A THEORETICAL INVESTIGATION OF USING N LOW THRUST IMPULSES TO ESCAPE FROM A CIRCULAR SATELLITE ORBIT

Thesis by

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ABSTRACT

An investigation was made into the advantages of using a discontinuous thrust program to escape from a cricular satellite orbit when using a low thrust propulsion unit. To make maximum use of the applied thrust, the perigee distance was chosen at a point where the atmospheric drag was negligible and this distance was held constant throughout the escape maneuver. A numerical integration was made of this method and the spiralling method. The appendixes show a comparison of these two methods.

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TABLE OF SYMBOLS

a	semi major axis of orbit
g_o	gravity at the surface of the earth (32.3 ft/sec^2)
R_o	mean radius of the earth (4000 miles)
r	radius to any point in the orbit
r_a	radius at apogee
r_p	radius at perigee
μ	gravitational force per unit mass
ℓ	angular momentum of satellite
T	kinetic energy
V	potential energy
T	thrust time
V	velocity at any point
ϵ	eccentricity of the ellipse
$\dot{r}$	derivative with respect to time
$\dot{\theta}$	derivative with respect to time
θ	angle of the radius measured from the periqpsis
δ_1	angle the velocity vector makes with angular velocity vector
δ_2	thrust angle
U_θ	angular velocity component
U_r	radial velocity component
$\lvert\dot{V}\rvert$	absolute value of acceleration due to thrust

INTRODUCTION

As man's scientific and engineering advances in space sciences expand, the ageless desire to explore interplanetary space has become not only a possibility, but a distinct probability. To provide the propulsion for such flight many new types of engines are being developed. Most of these engines are of the low thrust to field force type (i.e. of the order of 10^{-3} to 10^{-4}g). These thrust devices are especially adapted to long space flight because of the high specific impulse and long burning times that they can provide. Specific impulses of 500 to 20,000 seconds can be developed by certain low thrust engines, and burning times are practically unlimited. These features allow for more flexibility in space flight than high thrust chemically powered rockets. Using a low thrust vehicle, adjustments in trajectory can be made easily thereby reducing the need for extremely high precision in establishing the original flight path. Other advantages of the low thrust engines over the chemical engines are their small size and low propellant consumption. In certain types of low thrust propulsion units the power can be taken directly from the sun and thereby reduce the amount of propellant which must be carried from the earth. With this practically unlimited power source it will be possible to discard the present concepts of using minimum energy orbits. Also, by having power available for the whole flight, long space flights can be accomplished in much shorter time than with a chemical engine. For example a vehicle traveling to the vicinity of Saturn using a low thrust for most of the flight can make the trip in about two and a half years, whereas the same trip using a high thrust chemically powered rocket and minimum energy transfer trajectory would take about six and a half years.

The one important job that the low thrust engine cannot accomplish is that of lifting off from the launching site and reaching the initial parking orbit. This job must be left to the high thrust chemically powered engines.

Once the vehicle is placed in the parking circular orbit, the first major task that the low thrust engine must accomplish is to escape from the gravitational sphere of the earth. It is the investigation of this phase of the space flight with which this thesis is concerned.

The problem of providing escape velocity for a vehicle from a circular orbit has been investigated by many people. It has been shown that the least energy is required when a single high thrust impulse is applied tangentially to the flight path of the vehicle. This requires a thrust of the order of 1 g. Also, various authors have investigated the use of a continuous low thrust applied tangentially producing a circular spiral of ever increasing radius. This spiralling technique although requiring less time to escape is not the most efficient from an energy standpoint.

A somewhat more efficient method to use to reach escape velocity with a low thrust propulsion device, is to apply an infinite number of infinitesimal impulses at perigee. This would result in an elliptical trajectory with an ever increasing semi major axis. It would require the same amount of energy as the single high thrust impulse method previously discussed. However, the infinite number of impulses would require an infinite time to produce escape velocity.

This paper is devoted to developing a technique which lies somewhere between the spiralling and infinite impulse methods. Since it takes more than twice the energy for escape using a 10^{-4} g thrust spiralling method than using the infinitesimal impulses at perigee method, it is apparent that there is a method of applying a 10^{-4} g thrust over a finite portion of the elliptical path near perigee which is more efficient than the spiralling method.

To obtain maximum effectiveness of the applied thrust, the perigee is held constant since in any low thrust acceleration program emphasis should be placed on maximizing the rate of increase of kinetic energy. This can best be accomplished by keeping the perigee as low as possible. In this paper a parking orbit of 300 miles is chosen to insure that no energy is lost due to atmospheric drag. This is an arbitrary choice but does not alter the basic theory. The perigee is maintained at 300 miles, because allowing it to decrease would put it into an area where atmospheric drag would use up the applied energy.

1. BASIC ORBITAL MECHANICS EQUATIONS

In the development of the basic orbital equations of motion and resulting parameter relationships, the usual assumption of the earth being spherically shaped with homogeneous mass distribution is used. Also, the assumption is made that the other celestial bodies do not exert a gravitational perturbing force on the vehicle. The equations can then be derived for a point mass (vehicle) moving in an inverse square gravitational field of another larger point mass (the earth).

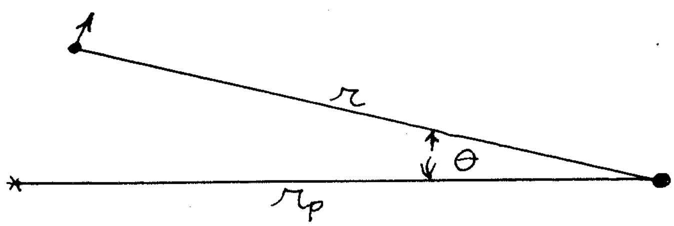

Fig. 1

Using the Lagrangian method, the equations of motion are:

$$L = T - V = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{\mu}{r}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right) = 0 = \frac{d}{dt}\left(r^2\dot{\theta}\right) \qquad 1\text{-}1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \left(\frac{\partial L}{\partial r}\right) = 0 = \ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2} \qquad 1\text{-}2$$

where $\mu = g_0 R_0^2$ (the gravitational force per unit mass).

The solution of equation 1-1 gives

$$r^2 \dot{\theta} = \text{Const.} = l \qquad 1\text{-}3$$

That is, the angular momentum is conserved. Now using equation 1-3 and solving for $\dot{\theta}$

$$\dot{\theta} = \frac{l}{r^2} \qquad 1\text{-}3a$$

which upon substitution into 1-2 gives

$$\ddot{r} - \frac{l^2}{r^3} = -\frac{\mu}{r^2} \qquad 1\text{-}4$$

Since the two most appropriate generalized coordinates for this problem are r and θ it is best to eliminate the variable t explicitly from the equations. This can be done by writing

$$\ddot{r} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{d\theta}{dt}\frac{d}{d\theta}\left(\frac{dr}{d\theta}\frac{d\theta}{dt}\right)$$

$$\ddot{r} = \dot{\theta}\frac{d}{d\theta}\left(\frac{dr}{d\theta}\dot{\theta}\right) = \frac{l^2}{r^2}\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)$$

equation 1-4 then becomes

$$\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) - \frac{1}{r} = -\frac{\mu}{l^2} \qquad 1\text{-}5$$

If a change of variable is made, i.e. $U = \frac{1}{r}$, equation 1-5 becomes

$$\frac{d^2 u}{d\theta^2} + u = \frac{\mu}{l^2}$$

the solution of which is

$$U = \frac{\mu}{\ell^2} + A \cos(\theta - \theta_0) = \frac{1}{r} \qquad 1\text{-}6$$

Now the general equation of a conic section is

$$r = \frac{P}{1 + \epsilon \cos(\theta - \theta_0)} \qquad 1\text{-}7$$

If we compare equations 1-6 and 1-7 we see that by letting be $A = \frac{\epsilon \mu}{\ell^2}$, equation 1-6 is in the form of a conic section, i.e.

$$r = \frac{\ell^2 / \mu}{1 + \epsilon \cos(\theta - \theta_0)} \qquad 1\text{-}7a$$

From elementary analytical geometry the parameter P for an ellipse can be written as

$$P = a(1 - \epsilon^2)$$

Combining this with equation 1-7 we get the general equation for an elliptical orbit

$$r = \frac{a(1 - \epsilon^2)}{1 + \epsilon \cos(\theta - \theta_0)} \qquad 1\text{-}8$$

where a is the semi-major axis, ϵ the eccentricity, θ the angle measured from the periapsis, and r the distance from the central focus (the earth). Using this equation and the solution to the general orbital equation (1-6) certain identities can be expressed. These identities will be used in the derivation of the equations for the solution of the problem with which this thesis is concerned.

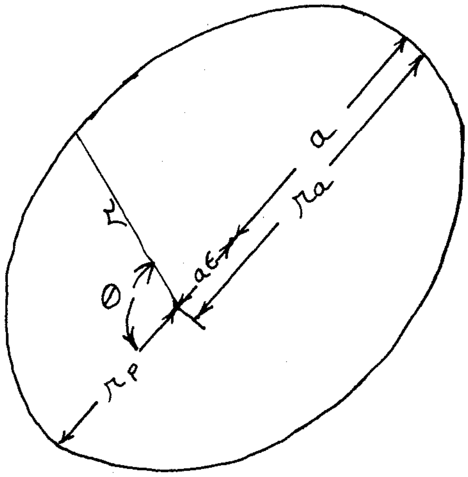

Fig. 2.

$$2a = r_a + r_p \qquad 1\text{-}9$$

$$r_a = a(1+\epsilon) \qquad 1\text{-}10$$

$$r_p = a(1-\epsilon) \qquad 1\text{-}11$$

$$l = \sqrt{\mu a(1-\epsilon)} \qquad 1\text{-}12$$

let $\theta_0 = 0 \longrightarrow$

$$r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos\theta} \qquad 1\text{-}13$$

The one remaining basic equation needed to begin the solution of the thesis problem is an energy equation. Writing the equation for total energy we get

$$E = T + V = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{\mu}{r} \qquad 1\text{-}14$$

but $(\dot{r}^2 + r^2\dot{\theta}^2)^{\frac{1}{2}}$ is simply the absolute value of the velocity vector. Therefore this equation can be written

$$E = \frac{1}{2}V^2 - \frac{\mu}{r} \qquad 1\text{-}15$$

It can be shown that*

$$E = \frac{\mu^2}{2l^2}(\epsilon^2 - 1) \qquad 1\text{-}16$$

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* Goldstein - Classical Mechanics, page 78.

Therefore the final energy relationship can be obtained by combining equation 1-12 into equation 1-16 giving an equation for the total energy of any orbit, i.e.

$$E = -\frac{\mu}{2a} \qquad 1\text{-}17$$

If 1-17 is then substituted into 1-15 we get the familiar Vis Viva equation which relates velocity to the radius and semi major axis of a conic section, namely

$$V^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \qquad 1\text{-}18$$

2. THRUST VECTOR ORIENTATION FOR CONSTANT Rp

Using the basic orbital equations the problem now becomes one of orienting the thrust vector such that the perigee distance remains constant. In this way the optimum use of the applied energy will be accomplished. It should be recalled that the particular perigee distance was chosen as the distance where atmospheric drag can be neglected. Therefore, we do not want to decrease r_p below its initial value. Allowing it to increase would be less efficient also. Therefore the optimum energy trajectory requires that the perigee distanceremain constant. In the derivation of the equations that follow it is assumed that the change in the orbital elements is very small on each orbit. From figure 3

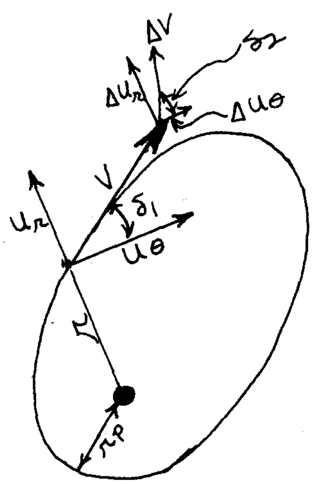


Fig. 3.

we see that the velocity vector can be expressed by the two components:

$$U_\theta = V \cos \delta_1$$

$$U_r = V \sin \delta_1$$

Then to develop an equation which will make an incremental increase in the velocity vector while keeping the perigee distance constant, we can take the total differential of r_p as a function of the velocity components, and set it equal to zero. This gives:

$$dr_p = \frac{\partial r_p}{\partial U_r} dU_r + \frac{\partial r_p}{\partial U_\theta} dU_\theta = 0$$

which leads to

$$\frac{dU_r}{dU_\theta} = - \frac{\frac{\partial r_p}{\partial U_\theta}}{\frac{\partial r_p}{\partial U_r}} = \tan \delta_2$$

The problem is now reduced to one of finding a relationship between r_p, Ur, and U_θ, and from this determining the angle δ_2, which is the orientation of the thrust vector to give a constant perigee distance.

Using the energy equation 1-18 we can write

$$2a - \frac{aV^2r}{\mu} = r \qquad 2\text{-}1$$

If we now substitute equation 1-9, 1-10 and 1-11 into 2-1 we get

$$r_p = r - a\left[1 + \epsilon - \frac{V^2r}{\mu}\right] \qquad 2\text{-}2$$

Then using the conservation of angular momentum equation, ϵ can be expressed in terms of the angle δ_1, i.e.

$$r^2(V\cos\delta_1)^2 = r^4\dot{\theta}^2 = \mu a(1-\epsilon^2)$$

and substituting for a in terms of r and v gives:

$$\epsilon = \left\{1 - \frac{2V^2r}{\mu}\cos^2\delta_1 - \left(\frac{V^2r}{\mu}\right)^2\cos^2\delta_1\right\}^{\frac{1}{2}}$$

which can be written as

$$\epsilon = \left\{\sin^2\delta_1 + \cos^2\delta_1\left(1 - \frac{V^2r}{\mu}\right)^2\right\}^{\frac{1}{2}} \qquad 2\text{-}3$$

Now substituting equation 2-3 into equation 2-2 gives:

$$r_p = r\left\{1 - \frac{1}{2-\frac{V^2 r}{\mu}}\left[1 + \left(\sin^2\delta_1 + \cos^2\delta_1\left[1-\frac{V^2 r}{\mu}\right]^2\right)^{\frac{1}{2}} - \frac{V^2 r}{\mu}\right]\right\} \qquad 2\text{-}4$$

Then by replacing $\sin^2\delta_1$ by $\frac{U_r^2}{V^2}$ and $\cos^2\delta_1$ by $\frac{U_\theta^2}{V^2}$ equation 2-4 can be written in terms of r_p, U_r, U_θ and V i.e.

$$r_p = \frac{r}{2-\frac{V^2 r}{\mu}}\left\{1 - \left[1 - \frac{2 r U_\theta^2}{\mu} + \frac{V^2 r^2}{\mu^2} U_\theta^2\right]^{\frac{1}{2}}\right\} \qquad 2\text{-}5$$

Differentiating equation 2-5 with respect to U_r and U_θ gives (see appendix 1 for detailed solution)

$$\frac{\partial r_p}{\partial U_\theta} = \frac{a^2 U_\theta}{\epsilon \mu}(1-\epsilon)^2\left[\frac{(1+\epsilon)^2 - (1+\epsilon\cos\theta)^2}{(1+\epsilon\cos\theta)^2}\right] \qquad 2\text{-}6$$

$$\frac{\partial r_p}{\partial U_r} = -\frac{a^2 U_r}{\epsilon \mu}(1-\epsilon)^2 \qquad 2\text{-}7$$

dividing the negative of 2-6 by 2-7 gives

$$\frac{dU_r}{dU_\theta} = \tan\delta_2 = \frac{U_\theta}{U_r}\left[\frac{(1+\epsilon)^2 - (1+\epsilon\cos\theta)^2}{(1+\epsilon\cos\theta)^2}\right] \qquad 2\text{-}8$$

It is now convenient to substitute for U_θ and U_r in terms of the orbital parameters r, ϵ and θ. This can be done by recalling that $U_r = \dot{r}$ and $U_\theta = r\dot{\theta}$. An expression for $\dot{r}$ in terms of ϵ, a and θ can be obtained by differentiating equation 1-13 with respect to time. This gives

$$\dot{r} = \frac{a(1-\epsilon^2)\ \epsilon \sin\theta}{(1+\epsilon\cos\theta)^2}\dot{\theta}$$

Eliminating a from this equation by the use of 1-3a gives

$$\dot{r} = \frac{l}{r}\ \frac{\epsilon \sin\theta}{1+\epsilon\cos\theta} \qquad 2\text{-}9$$

Therefore

$$U_r = \frac{l}{r}\ \frac{\epsilon \sin\theta}{1+\epsilon\cos\theta} \qquad 2\text{-}10$$

and

$$U_\theta = \frac{l}{r} \qquad 2\text{-}11$$

With these relationships it is now possible to solve for the angle δ_2. This can be done by substituting 2-10 and 2-11 into equation 2-8 giving the final relationship between the angle δ_2 and the orbital elements, i.e.

$$\tan\delta_2 = \frac{\epsilon\sin\theta}{1+\epsilon\cos\theta} + \frac{2(1-\cos\theta)}{\sin\theta(1+\epsilon\cos\theta)} \qquad 2\text{-}12$$

The first term of the right hand side of this equation is simply the tangent of the angle formed by the components of the original velocity vector, i.e. $\tan\delta_1$. This then gives a relationship between δ_2, δ_1, and θ namely

$$\tan\delta_2 = \tan\delta_1 + \frac{2\tan\frac{\theta}{2}}{1+\epsilon\cos\theta} \qquad 2\text{-}13$$

Equation 2-12 gives the orientation of the thrust vector at any point in the orbit which will maintain a constant perigee distance.

3. EQUATIONS FOR A NUMERICAL ANALYSIS OF A TYPICAL FLIGHT

To make an analysis of a mission employing this method it is necessary to develop three more equations. They are (a) the number of turns required to escape, (b) the total thrust time, (c) the total elapsed time.

Starting with the basic energy equation (1-18), we can write

$$\frac{\mu}{2a} = \frac{\mu}{r} - \frac{V^2}{2} \qquad \text{1-18}$$

or $$\frac{1}{a} = \frac{2}{r} - \frac{V^2}{\mu}$$

Then to determine the change in energy caused by an applied thrust at a given r we can write

$$d\left(\frac{1}{a}\right) = \frac{2}{\mu} V \cos(\delta_2 - \delta_1)\, dt \left|\frac{dv}{dt}\right| \qquad \text{3-1}$$

where $\cos(\delta_2 - \delta_1)\left|\frac{dv}{dt}\right|$ is the component of the force per unit mass in the direction tangent to the orbit and $V\,dt$ is the distance along the orbit. This is to say the incremental change in total energy is equal to the work done on the vehicle over an incremental distance. We therefore can use $V dt = ds$ and from fig. 4 we see that

$$ds = \frac{r\, d\theta}{\cos \delta_1}$$

If this relationship is put into 3-1 the resulting equation is

$$d\left(\frac{1}{a}\right) = -\frac{2|\dot{V}|}{\mu} \frac{\cos(\delta_2 - \delta_1)}{\cos \delta_1} r\, d\theta \qquad \text{3-2}$$

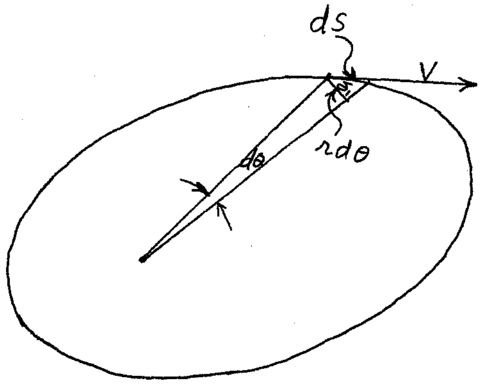


Fig. 4.

However if we substitute 1-13 and subsequently 1-11 into this equation we get

$$d\left(\frac{1}{a}\right) = -\frac{2r_p}{\mu}\left|\dot{V}\right|\left[\frac{\cos(\delta_2-\delta_1)}{\cos\delta_1}\right]\left(\frac{1+\epsilon}{1+\epsilon\cos\theta}\right)d\theta \qquad 3\text{-}3$$

Integrating this equation gives the change in energy per turn, i.e.

$$\Delta\frac{1}{a} = -\frac{4r_p}{\mu}\left|\dot{V}\right|\int_0^\theta\left\{\frac{\cos(\delta_2-\delta_1)}{\cos\delta_1}\left(\frac{1+\epsilon}{1+\epsilon\cos\theta}\right)\right\}d\theta \qquad 3\text{-}4$$

where the limits of integration describe the portion of the ellipse over which the thrust is applied. Since the ellipse is symmetrical about the periapsis, the integral is evaluated over one half the thrust period and doubled to give the total change in energy.

To make an accurate analysis of this process requires that it be done on a computer. However a fairly good approximation can be made by a simple numerical integration.

The number of turns required to increase the eccentricity by 10% can be determined by

$$n = \frac{\Delta \frac{(1-\epsilon)}{r_p}}{\Delta\left(\frac{1}{a}\right)} = \frac{0.1}{r_p \Delta\left(\frac{1}{a}\right)}$$

$$n = \frac{.1\,\mu}{4 r_p^2 |\dot{V}| \int_0^\theta \left\{ \frac{\cos(\delta_2 - \delta_1)}{\cos \delta_1} \left(\frac{1+\epsilon}{1+\epsilon \cos\theta} \right) \right\} d\theta}$$

$$n = \frac{2.16 \times 10^{-2}}{|\dot{V}| \int_0^\theta \left\{ \frac{\cos(\delta_2 - \delta_1)}{\cos \delta_1} \left(\frac{1+\epsilon}{1+\epsilon \cos\theta} \right) \right\} d\theta} \qquad 3\text{-}5$$

The thrust time per turn can be written as

$$T_i = 2 \int_0^\theta \frac{dt}{d\theta} d\theta \qquad 3\text{-}6$$

but

$$\frac{dt}{d\theta} = \frac{r^2}{r^2 \dot{\theta}} = \frac{r^2}{l}$$

Therefore the thrust time per turn is

$$T_i = \frac{2}{l} \int_0^\theta r^2 \, d\theta$$

If we now put 1-13 and subsequently 1-11 and 1-12 in this equation we get

$$T_i = \frac{2 r_p^{3/2}}{\sqrt{\mu(1-\epsilon)}} \int_0^\theta \left(\frac{1+\epsilon}{1+\epsilon \cos\theta} \right)^2 d\theta$$

but $\left(\frac{r_p}{\mu}\right)^{\frac{1}{2}} = \frac{1}{V_{co}}$ where V_{co} is circular velocity at perigee.

Thus the final equation for the thrust per turn can be written as

$$T_i = \frac{1807}{(1+\epsilon)^{\frac{1}{2}}} \int_0^{\theta} \left(\frac{1+\epsilon}{1+\epsilon \cos\theta} \right)^2 d\theta \qquad 3\text{-}7$$

Finally, the total elapsed time per orbit is

$$t = \frac{2\pi a^{3/2}}{\mu^{1/2}} = \frac{2\pi r_p^{3/2}}{(1-\epsilon)^{3/2} \mu^{1/2}} \qquad 3\text{-}8$$

which for our problem gives

$$t = \frac{.0658}{(1-\epsilon)^{3/2}} \text{ days} \qquad 3\text{-}9$$

4. NUMERICAL ANALYSIS

In order to make a comparative analysis between the circular spiral method and the elliptical method it is necessary to compute the energy applied and total time for the corresponding spiralling orbits. Again as in the elliptical method a rough approximation will be made.

Using the change in energy equation 3-1, and observing that since the thrust is applied tangentially in the spiralling method $(\delta_2 - \delta_1)$ is zero, we can write the energy equation as:

$$\Delta\left(\frac{1}{a}\right) = -\frac{2|\dot{V}|}{\mu} 2\pi a \qquad \text{(for one turn)}$$

or for a number of turns

$$\int_{a_1}^{a_2} \frac{1}{a} d\left(\frac{1}{a}\right) = -\int_{n_1}^{n_2} \frac{4\pi|\dot{V}|}{\mu} dn$$

which can be written as ($|\dot{V}|$ = const.)

$$\int_{a_1}^{a_2} d\left(\frac{1}{a}\right)^2 = -\frac{8\pi|\dot{V}|}{\mu}\int_{n_1}^{n_2} dn \qquad 4\text{-}1$$

Integrating this equation will give the number of turns required to go from one energy level to another.

To determine the energy applied to go from one energy level to another we simply need to determine the total time, since the thrust is applied constantly. The total time for any portion of the flight is:

$$\int_{t_1}^{t_2} dt = \int_{n_1}^{n_2} T\, dn \qquad 4\text{-}2$$

where T is the period of the orbit for any given radius a. Therefore, the time equation becomes

$$\int_{t_1}^{t_2} dt = \frac{2\pi}{\sqrt{\mu}} \int_{n_1}^{n_2} a^{3/2}\, dn \qquad 4\text{-}2a$$

Now using equation 4-1 to simplify this equation gives

$$\int_{t_1}^{t_2} dt = -\frac{\sqrt{\mu}}{|\dot{V}|} \int_{a_1}^{a_2} d\left(\sqrt{\frac{1}{a}}\right) \qquad 4\text{-}3$$

With these equations it is now possible to compare directly with the elliptical method since the total energy of an elliptical orbit with a given semi major axis a is equal to the energy in a circular orbit of radius a. Now if we compare the energy required to go from the original circular orbit with a major axis a_0 to an ellipse with a major axis of say 10 a_0 using the elliptical method, to that required to go from the original orbit to a circular orbit of radius 10 a_0 using the spiralling method, the relative efficiencies can be seen.

A graphical comparison is made in appendix II.

5. RESULTS AND DISCUSSION

A general method has been presented for determining the thrust vector orientation of a low thrust vehicle which will allow for a finite thrust program while maintaining a constant perigee distance. In the case considered in this paper it is shown that orientation of the thrust vector such that the resulting trajectory is an increasing elliptical orbit requires less energy than the constant thrust spiralling method, even though the time to escape is larger.

As can be observed from fig. 8, the efficiency of the elliptical and spiralling methods are practically the same when the maximum energy input angle $(\delta_2 - \delta_1)$ is approximately 48°. When this angle is less than 45° the elliptical, discontinuous thrust program becomes more efficient from the standpoint of total energy input. Even though the rough approximation tends to get very poor for a complete numerical comparison above an eccentricity of about .8 - .9, it can be seen that there is an appreciable savings by using this method. If an extrapolation is made of the curves in fig. 5 from .8 to 1.0 the savings in total energy is even more apparent. The numerical calculations were carried out to an eccentricity of .95 to show that even with rough approximations the elliptical method still shows more efficiency. It can also be seen that as the energy input angle is reduced still further the savings in energy input becomes even greater.

The numerical integrations were carried out by taking 10°

increments in Θ , then taking the average value of the particular parameter at the two limits of this interval, then multiplying by the interval (in radians) and summing over the total thrust interval for that particular orbit. Naturally as the eccentricity increases above about .8 the thrust periods become large and consequently the value of the limits over the 10 degree interval begin to get a very large spread and this method of integration tends to become less accurate. However, the smoothness of the curves up to this point and beyond seem to indicate that a considerably accurate extrapolation can be made on the values of the various parameters at the point of escape (i.e. $\epsilon = 1.0$). An extrapolation of the spiralling method gives very close correlation with the machine solutions. This should tend to add confidence to an extrapolation of the curves obtained for the elliptical method in this paper.

It should be mentioned that the elliptical method can also be used very effectively for transferring between orbits in the low eccentricity area. It is obvious that a transfer from say a 500 to a 4000 mile orbit can be accomplished more efficiently by this method than the spiralling method. Also, it is important in a vehicle rendezvous mission, since in this type of mission a discontinuous thrust program is very essential. One other type of mission in which the elliptical method has a distinct advantage over the spiralling method is in satellite reentry. In this case the reverse thrust is applied near apogee causing a decrease in the perigee distance which in turn results in atmospheric drag dissipating some of the energy, resulting in an even greater savings over the spiralling method.

Finally it should be pointed out that the elliptical technique used in this thesis is restricted to low thrust propulsion systems. All of the equations were derived and the calculations made without explicitly using the thrust values until the final charts were made for comparison of the 10^{-4} g vehicles. However, the final equations were arrived at by linearizing the basic equations with respect to the orbital perturbations due to the thrust. The final integrations neglected the changes in the orbital elements for any one orbit.

6. CONCLUSIONS

From the results of this study it can be concluded that there are distinct advantages in using a discontinuous thrust program to transfer from one energy level to another and finally to escape energy. The principal advantage being in the saving in total energy input. The advantage gained in total energy expenditure is made at the expense of total time to escape. However on the long missions on which low thrust engines are practical the loss in time appears to be justifiable.

APPENDIX I

APPENDIX I

Starting with equation 2-5

$$r_p = \frac{r}{2 - \frac{V^2 r}{\mu}} \left\{ 1 - \left[1 - \frac{2 r U_\theta^2}{\mu} + \frac{V^2 r^2}{\mu^2} U_\theta^2 \right]^{\frac{1}{2}} \right\}$$

but if we let

$$\alpha = 2 - \frac{V^2 r}{\mu} \qquad \text{and} \qquad \beta = \left[1 - \frac{2 r U_\theta^2}{\mu} + \frac{V^2 r^2}{\mu^2} U_\theta^2 \right]$$

we can write

$$r_p = \frac{r}{\alpha} \left\{ 1 - \beta^{\frac{1}{2}} \right\}$$

Then substituting for V^2 in terms of $U_\theta^2 + U_r^2$

$$\alpha = 2 - \frac{r}{\mu}\left(U_r^2 + U_\theta^2\right) \qquad \text{and} \qquad \beta = \left[1 - \frac{2 r U_\theta^2}{\mu} + \frac{r^2}{\mu^2}\left(U_\theta^2 U_r^2 + U_\theta^4\right) \right]$$

Differentiating gives

$$\frac{\partial \alpha}{\partial U_r} = -\frac{2 r U_r}{\mu} \qquad \text{I-1}$$

$$\frac{\partial \alpha}{\partial U_\theta} = -\frac{2 r U_\theta}{\mu} \qquad \text{I-2}$$

$$\frac{\partial \beta}{\partial U_r} = \frac{2 r^2 U_\theta^2 U_r}{\mu^2} \qquad \text{I-3}$$

$$\frac{\partial \beta}{\partial U_\theta} = \frac{2 r U_\theta}{\mu}\left[2 - \frac{2 U_\theta^2 r}{\mu} - \frac{U_r^2 r}{\mu} \right] \qquad \text{I-4}$$

$$\frac{\partial r_p}{\partial U_r} = -\frac{r}{\alpha^2}\frac{\partial \alpha}{\partial U_r} - \frac{r}{2\alpha\beta^{\frac{1}{2}}}\frac{\partial \beta}{\partial U_r} + \frac{r\beta^{\frac{1}{2}}}{\alpha^2}\frac{\partial \alpha}{\partial U_r}$$

or $$\frac{\partial r_p}{\partial U_r} = -\frac{r}{2\alpha^2\beta^{\frac{1}{2}}}\left[2\beta^{\frac{1}{2}}\frac{\partial \alpha}{\partial U_r}\left(1 - \beta^{\frac{1}{2}}\right) + \alpha\frac{\partial \beta}{\partial U_r} \right] \qquad \text{I-5}$$

$$\frac{\partial r_p}{\partial U_\theta} = -\frac{r}{\alpha^2}\frac{\partial \alpha}{\partial U_\theta} + \frac{r\,\beta^{\frac{1}{2}}}{\alpha^2}\frac{\partial \alpha}{\partial U_\theta} - \frac{1}{2}\frac{r}{\alpha\,\beta^{\frac{1}{2}}}\frac{\partial \beta}{\partial U_\theta}$$

or

$$\frac{\partial r_p}{\partial U_\theta} = -\frac{r}{2\alpha^2\beta^{\frac{1}{2}}}\left[2\beta^{\frac{1}{2}}\frac{\partial \alpha}{\partial U_\theta}\left(1-\beta^{\frac{1}{2}}\right) + \alpha\frac{\partial \beta}{\partial U_\theta}\right] \qquad \text{I-6}$$

if we now substitute for α, β, $\frac{\partial \alpha}{\partial U_r}$, $\frac{\partial \alpha}{\partial U_\theta}$, $\frac{\partial \beta}{\partial U_r}$, $\frac{\partial \beta}{\partial U_\theta}$ and recall that $\alpha = \frac{r}{a}$ and $r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos\theta}$ we get

$$\frac{\partial r_p}{\partial U_\theta} = -\frac{r}{\frac{r^2}{a^2}}\left(-\frac{2rU_\theta}{\mu}\right) + \frac{r\epsilon}{\frac{r^2}{a^2}}\left(-\frac{2rU_\theta}{\mu}\right) -$$

$$-\frac{1}{2}\frac{r}{\frac{r}{a}\epsilon}\left(-\frac{2rU_\theta}{\mu}\right)\left[2-\frac{U_\theta^2 r}{\mu}+\frac{U_r^2 r}{\mu}\right] = \frac{a^2}{r}\left(\frac{2rU_\theta}{\mu}\right)\left[1-\epsilon+\frac{1}{2}\frac{r}{2a\epsilon}\left(\frac{U_\theta^2 r}{\mu}+\frac{r}{a}\right)\right]$$

$$\frac{\partial r_p}{\partial U_\theta} = \frac{a^2 U_\theta}{\epsilon\mu}(1-\epsilon^2)^2\left[\frac{(1-\epsilon)^2-(1+\epsilon\cos\theta)^2}{(1+\epsilon\cos\theta)^2}\right] \qquad \text{I-7}$$

$$\frac{\partial r_p}{\partial U_r} = -\frac{r}{2\frac{r^2}{a^2}\epsilon}\left[2\epsilon\left(-2\frac{rU_r}{\mu}\right)(1-\epsilon) + \frac{r}{a}\left(\frac{2rU_r}{\mu}\right)\frac{rU_\theta^2}{\mu}\right]$$

$$\frac{\partial r_p}{\partial U_r} = -\frac{a^2 U_r}{\epsilon\mu}\left[-2\epsilon+2\epsilon^2+1-\epsilon^2\right]$$

$$\frac{\partial r_p}{\partial U_r} = -\frac{a^2 U_r}{\epsilon\mu}(1-\epsilon^2)^2 \qquad \text{I-8}$$

Now dividing the negative of I-7 by I-8 gives

$$\frac{dU_r}{dU_\theta} = \frac{\frac{a^2 U_\theta(1-\epsilon^2)^2}{\epsilon\mu}\left[\frac{(1+\epsilon)^2-(1+\epsilon\cos\theta)^2}{(1+\epsilon\cos\theta)^2}\right]}{\frac{a^2 U_r}{\epsilon\mu}(1-\epsilon)^2}$$

if we now substitute for U_r & U_θ, i.e.

$$U_\theta = \frac{l}{r} \qquad \text{and} \qquad U_r = \frac{l}{r}\,\frac{\epsilon\sin\theta}{1+\epsilon\cos\theta}$$

we get

$$\frac{dU_r}{dU_\theta} = \frac{(1+\epsilon)^2 - (1+\epsilon \cos\theta)^2}{(\epsilon \sin\theta)(1+\epsilon \cos\theta)}$$

which can be reduced further to give

$$\frac{dU_r}{dU_\theta} = \frac{\epsilon \sin\theta}{(1+\epsilon \cos\theta)} + \frac{2(1-\cos\theta)}{\sin\theta(1+\epsilon \cos\theta)}$$

or finally

$$\frac{dU_r}{dU_\theta} = \frac{\epsilon \sin\theta}{1+\epsilon \cos\theta} + \frac{2\tan\frac{\theta}{2}}{1+\epsilon \cos\theta} = \tan\delta_2 \qquad 2\text{-}12$$

$$\tan\delta_2 = \tan\delta_1 + \frac{2\tan\frac{\theta}{2}}{1+\epsilon \cos\theta} \qquad 2\text{-}13$$

APPENDIX II

THRUST TIME (|v|)
(energy input)

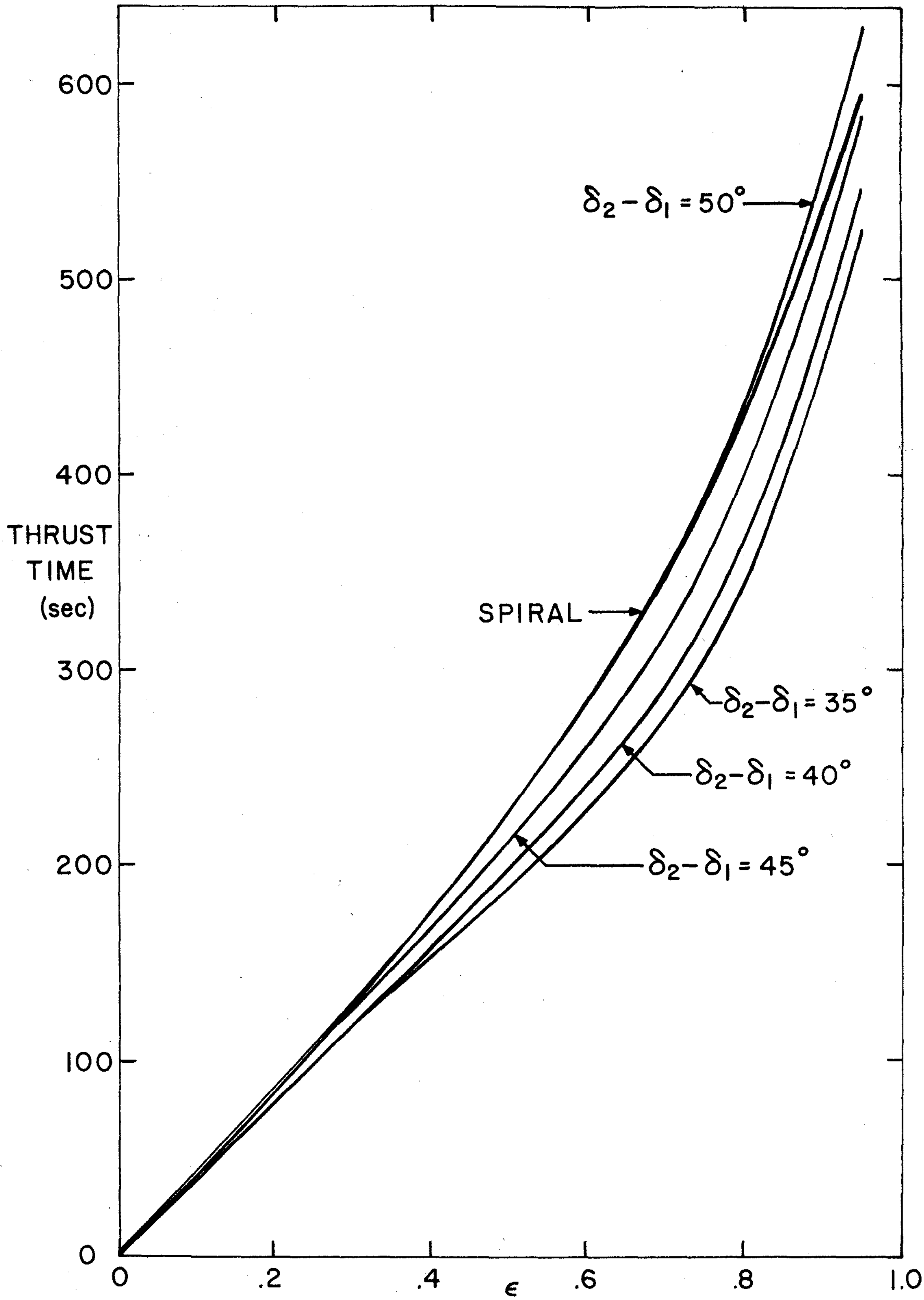


Fig. 5

Thrust Time (Elliptical Method)

	50^o	Total	45^o	Total	40^o	Total	35^o	Total
0-.1	45.4		43.4		41.8		40.7	
		45.4		43.4		41.8		40.7
.1-.2	44.4		42.2		40.8		39.4	
		89.8		85.6		82.6		80.1
.2-.3	44.2		41.9		40.1		39.2	
		134.0		127.5		122.7		119.3
.3-.4	44.4		41.3		39.4		38.2	
		178.4		168.8		162.1		157.5
.4-.5	49.2		43.3		39.2		36.8	
		227.6		212.1		201.3		194.3
.5-.6	59.4		49.5		41.5		36.4	
		287.0		261.6		242.8		230.7
.6-.7	68.3		58.9		48.5		48.3	
		355.3		320.5		291.3		279.0
.7-.8	81.8		77.6		74.8		68.0	
		437.1		398.1		366.1		347.0
.8-.9	119.1		118.0		116.5		112.8	
		556.2		516.1		482.6		459.8
.9-.95	74.3		73.2		71.5		70.1	
		630.5		589.3		554.1		529.9

TABLE 1

Thrust Time (Spiralling Method)

	T	total	a
0-.1	40.7		
		40.7	$1.11a_o$
.1-.2	42.3		
		83.0	$1.25a_o$
.2-.3	45.5		
		128.5	$1.43a_o$
.3-.4	48.5		
		177.0	$1.67a_o$
.4-.5	52.5		
		229.5	$2.0a_o$
.5-.6	58.5		
		288.0	$2.5a_o$
.6-.7	65.5		
		353.5	$3.33a_o$
.7-.8	79.0		
		432.5	$5.0a_o$
.8-.9	103.5		
		536.0	$10.1a_o$
.9-.95	74.0		
		610.0	$20.0a_o$

TABLE 1a

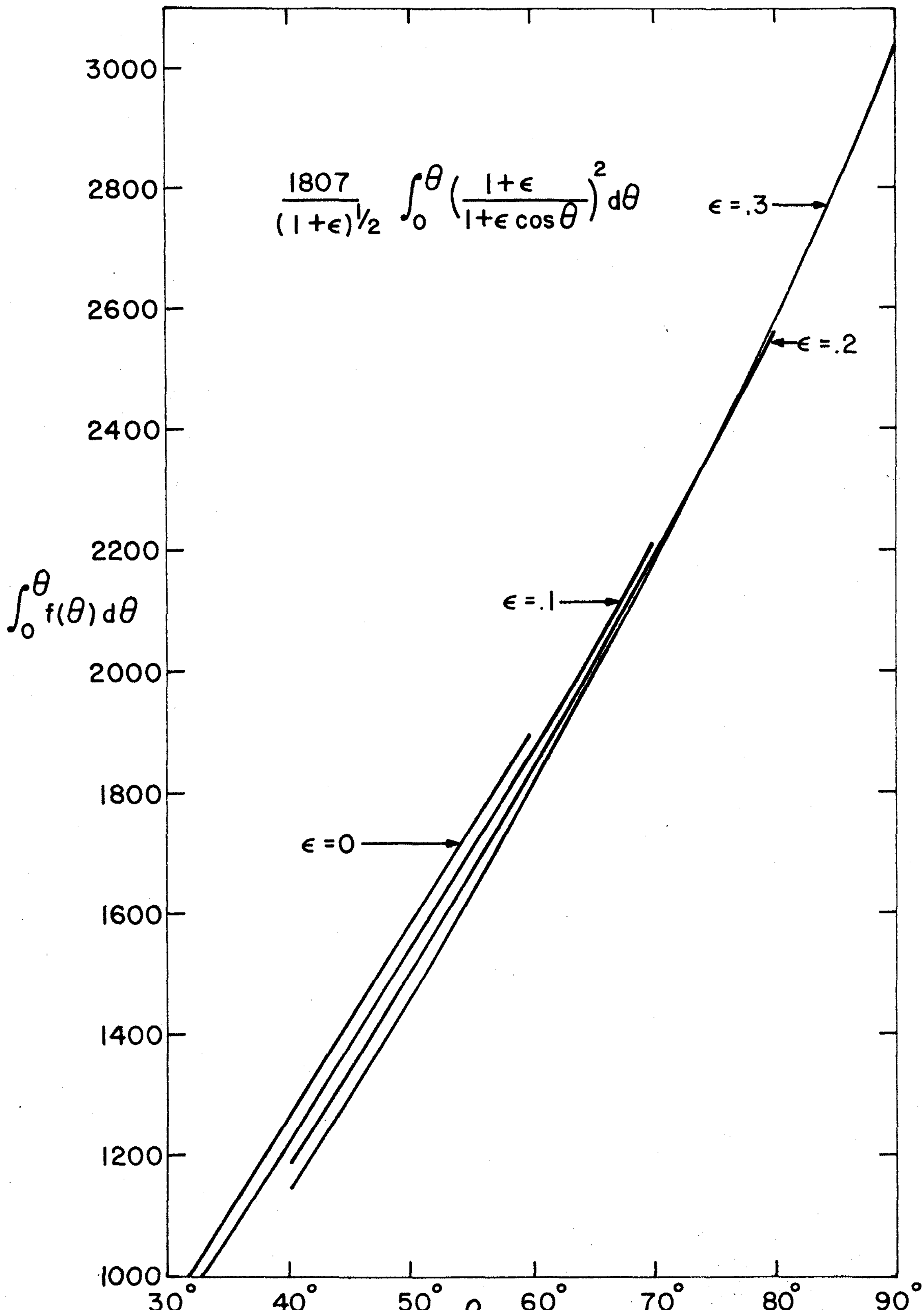

$\frac{1807}{(1+\epsilon)^{1/2}} \int_0^\theta \left(\frac{1+\epsilon}{1+\epsilon\cos\theta}\right)^2 d\theta$
$\epsilon = .3$
$\epsilon = .2$
$\epsilon = .1$
$\epsilon = 0$
$\int_0^\theta f(\theta)\,d\theta$
3000
2800
2600
2400
2200
2000
1800
1600
1400
1200
1000
30°
40°
50°
60°
70°
80°
90°
θ

Fig. 6

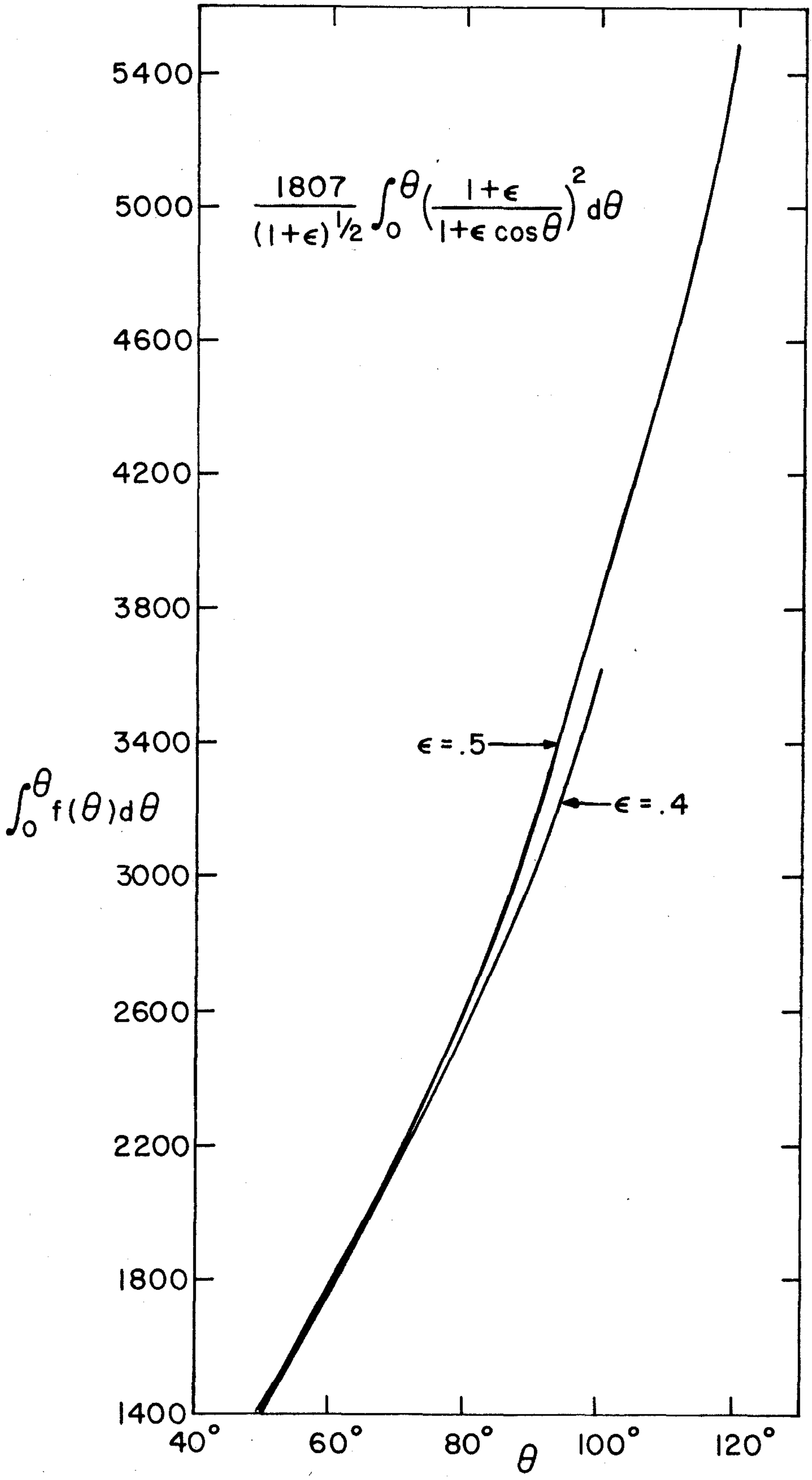

5400
5000
4600
4200
3800
3400
3000
2600
2200
1800
1400
$\frac{1807}{(1+\epsilon)^{1/2}}\int_0^\theta\left(\frac{1+\epsilon}{1+\epsilon\cos\theta}\right)^2 d\theta$
$\int_0^\theta f(\theta)d\theta$
$\epsilon = .5$
$\epsilon = .4$
40°
60°
80°
100°
120°
θ

Fig. 7

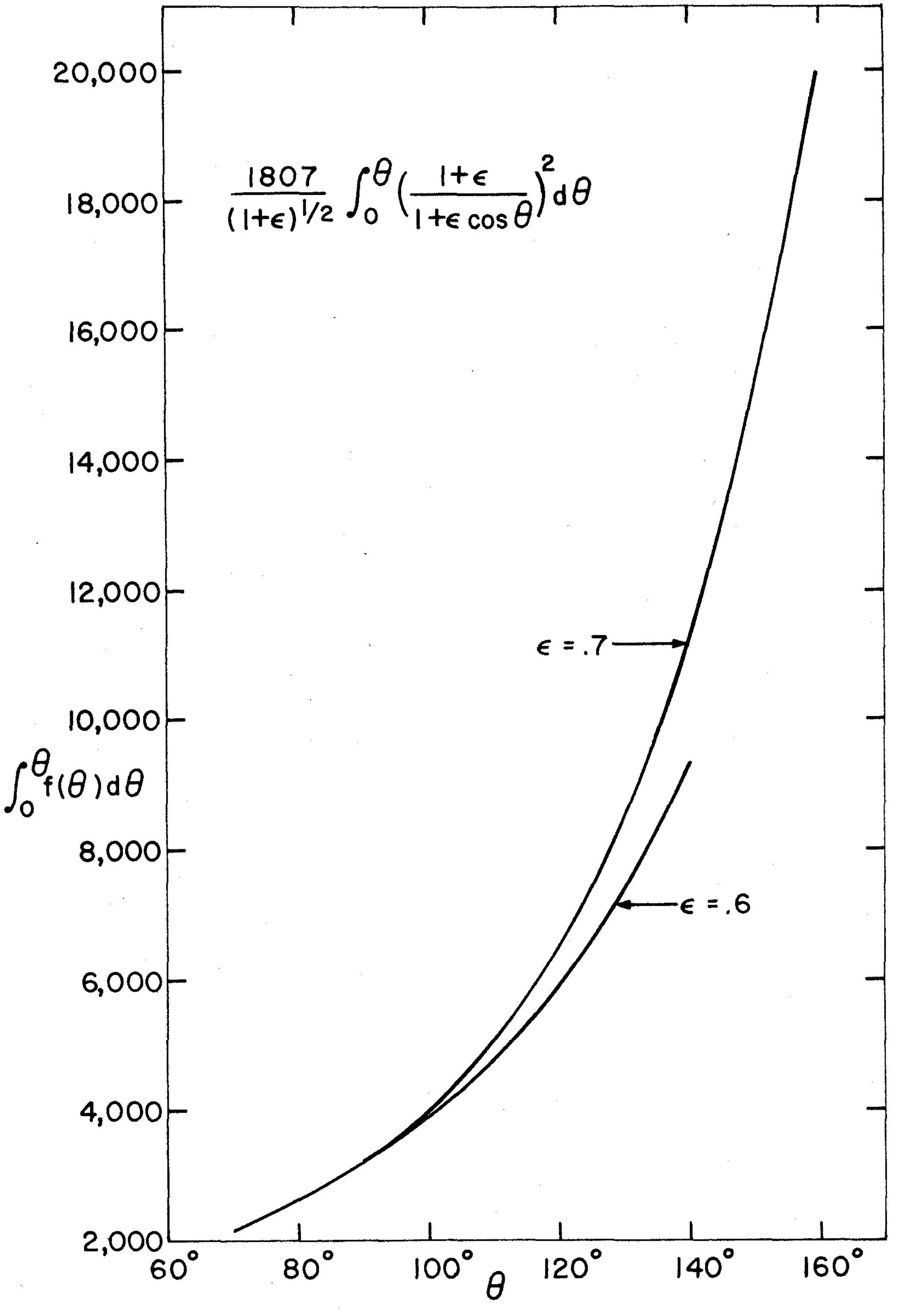

20,000
18,000
16,000
14,000
12,000
10,000
8,000
6,000
4,000
2,000
$\frac{1807}{(1+\epsilon)^{1/2}} \int_0^\theta \left(\frac{1+\epsilon}{1+\epsilon\cos\theta}\right)^2 d\theta$
$\epsilon = .7$
$\epsilon = .6$
$\int_0^\theta f(\theta)\,d\theta$
60°
80°
100°
120°
140°
160°
θ

Fig. 8

Cumulative Thrust Time / Turn

θ	$\epsilon=.0$	$\epsilon=.1$	$\epsilon=.2$	$\epsilon=.3$	$\epsilon=.4$	$\epsilon=.5$	$\epsilon=.6$	$\epsilon=.7$	$\epsilon=.8$	$\epsilon=.9$	$\epsilon=.915$	$\epsilon=.95$	$\epsilon=1.0$
0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	316	302	286	277	267	258	250	242	236	230	228	228	225
20	633	606	581	558	539	522	506	491	478	467	464	462	453
30	949	912	879	846	821	796	774	753	736	720	717	709	700
40	1268	1222	1187	1148	1118	1104	1162	1034	1014	994	991	982	970
50	1581	1542	1504	1464	1437	1420	1378	1350	1328	1308	1301	1291	1281
60	1900	1870	1840	1820	1785	1770	1734	1710	1690	1672	1666	1651	1641
70		2219	2200	2185	2170	2164	2141	2128	2120	2112	2102	2105	2088
80			2568	2588	2600	2622	2652	2635	2644	2664	2655	2675	2660
90				3035	3090	3162	3258	3260	3324	3400	3382	3420	3435
100					3620	3882	3922	4060	4220	4400	4400	4470	4530
110						4560	4780	5120	5465	5820	5865	6080	6210
120						5498	5910	6560	7250	8040	8180	8530	9040
130						6600	7410	8470	9910	11790	12080	12940	14350
140							9360	11200	14100	18480	19270	21580	27400
150							11660	15100	20810	31400	33760	41300	60900
160								19950	31040	57400	64800	92700	212500
170									45600	112800	138800	235000	∞

TABLE 2

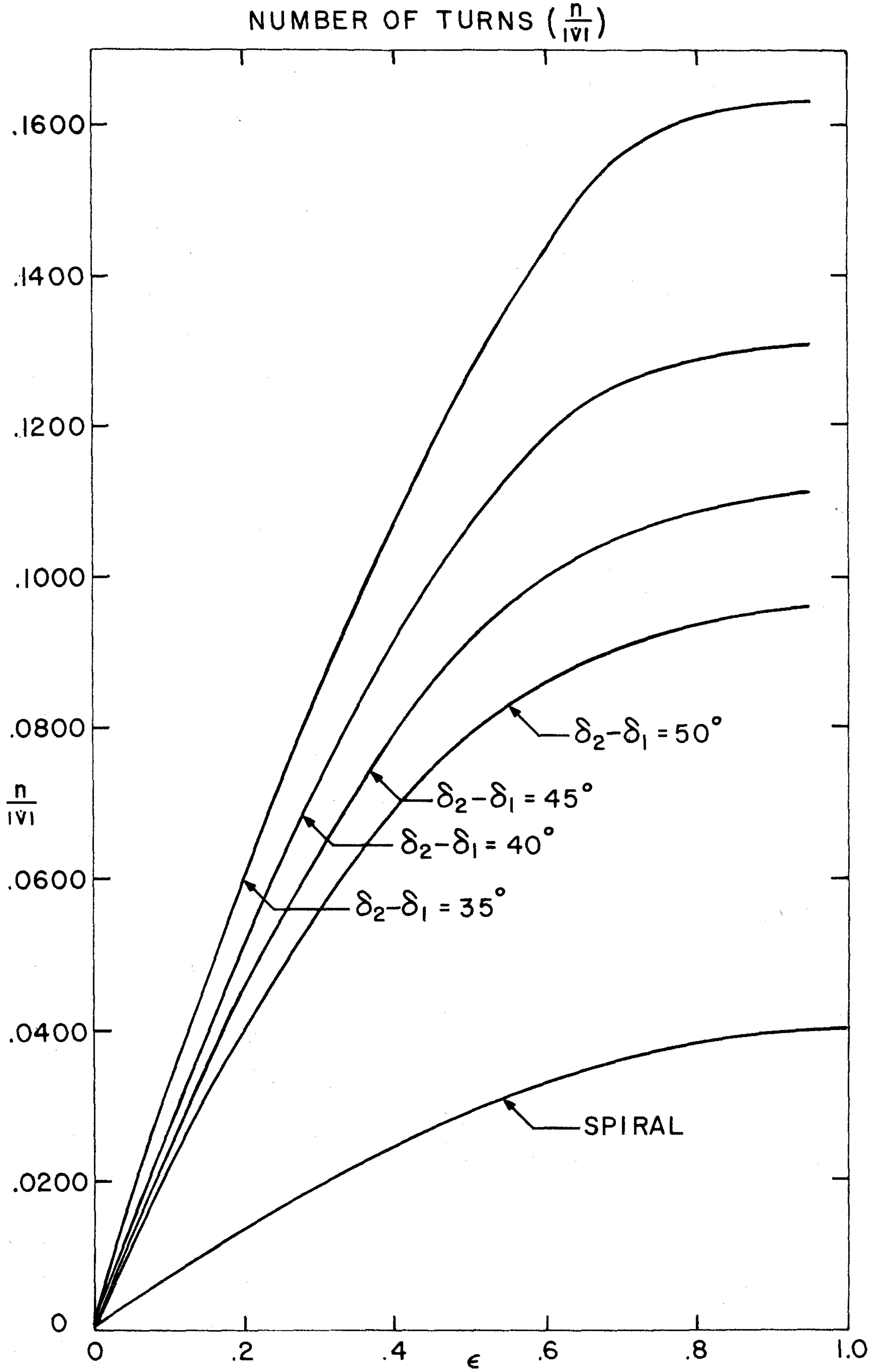
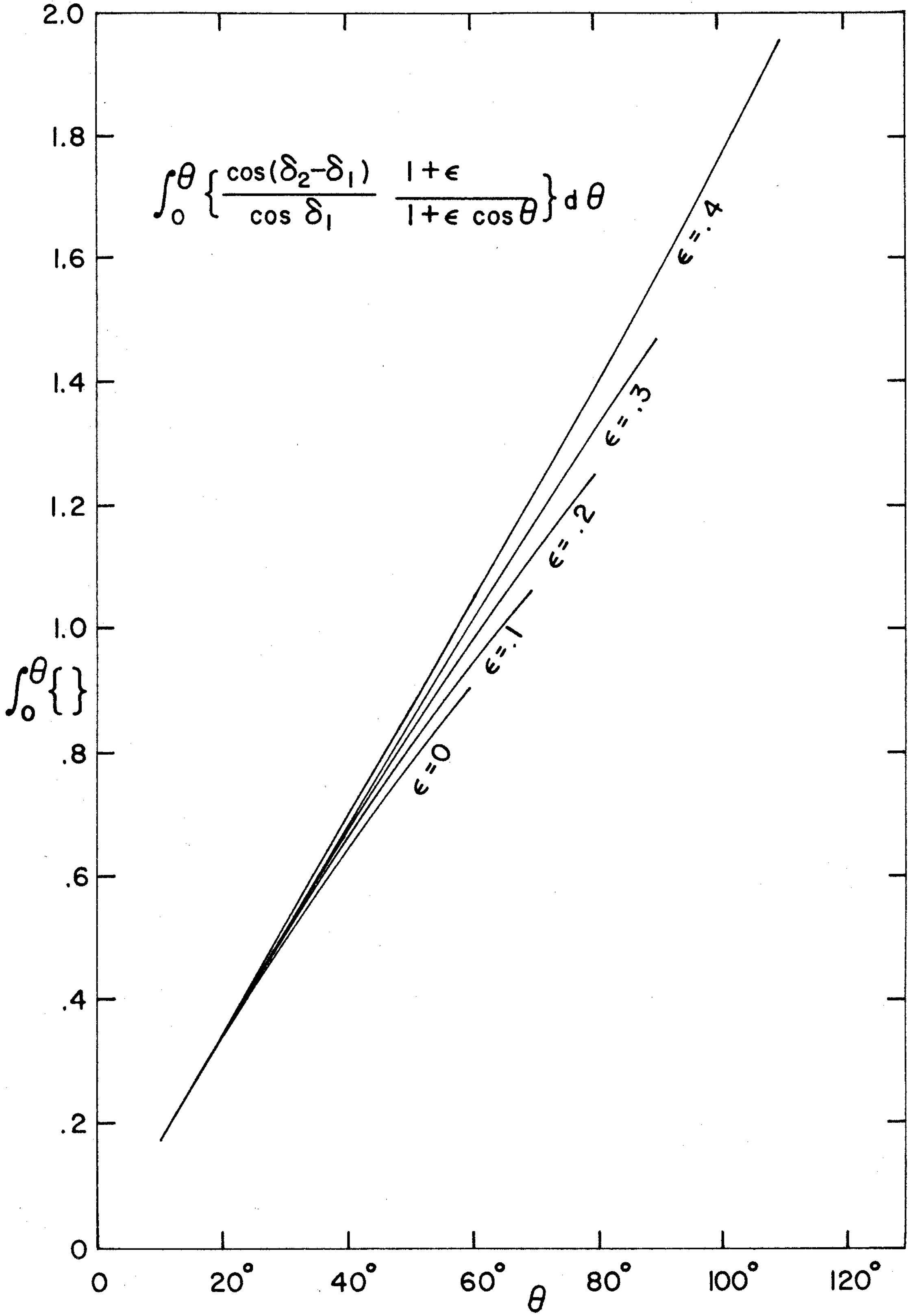

NUMBER OF TURNS $\left(\frac{n}{|V|}\right)$
.1600
.1400
.1200
.1000
.0800
.0600
.0400
.0200
0
$\frac{n}{|V|}$
$\delta_2-\delta_1 = 50°$
$\delta_2-\delta_1 = 45°$
$\delta_2-\delta_1 = 40°$
$\delta_2-\delta_1 = 35°$
SPIRAL
0
.2
.4
.6
.8
1.0
ϵ

Fig. 9

NUMBER OF TURNS $\frac{n}{|\dot{V}|}$

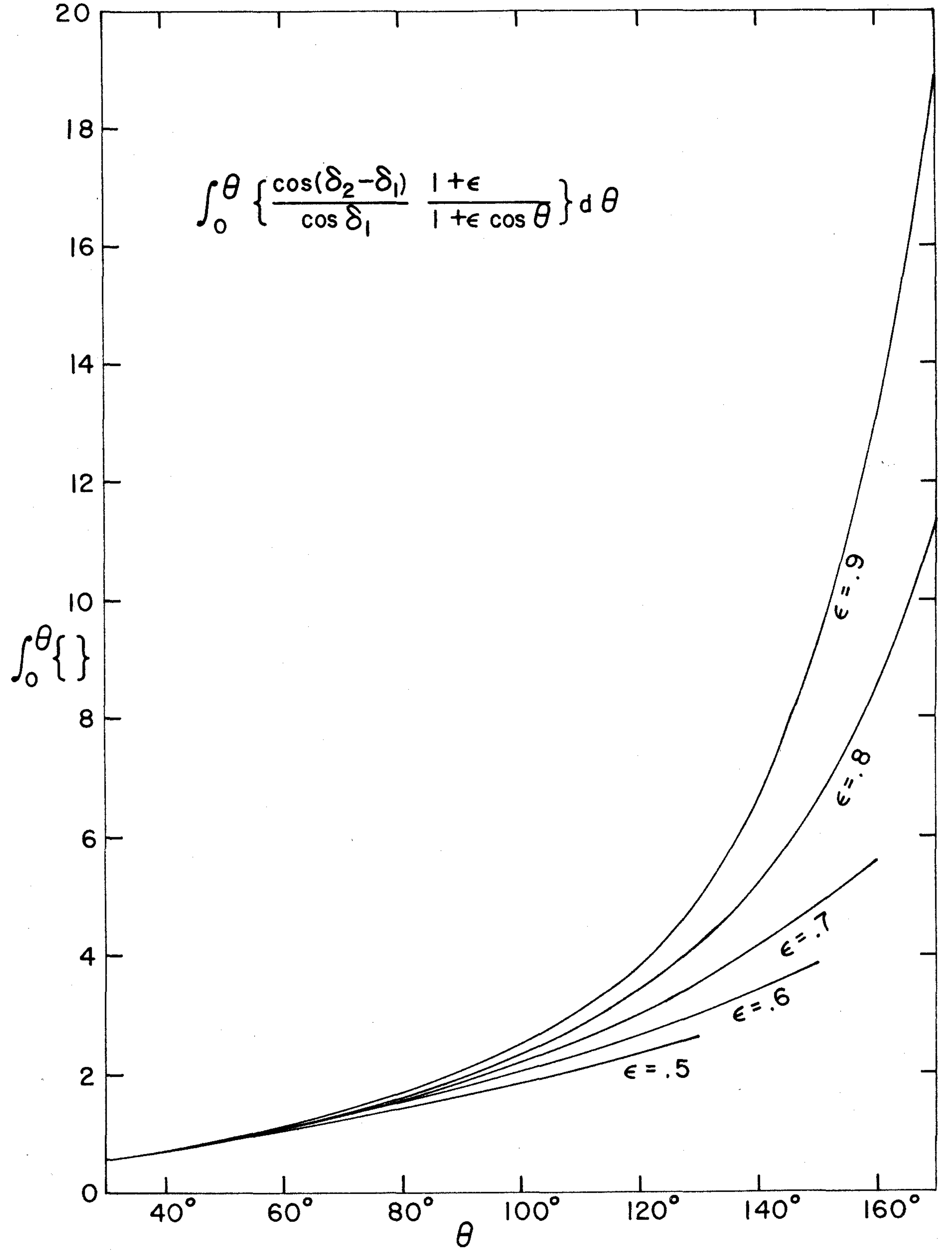
	50°	total	45°	total	40°	total	35°	total
0.-1	.0217	.0217	.0245	.0245	.0280	.0280	.0324	.0324
.1-.2	.0187	.0404	.0213	.0458	.0244	.0524	.0285	.0609
.2-.3	.0159	.0563	.0184	.0642	.0213	.0737	.0253	.0862
.3-.4	.0129	.0692	.0153	.0795	.0183	.0920	.0223	.1085
.4-.5	.00986	.0791	.0121	.0916	.0151	.1071	.0194	.1279
.5-.6	.00722	.0863	.00875	.1004	.0117	.1188	.0161	.1440
.6-.7	.00478	.0911	.00562	.1060	.00715	.1260	.0127	.1567
.7-.8	.00273	.0938	.00293	.1089	.00333	.1293	.00501	.1617
.8-.9	.00138	.0952	.00142	.1103	.00152	.1308	.00167	.1634
.9-.95	.00045	.0957	.00046	.1108	.00049	.1313	.00051	.1639

TABLE 3



$\int_0^\theta \left\{ \frac{\cos(\delta_2-\delta_1)}{\cos\delta_1} \frac{1+\epsilon}{1+\epsilon\cos\theta} \right\} d\theta$
$\int_0^\theta \{\ \}$
$\epsilon = .4$
$\epsilon = .3$
$\epsilon = .2$
$\epsilon = .1$
$\epsilon = 0$
2.0
1.8
1.6
1.4
1.2
1.0
.8
.6
.4
.2
0
0
20°
40°
60°
80°
100°
120°
θ

Fig. 11



$\int_0^\theta \left\{ \frac{\cos(\delta_2-\delta_1)}{\cos\delta_1} \; \frac{1+\epsilon}{1+\epsilon\cos\theta} \right\} d\theta$
$\int_0^\theta \{ \}$
$\epsilon = .9$
$\epsilon = .8$
$\epsilon = .7$
$\epsilon = .6$
$\epsilon = .5$
θ

Fig. 10

θ	$\epsilon = 0$	$\epsilon = .1$	$\epsilon = .2$	$\epsilon = .3$	$\epsilon = .4$	$\epsilon = .5$	$\epsilon = .6$	$\epsilon = .7$	$\epsilon = .8$	$\epsilon = .9$	$\epsilon = .915$	$\epsilon = .95$	$\epsilon = 1.0$
0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	.173	.174	.174	.174	.174	.174	.175	.175	.175	.175	.175	.176	.175
20	.341	.344	.346	.347	.348	.350	.351	.352	.352	.352	.352	.354	.353
30	.500	.508	.514	.516	.521	.527	.529	.534	.533	.540	.534	.537	.538
40	.648	.662	.676	.681	.693	.704	.710	.721	.721	.732	.725	.730	.733
50	.782	.807	.832	.844	.866	.883	.896	.914	.920	.937	.933	.937	.940
60	.904	.942	.981	1.005	1.040	1.067	1.097	1.120	1.161	1.162	1.161	1.165	1.170
70		1.067	1.123	1.164	1.215	1.259	1.304	1.343	1.398	1.416	1.416	1.425	1.436
80			1.258	1.320	1.395	1.460	1.537	1.590	1.668	1.708	1.714	1.728	1.760
90				1.474	1.578	1.672	1.791	1.869	1.983	2.061	2.073	2.100	2.163
100					1.765	1.896	2.050	2.186	2.363	2.507	2.526	2.588	2.680
110					1.956	2.136	2.342	2.559	2.841	3.085	3.034	3.233	3.398
120						2.394	2.684	3.002	3.452	3.877	3.827	4.148	4.466
130						2.666	3.066	3.532	4.237	5.012	5.008	5.547	6.194
140							3.474	4.260	5.255	6.713	6.866	7.845	9.354
150							3.902	4.951	6.679	9.323	9.828	11.940	16.354
160								5.606	8.389	13.063	14.348	19.440	37.734
170									9.759	17.603	20.378	31.150	455.734

TABLE 4

TOTAL ELAPSED TIME (DAYS)

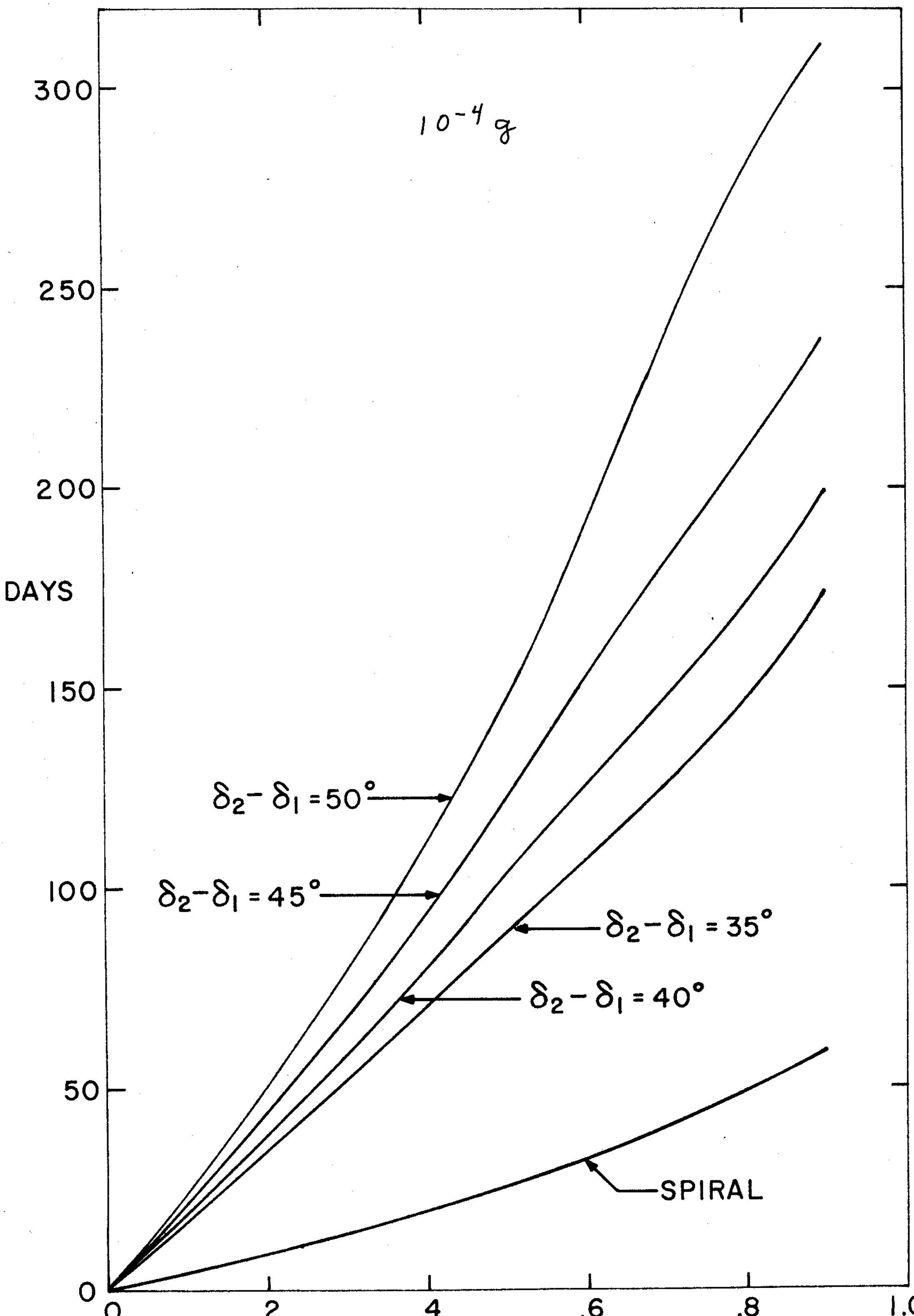


Fig. 12

TOTAL ELAPSED TIME IN DAYS

ϵ	50°	total	45°	total	40°	total	35°	total	Spiral	total
0-.1	16.6	16.6	18.9	18.9	21.6	21.6	25.1	25.1	4.72	4.72
.1-.2	17.2	33.8	19.6	38.5	22.5	44.1	25.3	50.4	4.8	9.52
.2-.3	17.9	51.7	20.7	59.2	24.0	68.1	28.5	78.9	5.27	14.79
.3-.4	18.3	70.0	21.7	80.9	25.9	94.0	31.6	110.5	5.62	20.41
.4-.5	18.4	88.4	22.6	103.5	28.2	122.2	36.2	146.7	6.08	26.49
.5-.6	18.8	107.2	22.8	126.3	30.6	152.8	42.1	188.8	6.77	33.26
.6-.7	19.2	126.4	22.6	148.9	28.7	181.5	51.0	239.8	7.58	40.84
.7-.8	20.1	146.5	21.6	170.5	24.5	206.0	36.9	276.7	9.15	49.99
.8-.9	28.8		29.6		31.6		34.8		12.01	
TOTALS		175.3		200.1		237.6		311.5		60.00

TABLE 5

APPENDIX III

ENERGY ANGLE

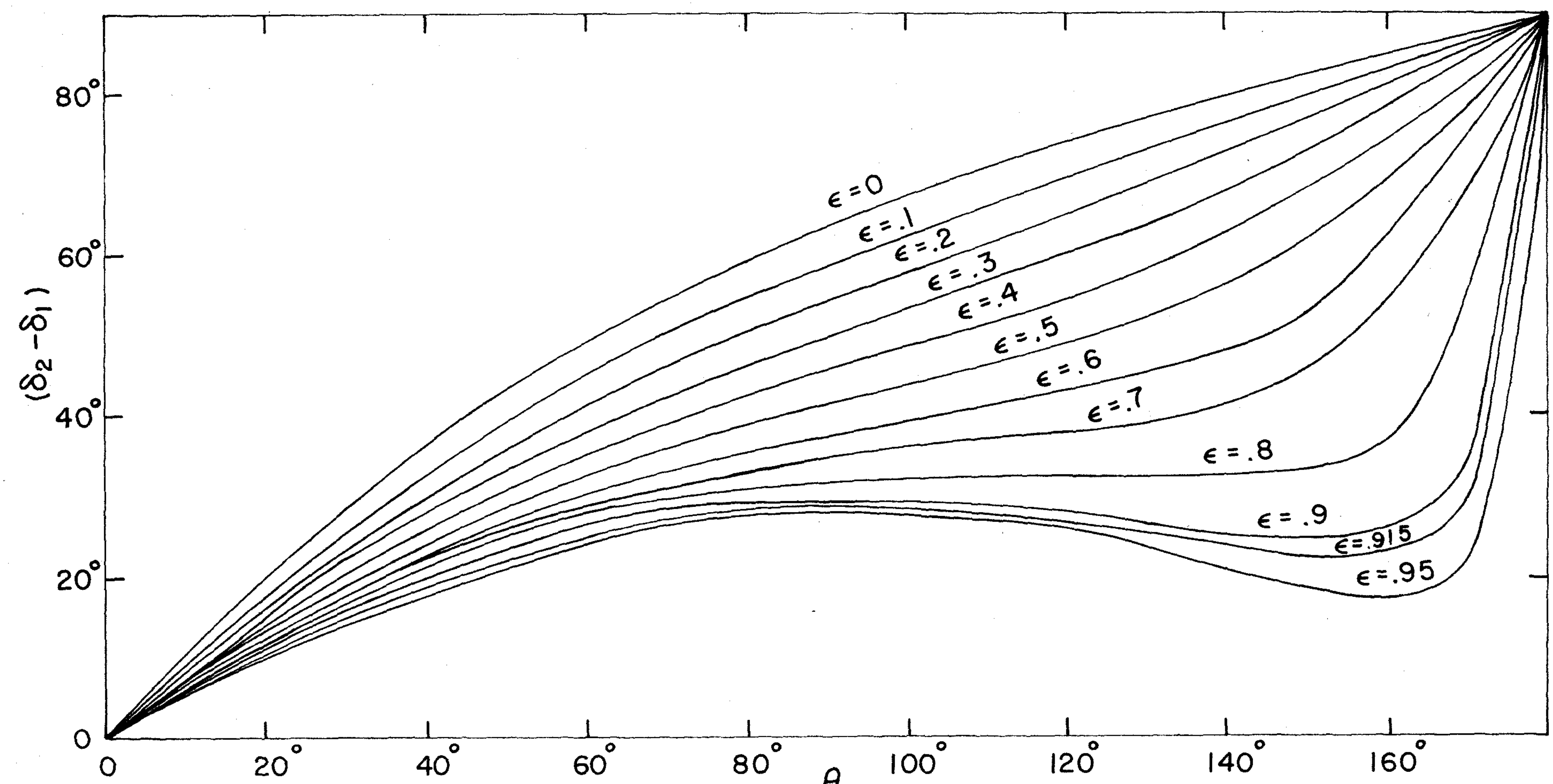


Fig. 13

VELOCITY VECTOR ANGLE

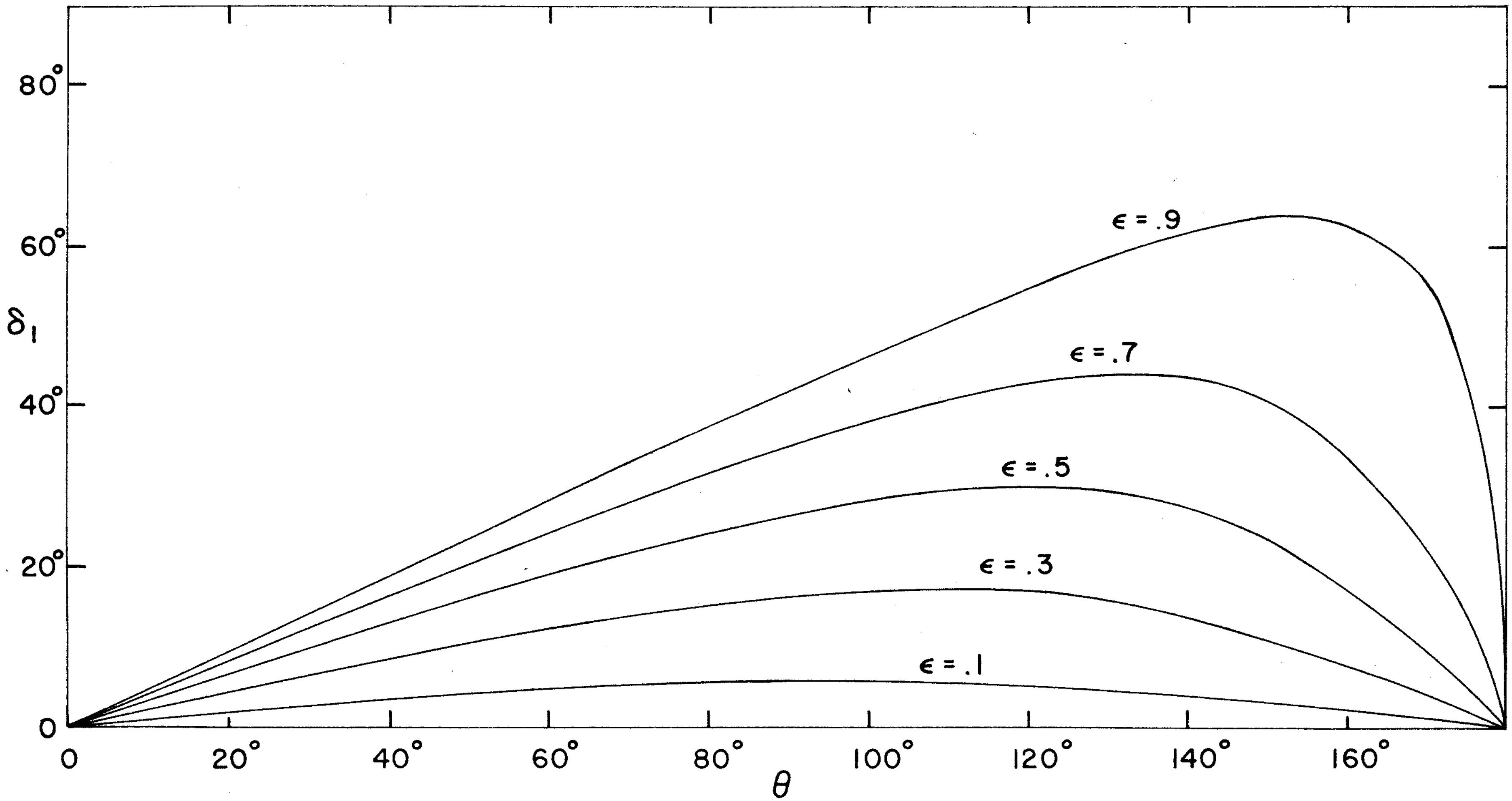


Fig. 14

THRUST ANGLE

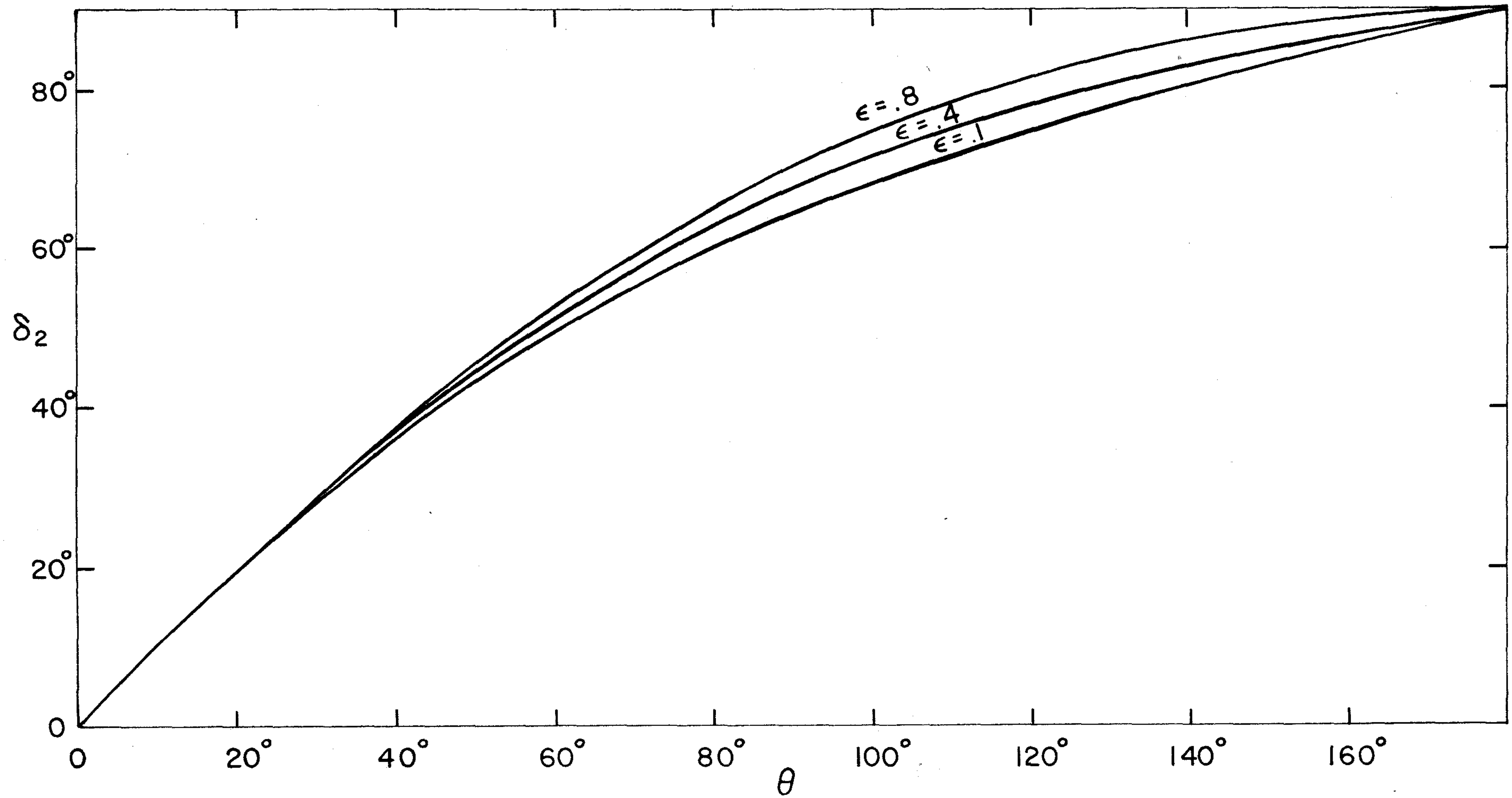


Fig. 14

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