Discrete Approximation of Continuous Allocation Mechanisms

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Dedication

To Joe and my parents

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plishment of us.

Abstract

This dissertation discusses two allocation mechanisms through which prices are set in markets.

The first chapter presents theories on discrete-bid auctions. In particular, we focus on four common auction institutions: the sealed-bid first-price auction, the sealed-bid second-price auction, the English auction and the Dutch auction, in a single-obj ect, independent-private-value setting in which bids can only be multiples of some fixed increment. Two different models of English auction, the pay-your-bid and the penultimate-bid English auction are introduced. It is shown that when bids are discrete , second-price auctions and English auctions are no longer dominance solvable as bidding games. Bidding is more aggressive in the penultimate-bid English auction than that in the pay-your-bid English auction. Nevertheless, first-price auctions and Dutch auctions are still strategically equivalent. The equivalence of expected revenues in the continuous case breaks down when bids are discrete. As the number of bidders participating in the auction increases, auctions in which the winner pays the next hig hest bid (second-price auctions and penultimate-bid English auctions) are more likely to yield higher expected revenues than auctions in which the winner pays his own bid (first-price auctions and pay-your-bid English auctions). The probability of tie in discrete-bid auctions is strictly positive and hence resulting allocations can be Pareto inefficient.

Chapter 2 reports the laboratory observations of bidders' behavior in the pay-your-bid and penultimate-bid English auctions. Results of six experiments show that theories developed in the first chapter in general perform very well in predicting the biddi ng behavior and the price range. However, observations of bidding that is significantly lower than what has been predicted by theory do exist in experiments with small increment. Two possible explanations are discussed.

Chapter 3 discusses a situation in which a monopolist seeks to sell a quality-differentiated spectrum of products of the same generic type to consumers of different characteristics that he cannot observe. The main difference between this framework and the previous literature is that there is a fixed set-up cost of each type of product. The presence of set-up cost makes it impossible for the monopolist to fully separate different types of consumers. The main purpose of this paper is to discuss the monopolist's profit maximization problem and characterize the optimal solution. It is shown that the lowest type in the consumer group consuming the highest quality level would be served efficiently in that the consumer's marginal rate of substitution between price and quality equals that of the monopolist. All other consumers will be served inefficiently and quality distortion takes the form of degradation. The monopolist's profit margin increases with the quality level and an upward shift of the distribution of consumer preference brings higher profit to the monopolist.

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Chapter 1 Theories of Discrete-Bid Auctions

1.1 Introduction

1.1.1 A Brief Overview of Auctions

With the history of around 2500[1]years, auctions have long been used as methods for allocating and procuring goods and services. In the most common form of auctions, there is one seller with one or more items and a large number of interested buyers. Prospective buyers compete by submitting bids that specify quantities and maximum prices. The available supply is then allocated among those offering the highest prices exceeding the seller's reserve price. The actual price paid by a successful bidder depends on a pricing rule, usually selected by the seller and announced before the auction starts.

Auctions are used in a variety of settings. They are used to sell art, antiques, wine, stamps and other collectible items. There are auctions for animal stock and perishable commodities. The U.S. Treasury auctions off weekly the 91-day and 182-day government bonds. Land and buildings, rights for timber and minerals, including coal and oil are usually sold via sealed bids. In many countries, large firms and government agencies use sealed-bid auctions to select vendors and to procure services. An individual who wants to get a house built typically solicits bids from many different builders and chooses the builder offering the lowest price. Recently, the U.S. and Australian governments used auctions to sell spectrum licenses.

Apart from their remarkable applicability, auctions are of considerable theoretical importance. Auctions play an important role in the theory of exchange as they remain one of the simplest and most familiar means of price determination in the absence of market intermediaries. The analysis of auctions has been one of the most fruitful

[1]Herodotus reports the use of auctions as early as 500 BC in Babylon (see Cassady (1967) pp. 26-40 for references) in which men bid for women to wed.

applications of theories of games with incomplete information as bidders' private information is the main factor determining their strategic behavior.

Auctions can be categorized according to the different institutional rules governing the exchange. Since the seminal work of Vickrey (1961), it has been recognized that these rules are important since they can affect bidding incentives and hence the expected revenues from and the efficiency of an exchange. Although there are many different auction institutions recorded in the literature (Cassady, 1967), four major types constitute the primary concerns of economists studying auctions in which there is only one single item to be sold: the English auction; the Dutch auction; the first-price sealed-bid auction; and the second-price sealed-bid auction. In the sealed-bid first-price auction, potential buyers submit sealed-bids to the auctioneer. The highest bidder wins the object for the price he bids. In the sealed-bid second-price auction, the highest bidder wins the object but pays a price equal to the second-highest bid. This auction is invented by Vickrey (1961) and hence is also named after him. In the English auction, the price is successively raised until only one bidder remains. This can be done by having an auctioneer announce prices, or by having bidders call the bids themselves, or by having bids submitted electronically with the current best bid posted. The last remaining bidder wins the object and pays the price at which the auction stops. The Dutch auction is the converse of the English auction. The price begins at some level thought to be higher than any buyer is willing to pay, and the auctioneer decreases the price in decrements until the first buyer accepts the current price. The object is then awarded to that buyer at the price accepted. The central issues involved in the studies of these auctions are the comparison of expected revenues and the efficiency of the allocation under different auction forms.

Vickrey (1961) was the first to solve the independent private-values model using a game theoretic formulation. He found that truth-telling is the dominant strategy for each bidder in the second-price auction as well as in the English auction, regardless of bidders' risk attitudes, distributions from which bidders' valuations of the object are drawn and the level of competition. Moreover, assuming risk neutrality, in the case where each bidder knows his valuation of the item with certainty and bidder's valuations are drawn independently from an identical distribution known to everyone. all four auctions yield the same amount of expected revenues to the seller with equilibrium behavior. Myerson (1981) extended Vickrey's revenue equivalence result to the case of asymmetric bidders, that is, bidders' valuations are drawn from independent. but not necessarily identical, probability distributions. He considered the whole class of possible auction mechanisms of selling the good rather than just a pre-specified set of auction mechanisms and formulated and solved the "optimal auction design problem": among all possible ways of auctioning the object, which one should the auctioneer adopt to maximize her expected net revenues? It is shown that any two mechanisms that always lead to the same allocation of the good would yield the same expected revenue to the seller.

When the bidders' valuations are statistically dependent or when the bidders are risk averse, the revenue equivalence result breaks down. Riley (1989) found that in the affiliated-values model in which bidders' valuations are positively correlated, the seller extracts more revenue in the English auction. More specifically, Milgrom and Weber (1982a) provided complete revenue rankings of the four auctions in this case: The English auction generates the highest revenue followed by the second-price auction and then the Dutch auction and finally, the first-price auction. With risk-averse bidders, the first-price sealed-bid auction produces larger expected revenue than the English or second-price auction (Harris and Raviv 1981b).

Information in auctions plays two important roles: information about the physical state of the world can indicate the quality of the good a bidder considers buying; information about one's potential rivals can signal the level of competition one may expect to encounter. Milgrom and Weber's (1982b) Linkage Principle provides researchers with great insight into the role of information in auctions. They showed that when the bidders' valuations are affiliated, in all of the four types of auctions the seller could raise the expected revenue by adopting a policy of providing expert appraisals of the quality of the objects he sells. They further explored the case where bidders are asymmetrically informed. They found that in most cases the seller can raise the expected revenue by making the better-informed bidder's information public.

Experimental studies of auctions provide ample chance to examine theory predictions in settings in which the experimentor has complete information about the objective economic data. Laboratory observations have shown a systematic failure of the strategic equivalence of second-price and English (Kagel et al. 1987) and of first-price and Dutch auctions (Coppinger et al. 1980, Cox et al. 1982), with higher revenues in sealed-bid auctions in both cases. More aggressive bidding in the first-price and Dutch auctions has been attributed to the risk aversion of subjects and other things.

1.1.2 Why Discrete-Bid Auctions

Most of the existing theoretical work on single-object auctions focuses attention on the information structures of the auction games while assuming no restrictions on the bids. In the majority of the environments considered by auction theorists, a bid is allowed to be any arbitrary function of a bidder's observed information. That is, a bidder may choose to bid any number out of a continuum of choices, based on the information he actually observes and his conjectures of his rivals' bidding strategies, to maximize his benefit. However, restrictions on bids are not unusual in many real-world auctions. For example, in an English auction, the auctioneer sometimes sets a minimum acceptable increment to the highest existing bid to speed up the bidding process. According to Cassady (1967), "often, but not always, the auctioneer announces not only the amount of the various bids as they are made, but also the amount he hopes to obtain. ... In this type of bid calling the auctioneer overtly establishes the amount of the increase he wants." Escalations are standard in American tobacco auctions, and all buyers are familiar with them. "At London antique auctions, the bidders are supposed to be sophisticated enough to know what the amount of the advance will be at various price levels." (Cassady, 1967) California's bankruptcies court uses 5 percent increment in their auctions. In the recent auctions of spectrum licenses by FCC, a minimum bid increment was set initially at the greater of 5 percent of the previous high bid or $0.02 per MHz-pop (FCC, 1995). After all, a continuous bidding space is simply not feasible in reality. Any actual bid can only be some multiple of a currency unit. In many experimental studies, the level of increment or decrement in the English or Dutch auction is a very important technical parameter often ignored by theory. However, as we will illustrate in this chapter and the companion chapter on experiments of discrete-bid auctions, this parameter can conceivably affect bidding behavior.

Cases in which bids can vary by increments are not quite the same thing as the ones of mathematical continuum. As we will show in this chapter, with discreteness in bids, there is no longer a dominant strategy in the second-price auction and the English auction. Even when bidders' valuations are identically and independently distributed, in general the four basic types of auctions do not generate the same amount of expected revenues. The attractiveness of a particular auction form varies by context. One of the simplest explanations of the continuing popularity of auctions is that auctions often lead to outcomes that are Pareto efficient. However, with discreteness in bids, if the bidders' valuations are continuously distributed, the probability of having inefficient allocations is strictly positive.

1.1.3 What Has Been Done

Although auctions with bid increments haven't drawn enough attention from auction theorists, there do exist a couple of works studying bidding strategies and hence the expected prices in discrete-bid auctions. Vickrey (1962) considered the situation where the single object to be bid for can have, for each bidder, one of the only two distinct values and where only two bidding levels are permitted. He found that the seller could extract more expected revenues in the English auction than in the second-price auction by selecting the two bidding levels appropriately. Chwe (1989) discussed the first-price auction within the independent private-values model in which each buyer's valuation is continuously distributed. When buyer values are uniformly distributed and bid possibilities are multiples of an increment, he showed that a symmetric Nash equilibrium bidding strategy exists uniquely and converges to the equilibrium of the continuous-bid auctions as the bid increment goes to zero. He also found that the expected price in discrete-bid auctions is always less than the one in continuous-bid auctions and thus the seller has an incentive to make bid increments small.

1.1.4 Organization of This Chapter

Following Chwe's (1989) framework, we further explore in this chapter the symmetric equilibrium bidding strategies in the other three commonly studied auctions: the sealed-bid second-price auction, the English auction and the Dutch auction. The following section describes the general set-up of the model. Section 1.3 provides the definition of a symmetric equilibrium bidding strategy along with equilibrium conditions. Existence of such equilibrium strategy is also proved in this section. We then provide a brief review of Chwe's work on the first-price auction in section 1.4. Section 1.5, 1.6, 1.7 discuss the symmetric equilibrium bidding strategies in the sealed-bid second-price auction, the English auction and the Dutch auction respectively. We particularly study two different models of English auction in section 1.6: the penultimate-bid English auction and the pay-your-bid English auction. Although the two models of English auction lead to the same prediction of the dominant bidding strategy in the continuous-bid case, bidding behavior is different in the case with discrete bids. Section 1.8 provides a numerical example. Remarks on revenue comparisons are in section 1.9. Section 1.10 collects all the results which shows that the symmetric equilibrium bidding strategies of all the auctions studied converge to their counterparts in the continuous-bid frameworks. Section 1.11 briefly covers the issue of efficiency. The last section concludes this chapter and discusses possible directions of future research.

1.2 The Model

We consider an auction situation in which the auctioneer auctions an object to N buyers. Each bidder's value of the object denoted by v is distributed independently over $[0, 1]$ with cumulative distribution function $F(v)$. Let $f(v)$ be the corresponding density function which is positive everywhere in the domain. Each bidder only knows his own value of the object and picks a b_i from the set $\{b_1, b_2, \ldots, b_{M+1}\}$ where $b_i = \frac{i-1}{M}$. That is, bid possibilities are multiples of the increment $\frac{1}{M}$. In this paper, we follow Chwe's (1989) assumption and assume that $b_1 = 0$ and $b_{M+1} \leq 1$.

1.3 The Equilibrium Strategy

A bidder's strategy $b(v) : [0, 1] \longrightarrow \{b_1, b_2, \ldots, b_{M+1}\}$ is an *equilibrium strategy* if

$$E\Pi(v, b_i) \geq E\Pi(v, b_j) \text{for all } b_j \in \{b_1, b_2, \ldots, b_{M+1}\}, 1 \leq i \leq r \qquad \text{(IC)} \qquad (1.1)$$

and

$$E\Pi(v, b_i) \geq 0 \qquad \text{(IR)} \qquad (1.2)$$

where $E\Pi(v, b_i)$ denotes the expected payoff to a bidder whose value is v and bids b_i.

In this paper, we focus our attention on the symmetric Nash or Baysian Nash equilibrium strategies in the four common auction institutions: 1)the sealed-bid first-price auction; 2)the sealed-bid second-price auction; 3)the English auction and 4)the Dutch auction. We wish to discuss the equilibrium point of these auction games and evaluate the expected price under different circumstances. In any of these auctions, the object is awarded to the bidder who submits the highest bid and in the case of a tie, the winner is chosen randomly. Let b_{-i} denote the bids submitted by the other bidders and $p(b_i, b_{-i})$ denote the payment that a bidder bidding b_i needs to pay, we have

$$\begin{aligned} E\Pi(v, b_i) &= (v - p(b_i, b_{-i}))Prob(\text{winning}|b_i) \\ &= vPr_i - p_iPr_i \\ &= vPr_i - p_i^e \end{aligned} \qquad (1.3)$$

For simplicity, we use p_i to denote $p(b_i, b_{-i})$, P_i to denote Prob(winning$|b_i$) and p_i^e

to denote the expected payment conditional on b_i. To characterize the equilibrium bidding strategy $b(v)$, we need to first state the following lemmas (proofs are straightforward and can be found in Appendix).

Lemma 1 $B_i = \{v \in [0,1] : b(v) = b_i\}$ *is convex in* v*;*

Lemma 2 $b(v) = 0$ *for all* $v \in [0,1]$ *is NOT an equilibrium strategy;*

Lemma 3 $b(v)$ *is monotonically increasing in* v.

It is obvious that in any of the four auction mechanisms, $b(0) = 0$. Therefore, the above three lemmas implies that the symmetric equilibrium strategy must be of the following form:

$$b(v) = \begin{cases} b_i & \text{if } v \in [s_{i-1}, s_i),\ 1 \le i \le r \\ b_r & \text{if } v = s_r \end{cases} \tag{1.4}$$

where $0 = s_0 < s_1 < s_2 < \ldots < s_r = 1$, and r is an integer which satisfies $1 \le r \le M + 1$.

Considering auctions as games of incomplete information, given his value estimate of the object v, a bidder chooses his bid as the best response to his rivals' bidding strategies. The equilibrium bidding strategy of the form (1.4) is a symmetric pure strategy equilibrium. The next proposition shows that such pure strategy equilibrium exists.

Proposition 1 *There exists an equilibrium point in symmetric pure strategies in the four basic types of auctions if bidders' type space and bidding space are defined as in section 1.3.*

This existence result follows from the Purification Theorem in Milgrom and Weber (1985) and the existence of a symmetric equilibrium strategy in any finite symmetrical strategic-form game. The four common auction institutions are all symmetrical games. The sealed-bid first-price auction and second-price auction are strategic-form games while the English auction and the Dutch auction are extensive-form games with perfect recall. By Kuhn's (1953) theorem[2], the existence of symmetric equilibrium applies to all four auction forms. According to the assumptions we made in our model, the equilibrium strategy has a purification and hence a symmetric pure strategy equilibrium exists. A detailed proof of Proposition 1 is in Appendix.

The bidding strategy $b(v)$ is an equilibrium strategy if both the incentive compatibility constraint (1.1) and participation constraint (1.2) are satisfied. In fact, in any of the four auction mechanisms, $E\Pi(v, b_i) \geq 0$ for all $v \in [0, 1]$. Hence, we can restrict our attention to the incentive compatibility constraint. Note, if a bidder chooses to bid b_r instead of b_{r+1}, it must be either $E\Pi(v, b_r) \geq E\Pi(v, b_{r+1})$ or simply b_{r+1} is not available, i.e., $b_r = 1$ and $r = M + 1$. Before we discuss the equilibrium strategy, we can use the following two lemmas (proofs are in Appendix) to further simplify the problem.

Lemma 4 *Given the incentive compatibility constraints, for each* $i = 1, \ldots, r-1$, *type* s_i *is indifferent between bidding* b_{i+1} *and* b_i. *That is,* $E\Pi(s_i, b_i) = E\Pi(s_i, b_{i+1})$.

Lemma 5

$$E\Pi(s_r, b_r) \geq E\Pi(s_r, b_{r+1})$$

for $1 \leq r < M + 1$, *and,*

$$E\Pi(s_i, b_i) = E\Pi(s_i, b_{i+1})$$

for $i = 1, 2, \ldots, r-1$ *are the only binding constraints.*

Immediately following from these two lemmas are the following two propositions:

Proposition 2 *The bidding strategy* $b(v)$ *of the form (1.4) is an equilibrium strategy if the following* r *conditions are satisfied:*

$$E\Pi(s_r, b_r) \geq E\Pi(s_r, b_{r+1}) \tag{1.5}$$

[2] In a game of perfect recall, mixed and behavior strategies are equivalent.

for $1 \leq r < M+1$, *and,*

$$E\Pi(s_i, b_i) = E\Pi(s_i, b_{i+1}) \tag{1.6}$$

for $i = 1, 2, \ldots, r-1$

Proposition 3 *The following two conditions are necessary and sufficient conditions for the incentive compatibility constraints of all types to be satisfied:*

$$p_i^e = s_i(Pr_i - Pr_{i-1}) + s_{i-1}(Pr_{i-1} - Pr_{i-2}) + \ldots + s_1(Pr_2 - Pr_1) + s_0 Pr_1 \tag{1.7}$$

for all $i = 1, 2, \ldots, r$ *and*

$$s_r Pr_r - p_r^e \geq s_r Pr_{r+1} - p_{r+1}^e \tag{1.8}$$

for $1 \leq r < M+1$.

Proposition 3 provides an algorithm to compute the equilibrium strategy by identifying $\{s_i \in (0,1) : 1 \leq i \leq M\}$ and r that satisfy (1.7) and (1.8). Given any auction mechanism with allocation rules Pr_i and payment rules p_i^e, we can start from s_0 and compute $s_1, \ldots, s_i, \ldots$ iteratively using (1.7). To identify r, we need to check if (1.8) holds for each b_i. The first b_i with which (1.8) holds is b_r. If $s_r Pr_r - p_r^e < s_r Pr_{r+1} - p_{r+1}^e$ for all b_i, $r = M+1$ and $b_r = 1$. In the following sections we discuss the symmetric equilibrium strategies in the sealed-bid first-price auction, the sealed-bid second-price auction, the English auction and the Dutch auction respectively.

1.4 Sealed-Bid First-Price Auctions

Chwe (1989) first studied the equilibrium bidding strategy in the sealed-bid first-price auction when bidders can only bid multiples of an increment.

In this section, we provide a brief review of his findings. In the first-price auction, every bidder submits a sealed bid to the auctioneer. The bidder submitting the highest bid wins the object at the price equal to the amount he bids. If more than one bidder submits the same highest bid, the winner is chosen randomly. Therefore,

$$
\begin{aligned}
E\Pi(v,b_i) &= (v-b_i)\sum_{t=1}^{N}\frac{1}{t}Prob(b_i \text{ is the highest bid and } t \text{ buyers bid } b_i) \\
&= \sum_{t=1}^{N}\frac{1}{t}(v-b_i)\begin{pmatrix} N-1 \\ t-1 \end{pmatrix}[F(s_i)-F(s_{i-1})]^{t-1}[F(s_{i-1})]^{N-t} \\
&= \frac{v-b_i}{N[F(s_i)-F(s_{i-1})]}\sum_{t=1}^{N}\begin{pmatrix} N \\ t \end{pmatrix}[F(s_i)-F(s_{i-1})]^{t}[F(s_{i-1})]^{N-t} \\
&= \frac{v-b_i}{N[F(s_i)-F(s_{i-1})]}[F(s_i)-F(s_{i-1})+F(s_{i-1})]^{N}-\begin{pmatrix} N \\ 0 \end{pmatrix}F^{N}(s_{i-1}) \\
&= \frac{v-b_i}{N[F(s_i)-F(s_{i-1})]}[F^{N}(s_i)-F^{N}(s_{i-1})]
\end{aligned}
\tag{1.9}
$$

Chwe (1989) showed that, when F is the uniform distribution, there exists a unique symmetric Nash equilibrium in the first-price auction and the equilibrium converges to that of the continuous bid auction as the bid increment goes to zero. He also found that, however, the expected price in the discrete bid auction is always less than the continuous bid expected price, and thus the seller has an incentive to make bid increments small. These are summarized in the next proposition.

Proposition 4 ***(Chwe)*** *If v is uniformly distributed, then i) there exist unique r and $s_0, s_1, \ldots, s_r$ satisfying the equilibrium conditions; ii) as $M \to \infty$, $b_{N,M}(v) \to \frac{N-1}{N}v$, where $b_{N,M}(v)$ denotes the symmetric Nash equilibrium strategy when the number of bidders is N and the bid increment is $\frac{1}{M}$; iii) for all $M \geq 2$ and $N \geq 2$, $\frac{N-1}{N+1} > Ep_{N,M}$, where $Ep_{N,M}$ denotes the expected price at the equilibrium; iv) $\frac{N}{N-1}b_i < s_i \leq \frac{N}{N-1}b_{i+1}$; v) $b_r = \max\{b_i : b_i < \frac{N-1}{N}\}$.*

1.5 Sealed-Bid Second-Price Auctions

In this section, we consider the sealed-bid second-price auction invented by Vickrey (1961) in which every bidder submits a sealed bid to the auctioneer. The bidder whose bid is the highest wins the object and the price he pays to the seller equals to the second highest bid. In the case of a tie, the winner is chosen randomly and fairly.

Hence,

$$\begin{aligned}
E\Pi(v,b_i) &= (v-b_1)Prob(b_i \text{ is the highest bid and the second highest bid} = b_1 = 0) \\
&+ (v-b_2)Prob(b_i \text{ is the highest bid and the second highest bid} = b_2) \\
&+ \\
&\vdots \\
&+ (v-b_{i-1})Prob(b_i \text{ is the highest bid and the second highest bid} = b_{i-1}) \\
&+ (v-b_i)\sum_{t=2}^{N}\frac{1}{t}Prob(b_i \text{ is the highest bid and } t \geq 2 \text{ buyers bid } b_i)
\end{aligned} \quad (1.10)$$

$$\begin{aligned}
E\Pi(v,b_{i+1}) &= (v-b_1)Prob(b_{i+1} \text{ is the highest bid and the second highest bid} = b_1 = 0) \\
&+ (v-b_2)Prob(b_{i+1} \text{ is the highest bid and the second highest bid} = b_2) \\
&+ \\
&\vdots \\
&+ (v-b_{i-1})Prob(b_{i+1} \text{ is the highest bid and the second highest bid} = b_{i-1}) \\
&+ (v-b_i)Prob(b_{i+1} \text{ is the highest bid and the second highest bid} = b_i) \\
&+ (v-b_{i+1})\sum_{t=2}^{N}\frac{1}{t}Prob(b_{i+1} \text{ is the highest bid and } t \geq 2 \text{ buyers bid } b_{i+1})
\end{aligned} \quad (1.11)$$

To discuss the equilibrium strategy in this auction form, we need to examine the equilibrium conditions (1.6) for $i = 1, \ldots, r-1$ and (1.5) for $i = r$. Plug (1.10) and (1.11) into (1.6) and rearrange the terms, we get

$$
\begin{aligned}
& (s_i - b_{i+1}) \sum_{t=2}^{N} \frac{1}{t} Prob(b_{i+1} \text{ is the highest bid and } t \geq 2 \text{ buyers bid } b_{i+1}) \\
+ \quad & (s_i - b_i)[Prob(b_{i+1} \text{ is the highest bid and the second highest bid} = b_i) \\
& - \sum_{t=2}^{N} \frac{1}{t} Prob(b_i \text{ is the highest bid and } t \geq 2 \text{ buyers bid } b_i)] \\
= \quad & (s_i - b_{i+1}) \sum_{t=2}^{N} \frac{1}{t} \binom{N-1}{t-1} [F(s_{i+1}) - F(s_i)]^{t-1} [F(s_i)]^{N-t} \\
+ \quad & (s_i - b_i)[F^{N-1}(s_i) - F^{N-1}(s_{i-1}) - \sum_{t=2}^{N} \frac{1}{t} \binom{N-1}{t-1} [F(s_i) - F(s_{i-1})]^{t-1} [F(s_{i-1})]^{N-t}] \\
= \quad & 0
\end{aligned}
\tag{1.12}
$$

Note,

$$
\begin{aligned}
& \sum_{t=2}^{N} \frac{1}{t} \binom{N-1}{t-1} (a-b)^{t-1} b^{N-t} \\
= \quad & \frac{\sum_{t=2}^{N} \binom{N}{t} (a-b)^t b^{N-t}}{N(a-b)} \\
= \quad & \frac{1}{N(a-b)} [(a-b+b)^N - \binom{N}{0} b^N - \binom{N}{1} (a-b) b^{N-1}] \\
= \quad & \frac{1}{N(a-b)} [a^N - b^N - N(a-b) b^{N-1}] \\
= \quad & \frac{1}{N(a-b)} [a^N - b^N] - b^{N-1}
\end{aligned}
\tag{1.13}
$$

Hence, we can rewrite (1.12) as follows:

$$\begin{aligned}
& (s_i - b_{i+1})(\frac{1}{N[F(s_{i+1}) - F(s_i)]}[F^N(s_{i+1}) - F^N(s_i)] - F^{N-1}(s_i)) \\
+ \quad & (s_i - b_i)(F^{N-1}(s_i) - F^{N-1}(s_{i-1}) - \frac{1}{N[F(s_i) - F(s_{i-1})]}[F^N(s_i) - F^N(s_{i-1})] + F^{N-1}(s_{i-1})) \\
= \quad & (s_i - b_{i+1})[\frac{F^N(s_{i+1}) - F^N(s_i)}{N[F(s_{i+1}) - F(s_i)]} - F^{N-1}(s_i)] + (s_i - b_i)[F^{N-1}(s_i) - \frac{F^N(s_i) - F^N(s_{i-1})}{N[F(s_i) - F(s_{i-1})]}] \\
= \quad & 0
\end{aligned} \tag{1.14}$$

Plug (1.10) and (1.11) into (1.5) and rearrange the terms (Note, we assume that no one else would bid b_{r+1}), we get

$$\begin{aligned}
& (1 - b_r)[\frac{F^N(s_r) - F^N(s_{r-1})}{N[F(s_r) - F(s_{r-1})]} - F^{N-1}(s_{r-1}) - (F^{N-1}(s_r) - F^{N-1}(s_{r-1}))] \\
= \quad & (1 - b_r)[\frac{1 - F^N(s_{r-1})}{N[1 - F(s_{r-1})]} - 1] \\
\geq \quad & 0
\end{aligned} \tag{1.15}$$

Hence, for $b(v)$ to be an equilibrium strategy, (1.14) and (1.15) need to be satisfied. The following is implied by these two conditions:

Proposition 5 *In the sealed-bid second-price auction, there is no dominant strategy. Moreover, the equilibrium bidding strategy varies with the number of bidders and the distribution function of the value estimates.*

Before proceeding further, we would like to introduce the following two lemmas (The proofs are straightforward and can be found in Appendix):

Lemma 6

$$\frac{F^N(s_i) - F^N(s_{i-1})}{N[F(s_i) - F(s_{i-1})]} < F^{N-1}(s_i) < \frac{F^N(s_{i+1}) - F^N(s_i)}{N[F(s_{i+1}) - F(s_i)]}$$

for $i = 1, \ldots, r-1$, *and*

$$\frac{1 - F^N(s_{r-1})}{N[1 - F(s_{r-1})]} < 1$$

Lemma 7 *In the sealed-bid second-price auction,* $r = M + 1$ *and* $b_r = 1$.

We are now ready to discuss the equilibrium strategy of the sealed-bid second-price auction.

Proposition 6 *For* $i = 1, \ldots, r-1$, *the* s_i *that satisfies (1.14) must lie between* b_i *and* b_{i+1}. *That is,* $b_i < s_i < b_{i+1}$.

Proof: i) Suppose $s_i \geq b_{i+1}$, then by Lemma 6,

$$\begin{aligned}
&(s_i - b_{i+1})\left[\frac{F^N(s_{i+1}) - F^N(s_i)}{N[F(s_{i+1}) - F(s_i)]} - F^{N-1}(s_i)\right] + (s_i - b_i)\left[F^{N-1}(s_i) - \frac{F^N(s_i) - F^N(s_{i-1})}{N[F(s_i) - F(s_{i-1})]}\right] \\
\geq\ & 0 + (s_i - b_i)\left[F^{N-1}(s_i) - \frac{F^N(s_i) - F^N(s_{i-1})}{N[F(s_i) - F(s_{i-1})]}\right] \\
>\ & 0
\end{aligned}$$

Since $b_{i+1} > b_i$ implies $s_i > b_i$. Contradiction.

ii) Recall equation (1.14):

$$(s_i - b_{i+1})\left[\frac{F^N(s_{i+1}) - F^N(s_i)}{N[F(s_{i+1}) - F(s_i)]} - F^{N-1}(s_i)\right] + (s_i - b_i)\left[F^{N-1}(s_i) - \frac{F^N(s_i) - F^N(s_{i-1})}{N[F(s_i) - F(s_{i-1})]}\right] = 0$$

From i), $s_i - b_{i+1} < 0$. From Lemma 6, $\frac{F^N(s_{i+1}) - F^N(s_i)}{N[F(s_{i+1}) - F(s_i)]} - F^{N-1}(s_i) > 0$ and $F^{N-1}(s_i) - \frac{F^N(s_i) - F^N(s_{i-1})}{N[F(s_i) - F(s_{i-1})]} > 0$. Therefore, $s_i - b_i > 0$. Or, $s_i > b_i$. Q.E.D.

Proposition 6 implies a trade-off between the probability of winning and the amount of payoff conditional on winning. By bidding lower, a bidder commits to a lower price that he needs to pay if he wins. By bidding higher, a bidder enhances his chance to win. For all $i = 1, 2, \ldots, r-1$, any bidder with value $v \in [b_i, s_i)$ underbids in a second-price auction. That is, he bids some amount lower than his valuation. Any bidder with value $v \in [s_i, b_{i+1}]$ overbids in the auction.

1.6 English Auction

Most people are more familiar with the English auction employing electronic equipment. With the electronic equipment, "the price indicator moves clockwise on an ascending basis ... from zero until the transaction is completed. As long as two or more bidders are pressing the keys, the hand of the clock moves and prices advance. It continues to move until all bidders except one withdraw, at which time the clock stops ...." (Cassady, 1967, pp:196) However, before the electronic device was introduced, people ran English auctions as open ascending-bid selling schemes: "the auctioneer seeks an initial bid from one of the assembled buyers with the expectation that those interested in the item or lot will bid against one another until all but the highest bidder are eliminated." (Cassady, 1967, p:57)

From the above description, it is not very hard to tell that these two different versions of English auctions adopt different rules to determine the price that the winner pays. In the case with electronic device, since the price indicator stops when all bidders except one withdraw, the price that the last remaining bidder pays is the price at which the last bidder but one withdraws. On the other hand, in the open ascending-bid scheme, the winning bidder wins by outbidding all other bidders and he pays the amount that he bids. When the bid space is continuous, the standard argument on bidding strategies in English auctions holds for both cases and there is no difference in equilibrium bidding strategies and hence expected revenues in these two different English auctions. However, with the introduction of discreteness in bids, different pricing rules will induce different equilibrium bidding strategies. In fact, in the open ascending-bid scheme, if there is some time interval during which each increment of bids can be made and during which each bidder can observe the other bidders' decisions, a bidder may also decide strategically how quickly to respond with a higher bid during this interval. Hence, timeliness in response would also be a strategic variable in this case. In this paper, we restrict our attention to the difference in bidding strategies resulting from different pricing rules. Discussions on the bidders' strategic choice of the timing of the response are out of the scope of this paper.

To achieve the research purpose of this paper, we will examine two different models of English auctions, both with electronic devices. In the first case, the price indicator stops when the *last* bidder withdraws and therefore the winning bidder pays the price at which he withdraws. We call this pay-your-bid English auction. In the second case, the price indicator stops when *all bidders except one* withdraw and the winning bidder pays the price at which the second highest bidder withdraws. We call this penultimate-bid English auction. We will discuss the equilibrium strategies in these two different models separately in the following subsections.

1.6.1 Pay-Your-Bid English Auctions

In this version of English auction, the price and the number of bidders who are currently staying in the auction are posted on an electronic display. The price is raised by $\frac{1}{M}$ every time starting from 0 and the number of bidders currently staying in is updated at the same time. A bidder remains active in the auction by holding his button down. When he releases, he has withdrawn from the auction. The auction stops when the last bidder releases his button. The bidder who is the last one to drop out from the auction wins the object and he pays the price at which he withdraws from the auction. In the case where there are at least two bidders in the last group of bidders withdrawing from the auction, the winner will be selected among them randomly. We use p_i to denote the price and k_i to denote the number of bidders who decide to stay at p_{i-1}. Hence, $p_1 = 0$ and $k_1 = N$. When the price is raised to p_i, a bidder's strategy is to decide whether to remain or to withdraw from the auction at this price. The decision is made based on his value of the object, the number of currently active bidders k_i and the current price p_i. Therefore, the bidding strategy $b_i : v \times p_i \times k_i \longrightarrow \{0, 1\}$. If he chooses 0, he withdraws from the auction and if he chooses 1, he decides to stay. Intuitively, if the price is raised to the amount such that $p_i > v - \frac{1}{M}$, the bidder wouldn't want to stay further in the auction. On the other hand, when the price is very low relative to the bidder's value, the bidder would like to stay in the auction to increase his probability of winning provided there is at least one more other active bidder. This intuition can be formalized by the following two lemmas.

Lemma 8 *For all i, if $v - \frac{1}{M} < p_i \leq v$, or $p_i \leq v < p_{i+1}$, $b_i(v, p_i, k_i) = 0$.*

Proof: The proof is very straightforward.

$$\begin{aligned} E\Pi(v, b_i = 0) &= \frac{1}{k_i}(v - p_i)\text{Prob}\{\text{all } k_i \text{ bidders release their buttons at} \\ &\quad p_i | k_i \text{ bidders decided to stay at } p_{i-1}\} \\ &\geq 0. \end{aligned}$$

However,

$$\begin{aligned} E\Pi(v, b_i = 1) &= \frac{1}{k_{i+1}}(v - p_{i+1})\text{Prob}\{\text{all } k_{i+1} \text{ people release their buttons} \\ &\quad \text{at } p_{i+1} | k_i \text{ bidders decided to stay at } p_{i-1}\} \\ &\quad + E\Pi(v, b_{i+1} = 1) \\ &< 0. \end{aligned}$$

Therefore, the bidder is better off withdrawing at p_i rather than staying until p_{i+1}.

Lemma 9 *For all i, if $p_i \leq v - 2\frac{1}{M}$, or $v \geq p_{i+2}$ and $k_i \geq 2$, $b_i(v, p_i, k_i) = 1$.*

Proof:

$$E\Pi(v, b_i = 0) = \frac{1}{k_i}(v - p_i)\text{Prob}\{\text{all } k_i \text{ people release their buttons at } p_i\}$$

$$\begin{aligned} E\Pi(v, b_i = 1) &= E\Pi(v, b_i = 1 | \text{all } k_i \text{ bidders release their buttons at } p_i) \\ &\quad + E\Pi(v, b_i = 1 | \text{not all } k_i \text{ bidders release their buttons at } p_i) \\ &= (v - (p_i + \frac{1}{M}))\text{Prob}\{\text{all } k_i \text{ people release their buttons at } p_i\} \\ &\quad + E\Pi(v, b_i = 1 | \text{not all } k_i \text{ bidders release their buttons at } p_i) \end{aligned}$$

Since the second term in the above equation is nonnegative, $(v - (p_i + \frac{1}{M})) \geq \frac{1}{k_i}(v - p_i)$ would imply $E\Pi(v, b_i = 1) \geq E\Pi(v, b_i = 0)$.

$$(v - (p_i + \frac{1}{M})) \geq \frac{1}{k_i}(v - p_i) \Longleftrightarrow p_i \leq v - \frac{k_i}{k_i - 1}\frac{1}{M}$$

But $\frac{k_i}{k_i-1} \leq 2$ for $k_i \geq 2$. Obviously, if $k_i = 1$, $b_i(v, p_i, k_i) = 0$ and the auction stops. Q.E.D.

Note, as long as each bidder's utility function is non-decreasing in his monetary payoff[3], the results stated in the above two lemmas hold regardless of the initial level of competition N, the size of increment M, the distribution function F and the bidders' risk attitudes.

With the above two lemmas, the only price at which that we need to study the bidder's decision is p_i such that $v - \frac{2}{M} \leq p_i \leq v - \frac{1}{M}$, or $v \in [p_{i+1}, p_{i+2}]$. At p_i, we know that bidders with value $v \in [p_{i+1}, p_{i+2}]$ are staying in the auction up to this price given lemma 8 and these bidders will drop out from the auction when price is raised to p_{i+1} given lemma 9. With this p_i,

$$\begin{aligned} E\Pi(v, b_i = 0) &= \frac{1}{k_i}(v - p_i)Prob(\text{All } k_i \text{ bidders drop out at } p_i | \text{All } k_i \\ &\quad \text{bidders didn't drop out at } p_{i-1}) \\ &= \frac{1}{k_i}(v - p_i)\frac{[F(s_i) - F(s_{i-1})]^{k_i - 1}}{[1 - F(s_{i-1})]^{k_i - 1}} \end{aligned} \tag{1.16}$$

$$\begin{aligned} E\Pi(v, b_i = 1) &= (v - p_{i+1})Prob(\text{All } k_i \text{ bidders drop out at } p_i | \text{All } k_i \\ &\quad \text{bidders didn't drop out at } p_{i-1}) \\ &\quad + \sum_{k_{i+1}=2}^{k_i} \frac{1}{k_{i+1}}(v - p_{i+1})Prob(\text{All } k_{i+1} \text{ stays at } p_i \text{ and all } k_{i+1} \text{ drop out} \\ &\quad \text{at } p_{i+1} | \text{All } k_i \text{ didn't drop out at } p_{i-1}) \end{aligned}$$

[3]This includes the case of risk-aversion in which a bidder's utility may be some non-decreasing concave function of his monetary payoffs.

$$
\begin{aligned}
&= (v - p_i - \frac{1}{M})\frac{[F(s_i) - F(s_{i-1})]^{k_i - 1}}{[1 - F(s_{i-1})]^{k_i - 1}} \\
&\quad + \sum_{k_{i+1}=2}^{k_i} \frac{1}{k_{i+1}}(v - p_i - \frac{1}{M}) \\
&\quad * \begin{pmatrix} k_i - 1 \\ k_{i+1} - 1 \end{pmatrix} \frac{[F(s_{i+1}) - F(s_i)]^{k_{i+1}-1}[F(s_i) - F(s_{i-1})]^{k_i - k_{i+1}}}{[1 - F(s_{i-1})]^{k_i - 1}} \\
&= \sum_{k_{i+1}=1}^{k_i} \frac{1}{k_{i+1}}(v - p_i - \frac{1}{M}) \\
&\quad * \begin{pmatrix} k_i - 1 \\ k_{i+1} - 1 \end{pmatrix} \frac{[F(s_{i+1}) - F(s_i)]^{k_{i+1}-1}[F(s_i) - F(s_{i-1})]^{k_i - k_{i+1}}}{[1 - F(s_{i-1})]^{k_i - 1}} \\
&= \frac{(v - p_i - \frac{1}{M})}{k_i[1 - F(s_{i-1})]^{k_i - 1}[F(s_{i+1}) - F(s_i)]} * \\
&\quad ([F(s_{i+1}) - F(s_{i-1})]^{k_i} - [F(s_i) - F(s_{i-1})]^{k_i}) \qquad (1.17)
\end{aligned}
$$

By the definition of s_i, we have the following equation

$$
\begin{aligned}
&(s_i - p_i)[F(s_i) - F(s_{i-1})]^{k_i - 1} \\
&= \frac{(s_i - p_i - \frac{1}{M})}{[F(s_{i+1}) - F(s_i)]}([F(s_{i+1}) - F(s_{i-1})]^{k_i} - [F(s_i) - F(s_{i-1})]^{k_i}) \qquad (1.18)
\end{aligned}
$$

Note that to figure out the equilibrium strategy at p_i, one needs to solve (1.18) $r - i$ times to find $\{s_i, s_{i+1}, \ldots, s_{r-1}\}$. However, at p_i, one only observes k_i but has no information on $\{k_{i+1}, \ldots, k_{r-1}\}$. Therefore, at p_i, each bidder needs to guess how many people would stay or drop out in later rounds based on his current observation and solve for $\{s_i, s_{i+1}, \ldots, s_{r-1}\}$ based on the guesses. Hence, the $r - i$ equations will take the following form:

For $j = i$, the equation takes the form of (1.18). For all $i < j \le r - 1$,

$$
\begin{aligned}
&(s_j - p_j) \sum_{k_{i+1}=2}^{k_i} \sum_{k_{i+2}=2}^{k_{i+1}} \cdots \sum_{k_j=2}^{k_{j-1}} \frac{1}{k_j}\text{Prob}(k_{\tilde{j}} \text{ bidders decide to stay at } p_{\tilde{j}-1} \text{ for } \tilde{j} = i+1, \ldots, j \\
&\text{and } k_j \text{ bidders decide to withdraw at } p_j | k_i \text{ bidders decided to stay at } p_{i-1})
\end{aligned}
$$

$$= \quad (s_j - p_j - \frac{1}{M}) \sum_{k_{i+1}=2}^{k_i} \sum_{k_{i+2}=2}^{k_{i+1}} \cdots \sum_{k_j=2}^{k_{j-1}} \sum_{k_{j+1}=1}^{k_j} \frac{1}{k_{j+1}} \text{Prob}(k_{\bar{j}} \text{ bidders decide to stay at } p_{\bar{j}-1} \text{ for}$$

$\bar{j} = i+1, \ldots, j+1$and k_{j+1} bidders decide to withdraw at $p_{j+1}|k_i$ bidders

decided to stay at p_{i-1})

(1.19)

That is,

$$(s_j - p_j) \sum_{k_{i+1}=2}^{k_i} \sum_{k_{i+2}=2}^{k_{i+1}} \cdots \sum_{k_j=2}^{k_{j-1}} \frac{1}{k_j} \frac{1}{[1 - F(s_{i-1})]^{k_i - 1}}$$

$$* \begin{pmatrix} & & k_i - 1 & & \\ k_j - 1 & k_{j-1} - 1 & \ldots & k_{i+2} - 1 & k_{i+1} - 1 \end{pmatrix}$$

$$*[F(s_i) - F(s_{i-1})]^{k_i - k_{i+1}} [F(s_{i+1}) - F(s_i)]^{k_{i+1} - k_{i+2}} \ldots [F(s_{j-1}) - F(s_{j-2})]^{k_{j-1} - k_j}$$

$$*[F(s_j) - F(s_{j-1})]^{k_j - 1}$$

$$= \quad (s_j - p_j - \frac{1}{M}) \sum_{k_{i+1}=2}^{k_i} \sum_{k_{i+2}=2}^{k_{i+1}} \cdots \sum_{k_j=2}^{k_{j-1}} \sum_{k_{j+1}=1}^{k_j} \frac{1}{k_{j+1}} \frac{1}{[1 - F(s_{i-1})]^{k_i - 1}}$$

$$* \begin{pmatrix} & & & k_i - 1 & & \\ k_{j+1} - 1 & k_j - 1 & k_{j-1} - 1 & \ldots & k_{i+2} - 1 & k_{i+1} - 1 \end{pmatrix}$$

$$*[F(s_i) - F(s_{i-1})]^{k_i - k_{i+1}} [F(s_{i+1}) - F(s_i)]^{k_{i+1} - k_{i+2}} \ldots [F(s_{j-1}) - F(s_{j-2})]^{k_{j-1} - k_j}$$

$$*[F(s_j) - F(s_{j-1})]^{k_j - k_{j+1}} [F(s_{j+1}) - F(s_j)]^{k_{j+1} - 1}$$

where

$$\begin{pmatrix} & & & k_i - 1 & & \\ k_{j+1} - 1 & k_j - 1 & k_{j-1} - 1 & \ldots & k_{i+2} - 1 & k_{i+1} - 1 \end{pmatrix}$$

$$= \frac{(k_i - 1)!}{(k_i - k_{i+1})!(k_{i+1} - k_{i+2})! \ldots (k_{j-1} - k_j)!(k_{j+1} - k_j)!(k_{j+1} - 1)!}$$

and $\begin{pmatrix} & & k_i - 1 & & \\ k_j - 1 & k_{j-1} - 1 & \ldots & k_{i+2} - 1 & k_{i+1} - 1 \end{pmatrix}$ is defined similarly.

Note, $\frac{1}{[1-F(s_{i-1})]^{k_i-1}}$ cancels on both sides of the equation; therefore, the simplified version of the above equation is

$$
\begin{aligned}
& (s_j - p_j) \sum_{k_{i+1}=2}^{k_i} \sum_{k_{i+2}=2}^{k_{i+1}} \cdots \sum_{k_j=2}^{k_{j-1}} \frac{1}{k_j} \begin{pmatrix} & & k_i - 1 & & \\ k_j - 1 & k_{j-1} - 1 & \ldots & k_{i+2} - 1 & k_{i+1} - 1 \end{pmatrix} \\
& *[F(s_i) - F(s_{i-1})]^{k_i - k_{i+1}} [F(s_{i+1}) - F(s_i)]^{k_{i+1} - k_{i+2}} \ldots [F(s_{j-1}) - F(s_{j-2})]^{k_{j-1} - k_j} \\
& *[F(s_j) - F(s_{j-1})]^{k_j - 1} \\
= & (s_j - p_j - \frac{1}{M}) \sum_{k_{i+1}=2}^{k_i} \sum_{k_{i+2}=2}^{k_{i+1}} \cdots \sum_{k_j=2}^{k_{j-1}} \sum_{k_{j+1}=1}^{k_j} \frac{1}{k_{j+1}} \\
& * \begin{pmatrix} & & & k_i - 1 & & \\ k_{j+1} - 1 & k_j - 1 & k_{j-1} - 1 & \ldots & k_{i+2} - 1 & k_{i+1} - 1 \end{pmatrix} \\
& *[F(s_i) - F(s_{i-1})]^{k_i - k_{i+1}} [F(s_{i+1}) - F(s_i)]^{k_{i+1} - k_{i+2}} \ldots [F(s_{j-1}) - F(s_{j-2})]^{k_{j-1} - k_j} \\
& *[F(s_j) - F(s_{j-1})]^{k_j - k_{j+1}} [F(s_{j+1}) - F(s_j)]^{k_{j+1} - 1}
\end{aligned} \tag{1.20}
$$

Then, at p_{i+1}, the bidder observes k_{i+1} and uses this piece of new information and s_i solved at p_i to solve for a new set of $\{s_{i+1}, \ldots, s_{r-1}\}$.

Therefore, at each p_i, based on the observation of k_i, the number of bidders who decided to stay at p_{i-1}, all bidders who decided to stay at p_{i-1} with $v < s_i$ will drop out from the auction while all $v \geq s_i$ will stay. From lemma 8 and lemma 9, $s_i \in [p_{i+1}, p_{i+2}]$. A bidder will stay at most up to the price within one increment below his value. Just like what we have learned from the first-price auction in the continuous case, bidders in the pay-your-bid English auction have incentives to shade their bids to a certain extent, trading the probability to win for a higher payoff. Even a bidder with the highest possible value 1 is willing to stay in the auction only up to $\frac{M-1}{M}$.

Proposition 7 *In the pay-your-bid English auction, if $k_i \geq 2$ for all $i = 1, \ldots, r-1$, then $r = M$ and the highest achievable price $p_r = \frac{M-1}{M}$.*

Proof: First, when $i = M$, since

$$E\Pi(1, b_M = 0) = \frac{1}{k_M}(1 - \frac{M-1}{M})\text{Prob}\{\text{all } k_M \text{ bidders release their buttons at } p_M\} > 0$$

and

$$E\Pi(1, b_M = 1) = 1 - 1 = 0$$

we have $r \neq M + 1$. Secondly, from lemma 9, when $i = M - 1$, $b_i(1) = 1$ if $k_i \geq 2$. Of course, if for any $i \leq M$, $k_i = 1$, the auction will stop at that i. Therefore, if $k_i \geq 2$ for all $i = 1, \ldots, r-1$, $r = M$ and $p_r = \frac{M-1}{M}$. Q.E.D.

The next proposition illustrates the non-uniqueness of the equilibrium strategy in the pay-your-bid English auction.

Proposition 8 *The equilibrium strategy in the pay-your-bid English auction is not unique.*

Proof:

We can illustrate this with a simple example in which $N = 2, M = 3, F(v) = v$. In this case,(1.20) gives two sets of solutions $\{\frac{2}{3}, \frac{2}{3}\}$ and $\{\frac{1}{2}, 1\}$. That is, there are two equilibria in the pay-your-bid English auction. In one equilibrium, bidders with values less than $\frac{2}{3}$ will decide to drop out from the auction at $p_1 = 0$ while bidders with values higher than $\frac{2}{3}$ will raise the price up to $\frac{2}{3}$ and no bidder will further raise the price up. In the other equilibrium, bidders with values less than $\frac{1}{2}$ will decide to drop out from the auction at $p_1 = 0$ while bidders with values higher than $\frac{1}{2}$ will raise the price to $\frac{1}{3}$ and no bidder will further raise the price up. Q.E.D.

1.6.2 Penultimate-Bid English Auctions

This version of the English auctions is similar to the pay-your-bid English auction except that the auction stops when all bidders except one release their buttons. The last remaining bidder wins the object and pays the price at which the auction stops, i.e., the price at which the last bidder except him withdraws. If more than one bidder decide to stay at a posted price but all decide to withdraw at the next price, a winner will be chosen randomly from these bidders and the winner pays the price at which he withdraws.

If the current posted price is higher than a bidder's value and at least two bidders decided to stay at p_{i-1}, the bidder needs to pay at least the current price if he stays and wins the object. In this case, he incurs a negative payoff. In the case where the current price is more than one increment less than his value, if the bidder withdraws, the only chance that he can own the object is when all other bidders withdraw at p_i and he is the lucky one chosen to be the winner. On the other hand, if he stays, he can win the object for sure if all other bidders withdraw at p_i and pays the same price as he would if he withdraws and wins. As a consequence, the bidder is always better off staying if $p_i < v - \frac{1}{M}$. The following two lemmas formally summarize this intuition.

Lemma 10 *For all i, if $p_i \leq v - \frac{1}{M}$ and $k_i \geq 2$, then $b_i(v, p_i, k_i) = 1$.*

Proof:

Since $p_i \leq v - \frac{1}{M}$, $v - p_{i+1} = v - (p_i + \frac{1}{M}) \geq 0$. Hence,

$$\begin{aligned}
E\Pi(v, b_i = 1) &= (v - p_i)\text{Prob}\{\text{All } k_i \text{ bidders withdraw at} \\
&\quad p_i | k_i \text{ bidders didn't withdraw at } p_{i-1}\} \\
&\quad + \frac{1}{k_{i+1}}(v - p_{i+1})\text{Prob}\{k_{i+1} \text{ out of } k_i \text{ bidders decide to stay at } p_i \\
&\quad \text{but all withdraw at } p_{i+1} | k_i \text{ bidders didn't withdraw at } p_{i-1}\} \\
&> \\
E\Pi(v, b_i = 0) &= \frac{1}{k_i}(v - p_i)\text{Prob}\{\text{All } k_i \text{ bidders withdraw at} \\
&\quad p_i | k_i \text{ bidders didn't withdraw at } p_{i-1}\} \\
&> 0
\end{aligned}$$

Lemma 11 *For all i, if $p_i \geq v$, then $b_i(v, p_i, k_i) = 0$.*

Proof:

Since $p_i \geq v$, $v - p_i \leq 0$ and $v - p_{i+1} < 0$. Hence,

$$\begin{aligned} E\Pi(v, b_i = 1) &= (v - p_i)\text{Prob}\{\text{All } k_i \text{ bidders withdraw at} \\ &\quad p_i | k_i \text{ bidders didn't withdraw at } p_{i-1}\} \\ &\quad + \frac{1}{k_{i+1}}(v - p_{i+1})\text{Prob}\{k_{i+1} \text{ out of } k_i \text{ bidders decide to stay at } p_i \\ &\quad \text{but all withdraw at } p_{i+1} | k_i \text{ bidders didn't withdraw at } p_{i-1}\} \\ &< \\ E\Pi(v, b_i = 0) &= \frac{1}{k_i}(v - p_i)\text{Prob}\{\text{All } k_i \text{ bidders withdraw at} \\ &\quad p_i | k_i \text{ bidders didn't withdraw at } p_{i-1}\} \\ &\leq 0 \end{aligned}$$

Similar to our discussions in the pay-your-bid English auction, the above two lemmas hold for any arbitrary N, M, continuous F as long as each bidder's utility is non-decreasing in his monetary payoffs. With these two lemmas, for each bidder, the only price at which that we need to study the bidder's decision is p_i such that $v \in (p_i, p_{i+1})$. At each p_i, we know that bidders with value $v \in (p_i, p_{i+1})$ are staying in the auction up to this price given lemma 10 and these bidders will withdraw for sure from the auction when price is raised to p_{i+1} given lemma 11. With this p_i,

$$\begin{aligned} E\Pi(v, b_i = 0) &= \frac{1}{k_i}(v - p_i)Prob(\text{All } k_i \text{ bidders withdraw at } p_i | \text{All } k_i \\ &\quad \text{bidders didn't withdraw at } p_{i-1}) \\ &= \frac{1}{k_i}(v - p_i)\frac{[F(s_i) - F(s_{i-1})]^{k_i - 1}}{[1 - F(s_{i-1})]^{k_i - 1}} \end{aligned} \tag{1.21}$$

$$\begin{aligned} E\Pi(v, b_i = 1) &= (v - p_i)Prob(\text{All } k_i \text{ bidders drop out at } p_i | \text{All } k_i \\ &\quad \text{bidders didn't drop out at } p_{i-1}) \end{aligned}$$

$$
\begin{aligned}
& + \sum_{k_{i+1}=2}^{k_i} \frac{1}{k_{i+1}}(v - p_{i+1}) Prob(k_{i+1} \text{ out of } k_i \text{ bidders stays at } p_i \\
& \text{and all } k_{i+1} \text{ bidders withdraw at } p_{i+1} | \text{All } k_i \text{ didn't drop out at } p_{i-1}) \\
= \; & (v - p_i) \frac{[F(s_i) - F(s_{i-1})]^{k_i - 1}}{[1 - F(s_{i-1})]^{k_i - 1}} \\
& + \sum_{k_{i+1}=2}^{k_i} \frac{1}{k_{i+1}}(v - p_i - \frac{1}{M}) \\
& * \begin{pmatrix} k_i - 1 \\ k_{i+1} - 1 \end{pmatrix} \frac{[F(s_{i+1}) - F(s_i)]^{k_{i+1}-1}[F(s_i) - F(s_{i-1})]^{k_i - k_{i+1}}}{[1 - F(s_{i-1})]^{k_i - 1}} \\
= \; & (v - p_i) \frac{[F(s_i) - F(s_{i-1})]^{k_i - 1}}{[1 - F(s_{i-1})]^{k_i - 1}} \\
& + \frac{(v - p_i - \frac{1}{M})}{k_i[1 - F(s_{i-1})]^{k_i - 1}[F(s_{i+1}) - F(s_i)]} * \\
& ([F(s_{i+1}) - F(s_{i-1})]^{k_i} - [F(s_i) - F(s_{i-1})]^{k_i} - k_i[F(s_{i+1}) - F(s_i)] \\
& *[F(s_i) - F(s_{i-1})]^{k_i - 1}) \qquad (1.22)
\end{aligned}
$$

By the definition of s_i, we have the following equation

$$
\begin{aligned}
& (s_i - p_i)(1 - \frac{1}{k_i})[F(s_i) - F(s_{i-1})]^{k_i - 1} \\
+ \; & \frac{(s_i - p_i - \frac{1}{M})}{[F(s_{i+1}) - F(s_i)]}([F(s_{i+1}) - F(s_{i-1})]^{k_i} - [F(s_i) - F(s_{i-1})]^{k_i} - k_i[F(s_{i+1}) - F(s_i)] \\
& *[F(s_i) - F(s_{i-1})]^{k_i - 1}) \\
= \; & 0 \qquad (1.23)
\end{aligned}
$$

Again, to figure out the equilibrium strategy at p_i, one needs to solve (1.23) $r - i$ times to find $\{s_i, s_{i+1}, \ldots, s_{r-1}\}$. However, at p_i, one only observes k_i but has no information on $\{k_{i+1}, \ldots, k_{r-1}\}$. Therefore, at p_i, each bidder needs to guess how many people would stay or withdraw in later rounds based on his current observation and solve for $\{s_i, s_{i+1}, \ldots, s_{r-1}\}$ based on the guesses. Equation (1.23) is very similar to equation (1.18) with the only difference in the case of no tie when the auction stops. If the auction stops at p_{j+1} and only the winning bidder stays at p_j, the winner pays p_{j+1} in the pay-your-bid English auction and p_j in the penultimate-bid English auction. Hence, the $r-i$ equations will take the following forms:

For $j=i$, the equation takes the form of (1.23).

For all $i < j \leq r-1$,

$$
\begin{aligned}
&(s_j - p_j) \sum_{k_{i+1}=2}^{k_i} \sum_{k_{i+2}=2}^{k_{i+1}} \cdots \sum_{k_j=2}^{k_{j-1}} \frac{1}{k_j} \text{Prob}(k_{\tilde{j}} \text{ bidders decide to stay at } p_{\tilde{j}-1} \text{ for } \tilde{j} = i+1,\ldots,j \\
&\text{and } k_j \text{ bidders decide to withdraw at } p_j | k_i \text{ bidders decided to stay at } p_{i-1}) \\
= \; &(s_j - p_j - \frac{1}{M}) \sum_{k_{i+1}=2}^{k_i} \sum_{k_{i+2}=2}^{k_{i+1}} \cdots \sum_{k_j=2}^{k_{j-1}} \sum_{k_{j+1}=2}^{k_j} \frac{1}{k_{j+1}} \text{Prob}(k_{\tilde{j}} \text{ bidders decide to stay at } p_{\tilde{j}-1} \text{ for} \\
&\tilde{j} = i+1,\ldots,j+1 \text{ and } k_{j+1}-1 \text{ bidders decide to withdraw at } p_{j+1} | k_i \text{ bidders} \\
&\text{decided to stay at } p_{i-1}) \\
&+ (s_j - p_j) \sum_{k_{i+1}=2}^{k_i} \sum_{k_{i+2}=2}^{k_{i+1}} \cdots \sum_{k_j=2}^{k_{j-1}} \sum_{k_{j+1}=1}^{1} \text{Prob}(k_{\tilde{j}} \text{ bidders decide to stay at } p_{\tilde{j}-1} \text{ for} \\
&\tilde{j} = i+1,\ldots,j \text{ and } k_j - 1 \text{ bidders decide to withdraw at } p_j | k_i \text{ bidders} \\
&\text{decided to stay at } p_{i-1})
\end{aligned}
\tag{1.24}
$$

That is,

$$
\begin{aligned}
&(s_j - p_j) \sum_{k_{i+1}=2}^{k_i} \sum_{k_{i+2}=2}^{k_{i+1}} \cdots \sum_{k_j=2}^{k_{j-1}} \frac{1}{k_j} \frac{1}{[1-F(s_{i-1})]^{k_i-1}} \\
&* \left(\begin{array}{c} k_i - 1 \\ k_j - 1 \quad k_{j-1} - 1 \quad \ldots \quad k_{i+2} - 1 \quad k_{i+1} - 1 \end{array} \right) \\
&* [F(s_i) - F(s_{i-1})]^{k_i - k_{i+1}} [F(s_{i+1}) - F(s_i)]^{k_{i+1} - k_{i+2}} \ldots [F(s_{j-1}) - F(s_{j-2})]^{k_{j-1} - k_j} \\
&* [F(s_j) - F(s_{j-1})]^{k_j - 1} \\
= \; &(s_j - p_j - \frac{1}{M}) \sum_{k_{i+1}=2}^{k_i} \sum_{k_{i+2}=2}^{k_{i+1}} \cdots \sum_{k_j=2}^{k_{j-1}} \sum_{k_{j+1}=2}^{k_j} \frac{1}{k_{j+1}} \frac{1}{[1-F(s_{i-1})]^{k_i-1}} \\
&* \left(\begin{array}{c} k_i - 1 \\ k_{j+1} - 1 \quad k_j - 1 \quad k_{j-1} - 1 \quad \ldots \quad k_{i+2} - 1 \quad k_{i+1} - 1 \end{array} \right) \\
&* [F(s_i) - F(s_{i-1})]^{k_i - k_{i+1}} [F(s_{i+1}) - F(s_i)]^{k_{i+1} - k_{i+2}} \ldots [F(s_{j-1}) - F(s_{j-2})]^{k_{j-1} - k_j}
\end{aligned}
$$

$$*[F(s_j)-F(s_{j-1})]^{k_j-k_{j+1}}[F(s_{j+1})-F(s_j)]^{k_{j+1}-1}$$

$$+(s_j-p_j)\sum_{k_{i+1}=2}^{k_i}\sum_{k_{i+2}=2}^{k_{i+1}}\cdots\sum_{k_j=2}^{k_{j-1}}\frac{1}{[1-F(s_{i-1})]^{k_i-1}}$$

$$*\begin{pmatrix} & & k_i-1 & & \\ k_j-1 & k_{j-1}-1 & \ldots & k_{i+2}-1 & k_{i+1}-1\end{pmatrix}$$

$$*[F(s_i)-F(s_{i-1})]^{k_i-k_{i+1}}[F(s_{i+1})-F(s_i)]^{k_{i+1}-k_{i+2}}\ldots[F(s_{j-1})-F(s_{j-2})]^{k_{j-1}-k_j}$$

$$*[F(s_j)-F(s_{j-1})]^{k_j-1}$$

where

$$\begin{pmatrix} & & & k_i-1 & & \\ k_{j+1}-1 & k_j-1 & k_{j-1}-1 & \ldots & k_{i+2}-1 & k_{i+1}-1\end{pmatrix}$$

$$=\frac{(k_i-1)!}{(k_i-k_{i+1})!(k_{i+1}-k_{i+2})!\ldots(k_{j-1}-k_j)!(k_{j+1}-k_j)!(k_{j+1}-1)!}$$

and $\begin{pmatrix} & & k_i-1 & & \\ k_j-1 & k_{j-1}-1 & \ldots & k_{i+2}-1 & k_{i+1}-1\end{pmatrix}$ is defined similarly.

Note, $\frac{1}{[1-F(s_{i-1})]^{k_i-1}}$ cancels on both sides of the equation; therefore, the simplified version of the above equation is

$$(s_j-p_j)\sum_{k_{i+1}=2}^{k_i}\sum_{k_{i+2}=2}^{k_{i+1}}\cdots\sum_{k_j=2}^{k_{j-1}}\left(\frac{1}{k_j}-1\right)\begin{pmatrix} & & k_i-1 & & \\ k_j-1 & k_{j-1}-1 & \ldots & k_{i+2}-1 & k_{i+1}-1\end{pmatrix}$$

$$*[F(s_i)-F(s_{i-1})]^{k_i-k_{i+1}}[F(s_{i+1})-F(s_i)]^{k_{i+1}-k_{i+2}}\ldots[F(s_{j-1})-F(s_{j-2})]^{k_{j-1}-k_j}$$

$$*[F(s_j)-F(s_{j-1})]^{k_j-1}$$

$$=\ (s_j-p_j-\frac{1}{M})\sum_{k_{i+1}=2}^{k_i}\sum_{k_{i+2}=2}^{k_{i+1}}\cdots\sum_{k_j=2}^{k_{j-1}}\sum_{k_{j+1}=2}^{k_j}\frac{1}{k_{j+1}}$$

$$*\begin{pmatrix} & & & k_i-1 & & \\ k_{j+1}-1 & k_j-1 & k_{j-1}-1 & \ldots & k_{i+2}-1 & k_{i+1}-1\end{pmatrix}$$

$$*[F(s_i)-F(s_{i-1})]^{k_i-k_{i+1}}[F(s_{i+1})-F(s_i)]^{k_{i+1}-k_{i+2}}\ldots[F(s_{j-1})-F(s_{j-2})]^{k_{j-1}-k_j}$$

$$*[F(s_j)-F(s_{j-1})]^{k_j-k_{j+1}}[F(s_{j+1})-F(s_j)]^{k_{j+1}-1} \tag{1.25}$$

Then, at p_{i+1}, the bidder observes k_{i+1} and uses this piece of new information and s_i solved at p_i to solve for a new set of $\{s_{i+1}, \ldots, s_{r-1}\}$.

Therefore, at each p_i, based on the observation of k_i, the number of bidders who decide to stay at p_{i-1}, all bidders who decided to stay at p_{i-1} with $v < s_i$ will drop out from the auction while all $v \geq s_i$ will stay. From lemma 11 and lemma 10, $s_i \in (p_i, p_{i+1})$. Similar to what we have observed in the second-price auction, the trade-off between payoff and probability to win makes some bidders overbid while some others underbid in the penultimate-bid English auction.

1.7 Dutch Auctions

Dutch auctions are the converse of English auctions. In a Dutch auction, the auctioneer calls an initial high price and then lowers the price until one bidder raises his hand and accepts the current price. If there is more than one bidder raising hands, the object will be awarded randomly. We use p_i to denote the current price. When the price is lowered to p_i, a bidder's strategy is to decide whether to raise hand and accepts the price or to remain silent. The decision is made based on how he values the object and the current price p_i. Therefore, the bidding strategy $b_i : v \times p \longrightarrow \{0, 1\}$, $b_i(v) = 0$ meaning the bidder remains silent and $b_i(v) = 1$ meaning he decides to raise hand.

Given the current price p_i, if a bidder with v raises his hand, he gets

$$\begin{aligned}
E\Pi(v, b_i = 1|p_i) &= (v - p_i) \sum_{t=0}^{N-1} \frac{1}{t+1} \text{Prob}\{t \text{ other bidders raise their hands} \\
&\quad |\text{no one raised hand at } p_{i+1}\} \\
&= (v - p_i) \sum_{t=0}^{n-1} \frac{1}{t+1} \frac{\binom{N-1}{t-1} [F(s_i) - F(s_{i-1})]^{t-1} F^{N-t}(s_{i-1})}{F^{N-1}(s_i)} \\
&= \frac{v - p_i}{[F(s_i) - F(s_{i-1})]F^{N-1}(s_i)} \sum_{t=1}^{N} \frac{1}{N} \binom{N}{t} [F(s_i) - F(s_{i-1})]^t F^{N-t}(s_{i-1})
\end{aligned}$$

$$= \frac{v - p_i}{N[F(s_i) - F(s_{i-1})]F^{N-1}(s_i)}[F^N(s_i) - F^N(s_{i-1})] \tag{1.26}$$

If he remains silent, he can wait and raise his hand at p_{i-1}, or $p_{i-2}, \ldots, p_1$. His expected payoff of waiting to raise hand at p_{i-1} is

$$\begin{aligned} E\Pi(v, b_{i-1} = 1|p_i) &= (v - p_{i-1}) \sum_{t=0}^{N-1} \frac{1}{t+1} \text{Prob}\{t \text{ other bidders raise their hands at } p_{i-1} \\ &\quad |\text{no one raised hand at } p_{i+1}\} \\ &= \frac{v - p_i}{N[F(s_{i-1}) - F(s_{i-2})]F^{N-1}(s_i)}[F^N(s_{i-1}) - F^N(s_{i-2})] \qquad (1.27) \end{aligned}$$

Similarly, we can calculate the expected payoffs of waiting to raise a hand at $p_{i-2}, \ldots, p_1$ respectively. A bidder with value v will raise a hand at p_i if $E\Pi(v, b_i = 1|p_i) \geq E\Pi(v, b_j = 1|p_i)$ for all $j = 1, \ldots, r$. In fact, equation 1.26 and equation 1.27 imply that for any $i, j = 1, \ldots, r$, if $E\Pi(v, b_i = 1|p_k) \geq E\Pi(v, b_j = 1|p_k)$ for any $k = 1, \ldots, r$, $E\Pi(v, b_i = 1|p_i) \geq E\Pi(v, b_j = 1|p_i)$. Therefore, if at the very beginning of the auction, before any observation of other bidders' decisions, a bidder prefers raising a hand at p_i, he will still prefer raising a hand at p_i after he observes other bidders' decision up to p_{i+1}. Hence, to determine the equilibrium strategy and s_i's, we can restrict our attention to the bidder's choice among different prices at the beginning of the auction. At the beginning of the auction,

$$\begin{aligned} E\Pi(v, b_i = 1) &= (v - p_i) \sum_{t=0}^{N-1} \frac{1}{t+1} \text{Prob}\{t \text{ other bidders raise their hands at } p_i\} \\ &= \frac{v - p_i}{N[F(s_i) - F(s_{i-1})]}[F^N(s_i) - F^N(s_{i-1})] \qquad (1.28) \end{aligned}$$

Equation 1.28 is the same expression as we see in the sealed-bid first-price auction. Hence, the equilibrium conditions which determine the sequence $\{s_0, s_1, \ldots, s_r\}$ are the same as those in the sealed-bid first-price auction. This brings us the following proposition:

Proposition 9 *The Dutch auction is strategically equivalent to the sealed-bid first-price auction.. Consequently, these two auction mechanisms generate the same amount of expected revenues.*

1.8 An Example

In this section, we provide an example in which $N = 2, M = 3, F(v) = v$. As we can see from this example, different from the continuous case, different auction mechanisms in general generate different amount of expected revenues even when the bidders' valuations are independently and identically distributed. We can also tell from the example that the pay-your-bid English auction has non-unique symmetric equilibrium bidding strategies.

Example:

Sealed-Bid Second-Price Auction: Equation (1.14) yields $s_1 = \frac{1}{5}, s_2 = \frac{1}{2}, s_3 = \frac{4}{5}$. Hence, we have

$$b(v) = \begin{cases} 0 & \text{if } v \in [0, \frac{1}{5}) \\ \frac{1}{3} & \text{if } v \in [\frac{1}{5}, \frac{1}{2}) \\ \frac{2}{3} & \text{if } v \in [\frac{1}{2}, \frac{4}{5}) \\ 1 & \text{if } v \in [\frac{4}{5}, 1] \end{cases} \tag{1.29}$$

$$\begin{aligned} Ep_{N,M} &= \sum_{i=1}^{r} b_i(\begin{pmatrix} N \\ 1 \end{pmatrix} [1 - s_i][s_i^{N-1} - s_{i-1}^{N-1}] \\ &\quad + \sum_{t=2}^{N} \begin{pmatrix} N \\ t \end{pmatrix} [s_i - s_{i-1}]^t s_{i-1}^{N-t}) \\ &= \sum_{i=1}^{r} b_i[(N-1)(s_{i-1}^N - s_i^N) + N(s_i^{N-1} - s_{i-1}^{N-1})] \\ &= 0.31 \end{aligned}$$

Sealed-Bid First-Price Auction And Dutch Auction: From equation (1.9) and

Proposition 4 we have $s_1 = \frac{1}{2}$ and $r = 2$. That is,

$$b(v) = \begin{cases} 0 & \text{if } v \in [0, \frac{1}{2}) \\ 1 & \text{if } v \in [\frac{1}{2}, 1] \end{cases} \tag{1.30}$$

$$Ep_{N,M} = \sum_{i=1}^{r} b_i[s_i^N - s_{i-1}^N] = 0.25$$

Pay-Your-Bid English Auction: From equation (1.20), we get two equilibria $\{\frac{2}{3}, \frac{2}{3}\}$ and $\{\frac{1}{2}, 1\}$. In one equilibrium, bidders with values less than $\frac{2}{3}$ will decide to drop out from the auction at $p_1 = 0$ while bidders with values higher than $\frac{2}{3}$ will raise the price up to $\frac{2}{3}$ and no bidder will further raise the price. The expected revenue generated by this equilibrium is $\frac{2}{9}$. In the other equilibrium, bidders with $v < \frac{1}{2}$ will decide to drop out from the auction at $p_1 = 0$ while bidders with values higher than $\frac{1}{2}$ will raise the price to $\frac{1}{3}$ and no bidder will further raise the price. The expected revenue generated by this equilibrium is $\frac{1}{4}$.

Penultimate-Bid English Auction: From equation (1.25), we have $s_1 = 0.279, s_2 = 0.579, s_3 = 0.846$, that is

$$b(v) = \begin{cases} 0 & \text{if } v \in [0, 0.279) \\ \frac{1}{3} & \text{if } v \in [0.279, 0.579) \\ \frac{2}{3} & \text{if } v \in [0.579, 0.846) \\ 1 & \text{if } v \in [0.846, 1] \end{cases} \tag{1.31}$$

As a result, $Ep_{N,M} = 0.239$.

1.9 On Revenue Rankings

The most important result in Vickrey's work is that if bidders' valuations are identically and independently distributed, the expected revenues to the auctioneer in the sealed-bid first-price auction, the sealed-bid second-price auction, the Dutch auction and the English auction are the same. However, this is the case only if the bidding strategy is a continuous function of the bidders' valuations. As we see from last section, this revenue-equivalence result no longer holds when the bidding space is discrete. In general, different auction mechanisms lead to different amount of revenues except that the sealed-bid first-price auction and the Dutch auction are still revenue equivalent when the value distributions are independent and identical. The following proposition is the formal statement of this observation.

Proposition 10 *When the bidding strategy is a discrete function of the bidders' valuations, the sealed-bid second-price auction, the sealed-bid first-price auction, the pay-your-bid English auction and the penultimate-bid English auction yield different amount of expected revenues to the auctioneer. The Dutch auction is revenue equivalent to the sealed-bid first-price auction.*

Except in some special cases, calculations of equilibrium strategies and hence expected revenues in the discussed auctions are computationally difficult. However, with the properties that we have learned about the s_i's in these auctions, we can claim the following results. We denote the revenues generated by and the s_i's of the sealed-bid first-price auction, the sealed-bid second-price auction, the pay-your-bid English auction, the penultimate-bid English auction and the Dutch auction by $P^{SBF}_{N,M}$, $P^{SBS}_{N,M}$, $P^{PUBE}_{N,M}$, $P^{PBE}_{N,M}$, $P^{D}_{N,M}$, s_i^{SBF}, s_i^{SBS}, s_i^{PUBE}, s_i^{PBE} and s_i^D respectively. We also denote the highest and the second highest values among all the bidders by v_1 and v_2 respectively.

Lemma 12 *i)*$p_{[v_2M]} \leq P^{PUBE}_{N,M} \leq p_{[v_2M]+2}$;
ii)$p_{[v_2M]+1} \leq P^{PBE}_{N,M} \leq p_{[v_2M]+2}$;
iii)$b_{[v_2M]} \leq P^{SBS}_{N,M} \leq b_{[v_2M]+2}$;
iv)$P^{SBF}_{N,M} = P^{D}_{N,M} \leq \frac{M-1}{M}$;
where $[v_2M]$ *is the integer part of* v_2M.

Proof:

Note, by our definitions of b_i's and p_i's, $b_{[v_2M]+1} = p_{[v_2M]+1} = \frac{[v_2M]}{M}$ and $v_2 \in$

$[p_{[v_2M]+1}, p_{[v_2M]+2})$ in the English auction and the Dutch auction or $v_2 \in [b_{[v_2M]+1}, b_{[v_2M]+2})$ in sealed-bid auctions.

i) From lemma 8 and lemma 9, we have $s^{PUBE}_{[v_2M]-1} \in [p_{[v_2M]}, p_{[v_2M]+1}]$ and $s^{PUBE}_{[v_2M]+1} \in [p_{[v_2M]+2}, p_{[v_2M]+3}]$. Since $v_2 \geq s^{PUBE}_{[v_2M]-1}$, v_2 doesn't drop out from the auction before the price is raised to $p_{[v_2M]}$. Therefore, $P^{PUBE}_{N,M} \geq p_{[v_2M]}$. On the other hand, since $v_2 < s^{PUBE}_{[v_2M]+1}$, v_2 at most stays up to $p_{[v_2M]+1}$ and hence the auction stops no later than $p_{[v_2M]+2}$. Therefore, $P^{PUBE}_{N,M} \leq p_{[v_2M]+2}$.

ii) From lemma 10 and lemma 11, we have $s^{PBE}_{[v_2M]} \in (p_{[v_2M]}, p_{[v_2M]+1})$ and $s^{PBE}_{[v_2M]+2} \in (p_{[v_2M]+2}, p_{[v_2M]+3})$. Since $v_2 > s^{PBE}_{[v_2M]}$, v_2 doesn't drop out from the auction before the price is raised to $p_{[v_2M]+1}$. Therefore, $P^{PBE}_{N,M} \geq p_{[v_2M]+1}$. On the other hand, since $v_2 < s^{PBE}_{[v_2M]+2}$, v_2 at most stays up to $p_{[v_2M]+2}$. Therefore, $P^{PUBE}_{N,M} \leq p_{[v_2M]+2}$.

iii) The proof is similar to that of part i).

iv) Since

$$\begin{aligned}
E\Pi(v = 1, 1) &= (1-1)\sum_{t=1}^{N} \frac{1}{t} Prob(1 \text{ is the highest bid and } t \text{ buyers bid } 1) \\
&= 0 \\
&< \\
E\Pi(v = 1, b(v) < 1) &= (1 - b(v))\sum_{t=1}^{N} \frac{1}{t} Prob(b(v) \text{ is the highest bid and } t \text{ buyers bid } b(v)) \\
&> 0,
\end{aligned}$$

$b(v_1) < 1$ and hence $P^{SBF}_{N,M} = P^{D}_{N,M} \leq \frac{M-1}{M}$; Q.E.D.

Lemma 12 provides the upper bounds and the lower bounds of revenues that would be generated by different auction mechanisms. In the example provided in the last section, there are two symmetric Baysian Nash equilibrium bidding strategies in the pay-your-bid English auction. One yields higher expected revenues than the penultimate-bid English auction does while the other one yields lower expected revenues. As we can see from the proof of next proposition, only if the values of v_1 and v_2 are such that v_2 drops out at $p_{[v_2M]+1}$ while v_1 stays, will the pay-your-bid English auction yield higher revenue than the penultimate-bid English auction does. The probability that this case occurs shrinks when the number of bidders goes up. That is, it is less likely for the pay-your-bid English auction to outperform the penultimate-bid English auction in terms of revenue when we have more bidders participating in the auctions. The same rationale holds for the revenue comparisons between the pay-your-bid English and the sealed-bid second-price auctions, between the sealed-bid first-price auction and the sealed-bid second-price auction, between the penultimate-bid English auction and the sealed-bid first-price auction. The analysis is formally stated in following four propositions.

Proposition 11 *For any F, M and ϵ, there exists an $N_0(F, M, \epsilon)$ such that for all $N \geq N_0(F, M, \epsilon)$, Prob($P_{N,M}^{PBE} \geq P_{N,M}^{PUBE}$) $> 1 - \epsilon$.*

Proof:

By our definitions of p_i's, $p_{[v_2M]+1} = \frac{[v_2M]}{M}$ and $v_2 \in [p_{[v_2M]+1}, p_{[v_2M]+2})$. From lemma 8 and lemma 9, we have $s_{[v_2M]-1}^{PUBE} \in [p_{[v_2M]}, p_{[v_2M]+1}]$ and $s_{[v_2M]+1}^{PUBE} \in [p_{[v_2M]+2}, p_{[v_2M]+3}]$. Therefore, $s_{[v_2M]-1}^{PUBE} \leq v_2 < s_{[v_2M]+1}^{PUBE}$. From lemma 10 and lemma 11, we have $s_{[v_2M]}^{PBE} \in (p_{[v_2M]}, p_{[v_2M]+1})$ and $s_{[v_2M]+2}^{PBE} \in (p_{[v_2M]+2}, p_{[v_2M]+3})$. Therefore, $s_{[v_2M]}^{PBE} < v_2 < s_{[v_2M]+2}^{PBE}$. Hence, we have the following four cases:

Case1: $s_{[v_2M]+1}^{PBE} \leq v_2 < s_{[v_2M]}^{PUBE}$;

Case2: $s_{[v_2M]+1}^{PBE} \leq s_{[v_2M]}^{PUBE} \leq v_2$ or $s_{[v_2M]}^{PUBE} \leq s_{[v_2M]+1}^{PBE} \leq v_2$;

Case3: $s_{[v_2M]}^{PUBE} \leq v_2 < s_{[v_2M]+1}^{PBE}$;

Case4: $v_2 < s_{[v_2M]}^{PUBE} \leq s_{[v_2M]+1}^{PBE}$ or $v_2 < s_{[v_2M]+1}^{PBE} \leq s_{[v_2M]}^{PUBE}$;

Case1:

Case1A: $v_1 < s_{[v_2M]}^{PUBE}$;

In the penultimate-bid English auction, v_2 stays in the auction until the price is raised to $p_{[v_2M]+2}$. Therefore, $P_{N,M}^{PBE} = p_{[v_2M]+2}$. In the pay-your-bid English auction, both v_1 and v_2 stay in the auction until the price is raised to $p_{[v_2M]}$. Therefore, $P_{N,M}^{PUBE} = p_{[v_2M]}$. As a result, $P_{N,M}^{PBE} > P_{N,M}^{PUBE}$.

Case1B: $v_1 \geq s_{[v_2M]}^{PUBE}$;

v_2 stays in the auction until the price is raised to $p_{[v_2M]+2}$ in the penultimate-bid English auction. In the pay-your-bid English auction, both v_1 and v_2 stay in the auction before the price is raised to $p_{[v_2M]}$. v_2 withdraws from the auction at $p_{[v_2M]}$ while v_1 drops out at $p_{[v_2M]+1}$. Hence, $P_{N,M}^{PBE} = p_{[v_2M]+2} > P_{N,M}^{PUBE} = p_{[v_2M]+1}$.

Case2:

Case2A: $v_1 < s_{[v_2M]+1}^{PUBE}$;

In the penultimate-bid English auction, v_2 stays in the auction until the price is raised to $p_{[v_2M]+2}$. In the pay-your-bid English auction, both v_1 and v_2 stay in the auction until the price is raised to $p_{[v_2M]+1}$. Therefore, $P_{N,M}^{PBE} = p_{[v_2M]+2} > P_{N,M}^{PUBE} = p_{[v_2M]+1}$.

Case2B: $v_1 \geq s_{[v_2M]+1}^{PUBE}$;

v_2 stays in the auction until the price is raised to $p_{[v_2M]+2}$ in the penultimate-bid English auction. In the pay-your-bid English auction, both v_1 and v_2 stay in the auction before the price is raised to $p_{[v_2M]+1}$. v_2 withdraws at $p_{[v_2M]+1}$ while v_1 drops out at $p_{[v_2M]+2}$. Hence, $P_{N,M}^{PBE} = p_{[v_2M]+2} = P_{N,M}^{PUBE} = p_{[v_2M]+2}$.

Case3:

Case3A: $v_1 < s_{[v_2M]+1}^{PUBE}$;

In the penultimate-bid English auction, v_2 stays in the auction until the price is raised to $p_{[v_2M]+1}$. In the pay-your-bid English auction, both v_1 and v_2 stay in the auction until the price is raised to $p_{[v_2M]+1}$. Therefore, $P_{N,M}^{PBE} = p_{[v_2M]+1} = P_{N,M}^{PUBE} = p_{[v_2M]+1}$.

Case3B: $v_1 \geq s_{[v_2M]+1}^{PUBE}$;

v_2 stays in the auction until the price is raised to $p_{[v_2M]+1}$ in the penultimate-bid English auction. In the pay-your-bid English auction, both v_1 and v_2 stay in the auction until the price is raised to $p_{[v_2M]+1}$. v_2 withdraws at $p_{[v_2M]+1}$ while v_1 drops out at $p_{[v_2M]+2}$. Hence, $P_{N,M}^{PBE} = p_{[v_2M]+1} < P_{N,M}^{PUBE} = p_{[v_2M]+2}$.

Case4:

Case4A: $v_1 < s_{[v_2M]}^{PUBE}$;

In the penultimate-bid English auction, v_2 stays in the auction until the price is raised to $p_{[v_2M]+1}$. In the pay-your-bid English auction, both v_1 and v_2 stay in the auction until the price is raised to $p_{[v_2M]}$. Therefore, $P_{N,M}^{PBE} = p_{[v_2M]+1} > P_{N,M}^{PUBE} = p_{[v_2M]}$.

Case4B: $v_1 \geq s_{[v_2M]}^{PUBE}$;

v_2 stays in the auction until the price is raised to $p_{[v_2M]+1}$ in the penultimate-bid English auction. In the pay-your-bid English auction, both v_1 and v_2 stay in the auction until the price is raised to $p_{[v_2M]}$. v_2 withdraws from the auction at $p_{[v_2M]}$ while v_1 drops out at $p_{[v_2M]+1}$. Hence, $P^{PBE}_{N,M} = p_{[v_2M]+1} = P^{PUBE}_{N,M} = p_{[v_2M]+1}$. Therefore, the only case in which $P^{PBE}_{N,M} = p_{[v_2M]+1} < P^{PUBE}_{N,M} = p_{[v_2M]+2}$ is Case3B in which $s^{PUBE}_{[v_2M]} \leq v_2 < s^{PBE}_{[v_2M]+1} < s^{PUBE}_{[v_2M]+1} \leq v_1$. This condition is stronger than $v_1 \geq p_{[v_2M]+2}$. That is,

$$\begin{aligned} Prob(Case3B) &\leq Prob(v_1 \geq p_{[v_2M]+2}) \\ &= Prob(v_2M \leq [v_1M]) \\ &= Prob(v_2 \leq \frac{[v_1M]}{M}). \end{aligned}$$

Or,

$$Prob(P^{PBE}_{N,M} \geq P^{PUBE}_{N,M}) \geq 1 - Prob(v_2 \leq \frac{[v_1M]}{M}).$$

Since $Prob(v_2 < x|v_1) = \frac{F^{N-1}(x)}{F^{N-1}(v_1)}$, we have

$$\begin{aligned} Prob(v_2 \leq \frac{[v_1M]}{M}) &= \int_0^1 \frac{F^{N-1}(\frac{[v_1M]}{M})}{F^{N-1}(v_1)} NF^{N-1}(v_1)dv_1 \\ &= \sum_{i=0}^{M-1} F^{N-1}(\frac{i}{M})N\frac{1}{M} \end{aligned}$$

Since $\lim_{n\to\infty} \sum_{i=0}^{M-1} F^{N-1}(\frac{i}{M})N\frac{1}{M} = 0$, there exists an $N_0(F, M, \epsilon)$ such that for all $N \geq N_0$, $\sum_{i=0}^{M-1} F^{N-1}(\frac{i}{M})N\frac{1}{M} < \epsilon$. Hence, $Prob(P^{PBE}_{N,M} \geq P^{PUBE}_{N,M}) > 1-\epsilon$. Q.E.D.

Proposition 12 *For any F, M and ϵ, there exists an $N_0(F, M, \epsilon)$ such that for all $N \geq N_0(F, M, \epsilon)$, $Prob(P^{SBS}_{N,M} \geq P^{PUBE}_{N,M}) > 1 - \epsilon$.*

The proof is similar to that of Proposition 11.

In spite of the fact that bidders bid more aggressively in the second-price auction and the penultimate-bid English auction than they do in the pay-your-bid English auction, the high bidder pays his bid only in the case of tie. Therefore, if $b_{N,M}(v_1)$ in the pay-your-bid English auction happens to be higher than $b_{N,M}(v_2)$ in the penultimate-bid English auction, the previous auction mechanism leads to a higher revenue than the latter one does. However, as more and more bidders participate in the auctions, the chance of tie in the penultimate-bid English auction and the second-price auction gets larger and larger. Consequently, with higher and higher probability, the winning bidder pays his bid that is more aggressive than the one in the pay-your-bid English auction. As a result, the second-price auction and the penultimate-bid English auction are more likely to generate higher revenues.

Proposition 13 *For any F, M and ϵ, there exists an $N_0(F, M, \epsilon)$ such that for all $N \geq N_0(F, M, \epsilon)$, $Prob(P_{N,M}^{SBS} \geq P_{N,M}^{SBF}) > 1 - \epsilon$.*

Proof:

From Lemma 12, $P_{N,M}^{SBF} \leq \frac{M-1}{M}$. Therefore, $Prob(P_{N,M}^{SBS} \geq P_{N,M}^{SBF}) \geq Prob(v_2 \geq \frac{M-1}{M})$. By the definition of v_2, we know

$$
\begin{aligned}
& Prob(v_2 \geq x) \\
= & \sum_{t=2}^{N} \binom{N}{t} [1 - F(\frac{M-1}{M})]^t F(\frac{M-1}{M})^{N-t} \\
= & 1 - F^N(\frac{M-1}{M}) - NF^{N-1}(\frac{M-1}{M})[1 - F(\frac{M-1}{M})] \\
\rightarrow & 1 \\
as & \quad N \rightarrow \infty
\end{aligned}
$$

Therefore, there exists an $N_0(F, M, \epsilon)$ such that for all $N \geq N_0$, $Prob(v_2 \geq x) > 1 - \epsilon$. Hence, $Prob(P_{N,M}^{SBS} \geq P_{N,M}^{SBF}) > 1 - \epsilon$. Q.E.D.

Proposition 14 *For any F, M and ϵ, there exists an $N_0(F, M, \epsilon)$ such that for all $N \geq N_0(F, M, \epsilon)$, $Prob(P_{N,M}^{PBE} \geq P_{N,M}^{SBF}) > 1 - \epsilon$.*

The proof is similar to that of Proposition 13.

Similar to the rationale behind the price comparisons of the second-price auction and the penultimate-bid English auction as opposed to the pay-your-bid English auction, bidders bid more aggressively in the first two auctions than they do in the first-price auction. As N increases, the chance of tie in the first two auctions gets larger. The winning bidder is more likely to pay his more aggressive bid than he does in the first-price auctions and this therefore contributes to a higher revenue in these two auctions.

Recall, Lemma 8, 9, 10 and 11 hold for any arbitrary N, M, and continuous F so long as each bidder's utility is non-decreasing in his monetary payoffs and Lemma 12 is derived from these four lemmas, the four revenue-comparing propositions in this section also hold regardless of N, M, actual functional forms of F and the utility function.

1.10 Continuous Model As A Special Case

As we mentioned in section 1.4, Chwe (1989) proved that in the sealed-bid first-price auction, if $F(v)$ is uniform, the symmetric Nash equilibrium strategy converges to the one in the continuous-bid auctions as M goes to infinity. Similarly, we can prove the convergence results for all the auction mechanisms that we have discussed in previous sections. Denoting $b_{N,M}(v)$ in the sealed-bid second-price auctions, the pay-your-bid English auction and the penultimate-bid English auction by $b^{SBS}_{N,M}(v)$, $b^{PUBE}_{N,M}(v)$ and $b^{PBE}_{N,M}(v)$ respectively, we have the following proposition:

Proposition 15 *For any F, as $M \to \infty$,*

i) $b^{SBS}_{N,M}(v) \xrightarrow{u} b(v) = v$;

ii) $b^{PUBE}_{N,M}(v) \xrightarrow{u} b(v) = v$;

iii) $b^{PBE}_{N,M}(v) \xrightarrow{u} b(v) = v$.

<u>Proof:</u>

We will prove i) below. Proofs of ii) and iii) are similar.

If $v \in [s_0^{SBS}, s_1^{SBS}]$, $b_{N,M}^{SBS}(v) = b_1 = 0$. Hence $b_1 \leq v < b_2$. If $v = s_r^{SBS} = 1$, $b_{N,M}^{SBS}(v) = b_r = 1$. Hence $b_{r-1} < v \leq 1$. If $v \in [s_{i-1}^{SBS}, s_i^{SBS}]$, $2 \leq i \leq r$, $b_{N,M}^{SBS}(v) = b_i$. From Proposition 6, $b_{i-1} < v < b_{i+1}$. Therefore, for $v \in [0, 1]$, $|v - b_{N,M}^{SBS}(v)| \leq \frac{2}{M} \to 0$ as $M \to \infty$. Q.E.D.

Therefore, we can treat the case where the bidding space is continuous as a special case or limiting case of the model with discrete bids for the second-price auction, the pay-your-bid English auction and the penultimate-bid English auction. When the increment gets infinitely small, a bidder will report her true value in the sealed-bid second-price auction and stay until the price is raised above her valuation as long as there are at least two bidders remaining active in English auctions. Note, lemma 12 implies that $|P_{N,M}^{PBE} - P_{N,M}^{PUBE}| \leq \frac{2}{M} \to 0$ as $M \to \infty$. Hence, when the bidding space is continuous, there is no difference in equilibrium bidding strategies and hence expected revenues among the sealed-bid second-price auction and the two different English auctions. Moreover, as stated in Proposition 4, Chwe (1989) showed the convergence of bidding strategy and expected revenue of discrete-bid auctions to those of continuous ones when $F(v)$ is uniform. Therefore, if the distribution of the private valuations is uniform, all of the five auction mechanisms yield the same amount of expected revenues. Or, we can state it formally in the following proposition:

Proposition 16 ***(Revenue Equivalence)*** *When* $F(v) = v$, $lim_{M\to\infty} Ep_{N,M}^{SBF} = lim_{M\to\infty} Ep_{N,M}^{SBS} = lim_{M\to\infty} Ep_{N,M}^{PUBE} = lim_{M\to\infty} Ep_{N,M}^{PBE} = lim_{M\to\infty} Ep_{N,M}^{D} = \frac{N-1}{N} v$, *where* $Ep_{N,M}^{SBF}$, $Ep_{N,M}^{SBS}$, $Ep_{N,M}^{PUBE}$, $Ep_{N,M}^{PBE}$ *and* $Ep_{N,M}^{D}$ *denote the expected prices of the sealed-bid first-price auction, the sealed-bid second-price auction, the pay-your-bid English auction, the penultimate-bid English auction and the Dutch auction respectively.*

1.11 Note On Efficiency

One convincing explanation of the longevity of auctions as an institution is that auctions often lead to outcomes that are efficient, that is, the object is always awarded to the bidder with the highest valuation. More technically, the set of equilibrium outcomes in an auction coincides with the set of core allocations. However, in our set-up, the bidding strategy maps a continuously distributed valuation to a finite choice set. As a consequence, there is always a positive probability of a tie. Since $F(v)$ is assumed to be continuous and the tie-breaking rule that we adopt throughout the analysis is random, the probability of an inefficient allocation in any auction mechanism is strictly positive[4]. In other words, there is always a chance of awarding the object to some bidder whose valuation is not the highest. Therefore, an auctioneer considering the use of bid increment to speed up the bidding process needs to be aware that she will most likely sacrifice allocative efficiency.

1.12 Concluding Remarks

This paper presents theories on discrete-bid auctions. In particular, we focus on the four common auction institutions: the sealed-bid first-price auction, the sealed-bid second-price auction, the English auction and the Dutch auction, in a single-object, independent-private-value setting in which bids can only be multiples of some fixed increment. Two different models of the English auction, the pay-your-bid and the penultimate-bid English auction are introduced. It is shown that with the discreteness of bids, second-price auctions and English auctions are no longer dominance solvable as bidding games. In the continuous-bid case, different pricing rules of the English auction lead to the same dominant bidding strategy and hence the same expected revenue. However, we have shown that, in discrete-bid auctions, bidding is more aggressive in the penultimate-bid English auction than that in the pay-your-bid English auction. Nevertheless, first-price auctions and Dutch auctions are still strategically equivalent. The equivalence of expected revenues in the continuous case breaks down when bids are discrete. As the number of bidders participating in the auction increases, auctions in which the winner pays the next highest bid (second-price auctions and penultimate-bid English auctions) are more likely to yield higher expected revenues than auctions in which the winner pays his own bid (first-price

[4]However, it is easy to show that, in the second-price auction, the pay-your-bid English auction and penultimate-bid English auction, the inefficiency is bounded by $\frac{2}{M}$. Hence, as the size of increment gets very small, the level of inefficiency is negligible in these auctions.

auctions and pay-your-bid English auctions). The probability of tie in discrete-bid auctions is strictly positive and hence resulting allocations can be Pareto inefficient.

The assumption of identical and independent distributions of private values is very restrictive. It requires that there are no resale possibilities and that each bidder knows exactly how much the object would be worth to him. A natural extension of our work is a further study with the assumptions of affiliated values and asymmetric distributions.

The appeal of using bidding increment is to speed up the bidding process in dynamic auctions like English auctions and Dutch auctions. However, this is at the expense of possible inefficient allocations and lower expected revenues[5]. Studies of the optimal choice of the size of the increment and comparisons of the degree of inefficiency in different auctions would provide very important insights both to auction theorists and practitioners.

[5]Recall Proposition 4, the expected price in the first-price auction when bids are discrete is always lower than that in the continuous case. By Proposition 9, this also applies to the Dutch auction.

1.13 Appendix

1.13.1 Proof of Lemma 1

Suppose $b(v_1) = b(v_2) = b_i$ for some $v_1, v_2 \in [0,1]$, we have

$$E\Pi(v_1, b_i) \geq E\Pi(v_1, b_j), \forall b_j$$

$$E\Pi(v_1, b_i) \geq 0$$

and

$$E\Pi(v_2, b_i) \geq E\Pi(v_2, b_j), \forall b_j$$

$$E\Pi(v_2, b_i) \geq 0$$

Recall the definition of $E\Pi$, we have

$$v_1 P_i - p_i P_i \geq v_1 P_j - p_j P_j, \forall j$$

$$v_2 P_i - p_i P_i \geq v_2 P_j - p_j P_j, \forall j$$

Therefore,

$$[\lambda v_1 + (1-\lambda)v_2]P_i - (\lambda + 1 - \lambda)p_i P_i \geq [\lambda v_1 + (1-\lambda)v_2]P_j - (\lambda + 1 - \lambda)p_j P_j, \forall j$$

That is,

$$vP_i - p_i P_i \geq vP_j - p_j P_j, \forall j$$

for all $v = \lambda v_1 + (1-\lambda)v_2$, where $\lambda \in [0,1]$.

Similarly,

$$E\Pi(v, b_i) \geq 0$$

for all $v = \lambda v_1 + (1-\lambda)v_2$, where $\lambda \in [0,1]$. Q.E.D.

1.13.2 Proof of Lemma 2

Suppose $b(v) = 0$ for all v, then $E\Pi(v, b(v)) = \frac{1}{N}$ for all v. If some bidder with $v = \frac{1}{N} + \epsilon$ ($\epsilon > 0$) deviates and bids $\frac{\epsilon}{2}$, he will win the object with probability 1. Since in any of the four auction formats, the price that he needs to pay for the object never exceeds his bid, he gets

$$\begin{aligned} E\Pi(\frac{1}{N} + \epsilon, \frac{\epsilon}{2}) &\geq \frac{1}{N} + \epsilon - \frac{\epsilon}{2} \\ &= \frac{1}{N} + \frac{\epsilon}{2} \\ &> \frac{1}{N} \\ &= E\Pi(\frac{1}{N} + \epsilon, 0) \end{aligned}$$

Q.E.D.

1.13.3 Proof of Lemma 3

From (IC), we have

$$vP_i - p_iP_i \geq vP_j - p_jP_j, \forall j \neq i$$

Or,

$$v(P_i - P_j) \geq p_iP_i - p_jP_j, \forall j \neq i$$

It's obvious that the above inequality holds for any $v' > v$. Therefore, $b(v') \geq b(v)$. Q.E.D.

1.13.4 Proof of Proposition 1

The four auction institutions are all symmetrical games. The sealed-bid first-price auction and the second-price auction are strategic-form games while the English auction and the Dutch auction are extensive-form games with perfect recall. We will first show that any finite symmetrical strategic-form game has symmetric equilibrium strategy. By Kuhn's (1953) theorem (In a game of perfect recall, mixed and behavior strategies are equivalent.), this existence result applies to all four auction forms. We will then apply Milgrom and Weber (1985)'s purification theorem to show that the equilibrium strategy has a purification and hence a symmetric pure strategy equilibrium exists.

To facilitate the proof, we adopt the following notations: $C_i \equiv$ Player i's strategy space. In our model, $C_i = C_j = C = \{b_1, \ldots, b_{M+1}\}$. $\Sigma = \Delta C \equiv$ Set of all mixed strategies over C. $r : \Sigma \rightarrow\rightarrow \Sigma \equiv$ The reaction (best response) correspondence for each player. By the symmetry of the games, $r_i(\sigma_j, \sigma_{N\setminus\{i,j\}}) = r_j(\sigma_i, \sigma_{N\setminus\{i,j\}})$. $u_i(\sigma_i, \sigma_{-i}) \equiv$ Player i's payoff when he plays σ_i and his opponents play σ_{-i}.

From Kakutani's theorem, the following are sufficient conditions for $r : \Sigma \rightarrow\rightarrow \Sigma$ to have a fixed point: (i) Σ is a compact, convex, nonempty subset of a finite-dimensional Euclidean space; (ii) $r(\sigma)$ is nonempty for all σ; (iii) $r(\sigma)$ is convex for all σ; (iv) $r(\cdot)$ has a closed graph.

(i) follows from the fact that C is finite. (ii) follows from Weierstrass theorem. (A continuous function u_i attains a maximum over a compact set Σ.) (iii) follows from the fact that every convex combination of best response is a best response. (iv) follows from the Theorem of the Maximum. Therefore, by Kakutani's fixed point theorem, there exists a σ^* such that $r_i(\sigma^*_{-i}) = \sigma^*$ for all $i \in N$. This proves the existence of symmetric equilibrium strategy.

Next, we invoke Milgrom and Weber's purification theorem to prove the existence of symmetric equilibrium in pure strategies.

According to Milgrom and Weber, if (i) the player's types are independent, (ii) $F(v)$ is atomless[6], (iii) each player's payoff depends only on his own type and the list of strategies, that is, $u_i = u_i(v_i, \sigma_i, \sigma_{-i})$. (iv) each player's strategy set is finite, and (v) payoffs are equicontinuous[7], then each strategy of each player has a purification.

(i), (ii) and (v) are satisfied according to the assumptions we employed in our

[6] A probability measure η is *atomless* if for every B with $\eta(B) > 0$, there is a $C \subset B$ for which $\eta(B) > \eta(C) > 0$.

[7] A family of functions $\{f_\alpha\}$ is *equicontinuous* if for every x and every $\epsilon > 0$, there is a $\delta > 0$ such that $|x - y| < \delta$ implies $|f_\alpha(x) - f_\alpha(y)| < \epsilon$ for every α.

model. (iii) is satisfied in all of the four auction forms. Hence, every strategy in the auction games that we are concerned with has a purification. Furthermore, these games have symmetric equilibrium points and hence have symmetric equilibrium points in pure strategies. Q.E.D.

1.13.5 Proof of Lemma 4

From the incentive compatibility constraint and the definition of s_i, we know $E\Pi(s_i, b_{i+1}) \geq E\Pi(s_i, b_i)$. Suppose $E\Pi(s_i, b_{i+1}) > E\Pi(s_i, b_i)$, then by continuity of $E\Pi$ in v, there exists $\epsilon > 0$ such that $E\Pi(s_i - \epsilon, b_{i+1}) > E\Pi(s_i - \epsilon, b_i)$. Therefore, $s_i - \epsilon$ would be better off bidding b_{i+1} instead of b_i. Contradiction.

1.13.6 Proof of Lemma 5

For all $v \in [s_{i-1}, s_i]$,

$$E\Pi(v, b_i) \geq E\Pi(v, b_{i+1})$$

$\Longrightarrow$

$$vPr_i - p_i^e \geq vPr_{i+1} - p_{i+1}^e$$

or,

$$v(Pr_{i+1} - Pr_i) \leq p_{i+1}^e - p_i^e$$

Since both p_i^e and Pr_i are nondecreasing, if this holds for s_{i-1}, it also holds for all $v \leq s_{i-1}$.

Also, for all $v \in [s_i, s_{i+1}]$,

$$E\Pi(v, b_{i+1}) \geq E\Pi(v, b_i)$$

$\Longrightarrow$

$$vPr_{i+1} - p_{i+1}^e \geq vPr_i - p_i^e$$

or,

$$v(Pr_{i+1} - Pr_i) \geq p_{i+1}^e - p_i^e$$

If this is true for s_{i+1}, it is also true for all $v \geq s_{i+1}$. Therefore,

$$E\Pi(v, b_i) = E\Pi(v, b_{i+1})$$

for $i = 1, 2, \ldots, r-1$ and $E\Pi(s_r, b_r) \geq E\Pi(s_r, b_{r+1})$ for $1 \leq r < M+1$ are the only binding constraints. Q.E.D.

1.13.7 Proof of Proposition 3

($\Longleftarrow$) First, in all of the four auction formats, we have for all $v \in [s_0, s_1]$,

$$E\Pi(v, b_1) = vPr_1 - p_1^e \geq 0$$

and in particular

$$0Pr_1 - p_1^e = 0$$

which implies $p_1^e = 0$ and $Pr_1 = 0$. Also, by Lemma 4, we have

$$s_1Pr_1 - p_1^e = s_1Pr_2 - p_2^e \Longrightarrow p_2^e = s_1Pr_2 = s_1(Pr_2 - Pr_1) + s_0Pr_1$$

Suppose that the incentive compatibility constraints of all $v \in [0, 1]$ are satisfied, for all $i = 1, \ldots, r-1$, for all $v \in [s_i, s_{i+1}]$,

$$vPr_{i+1} - p_{i+1}^e \geq vPr_i - p_i^e$$

with equality for s_i. Suppose

$$p_i^e = s_{i-1}(Pr_i - Pr_{i-1}) + s_{i-2}(Pr_{i-1} - Pr_{i-2}) + \ldots + s_1(Pr_2 - Pr_1) + s_0Pr_1$$

Then,

$$p^e_{i+1} = s_i(Pr_{i+1}-Pr_i)+p^e_i = s_i(Pr_{i+1}-Pr_i)+s_{i-1}(Pr_i-Pr_{i-1})+...+s_1(Pr_2-Pr_1)+s_0Pr_1$$

($\Longrightarrow$)

Given equation (1.7), we have $p^e_i = s_{i-1}(Pr_i - Pr_{i-1}) + p^e_{i-1}$ for all $i = 1, 2, \ldots, r$. Which implies $s_{i-1}Pr_i - p^e_i = s_{i-1}Pr_{i-1} - p^e_{i-1}$ for all $i = 1, 2, \ldots, r-1$.

By Lemma 5, these are the only binding constraints for $v \in [0, s_{r-1}]$. From the proof of Lemma 5, $s_{r-1}Pr_r - p^e_r = s_{r-1}Pr_{r-1} - p^e_{r-1}$ together with $s_rPr_r - p^e_r \geq s_rPr_{r+1} - p^e_{r+1}$ are sufficient for incentive compatibility for all $v \in [s_{r-1}, s_r]$. The proof is finished.

1.13.8 Proof of Lemma 6

We'll prove the first inequality here. The other two inequalities can be proved similarly.

$$\begin{aligned} & \frac{F^N(s_i) - F^N(s_{i-1})}{N[F(s_i) - F(s_{i-1})]} \\ = & \frac{F^{N-1}(s_i) + F^{N-2}(s_i)F(s_{i-1}) + \ldots + F(s_i)F^{N-2}(s_{i-1}) + F^{N-1}(s_{i-1})}{N} \end{aligned}$$

Since $s_{i-1} < s_i$, $F(s_{i-1}) < F(s_i)$. Therefore, $F^{N-2}(s_i)F(s_{i-1}) < F^{N-1}(s_i)$, ..., $F(s_i)F^{N-2}(s_{i-1}) < F^{N-1}(s_i)$, $F^{N-1}(s_{i-1}) < F^{N-1}(s_i)$. This implies that

$$F^{N-1}(s_i) + F^{N-2}(s_i)F(s_{i-1}) + \ldots + F(s_i)F^{N-2}(s_{i-1}) + F^{N-1}(s_{i-1}) < NF^{N-1}(s_i)$$

and hence

$$\frac{F^N(s_i) - F^N(s_{i-1})}{N[F(s_i) - F(s_{i-1})]} < F^{N-1}(s_i)$$

Q.E.D.

1.13.9 Proof of Lemma 7

From Lemma 6, we have $E\Pi(1, b_r) < E\Pi(1, b_{r+1})$ for all admissible r. Therefore, a bidder with $v = 1$ would prefer the highest possible bid. Therefore, $b_r = 1$. Recall $b_i = \frac{i-1}{M}$, hence $r = M + 1$. Q.E.D.

Chapter 2 An Experimental Study of Discrete-Bid Auctions

In last chapter, we discussed theories of the equilibrium strategies in four common auction institutions with bidding possibilities being some multiple of an increment. In particular, we introduced two different models of the English auction, the pay-you r-bid English auction and the penultimate-bid English auction and showed that rules matter. The symmetric Baysian Nash equilibrium predicts different bidding behavior in these two versions of the auction. In this chapter, we report our laboratory observ ations on the behavioral properties of these two auction rules.

2.1 Pay-Your-Bid And Penultimate-Bid English Auctions: Theory Revisited

Recall that both of these two versions of English auction are conducted in multiple rounds. The price starts from zero and goes up by a constant after each round. At the beginning of the auction, there are N potential buyers. Each of them has a priva te valuation v which is a random draw from $[0,1]$ according to a common distribution function $F(v)$. Although a bidder only observes his own v, the way in which private valuations are determined is common knowledge. In each round, a bidder's decis ion is whether to stay or withdraw from the auction. After each round, the number of bidders who are still in the auction is made available to all the participating bidders. In the pay-your-bid English auction, the auction stops when the last bidder wit hdraws. The object is awarded to the last bidder at the price at which he withdraws. In the penultimate-bid English auction, the auction stops when all bidders but one have withdrawn. The last remaining bidder wins the object and pays the price at which the auction stops. A random device is employed to determine the winner if there is more than one bidder in the last group withdrawing from the auction.

We use a slightly different notation in this chapter. Let $b(v)$ denote the price at which a bidder stays up to[1] if the bidder's valuation is v. In the pay-your-bid English auction, when the price is higher than one increment below v, if a bidder decides not to withdraw, he leaves himself a non-positive profit. If he wins, the price he needs to pay is at lea st the current price plus the increment which is higher than v. His profit is negative in this case. Hence, staying is always dominated by withdrawing when $p > v - \frac{1}{M}$. On the other hand, if the price is lower than two increments below v and there is at least one more other bidder remaining in the auction, a bidder is always better off staying than withdrawing. If he withdraws at the current price, the only case in which he can earn a positive profit is when all the other remaining bidders also withdraw at this price. In this cas e, he needs to share the luck with the other bidders to be chosen as the winner expecting a profit of $\frac{1}{k}(v-p)^2$. However, if he stays and withdraws in the next round, he can be a guaranteed winner with a profit of $v - (p + \frac{1}{M})$! This amount is bigger than $\frac{1}{k}(v-p)$ when $k \geq 2$. In the case where some of his rivals decide to stay in current round, he will not be able to win by withdrawing at current price, but he can expect some positive profit by staying. As a result, $v - \frac{2}{M} \leq b(v) < v$ dominates the other choices of $b(v)$ as long as there is more than one other bidder staying in the auction. The price at which a bidder drops out must satisfy one of the following two conditions:

$$b(v) + \frac{1}{M} < v < b(v) + \frac{2}{M}$$

or

$$b(v) < v \leq b(v) + \frac{1}{M}. \tag{2.1}$$

[1]For most of the bidders, $b(v)$ is the price at which they withdraw. For the winner, $b(v)$ is the price at which the auction stops.

[2]Recall from last chapter k denotes the number of bidders who haven't dropped out yet up to the current round.

The only exception is that the auction stops earlier.

In the penultimate-bid English auction, if the current price is already higher than v and the auction has not stopped yet, withdrawing definitely dominates staying because a bidder expects a non-positive profit from staying. On the other hand, if the p rice is lower than one increment below v, withdrawing is dominated by staying. If the bidder withdraws at the current price, he earns $v-p$ if all the other remaining bidders also withdraw in this round and he luckily gets chosen to be the winner. However, if he stays he will be a sure winner and still earn a positive profit $v-p$. In the case where some of the other bidders decide to stay to next round, he can still get away next round with a positive expected profit. Therefore, the price up to which a bidder stays must satisfy the following condition:

$$|b(v) - v| \leq \frac{1}{M} \tag{2.2}$$

unless the auction ends earlier.

As we have mentioned in last chapter, because of the discreteness of bids, the probability of having a tie and hence a possible inefficient allocation resulting from the auctions is strictly positive. If we denote the highest valuation by v_1, this pro bability is equivalent to the probability of having at least one more bidder whose private value v is such that $b(v) = b(v_1)$. Obviously, this probability increases with the size of increment $\frac{1}{M}$.

We also showed in last chapter that the expected revenues generated from the pay-your-bid English auction and the penultimate-bid English auction are in general different. As we showed in the proof of Proposition 11, when $s^{PUBE}_{[v_2M]} \leq v_2 < s^{PBE}_{[v_2M]+1} < s^{PUBE}_{[v_2M]+1} < v_1$[3], the price in the pay-your-bid English auction is higher than that in the penultimate-bid English auction. This is the case when $v_2 \leq \frac{[v_1M]+1}{M}$. It is easy to show that the probability of $v_2 \leq \frac{[v_1M]+1}{M}$ decreases with the size o f increment $\frac{1}{M}$.

Bidding rules (2.1) and (2.2) depend on the assumption that bidder's utility de-

[3]Bidders with valuation s^j_i are indifferent between dropping out at p_i and at p_{i+1} in auction j. v_1 and v_2 denotes the highest and the second highest valuations amo ng all bidders. $[v_2M]$ is the integer part of v_2M.

pends only on his monetary gain from the auction and is non-decreasing in this gain. Note, the dominance arguments do not depend on the init ial level of competition N, the distribution function $F(v)$[4], the size of increment $\frac{1}{M}$ and bidders' risk attitudes. We wish our experimental examination of these two auction institutions could provide some insights on people's behaviors in "real world" situations.

2.2 Experimental Design and Procedures

The experiments we report here consist of six[5] experiments conducted a t the Caltech Laboratory for Experimental Economics and Political Science. They are indexed by the date of the experiment. Subjects for these experiments were students at the California Institute of Technology. All experiments were implemented on aucti on software developed by Anil Roopnarine and Dave Porter. Experimental instructions can be found in Appendix.

Each experiment had 35[6]auction periods. Each auction period is a stand-alone auction during which five subjects competed for one unit of a fictional object. In each period, the high bidder winning the object earned profit equal to his/her valuation (referred as redemption value in the instructions) minus the price paid. Other subjects earned zero profit. The computer randomly picked a winner in the case of tie.

Subjects were given participation bonuses to cover the possibility of losses. Although rational (Baysian Nash equilibrium) bidders will not suffer losses in the pay-your-bid English auction, they may suffer losses in the penultimate-bid English auction. Earnings from the experiment by each subject were paid in cash at the end of the experiment together with the participation bonuses.

The currency used in the experiment is "francs." All earnings in francs were con-

[4]In the special case where the distribution of v is discrete and the value space is the same as the bidding space (possible values for v are multiples of $\frac{1}{M}$), a bidder's dominant strategy is to stay up to $v - \frac{1}{M}$ in the pay-your-bid English auction and up to v in the penultimate-bid English auction.

[5]Indeed, eight experiments were conducted. The first two experiments not reported were pilot experiments with fewer periods testing the software and experimental instructions.

[6]36 periods were actually conducted. The first period of each experiment was practice period to help subjects familiarize with the rules and software. Subjects didn't earn anything in these periods. All the data we report for each experiment is data from the other 35 periods.

verted to dollars according to the conversion rate specified on the instructions.

Private valuations, v, were integers independently drawn in each auction period from a discrete uniform distribution over the interval $[0, 24]$. Subjects knew their own valuation, the distribution from which others' values were drawn, and the number of participants. A new set of random draws preceded each auction period.

There were several rounds in each auction period. At the beginning of each round, the private valuation and the posted price for the current round were printed on each subject's computer screen. Each subject was then asked to submit his/her decision whe

ther to accept the posted price. A subject who did not accept the posted price was labeled "inactive" and was not allowed to participate in later rounds of the same period. His/her eligibility was resumed at the beginning of the next period. When a ro

und was over, the number of active bidders was posted on each subject's screen. The posted price for each round was increased by an increment of $\frac{24}{M}$ per round starting with $\frac{24}{M}$ for the first round. Stopping rules for each period an

d the prices charged to winning bidders were applied according to those specified in our set-up of the pay-your-bid English auction and the penultimate-bid English auction respectively. The values of M used in the experiments were $M = 3, 6$ and 24. Table 2.1 summarizes the parameters used in the experiments.

2.3 Experimental Results

2.3.1 Individual Behavior

Table 2.2 and 2.3 summarize the bidding behavior observed in all the experiments of the pay-your-bid and penultimate-bid English auctions. Over 90 percent of the bids comply with conditions (2.1) and 2.2) in both the pa y-your-bid and penultimate-bid English auctions for both $M = 3$ and $M = 6$, supporting the dominance arguments. However, the data are much noisier for the smallest bid increment experiments, i.e.,

$\frac{24}{M} = 1$. About 50 percent[7] and 30 percent of the bids when $M = 24$ are out of the range predicted by theory in the pay-your-bid and penultimate-bid English auctions respectively. Among these, m ost are underbidding (bids are below the predicted range).

As Kagel (1991) pointed out, the structure of the English auction makes it relatively transparent to bidders that they should not bid above their valuations. A bidder necessarily loses money in any case if he wins by staying up to a price too high ($p > v$ in the pay-your-bid auction and $p > v + \frac{24}{M}$ in the penultimate-bid auction). The "real time" nature of English auctions provides subjects a good chance of observational learning. By comparing the going price with their private values, subjects are likely to see that they will lose money if they win whenever the price exceeds v or $v + \frac{24}{M}$ depending on the specific auction format. We believe this interpretation is supported by our data for $M = 3$ and $M = 6$.

There are two possible explanations for the significant underbidding when $M = 24$. First, with the small size of increment, the auction usually lasts fairly long and consists of many more rounds than those with larger increments. Subjects with small valu ations may feel bored watching the price movements and waiting to update their bidding decisions and hence withdraw in earlier rounds, considering their chance of winning is very slim. Secondly, auctions with many rounds and small increments make it easi er to signal a willingness to cooperate by an individual subject through withdrawing at unusually low price[8].

From Table 2.2 and Table 2.3, we can see that underbidding when the winning probability is less than 50 percent ($v < 12$) accounts for 50 percent of all the underbids in the pay-your-bid and penultimate-bid English auctions respectively for $M = 24$. Among them, many were withdrawals at the beginning of the auctions ($b(v)=0$). These observations partially support our two explanations.

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[7]This number do es not include those bids lower than the predicted range due to the ending of auctions.

[8]We talked to the subjects who tended to withdraw at prices much lower than their valuations after the experiments. They acknowledged that signaling willingness to collaborate was the motivation.

2.3.2 Efficiencies

Table 2.4 summarizes allocative efficiencies in the experiments. Following Coppinger et al. (1980), we use the percentage of sales that were Pareto optimal, meaning the sale was made to the highest value bidder, as a measure of efficiency. More than 97 percent of the auctions were Pareto efficient in the pay-your-bid auction when $M = 24$. The efficiency was about 89 percent for $M = 6$. However, this percentage dropped to 69 percent in those pay-your-bid English auction for $M = 3$. All of the inefficienc ies occurred in the situations in which the highest value bidder failed to win in the case of tie. Allocations to the third highest value bidder occurred when $M = 3$. Observations in penultimate-bid English auctions are similar.

2.3.3 Revenues

Table 2.5 shows that most of the auction prices are within the range predicted by Lemma 12. Figure 2.1-2.3 are charts of price comparisons between the pay-your-bid English auction and the penultimate-bid English auction for different i ncrements. The percentage of pay-your-bid English auctions generating higher prices decreased with the size of increment. This is consistent with our conjecture.

In their experiments, Coppinger et al. (1980) observed that English auction prices in all experiments tended to be slightly above the second highest valuation. They reasoned that this was caused by the discrete bid increments they employed. This conject ure is supported by our data. As Table 2.5 shows, most of the prices in our experiments are within the range predicted after considering the discrete bid increments. Even with $M = 24$, majority of prices are within the predicted range. We therefore beli eve that there is no strong evidence of collusion in our experiments even if some bidders signaled their willingness to collaborate by underbidding.

2.4 Summary and Conclusions

In this chapter, we report the results of six experiments on the pay-your-bid and penultimate-bid English auctions. Our findings can be summarized as follows:

(1) Over 90 percent of bids in both the pay-your-bid and penultimate-bid English auctions comply with conditions (2.1) and 2.2), supporting the dominance arguments, when $M = 3$ and $M = 6$. However, underbidding behavior i s serious in the case of small increment, i.e., $M = 24$. Subjects with low valuations tend to withdraw at the beginning of the auctions.

(2) The frequency of a tie goes up with the size of the increments. As a result, efficiency, measured as the percentage of sales that were Pareto Optimal, deteriorates as the size of increment increases.

(3) Most of the auction prices fall in the theory-predicted range, even with the smallest increment, the case in which underbidding was significant. The percentage of pay-your-bid English auctions generating higher prices, compared to penultimate-bid Eng lish auctions, increases as the increment gets smaller.

We provided two possible explanations for the underbidding behavior observed in the auction with a small increment. First, since auctions with small increment consisted of many rounds, subjects with small valuations might feel bored waiting to drop out u ntil the price went close to their values and hence withdraw in earlier rounds, considering their chance of winning is very slim. Secondly, subjects may withdraw at unusually low prices to signal a willingness to cooperate. Further experiments with rand om groupings of potential bidders would help differentiate the two explanations. A finding that the degree of underbidding is not significantly different with random groupings will support the first explanation. Otherwise, the second explanation is more favorable. Moreover, if the second explanation is supported, experiments with the same group of subjects for different sizes of increments would provide insights on how sensitive the size of increment is to the incentive to collude.

Finally, although the bidding range predicted by dominance arguments holds regardless of the initial level of competition, the distribution of private valuations, the size of increment and bidders' risk attitudes, the Baysian Nash equilibrium strategies d o depend on these parameters. Experiments examining the comparative statics could potentially provide interesting results.

2.5 Appendix

2.5.1 Experimental Instructions

Experimental Instructions For Pay-Your-Bid English Auctions

Introduction:

This is an experiment in market decision making in which you may earn money. What you earn depends partly on your decisions and partly on the decisions of others. All earnings you make are yours to keep and will be paid to you in cash at the end of the experiment. The currency in this experiment is francs. All francs will be converted to dollars at the end of the experiment, at a rate of $0.4 per Frac. In addition to your earnings from the experiment, you will also be paid a $5 participation bonus.

This experiment will consist of several periods. In each period, you will be participating in a market in which a single fictional object will be sold to one of the experiment participants. You will each receive a sequence of numbers from the computer, one for each period, which describes t he value to you of any decisions you might make. These numbers may differ among individuals. You are not to reveal this information to anyone. It is your own private information. From this point forward, you will be referred to by your bidder number. You are bidder number ____ in this experiment. In each period you will be able to place bids to purchase the object.

We will start with a detailed instruction period. During the instruction period, you will be given a complete description of the experiment and of how your earnings will be determined. Please follow along with these instructions as they are read aloud a nd please do not read ahead. Please also do not touch the computer until you are told to do so.

It is important that you not talk or in any way try to communicate with other participants during the experiment. If you disobey the rules, we will have to ask you to leave the experiment.

If you have any questions during the instruction period, raise your hand and your question will be answered so everyone can hear. If any difficulties arise after the experiment has begun, raise your hand, and a monitor will come and assist you.

The first period will be practice. You will receive no earnings for this period.

Redemption Values and Earnings:

During each market period you are free to purchase a unit if you want. If you successfully purchase a unit in a period, you will receive the redemption value indicated as "Value" in your Bidbook. Your earnings from a unit purchased is the difference be tween your redemption value for that unit and the price you paid for the unit. That is:

$$\text{Your earnings} = (\text{redemption value}) - (\text{purchase price})$$

Suppose for example that you buy a unit and that your redemption value is 200. If you pay 150 for the unit then your earnings are

$$\text{Earnings from unit} = 200 - 150 = 50.$$

If the object is not sold to you, then your earnings are zero for that period.

You can calculate your earnings on your Summary of Experiment Record sheet at the end of each period. Notice that if the price paid is above the redemption value, you experience a loss. Anyone with a net loss at the end of the experiment is allowed to w ork to pay the loss at a rate of $6 per hour.

Determination of Redemption Values:

For each buyer the redemption value for the object in each period will be between 1 and 24. Each whole number from 1 to 24 has an equal chance of being selected. It is as if each integer value from 1 to 24 is stamped on a single ball and pla ced in an urn. A draw from the urn determines the redemption value for an individual. The ball is replaced and a second draw determines the redemption value for another individual. The redemption values for each period are determined the same way. Not e that the redemption value for any participant is not affected by the redemption values of the other participants. The following is a table which lists the probability of getting a value in a certain range: (It is for your reference)

Range of Redemption Vale	Probability of A Value in This Range	Range of Redemption Value	Probability of A Value in This Range
1-6	25%	1-6	25%
7-12	25%	1-12	50%
13-18	25%	1-18	75%
19-24	25%	1-24	100%

Market Organization - What You're Bidding For:

In each period, there will be 5 participants in the market. Every participant will be bidding for one unit of the object and only ONE person will win it. The process of purchasing units will be conducted in a market which consists of several rounds.

At the beginning of each round, a price will be posted. The determination of the posted price will be explained later. If you are willing to purchase the object at the posted price or higher, please submit your bid by accepting the price. If you accept the price, you will be considered as being ACTIVE in that round. If you are NOT willing to purchase the object at the posted price or higher, please do NOT accept the price. However, if you choose NOT to accept the price in a round, you will NOT be eli gible to participate in later rounds of the same period and you will be considered as being INACTIVE for this round and later rounds of the same period. You will be eligible to participate again when a new period starts.

After the closing of each round, the number of active bidders will be posted on your main page. The market will continue to the next round if there is more than one active bidder by the end of that round. If there is only ONE active bidder in a round, t he period closes at the end of that round. The object will be sold to that bidder at the last posted price of that period.

For example, if by the end of round 4 there are three active bidders, round 5 will be conducted. On the other hand, if in round 5 only bidder No. 1 remains active, the market will close at the end of round 5. The object will be sold to bidder No . 1 at the posted price of round 5.

If more than one bidder remains active in a round but all decide to be inactive in the following round, one of these bidders will be chosen randomly to be the buyer. That is, each of these bidders has an equal chance of being selected as the buyer. The object will be sold to the buyer at the posted price in the *LAST* round in which he or she remains active.

For example, if three bidders remain active in round 5 but all decide NOT to accept the price in round 6, the market will close at the end of round 6. One of the three bidders will be chosen randomly as the buyer. He or she will then be sold the o bject at the posted price of round 5.

Unless you are the buyer, you receive no redemption value and pay nothing, and so have earnings of zero for that period.

Submitting Bids:

On your screen you will see an icon labeled *Bidbook*. Please click this now. Clicking on this icon will bring you a window in which you can submit your bidding decisions. The number under "Price" in your bidbook is the posted price for the cur rent round. If you are willing to accept the posted price, please check "*Yes*" under "Decisions". If you are not willing to accept the posted price, please check "*No*" under "Decisions". No matter what decision you've made, please s ubmit your decision by clicking on "*Submit*". You will have approximately one minute in order to submit your decisions.

Now, please click "Yes" or "No" according to the chart below:

Yes	No
Bidder #1	Bidder #4
Bidder #2	Bidder #5
Bidder #3	

Then click "*Submit*". Once all bidders have submitted their decisions, the round will be closed. I will now solve the round. You can now see the results on the main

Netscape window. The results show how many bidders decided to remain active i n the last round. You may also view the results of previous rounds by clicking on *Previous Round Results.*

Determination of The Posted Price:

The posted price of each round is determined in the following way:

The posted price of each round
=the posted price of the previous round + increment.

Where,

the increment is the posted price of the first round.

For example, if the posted price of the first round is 4, then the posted prices of the following three rounds will be

Round	Posted Price
2	8
3	12
4	16

The posted price of the first round may be different for different periods.

Any Questions?

If there are no more questions, let's start the experiment with a practice period. You will NOT be paid for this period.

Experimental Instructions For Penultimate-Bid English Auctions

Introduction:

This is an experiment in market decision making in which you may earn money. What you earn depends partly on your decisions and partly on the decisions of others. All earnings you make are yours to keep and will be paid to you in cash at the end of the experiment. The currency in this experiment is francs. All francs will be converted to dollars at the end of the experiment, at a rate of $0.4 per Franc. In addition to your earnings from the experiment, you will also be paid a $5 participation bonus.

This experiment will consist of several periods. In each period, you will be participating in a market in which a single fictional object will be sold to one of the experiment participants. You will each receive a sequence of numbers from the computer, one for each period, which describes the value to you of any decisions you might make. These numbers may differ among individuals. You are not to reveal this information to anyone. It is your own private information. From this point forward, you will be referred to by your bidder number. You are bidder number ____ in this experiment. In each period you will be able to place bids to purchase the object.

We will start with a detailed instruction period. During the instruction period, you will be given a complete description of the experiment and of how your earnings will be determined. Please follow along with these instructions as they are read aloud a nd please do not read ahead. Please also do not touch the computer until you are told to do so.

It is important that you not talk or in any way try to communicate with other participants during the experiment. If you disobey the rules, we will have to ask you to leave the experiment.

If you have any questions during the instruction period, raise your hand and your question will be answered so everyone can hear. If any difficulties arise after the experiment has begun, raise your hand, and a monitor will come and assist you.

The first period will be practice. You will receive no earnings for this period.

Redemption Values and Earnings:

During each market period you are free to purchase a unit if you want. If you successfully purchase a unit in a period, you will receive the redemption value indicated as "Value" in your Bidbook. Your earnings from a unit purchased is the difference be tween your redemption value for that unit and the price you paid for the unit. That is:

$$\text{Your earnings} = (\text{redemption value}) - (\text{purchase price})$$

Suppose for example that you buy a unit and that your redemption value is 200. If you pay 150 for the unit, then your earnings are

$$\text{Earnings from unit} = 200 - 150 = 50.$$

If the object is not sold to you, then your earnings are zero for that period.

You can calculate your earnings on your Summary of Experiment Record sheet at the end of each period. Notice that if the price paid is above the redemption value, you experience a loss. Anyone with a net loss at the end of the experiment is allowed to w ork to pay the loss at a rate of $6 per hour.

Determination of Redemption Values:

For each buyer the redemption value for the object in each period will be between 1 and 24. Each whole number from 1 to 24 has an equal chance of being selected. It is as if each integer value from 1 to 24 is stamped on a single ball and pla ced in an urn. A draw from the urn determines the redemption value for an individual. The ball is replaced and a second draw determines the redemption value for another individual. The redemption values for each period are determined the same way. Not e that the redemption value for any participant is not affected by the redemption values of the other participants. The following is a table which lists the probability of getting a value in a certain range: (It is for your reference.)

Range of Redemption Value	Probability of A Value in This Range	Range of Redemption Value	Probability of A Value in This Range
1-6	25%	1-6	25%
7-12	25%	1-12	50%
13-18	25%	1-18	75%
19-24	25%	1-24	100%

Market Organization - What You're Bidding For:

In each period, there will be *five* participants in the market. Every participant will be bidding for one unit of the object and only ONE person will win it. The process of purchasing units will be conducted in a market which consists of several rounds.

At the beginning of each round, a price will be posted. The determination of the posted price will be explained later. If you are willing to purchase the object at the posted price or higher, please submit your bid by accepting the price. If you accept the price, you will be considered as being ACTIVE in that round. If you are NOT willing to purchase the object at the posted price or higher, please do NOT accept the price. However, if you choose NOT to accept the price in a round, you will NOT be eli gible to participate in later rounds of the same period and you will be considered as being INACTIVE for this round and later rounds of the same period. You will be eligible to participate again when a new period starts.

After the closing of each round, the number of active bidders will be posted on your main page. The market will continue to the next round if there is more than one active bidder by the end of that round. If there is only ONE active bidder in a round, t he period closes at the end of that round. The object will be sold to that bidder at the posted price of the *PREVIOUS* round.

For example, if by the end of round 4 there are three active bidders, round 5 will be conducted. On the other hand, if in round 5 only bidder No. 1 remains active, the market will close at the end of round 5. The object will be sold to bidder No . 1 at the posted price of round 4.

If more than one bidder remains active in a round but all decide to be inactive in the following round, one of these bidders will be chosen randomly to be the buyer. That is, each of these bidders has an equal chance of being selected as the buyer. The object will be sold to the buyer at the posted price in the *LAST* round in which he or she remains active.

For example, if three bidders remain active in round 5 but all decide NOT to accept the price in round 6, the market will close at the end of round 6. One of the three bidders will be chosen randomly as the buyer. He or she will then be sold the object at the posted price of round 5.

Unless you are the buyer, you receive no redemption value and pay nothing, and so have earnings of zero for that period.

Submitting Bids:

On your screen you will see an icon labeled *Bidbook*. Please click this now. Clicking on this icon will bring you a window in which you can submit your bidding decisions. The number under "Price" in your bidbook is the posted price for the current round. If you are willing to accept the posted price, please check "*Yes*" under "Decisions." If you are not willing to accept the posted price, please check "*No*" under "Decisions." No matter what decision you've made, please submit your decision by clicking on "*Submit*". You will have approximately one minute in order to submit your decisions.

Now, please click "Yes" or "No" according to the chart below:

Yes	No
Bidder #1	Bidder #4
Bidder #2	Bidder #5
Bidder #3	

Then click "*Submit.*" Once all bidders have submitted their decisions, the round will be closed. I will now solve the round. You can now see the results on the main Netscape window. The results show how many bidders decided to remain active in the last round. You may also view the results of previous rounds by clicking on *Previous Round Results.*

Determination of The Posted Price:

The posted price of each round is determined in the following way:

The posted price of each round

=the posted price of the previous round + increment.

Where,

the increment is the posted price of the first round.

For example, if the posted price of the first round is 4, then the posted prices of the following three rounds will be

Round	Posted Price
2	8
3	12
4	16

The posted price of the first round may be different for different periods.

Any Questions?

If there are no more questions, let's start the experiment with a practice period. You will NOT be paid for this period.

2.5.2 Tables and Figures

Table 2.1: Summary of Experimental Data

Experiment	No. of Subjects in Each Period	No. of Periods	Subject Pool	M	Auction Format
06/04/98	5	35	Caltech	6	Penultimate-Bid
06/09/98	5	35	Caltech	6	Pay-Your-Bid
06/10/98	5	35	Caltech	24	Penultimate-Bid
06/10/98	5	35	Caltech	3	Penultimate-Bid
06/11/98	5	35	Caltech	24	Pay-Your-Bid
06/11/98	5	35	Caltech	3	Pay-Your-Bid

Table 2.2: Summary of Bidding Patterns: Pay-Your-Bid English Auctions

	$M = 3$	$M = 6$	$M = 24$
$b(v) + \frac{1}{M}24 < v \leq b(v) + \frac{2}{M}24$	5%	9%	0%
$b(v) < v \leq b(v) + \frac{1}{M}24$	92%	87%	34%
$v \geq b(v) + \frac{2}{M}24$	0%	2%	43%
$v \leq b(v) \leq v + \frac{1}{M}$	3%	2%	14%
$b(v) > v + \frac{1}{M}24$	0%	0%	9%
$v \geq b(v) + \frac{2}{M}24$ because of winning	0%	1%	16%
$v \geq b(v) + \frac{2}{M}24$ when $v \leq 12$	0%	1%	17%
$b(v) = 0$ when $\frac{2}{M}24 \leq v \leq 12$	0%	1%	8%

Remarks: $b(v) + \frac{1}{M}24 < v \leq b(v) + \frac{2}{M}24$ and $b(v) < v \leq b(v) + \frac{1}{M}24$ form the range predicted by dominance arguments. $v \geq b(v) + \frac{2}{M}24$ is considered as underbidding. $v \leq b(v) \leq v + \frac{1}{M}$ and $b(v) > v + \frac{1}{M}24$ are considered as overbidding. However, $v \leq b(v) \leq v + \frac{1}{M}$ implies staying up to the price just above his/her value. We consider this as slight overbidding.

Table 2.3: Summary of Bidding Patterns: Penultimate-bid English Auctions

	$M = 3$	$M = 6$	$M = 24$
$\lvert v - b(v)\rvert \leq \frac{1}{M}$	97%	93%	67%
$v > b(v) + \frac{1}{M}24$	2%	6%	29%
$b(v) > v + \frac{1}{M}$	1%	1%	4%
$v > b(v) + \frac{1}{M}24$ because of winning	0%	0%	0%
$v > b(v) + \frac{1}{M}24$ when $v \leq 12$	1%	4%	15%
$b(v) = 0$ when $\frac{1}{M}24 \leq v \leq 12$	1%	3%	13%

Remarks: $\lvert v - b(v)\rvert \leq \frac{1}{M}$ is the range predicted by dominance arguments. $v > b(v) + \frac{1}{M}24$ is considered as underbidding. $b(v) > v + \frac{1}{M}$ is considered as overbidding.

Table 2.4: Summary of Ties and Efficiencies

	$M = 3$		$M = 6$		$M = 24$	
	PUB	PB	PUB	PB	PUB	PB
Observed Efficiency	69%	66%	89%	83%	97%	86%
Frequency of Tie	57%	54%	29%	34%	9%	9%
Percentage of Inefficiency Because of Tie	91%	67%	75%	100%	100%	40%
Random Allocation Efficiency	50%	58%	70%	50%	67%	33%
Allocation to The Highest Value Bidder	69%	66%	89%	83%	97%	86%
Allocation to The 2nd Highest Value Bidder	14%	28%	11%	11%	3%	14%
Allocation to The 3rd Highest Value Bidder	14%	6%	0%	3%	0%	0%
Allocation to The 4th Highest Value Bidder	3%	0%	0%	3%	0%	0%

Note: Random Allocation Efficiency is the percentage of auctions with ties in which the object was awarded to the highest value bidder. PUB and PB stand for the pay-your-bid and the penultimate-bid English auctions respectively.

Table 2.5: Percentage of Prices Within The Predicted Range

	Pay-Your-Bid Auction	Penultimate-Bid Auction
M=3	100%	97%
M=6	100%	94%
M=24	83%	89%

Note: Predicted Range as specified by Lemma 12:
$P^{PUBE} \in [p_{[v_2M]}, p_{[v_2M]+2}]$ and $P^{PBE} \in [p_{[v_2M]+1}, p_{[v_2M]+2}]$.

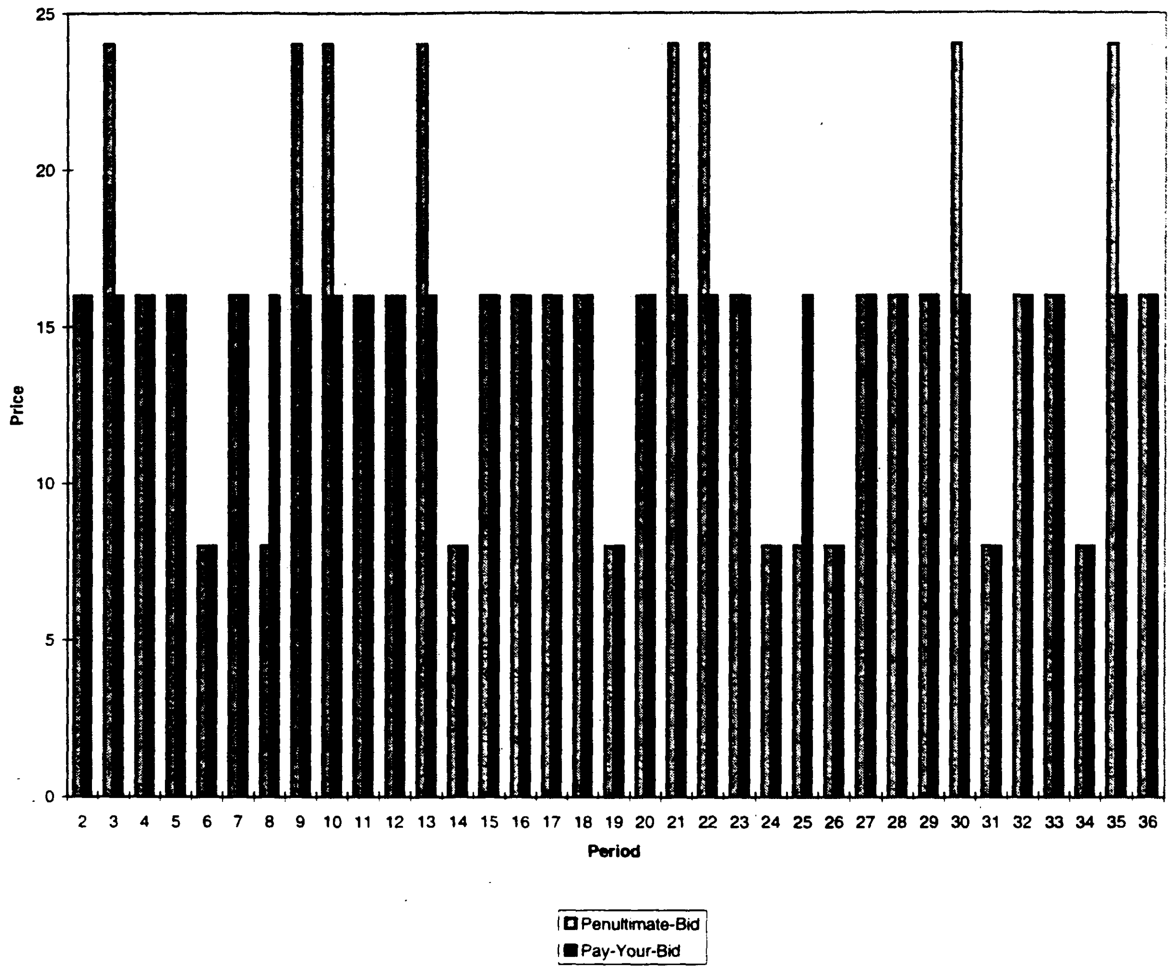

Figure 2.1: Price Comparisons: M=3

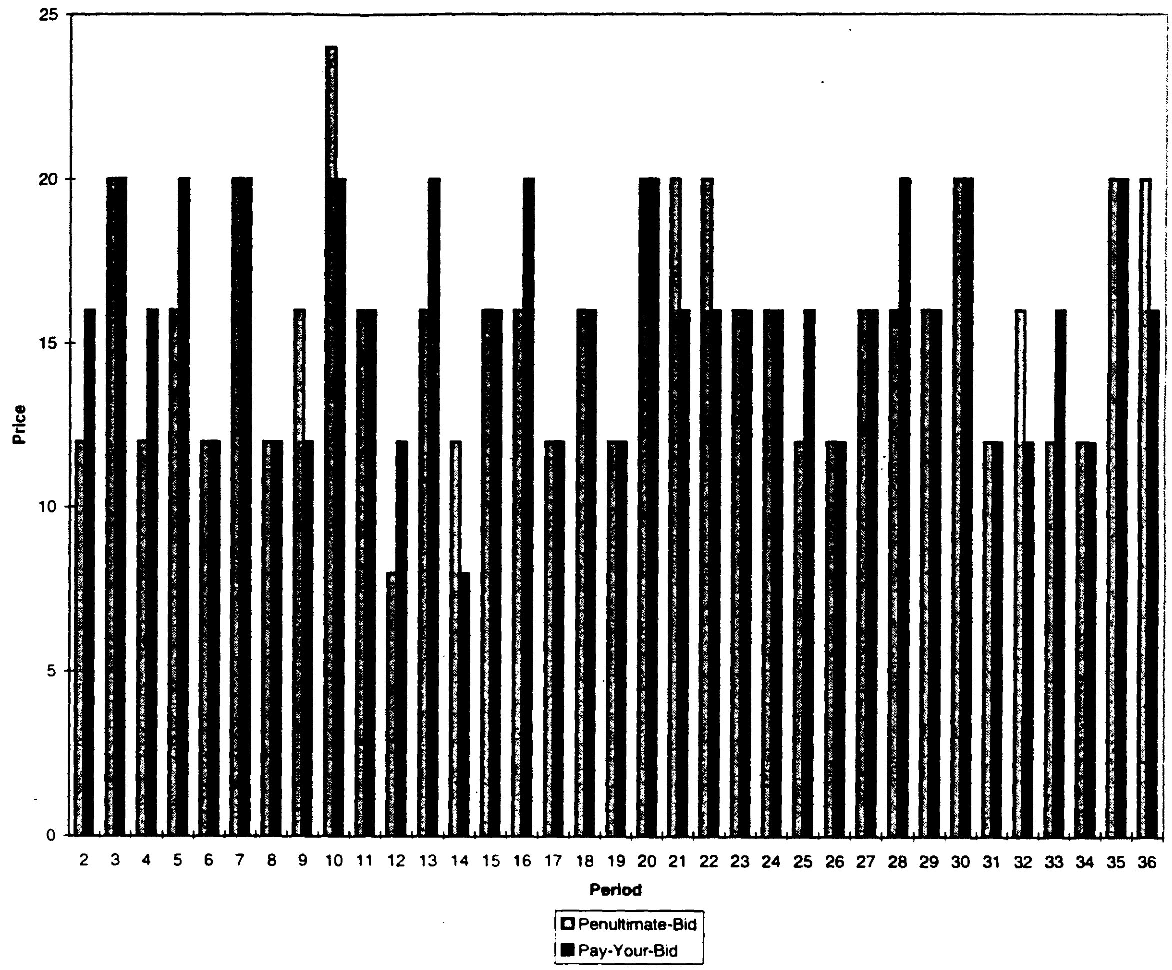

Figure 2.2: Price Comparisons: M=6

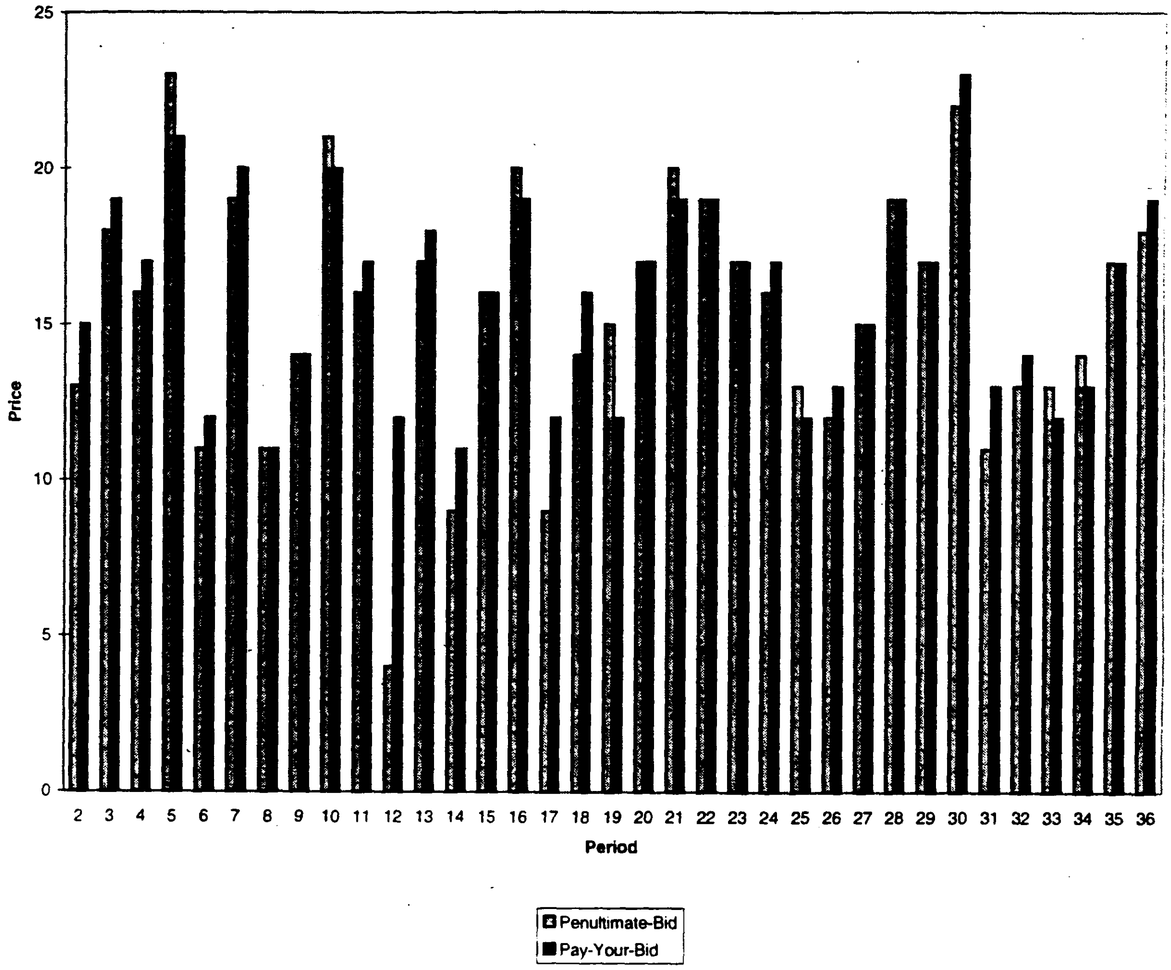

Figure 2.3: Price Comparisons: M=24

Chapter 3 Product Differentiation By A Quality Discriminating Monopolist

3.1 Introduction

There is a rich literature studying the monopoly pricing problem under the situation in which the monopolist has incomplete information of consumers' preferences for certain attributes of the product produced by the monopolist. Examples include situations in which a monopolist attempts to separate consumers by offering: different price-quality bundles as in Mussa and Rosen (1978) and Srinagesh and Bra dburd (1989); different price-quantity bundles as in Spence (1980) and Maskin and Riley (1984); different price-time bundles as in Stokey (1979) and Chiang and Spatt (1982). For each of these problems, the central issue is how to construct a sorting mech anism to extract the greatest possible private gain for the monopolist. Two of the most important results of these works can be summarized as follows: 1) In most cases, the optimal strategy for a monopolist is to use non-linear pricing to induce self-selection among heterogeneous customers[1]. 2) When the monopolist employs imperfect price discrimination to induce heterogeneous consumers to self-select among different bundles, the consumers who derive the highest total utility from the attribute of interest will be served efficie ntly while the problem of allocative distortions arises for all the other consumers[2].

Some common assumptions adopted in these papers are as follows: 1) Consumers can be ordered by their tastes and marginal utilities derived from an increment of the attribute of interest; 2) The attribute of interest is in general restricted to one dimension; 3) The unit cost and marginal cost of producing the monopoly product is constant. That is, there is no fixed cost in the production. Although these assumptions are made for simplicity, the results derived based on these assumptions shed

[1]Salant (1989) derives a condition under which inducing self-selection is optimal.

[2]The allocative distortions arising in problems of self-selection are characterized in Cooper (1984).

light on more general frameworks. In this paper, we introduce a model similar to the one in Mussa and Rosen (1978) in which the monopoly product is differentiated by its quality level. However, we atte mpt to generalize the problem by introducing a constant set-up cost for each type of product the monopolist produces. The assumption of set-up costs immediately brings us two important differences between the current model and the previous ones: 1) In pr evious models without fixed production cost, when the distribution of consumer's type is regular[3]only consumers of low types may be bunched onto one offering. In general, this offering makes this group of consumers of different types choose not to consume the monopoly product. All the other consumer s are fully separated and each type is provided with a unique offering. However, with a set-up cost, it is impossible to have a complete separation even among higher consumer types in the optimal solution. 2) Without the assumption of set-up cost, if th e consumers' types are continuously distributed, the monopolist chooses to provide a continuous spectrum of quality levels. If consumers' types are discrete, the number of different product types served by the monopolist is the same as the number of cons umers' types. However, with the set-up cost, the number of different quality levels provided by the monopolist is finite. In fact, the number of different quality levels becomes an important choice variable of the monopolist in this case.

This paper discusses the monopolist's optimization problem in the model with fixed set-up cost and provides characterizations of the optimal solution. The following section describes the model. Section 3.3 studies the monopolist's optimization problem and provides the solution algorithm. In this section, we find that the lowest type in the consumer group consuming the highest quality level will be served efficiently in that the consumer's marginal rate of substitution between price and quality equals that of the monopolist. There is quality distortion for all the other consumer types and the distortion takes the form of degradation. Profitability of different quality levels is discussed in section 3.4. It is shown that the average profit increases with the quality level. Section 3.5 examines the effects of changes in the distribution

[3]Following Myerson (1981), the distribution is regular if the harzard rate condition is satisfied. That is, the hazard rate $\theta - \frac{1-F(\theta)}{f(\theta)}$ is strictly increasing in θ.

of consumer preference on the onopolist's profit. A first order stochastic dominant shift of the distribution function of consumer preference yields higher profit for the monopolist. We conclude in the last section with a discussion of further applications of the theory and possible directions of future research.

3.2 The Model

We consider a model similar to the one in Mussa and Rosen (1978). The problem involves the optimal strategy of a monopolist who seeks to sell a quality-differentiated spectrum of goods of the same generic type to consumers of different characteristics th at he cannot observe.

Following Mussa and Rosen (1978), we assume that the market contains a continuum of consumers indexed by a variable θ that takes on values in the interval $\Theta=[0,\bar{\theta}]$ according to a continuous density function $f(\theta)$ that is st rictly positive everywhere in the domain. $F(\theta)$ is the associated distribution function and is known by the monopolist. The commodity we are considering can be produced by the monopolist in a number of different varieties. Let q represent the underlying attributes of a particular variety. In this paper, we refer q as "product quality," which is restricted to one dimens ion. Larger values of q indicate higher quality varieties. We treat $q = 0$ as not consuming the commodity. Each consumer is assumed to purchase at most one unit of the commodity. The same price $p(q)$ is charged to all consumers who buy the commodity of quality q.

The utility of a consumer of type θ derived from consuming one unit of the commodity of quality q after paying price p is

$$u(\theta, q, p) = \theta q - p \tag{3.1}$$

From (3.1), we can tell θ in fact parameterizes the intensity of a consumer's demand for quality. That is, higher θ's have stronger preference for quality. It is assumed that

$$u(\theta, 0, p) = 0$$

for all θ.

Since the consumers can always choose not to buy the product, the above equation tells us that $p(0) = 0$.

To produce any variety of the good, the monopolist has a fixed set-up cost denoted by c_0. The unit variable cost is $c(q)$ which is assumed to be a continuous, strictly convex, increasing function of q.

The monopolist offers a spectrum of quality-price bundles (q_i, p_i) targeted at different consumer types. A consumer of type θ chooses q_i over q_j if the following self-selection constraint is satisfied:

$$\theta q_i - p_i \geq \theta q_j - p_j \tag{3.2}$$

for all i, j.

Let $\bar{q}$ be the highest quality level the monopolist can supply, then the maximum total amount of money that consumers are willing to pay is bounded by $\int_0^{\bar{\theta}} \bar{q}\theta f(\theta) d\theta$. Therefore, the total number of different quality levels the monopolist is willing to produce is bounded by $\frac{\int_0^{\bar{\theta}} \bar{q}\theta f(\theta) d\theta}{c_0}$. That is, with the set-up cost, it does not pay the monopolist to perfectly discriminate among all the consumer types and produce infinite number of diff erent varieties. Rather it is more profitable to bunch some customers of different tastes onto the same product. Therefore, if we let $A(q)$ be the set of consumers consuming q, we can write the monopolist's cost function of q as follows:

$$C(q) = c_0 + c(q) \int_{A(q)} f(\theta) d\theta \tag{3.3}$$

Equation (3.1) implies that higher θ's are willing to pay more for a given q than lower θ's. Therefore, that q must be nondecreasing in θ follows from the self-selection constraint. Since $c(q)$ is increasing in q, it is obvious that p charged by the profit-maximizing monopolist is nondecreasing in q and hence nondecreasing in θ.

3.3 The Monopolist's Problem

As we mentioned in previous section, the presence of the set-up cost makes it unprofitable for the monopolist to perfectly discriminate among all the consumer types. Since θ is continuously distributed over $[0, \bar{\theta}]$ and $f(\theta)$ is strictly positive everywhere in the domain, there are an infinite number of consumer types in the economy. However, as we have seen in last section, th e total number of different quality levels the monopolist is willing to produce is bounded by a finite number. Therefore, the monopolist has to bunch the infinite number of consumer types into a finite number of groups and offer distinguished bundles to cater to different groups. This brings us the first interesting question: how would consumers be grouped together? Or, what is the property of $A(q)$ as defined in section 3.2?

Proposition 17 *$A(q)$ is convex for all q.*

Proof:

Suppose consumers of both θ_1 and θ_2 choose q_i. From self-selection constraints of both θ_1 and θ_2, we have

$$\theta_1 q_i - p_i \geq \theta_1 q_j - p_j$$

for all j,

and

$$\theta_2 q_i - p_i \geq \theta_2 q_j - p_j$$

for all j.

Then for all $\theta = \lambda\theta_1 + (1-\lambda)\theta_2$, where $\lambda \in [0,1]$, we have

$$[\lambda\theta_1 + (1-\lambda)\theta_2]q_i - (\lambda + 1 - \lambda)p_i \geq [\lambda\theta_1 + (1-\lambda)\theta_2]q_j - (\lambda + 1 - \lambda)p_j$$

That is, $\theta q_i - p_i \geq \theta q_j - p_j$, for all j and the proof is finished.

Proposition 17 tells us that the set of consumers consuming the same quality level is convex. In our case, since θ is restricted to one dimension, the set is just an interval. Therefore, the monopolist's problem is to choose the optimal way of dividing $[0. \bar{\theta}]$ into subintervals and the corresponding q to cater to each subinterval[4].

One immediate result that follows from Proposition 17 is that the monopolist cannot fully separate any subinterval of Θ.

Corollary 1 *There is no full separation among any subinterval of* Θ.

Proof:

Suppose there is a subinterval of Θ in which types are fully separated. Since $f(\theta) > 0$ for all $\theta \in \Theta$, there are an infinite number of values of θ in any such subinterval. This contradicts with the fact that the total number of different quality levels offered by the monopolist is finite. This completes the proof.

Corollary 1 yields an important difference between the model we are studying and the previous models with no fixed costs in production. In those models, some types may be fully separated even if there is a continuum of consumers.

Let θ_i denote the lowest type in the ith interval, or, A_i, and q_i denote the corresponding quality level that the monopolist chooses to serve all θ's in A_i. p_i is then the price that the monopolist would charge for each unit of commodity of quality q_i. Now we can write the monopolist's problem as follows:

$$\max_{\{\theta_i\},\{q_i,p_i\},n} \Pi = \sum_{i=1}^{n} \int_{A_i} (p_i - c(q_i))dF(\theta) - nc_0 = \sum_{i=1}^{n} (p_i - c(q_i))[F(\theta_{i+1}) - F(\theta_i)] - nc_0 \tag{3.4}$$

subject to $\theta q_i - p_i \geq \theta q_j - p_j$, for all $\theta \in [\theta_i, \theta_{i+1}]$,

where n is the total number of different quality levels that the monopolist would choose to produce.

[4]It follows immediately that q is a step function of θ and that the value function for a consumer of type θ, $V(\theta, q(\theta), p(\theta))$ (where $q(\theta)$ and $p(\theta)$ are optimal choices of the monopolist), is piecewise linear in θ.

Before looking at the monopolist's optimal choice, we can use the following proposition to simplify the problem.

Proposition 18 *Given the profit-maximizing behavior of the monopolist, the following equation is a necessary and sufficient condition for the self-selection constraints of all types to be satisfied:*

$$p_i = \theta_i(q_i - q_{i-1}) + \theta_{i-1}(q_{i-1} - q_{i-2}) + ... + \theta_2(q_2 - q_1) + \theta_1 q_1 \tag{3.5}$$

for all $i = 1, 2, \ldots, n$.

To prove this proposition, we need to prove two lemmas first.

Lemma 13 *Given the self-selection constraints, for each* i*, type* θ_i *is indifferent between* q_i *and* q_{i-1}*. That is,* $\theta_i q_i - p_i = \theta_{i-1}q_{i-1} - p_{i-1}$*. In particular, if the monopolist is profit-maximizing,* θ_1 *is indifferent between consuming* q_1 *and not consuming at all.*

Proof:

From self-selection constraint, we know $u(\theta_i, q_i, p_i) \geq u(\theta_i, q_{i-1}, p_{i-1})$. Suppose $u(\theta_i, q_i, p_i) > u(\theta_i, q_{i-1}, p_{i-1})$, then by continuity of u, there exists $\epsilon > 0$ such that $u(\theta_i - \epsilon, q_i, p_i) > u(\theta_i - \epsilon, q_{i-1}, p_{i-1})$. Therefore, $\theta_i - \epsilon$ would be better off choosing q_i instead of q_{i-1}. Contradiction.

Suppose $u(\theta_1, q_1, p_1) > u(\theta_1, 0, 0) = 0$, then by continuity of u, there exists $\epsilon > 0$, such that $u(\theta_1, q_1, p_1 + \epsilon) > 0$. Therefore, the monopolist is not maximizing profit by charging p_1. Contradiction. We are done with the proof.

From Lemma 13, we can tell that the lowest type, $\theta_1 = 0$, enjoys no consumer surplus but all higher θ's enjoy positive surplus[5]. To provide an incentiv e for high-taste-type consumers to buy higher quality product, the monopolist must offer them a surplus in excess of that obtained by the low-taste-type consumers.

[5]By continuity of u, $u(\theta_1 + \epsilon, q_1, p_1) > u(\theta_1, q_1, p_1)$ for all $\epsilon > 0$. Therefore, for all $\theta > \theta_1$, $u(\theta, q_1, p_1) > u(\theta_1, q_1, p_1) = 0$. By self-selection constraint, $u(\theta, q_i, p_i) \geq u(\theta, q_1, p_1) > 0$ for all $\theta \in [\theta_i, \theta_{i+1}]$, for all $i > 1$.

Lemma 14

$$\theta_i q_i - p_i = \theta_i q_{i-1} - p_{i-1}$$

for $i = 1, 2, \ldots, n$[6] *are the only binding constraints.*

Proof:

For all $\theta \in [\theta_{i-1}, \theta_i]$, we have

$$\theta q_{i-1} - p_{i-1} \geq \theta q_i - p_i$$

or,

$$\theta(q_i - q_{i-1}) \leq p_i - p_{i-1}$$

Since both p_i and q_i are nondecreasing, if this holds for θ_i, it also holds for all $\theta \leq \theta_i$. Also, for all $\theta \in [\theta_i, \theta_{i+1}]$, we have

$$\theta q_i - p_i \geq \theta q_{i-1} - p_{i-1}$$

or,

$$\theta(q_i - q_{i-1}) \geq p_i - p_{i-1}$$

If this is true for θ_i, it is also true for all $\theta \geq \theta_i$. Therefore,

$$\theta_i q_i - p_i = \theta_i q_{i-1} - p_{i-1}$$

for $i = 1, 2, \ldots, n$ are the only binding constraints[7].

Lemma 14 reduces the number of constraints in (3.4) to n. Now, we are ready to prove Proposition 18.

[6] This is what Maskin and Riley(1984) called "the local downward" constraint.

[7] The logic we have used here in the proof is in essence the same as the adjacency condition defined by Cooper (1984).

Proof of Proposition 18:

($\Longleftarrow$)

Suppose the self-selection constraints of all types are satisfied, then we have for all $\theta \in [0, \theta_2]$,

$$\theta q_1 - p_1 \geq 0$$

By Lemma13, we have

$$0 q_1 - p_1 = 0$$

which implies $p_1 = 0$ and $q_1 = 0$. Also, by Lemma 1, we have

$$\theta_2 q_1 - p_1 = \theta_2 q_2 - p_2 \Longrightarrow p_2 = \theta_2 q_2 = \theta_2(q_2 - q_1) + \theta_1 q_1$$

In fact, for all i, for all $\theta \in [\theta_{i+1}, \theta_{i+2}]$, we have

$$\theta q_{i+1} - p_{i+1} \geq \theta q_i - p_i$$

with equality for θ_{i+1}. Suppose

$$p_i = \theta_i(q_i - q_{i-1}) + \theta_{i-1}(q_{i-1} - q_{i-2}) + ... + \theta_2(q_2 - q_1) + \theta_1 q_1$$

Then,

$$p_{i+1} = \theta_{i+1}(q_{i+1} - q_i) + p_i = \theta_{i+1}(q_{i+1} - q_i) + \theta_i(q_i - q_{i-1}) + ... + \theta_2(q_2 - q_1) + \theta_1 q_1$$

($\Longrightarrow$)

Given equation (3.5), we have $p_i = \theta_i(q_i - q_{i-1}) + p_{i-1}$ for all $i = 1, 2, \ldots, n$. Which implies $\theta_i q_i - p_i = \theta_{i-1} q_{i-1} - p_{i-1}$ for all $i = 1, 2, \ldots, n$. By Lemma 14, these are the only binding constraints. That is, if these constraints hold, all the other self-selection constraints are trivially satisfied. The proof is finished.

With Proposition 18 and equation (3.4), we can rewrite the monopolist's problem as

$$\begin{aligned}
\max_{\{\theta_i\},\{q_i\},n} \Pi &= ([\theta_n(q_n - q_{n-1}) + \theta_{n-1}(q_{n-1} - q_{n-2}) + \ldots + \theta_2(q_2 - q_1) + \theta_1 q_1] - c(q_n)) \\
&\quad *(F(\bar{\theta}) - F(\theta_n)) \\
&\quad +([\theta_{n-1}(q_{n-1} - q_{n-2}) + \ldots + \theta_2(q_2 - q_1) + \theta_1 q_1] - c(q_{n-1})) \\
&\quad *(F(\theta_n) - F(\theta_{n-1})) \\
&\quad + \ldots \\
&\quad +([\theta_2(q_2 - q_1) + \theta_1 q_1] - c(q_2))(F(\theta_3) - F(\theta_2)) \\
&\quad +(\theta_1 q_1 - c(q_1))(F(\theta_2) - F(\theta_1)) \\
&\quad -nc_0 \qquad (3.6)
\end{aligned}$$

where n is the total number of different quality levels that the monopolist chooses to offer.

The monopolist's problem is now reduced to an unconstrained maximization problem. Let $\Pi_n(\theta_1, q_1, \ldots, \theta_n, q_n)$ denote the profit function for fixed n and $\Pi_n^*(\theta_1, q_1, \ldots, \theta_n, q_n)$ denote the maximal profit the monopolist can achieve for fixed n. Then the first order conditions for maxim izing Π_n are as follows:

For all $i = 1, 2, \ldots, n$,

$$\frac{\partial \Pi_n}{\partial \theta_i} = (q_i - q_{i-1})[F(\bar{\theta}) - F(\theta_i)] - f(\theta_i)[\theta_i(q_i - q_{i-1}) - c(q_i) + c(q_{i-1})] = 0^8 \qquad (3.7)$$

For all $i = 1, 2, \ldots, n-1$,

$$\begin{aligned}
\frac{\partial \Pi_n}{\partial q_i} &= \theta_i[F(\bar{\theta}) - F(\theta_i)] - \theta_{i+1}[F(\bar{\theta}) - F(\theta_{i+1})] \\
&\quad -c'(q_i)[F(\theta_{i+1}) - F(\theta_i)] \\
&= (\theta_i - \theta_{i+1})(F(\bar{\theta}) - F(\theta_{i+1})) \\
&\quad +(\theta_i - c'(q_i))(F(\theta_{i+1}) - F(\theta_i))
\end{aligned}$$

[8] $q_0 = 0$.

$$= 0 \tag{3.8}$$

and

$$\frac{\partial \Pi_n}{\partial q_n} = (\theta_n - c'(q_n))(F(\bar{\theta}) - F(\theta_n)) = 0 \tag{3.9}$$

The set of θ_i's and q_i's that the monopolist chooses for fixed n must satisfy above equations. Set $M = [\frac{\int_0^{\bar{\theta}} \bar{q}\theta f(\theta)d\theta}{c_0}]$, that is, the largest integer which is less than or equal to $\frac{\int_0^{\bar{\theta}} \bar{q}\theta f(\theta)d\theta}{c_0}$. To solve (3.6), the monopolist needs to calculate Π_n^* for all $\{n \in \mathcal{N} : n \leq M\}$. Let n^* be the n such that $\Pi_{n^*}^*=\max\{\Pi_1^*, \Pi_2^*, \ldots, \Pi_M^*\}$ and θ_{in^*} and q_{in^*} denote the optimal θ_i and q_i corresponding to n^* for $i = 1, \ldots, n^*$. $\{n^*, \theta_{1n^*}, \ldots, \theta_{n^*n^*}, q_{1n^*}, \ldots, q_{n^*n^*}\}$ is then the solution to (3.6). From now on, the optimal θ_i's and q_i's we talk about are corresponding to n^*.

The above discussion gives us a general algorithm to solve the monopolist's problem. In the latter part of this section and following sections, we derive some characterizations of the monopolist's optimal solution.

The first question we would like to ask about the monopolist's choice is: Does the problem of allocative distortion arise in this problem? That is, comparing to the efficient level, how would the quality level served to customers be distorted by the profit-maximizing monopolist?

Here we follow Cooper (1984) and Srinagesh and Bradburd (1989)'s notion of efficiency. A q is considered to be efficient if the marginal rate of substituion between quality and price of the consumer equals that of the monopolist at that q. Under the assumptions we have, the MRS of the consumer of type θ is θ and the MRS of the monopolist is simply $c'(q)$. Therefore, for a q to be efficient, the condition $\theta = c'(q)$ must be satisfied. With this condition, we then can get the next proposition.

Proposition 19 *There is no quality distortion by the monopolist for the lowest type in the highest consumer group. The quality distortion takes the form of degradation for all the other consumers.*

Proof:

This follows immediately from the first order conditions of the monopolist's maximization problem.

(3.9) $\Longrightarrow \theta_n = c'(q_n)$. So all the consumers of type θ_n are served efficiently.

(3.9) also implies $\theta > c'(q_n)$, for all $\theta \in (\theta_n, \bar{\theta}]$. Since $c(q)$ is strictly convex in q, all $\theta \in (\theta_n, \bar{\theta}]$ consume the quality that is lower than the efficient level.

Since $\theta_i - \theta_{i+1} < 0$, $F(\bar{\theta}) - F(\theta_{i+1}) > 0$ and $F(\theta_{i+1}) - F(\theta_i) > 0$; (3.8) implies $\theta_i > c'(q_i)$ which in turn implies for all $\theta \in [\theta_i, \theta_{i+1}]$, $\theta > c'(q_i)$. Therefore, for all i, all $\theta \in [\theta_i, \theta_{i+1}]$ consume the quality that is lower than the efficient level. This completes the proof.

Since the above results hold for every n, the lowest-type consumers consuming the highest quality level are always served efficiently for each fixed n. However, all other types always consume lower qualities than they would at the efficient level.

3.4 Profit Margin and The Quality Level

The next important question this model is designed to answer is: What is the relationship between the profit margin and the quality level? This brings us the fourth proposition.

Proposition 20 *At the monopolist's optimal solution, the profit margin is increasing in quality level. That is, for all i,*

$$p_i - c(q_i) - \frac{c_0}{\int_{A_i} f(\theta)d\theta} \geq p_{i-1} - c(q_{i-1}) - \frac{c_0}{\int_{A_{i-1}} f(\theta)d\theta}, \forall i$$

In addition, $p_i - c(q_i) > p_{i-1} - c(q_{i-1})$ for all i.

Proof:

From self-selection constraint, $\theta \in [\theta_{i-1}, \theta_i)$ strictly prefers q_{i-1} to q_i. Since p is nondecreasing in θ, all $\theta \geq \theta_i$ can at least choose q_{i-1}. Suppose q_{i-1} is more profitable than q_i. That is,

$$p_i - c(q_i) - \frac{c_0}{\int_{A_i} f(\theta)d\theta} < p_{i-1} - c(q_{i-1}) - \frac{c_0}{\int_{A_{i-1}} f(\theta)d\theta}$$

From Lemma 13, we have $u(\theta_i, q_{i-1}, p_{i-1}) = u(\theta_i, q_i, p_i)$. By continuity of u, $u(\theta_i, q_{i-1}, p_{i-1}) > u(\theta_i, q_i, p_i+\epsilon)$ for all $\epsilon > 0$. So if the monopolist increases p_i to $p_i+\epsilon$, s ome $\theta \in [\theta_i, \theta_{i+1}]$ will switch to q_{i-1}. Let A'_{i-1} and A'_i denote the new sets of θ's choosing q_{i-1} and q_i respectively. We have

$$\int_{A'_{i-1}} f(\theta)d\theta > \int_{A_{i-1}} f(\theta)d\theta$$

and

$$\int_{A'_i} f(\theta)d\theta < \int_{A_i} f(\theta)d\theta.$$

$\Longrightarrow$

$$\begin{aligned} p_{i-1} - c(q_{i-1}) - \frac{c_0}{\int_{A'_{i-1}} f(\theta)d\theta} & \\ &> p_{i-1} - c(q_{i-1}) - \frac{c_0}{\int_{A_{i-1}} f(\theta)d\theta} \\ &> p_i - c(q_i) - \frac{c_0}{\int_{A_i} f(\theta)d\theta} \\ &> p_i - c(q_i) - \frac{c_0}{\int_{A'_i} f(\theta)d\theta} \end{aligned}$$

This implies q_{i-1} is even more profitable than q_i. Therefore, to make more profit, the monopolist can keep increasing p_i until $[\theta_{i-1}, \theta_i]$ is merged with $[\theta_i, \theta_{i+1}]$. This contradicts the assumption that the p_i' s and q_i's are optimal solutions to the monopolist's problem and the monopolist is producing optimal number of quality levels.

Similarly, if $p_i - c(q_i) \leq p_{i-1} - c(q_{i-1})$ for some i, then the monopolist can make more profit by merging $[\theta_{i-1}, \theta_i]$ with $[\theta_i, \theta_{i+1}]$ and paying just one set-up cost for this whole group instead of two.

This completes the proof.

This result is consistent with the observation of quality distortions at the monopolist's optimum. As indicated by Mussa and Rosen (1978), serving customers who place smaller valuations on quality creates negative externalities for the monopolist that li mit possibilities for capturing consumer surplus from those who do value quality highly and have more potentially extractable consumer surplus. The monopolist internalizes this externality by lowering the prices and qualities offered to lower-type groups to make the degraded good less attractive to higher-type groups. This allows the monopolist to charge higher prices to higher-type groups without having them switch to lower quality products. The cost to the monopolist of serving lower-type customers l ess efficiently is paid for by charging higher prices to higher-type customers. A wave of such distortions ripples down the quality spectrum from the highest group to the lowest group in order to preserve market segmentation.

3.5 Change In The Distribution of Consumer Preference

Another important qualitative issue is how a change in the underlying distribution of consumer preference affects the monopolist's profit. Suppose that there is a rightward shift in the distribution, so that the new distribution $G(\theta)$ is strictly less than $F(\theta)$ for all $\theta \in [0, \bar{\theta})$. Would this lead to higher monopolist profit? With the help of Proposition 20, we can answer this question easily.

Proposition 21 *Let Π^G and Π^F denote the monopolist's profits associated with $G(\theta)$ and $F(\theta)$ respectively. If $G(\theta)$ first order stochastically dominates $F(\theta)$*[9]*, then $\Pi^G \geq \Pi^F$*[10].

Proof:

From (3.4),

$$\Pi^F = \sum_{i=1}^{n}(p_i - c(q_i))[F(\theta_{i+1}) - F(\theta_i)] - nc_0$$

[9] Maskin and Riley (1984) indicated that one condition sufficient to ensure that $G(\theta)$ first order stochastically dominates $F(\theta)$ is that the hazard rate for F exceeds the hazard rate for G.

[10] If $G(\theta) = \Phi(F(\theta))$, with $\Phi(0) = 0$, $\Phi(\bar{\theta}) = 1$ and $\Phi(\cdot)$ is convex, then $G(\theta)$ is in fact a first order stochastically dominant shift of $F(\theta)$. Therefore, a convex transformation of $F(\theta)$ brings higher profit to the monopolist.

$$
\begin{aligned}
&= (p_1 - c(q_1))[F(\theta_2) - F(\theta_1)] + (p_2 - c(q_2))[F(\theta_3) - F(\theta_2)] \\
&\quad + \ldots + (p_{n-1} - c(q_{n-1})[F(\theta_n) - F(\theta_{n-1})] + (p_n - c(q_n))[F(\bar{\theta}) - F(\theta_n)] - nc_0 \\
&= F(\theta_1)(c(q_1) - p_1) + F(\theta_2)[(p_1 - c(q_1)) - (p_2 - c(q_2))] \\
&\quad + F(\theta_3)[(p_2 - c(q_2)) - (p_3 - c(q_3))] + \ldots \\
&\quad + F(\theta_n)[(p_{n-1} - c(q_{n-1})) - (p_n - c(q_n))] + (p_n - c(q_n))F(\bar{\theta}) - nc_0
\end{aligned}
$$

From proof of Proposition 18, we know $q_1 = p_1 = 0$. Let $m_i = p_i - c(q_i)$. Using the fact $F(\bar{\theta}) = 1$, we can rewrite Π^F as:

$$\Pi^F = \sum_{i=2}^{n} F(\theta_i)(m_{i-1} - m_i) + m_n - nc_0 \tag{3.10}$$

By Proposition 20, $m_{i-1} - m_i \leq 0$.

$G(\theta)$ first order stochastically dominates $F(\theta) \Longrightarrow G(\theta_i) \leq F(\theta_i)$, $\forall \theta_i$.

This implies that with the same set of θ_i's and q_i's, the monopolist makes larger profit with $G(\theta)$ than he would with $F(\theta)$.

Therefore, with $G(\theta)$, the monopolist can at least choose the same θ_i's and q_i's as he would with $F(\theta)$ and be no worse off. Hence, $\Pi^G \geq \Pi^F$ and the proof is completed.

In fact, Proposition 21 is very intuitive. Since higher values of θ are associated with higher demanders of quality and from Proposition 20 we know higher quality products are more profitable, the upward shift of the distribution of θ obviously will make the monopolist better off.

3.6 Concluding Remarks

In this paper, we have presented a model in which a monopolist seeks to sell a spectrum of quality differentiated products to consumers whose preferences for quality are not observable to him. The monopolist only knows the distribution of the consumer pr eference and he has to pay a constant amount of set-up cost for each type of products he produces. We have proposed a solution algorithm to the monopolist's profit-maximization problem and have characterized some aspects of the optimal solution. However , there are some more questions that merit further study.

We have shown that except the lowest type of the group of consumers consuming the product of the highest quality, all consumers are served inefficiently in the sense that the marginal rate of substitution between quality and price of the consumer exceeds that of the monopolist. But as we have indicated earlier, in the case with set-up cost, how many different types of products to produce becomes an important choice variable of the monopolist. Therefore, in this case, it is natural to consider the proble m of allocative distortions in two ways: one is the notion of efficiency as we have discussed; another one is the range of qualities provided by the monopolist. Comparing to social optimum which maximizes aggregate surplus, whether the monopolist is prod ucing too many or too few quality levels should be an important question.

The model we have discussed in this paper is lifted from Mussa and Rosen (1978) with an additional assumption of set-up cost. In Mussa and Rosen (1978)'s case with no set-up cost, the monopolist provides a continuous spectrum of quality levels when the c onsumers' preference is continuously distributed. As we have mentioned several times, with the set-up cost, the total number of different quality levels served by the monopolist must be finite. However, an interesting question we may ask is: as the set- up cost becomes arbitrarily small, will the n^* chosen by the monopolist go to infinity? Would Mussa and Rosen's (1978) solution constitute the limiting case of our solution?

Further study on comparative statics of the monopolist's choice of $\{\theta_i\}$, $\{q_i\}$ and n^* needs to be done. It would be interesting to see how changes in the cost function, distribution of consumer's preference and other parameters affec t the monopolist's choices.

We have focused on the profit-maximizing behavior of a monopolist. However, there are reasonable situations in which there are several producers competing on selling the product of the same generic type. We may want to know in the framework of oligopoly , whether all the producers will choose to sell the same spectrum of products and compete on prices or they will divide the spectrum and each sell quality levels different from others. Furthermore, if there is free entry in the market, what would be the competitive equilibrium?

In this paper, we have adopted the standard assumption of consumer preference in the literature. Namely, the consumers can be ordered by their intensities of demand for the quality[11]. Another extension of this framework is to co nsider preferences that do not permit the ordering of tastes. This would imply that the simplifications of the monopolist's optimization problem via reducing the number of self-selection constraints would not hold any longer.

Finally, our analysis is restricted to the one-dimensional case. In fact, Proposition 17, 18, 19 and Corollary 1 do hold for the general case of multi-dimensions. That is, even when θ and q are multi-dimensional, we can simplify the monopolist's problem in the same way. However, whether our characterizations of the monopolist's optimum hold in multi-dimensional case is left to be checked.

[11]This follows from the "single-crossing property" of the indiff erence curve. For example, if the marginal rate of substitution is monotonically increasing in type, then the indifference curves of two different types cross at most once. This property holds in our case.

Bibliography

Avery, Christopher (1998), "Strategic Jump Bidding in English Auctions," *Review of Economic Studies*, 65:1-26.

Besanko, David, Shabtai Donnenfeld, and Lawrence White (1967), "Monopoly and Quality Distortion: Effects and Remedies," *Quarterly Journal of Economics*, Vol. 102:743-68.

Border, K.C. (1995), "Notes on Product Differentiation," California Institute of Technology.

Bulow, Jeremy and D. John Roberts (1989), "The Simple Economics of Optimal Auctions," *Journal of Political Economy*, 97:1060-1090.

Cassady, Ralph Jr. (1967), *Auctions and Auctioneering*, Berkeley: University of California Press.

Chiang, R. and Chester S. Spatt (1982), "Imperfect Price Discrimination and Welfare," *Review of Economic Studies*, 49:155-82.

Chwe, Michael S. Y. (1989), "The Discrete Bid First Auction," *Economics Letters*, 31:303-306.

Cooper, Russell (1984), "On Allocative Distortion in Problems of Self-Selection," *Rand Journal of Economics*, 15:568-77.

Coppinger, Vicki M., Vernon L. Smith and Jon A. Titus (1980), "Incentives and Behavior in English, Dutch and Sealed-Bid Auctions," *Economic Inquiry* Vol. XVIII, Jan. 1-22.

Cox, James C. and Vernon L. Smith (1982), "Theory And Behavior of Single Object Auctions," in *Research In Experimental Economics*, Vernon L. Smith, ed., Greenwich, Conn.: JAI Press.

Cox, James C., Vernon L. Smith and James M. Walker (1983), "A Test That Discriminates Between Two Models of The Dutch-First Auction Non-Isomorphism," *Journal of Economic Behavior and Organization*, 4:205-219.

Cox, James C., Vernon L. Smith and James M. Walker (1986), "Theory and Individual Behavior of First Price Auctions," Discussion paper No. 86-6, Department of Economics, University of Arizono, Tuscon, AZ.

D'Aspremont, C., J. J. Gabszewicz and J. F. Thisse (1979), "On Hotelling's 'Stability In Competition'," *Econometrica*, Vol. 47, No. 5.

Dixit, Avinash K. and Joseph E. Stiglitz (1977), "Monopolistic Competition And Optimum Product Diversity," *The American Economic Review*, Vol. 67, No. 3:297-308.

Federal Communications Commission (U.S.), *FCC Entrepreneurs' Auction, Broadband Personal Communications Services (PCS)*, Aug., 1995.

Fudenberg, Drew and Jean Tirole (1993), *Game Theory*, The MIT Press.

Guesnerie, R. and Jesus Seade (1982), "Nonlinear Pricing in a Finite Economy," *Journal of Public Economics*, 17:157-79.

Harris, Milton and Arthur Raviv (1981a), "A Theory of Monopoly Pricing Schemes with Demand Uncertainty," *The American Economic Review*, 71:347-365.

Harris, Milton and Arthur Raviv (1981b), "Allocation Mechanisms and The Design of Auctions," *Econometrica*, 49:1477-1499.

Hart, O. D. (1979), "Monopolistic Competition in a Large Economy with Differentiated Commodities," *Review of Economic Studies*, 46:1-30.

Kagel, John H. (1991), "Auctions: A Survey of Experimental Research," in: *Handbook of Experimental Economics.* Amsterdam: Elsevier Science Publishers.

Kagel, John H., Ronald Harstad and Dan Levin (1987), "Information Impact and Allocation Rules in Auctions With Affiliated Private Values: A Laboratory Study," *Econometrica*, 55:1275-1304.

Kagel, John H. and Dan Levin (1993), "Independent Private Value Auctions: Bidder Behavior In First, Second, and Third-Price Auctions With Varying Numbers of Bidders," *The Economic Journal*, 103:868-879.

Kuhn, H. (1953), "Extensive Games and The Problem of Information," *Annals of Mathematics Studies*, No. 28., Princeton University Press.

Maskin, Eric and John Riley (1984), "Monopoly with Incomplete Information," *Rand Journal of Economics*, 15:171-96.

Matthews, Steven A. (1995), "A Technical Primer On Auction Theory I: Independent Private Values," Discussion Paper No. 1096, Northwestern University.

McAfee, R. Preston and John McMillan (1987), "Auctions and Bidding," *Journal of Economic Literature*, 25:699-738.

Milgrom, Paul R. and Robert J. Weber (1982a), "A Theory of Auctions And Competitive Bidding," *Econometrica*, 50:1089-1122.

Milgrom, Paul R. and Robert J. Weber (1982b), "The Value of Information In A Sealed Bid Auction," *Journal of Mathematical Economics*, 10:105-114.

Milgrom, Paul R. and Robert J. Weber (1985) "Distributional Strategies for Games

With Incomplete Information," *Mathematics of Operations Research*, 10:619-632.

Mirman, Leonard J., and David Sibley (1980), "Optimal Nonlinear Prices for Multiproduct Monopolies," *The Bell Journal of Economics*, 11:659-670.

Mussa, M. and S. Rosen (1978), "Monopoly and Product Quality," *Journal of Economic Theory*, Vol. 18:301-317.

Myerson, R.B. (1981), "Optimal Auction Design," *Mathematics of Operations Research*, Vol. 6:58-73.

Perloff, J. M. and S. C. Salop (1985), "Equilibrium with Product Differentiation," *Review of Economic Studies*, LII:107-120.

Phlips, Louis (1983), *The Economics of Price Discrimination.* Cambridge, UK: Cambridge University Press.

Riley, John G. (1989), "Expected Revenue From Open And Sealed Bid Auctions," *Journal of Economic Perspectives*, 3:41-50.

Riley, John G. and William F. Samuelson (1981), "Optimal Auctions," *The American Economic Review*, 71:381-392.

Rothkopf, Michael H. and Ronald M. Harstad (1994), "Modeling Competitive Bidding: A Critical Essay," *Management Science*, Vol. 40, No. 3:364-384.

Salant, Stephen W. (1989), "When Is Inducing Self-Selection Suboptimal for a Monopolist?" *Quarterly Journal of Economics*, 104:391-7.

Schmalensee, Richard (1981a), "Monopolistic Two-Part Pricing Arrangements," *Bell Journal of Economics*, 12:445-66.

Schmalensee, Richard (1981b), "Output and Welfare Implications of Monopolistic

Third Degree Price Discrimination," *The American Economic Review*, 71: 242-247.

Spence, A. Michael (1980), "Multiproduct Quantity-Dependent Prices and Profitability Constraints," *Review of Economic Studies*, 47:821-41.

Srinagesh, P. (1990), "Self-Rationing and Nonlinear Prices," Livingston, NJ: Bell Communications Research.

Srinagesh, P. and Ralph M. Bradburd (1989), "Quality Distortion by a Discriminating Monopolist," *The American Economic Review*, 78:96-105.

Stockey, Nancy (1979), "Intertemporal Price Discrimination," *Quarterly Journal of Economics*, 93:355-71.

Vickrey, William (1961), "Counterspeculation, Auctions, And Sealed Tenders," *Journal of Finance*, 16:8-37.

Vickrey, William (1962), "Auctions and Bidding Games," in: O. Morgenstern and A. Tucker, eds., *Recent Advances In Game Theory*, Princeton: Princeton University Press.

White, Lawrence (1977), "Market Structure and Product Varieties," *Quarterly Journal of Economics*, 93:355-371.

Wilson, Robert (1992), "Strategic Analysis of Auctions," in Robert J. Aumann and Sergiu Hart, eds. *Handbook of Game Theory*, Chap. 8.