega = \Omega_R = 1 - M^2 c_R^{\,2} = 1 - \frac{(c^* - U_e^*)^2}{a_e^{*2}} \tag{5.19}$$



Sketch 5.2



Sketch 5. 3

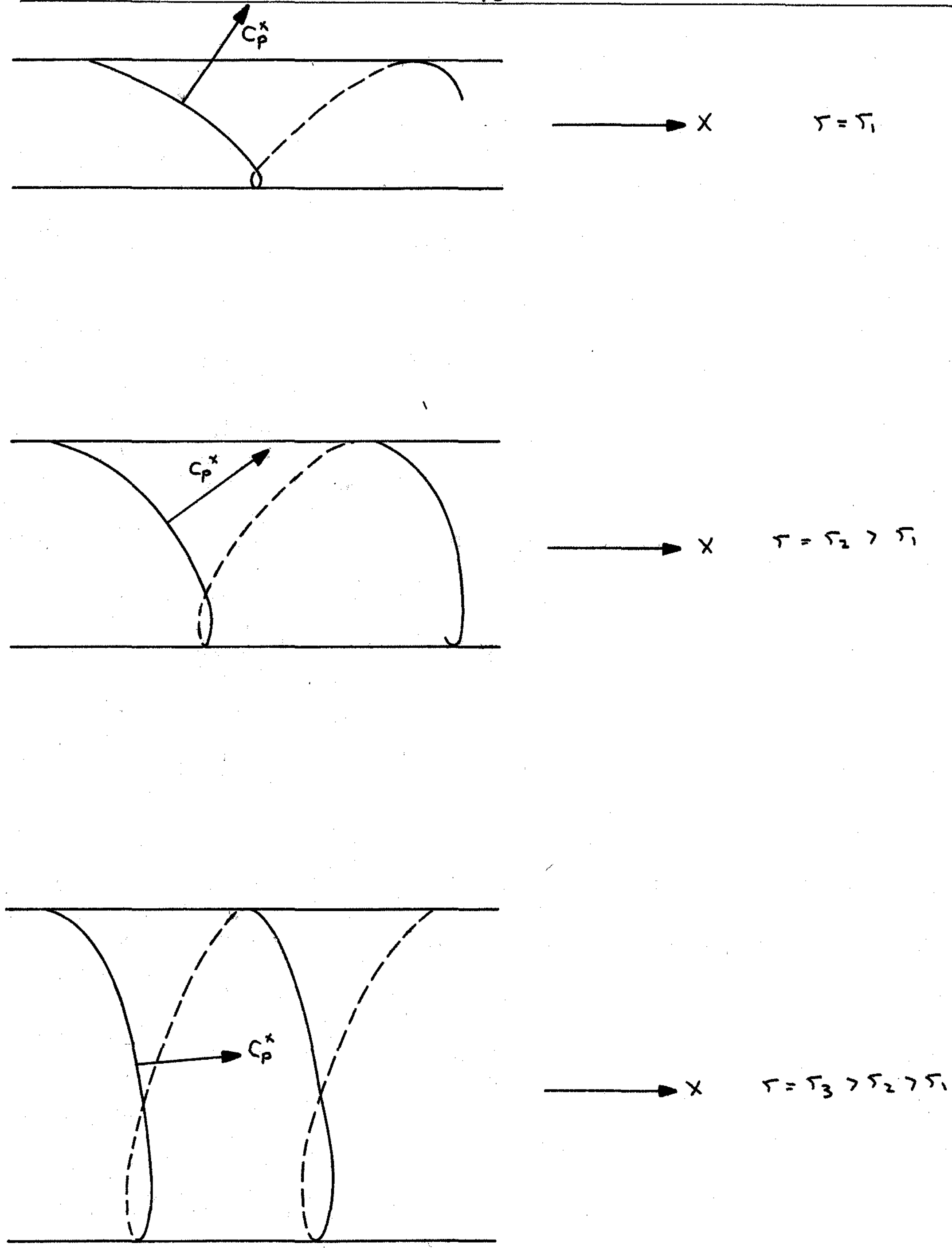
The disturbances can be classified according to whether $\Omega_R \gtreqless 0$; corresponding to subsonic $\left[U_e^* - a_e^* < c_R^* \right]$, sonic $\left[U_e^* - a_e^* = c_R^* \right]$ and supersonic $\left[c_R^* < U_e^* - a_e^* \right]$ disturbances respectively. This classification has the following interpretation. Consider two successive positions of the helix of constant phase, at time t and t + Δt (Sketch 5. 3). The local relative propagation velocity of the front is equal to

$$C_p^* = (U^* - C_R^*) \cos\left(\frac{\pi}{2} - \delta\right) \mp \frac{U^* - C_R^*}{\sqrt{1 + \frac{n^2}{\alpha^2 r^2}}} \tag{5. 20}$$

As $r \longrightarrow \infty$, $U^* \longrightarrow U_e^*$ and $c_p^* \longrightarrow U_e^* - c_R^*$. Thus only disturbances propagating at subsonic velocities relative to the free stream $\left[U_e^* - c_R^* < a_e^* \right]$ will have amplitudes that vanish exponentially as $r \longrightarrow \infty$. The helix of constant phase becomes more and more like a circle in the x- plane as $r \longrightarrow \infty$, i. e., $\delta = \tan^{-1} (\alpha r/n) \longrightarrow (\pi/2)$. This situation is illustrated in Sketch 5. 4. Eventually, the local wave front propagates almost parallel to the x- axis, and the outer boundary condition is the same for axi-symmetric flow as it is for two-dimensional flow.

The local Mach number of the disturbance is defined as the local propagating velocity of the disturbance divided by the local speed of sound

$$\begin{aligned} M_L^2 = \frac{C_p^{*2}}{a^{*2}} &= \frac{\left[U^* - c_R^*\right]^2}{\left[1 + \frac{n^2}{\alpha^2 r^2}\right] a^{*2}} \\ &= \frac{(W - C_R)^2}{\left(1 + \frac{n^2}{\alpha^2 r^2}\right)} \frac{M^2}{T} \end{aligned} \tag{5. 21}$$



$$\tan \delta = \frac{\alpha \tau}{\eta} \qquad \delta \longrightarrow \frac{\pi}{2} \qquad \tau \longrightarrow \infty$$

Sketch 5. 4

If

$$T = \frac{M^2 (w - c_R)^2}{\left(1 + \frac{n^2}{\alpha^2 r^2}\right)} ,$$

the local Mach number is unity, and the denominator of ξ [Eq. (5.10)] vanishes. This sonic line is only an apparent singularity of the differential equation, Eq. (5.9). If

$$T > \frac{M^2 (w - c_R)^2}{\left(1 + \frac{n^2}{\alpha^2 r^2}\right)}$$

the flow is everywhere subsonic with respect to the wave. If the local Mach number is unity somewhere within the wake then subsonic and supersonic flow exists within the flow region.

For neutral supersonic disturbances, the radial velocity disturbance and the pressure disturbance are composed of both incoming and outgoing waves of unequal amplitudes, in general. Therefore, unless the Sommerfeld radiation condition (pure oncoming or outgoing waves) is imposed as the boundary condition for $r \longrightarrow \infty$, the characteristic values will not be discrete. This problem and the problem of neutral sonic disturbances are discussed by Lees and Lin[32] and will not be treated here.

V. 3. Singularities of the Inviscid Disturbance Equation

The inviscid equation, Eq. (5.9) has a regular singularity at the critical annulus, $w = c$, $r = r_c \neq 0$ in the complex r-plane. The solution of the equation in the neighborhood of this "singular point" (annulus) is obtained by the method of Frobenius. The two linearly independent solutions of Eq. (5.9) valid in the neighborhood of the critical point are (Appendix E)

$$\psi_1 = \chi\left[1 + \frac{W_c''}{2W_c'}\chi + \left(\frac{\alpha^2 X_c}{6} + \frac{W_c'''}{6W_c'}\right)\chi^2 + \cdot\cdot\right] \tag{5.22}$$

$\underline{\chi > 0}$

$$\psi_2 = K\,\psi_1 \ln\chi + 1 + b_1\chi + \cdot \tag{5.23}$$

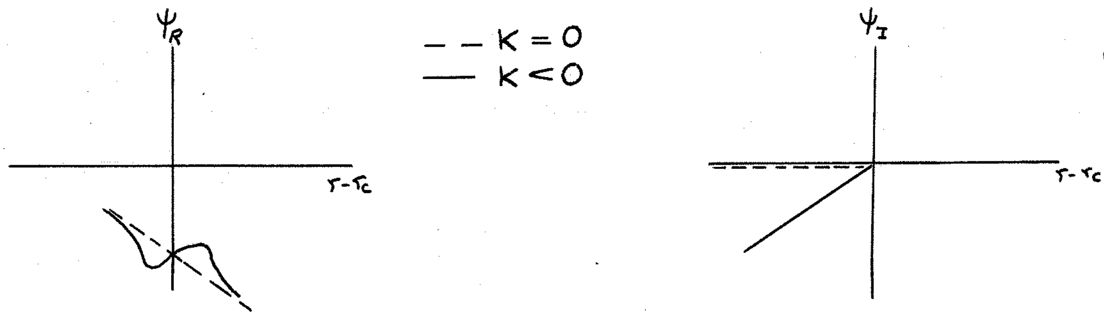
$\underline{\chi < 0}$

$$\psi_2 = K\,\psi_1\left[\ln|\chi| - i\pi\right] + 1 + b_1\chi + \cdot$$

where

$$\begin{aligned}
\chi &= r - r_c \\
K &= \frac{1}{W_c'\xi_c}\frac{d}{dr}\left[\xi W'\right]_c \\
X_c &= 1 + \frac{n^2}{\alpha^2 r_c^2} = \frac{m_c^2}{\alpha^2} \\
\xi_c &= \frac{1}{T_c r_c X_c} \\
\xi_c'/\xi_c &= -\left[\frac{1 - n^2/\alpha^2 r_c^2}{r_c X_c} + \frac{T_c'}{T_c}\right]
\end{aligned} \tag{5.24}$$

The coefficient b_1 is not determined by this method. In going from $Rl\,(r - r_c) > 0$ to $Rl\,(r - r_c) < 0$ the proper path lies below the point $r = r_c$ for $w_c' > 0$ and hence, for proper analytical continuation, the term $\ln\chi\ (\chi > 0)$ must be modified to $(\ln|\chi| - i\pi)\ (\chi < 0)$ in Eq. (5.23) [Appendix G]. It will be shown in Section V.4 that k, which is the gradient of a density-vorticity product at $r = r_c$, must vanish if a solution of Eq. (4.1) is to exist for neutral disturbances. The behavior of the disturbance amplitudes in the neighborhood of the critical point is sketched on the next page in Sketch 5.5. For $k = 0$, ψ and all its derivatives are continuous. For $k \neq 0$, ψ is continuous,

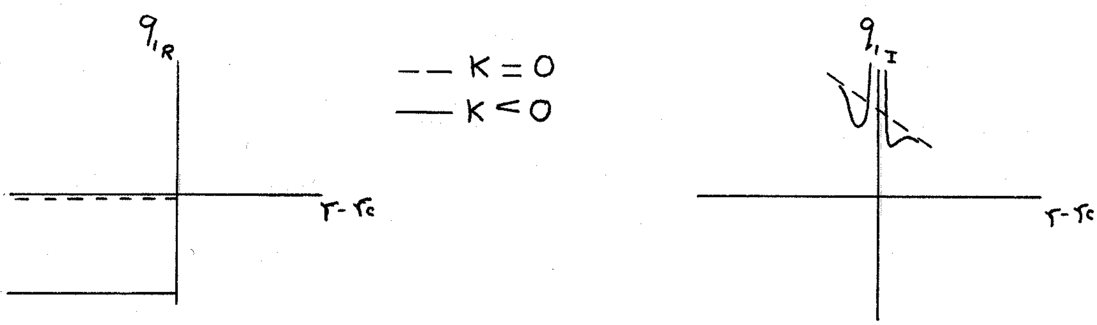


Sketch 5. 5

ψ_R has a continuous derivative but a discontinuous curvature at the critical point, while ψ_I has a discontinuity in slope. This result is analogous to that for the two-dimensional case [Reshotko[59], page 42]. *

$\underline{q_1}$

$$q_1 = \frac{i\alpha}{r_c m_c}\left[K\left\{\begin{matrix}\ln \chi \\ \ln \chi - i\pi\end{matrix}\right\} + \cdots\right] \quad \begin{matrix}\chi > 0 \\ \chi < 0\end{matrix} \tag{5.25}$$



Sketch 5. 6

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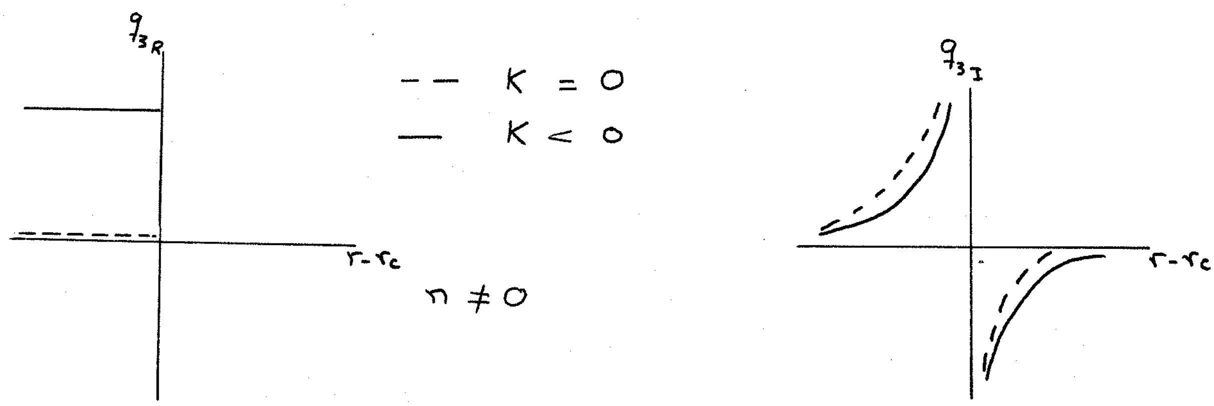
* This notation is slightly different than in Reference 59. ψ_R (Gold) $\longrightarrow$ Φ_I (Reshotko), ψ_I (Gold) $\longrightarrow$ Φ_R (Reshotko).

This case is identically the same as the two dimensional case (where q_1 is similar to f) in which there are no discontinuities in q_1 for $k = 0$, but for $k \neq 0$ there is a jump discontinuity in q_R and a logarithmic discontinuity in q_I.

$\underline{q_3}$

$$q_3 = -\frac{in}{m_c r_c^2}\left[\frac{1}{\chi} + K\begin{Bmatrix}\ln \chi \\ \ln|\chi| - i\pi\end{Bmatrix} + \cdots\right] \begin{matrix}\chi > 0 \\ \chi < 0\end{matrix} \tag{5.26}$$

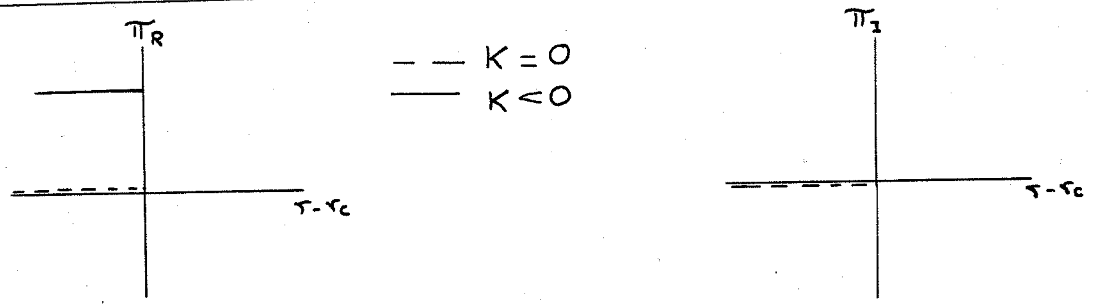


Sketch 5.7

For $n = 0$, $q_3 \equiv 0$. When $n \neq 0$ and $k = 0$ there is a hyperbolic-type discontinuity in q_{3_I}. The real part of q_3 has a jump discontinuity, $n \neq 0$ and $k \neq 0$. There is somewhat analogous to the behavior of θ [Sketch 5.9]

$\underline{\pi}$

$$\frac{\pi}{\gamma M^2} = i W_c' \xi_c \left[1 - \frac{K\alpha^2 X_c}{3}\chi^3\begin{Bmatrix}\ln \chi \\ \ln|\chi| - i\pi\end{Bmatrix} + \cdot\right] \begin{matrix}\chi > 0 \\ \chi < 0\end{matrix} \tag{5.27}$$

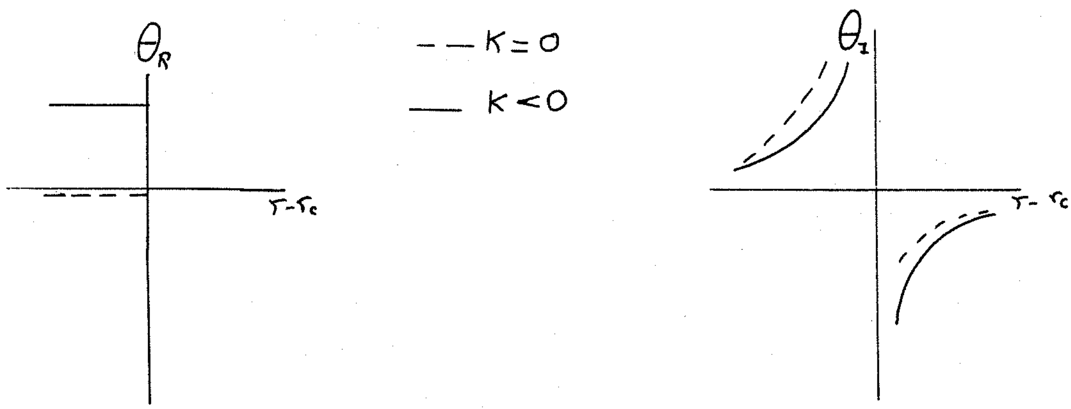


Sketch 5. 8

π_I is continuous for all k for π_R has a jump discontinuity for k $\neq$ 0.

$\underline{\theta}$

$$\theta \doteq -\frac{i\,T_c'}{r_c W_c'}\left[\frac{1}{\chi} + k\begin{Bmatrix}\ln\chi \\ \ln|\chi| - i\pi\end{Bmatrix} + \cdots\right]_{\chi < 0}^{\chi > 0} \tag{5. 28}$$



Sketch 5. 9

The temperature perturbation amplitude behaves like the q_x velocity perturbation near the critical point for $n \neq 0$. This discontinuity must be modified by the introduction of conductivity in the neighborhood of the critical point. [Reshotko[59]]

The density fluctuation behaves exactly like the temperature fluctuation in this region.

Summarizing, even when $k = 0$* the fluid viscosity, however small, must be taken into account when $n \neq 0$ to smooth out the discontinuity in q_3. The effects of viscosity are limited to a thin annulus of the order of $(\alpha R)^{-1/3}$ in thickness [Appendix G] . The disturbance is inviscid in the sense that the gross features of the disturbance amplitudes can be found outside of the viscous layer without considering the effects of viscosity. The temperature and density fluctuations in the neighborhood of the critical point can be smoothed out by introducing conductivity, analogous to the two-dimensional case [Reshotko[59]] . When $k = 0$, viscosity smooths out the disturbance amplitudes between two equal "inviscid" values for $r - r_c \lessgtr 0$ outside the viscous layer. However, when $k \neq 0$, the "jump" in the "inviscid" value of q_{1_R} persists; viscosity merely insures that q_{1_R} changes continuously over a small but finite layer. Thus one suspects that the vanishing of k at some point in the shear flow is a necessary and sufficient condition for the existence of a neutral, inviscid subsonic disturbance with $c = c_s \neq 0$, $\alpha = \alpha_s \neq 0$ in the limit $\alpha R \longrightarrow \infty$ [Section V. 4] .

---

* The quantity k is analogous to the gradient of the density vorticity product in two-dimensional flow [Section V. 5] .

There is another singularity at the origin, $r = 0$, when $c = w(0)$. In the neighborhood of this point

$$\psi \sim 1 \quad , \quad r^2 \qquad\qquad n = 0$$
$$\psi \sim r^{-\sqrt{n^2+4}} \quad , \quad r^{\sqrt{n^2+4}} \qquad\qquad n \neq 0 \quad . \tag{5.29}$$

Integrate Eq. (5. 9) to obtain

$$\xi \psi' = -\int_r^\infty \left(\rho + \frac{\alpha^2}{Tr}\right) \psi \, dr \quad . \tag{5.30}$$

The only solutions that satisfy Eq. (5.30) are

$$\psi \sim 1 \qquad\qquad n = 0$$
$$\psi \sim r^{-\sqrt{n^2+4}} \qquad\qquad n \neq 0 \tag{5.31}$$

and the boundary condition Eq. (5. 13) can never be satisfied.

As in the two dimensional case the only possible non-trivial solution is given by $\alpha = 0$, $n = 0$, $c = w(0)$ and $\psi = w(r) - w(0)$.

The singularity at the point $T\left(1 + \frac{n^2}{\alpha^2 r^2}\right) = M^2 (w-c)^2$ is an apparent singularity of the differential equation [Eq. (5. 9)], since a self-adjoint equation can be written for the pressure disturbance [Eq. (5. 12)] in which none of the coefficients of the equation become infinite at this point. At this point the local Mach number of the disturbance is equal to unity [Section V. 2] .

V. 4. Necessary and Sufficient Conditions for the Existence of Inviscid Subsonic Disturbances

The self-adjoint equation, Eq. (5. 9), and boundary conditions, Eqs. (5. 13) and (5. 17), constitute a Sturm-Liouville system which is analogous to that treated by Lees and Lin[32] for two-dimensional flows. This fact suggests that results similar to those for the two-dimensional case can be obtained for the axi-symmetric flow problem. Some general conditions for instability will now be derived using the properties of this system.

Multiply the self-adjoint equation, Eq. (5. 9), by the complex conjugate of Ψ (denoted by the superscript $\hat{}$) and subtract the complex conjugate of Eq. (5. 9) multiplied by Ψ from it to obtain

$$\left[\xi \hat{\Psi} \Psi' - \hat{\xi} \Psi \hat{\Psi}'\right] = \left[\xi - \hat{\xi}\right] |\Psi'|^2 - \left[P - \hat{P}\right] |\Psi|^2 \tag{5.32}$$

Let

$$\underline{W} = -\frac{i}{2}\left[\xi \hat{\Psi} \Psi' - \xi \Psi \hat{\Psi}\right] = Imag\left[\xi \hat{\Psi} \Psi'\right] \tag{5.33}$$

so that Eq. (5. 32)

$$\underline{W}' = \xi_I \left[\Psi'\right]^2 + P_I |\Psi|^2 \tag{5.34}$$

where

$$\begin{aligned}
\xi_R &= T\left[1 + \frac{n^2}{\alpha^2 r^2}\right] - M^2\left[(W - C_R)^2 - C_I^2\right] r |\xi|^2 \\
\xi_I &= -\left[2M^2 (W - C_R)\, C_I\right] r |\xi|^2 \\
P_R &= \frac{1}{|W - C|^2}\left[(W - C_R)(\xi_R W')' - C_I(\xi_I W')'\right] \\
P_I &= \frac{1}{|W - C|^2}\left[(W - C_R)(\xi_I W')' + C_I(\xi_R W')'\right]
\end{aligned} \tag{5.35}$$

Since $\underline{W}$ vanishes at $r = 0$ and $r = \infty \left[\Psi = 0 \text{ at these end points}\right]$, $(d\underline{W}/dr)$ must change its sign in the interval $(0 < r < \infty)$. Therefore, a necessary condition for the existence of amplified subsonic disturbances $(c_I > 0)$ is that ξ_I and/or ρ_I must change their sign in the interval $0 < r < \infty$. For incompressible flow,

$$\xi_I = 0 \, , \; T = 1 \, , \; \xi_R = 1/\left(1 + \frac{n^2}{\alpha^2 r^2}\right) r \qquad \text{and} \qquad (\xi_R w')'$$

must vanish at some interior point of the interval if amplified disturbances are to exist.[36] There is no singularity along the real axis since the critical point lies above it when $c_I > 0$.

For a neutral inviscid disturbance, $c_I = 0$, $(d\underline{W}/dr) = 0$, and $\underline{W}$ = constant except possibly at the critical point $r = r_c \neq 0$, or in other words the Wronskien of the solutions, $\underline{W}/\xi_R$, is constant outside the critical layer $r = r_c$. Consider the jump in $\underline{W}$ across the critical layer, $w = c_R$:

$$[\underline{W}] = [\underline{W}(r_c + 0) - \underline{W}(r_c - 0)] = \underset{c_I \to 0}{\mathrm{Limit}} \left[\int_{r_c - 0}^{r_c + 0} \xi_I |\Psi'|^2 dr + \int_{r_c - 0}^{r_c + 0} \rho_I |\Psi|^2 dr \right] \tag{5.36}$$

The only contribution comes from the second integral

$$[\underline{W}] = \underset{c_I \to 0}{\mathrm{Limit}} \int_{r_c - 0}^{r_c + 0} \frac{c_I |\Psi|^2}{[(w - c_R)^2 + c_I^2]} (\xi_R w')' \, dr \tag{5.37}$$

The integrand is essentially a delta function and in the limit $c_R \longrightarrow 0$

$$[\underline{W}] = \frac{\pi}{w_c'} |\Psi_c|^2 (\xi_R w')_c' = \pi \, \xi_c K |\Psi_c|^2 \tag{5.38}$$

for $w_c' > 0$. This same answer could also be obtained by using the expansions around the singular point in the expression for $\underline{W}$. [Eq. (5.23), Section V.3]. It is just the discontinuity in ψ_I' in passing from $r - r_c > 0$ to $r - r_c < 0$ that leads to the discontinuity in $\underline{W}$.

Now $\underline{W}$ must vanish at the end points. Therefore a necessary condition for the existence of inviscid, neutral, subsonic disturbances is that $(\xi_R W')'$ must vanish if $\psi_c \neq 0$. The quantity $(\xi_R W')$ is the density-vorticity product in the q_3 direction [Section V.5]. It must have a true extremum at $r = r_c$ and not a point of inflection [Appendix F]. Denote the value of $c = w$ at this point by c_s and let the corresponding value of α be α_s.

To make the previous statement complete it is necessary to prove that $\psi_c \neq 0$. Eq. (5.9) can be written in the following form

$$\left[\xi (w-c)^2 \left(\frac{\psi}{w-c}\right)'\right]' - \frac{\alpha^2}{Tr} (w-c)^2 \left(\frac{\psi}{w-c}\right) = 0 \qquad (5.39)$$

Integrate Eq. (5.39) to obtain

$$\xi (w-c)^2 \left(\frac{\psi}{w-c}\right)' = -\int_r^\infty \frac{\alpha^2 (w-c)^2}{Tr} \frac{\psi}{w-c} \, dr \qquad (5.40)$$

Now if $\xi \geqslant 0$, $\psi/(w-c)$ will increase or decrease monotonically as r decreases according to whether its sign is positive or negative, respectively, for large values of r. Then, $\psi = C_1 \psi_1 + C_2 \psi_2$ where ψ_1 and ψ_2 are given by Eqs. (5.22) and (5.23) in the neighborhood of $r = r_c$. If $\psi_c = 0$, then $C_2 = 0$ [since $\psi_{1_c} = 0$ and $\psi_{2_c} = 1$], and the left-hand side of Eq. (5.40) behaves like $(r - r_c)^2$ which is inconsistent with the right-hand side. If $C_2 \neq 0$ then the left-hand side approaches a constant. Therefore, if Ψ is a solution of the disturbance equation which satisfies the boundary conditions, then Ψ_c cannot vanish. This statement can also be proved if ξ changes sign in the interval (existence of a sonic point in the flow field).

Lees and Lin[32] have shown that this condition is not only necessary but also sufficient for the existence of inviscid, neutral, two-dimensional subsonic disturbances. The proof of inviscid disturbances of helical form is similar and will not be reproduced here.

In the region adjacent to the neutral disturbance, $\alpha = \alpha_s$, $c = c_s$, the condition $(\xi_R w')'_c = 0$ is also sufficient for the existence of damped or amplified subsonic disturbances. Since c is an analytic function of α^2 except in the neighborhood of $c = -1$, $\alpha = 0$, it can be expanded in a Taylor series expansion about the point $\alpha^2 = \alpha_s^2$, $c = c_s$

$$C = C_s + (\alpha^2 - \alpha_s^2)\left(\frac{dc}{d\alpha^2}\right)_c + \cdots \tag{5.41}$$

If $\Psi(r; c, \alpha^2)$ is a characteristic function, and c and α^2 are characteristic values of Eq. (5.9), then

$$\frac{d\Psi}{d\alpha^2} = \Psi_{\alpha^2} = \frac{\partial\Psi}{\partial\alpha^2} + \frac{\partial\Psi}{\partial c}\frac{dc}{d\alpha^2} \tag{5.42}$$

Differentiate Eq. (5.9) with respect to α^2 and multiply it by Ψ; multiply Eq. (5.9) by Ψ_{α^2} and subtract to obtain

$$\begin{aligned}\left[\xi\left(\Psi\Psi'_{\alpha^2} - \Psi'\Psi_{\alpha^2}\right)\right]' &= \frac{\Psi^2}{Tr} - \Psi\left[\frac{\partial\xi}{\partial\alpha^2}\Psi'\right]' \\ &+ \frac{dc}{d\alpha^2}\left[\frac{\partial\rho}{\partial c}\Psi^2 - \Psi\left(\frac{\partial\xi}{\partial c}\Psi'\right)'\right]\end{aligned} \tag{5.43}$$

Integrate Eq. (5.43) along any path in the complex r- plane between the end points r = 0 and r = ∞ and consider the neutral disturbance,

$\Psi = \Psi_s$, $c = c_s$, and $\alpha^2 = \alpha_s^2$,

$$\left(\frac{dc}{d\alpha^2}\right)_s = \frac{-\int_0^\infty \left[\frac{\Psi_s^2}{Tr} - \left(\frac{\partial \xi}{\partial \alpha^2}\right)_s |\Psi_s'|^2\right] dr}{\int_0^\infty \left[\left(\frac{\partial P}{\partial c}\right)_s \Psi_s^2 + \left(\frac{\partial \xi}{\partial c}\right)_s |\Psi_s'|^2\right] dr} \tag{5.44}$$

All of the integration can be carried out along the real axis, except for the term containing $(\partial P/\partial c)_s$, which has a pole at $r = r_c$ and must be evaluated by integrating along a path that passes below this point. Therefore,

$$\mathrm{Imag}\left[\int_0^\infty \left(\frac{\partial P}{\partial c}\right)_s |\Psi_s|^2 dr\right] = \frac{\pi |\Psi_s|_c^2}{W_c'^2} \left(\xi_s W'\right)_c'' \tag{5.45}$$

The numerator is always negative so that the imaginary part of takes the sign of $(\xi_s w')''$. Since $\Psi_{s_c} \neq 0$, Imag $(dc/d\alpha^2) \neq 0$, provided $(\xi_s w')'' \neq 0$; and c_I must be positive for some value of α slightly greater or less than α_s. This result proves the sufficiency condition for the existence of damped or amplified disturbances adjacent to the neutral disturbance.

The results derived so far correspond exactly to the results obtained for two-dimensional disturbances and are summarized below. The vanishing of the gradient of the density-vorticity product in the q_3 direction is a

(1) necessary and sufficient condition for the existence of neutral, inviscid, subsonic disturbances,

(2) sufficient condition for the existence of adjacent amplified or damped subsonic disturbances,

(3) necessary and sufficient condition for the existence of neutral and amplified disturbances in incompressible flow.

A necessary condition for the existence of amplified subsonic disturbances is that p_I and/or q_I change sign in the interval under consideration; a specific statement regarding the gradient of the density-vorticity product cannot be made for this case.

For most problems of interest, $\xi \geq 0$. By the oscillation theorem of Sturm (Ince[61], ¶ 10.6), if $\rho + (\alpha^2/Tr)$ is positive then Ψ is monotonic; if it is negative then Ψ is oscillatory. Since Ψ must vanish at the end points, a necessary condition for the existence of neutral inviscid subsonic disturbances is that

$$\mathrm{Max}\left[p + \frac{\alpha^2}{Tr}\right] < 0 \tag{5.46}$$

This condition restricts the value of (n/α) to a finite integer, which depends upon the mean flow profiles [Section V. 6]. If ξ changes sign in the interval, these results do not apply. This case has not been investigated.

V. 5. Energy and Vorticity Relations

For a physical understanding of the stability phenomenon, it is important to investigate the transfer of kinetic energy between the mean flow and the perturbed flow. Qualitatively the energy transfer mechanism for axisymmetric flow subsonic disturbances can be described by the following relation (in dimensionless form)

$$\frac{d}{dt}\int_0^\infty \frac{\rho}{2}\left[\overline{q_r'^2 + q_\phi'^2 + q_x'^2}\right] r\,dr \tag{5.47*}$$

$$= \int_0^\infty \tau \frac{dw}{dr} r\,dr \quad - \quad \left[\text{viscous dissipation}\right]$$

where

$$\tau = \text{Reynolds shear stress} = -\rho\,\overline{q_x' q_r'}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x} \quad ,$$

and the bar indicates an average over one wave length in the x and one period in ϕ. The term on the left side of Eq. (5.47) represents the rate of increase of the kinetic energy of the disturbance, while the first term on the right represents the conversion of energy from the basic flow to the disturbance by the action of the Reynolds shear stress.

For a neutral disturbance the time rate of change of the kinetic energy over one cycle must vanish and the viscous dissipation term must exactly balance the energy conversion term associated with the Reynolds stress. In order for a disturbance to be unstable the mean flow must feed energy into the disturbance. Clearly, if there is to be any instability, the Reynolds shear stress must have the same sign as the velocity gradient of the mean flow.

In the limit of zero viscosity, the dissipation terms vanish and the rate of change of the kinetic energy must exactly balance the energy conversion term. If this Reynolds stress term is positive, energy will

---

* The primes (') under the bars indicate fluctuation quantities.

be transferred from the mean flow to the disturbance and the flow will be unstable; if it is negative the mean flow will absorb energy from the disturbance and the flow will be stable; if it is zero there is no exchange of energy between the mean flow and the disturbance and the flow will be neutrally stable.

It will be shown that the Reynolds shear stress is composed of two components; one perpendicular to a helix of constant phase and the other tangent to this helix. A necessary and sufficient condition for the existence of a neutral subsonic disturbance is that the perpendicular component of the shear stress be zero everywhere in the flow field. However, the other component behaves like a delta function in the neighborhood of the critical point for neutral disturbances, and the contribution to the Reynolds stress term in Eq. (5.47) is finite. Therefore, viscous dissipation in a narrow layer [Appendix G] must balance this excess production of disturbance energy however small the viscosity may be. In the two-dimensional case there is only one component of the Reynolds stress, when this stress vanishes a neutral subsonic disturbance can exist, and viscous dissipation is not required. *

In dimensionless form, the Reynolds shear stress, τ , for quasi-parallel, axi-symmetric flow is

$$\tau = - \rho \overline{q_x' q_r'} \tag{5.48}$$

The longitudinal and radial velocity fluctuations are

---

* Thermal conduction, however, is required (Reference 59, page 44).

$$q_x' = Rl(q_x E) = \frac{1}{2}\left[q_x E + \hat{q}_x \hat{E}\right]$$

$$q_r' = Rl(\alpha q_r E) = \frac{\alpha}{2}\left[q_r E + \hat{q}_r \hat{E}\right] \qquad (5.49)*$$

where

$$E = \exp\left[i\,\alpha\,(x - ct) + i\,n\,\phi\right] \quad .$$

Therefore the shear stress is given by

$$\tau = -\frac{\alpha \rho}{4}\left[q_r q_x E^2 + \hat{q}_r \hat{q}_x \hat{E}^2 + \left(q_r \hat{q}_x + \hat{q}_r q_x\right)|E|^2\right]$$

But the average over one wavelength in x, and one cycle in ϕ of E^2 and $\hat{E}^2$ is zero so that

$$\tau = -\frac{\alpha \rho}{4}\left(q_r \hat{q}_x + \hat{q}_r q_x\right) e^{2\alpha c_I t} = -\frac{\alpha \rho e^{2\alpha c_I t}}{2r} Rl\left(\hat{\psi} q_x\right) \qquad (5.50)$$

Substituting Eq. (5.11) into Eq. (5.50), the shear stress is given in terms of ψ

$$\tau = \frac{\alpha e^{2\alpha c_I t}}{2r}\left[\underline{W} + \underline{G}\right] \qquad (5.51)$$

where

$$\underline{W} = \mathrm{Imag}\, \xi \hat{\psi} \psi'$$

$$\underline{G} = \frac{|\psi|^2 w'}{rT}\, \mathrm{Imag}\left[\frac{1}{w-c} - \frac{\xi r T}{w-c}\right] \qquad (5.52)$$

---

* The primed quantities are fluctuation amplitudes.

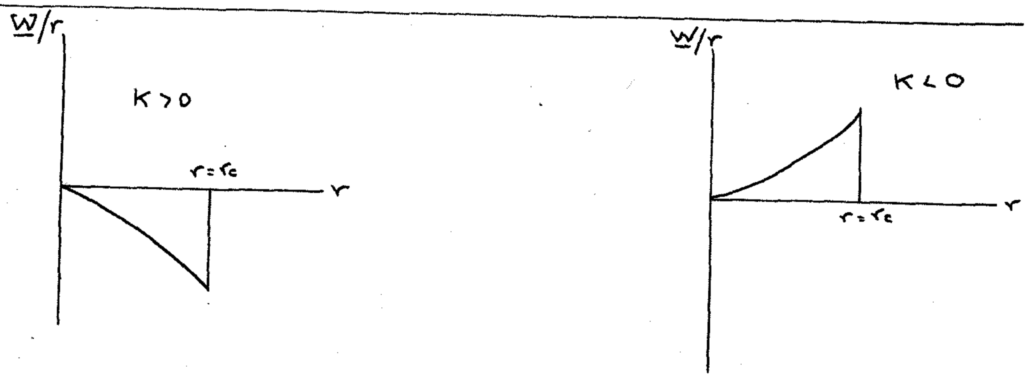
As $c_I \longrightarrow 0$, $\underline{W}$ has a jump of magnitude [Eq. (5.38)]

$$\begin{aligned} [\underline{W}] &= [\underline{W}(r_c + 0) - \underline{W}(r_c - 0)] \\ &= \pi \xi_c K |\psi_c|^2 \end{aligned} \tag{5.53}$$

across the critical layer, where

$$K = \frac{1}{W_c' \xi_c} (\xi W')_c' \qquad , \ \psi_c \neq 0$$

This result is analogous to the two-dimensional case[32]. However, τ varies as $(1/r)$ so that the $\underline{W}$ contribution to the shear stress will have the form



Sketch 5.10

G is singular at $r = r_c$ and behaves like a delta function, i.e., (Sketch 5.11)

$$\underset{\substack{C_I \to 0 \\ r \to r_c}}{\text{Limit}} \left[\frac{1}{W-C} - \frac{\xi r T}{W-C}\right] = \frac{n^2}{\alpha^2 r_c^2} \frac{1}{\left(1 + \frac{n^2}{\alpha^2 r_c^2}\right)} \underset{\substack{C_I \to 0 \\ r \to r_c}}{\text{Limit}} \frac{C_I}{(W-C_R)^2 + C_I^2} \tag{5.54}$$

The rate at which energy is transferred from the mean flow to the disturbance flow is given by

$$-\int_0^{\Lambda}\int_0^{2\pi}\int_0^{\infty} \rho q_r' q_x' \frac{dw}{dr}\, r\, dr\, d\phi\, dx = \int_0^{\infty} \tau \frac{dw}{dr}\, r\, dr$$

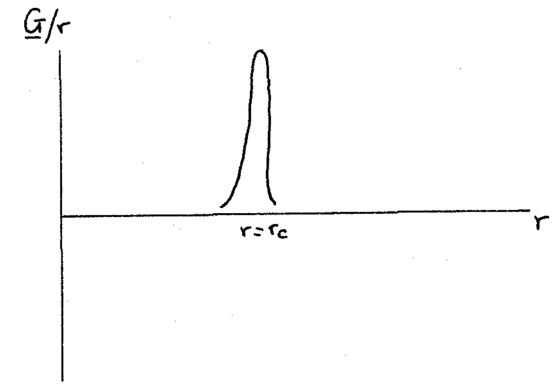
$$= \frac{\alpha e^{2\alpha c_I t}}{2} \int_0^{\infty} (\underline{W} + \underline{G}) \frac{dw}{dr}\, dr \tag{5.55}$$

$$= \frac{\alpha}{2} |\Psi_c|^2 \pi \left[-\xi_c (1 + C_R) K + \frac{n^2}{\alpha^2 r_c^2} \frac{1}{\left(1 + \frac{n^2}{\alpha^2 r_c^2}\right)} \frac{w_c'}{r_c T_c} \right]_{C_I \to 0}$$

where

$w_c' > 0 \qquad w(0) = -1$

$w(\infty) = 0 \qquad \underline{W}(r_c + 0) = 0$



Sketch 5.11

A necessary and sufficient condition for the existence of a neutral subsonic disturbance is that k = 0 (Eq. (5.53)) so that the first term in Eq. (5.55) vanishes but the second does not. Therefore, even for "inviscid" neutral disturbances, viscous dissipation must balance the excess production of disturbance energy however small the viscosity may be. The contribution from the $\underline{G}$ term is positive for $w_c' > 0$, $n \neq 0$ and is always a destabilizing influence. For rotationally symmetric disturbances, $n = 0$, the contribution to the $\underline{G}$ term is zero and the results are analogous to those for the two-dimensional case.

Batchelor and Gill[36] show that the Reynolds shear stress $\tau = -\rho\overline{q_r' q_x'}$ is composed of two components; one perpendicular to a helix of constant phase and the other parallel to the tangent to the helix (on r = constant) [Section V.1]. Using Eqs. (5.1) and (5.48), these stress terms are

$$
\begin{aligned}
-\rho\overline{q_1' q_r'} &= -\frac{\rho\alpha e^{2\alpha c_I t}}{2mr} Rl\left[\hat{\psi}\left[\frac{n}{r} q_\phi + \alpha q_x\right]\right] \\
&= \frac{\alpha e^{2\alpha c_I t}}{2r}\left[\frac{m}{\alpha}\underline{W} - \frac{\alpha}{m}|\psi|^2 w' \frac{M^2}{T} \mathrm{Imag}\, \xi(w-c)\right]
\end{aligned}
\tag{5.56}
$$

$$
\begin{aligned}
-\rho\overline{q_r' q_3'} &= -\frac{\rho\alpha e^{2\alpha c_I t}}{2mr} Rl\left[\hat{\psi}\left[\alpha q_\phi - \frac{n}{r} q_x\right]\right] \\
&= -\frac{\alpha e^{2\alpha c_I t}}{2r} \frac{n}{mr} \frac{|\psi|^2 w'}{rT} \mathrm{Imag} \frac{1}{w-c}
\end{aligned}
\tag{5.57}
$$

respectively. Therefore,

$$
\begin{aligned}
\tau = -\rho\overline{q_r' q_x'} &= \frac{\alpha}{m}\left[-\rho\overline{q_1' q_r'}\right] - \frac{n}{mr}\left[-\rho\overline{q_3' q_r'}\right] \\
&= \frac{\alpha e^{2\alpha c_I t}}{2r}\left[\underline{W} + \underline{G}\right]
\end{aligned}
$$

as before [Eq. (5.51)].

A necessary and sufficient condition for the existence of a neutral, subsonic disturbance is that $\underline{W} \equiv 0$ and, therefore, $-\rho\overline{q_1' q_r'} \equiv 0$. As $c_I \longrightarrow 0$, the $-\rho\overline{q_r' q_3'}$ shear stress component has the behavior of a delta function [Eq. (5.54)] near $r = r_c$, and gives a finite contribution to the Reynolds stress term in Eq. (5.47).

The quantity k and the singularity associated with the q_3 velocity component at the critical point can also be interpreted in terms of the transport of the mean and disturbance vorticity across the plane $w = c$.

The disturbance vorticity components in the r, ϕ and x directions are

$$
\begin{aligned}
\Gamma_r &= -\frac{i}{r}\left[\alpha r\, q_\phi - n\, q_x\right] \\
\Gamma_\phi &= i\alpha^2 q_r - q_x' \\
\Gamma_x &= \frac{1}{r}\left[(r\, q_\phi)' - i n \alpha\, q_r\right]
\end{aligned}
\tag{5.58}
$$

and the mean vorticity in the ϕ direction is

$$
\overline{\Gamma}_\phi = -W' \tag{5.59}
$$

in the transformed velocity space [Eq. (5.1)], Eqs. (5.58) and (5.59) become

$$
\begin{aligned}
\Gamma_1 &= \frac{n}{mr}\Gamma_\phi + \frac{\alpha}{m}\Gamma_x = \frac{1}{mr}\left[m r q_3\right]' \\
\Gamma_2 &= \Gamma_r = -i m\, q_3 \\
\Gamma_3 &= \frac{\alpha}{m}\Gamma_\phi - \frac{n}{mr}\Gamma_x = i m q_r - \frac{1}{m}\left[m q_1\right]' - \frac{2n}{mr^2}\, q_\phi
\end{aligned}
\tag{5.60}
$$

and

$$\bar{\Gamma}_1 = \frac{\alpha}{m}\bar{\Gamma}_\phi$$
$$\bar{\Gamma}_3 = -\frac{n}{mr}\bar{\Gamma}_\phi \tag{5.61}$$

The radial disturbance vorticity is related directly to the q_3 velocity component.

The vorticity equation in the r direction is

$$(w-c)\Gamma_r = -m\bar{\Gamma}_3 q_r \tag{5.62}$$

and in the 3 direction is

$$\alpha m r q_r\left[\frac{\rho\bar{\Gamma}_\phi}{m^2 r}\right] = i\rho(w-c)\left[\frac{1}{m}(m q_1)' + \frac{\rho'}{\rho}q_1 + \frac{2n^2}{m^2r^3}q_1 - im\alpha q_r\right]$$
$$- \frac{\rho\bar{\Gamma}_\phi}{m}\left[\frac{\alpha}{r}(r q_r)' + imq_1\right] \tag{5.63}$$

Eq. (5.62) indicates to an observer riding with the disturbance that the convection of the disturbance radial vorticity component (or tangential component of disturbance momentum) by the mean flow balances the convection of the mean vorticity (or momentum) by the disturbance radial velocity except at the point w = c. At this point, except for the case n = 0 , the disturbance radial vorticity is singular. The fluid viscosity, however small, must be taken into account to smooth out this discontinuity.

At w = c, the right-hand side of Eq. (5.63) vanishes; the transport of the quantity ($\rho\bar{\Gamma}_\phi/m^2 r$) (density of angular momentum of the mean flow or density-vorticity product) must also vanish since $\psi = rq_r$ cannot be equal to zero at this point [Section V. 4] . If the gradient of the density-vorticity product does not vanish at this point, then the transport of $(\rho \bar{\Gamma}_\phi / m^2 r)$ can only be balanced by the diffusion of $(\rho \bar{\bar{\Gamma}}_\phi / m^2 r)$ through viscosity. If a neutral disturbance exists, then $(\rho \bar{\bar{\Gamma}}_\phi / m^2 r)'$ must vanish and viscosity is not needed to smooth out the discontinuity in the density-vorticity product (in the 3 direction). This is analogous to the two-dimensional case where the direction perpendicular to the plane x-y is associated with the 3 direction.

These energy and vorticity considerations only emphasize the fact that even though the vanishing of the gradient of the density-vorticity product insures the existence of a solution to the neutral inviscid equations, the effect of viscosity, however small it might be, must still be taken into account to smooth out the discontinuities in both the velocity components and the Reynolds shear stress. The effect of viscosity is limited to a thin annulus of the order of $([\alpha R]^{-1/3})$ [Appendix G] ; outside this annulus the disturbance quantities can be described by the inviscid equations. This situation is analogous to Prandtl's treatment of the boundary layer, in which the external flow is calculated by first neglecting viscosity and conductivity, and then viscosity and thermal conductivity are taken into account in determining the structure of the boundary layer.

V. 6. Inviscid Stability of Axi-Symmetric Wake Flows

The results of Section IV. 13 will be used to discuss the stability of incompressible and compressible wake-type flows in the limit $\alpha R \longrightarrow \infty$. In order to fix the ideas of this section, it will be assumed that the mean flow velocity profile is Gaussian

$$w = -e^{-r^2} \tag{5.64}$$

[Section II. 4]. The effects of temperature upon the stability criteria will be deduced using "reasonable" profiles.

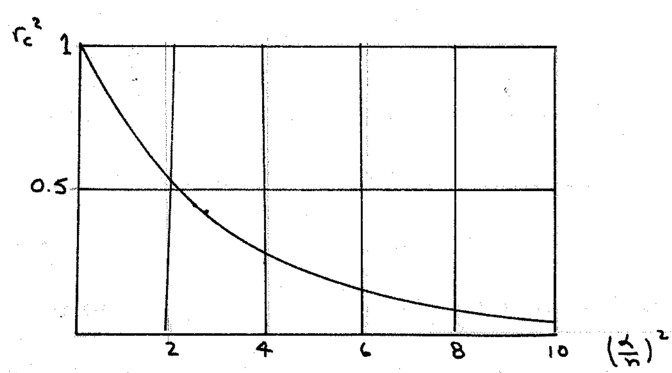
A necessary and sufficient condition for the existence of incompressible neutral and amplified disturbances (using Eq. (5.64)) is that

$$(\xi_R w')' = -\frac{4e^{-r^2}}{\left(1+\frac{n^2}{\alpha^2 r^2}\right)^2} \frac{n^4}{\alpha^4 r^3}\left[\left(\frac{\alpha r}{n}\right)^4 + \left(\frac{\alpha r}{n}\right)^2 - \left(\frac{\alpha}{n}\right)^2\right] \tag{5.65}$$

must vanish within the flow field, i.e.,

$$\left(\frac{\alpha r_c}{n}\right)^4 + \left(\frac{\alpha r_c}{n}\right)^2 - \left(\frac{\alpha}{n}\right)^2 = 0 \qquad n \neq 0 \tag{5.66}$$

If n = 0 then the quantity (5.65) can never vanish within the flow field and the only neutral inviscid disturbance is $\alpha = 0,\ C = \omega(0),\ \phi = \omega - C$. The locus of solutions of Eq. (5.66) is sketched below



Sketch 5.12

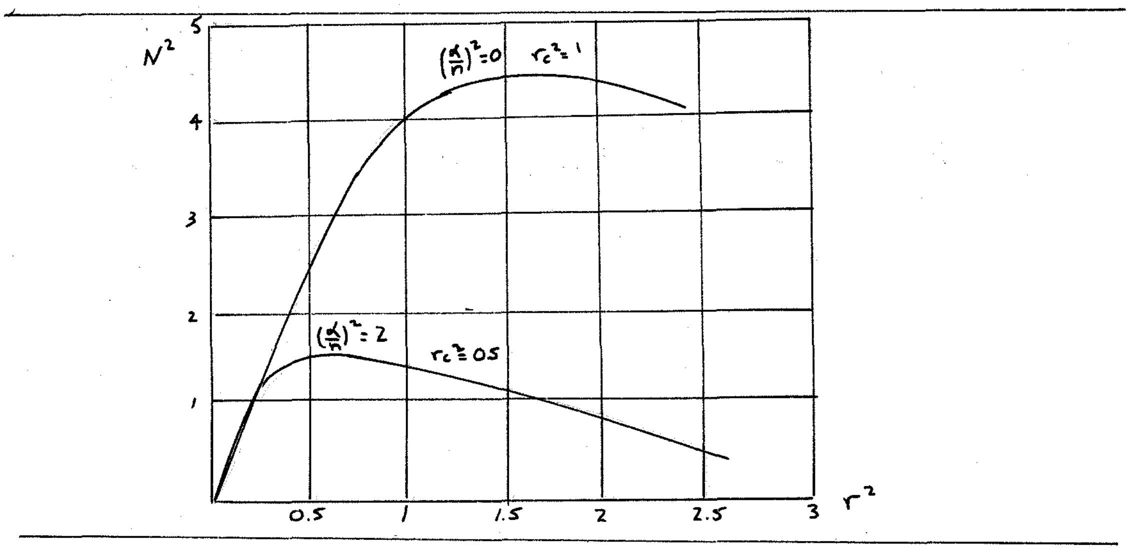
A necessary condition for the existence of a neutral disturbance is [Section V. 4].

$$Max\left(p + \frac{\alpha^2}{Tr}\right) < 0 \tag{5.67}$$

and for incompressible flow, using the Gaussian profile

$$Max \frac{4r^2}{(e^{r^2 r_c^2}-1)} \frac{\left(\frac{\alpha^2}{n^2}r^4 + r^2 - 1\right)}{\left(\frac{\alpha^2 r^2}{n^2}+1\right)^2} = N^2 > n^2 \tag{5.68}$$

where r_c^2 is determined from Eq. (5.66) as a function of $(\alpha/n)^2$. Eq. (5.68) was solved graphically as indicated in the following sketch.

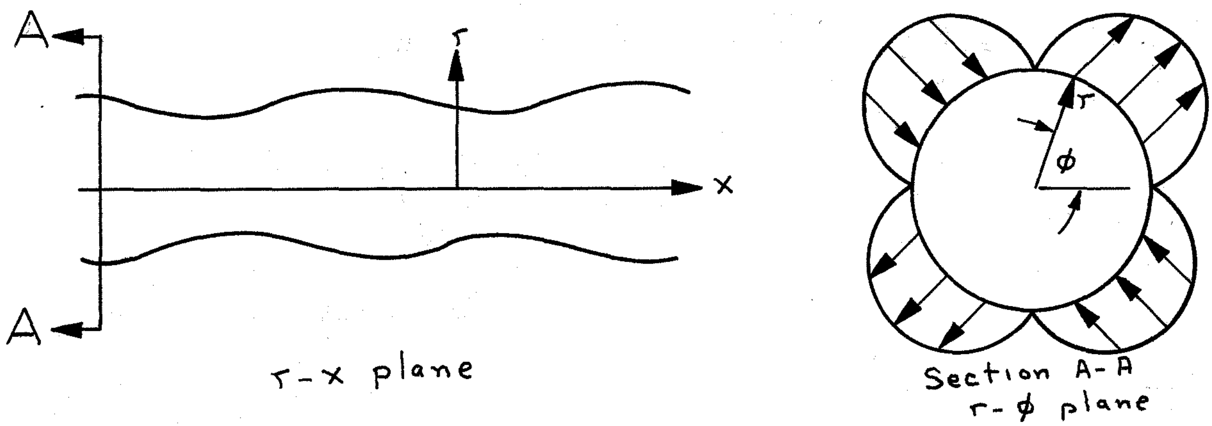


Sketch 5.13

The maximum value of n^2 occurs for $(\alpha^2/n^2) \longrightarrow 0$ and is approximately 4.7 at $r^2 = 1.82$. Therefore, for $n > 2$ the wake flow is always stable and the only modes of instability are those for $n = 2$.

The $n = 1$ mode represents the sinuous type instability, i.e., the nodal points for the radial disturbance are spaced 180° apart. The disturbance radial velocity component is not zero on the axis. This type of instability is shown in Figure 1 and is similar to the anti-symmetric oscillations in two dimensional flow [Section II. 2, Sketch 2. 3].

The n = 2 mode represents a varicose type instability, but with four nodal points spaced 90^{o} apart. In this case the radial velocity is zero on the centerline



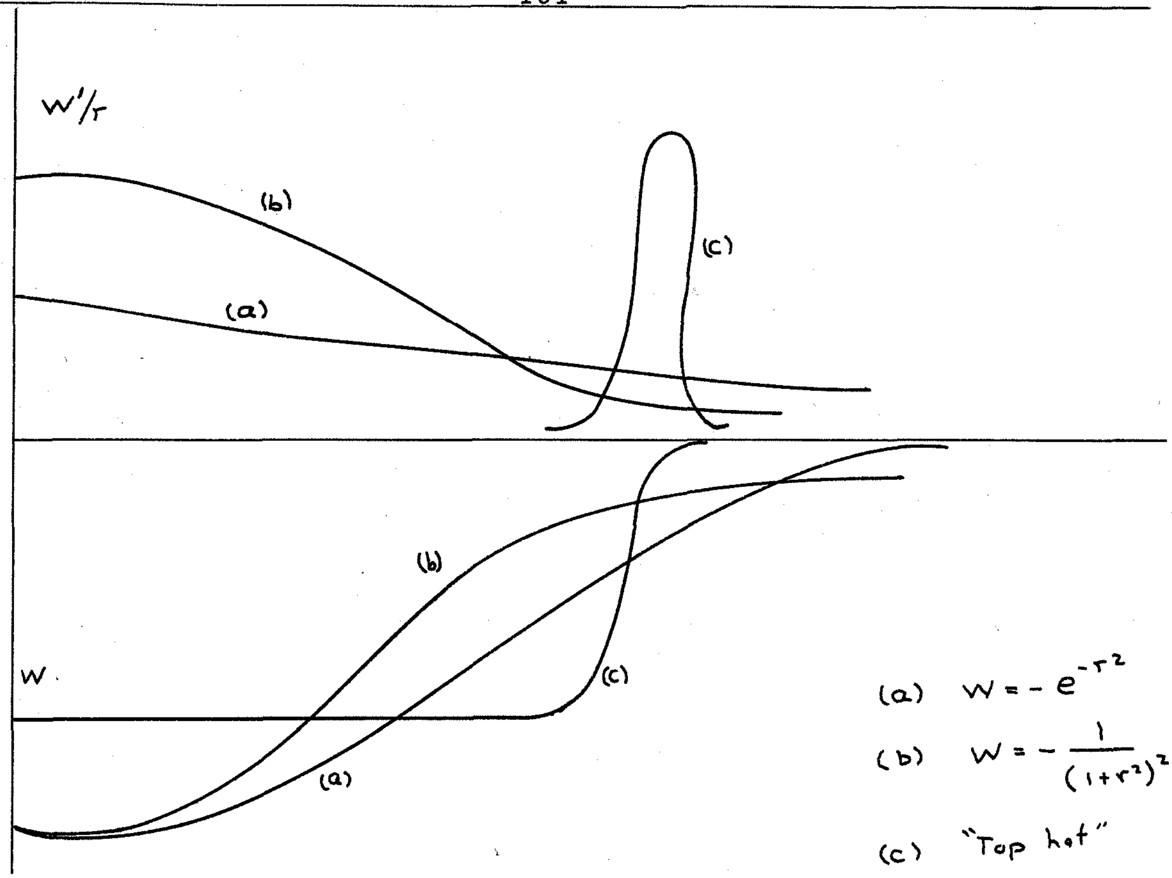
Sketch 5.14

Batchelor and Gill[36] have shown that only one neutral disturbance exists for the jet profile

$$W = -\frac{1}{(1+r^2)^2}$$

and is the sinuous type oscillation, n = 1. All other modes are stable. They also show that a "top-hat" profile, one in which the velocity is approximately constant in some central region and then falls rapidly to zero, is unstable to the n = 0 mode. These profiles and their derivatives together with the wake profile are sketched in Sketch 5.15. It is interesting to note that the sinuous type instability is common to the wake and jet profiles and is probably common to the "top-hat" profile.

It is reasonable to conjecture that the sinuous oscillations (n = 1)



Sketch 5. 15

are more unstable than the varicose ones for axi-symmetric flow. The radial velocity component on the axis is identically zero for $n \neq 1$ and arbitrary for $n = 1$. The flow is free to move normal to the axis only for the sinous mode and is restricted to zero motion in the radial direction on the axis for all other modes. This additional "degree of freedom" hints that the sinuous mode is the most unstable mode. It also seems likely that if an $n = 0$ mode is also unstable, the sinuous instability will still dominate, unlike Couette flow (flow between two rotating cylinders) instability, where the $n = 0$ mode always precedes any other unstable mode.

A necessary and sufficient condition for the existence of neutral subsonic disturbances and a sufficient condition for adjacent amplified disturbances is that the gradient of the density vorticity produce

$$\frac{d}{dr}\left[\frac{1}{rT\left(1+\frac{n^2}{\alpha^2 r^2}\right)}\frac{dw}{dr}\right] \tag{5.69}$$

vanish within the flow field for some $-w < (1/M)$. For slowly varying temperature and velocity profiles, the density-vorticity product will act like an equivalent incompressible far field wake or jet profile for purposes of stability, and the flow will be stable for the $n = 0$ mode and unstable for the $n = 1$ mode [Figure 1].

This can be illustrated by using the mean wake profiles of Section II. 4. If [Eqs. (2.55) and (2.56)]

$$w = -e^{-\eta} \qquad\qquad T = 1 + \Delta T\, e^{-\eta}$$

where

$$d\eta = (rdr/T)$$

then Eq. (5.69) becomes, for $n = 0$,

$$\frac{d}{d\eta}\left[\frac{1}{T^2}\frac{dw}{d\eta}\right]$$

which does not vanish within the flow field. For $n \neq 0$, Eq. (5.69) will vanish at some point in the flow field because of the term in the denominator $\left[1 + (n^2/\alpha^2 r^2)\right]$.

The stability or instability of the other modes will depend upon the exact profile shape. Now if the temperature profile has a "top-hat" shape and the velocity profile is slowly varying, or vice versa, then the $n = 0$ mode will be unstable. These cases are not physically unreasonable; for example, the velocity wake will decay more rapidly than the temperature wake in back of an axi-symmetric slender body flying at hypersonic speeds. The temperature profile will remain "top-hat" for many diameters downstream of the neck of the wake, even though the velocity defect becomes very small in the same region.

The remarks made in Section IV. 4 regarding the two-dimensional hypersonic wake problem apply equally to axi-symmetric wakes and will not be repeated. A typical Schlieren photograph of an axi-symmetric hypersonic slender body wake is shown in Figure 10.* The rear wake, downstream of the neck, is practically straight for about 10 - 15 base diameters and then starts to oscillate in the n = 1 mode. The wave length of the disturbance is of the order of the wake diameter and decreases as the wave progresses downstream. The wake then becomes turbulent at about 25 base diameters downstream of the neck. This Schlieren photograph certainly implies that transition is preceded by a wave-like motion which oscillates in the mode predicted by small disturbance theory. The oscillations that are visible in the near wake do not seem to originate in the neck but in a region between 10 - 15 base diameters downstream of the neck, indicating that subsonic disturbances do not exist until then.

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* These photographs were taken at NOL and were obtained through the courtesy of Dr. A. Pallone of the AVCO Corporation, Wilmington, Mass.

VI. SUGGESTIONS FOR FURTHER STUDY

1. The neutral stability characteristics of incompressible wake type flows have been determined for two limiting cases; (1) the "inviscid limit" ($\alpha R \longrightarrow \infty$, $\alpha \longrightarrow \alpha_s$, $R \longrightarrow \infty$) and (2) the "viscous limit" ($\alpha R \longrightarrow$ constant $<< 1$, $\alpha \longrightarrow 0$, $R \longrightarrow \infty$). It is suggested that the Orr-Sommerfeld equation be solved numerically to determine the complete neutral stability curve ($0 < \alpha R < \infty$).

An attempt should be made to clear up the problem of the "island of stability" within the neutral stability curve. [Section III. 2] Specifically, the fourth "inviscid" branch should be determined using smoothly varying profiles. If this branch exists then the extent of this "island of instability" should be found.

The behavior of the neutral stability curve is different for anti-symmetrical disturbances than for symmetrical ones. This fact, obtained by approximate methods, is not yet understood.

For large values of αR the "viscous solutions" do not enter the eigenvalue problem, yet they must be retained in determining the disturbance amplitudes around the critical point. Accordingly it is suggested that the viscous corrections around the singular point be found to smooth out any discontinuities in the inviscid disturbance amplitudes.

2. The problem of supersonic disturbances should be studied in great detail to determine if a mean flow is unstable to them, or not. Miles[34] has found that unstable supersonic disturbances may occur at low wave numbers (vortex sheet problem). A similar analysis should be made for smoothly varying profiles.

3. Actual wake profiles, starting from the neck of the wake, should be used in determining the streamwise location at which subsonic disturbances are possible. It would be of some interest to compare this distance with the "sticking distance" observed in hypersonic blunt and slender body wakes.

4. The effect of temperature and Mach number in the spatial amplification rate should be investigated more completely by using larger values of ΔT and M than was used in this investigation.

5. It is suggested that the equation for the amplification ratio [Eq. (4.31)] be examined in detail in order to furnish an explanation of some of the transition phenomena observed in hypersonic wakes.

6. The inviscid stability characteristics of axi-symmetric wake flows should be calculated by using a method similar to that described in Section IV. 2 and Appendix D. The amplification rates for the $n = 1$ mode should be compared with those for the $n = 2$ mode to determine which is more unstable. Numerical calculations for compressible wake flows should also be made.

VII. CONCLUDING REMARKS

1. The effect of temperature on the inviscid stability of two-dimensional wake flows is both stabilizing and destabilizing. As the temperature increases, the critical Mach number increases, and the range of Mach numbers over which subsonic disturbances can exist also increases. However, as long as the relative Mach number is below the critical Mach number the neutral inviscid wave number will decrease with increasing temperature.

2. The numerical calculations indicate that a heated wake will be more stable than a cool one if the relative Mach number is less than the critical Mach number. For a typical hypersonic blunt body wake, using Gaussian profiles for temperature and velocity in the Dorodnitsyn-Howarth variable, the maximum dimensionless spatial amplification rate is constant in the downstream direction and occurs at one preferred frequency. This result is similar to that for the incompressible flat plate wake.

3. The inviscid stability problem for axi-symmetric compressible wake flows is directly analogous to the two-dimensional problem in a transformed orthogonal velocity space, except for a delta function singularity associated with the Reynolds shear stress near the critical point. This stress is always a destabilizing influence. It is also found that a necessary and sufficient condition for the existence of neutral subsonic disturbances is that for some $w = -c_R < 1/M$ [in a system fixed in the fluid at rest], the gradient of the density-vorticity product in a certain direction must vanish.

4. For incompressible axi-symmetric wake flow (using a Gaussian), the only modes that are unstable are the $n = 1$ and $n = 2$ modes. For slowlying varying temperature profiles the same modes are unstable. However, for a "top-hat" temperature profile, the $n = 0, 1$ modes are unstable. By physical arguments it is shown that the $n = 1$ mode should be the most unstable mode for wake-type flows.

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APPENDIX A

BOUNDARY CONDITIONS FOR THE AXI-SYMMETRIC PROBLEM

For the axi-symmetric wake the boundary conditions on the axis are derived from the purely kinematic condition that all disturbance amplitudes and the vorticity disturbance must be finite there, regardless of the viscosity or the compressibility of the fluid. The three components of vorticity fluctuation are

$$\Gamma_r = \frac{in}{r} q_x - i\alpha q_\phi \tag{A. 1}$$

$$\Gamma_\phi = i\alpha^2 q_r - q_x' \tag{A. 2}$$

$$\Gamma_x = \frac{1}{r}(r q_\phi)' - \frac{in\alpha}{r} q_r \tag{A. 3}$$

For n = 0, the continuity equation [Eq. (2. 38)] shows that $q_r \sim r$ as $r \longrightarrow 0$ if q_z and s are to be finite on the axis and $q_\phi \sim r$ if Γ_x is to be finite on the axis [Eq. (A. 3)] . Therefore, $q_r(0) = q_\phi(0) = 0$; $q_x(0)$, $\pi(0)$, $s(0)$ and $\theta(0)$ are aribtrary.

For $n \neq 0$, let $q_r \longrightarrow r^\epsilon A$ as $r \longrightarrow 0$. Then from Eq. (2. 38),

$$q_\phi \rightarrow \frac{i\alpha}{n} A(1+\epsilon) r^\epsilon \tag{A. 4}$$

Substituting Eq. (A. 4) into Eq. (A. 3) one obtains

$$\Gamma_x \rightarrow \frac{i\alpha}{n} A\left[(1+\epsilon)^2 - n^2\right] r^{\epsilon - 1} = 0$$

Then $\epsilon = n - 1$ and for

$$n > 1 \quad , \qquad q_r(0) = q_\phi(0) = 0 \tag{A. 5}$$

$$n = 1 \quad , \qquad q_\phi(0) = i\,\alpha\, q_r(0) \quad . \tag{A. 6}$$

From the ϕ-momentum equation, $\pi(0) = 0$ when $n \neq 0$, and from Eq. (A. 1), $q_x \sim r$ $(n \neq 0)$ or $q_x(0) = 0$. Therefore, $s(0) = \theta(0) = 0$ when $n \neq 0$. In addition, Eq. (2. 40) shows that $\pi'(0) = 0$ when $n > 1$.

APPENDIX B

TWO-DIMENSIONAL WAKE MODEL

The mean flow quantities are assumed to satisfy the boundary layer equations. Using Kubota's method[55] for a zero external pressure gradient, the following set of equations are obtained for the compressible wake behind a flat plate or hypersonic vehicle:

Continuity

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0$$

Momentum (B.1)

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = \frac{\partial^2 U}{\partial Y^2}$$

Energy

$$U\frac{\partial h}{\partial X} + V\frac{\partial h}{\partial Y} = \frac{1}{\sigma}\frac{\partial^2 h}{\partial Y^2} + (\gamma-1)M_e^2\left(\frac{\partial U}{\partial Y}\right)^2$$

with the boundary conditions

$$\begin{aligned} &U(0,Y) = U_0(Y), \quad \frac{\partial U}{\partial Y}(X,0) = 0 \\ &h(0,Y) = h_0(Y), \quad \frac{\partial h}{\partial Y}(X,0) = 0 \\ &U \to 1, \quad h \to 0, \quad |Y| \to \infty \end{aligned} \qquad \text{(B. 2)}$$

where

$$U = \frac{U^*}{U_e^*}, \qquad h = \frac{h^* - h_e^*}{h_e^*} \qquad M_e = \frac{U_e^*}{a_e^*}$$

$$X(x^*) = \frac{\rho_e^* U_e^* \mu_e^*}{\rho_\infty^* U_\infty^* \mu_\infty^*}\frac{x^*}{d^*}$$

$$Y(Y^*) = \frac{U_e^*}{U_\infty^*}\sqrt{R_{\infty_{d^*}}}\int_0^{Y^*}\frac{\rho^*}{\rho_e^*}\frac{dY^*}{d^*}$$

σ = Prandtl number = constant

d^* = characteristic body dimension

$$R_{\infty_{d^*}} = \frac{U_\infty^* \rho_\infty^* d^*}{\mu_\infty^*}$$

$\rho^*\mu^* = \rho_e^*\mu_e^*$ = constant ; Chapman-Rubesin relation

h_e^*, U_e^* = constant

The above equations are linearized by using Oseen type variables

$$W = 1 - U \ll 1$$
$$h \ll 1 \qquad \text{(B. 3)}$$

Retaining the lowest order terms, the following equations are obtained

$$\frac{\partial W}{\partial X} = \frac{\partial^2 W}{\partial Y^2}$$
$$\sigma\frac{\partial h}{\partial X} = \frac{\partial^2 h}{\partial Y^2} \qquad \text{(B. 4)}$$

with the boundary conditions

$$W(0,Y) = W_0(Y) \qquad \frac{\partial U}{\partial Y}(X,0) = 0$$
$$h(0,Y) = h_0(Y) \qquad \frac{\partial h}{\partial Y}(X,0) = 0 \qquad \text{(B. 5)}$$

By using Laplace transforms, the solutions of Eqs. (B. 4) subject to the boundary conditions, Eqs. (B. 5) are obtained, as follows:

$$W(X,Y) = \frac{1}{2\sqrt{\pi X}} \int_{-\infty}^{\infty} W_o(\xi) \exp - \frac{(\xi + Y)^2}{4X} d\xi$$

$$h(X,Y) = \frac{1}{2} \sqrt{\frac{\sigma}{\pi X}} \int_{-\infty}^{\infty} h_o(\xi) \exp - \frac{\sigma(\xi + Y)^2}{4X} d\xi \tag{B. 6}$$

The momentum thickness (or drag coefficient, $C_D = \dfrac{\text{Drag}}{\frac{1}{2} \rho_e^* U_e^{*2} d^*}$) is given by

$$\Theta^*(x^*) = \frac{C_D d^*}{4} = \int_0^{\infty} \frac{\rho^* U^*}{\rho_e^* U_e^*} \left[1 - \frac{U^*}{U_e^*} \right] dy^*$$

$$= \frac{\rho_\infty^* U_\infty^* d^*}{\rho_e^* U_e^* \sqrt{R_{\infty_{d^*}}}} \int_0^{\infty} W_o(\xi) d\xi \tag{B. 7}$$

$$= \text{constant}$$

and the net heat transferred to the body by

$$\delta^* = - \int_{-\infty}^{\infty} \rho^* U^* [H^* - H_e^*] dy^*$$

$$= \frac{2 \rho_\infty^* U_\infty^* d^* h_e^*}{\sqrt{R_{\infty_{d^*}}}} \left[(\gamma - 1) M_e^2 \int_0^{\infty} W_o(\xi) d\xi - \int_0^{\infty} h_o(\xi) d\xi \right] \tag{B. 8}$$

$$= \text{constant}$$

where H^* = stagnation enthalpy.

Let

$$\Theta = \frac{\rho_e^* U_e^* \Theta^*}{\rho_\infty^* U_\infty^* d^*} \sqrt{\frac{R_{\infty_{d^*}}}{\pi}}$$

$$\Delta = \frac{\delta^*}{2 \rho_\infty^* U_\infty^* d^* h_e^*} \sqrt{\frac{\sigma R_{\infty_{d^*}}}{\pi}} \tag{B. 9}$$

Then from Eqs. (B. 7) and (B. 8)

$$\int_{-\infty}^{\infty} W_o(\xi)\, d\xi = 2\sqrt{\pi}\, \Theta$$
$$\int_{-\infty}^{\infty} h_o(\xi)\, d\xi = 2\sqrt{\frac{\pi}{\sigma}}\left[\sqrt{\sigma}\,(\gamma-1) M_e^2\, \Theta - \Delta\right] \tag{B. 10}$$

If the initial conditions are assumed to be point sources (delta functions), i. e.,

$$W_o(\xi) = A\,\delta(\xi)$$
$$h_o(\xi) = B\,\delta(\xi) \tag{B. 11}$$

then from Eqs. (B. 10) and (B. 11)

$$A = 2\sqrt{\pi}\, \Theta$$
$$B = 2\sqrt{\frac{\pi}{\sigma}}\left[\sqrt{\sigma}\,(\gamma-1) M_e^2\, \Theta - \Delta\right] \tag{B. 12}$$

The solutions [Eq. (B. 6)] then become

$$W(X,Y) = \frac{\Theta}{\sqrt{X}} \exp - \frac{Y^2}{4X}$$
$$h(X,Y) = \frac{\left[\sigma\,(\gamma-1) M_e^2\, \Theta - \Delta\right]}{X} \exp - \frac{\sigma Y^2}{4X} \tag{B. 13}$$

Let the characteristic length scale, L*, of the mean flow field be defined as

$$L^* = \frac{2\rho_\infty^* U_\infty^*}{\rho_e^* U_e^*}\sqrt{\frac{X}{R_{\infty_{d^*}}}}\; d^*$$

$$= \frac{2}{\sqrt{R_{\infty_{d^*}}}}\;\frac{U_\infty^*}{U_e^*}\;\sqrt{\frac{U_e^*\rho_e^*\mu_e^*}{U_\infty^*\rho_\infty^*\mu_\infty^*}}\sqrt{\frac{X^*}{d^*}}\; d^* \tag{B. 14}$$

For a flat plate incompressible wake,

$$L^* = \frac{2X^*}{\sqrt{Re_{X^*}}} \tag{B. 15}$$

For convenience, the following notation is adopted:

$$\Delta U = \frac{U_e^* - U^*(0)}{U_e^*} = \frac{\Theta}{\sqrt{X}}$$

$$\Delta T = \frac{T^*(0) - T_e^*}{T_e^*} = \frac{\left[\sqrt{\sigma}\,(\gamma-1)M_e^2\,\Theta - \Delta\right]}{\sqrt{X}} \tag{B. 16}$$

$$V^* = U_e^*\,\Delta U \quad , \quad \eta = \frac{Y}{2\sqrt{X}} = \int_0^Y \frac{dY}{T}$$

so that

$$w = \frac{U^* - U_e^*}{V^*} = -e^{-\eta^2}$$

$$T = \frac{h^*}{h_e^*} = 1 + \Delta T\, e^{-\eta^2}, \qquad \sigma = 1 \tag{B. 17}$$

V^* is the velocity defect of the wake and L^* is the Y position at which

$$\frac{U^* - U_e^*}{U_e^* - U^*(0)} = -\frac{1}{e}$$

The Reynolds number of the wake is

$$R = \frac{L^* V^*}{\nu_e^*} = \frac{2}{\sqrt{\pi}}\,Re_{\Theta^*} = \frac{Re_{d^*}\,C_D}{2\sqrt{\pi}} \tag{B. 18}$$

APPENDIX C

METHOD OF SOLUTION OF TATSUMI AND KAKUTANI[31] FOR SMALL αR

The Orr-Sommerfeld equation can be expressed in the following form [Eq. (3.2)] :

$$\phi^{IV} - \phi''\left[2\alpha^2 + i\alpha R(W-c)\right] + \phi\left[\alpha^4 + i\alpha R(W-c)\alpha^2 + i\alpha R W''\right] \qquad (C.1)$$

subject to the boundary conditions, Eqs. (3.3) and (3.7),

$$\phi \sim e^{-\sqrt{\alpha^2 - i\alpha Rc}\, Y},\ e^{-\alpha Y} \qquad Y \to \infty$$
$$-\pi < \arg(\alpha^2 - i\alpha Rc) < \pi \qquad (C.2)$$

Anti-symmetric Disturbances

$$\phi'(0) = \phi'''(0) = 0 \qquad (C.3)$$

Symmetric Disturbances

$$\phi(0) = \phi''(0) = 0 \qquad (C.4)$$

Tatsumi and Kakutani[31] expand the solution in powers of αR as follows:

$$\phi(Y) = \sum_{n=0}^{\infty} (i\alpha R)^n \phi^{(n)}(Y;\alpha,\beta) \qquad (C.5)$$

where

$$\beta^2 = \alpha^2 - i\alpha R c$$

Substituting Eq. (C.5) into Eq. (C.1), and matching powers of $i\alpha R$, the following equations relating the $\phi^{(n)}$'s , are obtained

$$(D^2-\alpha^2)(D^2-\beta^2)\,\phi^{(0)} = 0 \tag{C. 6}$$

$$(D^2-\alpha^2)(D^2-\beta^2)\,\phi^{(n)} = W\left[D^2\phi^{(n-1)} - \alpha^2\phi^{(n-1)}\right] - W''\phi^{(n-1)} \tag{C. 7}$$

The solutions of Eq. (C. 6) are

$$\phi_1^{(0)} = e^{-\alpha Y}, \quad \phi_2^{(0)} = e^{\alpha Y}, \quad \phi_3^{(0)} = e^{-\beta Y}, \quad \phi_4^{(0)} = e^{\beta Y} \tag{C. 8}$$

The solutions of Eq. (C. 7) can be found by the method of variation of parameters and are

$$\begin{aligned}\phi_j^{(n)} = \frac{1}{i\alpha Rc}\Bigg[&\left\{e^{-\alpha Y}\int^{Y} W e^{\alpha Y}\left[\phi_j^{(n-1)'} + \alpha\,\phi_j^{(n-1)}\right]dy\right\} \\ &+\left\{e^{\alpha Y}\int^{Y} W e^{-\alpha Y}\left[\phi_j^{(n-1)'} - \alpha\,\phi_j^{(n-1)}\right]dy\right\} \\ &-\left\{e^{-\beta Y}\int^{Y} W e^{\beta Y}\left[\phi_j^{(n-1)'} + \frac{\alpha^2+\beta^2}{2\beta}\,\phi_j^{(n-1)}\right]dy\right\} \\ &-\left\{e^{\beta Y}\int^{Y} W e^{-\beta Y}\left[\phi^{(n-1)'} - \frac{\alpha^2+\beta^2}{2\beta}\,\phi_j^{(n-1)}\right]dy\right\}\Bigg]\end{aligned} \tag{C. 9}$$

for $n \geqslant 1$, $j = 1, 2, 3, 4$. The general solution of Eq. (C. 1) is

$$\phi = \sum_{j=1}^{4} C_j\,\phi_j \tag{C. 10}$$

where the C_j's are arbitrary constants.

Since the solution Eq. (C. 8) must satisfy the outer boundary condition [Eq. (C. 2)], $C_2 = C_4 = 0$ and for a non-trivial solution, ϕ_1 and ϕ_3 must satisfy the following eigenvalue equations:

Anti-symmetric disturbances [Eq. (C. 3)]

$$\begin{vmatrix} \phi_1'(0) & \phi_3'(0) \\ \phi_1'''(0) & \phi_3'''(0) \end{vmatrix} = 0 \qquad \text{(C. 11)}$$

Symmetric disturbances [Eq. (C. 4)]

$$\begin{vmatrix} \phi_1(0) & \phi_3(0) \\ \phi_1''(0) & \phi_3''(0) \end{vmatrix} = 0 \qquad \text{(C. 12)}$$

For convenience, let $A(\phi_j^{(n)})$, $B(\phi_j^{(n)})$, $C(\phi_j^{(n)})$ and $D(\phi_j^{(n)})$ be the terms in the brackets of the solution $\phi_j^{(n)}$ [Eq. (C. 9)], respectively, where the lower limit is taken to be infinity, so that

$$\phi_j^{(n)} = A(\phi_j^{(n)}) + B(\phi_j^{(n)}) - C(\phi_j^{(n)}) - D(\phi_j^{(n)}) \qquad \text{(C. 13)}$$

and further introduce the notation

$$\begin{aligned} I^{(n)}(\phi_j) &= \left[B(\phi_j^{(n)}) - A(\phi_j^{(n)}) \right]_{Y=0} \\ J^{(n)}(\phi_j) &= \left[D(\phi_j^{(n)}) - C(\phi_j^{(n)}) \right]_{Y=0} \\ K^{(n)}(\phi_j) &= \left[B(\phi_j^{(n)}) + A(\phi_j^{(n)}) \right]_{Y=0} \\ L^{(n)}(\phi_j) &= \left[D(\phi_j^{(n)}) + C(\phi_j^{(n)}) \right]_{Y=0} \end{aligned} \qquad \text{(C. 14)*}$$

* This definition differs from that of Eq. (6. 3), Reference 31, by the factor (1/iαRc).

Then

$$\begin{aligned}
\phi_j^{(n)}(0) &= K^{(n)}(\phi_j) - L^{(n)}(\phi_j) \\
\phi_j^{(n)\prime}(0) &= \alpha I^{(n)}(\phi_j) - \beta J^{(n)}(\phi_j) \\
\phi_j^{(n)\prime\prime}(0) &= \alpha^2 K^{(n)}(\phi_j) - \beta^2 L^{(n)}(\phi_j) - \omega(0)\, \phi_j^{(n-1)}(0) \\
\phi_j^{(n)\prime\prime\prime}(0) &= \alpha^3 I^{(n)}(\phi_j) - \beta^3 J^{(n)}(\phi_j) + \omega(0)\, \phi_j^{(n-1)}(0)
\end{aligned} \tag{C. 15}$$

Substituting Eq. (C. 5) into Eqs. (C. 11) and (C. 12) and using Eq. (C. 15), the eigenvalue relations can be reduced to the following form

Anti-symmetric Disturbances

$$\begin{vmatrix} -1 + \sum_{n=1}^{\infty} (i\alpha R)^n I^{(n)}(\phi_1) & \sum_{n=1}^{\infty} (i\alpha R)^n I^{(n)}(\phi_3) \\ -\sum_{n=1}^{\infty} (i\alpha R)^n J^{(n)}(\phi_1) & 1 + \sum_{n=1}^{\infty} (i\alpha R)^n J^{(n)}(\phi_3) \end{vmatrix} = 0 \tag{C. 16}$$

Symmetric Disturbances

$$\begin{vmatrix} 1 + \sum_{n=1}^{\infty} (i\alpha R)^n K^{(n)}(\phi_1) & \sum_{n=1}^{\infty} (i\alpha R)^n K^{(n)}(\phi_3) \\ -\sum (i\alpha R)^n L^{(n)}(\phi_1) & 1 - \sum_{n=1} (i\alpha R)^n L^{(n)}(\phi_3) \end{vmatrix} = 0 \tag{C. 17}$$

Eqs. (C. 16) and (C. 17) are then expanded and only terms of the third and lower order in $i\alpha R$ are retained. The quantities in Eq. (C. 14) were evaluated using Eq. (C. 9) with $w = + e^{-y^2}$.

Since the complex wave speed is of order unity, or less, then $\beta - \alpha$ will be of the order of αR. In order to be consistent with the approximations used, the coefficients in the eigenvalue equations were expanded in powers of $\beta - \alpha = \alpha\sigma$ (σ complex) and terms of the order σ^3 and higher were neglected. The complex eigenvalue equations then become

Anti-Symmetric Disturbances

$$
\begin{aligned}
&-2\alpha[1+\sigma][2+\sigma] + i\alpha R\left[\sqrt{\pi}\,\sigma - 3\alpha(1+\sigma)(2+\sigma)\right] \\
&+(i\alpha R)^2\left[\frac{3\sqrt{\pi}}{2}\sigma - \sqrt{\frac{\pi}{2}}(1+2\sigma) - 3\alpha(1+\sigma)(2+\sigma)\right] \\
&+(i\alpha R)^2\left[\frac{\pi\alpha}{8}(2+\sigma)(11+7\sigma)\right] + (i\alpha R)^3\sqrt{\pi}\left[\frac{\pi}{48}(8-7\sigma)\right. \\
&\left.+\frac{3}{2}\sigma - \frac{3}{2\sqrt{2}}(1+2\sigma) + \frac{3}{8\sqrt{3}}(2+3\sigma)\right] = 0
\end{aligned}
\tag{C. 18}
$$

Symmetric Disturbances

$$
\begin{aligned}
&\alpha(2+\sigma) + (i\alpha R)\left[\sqrt{\pi} - \frac{3}{2}\alpha(2+\sigma)\right] + (i\alpha R)^2\left[\sqrt{\frac{\pi}{2}}\left(\frac{5}{\sqrt{2}} - 3\right)\right. \\
&\left.+\frac{3}{8}\alpha(2+\sigma)\right] + (i\alpha R)^3\sqrt{\pi}\left[-\frac{7\sqrt{\pi}}{48} + \frac{3}{4} - \frac{15}{4\sqrt{2}} + \frac{1}{8\sqrt{3}}\right] = 0
\end{aligned}
\tag{C. 19}
$$

where

$$\sigma_R = -1 + \sqrt{1 + \sigma_I^2 + \frac{R C_I}{\alpha}}$$

$$C_R = -\frac{2\alpha}{R}\,\sigma_I\left[1 + \sigma_R\right]$$

The asymptotic behavior of Eqs. (C. 18) and (C. 19) was determined by a trial and error method. The correct limiting processes and reduced equations are as follows for $c_I = 0$:

Anti-symmetric disturbances $\quad \sigma_I \to 0, \; \alpha \to 0, \; \sigma_R \to \frac{\sigma_I^2}{2}$

Real part: $-4 + \sigma_I \pi R + \alpha\sqrt{\frac{\pi}{2}}\, R^2 + \alpha^2 \sigma_I \sqrt{\pi}\, R^3 \Big[\frac{9}{8\sqrt{3}}$

$$- \frac{3}{\sqrt{2}} + \frac{3}{2} - \frac{7\pi}{48}\Big] = 0 \tag{C. 20}$$

Imaginary part: $- 6\sigma_I + \left[\frac{\sqrt{\pi}\,\sigma_I^2}{2} - 6\alpha\right] R - \alpha\sigma_I R^2\left[\frac{3\sqrt{\pi}}{2} - \frac{2\sqrt{\pi}}{\sqrt{2}}\right]$

$$- \alpha^2\sqrt{\pi}\, R^3\left[\frac{\pi}{6} - \frac{3}{2\sqrt{2}} + \frac{3}{4\sqrt{3}}\right] = 0$$

Symmetric disturbances $\quad \sigma_I \to -\infty, \; \sigma_R \to -\sigma_I, \; \alpha \to 0$

Real part $\quad \sigma_R + \frac{3}{2}\alpha\sigma_I R - \alpha\left[\frac{\sqrt{\pi}}{2}\left(\frac{5}{\sqrt{2}} - 3\right) + \frac{3}{8}\alpha\,\sigma_R\right] R^2 = 0$

Imaginary part $\quad \sigma_I + \left[\sqrt{\pi} - \frac{3}{2}\alpha\sigma_R\right] R - \alpha^2\sigma_I \frac{3}{8} R^2$ (C. 21)

$$- \alpha^2 R^3 \sqrt{\pi}\left[-\frac{7\sqrt{\pi}}{48} + \frac{3}{4} - \frac{15}{4\sqrt{2}} + \frac{1}{8\sqrt{3}}\right] = 0$$

The equations are solved simultaneously and the results are given in Table 3.2.* Since the coefficient of $(i\alpha R)^3$ in Eq. (C. 19) is real, the coefficient of R^3 in Eq. (C. 21) [Real part] is zero to the order of the approximation used.

Eq. (C. 18) was solved graphically and a minimum critical Reynolds number was found [Table 3.1].

---

* Since the profile $w = e^{-y^2}$ was used in these calculations, the sign of c_R as computed from Eqs. (C. 20) and (C. 21) must be changed to conform to the notation in the rest of the text. This was done in these tables.

APPENDIX D

SOLUTION OF THE INVISCID EQUATIONS FOR AMPLIFIED DISTURBANCES

For amplified subsonic disturbances, the solution of Eq. (4. 14) and Eq. (4. 17) is regular everywhere on the real η axis. Since G is singular at the axis $\left[G \sim (1/\pi)\right]$, it is convenient to make the following transformation

$$H = \eta G \tag{D. 1}$$

Eq. (4. 7) then becomes

$$H' = \eta\left[1 - \frac{M^2(W-c)^2}{T}\right] + \left[\frac{2w'}{w-c} - \frac{2T'}{T}\right]H - \alpha^2 T^2 \frac{H^2}{\eta} + \frac{H}{\eta} \tag{D. 2}$$

Eq. (D. 2) is a complex equation. Its real and imaginary parts are

$$H_R' = \eta\left[1 - \frac{M^2}{T}\left[(w-C_R)^2 - C_I^2\right]\right] + \left[\frac{2w'(W-C_R)}{(w-C_R)^2 + C_I^2} - \frac{2T'}{T}\right] H_R$$

$$- \frac{2w'C_I}{(w-C_R)^2 + C_I^2} H_I - \frac{\alpha^2 T^2}{\eta}\left[H_R^2 - H_I^2\right] + \frac{H_R}{\eta} \tag{D. 2}$$

$$H_I' = \eta\left[\frac{2M^2}{T}(W-C_R)\, C_I\right] + \frac{2w'C_I}{(w-C_R)^2 + C_I^2} H_R$$

$$+ \left[\frac{2w'(W-C_R)}{(w-C_R)^2 + C_I^2} - \frac{2T'}{T}\right] H_I - 2\alpha^2 T^2 \frac{H_R H_I}{\eta} + \frac{H_I}{\eta}$$

The boundary conditions as $\eta \longrightarrow \infty$ are

$$H_R = -\frac{\eta}{\sqrt{2}\,\alpha}\sqrt{1-M^2(C_R^2-C_I^2)}\sqrt{1+\sqrt{1+\frac{4M^4C_R^2C_I^2}{\left[1-M^2(C_R^2-C_I^2)\right]^2}}} \tag{D. 4}$$

$$H_I = -\frac{M^2 C_R C_I}{\alpha^2 H_R}\eta^2$$

Using a power series expansion about the axis, and satisfying the condition $\pi(0) = G$,

$$H_R = \frac{1}{\alpha^2 T_o^2} + a_2\eta^2 + \tag{D. 5}*$$

$$H_I = \eta^2\left[b_2 + b_4\eta^2 + \cdots\right]$$

where

$$a_o = \frac{1}{\alpha^2 T_o^2}$$

$$a_2 = \frac{1}{3}\left[A_o + a_o\left(B_o - \frac{T_o''}{T_o}\right)\right]$$

$$b_2 = \frac{1}{3}\left[D_o + a_o C_o\right]$$

$$b_4 = \frac{1}{5}\left[D_1 + C_o a_2 + C_1 a_o + b_2\left(B_o - 2\frac{a_2}{a_o} - 2\frac{T_o''}{T_o}\right)\right]$$

$$A_o = 1 - \frac{M^2}{T_o}\left[(1+C_R)^2 - C_I^2\right]$$

---

* Primes (') indicate differentiation with respect to η.

$$B_0 = -\frac{2\omega_0''(1+C_R)}{(1+C_R)^2 + C_I^2} - 2\frac{T_0''}{T_0}$$

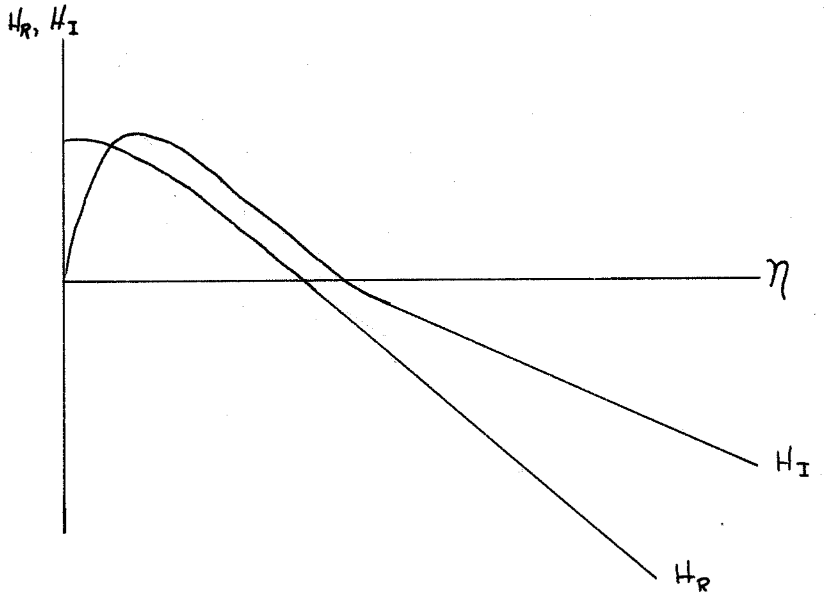
$$C_0 = \frac{2\omega_0'' C_I}{(1+C_R)^2 + C_I^2}$$

$$C_1 = \frac{1}{3}\frac{\omega_0^{IV} C_I}{(1+C_R)^2 + C_I^2} + \frac{1}{2}\frac{\omega_0''^2 (1+C_R) C_I}{\left[(1+C_R)^2 + C_I^2\right]^2}$$

$$D_0 = -\frac{2M^2(1+C_R)C_I}{T_0}$$

$$D_1 = \frac{M^2}{T_0} C_I\left[\omega_0'' + (1+C_R)\frac{T_0''}{T_0}\right]$$

There are only two integral curves that will simultaneously satisfy the boundary conditions at the axis and at infinity for a given set of eigen values; α, c_R and c_I. These are sketched below.



Sketch D. 1

If the given set is not consistent, the boundary conditions will not be satisfied and the integral curves oscillate very rapidly near the axis. For this reason, the integrations were started from the axis and infinity and the values of H_R and H_I were compared at a point within the domain. The matching point was taken to be the point at which $w = c_R$.

The calculation procedure used to obtain the inviscid amplified solution for the given profiles $w(\eta)$ and $T(\eta)$, the relative Mach number, M, and the wave speed, c_R, is as follows:

<u>Integration from Infinity to the Critical Point and from the Axis to the Critical Point</u>

(1) Assume a value of α and c_I and evaluate the boundary condition at infinity from Eq. (D. 4) and the boundary condition for a small positive value of η from Eq. (D. 5).

(2) Continue the calculation of H_R and H_I by the simultaneous integration of Eq. (D. 3) to the critical point, η_c.

(3) Compare the values of H_R and H_I at η_c obtained from the inner and outer integrations.

(4) Repeat steps (1) through (3) until the values of H_R and H_I are simultaneously matched at η_c.

APPENDIX E

EXPANSION ABOUT CRITICAL POINT - AXI-SYMMETRIC CASE

The solution of the inviscid equation [Eq. (5. 9)] in the neighborhood of the "singular point" in the complex r-plane (w = c) is obtained by a Taylor Series expansion (method of Frobenius). Eq. (5. 9) can be rewritten in the following form

$$\psi'' + \frac{\xi'}{\xi}\psi' - \left[\frac{Y}{w-c} + \alpha^2 X\right]\psi = 0 \tag{E. 1}$$

where

$$X = \frac{1}{\xi r T}$$
$$Y = \frac{1}{\xi}(\xi w')' \tag{E. 2}$$

Let $\chi = r - r_c$ and assume a series solution of the form

$$\psi = \chi^s\left[a_0 + a_1\chi + a_2\chi^2 + \cdots\right] \tag{E. 3}$$

Since (w-c) and T are analytic functions of r everywhere in the finite region of the complex r plane the coefficients of Eq. (E. 1) can be expanded in a Taylor Series about the point $r = r_c$ (w = c):

$$\frac{\xi'}{\xi} = \left(\frac{\xi'}{\xi}\right)_c + \left(\frac{\xi'}{\xi}\right)_c'\chi + \cdots$$
$$Y = Y_c + Y_c'\chi + \cdots \tag{E. 4}$$
$$\frac{Y}{w-c} = \frac{1}{w_c'\chi}\left[Y_c + \left(Y_c' - Y_c\frac{w_c''}{2w_c'}\right)\chi + \right]$$
$$\alpha^2 X = \alpha^2 X_c + \cdots$$

Eqs. (E. 3) and (E. 4) are substituted into Eq. (E. 1) and the coefficient of each power of χ is set equal to zero. The two linearly independent solutions, Ψ_1 and Ψ_2, valid in the neighborhood of the critical point along the real axis are as follows:

$$\Psi_1 = \chi\left[1 + \frac{w_c''}{2w_c'}\chi + \left(\frac{\alpha^2 X_c}{6} + \frac{w_c'''}{6w_c'}\right)\chi^2 + \cdots\right] \tag{E. 5}$$

$\chi > 0$

$$\Psi_2 = K\Psi_1 \ln\chi + b_1\Psi_1 + 1 + b_2\chi^2 + \cdot \tag{E. 6}$$

$\chi < 0$

$$\Psi_2 = K\Psi_1\left[\ln|\chi| - i\pi\right] + b_1\Psi_1 + 1 + b_2\chi^2 + \cdot$$

where

$$K = \frac{1}{w_c'\xi_c}(\xi w')_c'$$

$$X_c = \frac{1}{r_c T_c \xi_c} = 1 + \frac{n^2}{\alpha^2 r_c^2} = \frac{m_c^2}{\alpha^2} \tag{E. 7}$$

$$\frac{\xi_c'}{\xi_c} = -\left[\frac{T_c'}{T_c} + \frac{1 - n^2/\alpha^2 r_c^2}{r_c X_c}\right]$$

$$b_2 = \frac{1}{2}\left[\frac{Y_c}{w_c'} - K\left(\frac{1}{2}\frac{w_c''}{w_c'} + K\right) + \alpha^2 X_c\right]$$

The coefficient b_1 is not determined in this method. The proper path for analytical continuation of Ψ_2, in passing from $\chi > 0$ to $\chi < 0$, lies below the point $r = r_c$ for $w_c' > 0$ [Appendix G].

The other disturbance amplitudes can be found in the neighborhood of the critical point by using Eqs. (5.11), (E. 5) and (E. 6) :

$$
\begin{aligned}
\frac{i\,\pi}{\gamma M^2} &= -w_c'\,\xi_c\left[1 - \frac{K\alpha^2 X_c}{3}\chi^3 \ln\chi + \cdot\right] \\
i q_1 &= -\frac{m}{\alpha}\left[K \ln\chi + \cdots\right] \\
i q_3 &= \frac{n}{m_c r_c^2}\left[\frac{1}{\chi} + K \ln\chi + \cdot\cdot\right] \\
i s &= \frac{T_c'}{r_c w_c' T_c^2}\left[\frac{1}{\chi} + K \ln\chi + \cdot\right] \\
i\,\Theta &= \frac{T_c'}{r_c w_c'}\left[\frac{1}{\chi} + K \ln\chi + \cdot\cdot\right]
\end{aligned}
\qquad \text{(E. 8)}
$$

Note that for

$$
\begin{aligned}
\chi &> 0 \qquad \ln\chi \longrightarrow \ln\chi \\
\chi &< 0 \qquad \ln\chi \longrightarrow \ln|\chi| - i\pi
\end{aligned}
$$

in Eq. (E. 8).

APPENDIX F

EXTREMUM OF DENSITY-VORTICITY PRODUCT

For the case of neutral disturbances, $(\xi \omega')$ must have a true extremum at $r = r_c$ and not a point of inflection. This can be shown in exactly the same way as in the incompressible case[36] in the following way. Add the complex conjugate equations, instead of subtracting them (in derivation of Eq. (5.32)) to obtain

$$\int_0^\infty \xi_R \, |\psi'|^2 \, dr + \int_0^\infty \left(P_R + \frac{\alpha^2}{Tr}\right) |\psi|^2 \, dr = 0 \tag{F.1}$$

For most problems of interest, $\xi_R \geq 0$, so that

$$\int_0^\infty P_R \, |\psi|^2 \, dr < 0 \quad , \tag{F.2}$$

$$P_R = \frac{1}{\omega - C_R} \left(\xi_R \, \omega'\right)' \tag{F.3}$$

and

$$\int_0^\infty \frac{1}{\omega - C_R} \left(\xi_R \, \omega'\right)' |\psi|^2 \, dr < 0 \tag{F.4}$$

A necessary and sufficient condition for the existence of neutral disturbances is that

$$\left(\xi_R \, \omega'\right)'_{r = r_c} = 0 \tag{F.5}$$

and $c_R = w = c_s$ at this point.

Let $\qquad d\tilde{r} = dr/\xi_R \qquad \xi_R \geq 0 \qquad$ (F.6) so that

$$\int_0^{\infty} \frac{1}{w - c_R} \frac{d^2 w}{d\tilde{r}^2} \, d\tilde{r} < 0$$

$$(\xi_R w')'_{r=r_c} = \frac{1}{\xi_R}\left(\frac{d^2 w}{d\tilde{r}^2}\right)_{\tilde{r}=\tilde{r}_c} = 0 \qquad \text{(F. 7)}$$

$$(\xi_R w')''_{r=r_c} = \frac{1}{\xi_R^2}\left(\frac{d^3 w}{d\tilde{r}^3}\right)_{\tilde{r}=\tilde{r}_c}$$

For most profiles, $(\xi_r w')'$ and hence $d^2w/d\tilde{r}^2$ changes sign only once in the infinite interval and from Eq. (F. 7), $(w - c_R)$ and $d^2 w/d\tilde{r}^2$ must have opposite signs. Therefore, for neutral disturbances,

$|dw/d\tilde{r}|$ must have a maximum with respect to $\tilde{r}$, i. e.,

$(d^3 w/d\tilde{r}^3)_{\tilde{r}=\tilde{r}_c} \neq 0$ and consequently, $(\xi_R w')''_{r=r_c} \neq 0$.

This result cannot be shown for amplified disturbances except in the limiting case of incompressible flow[36].

APPENDIX G

VISCOUS CORRECTIONS IN THE CRITICAL LAYER

In considering inviscid neutral disturbances, a critical point occurs in the flow field, across which some of the disturbance amplitudes are singular [Section V. 3]. In a real fluid these singularities must be smoothed out by the action of viscosity and conductivity in the neighborhood of this critical point. These viscous corrections are important for the amplitude distributions but they do not affect the eigenvalue problem for $\alpha R >> 1$. However, if αR is not very much greater than unity, the viscous corrections around the critical layer may extend to the axis and the splitting of the solutions into inviscid and viscous types is not valid. In addition, the temperature and density fluctuations are singular at this point, and the thermal conductivity of the fluid must be included in the vicinity of this point to smooth out these discontinuities. It is to be expected that the viscous solutions for the axi-symmetric case are similar to those for the two-dimensional case except for the new element associated with the singularity in q_3 since the curvature effects in a thin annulus in the neighborhood of the critical point are unimportant. The incompressible case will be the only one considered here. The compressible problem is the same as the incompressible one in the Tollmien variable[59] and will not be discussed.

The solutions corrected for viscosity are given by

$$\begin{aligned}\left[\text{corrected solutions}\right] &= \left[\begin{array}{l}\left[\text{inviscid solution}\right] - \left[\text{singular terms}\right]\\ + \left[\text{viscous replacement term}\right]\end{array}\right]\\ &= \begin{array}{l}\left[\text{regular inviscid solution}\right]\\ + \left[\text{viscous replacement term}\right]\end{array}\end{aligned} ,$$

where the viscous replacement function is obtained by solving the full viscous disturbance equations in the vicinity of the critical point, i. e., retaining only the leading viscous terms in this region. This function must be such that it approaches the singular terms in the inviscid solution "far away" from the critical layer. The viscous replacement terms are found using the convergent series method[22]. Introduce the parameter

$$\epsilon = 1/(\alpha R)^{1/3} \quad , \tag{G. 1}$$

as in the two dimensional case, and the new independent variable

$$\eta = (r - r_c)/\epsilon \quad . \tag{G. 2}$$

The mean flow quantities are expanded in a Taylor series about the critical point

$$\begin{aligned} w - c &= w_c'(\epsilon\eta) + \frac{w_c''}{2}(\epsilon\eta)^2 + \cdots \\ w' &= w_c' + \end{aligned} \tag{G. 3}$$

Eqs. (2. 32) - (2. 35) then take the following forms

$$\frac{1}{\epsilon} q_r'\left[1+\cdot\right] + \frac{i m_c}{\alpha} q_1\left[1+\cdot\right] = 0 \tag{G. 4}$$

$$i\eta w_c' \epsilon q_r\left[1+\cdots\right] = -\frac{1}{\alpha^2\epsilon}\pi'\left[1+\cdot\cdot\right] + \epsilon\left[q_r'' + \cdots\right] \tag{G. 5}$$

$$i\eta w_c' \epsilon q_3\left[1+\cdots\right] - w_c' q_r \frac{n}{m_c r_c}\left[1+\cdot\right] = \epsilon\left[q_3'' + \cdot\cdot\right] \tag{G. 6}$$

$$i\eta w_c' \epsilon q_1\left[1+\cdot\cdot\right] + w_c' q_r \frac{\alpha}{m_c}\left[1+\cdot\cdot\right] = -\frac{m_c}{\alpha}\pi\left[1+\cdot\right] + \epsilon\left[q_1'' + \cdot\cdot\right] \tag{G. 7}$$

where $m_c = \sqrt{\alpha^2 + n^2/r_c^2}$

In order for Eqs. (G. 4) - (G. 7) to be consistent, the disturbance amplitudes must be of the following form

$$
\begin{aligned}
q_1 &= q_1^{(0)} + \epsilon q_1^{(1)} + \cdot \\
q_r &= \epsilon q_r^{(0)} + \epsilon^2 q_r^{(1)} + \cdot \\
\pi &= \epsilon^3 \pi^{(0)} + \epsilon^4 \pi^{(1)} + \cdot \\
q_3 &= q_3^{(0)} + \epsilon q_3^{(1)}
\end{aligned}
\tag{G. 8}
$$

Substituting Eq. (G. 8) into Eq. (G. 4) - (G. 7) and eliminating the pressure, the following zeroeth order equations are obtained:

$$q_r^{(0)\prime} = -\frac{i m_c}{\alpha} q_1^{(0)} \tag{G. 9a}$$

$$q_3^{(0)\prime\prime} - i\omega_c'\eta\, q_3^{(0)} = -\frac{n\,\omega_c'}{m_c r_c} q_r^{(0)} \tag{G. 9b}$$

$$q_1^{(0)\prime\prime\prime} - i\omega_c'\eta\, q_1^{(0)\prime} = 0 \tag{G. 10}$$

The solutions of Eq. (G. 10) are

$$
\begin{aligned}
q_{1_1}^{(0)} &= \int_{\infty}^{z} h_1(z)\, dz \\
q_{1_2}^{(0)} &= \int_{-\infty}^{z} h_2(z)\, dz \\
q_{1_3}^{(0)} &= 1 \\
q_{1_4}^{(0)} &= 0
\end{aligned}
\tag{G. 11}
$$

where

$$z = (w_c')^{1/3}\eta = (\alpha R w_c')^{1/3}(r - r_c)$$

$$h_1(z) = \left[\frac{2}{3}(iz)^{3/2}\right] H^{(1)}_{1/3}\left[\frac{2}{3}(iz)^{3/2}\right]$$

$$h_2(z) = \left[\frac{2}{3}(iz)^{3/2}\right] H^{(2)}_{1/3}\left[\frac{2}{3}(iz)^{3/2}\right] \qquad \text{(G. 12)}$$

and $H^{(1)}_{1/3}\left[\frac{2}{3}(iz)^{3/2}\right]$ and $H^{(2)}_{1/3}\left[\frac{2}{3}(iz)^{3/2}\right]$ are Hankel functions of order (1/3) and the first and second kind respectively. The asymptotic expansions of the Hankel functions of order (1/3) are valid in the following region (Lin[22]).

$$-7\pi/6 < \arg\,(w_c')^{1/3}\eta < \pi/6 \qquad \text{(G. 13)}$$

The solutions obtained by means of an asymptotic series (of the full viscous equations) can be formally related to the asymptotic expansions of the four solutions obtained by the method of convergent series (Lin[22]). The solutions of the inviscid equations are two of the asymptotic solutions of the full viscous equations. Therefore in order that the inviscid solutions represent valid asymptotic solutions of the full viscous equations, the correct path of integration around the singular point should follow the same criterion as Eq. (G. 13), and should lie below the singular point for $w_c' > 0$ and above for $w_c' < 0$.

If $c_I > 0$, the singular point of the inviscid equation lies above the real axis, and the effect of viscosity can be neglected inside the fluid for sufficiently large Reynolds numbers. If $c_I = 0$, the two lines

$$-7\pi/6 < \arg\,(w_c')^{1/3}\eta < \pi/6$$

intersect at a single point on the real axis, and the inviscid solutions can never hold along the entire real axis. Viscosity cannot be neglected at the singular point no matter how large the Reynolds number may be. For $c_I < 0$ the two lines intersect the real axis at two points and viscosity is important all along the real axis between the two intersections.

$q_r^{(0)}$ can be determined directly from Eq. (G. 9) and Eq. (G. 11)

$$
\begin{aligned}
q_{r_1}^{(0)} &= -\frac{i m_c}{\alpha (w_c')^{1/3}} \int_{\infty}^{z} dz \int_{\infty}^{z} h_1(z)\, dz \\
q_{r_2}^{(0)} &= -\frac{i m_c}{\alpha (w_c')^{1/3}} \int_{-\infty}^{z} dz \int_{-\infty}^{z} h_2(z)\, dz \\
q_{r_3}^{(0)} &= -\frac{i m_c}{\alpha (w_c')^{1/3}}\, z \\
q_{r_4}^{(0)} &=
\end{aligned}
\qquad \text{(G. 14)}
$$

The solutions Eqs. (G. 11) and (G.14) are identical to those for the two dimensional case.

Rewrite Eq. (G.9b) in terms of the independent variable z

$$
\frac{d^2 q_3^{(0)}}{dz^2} - i z\, q_3^{(0)} = \frac{-n\,(w_c')^{1/3}}{m_c r_c}\, q_r^{(0)} \qquad \text{(G. 15)}
$$

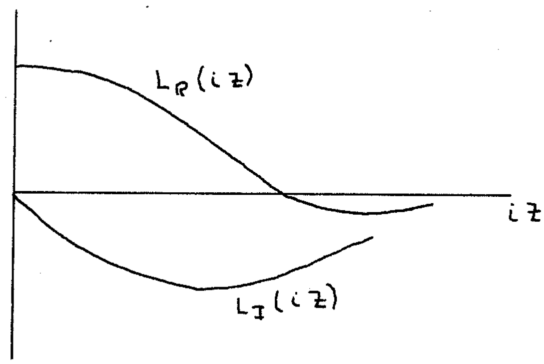
By the method of variation of constants, the solution of Eq. (G. 15) is

$$
q_3^{(0)} = \left[\frac{-n\,(w_c')^{1/3}/m_c r_c}{W\left(h_1(z), h_2(z)\right)} \right] \left[h_2(z) \int_{\infty}^{z} h_1(z)\, q_r^{(0)}\, dz \right.
\qquad \text{(G. 16)}
$$

$$
\left. - h_2(z) \int_{-\infty}^{z} h_2(z)\, q_r^{(0)}\, dz \right]
$$

where $W\left[\, h_1(z) \;\; h_2(z) \,\right]$ is the Wronskian of the functions $h_1(z)$ and $h_2(z)$, and $-(2/3)\,\pi < \arg\,(i\, z) < (2/3)\,\pi$ or $-(7\pi/6) < \arg z < (\pi/6)$.

For $q_r^{o} = q_{r_4}^{(0)} = 1$, $q_3^{(0)}$ is a Lommel function, L(Z), [Benney,[62]]. The real part of the Lommel function is even and the imaginary part is an odd function of z. The graphs of $L_R(iz)$ and $L_I(iz)$ are shown below

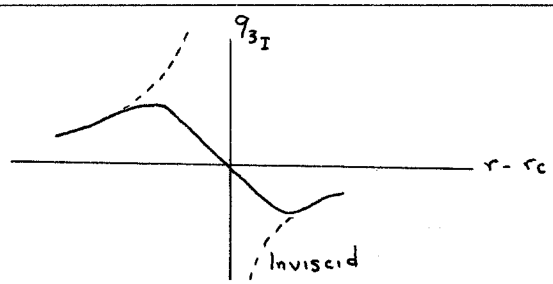


Sketch G. 1

For large values of Z

$$L_I\,(iz) \longrightarrow -\frac{1}{z} + \cdots \tag{G. 17}$$

The viscous corrections which apply for $z = 0(1)$, $r - r_c \sim 0\,(\alpha R)^{-1/3}$ will remove the singularity at the singular point, and the disturbance amplitude, q_{3_I}, in the vicinity of $r = r_c$ will look like



Sketch G. 2

The discontinuity is smeared out by the action of viscosity.

If the phase velocity is taken to be equal to the velocity of the mean flow on the axis, then the solution is singular at that point, and does not satisfy the boundary conditions. Again, a viscous replacement term must be found. The reader is referred to Lin[22] and Gill[37] for a discussion of this problem.

TABLE I

NEUTRAL, INVISCID STABILITY CHARACTERISTICS

T	c_R	η_c	M^2	a
0	-0.607	0.707	0 1 2 2.5	1.919 1.803 1.684 1.618
0.25	-0.565	0.753	0 1 2 3	1.564 1.467 1.370 1.266
0.50	-0.531	0.795	0 1 2 3.5	1.335 1.251 1.169 1.035
0.75	-0.502	0.830	0 1	1.202 1.126
1.0	-0.478	0.860	0 1 2 4.3	1.057 0.992 0.928 0.782
2.0	-0.405	0.951	0	0.849
3.0	-0.356	1.017	0	0.730
4.0	-0.320	1.068	0	0.674
5.0	-0.292	1.109	0	0.633

TABLE II

AMPLIFIED, INVISCID STABILITY CHARACTERISTICS

Case	ΔU	ΔT	M^2
1	0.692	0	0
2	0.285	0.50	0.42
3	0.160	0.38	0.17
4	0.083	0.30	0.07
5	0.049	0.20	0.03

Case	α	c_R	c_I	$\alpha \frac{dC_R}{d\alpha}$	$\frac{\alpha C_I}{\bar{C}_R}$	$\frac{\alpha C_I}{\bar{C}_g}$	$\alpha \bar{c}_R$
1	1.92	-0.61	0	-0.35	0	0	1.11
	1.80	-0.58	0.02	-0.32	0.06	0.10	1.08
	1.60	-0.55	0.06	-0.32	0.16	0.24	0.99
	1.40	-0.51	0.10	-0.28	0.22	0.31	0.91
	1.20	-0.47	0.14	-0.24	0.25	0.33	0.81
	1.00	-0.43	0.18	-0.20	0.26	0.32	0.70
	0.80	-0.39	0.23	-0.20	0.25	0.30	0.59
	0.60	-0.33	0.26	-0.15	0.20	0.23	0.46
	0.40	-0.26	0.28	-0.17	0.14	0.16	0.33
	0.20	-0.15	0.25	-0.13	0.06	0.06	0.18
2	1.32	-0.52	0	-0.34	0	0	1.12
	1.20	-0.49	0.04	-0.31	0.05	0.05	1.03
	1.00	-0.44	0.11	-0.26	0.12	0.13	0.88
	0.80	-0.39	0.18	-0.21	0.16	0.17	0.71
	0.60	-0.33	0.23	-0.17	0.15	0.16	0.54
	0.40	-0.27	0.26	-0.18	0.11	0.12	0.37
	0.20	-0.15	0.25	-0.15	0.05	0.06	0.19
3	1.43	-0.55	0	-0.36	0	0	1.30
	1.20	-0.49	0.07	-0.30	0.08	0.09	1.10
	1.00	-0.44	0.12	-0.25	0.13	0.14	0.93
	0.80	-0.39	0.18	-0.20	0.15	0.16	0.75
	0.60	-0.34	0.23	-0.18	0.15	0.15	0.57
	0.40	-0.26	0.27	-0.16	0.11	0.12	0.38
	0.20	-0.15	0.25	-0.12	0.05	0.05	0.20

TABLE II (CONTINUED)

Case	a	c_R	c_I	$\alpha \frac{dC_R}{d\alpha}$	$\frac{\alpha C_I}{\bar{C}_R}$	$\frac{\alpha C_I}{\bar{C}_g}$	$a\,\bar{c}_R$
4	1.51	-0.56	0	-0.38	0	0	1.44
	1.40	-0.53	0.03	-0.35	0.04	0.05	1.34
	1.20	-0.48	0.09	-0.30	0.11	0.11	1.16
	1.00	-0.43	0.14	-0.25	0.15	0.15	0.96
	0.80	-0.38	0.20	-0.20	0.16	0.17	0.77
	0.60	-0.33	0.24	-0.21	0.15	0.15	0.58
	0.40	-0.25	0.27	-0.15	0.11	0.11	0.39
	0.20	-0.16	0.25	-0.12	0.05	0.05	0.20
5	1.63	-0.58	0	-0.37	0	0	1.59
	1.40	-0.54	0.06	-0.32	0.08	0.08	1.36
	1.20	-0.48	0.11	-0.28	0.13	0.13	1.17
	1.00	-0.43	0.16	-0.23	0.16	0.16	0.98
	0.80	-0.39	0.20	-0.20	0.16	0.17	0.79
	0.60	-0.33	0.25	-0.19	0.15	0.15	0.59
	0.40	-0.25	0.28	-0.18	0.11	0.11	0.46
	0.20	-0.15	0.25	-0.13	0.05	0.05	0.20

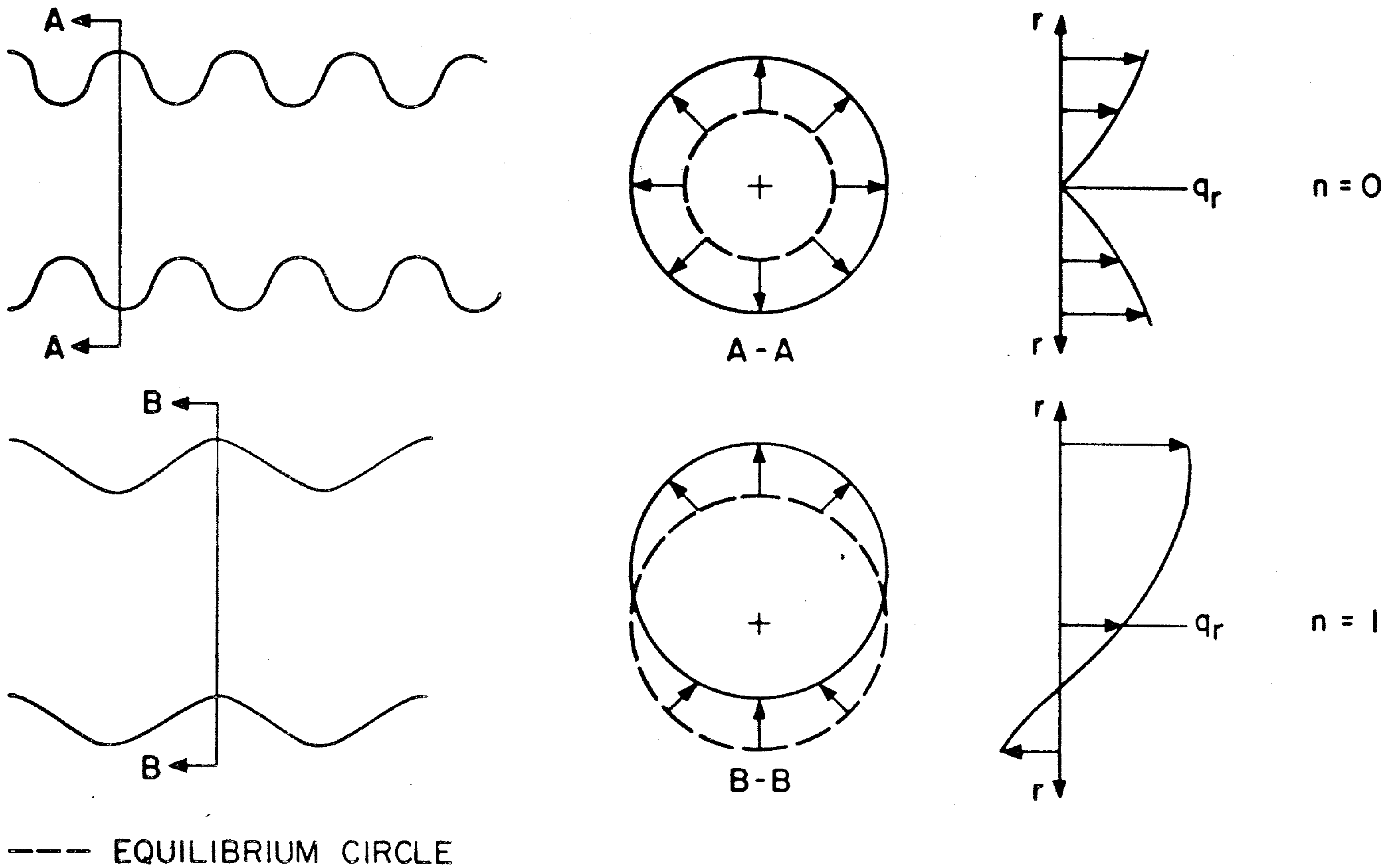


FIG. 1 MODES OF INSTABILITY

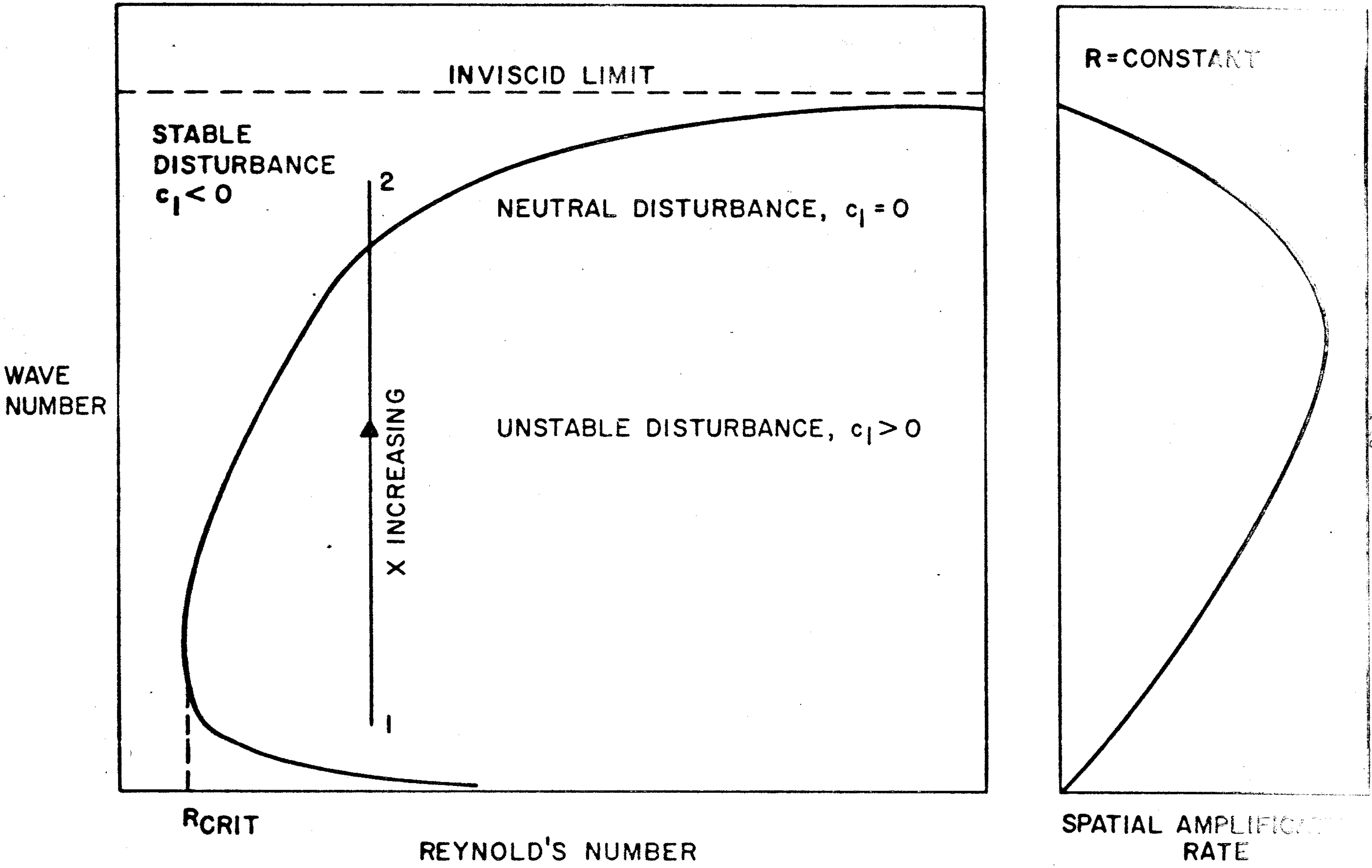


FIG. 2 TYPICAL NEUTRAL STABILITY CURVE FOR WAKE-TYPE PROFILES

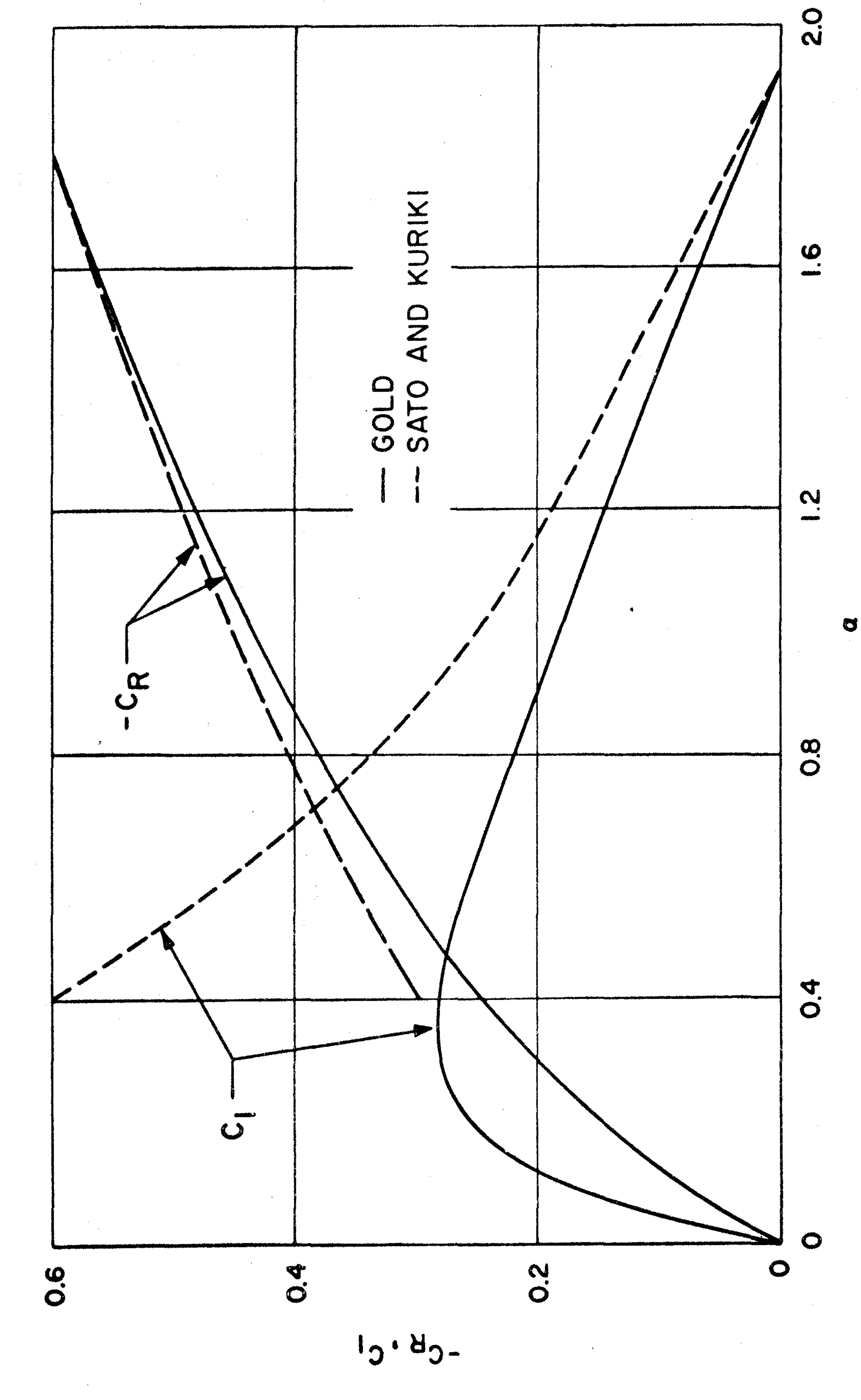


FIG. 3 WAVE NUMBER VS. COMPLEX WAVE VELOCITY

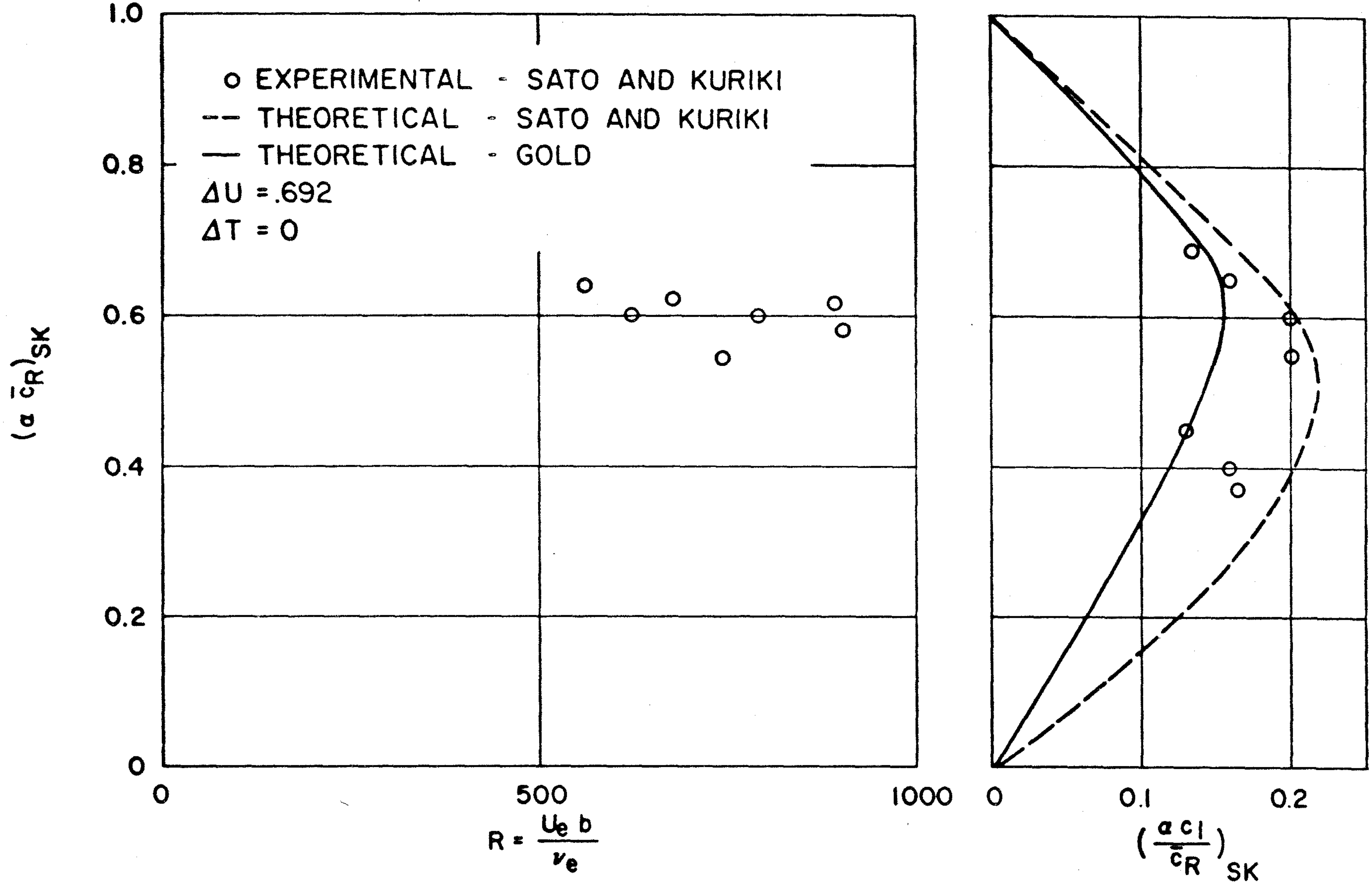


FIG. 4 FREQUENCY VS. REYNOLDS NO. AND SPATIAL AMPLIFICATION RATE

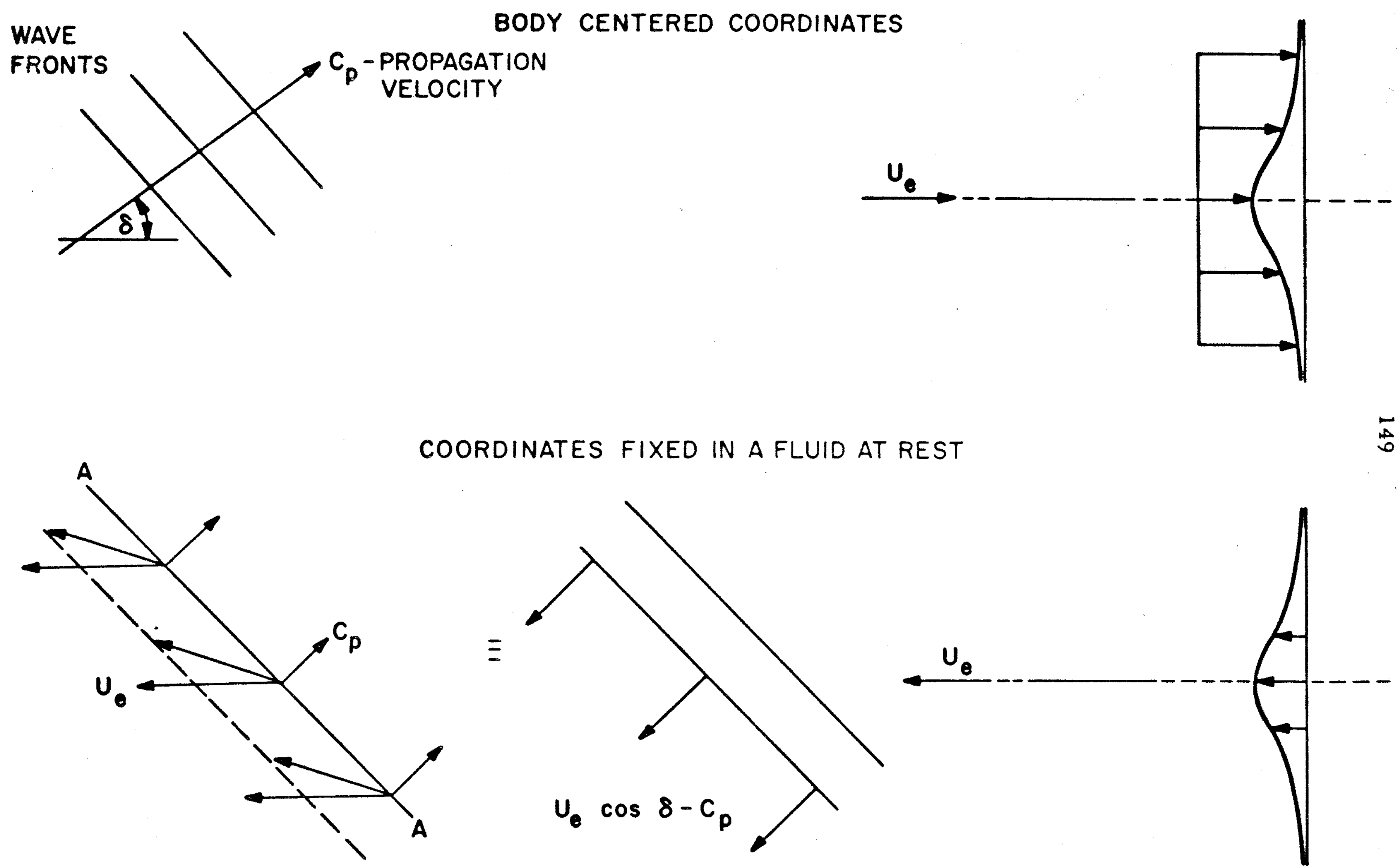


FIG. 5 DIRECTION OF PROPAGATION OF THE WAVE FRONTS

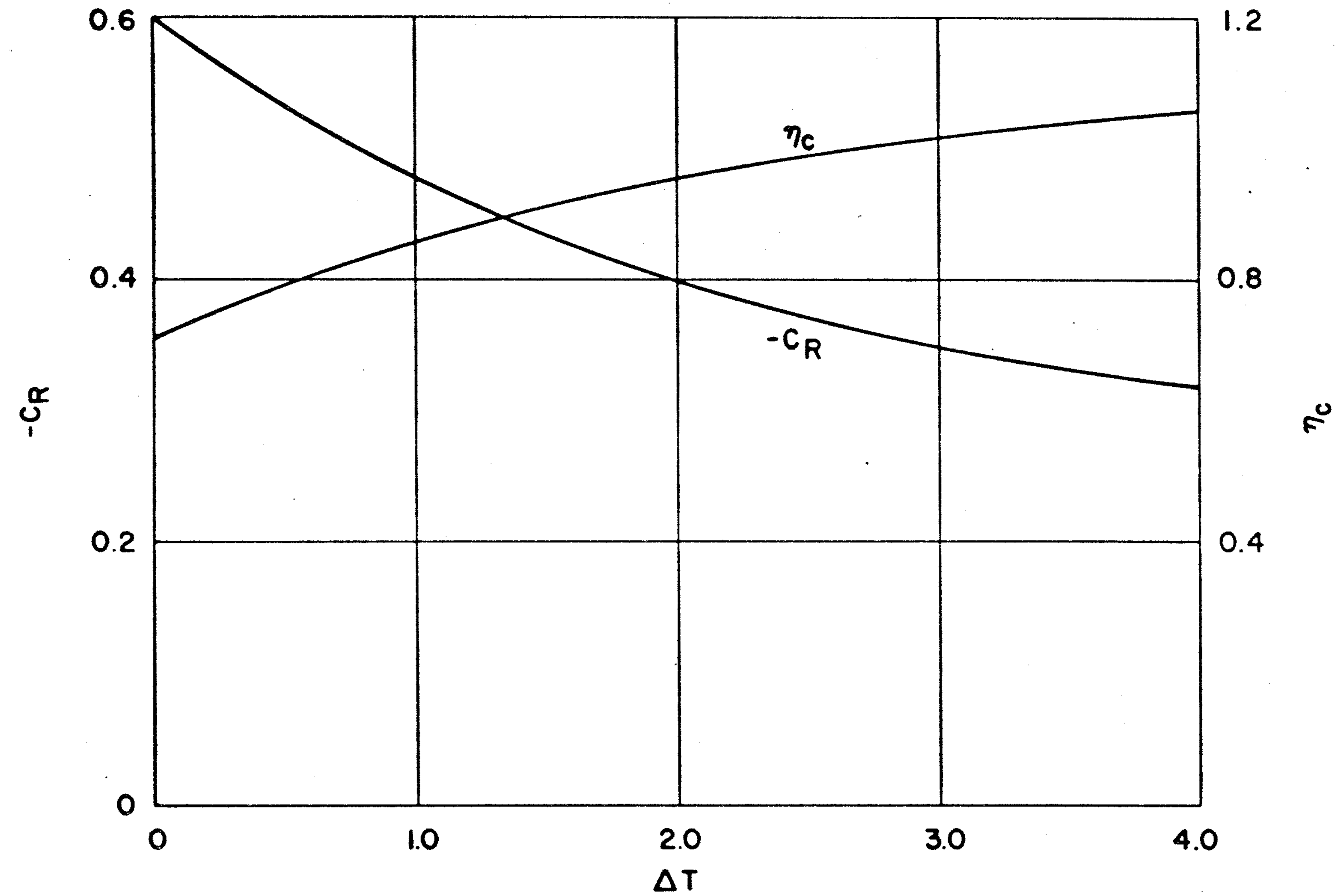


FIG. 6 RELATIVE WAVE VELOCITY AND CRITICAL POINT VS. TEMPERATURE DIFFERENCE

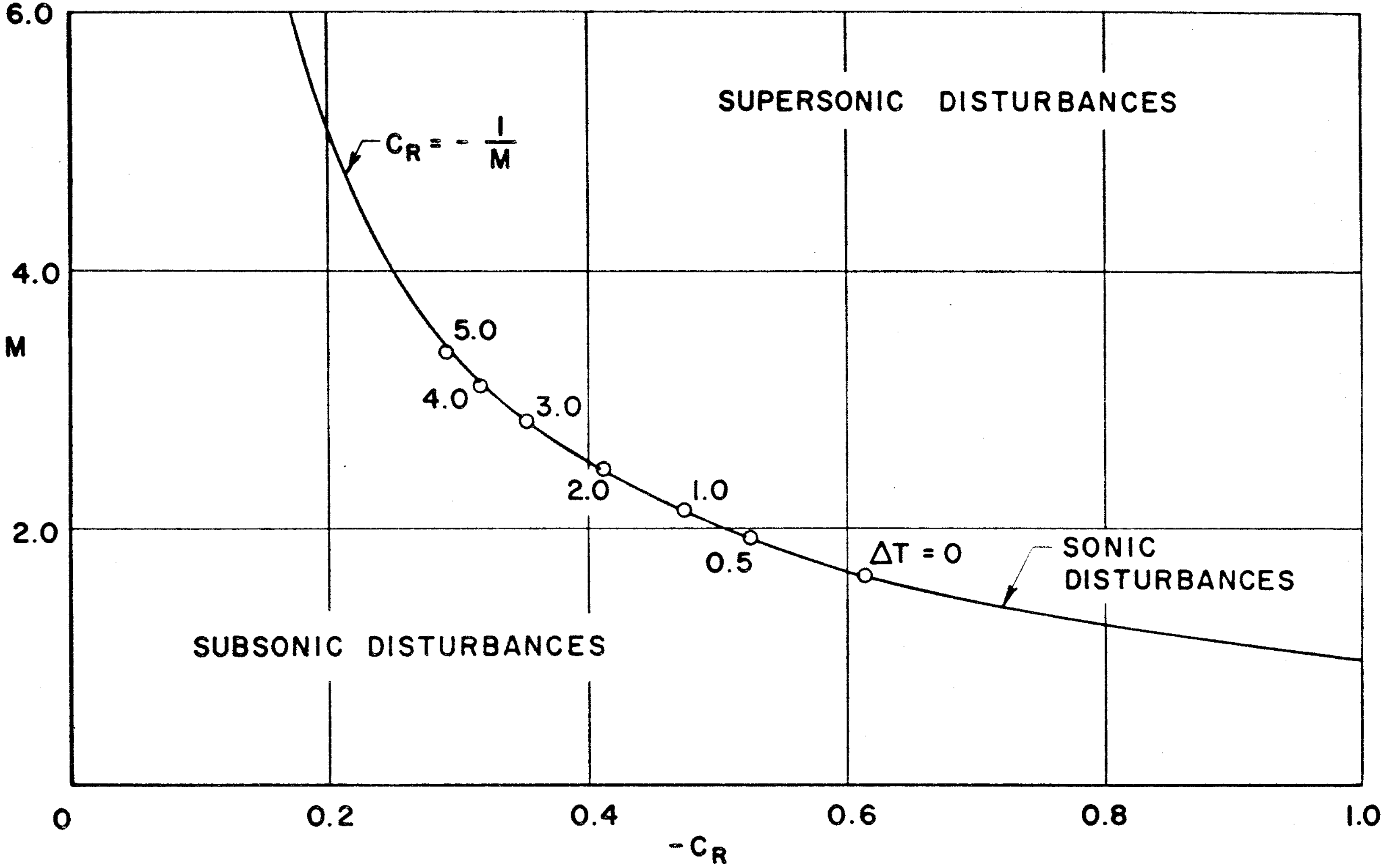


FIG. 7 RELATIVE WAVE VELOCITY VS. RELATIVE MACH NUMBER

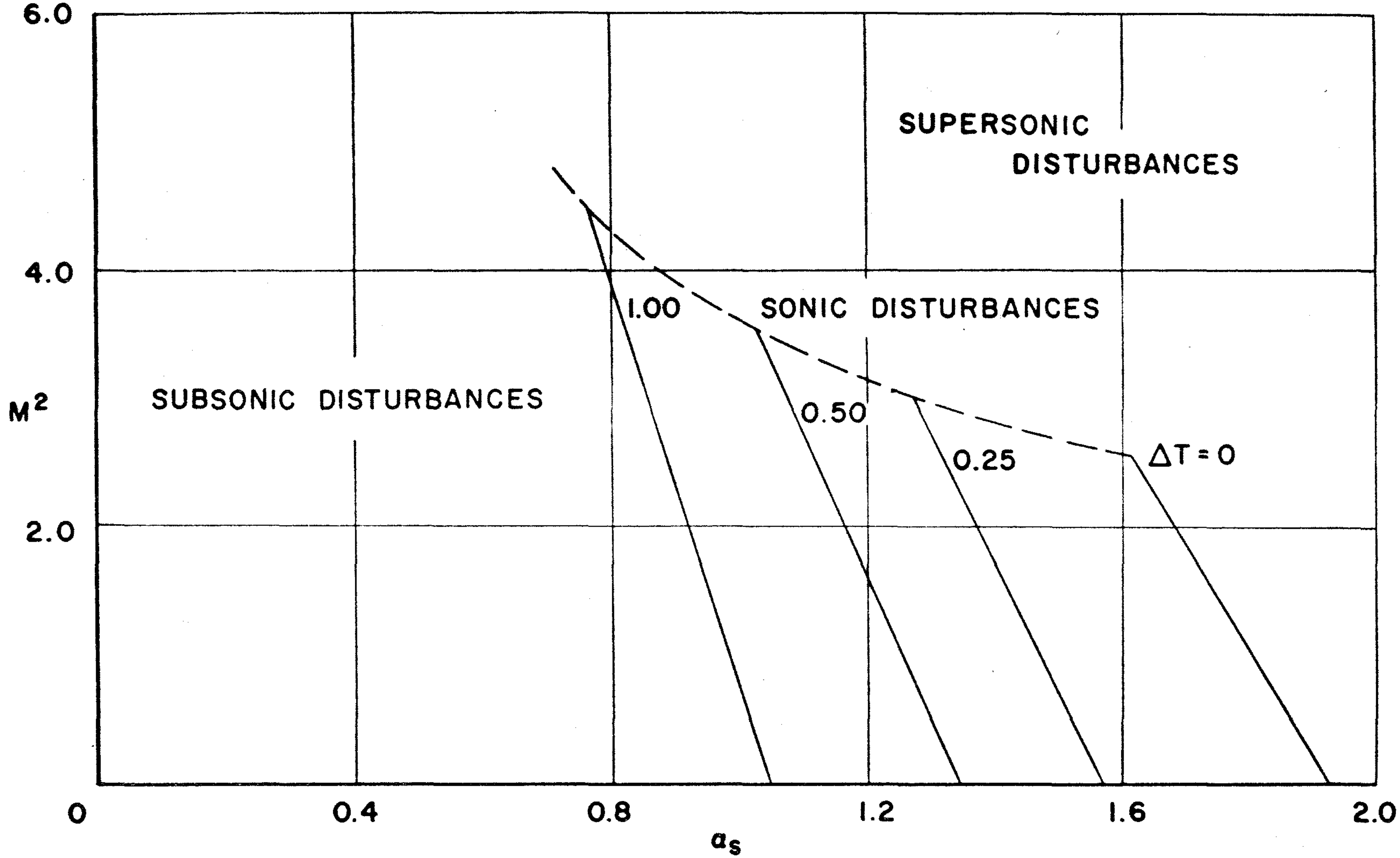


FIG. 8 INVISCID WAVE NUMBER VS. SQUARE OF RELATIVE MACH NUMBER

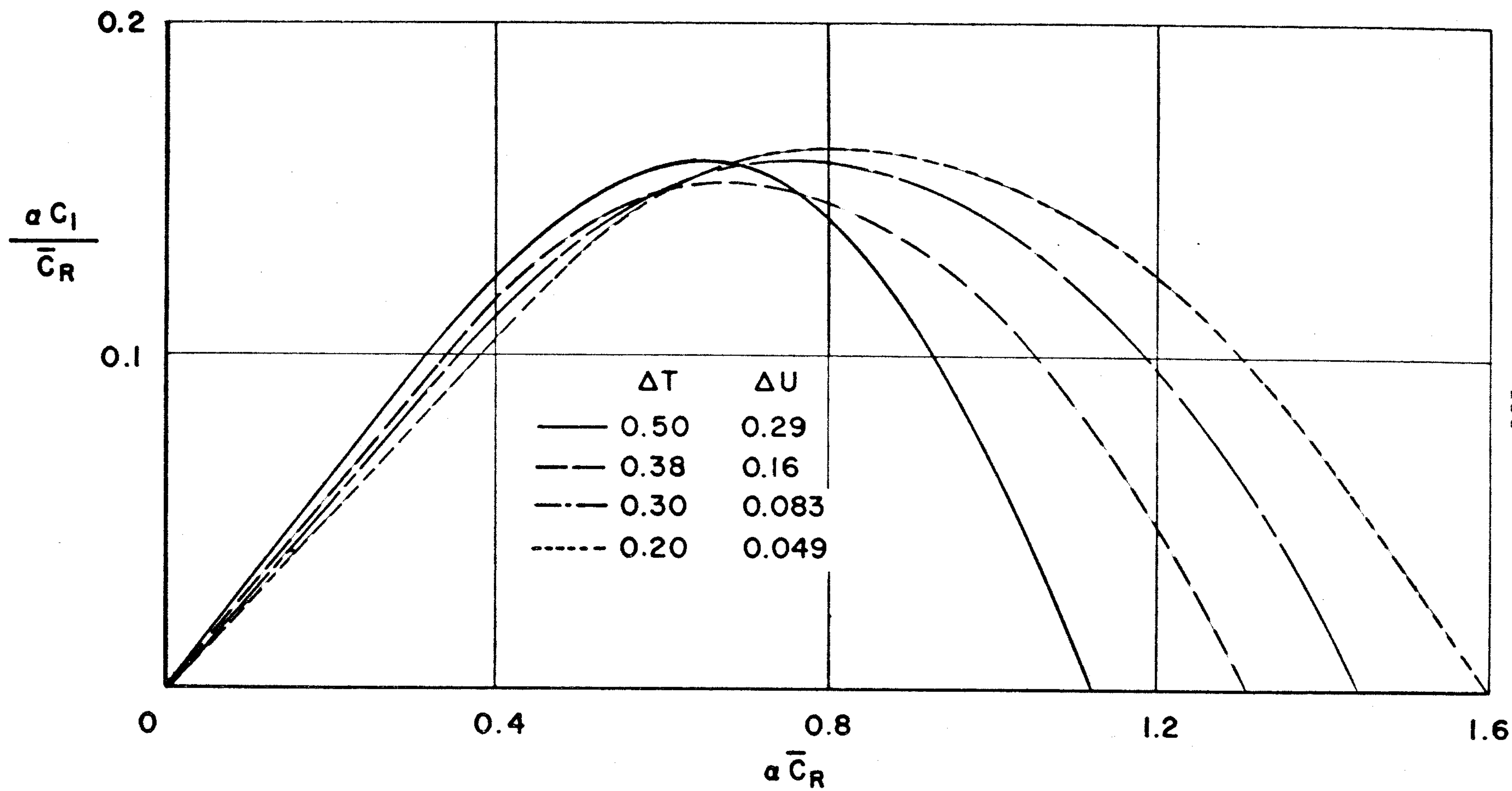
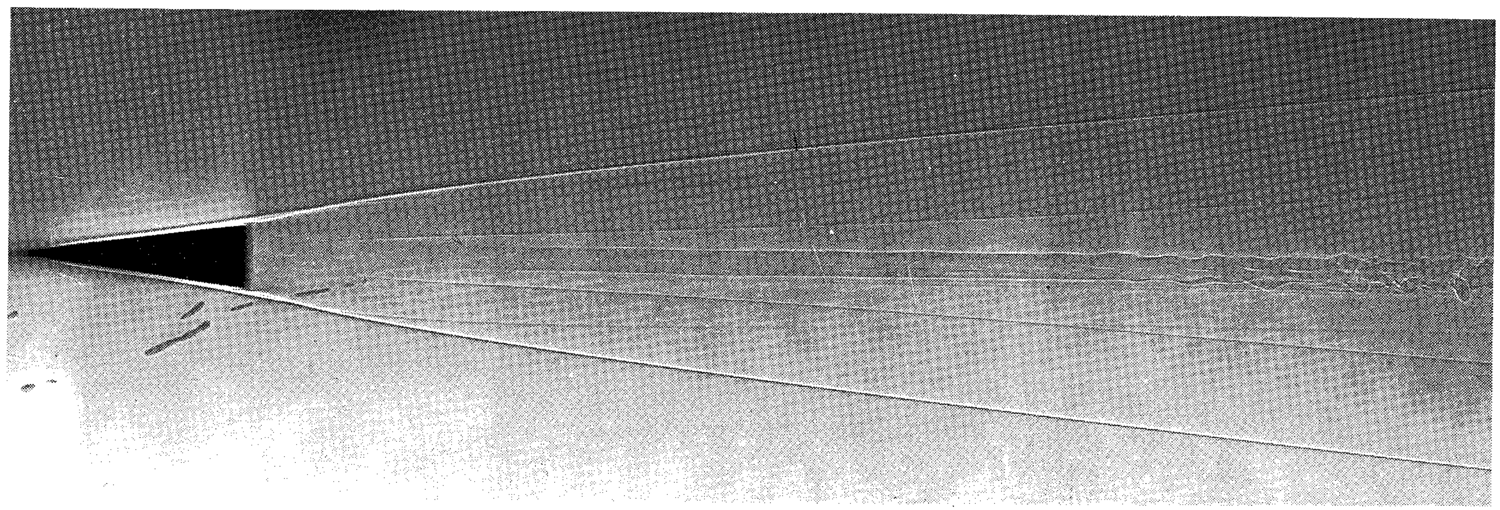


FIG. 9 DIMENSIONLESS FREQUENCY VS. SPATIAL AMPLIFICATION RATE



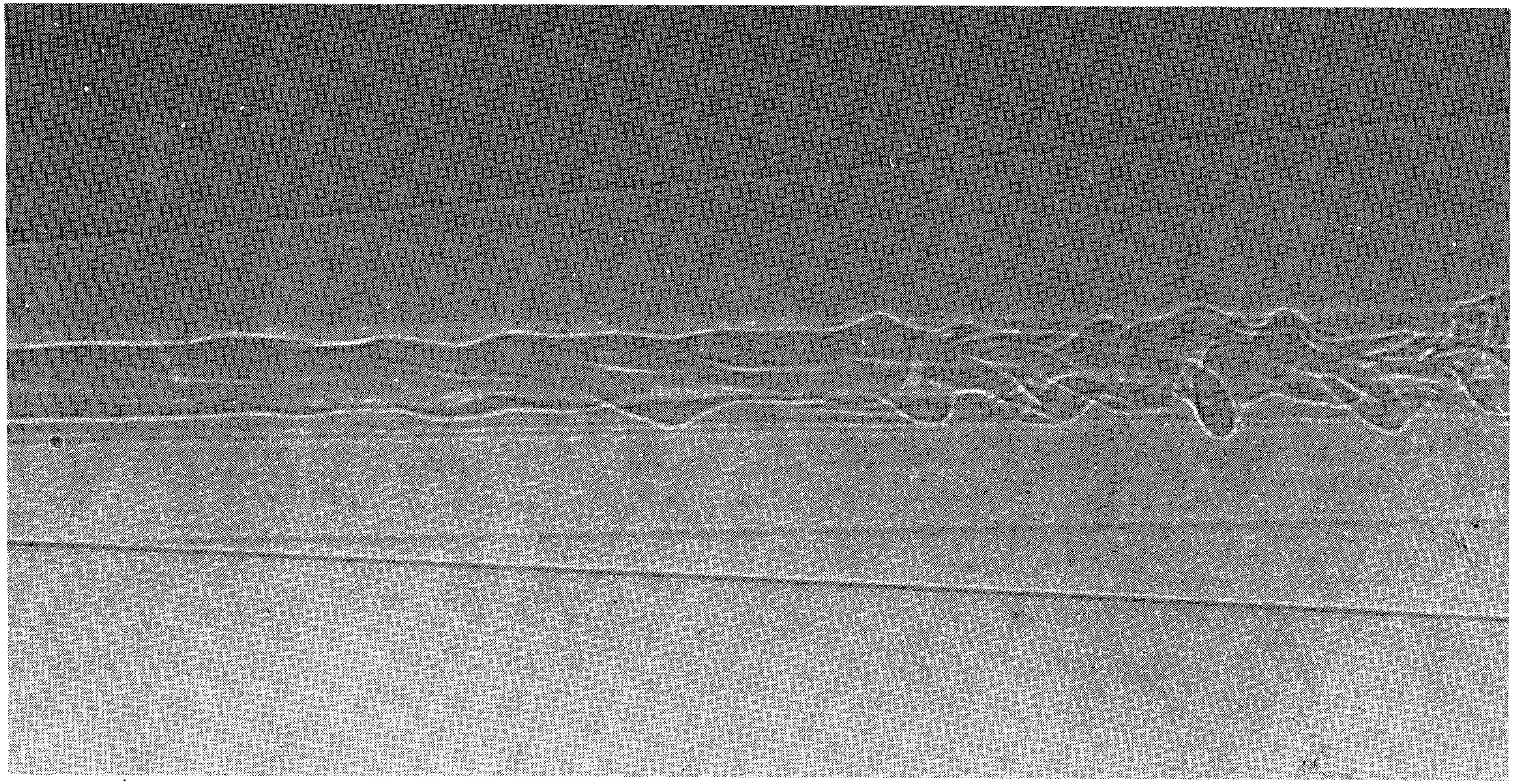
M_{∞} = 13

P = 100 mm. HG., AIR

BASE DIAMETER = 0.400"

CONE ANGLE = 16°

FIG. 10 (a) HYPERSONIC AXISYMMETRIC WAKE



n = 1 MODE OF INSTABILITY

FIG. 10 (b) HYPERSONIC AXISYMMETRIC WAKE