CALCULATION OF THE OPTIMUM PITCH DISTRIBUTION OF A PROPELLER WITH SWEEPBACK

Thesis

by

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In Partial Fulfillment of the Requirements for the

Degree of

Aeronautical Engineer

California Institute of Technology

Pasadena, California

1948

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ABSTRACT

The principal problem dealt with is the derivation of a method for computing the optimum pitch distribution of a lightly loaded, sweptback propeller. The method is based on an extension to the work of Glauert as presented in Volume IV of the Durand series. A complete calculation is carried out for a swept propeller in which the resultant velocities along the blade vary through the transonic regime. The analytical work is simplified by the use of the calculus of variations. A survey of the German literature on this topic is included along with a complete discussion of the aerodynamic parameters to be considered in making such a design.

DEFINITION OF SYMBOLS

$\frac{dT}{dr}$	thrust of a blade element of span dr at r. (T in pounds)
B	number of propeller blades
Γ	blade element circulation
ρ	air density in slugs per cu. foot.
Ω	rotational velocity of propeller (rad/sec)
ω	induced radial velocity of slipstream (rad/sec)
r	radius of a blade element dr or/and $d\xi$
$\frac{dQ}{dr}$	torque of a blade element of span dr at r (Q in lbs-feet)
u	translational velocity at a blade element (ft/sec)
V	forward speed of the airplane (ft/sec)
$\frac{dE}{dr}$	energy loss of a blade element of span dr at r (E in ft-lbs per second)
γ	circulation factor $= \frac{B\Gamma\Omega}{2\pi V^2}$
x	radial coordinate factor $= \frac{\Omega r}{V}$
$\frac{dE_D}{dr}$	profile energy loss of a blade element of span dr at r. (E_D in ft-lbs per second)
W	resultant velocity at a blade element = $\sqrt{u^2 + (\Omega r)^2}$ (in feet per second)
C_D	blade element drag coefficient
C_L	blade element lift coefficient
ϕ	advance angle $= \tan^{-1} \frac{V}{\Omega r}$
ϵ	drag-lift ratio $= C_D/C_L$
K	Goldstein tip loss reduction factor

SYMBOLS (continued)

ξ	coordinate along the peak pressure line of the swept profile
β	sweepback angle
R	radius of the swept propeller measured perpendicular to the blade center line (feet)
$\frac{dT}{d\xi}, \frac{dQ}{d\xi}, \frac{dE}{d\xi}$	values of thrust, torque, etc., of the blade element of span $d\xi$ at r.
W_e	W cos β
R_v	rotational velocity = $(\Omega - \frac{\omega}{Z})r$ (ft/sec)
c	blade chord measured perpendicular to ξ (feet)
u_e	u cos β
R_{v_e}	R_v cos β
δ	x / ϵ
μ	$(\gamma + Z)/2$
B, C, D	coefficients in the cubic in μ.
C_p	Power coefficient = $\frac{P}{\rho N^3 D^5}$
J	advance ratio = V/nD
D	propeller diameter in feet.
α_i	induced angle of attack of the blade element

I. INTRODUCTION

1.1 A Sweptback propeller--Pro and Con.

"Why should a propeller with sweepback be considered?" Such a proposal leads to complications, both aerodynamic and structural, which can be solved completely only by extensive and expensive testing programs. Before such a procedure is undertaken its logic must be proven and as much theoretical work as possible should be put forward to indicate that gains may be expected by such an unconventional design.

Introducing sweepback into a propeller is merely a continuation of the fight which propeller engineers have been making for some time against the losses of compressibility. One step in that fight was the simple expedient of limiting the rotational speed of the propeller. However, with high speed and cruising speed of flight moving nearer to Mach one the limitation of rotational speed has reached a definite barrier. Solidity ratios, prop diameters and gear ratios may no longer be altered sufficiently to prevent the operation of some part of the blade above critical Mach number. Therefore, if the high efficiencies ordinarily associated with propellers are to be maintained at these high speeds it is necessary to utilize some other, more drastic, design details.

A case in point is the turboprop. This installation is especially applicable to airplane designs which require long range, low specific fuel consumption, and a high cruising speed. If the basic propeller efficiency can be maintained in excess of 85% for airplane Mach numbers approaching

0.9 then the propeller driven aircraft will fulfill the above design conditions. The fuel consumption of jet driven airplanes is still much too high to fulfill the above requirements in their present state of advancement. Sweepback is just one of the features which may contribute to the establishment of the turboprop installation.

The physical way in which sweeping back the planform of the propeller will delay compressibility losses is simply explained and is well understood from similar proposals for sweeping back wings of high speed airplanes. With sweepback the propeller sections will be subjected to an effective velocity which is the product of the resultant velocity and a factor proportional to the cosine of the sweepback angle. Thus the propeller will be operating in a reduced velocity field and will not experience compressibility losses as soon as would conventional planform propellers. Drag and lift divergence at the tip sections will be delayed and the overall efficiency of the propeller will be correspondingly increased.

1.2 Survey of the German Literature

Some work has been done on the design and building of sweptback propellers. Such a propeller was given a full scale test flight by the Curtis-Wright Propeller Division but there has been no published report on the results of that test. The principal impetus to the idea of propellers of this configuration was originated in Germany. However, there is no indication in the literature that they ever got beyond the model testing stage in their designs. Reports of these tests are contained in several captured German documents. (See references 1, 2, and 3). These papers are very brief and almost all design and theoretical details have been ommitted. The substance of these references will be given here.

The tests were made in wind tunnels of relatively low velocity. Tip Mach numbers of approximately 0.98 were obtained in a flow of 157 mph. This gave a ratio of tip speed to forward speed of about 1.0 : 0.15. The geometrical angle of attack was 13 degrees on the pressure side of the blade sections. The following series of planforms was proposed.

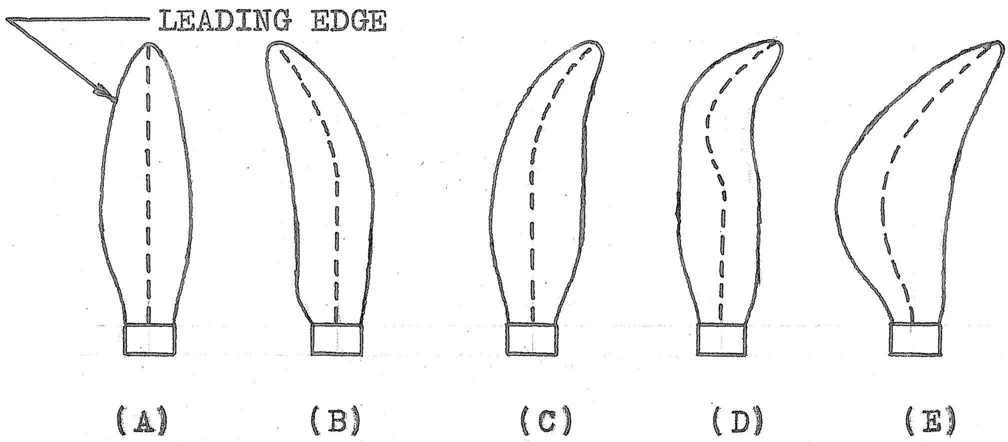

Fig. 1-1 Proposed propeller planforms.

The straight planform prop tested had a diameter of 3.6 feet. For the curved planform the curvature was obtained by displacing the blade elements. The curvature was in a plane defined by the axis of the straight blade and the zero lift direction at 0.7 radius. Sweepback at the tips was 45 degrees. The wind tunnel tests were actually only made on the first three configurations shown above. The results of these tests are indicated in the following graph (Fig. 1-2) where the efficiency indicated corresponds to the optimum operating condition of the model propellers.

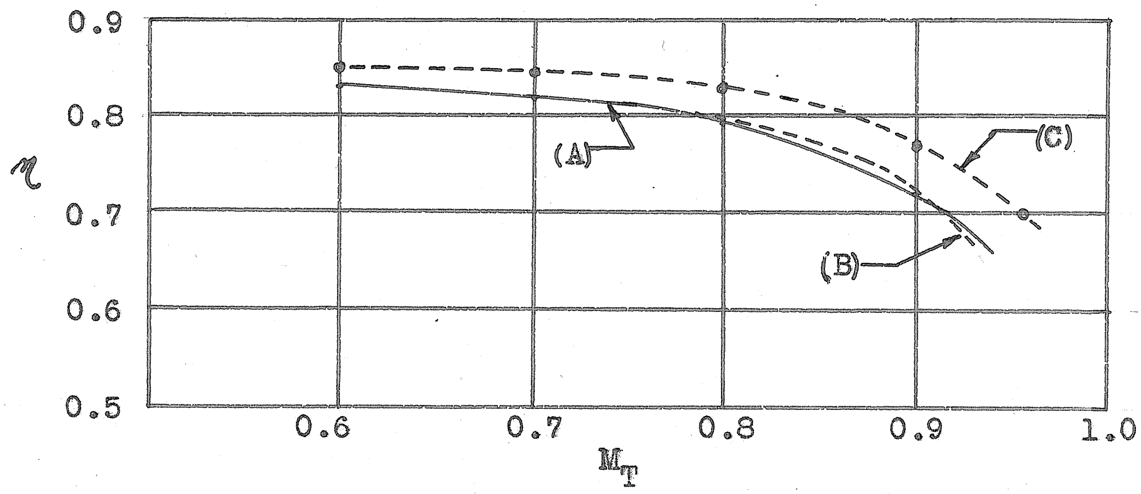

Fig. 1-2 Optimum efficiency for various planforms.

In addition to the results indicated in the above figure there was noted qualitatively that for the sweptback planform the thrust coefficient increased and the torque coefficient decreased with increasing Mach number. No attempt was made to evaluate the scale effect in these tests. A further interesting statement in the discussion was that the trailing edge was found to be the region of highest stress.

Referring to Figure 1-2 it is noted that the advantage shown by the sweptback blade is not evident with the swept forward blade. It must be concluded that the gain is not due solely to the delay of the compression shock. More significant are the radial pressure gradients due to the different directions of sweep. These radial pressure gradients are independent of shock wave development and react differently on the flow of the boundary layer. It must be concluded that this effect is in addition to the action of centrifugal force on the boundary layer air. Further, the inclined compression shock plane diverts the streamlines outwards in the case of the sweptback airfoil and inwards in the case of the sweptforward airfoil. This is because only the velocity component perpendicular to the plane of the shock suffers a discontinuous reduction at the shock while the component parallel to the shock plane remains unchanged.

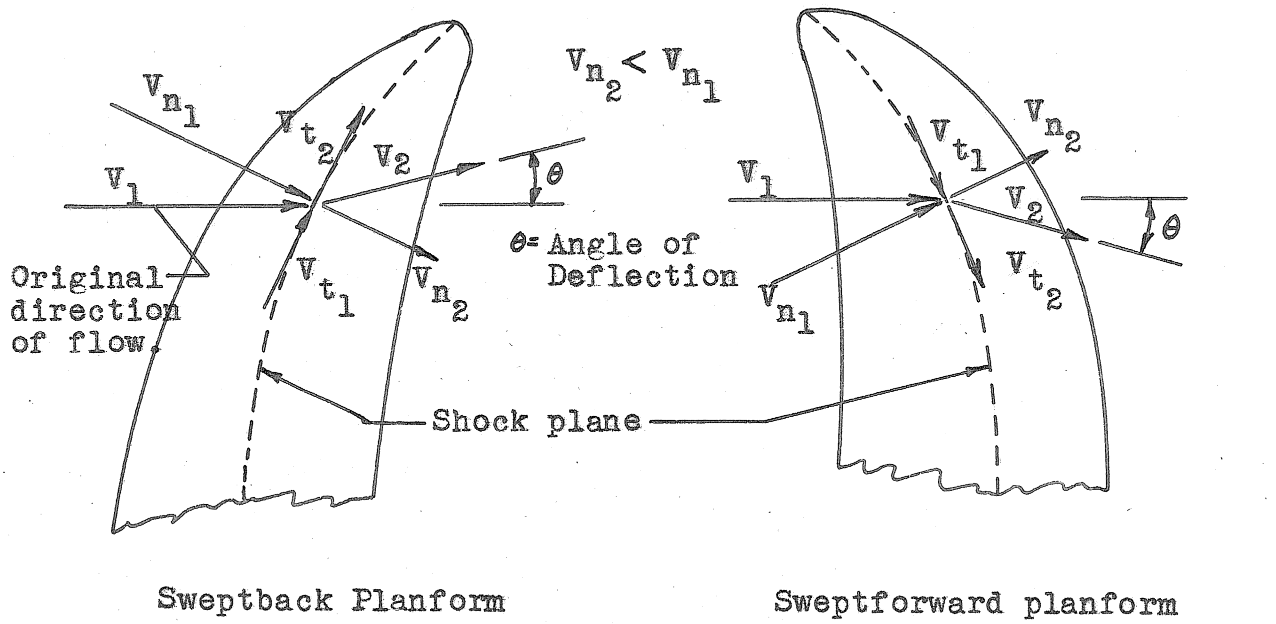

Fig. 1-3 Velocity gradients due to inclined shock fronts.

The above leads to changes in induced drag. The sweptback tip gives an effective increase in diameter and reduction in induced drag and the opposite to this effect is obtained with the sweptforward tip. Further, these flow conditions influence the boundary layer in such a way as to favor the sweptback propeller.

1.3 Statement of a Hypothetical Swept Propeller Design Problem

Let us assume a hypothetical propeller design problem. It is required to design a propeller incorporating sweepback which will provide high basic efficiencies (of the order of 85%) at the following operating conditions:

A.	Airplane velocity	550 mph
B.	Altitude	25,000 feet
C.	Prop rpm	1260
D.	Prop diameter	13 feet
E.	Engine BHP	2800
F.	Four bladed propeller	

This particular design condition imposes a Mach number variation along the blade ranging from sub- to supersonic. A reasonable amount of sweepback will reduce the entire velocity field to an effective value below the critical of the blade sections. (See Figure 1-4). In this paper we will only be concerned with the aerodynamic design of the propeller. The structural problems are present and are extensive but will not be treated in this material.

In designing the propeller aerodynamically we may assume complete freedom as to blade section, blade chord, activity factor, etc. The only physical features of the propeller which are prescribed are those on the diameter and number of blades.

II. AERODYNAMIC PARAMETERS IN THE PROBLEM

Having established the fact that the blade is to operate at transonic speeds, and being aware of the detrimental effects incurred in this regime, it is now necessary to investigate the design parameters which will delay or reduce these effects. In general, the following factors must be considered:

A. Blade Section

B. Activity Factor

C. Interference Effects

D. Planform

E. Optimum Design C_L Variation

These factors will now be discussed in the above order.

2.1 Blade Section

This division of the problem has had some attention in reference #6. The efforts in this reference have been, in the main, experimental and the results are tabulated therein. Since this paper is not fundamentally concerned with the investigation of airfoil sections and their characteristics we will settle upon the sections as recommended in the reference. Yet, at the same time, it would also seem advisable to understand the mechanics by which these particular airfoils have a lower M_{cr}. Therefore, the particular sections tested by the NACA will be examined in the light of existing information on this problem. The sections being considered are designated as the NACA 16-series.

The critical Mach number has been defined as corresponding to the free stream velocity when some local velocity on the airfoil reaches the local speed of sound. Associated with this critical Mach number is an increasingly large positive change in the drag coefficient. This drag increase is attributed to shock loss which occurs when the local velocity at any point on the airfoil reaches sonic velocity, i.e., when M_{cr} is reached. However, it has been shown theoretically that the fact that some local velocity is sonic, while a necessary condition for shock loss, is not necessarily a sufficient condition, reference #4. However, such a situation is thought to be an unstable one, and within engineering accuracies the two phenomena can be thought of as occurring simultaneously.

A very convenient method is available for theoretically predicting the M_{cr} of any section if $C_{P_{M \to 0}}$, the pressure coefficient for low velocities, is known, reference #4. This method involves the assumption of small perturbations, i.e., the local velocity, u, is assumed small such that its relation to the free stream velocity, V, at any point along the surface is of the order

$$\frac{u}{V} \ll 1$$

Such an assumption cannot hold very near the leading edge of an airfoil unless the radius of the leading edge is vanishingly small. Or, another way of expressing this criterion is that the thickness of the section must be small as compared with the chord. In accordance with the above assumptions the following relation may be derived.

$$M_{cr} = 1 - \frac{1}{2}\left[-\frac{\gamma+1}{2} C_{p_0}\right]^{2/3}$$

where,

M_{cr} is the critical Mach number

γ is the ratio of the specific heats

C_{p_0} is the maximum negative pressure coefficient on the airfoil.

For example, from reference #6, C_{p_0} for the NACA 16-009 section is -.225. Hence,

$$M_{cr} = 1 - \frac{1}{2}\left[-\frac{1.4+1}{2}(-.225)\right]^{2/3}$$
$$= 0.791$$

And for the NACA 07-009 section, C_{p_0} is -.210 for zero angle of attack for which

$$M_{cr} = 1 - \frac{1}{2}\left[-\frac{1.4+1}{2}(-.210)\right]^{2/3}$$
$$= 0.800$$

Experimental data as presented in reference #6 make possible the comparison with the theoretical results as calculated above. Thus for a 16-009 section the experimental value of M_{cr} is given as 0.802 for a variance of about 1.5% between the theoretically calculated and the experimental values.

From a perusal of the theoretical calculations above it would appear that the NACA 07-009 section has a higher M_{cr} than the 16- series section NACA 16-009. However, this is not supported experimentally, reference #6. The explanation seems to be in the fact that the 07-section has a more full pressure diagram near the leading edge where the compressibility effects are more critical. Thus the critical Mach number is not determined uniquely by the value of $(C_{P_0})_{min}$ and the form of the pressure distribution is important, at least insofar as the configuration near the leading edge is concerned. In the general case this would lead to a question as to which would bear the greater weight, a low value of $(C_{P_0})_{min}$ or low leading edge negative pressures. We may generalize to the extent that C_{P_0} must be minimized and with as little expense near the leading edge as possible. This means a very slight leading edge radius and small thickness.

It is evident that the local velocities corresponding to a given free stream velocity, V, can be reduced and, hence, M_{cr} increased by three means,

1. Decrease the thickness
2. Decrease the leading edge radius
3. Maintain the lowest possible design C_L (minimum camber)

Before concluding these remarks on blade section characteristics, it is necessary to discuss the desirability of using symmetrical blade sections in order to prevent a decrease in thrust at high values of M. This effect with cambered sections has been shown experimentally to begin at about $M = 0.8$, and its result is to decrease the angle of zero lift with increasing Mach number. The extent of this change can be severe enough to incur negative thrust at cambered tip sections. German data sustain this contention. This statement is based upon a personal conversation with the German aerodynamicist Goethert. It is further substantiated by CGD 483 wherein reference is made to measurements of this effect by Blankenburg (UM 1319). Of course, the desirable way to handle this in the design would be to utilize cambered sections with allowances in angle of attack to compensate for the change. Two things make this approach difficult. First, there are no design data available which specifically define the shift of the curve and, secondly, allowance for this effect at high speed would lead to excessive blade angles at the other operating conditions.

Therefore, at high forward design speeds it would appear that symmetrical sections would be desirable. However, it must be remembered that the M_{cr} of a blade section[4] is a function of the peak minimum pressure. And, while a symmetrical section will have a full pressure curve at $\alpha = 0^o$, it will peak seriously at the leading edge at angles of attack greater than zero. And it follows that the extent of this peak will increase with angle of attack. The writer has made theoretical calculations by the method of reference #5 which show that for the 16-series symmetrical sections angles of attack above 3^o will result in an advance pressure peak. For these sections an angle of attack of 3^o gives C_L values between .3 and .4 . The design condition for this blade is the high speed condition which infers a lightly loaded propeller. Therefore, angles of attack in excess of 3^o will not be required. Hence, it appears that when both the effect of the shift of the lift curve slope and the fact that small angles of attack are required are taken into account, the symmetrical section is the logical choice.

In a further effort to reduce compressibility effects it is important that the section thickness be kept to a minimum. This thickness variation will be dictated by structural considerations.

2.2 Activity Factor

The activity factor is a quasi-empirical factor based upon the blade area distribution of the propeller. It is defined by the relation,

$$AF = \frac{100,000}{16} \int_{0.2R}^{1.0R} \left(\frac{C}{D}\right)\left(\frac{r}{R}\right)^3 d\left(\frac{r}{R}\right)$$

where,

C is the chord at the radius r

D is the prop diameter

R is the radius = D/2

100,000 / 16 merely gives the AF a convenient magnitude.

An examination of the above relation shows that for a given D, AF will be increased by increasing the chord length C along the blade. Therefore, increased AF of a propeller will correspond to decreased aspect ratio. In evaluating this statement it must be recalled that propeller theory is derived on the basic assumption of two dimensional flow. Therefore the several paragraphs which follow are intended only to indicate qualitatively the changes in power absorbtion associated with changes in AF at near sonic Mach number operation. We are accordingly interested in determining the variation of M_{cr} with aspect ratio. This variation is as shown in the following figures.

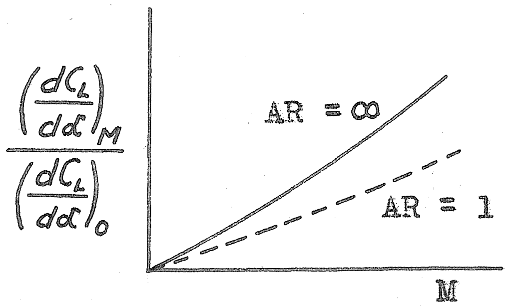

Fig. 2-1 Change of Lift curve slope with M and AR.

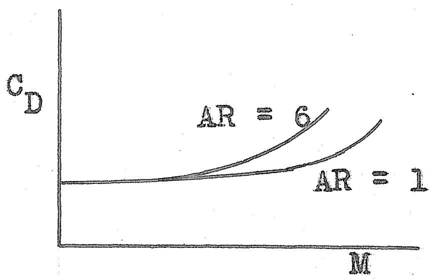

Fig. 2-2 Change of C_D with Mach number and AR.

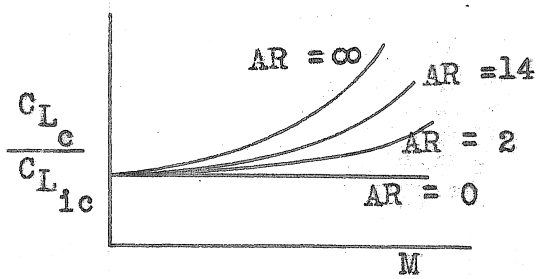

Fig. 2-3 Change of C_L with Mach number and AR.

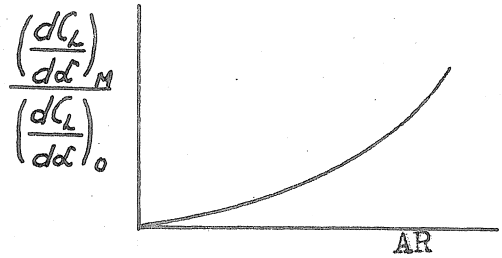

Fig. 2-4 Lift curve slope as a function of AR for a given Mach number.

Figure 2-1 shows the variation of the slope of the lift curve with change in Mach number with aspect ratio as the parameter. Thus for a given slope at $M_\infty \to 0$, the variation with aspect ratio will be as follows. For any given Mach number, M, the slope of the lift curve will increase with increasing aspect ratio. See figure 2-4. Qualitatively this graph is the same as for the case in which compressibility effects are ignored. And in this same respect, experimental results show no measurable effect of M on $dC_L/d\alpha$ for AR = 1 up to M = 0.80.

Figure 2-2 is of primary interest in that it clearly shows the delay in drag rise for the lower AR planforms. Theoretically there is no limit to which this effect may be carried to advantage. However, other considerations, e.g., skin friction and boundary layer extension will limit this reasoning. Also, Figure 2-3 shows that C_{L_C} decreases with decrease in AR so that the most efficient section does not necessarly correspond to the lowest AR which is structurally feasible.

High Mach number operation causes an extension of laminar flow over high speed sections which is characterized by the curve in Figure 2-5.

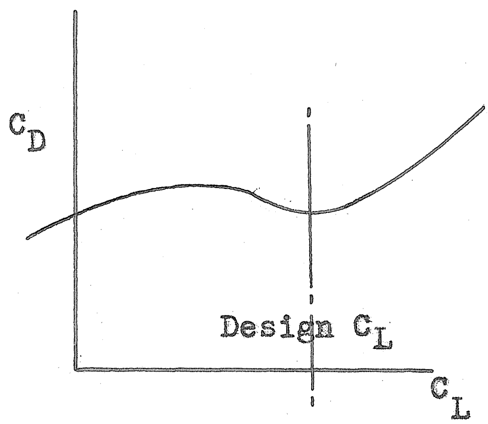

Fig. 2-5 The drag-lift curve in the "critical" range.

It is to be noted that this "drag bucket" occurs for one small range of C_L. With some sections, viz., the 16-sections, this effect is not indicated in the test data available. However it is felt that such a phenomenon would appear and that investigation is necessary if we are to rationally select design C_L's in the range above M_{cr}. As mentioned before, this bucket occurs over a very limited range of lift coefficients.

The qualitative conclusion to the above is that from a standpoint of drag rise in the compressibility range the AF should be as high as is structurally feasible. However, this will not lead to the most efficient planform since the lift of the planform will begreatly decreased.

In summary it is seen that an increasing AF leads to the following effects:

1. Higher thrust coefficient (incompressible)
2. Higher power coefficient (incompressible)
3. Loss of section lift coefficient (compressible)
4. Decrease in section drag coefficient (compressible), i.e., the value of M_{cr} is raised.
5. Little change in lift curve slope from compressibility considerations under M = 0.8.

Thus it is seen that an optimum AF must exist. However, its empirical definition precludes the possibility of arriving at such an optimum analytically. Therefore the procedure will have to be one of cut and try.

2.3 Interference Effects

There are indications that interference effects at high subsonic and transonic speeds may be of importance in determining the overall characteristics of aerodynamic performance in these speed ranges. There is no available material to indicate that this effect has been considered in the case of propellers. However, it is possible that interference of large proportions can occur both between the blade and the spinner and between the blade and the fuselage. This is especially so in the case of the swept propeller, which enters the spinner at either a swept forward or swept back angle and which curves back over the fuselage. In general, the result of this interference could be either favorable or unfavorable depending upon the configuration. There is no theoretical method available for predicting these effects, which means that resort must be made to experiment and that a number of configurations would have to be tested in order to evaluate the results.

Little light can be thrown on this problem by consideration of interference effects on other parts of the airplane, for two reasons. The principal one is that very little is known about the problem theoretically and little or no experimental information is available at this time. Such information may become available later, and at that time can be closely studied for analogies with the propeller interference problem. However, in this consideration we come upon the second difficulty, which is that the propeller interference problem is somewhat unigue. That is, it will bear little resemblance to other interfering parts of the airplane. If qualitative comparisonscan be obtained by considering interference, say at the wing and fuselage, such should be taken into account; but it cannot be hoped that anything quantitative will be obtained.

2.4 & 2.5 Planform and Optimum Design C_L Variation

Introduction of sweepback into the planform of the propeller is the basic factor underlying the presentation of this paper. The purpose and principles applicable to the use of sweepback have been discussed in the introduction and will not be treated further here.

In attempting to define the optimum design C_L variation along a swept propeller, it becomes evident that this is one aspect of the problem which is amenable to some theoretical investigation. Accordingly, the principal part of this work is the presentation of a means whereby the pitch distribution of a swept propeller may be calculated so as to give optimum efficiency at a given operating condition. At this point we will merely list the differences to be expected underlying the optimum pitch distribution of a conventional planform propeller versus a swept planform propeller.

1. The field of induced velocities of the propeller will be altered by compressibility and sweepback.
2. Profile drag coefficient will be decreased by sweepback.
3. Section thrust and energy loss will be altered.

4. All such effects will be expressable analytically.

Section III is concerned with the analytical treatment of this problem.

III DETERMINATION OF ANALYTICAL EXPRESSIONS FOR CALCULATING THE OPTIMUM PITCH DISTRIBUTION

3.1 Sources of the Method. Development by Glauert.

There is a wealth of material which may be searched in an effort to find a framework upon which to build a method for computing the optimum pitch distribution of a sweptback propeller. Up to the present nothing has been published covering such an extension. Thus the writer was completely free from "the power of suggestion" in the choice of the basic framework. British writers have done a great deal of work on the optimum propeller problem and these works appear in the R & M publications. Several are mentioned in the references at the end of this paper. All of the British works are adaptations of the Goldstien theory into tabular and graphical methods. Propeller engineers in this country also use Goldstein's work as a basis but in the large have developed their own methods of computation. All of these papers have as their purpose the procurement of the answer with minimum labor and as a result the original parameters in such developments are obscured. At the other extreme is the work of Glauert in Volume IV of the Durand series. There the problem is presented much more clearly and concisely. Therefore it was decided to use the fundamentals of this reference as a framework and to expand it to include a computing procedure which should not be unnecessarily complicated. As the method was enlarged upon it was found that the algebra became unwieldy and so the approach was generalized slightly to include methods of the calculus of variations. This step was inspired by references #8 and #9 and its adaptability will be evident upon inspection of the section in this paper which is devoted to it. A brief outline of the reference by Glauert follows after which this method will be applied to the swept propeller problem.

The problem dealt with in Glauert's paper is the determination of the optimum aerodynamic design of a prop of a given diameter operating at a given forward speed and power. Such a condition was first determined by Betz and his solution specified that for highest efficiency the vortex wake of the propeller should move axially aft from the propeller as rigid screw surfaces after undergoing an initial deformation. One of each such surfaces thereby corresponding to each blade. This condition is general but its calculation in specific cases is so difficult that simplifying assumptions must be made to make solutions tractable.

The first such assumptions that may be made are that the propeller does not experience profile drag, has an infinite number of blades, and is lightly loaded. Then the velocities induced by the propeller will be small in comparison with the translational and rotational velocity of the propeller. At a radius r we have the following expressions for the thrust and torque of a blade element of span dr

$$\frac{dT}{dr} = B\Gamma\rho\left(\Omega - \frac{\omega}{2}\right)r$$

and

$$\frac{dQ}{dr} = B\Gamma\rho u r$$

where the circulation has its value defined by

$$B\Gamma = 2\pi\omega r^2$$

The preceeding equation arises from the assumption of an infinite number of blades so that the vortex sheets lie close together.

Then if we combine with the above equations an expression for the thrust from momentum considerations,

$$\frac{dT}{dr} = 4\pi r \rho u (u - V)$$

and neglect second order terms in the induced velocities, the following relation for the axial velocity at the propeller disc is obtained.

$$u = V + \frac{B\Gamma\Omega}{4\pi V}$$

And, accordingly, the thrust is

$$\frac{dT}{dr} = B\Gamma\rho\Omega r$$

Now the energy loss for a frictionless propeller is

$$\frac{dE}{dr} = \Omega\frac{dQ}{dr} - V\frac{dT}{dr}$$

And in the order of this approximation

$$\frac{dE}{dr} = \frac{B^2\Gamma^2\rho\Omega}{4\pi}\left[\frac{\Omega r}{V} + \frac{V}{\Omega r}\right]$$

Now define

$$\gamma = \frac{B\Gamma\Omega}{2\pi V^2} \qquad \text{and} \qquad X = \frac{\Omega r}{V}$$

Then

$$\frac{dE}{dr} = \frac{2\pi\rho V^4}{\Omega}\gamma^2\frac{1+X^2}{2X}$$

and

$$V\frac{dT}{dr} = \frac{2\pi\rho V^4}{\Omega}\gamma X$$

Now the basis upon which the Betz condition was derived is that the ratio of the increments of VT and E due to an arbitrary increase in Γ at any station should be independent of the radial coordinate x . Our condition for optimum performance in this case then becomes

$$\gamma\frac{1+X^2}{X} = AX \qquad \text{or} \qquad \gamma = A\frac{X^2}{1+X^2}$$

A is an arbitrary constant. The effect of varying its magnitude is clearly defined by the power operating condition which is given in our problem.

The problem is now extended to account for the effect of the <u>profile drag</u> of the blade elements. It is evident that the efficiency will be altered and by examining the effect in detail we see that the expression for optimum distribution of circulation along the blade is also changed. In particular, the additional energy loss associated with profile drag is

$$\frac{dE_D}{dr} = \frac{1}{2}Bc\rho W^3 C_D$$

And, if we use the fact that,

$$\frac{dT}{dr} = \frac{1}{2}Bc\rho W^2(C_L\cos\phi - C_D\sin\phi)$$

the expression for profile drag loss may be written as

$$\frac{dE_D}{dr} = \epsilon\Omega r\frac{dT}{dr}$$

This relation contains several additional simplifying assumptions, namely,

1. Effect of profile drag on the thrust is ignored.
2. ϕ is assumed to be small.
3. W is approximately equal to Ωr.

We will find in extending this theory to the swept back propeller that these assumptions are too restrictive; especially so are 2 and 3, and accordingly such assumptions will not be made at that time.

But, for this present case, we assume that such conditions hold and the expression for the energy loss may then be written in the form

$$\frac{dE}{dr} = \frac{B^2\Gamma^2\rho\Omega}{4\pi}\left[\frac{\Omega r}{V} + \frac{V}{\Omega r}\right] + \epsilon B\Gamma\rho\Omega^2 r^2$$

The expression for thrust is unaltered and we obtain for the work and loss of each element the following relations in terms of the parameters γ and x.

$$V\frac{dT}{dr} = \frac{2\pi\rho V^4}{\Omega}\gamma x$$

$$\frac{dE}{dr} = \frac{2\pi\rho V^4}{\Omega}\left[\gamma^2\frac{1+x^2}{2x} + \epsilon\gamma x^2\right]$$

And, applying the optimum condition as before, we obtain

$$\gamma\frac{1+x^2}{x} + \epsilon x^2 = Ax \qquad \text{or,} \qquad \gamma = \frac{(A-\epsilon x)\,x^2}{1+x^2}$$

This relation could almost have been deduced directly. It was stated that A is defined by the power input. Then the above equation merely states that a portion of the power goes into the drag loss. Such loss must be proportional to the drag-lift ratio and to the relative velocity at each blade element. Since the velocity is proportional to the radius parameter, the validity of the above form is apparent.

All of that which has been said in the preceeding paragraphs applies only to a propeller of an infinite number of blades. We may now see the effect of considering our propeller to be comprised of a finite number of blades.

The obvious effect of considering only a finite number of blades is that the air between the vortex sheets will experience an important radial induced velocity which is not experienced by a propeller of an infinite number of blades. In addition, it is also apparent that this effect will become more pronounced near the blade tips. This consideration has led to the so-called tip loss correction factors (to be distinguished from compressibility loss). An approximate correction has been obtained by Prandtl and a more exact analysis by Goldstein. Goldstein's solution refers only to lightly loaded propellers. Neither approach will be given in detail here. It is sufficient at this point to say that the axial velocity is reduced and that this reduction requires that a factor be entered in the momentum equation for thrust. It is this reduction factor that is given by Prandtl and Goldstein in their analyses. The corrected form of the momentum equation then is

$$\frac{dT}{dr} = 4\pi r \rho u(u-V)K$$

K is a function of the advance angle ϕ. This functional relation is treated later. Then since the axial induced velocity has been decreased by a factor K the optimum condition for the case of the frictionless, lightly loaded propeller is

$$\gamma = A \frac{X^2 K}{1+X^2}$$

And, similarly for a propeller with drag, the optimum γ is given by

$$\gamma = \frac{(AK - \epsilon X)\, X^2}{1+X^2}$$

The reduction factor K is discussed in more detail in section 3.3 b.

3.2 Extension of Glauert's Analysis to the Swept Propeller

3.2a Coordinate system. Geometry.

Before carrying out the actual analysis it will be necessary to define some additional coordinates. Let us consider the following as a plan view of the propeller as seen if one were able to develop a helical surface into a plane.

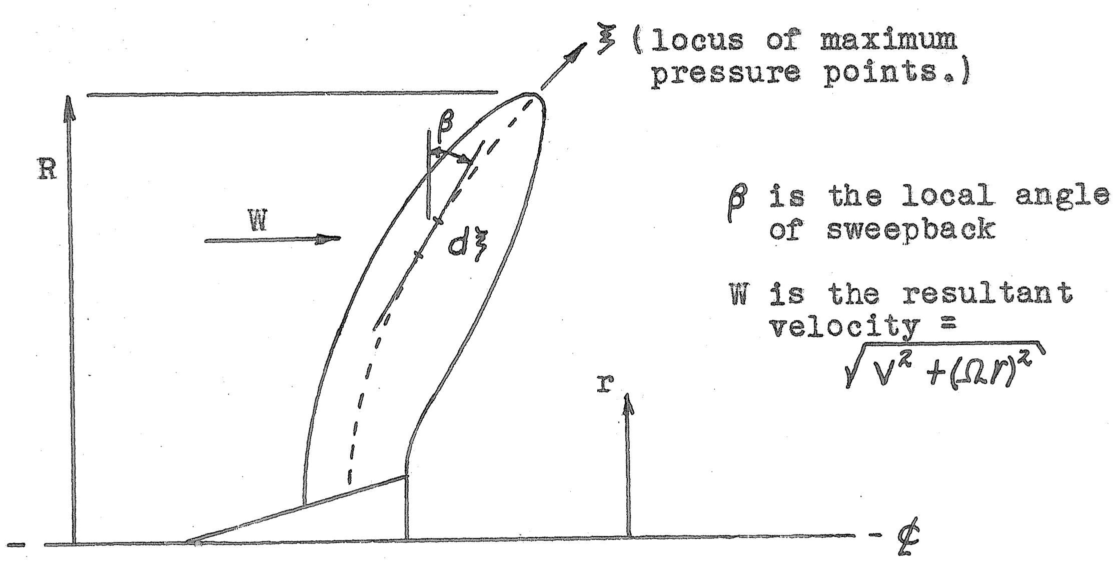

Fig. 3-1 Coordinate System

In addition, a great deal can be said about the magnitude of the effective velocity which will be experienced by the blade element $d\xi$.

Since our calculations are to be carried out on the basis of a strip theory the problem is two dimensional and may be likened to a wing of infinite span in side slip. Therefore, if the viscous nature of the flow is neglected, the aerodynamic characteristics of a section will clearly be completely defined by the component of the resultant velocity which is normal to $d\xi$. By such reasoning the resultant velocity is given by W cos β. Certainly this gives the correct lift of an element since any velocity along the blade in the spanwise direction cannot have a pressure gradient. By the same reasoning the induced drag will also be correctly described but not the friction drag. However, we will use the cosine factor here keeping in mind that it is not an exact description of what takes place. If a more satisfactory expression for the effective velocity is developed at any time, it will be a fairly simple matter to alter this analysis accordingly.

In the light of the above assumption covering the magnitude of the relative velocity, a few words should be said about the plane in which the sweepback is to be accomplished. We propose here that the blade should be twisted in such a fashion that the sweepback is accomplished in the plane of the helix which corresponds to the desired operating condition. The reason for this is best described by several sketches.

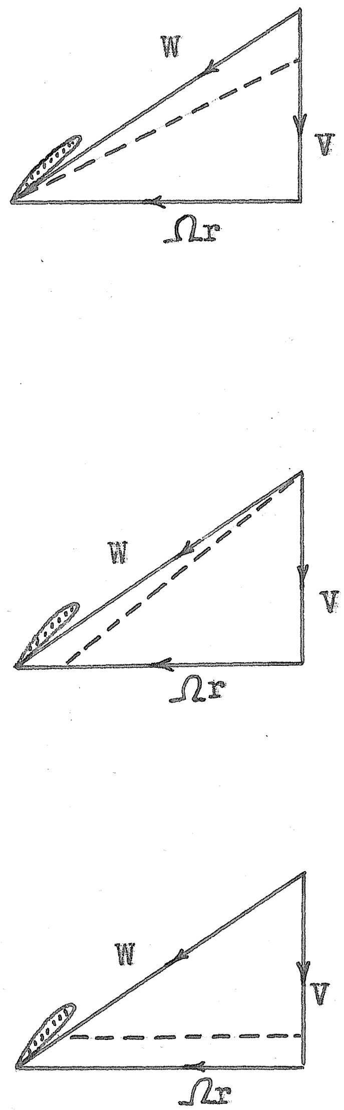

Fig. 3-2 Planes of sweepback.

Sweepback in the fore and aft plane: The dotted line corresponds to reduced resultant velocity and we notice that very large sweep back will be required to reduce the resultant velocity W by any satisfactory amount.

Sweepback in the rotational plane: It is obvious that here we would also need large sweepback to accomplish the desired purpose.

The logical solution to this problem then is to effect the sweep back in the plane of the helix. Then the sweepback is directly effective upon the resultant velocity and the forward and rotational velocities also have as their effective values

$$V \cos\beta \quad \text{and} \quad \Omega r \cos\beta$$

3.2b <u>Derivation of the Equations.</u>

Upon the basis of the preceeding conditions and assumptions we will now proceed to derive the equations for thrust, torque, etc., which will be required in the course of the analysis. For a propeller of straight planform the thrust may be written

$$\frac{dT}{dr} = 4\pi r \rho u (u - V)$$

This describes the momentum of the fluid in an annular element of radius r and width dr. Then the equivalent expression for a blade element $d\xi$ is

$$\frac{dT}{d\xi} = 4\pi r \rho u (u - V) \cos\beta$$

An expression for the circulation, Γ , experienced by any blade element will be needed. The value for a propeller of conventional planform is

$$B\Gamma = 2\pi r(\omega r) = 2\pi\omega r^2$$

Now this expression describes the circulation about a closed circular path of circumference $2\pi r$ corresponding to a velocity ωr. And, since this will be the value of the circulation in the flow irrespective of the type of planform the expression will be carried over unchanged. Accordingly, for any element $d\xi$ the circulation is given by

$$B\Gamma = 2\pi\omega r^2 \qquad 3\text{-}(1)$$

Another expression for thrust is obtained by application of the Kutta-Joukowsky equation which, for conventional planform propellers, gives

$$\frac{dT}{dr} = B\Gamma\rho(\Omega - \frac{\omega}{2})r$$

Recalling that we are applying the cosine of the sweepback angle as a measure of the reduction of the velocity components then it is clear that the equivalent expression for the swept propeller is

$$\frac{dT}{d\xi} = B\Gamma\rho\left(\Omega - \frac{\omega}{2}\right) r \cos\beta \qquad 3\text{-}(2)$$

And by similar reasoning, the expression for the torque which for conventional planform is

$$\frac{dQ}{dr} = B\Gamma\rho u r$$

will, in the case of the swept propeller become,

$$\frac{dQ}{d\xi} = B\Gamma\rho u r \cos\beta$$

The relation between the rotational and axial velocity should be unaltered by these 'transformations' over that obtained for the straight planform. This may be checked by equating the expressions for thrust.

$$4\pi r\rho u(u - V)\cos\beta = B\Gamma\rho\left(\Omega - \frac{\omega}{2}\right) r \cos\beta$$

Since the $\cos\beta$ factor cancels it is clear that the condition is sustained. Accordingly, the axial velocity will be obtained from the relation

$$u(u - V) = \frac{1}{2}\left[\Omega - \frac{\omega}{2}\right]\omega r^2$$

3.2c <u>Optimum Circulation for an Ideal Swept Propeller</u>

The analysis will now be made under the assumptions,

1. Flow is frictionless

2. The propeller has an infinite number of blades.

3. The propeller is lightly loaded.

Assumption 3 will be expressed by using (u - V) $\ll$ V and neglecting terms of ω in comparison to Ω . Then the equation defining the axial velocity may be written as,

$$V(u-V) = \frac{1}{2}\Omega\omega r^2$$

From equation 3-(1) we have that

$$\frac{B\Gamma}{2\pi} = \omega r^2$$

Therefore, the axial velocity is

$$u = \frac{B\Gamma\Omega}{4\pi V} + V$$

Also, equation 3-(2) for the thrust becomes

$$\frac{dT}{d\xi} = B\Gamma\rho\Omega r \cos\beta$$

And, to the same approximation, the equation for the torque is

$$\frac{dQ}{d\xi} = B\Gamma\rho r\cos\beta\, u = B\Gamma\rho r\cos\beta\left[\frac{B\Gamma\Omega}{4\pi V} + V\right]$$

Since the flow has been assumed frictionless the energy loss is described by

$$\frac{dE}{d\xi} = \Omega\frac{dQ}{d\xi} - V\frac{dT}{d\xi} = B\Gamma\rho r\cos\beta\left[\Omega(u-V) + \frac{\omega}{2}V\right]$$

and since,

$$\frac{\omega}{2} = \frac{B\Gamma}{4\pi r^2}$$

we obtain,

$$\frac{dE}{d\xi} = \frac{B^2\Gamma^2\rho\Omega}{4\pi}\left[\frac{\Omega r}{V} + \frac{V}{\Omega r}\right]\cos\beta$$

Using the notation of Glauert, we now write

$$\gamma = \frac{B\Gamma\Omega}{2\pi V^2} \qquad \text{and} \qquad X = \frac{\Omega r}{V}$$

So that,

$$V\frac{dT}{d\xi} = \frac{2\pi\rho V^4}{\Omega}\gamma X\cos\beta$$

and

$$\frac{dE}{d\xi} = \frac{2\pi\rho V^4}{\Omega}\gamma^2\left(\frac{X^2+1}{2X}\right)\cos\beta$$

As mentioned previously, the Betz condition for optimum operation is satisfied when the ratio of increments of VT and E caused by increasing the circulation is independent of the radius.

$$\frac{\Delta V\frac{dT}{d\xi}}{\Delta\frac{dE}{d\xi}} = \frac{\frac{2\pi\rho V^4}{\Omega}\cos\beta\; X\left[(\gamma+\Delta\gamma)-\gamma\right]}{\frac{2\pi\rho V^4}{\Omega}\cos\beta\;\frac{X^2+1}{2X}\left[(\gamma+\Delta\gamma)^2-\gamma^2\right]}$$

Therefore, the optimum condition is given by

$$\gamma = A\frac{X^2}{1+X^2}$$

It should be noticed that this is the same relation that was obtained for the conventional propeller under the same assumptions. This result was to be expected. Since the flow is frictionless and the propeller has an infinite number of blades it is obvious that both the thrust and energy will be altered by the same factor, viz., cos β , and it follows that in forming the ratio of increments of these two factors that the cosine function is cancelled. However, it cannot be expected that this will be the case when the analysis is extended to include profile drag. That, then is the next step.

3.2d Effect of Profile Drag

We will make an approximate check upon the effect of profile drag by including the following assumptions.

1. Neglect the effect of drag on the thrust
2. Assume that the advance angle ϕ is small
3. Let the resultant velocity $W = \Omega r \cos\beta$

The energy loss will now be given by,

$$\frac{dE}{d\xi} = \Omega \frac{dQ}{d\xi} - V \frac{dT}{d\xi} + \frac{dE_D}{d\xi}$$

where the energy loss due to profile drag is

$$\frac{dE_D}{d\xi} = B C_D \frac{\rho}{2} W^3 c = B C_D \frac{\rho}{2} \Omega^3 r^3 \cos^3\beta$$

The thrust may be expressed in the conventional manner as

$$\frac{dT}{d\xi} = \frac{1}{2} B c \rho W^2 \left[C_L \cos\phi - C_D \sin\phi \right]$$

and, if we write for the drag-lift ratio

$$\epsilon = \frac{C_D}{C_L}$$

then

$$\frac{dT}{d\xi} = \frac{1}{2} B \rho c \Omega^2 r^2 \cos^2\beta \frac{C_D}{\epsilon}$$

for small ϕ and $W = \Omega r \cos\beta$

It follows that

$$\frac{dE_D}{d\xi} = \epsilon \Omega r \cos\beta \frac{dT}{d\xi}$$

Then we can write the total energy loss as

$$\frac{dE}{d\xi} = \Omega\frac{dQ}{d\xi} - V\frac{dT}{d\xi}\left(1 - \frac{\epsilon\Omega r}{V}\cos\beta\right) = \Omega\frac{dQ}{d\xi} - V\frac{dT}{d\xi}(1-\epsilon X\cos\beta)$$

If this is written in terms of the parameters γ and x it becomes

$$\frac{dE}{d\xi} = \frac{2\pi\rho V^4}{\Omega}\cos\beta\left[\gamma^2\frac{X^2+1}{2X} + \gamma\epsilon X^2\cos\beta\right]$$

and the thrust force remains

$$V\frac{dT}{d\xi} = \frac{2\pi\rho V^4}{\Omega}\cos\beta\,(\gamma X)$$

Applying the condition for optimum performance as previously defined, we find that the desired ratio in this case is

$$\frac{\Delta\left(V\frac{dT}{d\xi}\right)}{\Delta\left(\frac{dE}{d\xi}\right)} = \frac{X\Delta\gamma}{\frac{2\gamma\Delta\gamma(1+X^2)}{2X} + \Delta\gamma\epsilon X^2\cos\beta}$$

Thus the optimum distribution of circulation in this simplified case where profile drag is included is given by

$$\gamma = \frac{(A-\epsilon X\cos\beta)X^2}{1+X^2}$$

Note that the corresponding equation for a propeller of conventional planform was

$$\gamma = \frac{(A-\epsilon X)X^2}{1+X^2}$$

Thus it is seen that the optimum distribution of circulation is altered when one includes the effect of profile drag. This effect, too, could have been deduced rather than derived for the energy loss due to this drag will be a function of β. The chief interest here is the power to which this factor enters into the relation.

Clearly, then, the next step to be taken is to extend the analysis with the simplifying assumptions withdrawn so that the results will be applicable to calculations. Also it will be necessary to include the tip effect in the equations. But before doing that it is of some interest to look at the preceeding results graphically.

Let us assume that the constant A = 0.5; that the drag-lift ratio ϵ is constant along the blade = 0.05; and that the sweepback angle is constant = 60°. For these values the following results are obtained, Table 3-1, and are plotted in figure 3-3.

Table 3-1

Optimum Circulation

$A = 0.5 \quad \epsilon = 0.05 \quad \beta = 60^\circ$

$X = \Omega r/V$	0.5	1.0	2.0	4.0	6.0	8.0	10.0
$\gamma = \frac{AX^2}{1+X^2}$	0.100	0.250	0.400	0.471	0.487	0.492	0.495
$\gamma = \frac{(A-\epsilon X)X^2}{1+X^2}$	0.095	0.225	0.320	0.284	0.195	0.099	0.0
$\gamma = \frac{(A-\epsilon X\cos\beta)X^2}{1+X^2}$	0.097	0.238	0.360	0.377	0.340	0.296	0.250

Note that when the profile drag is considered, sweeping the propeller allows for higher loadings on the outboard sections of the blade. This result is to be expected since the energy loss is reduced in relation to the effective velocity and the amount of this reduction is increased as one passes to the outboard sections of the blade. By inspection of Figure 3-3 we would conclude that the effects would be very slight for propellers operating at low Ω/V ratios. However it must be remembered that these results were based upon the assumption of low advance angle (high Ω/V ratio) so that conclusions in the high forward velocity range must be withheld until the analysis is extended beyond the present restrictive assumptions. This is done next.

3.3 Further Extension of the Analysis

3.3a Restrictive Assumptions Lifted. Introduction of the Calculus of Variations.

Let us first summarize the assumptions under which the preceeding analysis was made. They were that

1. The sweepback is accomplished in the plane of the relative velocity.
2. the propeller is lightly loaded
3. the effect of drag on thrust could be neglected
4. the advance angle ϕ is small
5. the relative velocity can be written $W = \Omega r \cos\beta$
6. and, that the drag-lift ratio is constant along the blade.

Now assumption 1 will be retained for the reasons put forth on page 30 . Also, this entire paper is meant to apply to propellers operating at high cruising speed conditions so that assumption 2 is applicable. By looking at a velocity diagram of a blade element it is easily seen that assumptions 3, 4, and 5 must be cancelled at high forward speeds. (Fig 3-4)

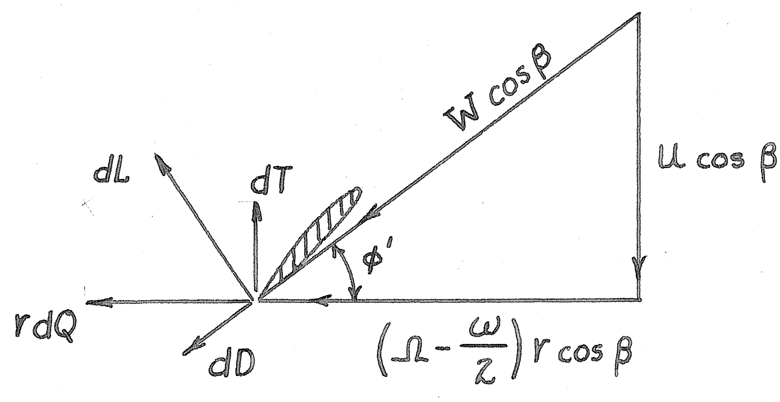

Fig 3-4 Blade element forces

Finally, it will be necessary to allow for changes in blade thickness and chord along the blade so that the drag-lift ratio cannot sensibly be considered constant. Thus, assumption 6 must also be lifted and the analysis will be carried out on this basis. For the present the propeller will be assumed to consist of an infinite number of blades and the correction for a finite number of blades will be applied later.

We now proceed to calculate the values of energy loss and thrust. In order to include the effect of drag the thrust will be described in the following manner. Denote the effective velocities in the forward, rotational and resultant directions as u_e, R_{v_e}, and W_e respectively. Then

$$\phi = \cos^{-1}\frac{R_{v_e}}{W_e} = \sin^{-1}\frac{U_e}{W_e}$$

We have,

$$\frac{dT}{d\xi} = \frac{1}{2} B c \rho W_e^2 C_L (\cos\phi - \epsilon \sin\phi)$$

and

$$C_L = \frac{2\Gamma}{W_e c}$$

so that we can write for the thrust,

$$\frac{dT}{d\xi} = B\rho W_e \Gamma (\cos\phi - \epsilon \sin\phi)$$

or, substituting for cos ϕ and sin ϕ their values in terms of the velocity components, we obtain

$$\frac{dT}{d\xi} = B\rho\Gamma(R_{V_e} - \epsilon u_e) = B\rho\Gamma(R_V - \epsilon u)\cos\beta$$

The velocity components may be expressed in our dimensionless parameters γ and x as

$$R_V = \Omega r \frac{V}{V} = xV$$

and

$$u = \frac{B\Gamma\Omega}{4\pi V} + V = \frac{V}{2}(\gamma + 2)$$

so that

$$\frac{dT}{d\xi} = B\rho\Gamma V \cos\beta \left[x - \frac{\epsilon(\gamma+2)}{2}\right]$$

Finally, the expression for the work obtained from the propeller is

$$V\frac{dT}{d\xi} = \frac{2\pi\rho V^4}{\Omega}\gamma\cos\beta\left[x - \frac{\epsilon(\gamma+2)}{2}\right]$$

The corresponding energy loss will be derived from the equation

$$\frac{dE}{d\xi} = \Omega\frac{dQ}{d\xi} - V\frac{dT}{d\xi} + \frac{dE_D}{d\xi} \qquad 3\text{-}(3)$$

We will use expressions for torque input and work output in the forms,

$$\Omega \frac{dQ}{d\xi} = B\Gamma\rho r \cos\beta \, \Omega \left[\frac{B\Gamma\Omega}{4\pi V} + V\right] \qquad 3\text{-}(4)$$

and

$$V \frac{dT}{d\xi} = B\Gamma\rho r \cos\beta \left(\Omega - \frac{\omega}{2}\right) \qquad 3\text{-}(5)$$

It will be necessary to derive a suitable expression for the energy loss due to drag. This loss is given by,

$$\frac{dE_D}{d\xi} = B C_D \frac{1}{2} \rho W_e^3 c$$

But, also

$$\frac{dT}{d\xi} = \frac{1}{2} B c \rho W_e^2 \left[C_L \cos\phi - C_D \sin\phi\right]$$

These equations may be combined to give

$$\frac{dE_D}{d\xi} = \frac{W_e^2 \, dT/d\xi}{(R_V/\epsilon - u)\cos\beta}$$

Then if we substitute the translation and rotational velocity components for W_e and recall that

$$u = \frac{V}{2}(\gamma + 2) \qquad \text{and} \qquad R_V = XV$$

the expression for the energy loss due to drag becomes

$$\frac{dE_D}{d\xi} = B\Gamma\rho\Omega r \cos\beta \, V \frac{\left[X^2 + \left(\frac{\gamma+2}{2}\right)^2\right]\cos\beta}{\frac{X}{\epsilon} - \frac{\gamma+2}{2}}$$

It will be convenient to write

$$\frac{X}{\epsilon} = \delta \qquad \text{and} \qquad \frac{\gamma+2}{2} = \mu$$

Then the energy loss due to drag can be written as,

$$\frac{dE_D}{d\xi} = \frac{2\pi\rho V^4}{\Omega}\gamma X \cos\beta \frac{(X^2+\mu^2)\cos\beta}{\delta-\mu} \qquad 3-(6)$$

The expression for the entire energy loss is obtained by substituting equations 3-(4), 3-(5), and 3-(6) into 3-(3) to give

$$\frac{dE}{d\xi} = \frac{2\pi\rho V^4}{\Omega}\cos\beta \left[\gamma^2\frac{X^2+1}{2X} + \gamma X\frac{(X^2+\mu^2)\cos\beta}{\delta-\mu}\right]$$

A more convenient expression is obtained by using the variable μ throughout. Then

$$\frac{dE}{d\xi} = \frac{2\pi\rho V^4}{\Omega}\cos\beta \; 2X\left[\frac{(\mu-1)^2(X^2+1)}{X^2} + \cos\beta\frac{\mu^3-\mu^2+X^2\mu-X^2}{\delta-\mu}\right]$$

And, in the same notation, the thrust work is

$$V\frac{dT}{d\xi} = \frac{2\pi\rho V^4}{\Omega}\cos\beta \; 2\left[\mu(X+\epsilon)-\epsilon\mu^2-X\right]$$

With these two expressions in hand we can calculate the ratio of increments of $V\frac{dT}{d\xi}$ and $\frac{dE}{d\xi}$ caused by an arbitrary increase in the circulation. Our optimum condition states that this ratio shall be a constant independent of the radial coordinate x. However, to do this in exactly the same manner as in the previous examples leads to algebra which is hopelessly complicated. The expression for optimum μ comes out in the form of a quartic with complicated coefficients involving the basic parameters. But this complication can largely be circumvented by resort to calculus of variations. Appendix I restates the problem in terms of this calculus and refers to other propeller calculations which have been made in this fashion. In this way the optimum condition is expressed by

$$\frac{\frac{\partial}{\partial \Gamma}\left(V\frac{dT}{d\xi}\right)}{\frac{\partial}{\partial \Gamma}\left(\frac{dE}{d\xi}\right)} = \frac{1}{A} = A'$$

The reciprocal of the constant is taken so as to give it a more convenient magnitude for computing purposes.

The next step then is to carry out the indicated differentiation. Upon doing this and then forming the ratio as indicated above we obtain

$$\frac{\frac{2\pi\rho V^4}{\Omega}(\cos\beta)2\epsilon\frac{\partial\mu}{\partial\Gamma}\left[(\delta+1)-2\mu\right]}{\frac{2\pi\rho V^4}{\Omega}(\cos\beta)\;2X\;\frac{\partial\mu}{\partial\Gamma}\left\{\frac{2(\mu-1)(X^2+1)}{X^2}+\cos\beta\left[\frac{(\mu^2+X^2)(\mu-1)}{(\delta-\mu)^2}+\frac{3\mu^2-2\mu+X^2}{\delta-\mu}\right]\right\}} = A' \qquad 3\text{-}(7)$$

The term $2\frac{2\pi\rho V^4}{\Omega}\cos\beta$ cancels identically.

Accordingly, the expression for optimum circulation is obtained in the form of a cubic,

$$\mu^3 + B\mu^2 + C\mu + D = 0 \qquad 3\text{-}(8)$$

where

$$\mu = \frac{\gamma+2}{2}$$

and where the coefficients of the cubic are

$$B = \frac{-(5\delta+1) - A\delta\left[2(2\delta+1)\frac{X^2+1}{X^2} - \cos\beta\,(3\delta+1)\right]}{2\left[1 + A\delta\left(\frac{X^2+1}{X^2} - \cos\beta\right)\right]}$$

$$C = \frac{2\delta(2\delta+1) + 2A\delta^2\left[(\delta+2)\frac{X^2+1}{X^2} - \cos\beta\right]}{2\left[1 + A\delta\left(\frac{X^2+1}{X^2} - \cos\beta\right)\right]}$$

$$D = \frac{-\delta^2(\delta+1) - A\delta\left[2\delta^2\frac{X^2+1}{X^2} - X^2\cos\beta\,(\delta-1)\right]}{2\left[1 + A\delta\left(\frac{X^2+1}{X^2} - \cos\beta\right)\right]}$$

A solution in this form means that calculations are going to be somewhat tedious. We can anticipate the results by inspecting equation 3-(7) and comparing with the similar expressions obtained for the simplified cases. The numerator which expresses the change in propulsive force with change in circulation has the additional factor $\epsilon(1 - 2\mu)$. This is a small quantity of the order of ϵ so that only for the smaller values of x will it have an effect upon the results. We also see that for large x the term in the denominator, which describes the change in profile drag energy, is reduced to $\epsilon x \cos\beta$ This agrees exactly with the result obtained from the simplified analysis for all x. So it is seen that the corrections included in the more exact expression are of second order. Their contribution is negligible for higher values of x. But for high forward speeds x will, in general, lie below 1 and thus it becomes necessary to retain the correction terms. Sample calculations have been made by equation 3-(8) and the results are indicated in Table 3-2 and Figure 3-3. It is seen at $x = 0.5$ that the optimum circulation as calculated here is about 25% below that indicated by the simplified solution. For higher values of x the results coincide for all practical purposes.

3.3b The effect of a Finite Number of Blades.

All of the preceeding theory has assumed the propeller to have an infinite number of blades. We want now to consider the changes which must be made in the analysis in order to account for the fact that the propeller actually has a

Table 3-2

Variation of Optimum Circulation with x

$A' = 1/A = 2.0 \quad \epsilon = 0.05 \quad \beta = 60^{\circ}$

x	B	C	D	μ	γ
.5	- 21.65	133.08	- 116	1.036	0.072
1	- 44.60	583.9	- 590	1.101	0.202
2	- 94.76	2,781.	- 3,109	1.164	0.328
4	-197.02	12,322	-14,242	1.178	0.356
6	-298.31	28,395	-32,625	1.167	0.334
8	-399.03	50,884	-57,608	1.142	0.284
10	-499.51	79,814	-88,732	1.120	0.240

finite number of blades. The physical fact is that the slipstream is composed of series of vortex sheets, one for each blade of the propeller. By the Biot Savart Law there are velocities induced by these vortices. Thinking of the propeller as composed of an infinite number of blades allowed us to substitute for this complex vortex pattern a geometrically much simpler pattern. We then could envisage a bound vortex upon the disc of the propeller with the horseshoe elements of this vortex passing backwards along the axis of the propeller and the cylinder generated by the blade tips. In such a system we were able to obtain expressions for induced velocities in the axial and rotational directions. There were no radial velocities induced by this vortex system. Passing to the actual fact of a finite number of blades, hence a finite number of vortex sheets, introduces a radial flow of the air between the vortex sheets. Immediately it is seen that this effect is such as to reduce the axially induced velocity and hence to reduce the thrust as computed from momentum considerations. Therefore, qualitatively, the effect is very clear but to carry out a quantitative analysis is not quite so straight forward. The problem has been solved by several people, chiefly Prandtl and Goldstein. Prandtl solves the problem in an approximate fashion by replacing the vortex sheets by a series of parallel lines. Goldstein, on the other hand, has solved the potential problem exactly. This solution exactly represents the Betz optimum condition. By reason of the complicated nature of the problem it is obvious that the exact solution must also be very complicated. Thus Goldstein was only able to obtain a solution for lightly loaded propellers. Both the analyses of Prandtl and Goldstein arrive at a reduction factor which must be applied to the thrust as calculated by momentum considerations. This factor which we shall denote by K, is a function of the station radius, the advance ratio λ, and the number of blades. Goldstein's analysis must be completely reformulated for each different number of blades while the approximate theory of Prandtl results in a general expression into which we need only substitute the parameters including the number of blades. Since it is exact, and since our problem fits the assumption of light loading, we will use the reduction factor as derived by Goldstein. Table 3-3 and Figures 3-5, 3-6, and 3-7 in the Appendix II give values of K for 2, 3, and 4 bladed propellers.

This is a convenient point at which to consider the effect of compressibility upon the equations which we have been writing. Both momentum theory and vortex theory have been used in this analysis and both are based upon an incompressible fluid. Tsien and Lees in reference #10 have evaluated the effect of compressibility upon the induced velocities and hence upon the thrust and torque for lightly loaded propellers. Upon such an assumption they applied the approximate Biot-Savart Law for compressible fluids and found that the axially induced velocities as calculated for an infinite number of blades is unaffected by compressibility. This is a direct consequence of the assumption that induced velocities are small and accordingly that density variations may be neglected in a first approximation. When one considers a finite number of blades the correction involved is increased by a factor proportional to the ratio $\frac{1}{\sqrt{1-M^2}}$. It has been found that the correction to the axially induced velocity by consideration of a finite number of blades is of major importance only for a propeller of a small number of blades (2) and then only on the outer portions of the blade. Thus for a four bladed propeller considerations of compressibility involve only a minor correction to a minor correction and therefore will be neglected here. It is important to note, however, that the section characteristics to be used must correspond to the effective Mach number at which the blade section will operate. It should also be remembered that the above statements will only apply within the assumptions of the Biot-Savart equation and for a lightly loaded propeller.

We now introduce the reduction factor K into the relations which define the optimum distribution of circulation along the blade. The expression for the propulsive work becomes

$$V \frac{dT}{d\xi} = \frac{2\pi\rho V^4}{\Omega} \cos\beta \left[Z(\mu-1)KX - Z\epsilon\mu(\mu-1)\right]$$

The expression for the energy will be unchanged since both the torque and the thrust are reduced by the factor K. Then in our cubic which defines the optimum circulation

$$\mu^3 + B\mu^2 + C\mu + D = 0 \qquad 3\text{-}(9)$$

the coefficients are given by

$$B = \frac{-\left[\delta(4K+1)+1\right] - A\delta\left[Z(1+Z\delta)(X^2+1)/X^2 - \cos\beta\,(3\delta+1)\right]}{Z\{1 + A\delta\left[(X^2+1)/X^2 - \cos\beta\right]\}}$$

$$C = \frac{Z\delta\left[\delta(1+K)+1\right] + ZA\delta^2\left[(\delta+Z)(X^2+1)/X^2 - \cos\beta\right]}{Z\{1 + A\delta\left[(X^2+1)/X^2 - \cos\beta\right]\}}$$

$$D = \frac{-\delta^2(\delta K+1) - A\delta\left[Z\delta^2(X^2+1)/X^2 - X^2(\cos\beta)(\delta-1)\right]}{Z\{1 + A\delta\left[(X^2+1)/X^2 - \cos\beta\right]\}}$$

3.3c <u>Evaluation of the constant A'</u>

The signifigance of A' and the means of determining its value can be shown by the following considerations. The power absorbed by the propeller is given by

$$P = \Omega \sum_{r=r_0}^{r=R} \frac{dQ}{d\xi} = \frac{Z\pi\rho V^4}{\Omega} \sum_{r=r_0}^{r=R} X\gamma\cos\beta\left(\frac{\gamma}{Z}+1\right)$$

Now,

$$C_P = \frac{P}{\rho n^3 D^5} = \frac{1}{D}\left(\frac{Z\pi V}{\Omega D}\right)^4 \sum_{r=r_0}^{r=R} X\gamma\cos\beta\left(\frac{\gamma}{Z}+1\right) \qquad 3\text{-}(10)$$

Or, this relation may very conveniently be written as

$$C_P = \frac{J^4}{D}\sum_{r=r_0}^{R} X\gamma\cos\beta\left(\frac{\gamma}{2}+1\right)$$

where

$$J = \frac{V}{nD}$$

and where γ must define the optimum circulation as previously obtained as a function of A, K, and the radius factor x. C_P is known numerically for a given operating condition.

Thus for a given distribution of ϵ and $\cos\beta$, the optimum γ is determinable as a function of A. Accordingly the summation is a definite number for each A and this constant can be adjusted to give the desired value of C_P. Having determined the correct A in this manner the desired range of is known. The method of incorporating this detail in the calculation procedure is outlined in section 3.3f.

3.3d Effect of the Drag-Lift Ratio Upon Optimum Distribution of Circulation.

It has been tacitly assumed that the optimum circulation does not depend upon the drag lift ratio; that is, in establishing the optimum μ, ϵ has been treated as a constant. Actually, of course, there is a dependence and to be strictly correct the derivative of ϵ with respect to the circulation should have been included. However, the operating blade section characteristics are the unknowns in this problem and to include the derivative $\frac{\partial \epsilon}{\partial \Gamma}$ in a general way leads to complications which are excessive. In fact, such a consideration is largely unnecessary as may be seen by violating a mathematical principle and allowing the ϵ to vary in the solutions previously derived upon the basis of its being constant. Let us consider the approximate solution corresponding to the case of the sweptback propeller with drag. The expression for optimum distribution of circulation in that instance is

$$\gamma = \frac{(A - \epsilon X \cos\beta) X^2}{1 + X^2}$$

In addition, we will restrict this investigation to values of $x \leqslant 1$ since this is the speed range of primary interest here. Then for constant sweepback of 60° and A = 0.5 the following variation is found between γ and ϵ. Values of γ are in the body of the table.

ϵ \ x	0.5	1.0
.01	0.099	0.247
.02	0.099	0.245
.03	0.098	0.242
.05	0.097	0.237
.1	0.095	0.225
.2	0.090	0.200

Note that the variation of γ becomes of important magnitude only as ϵ exceeds 0.05 and then only for $x \longrightarrow 1$. Propeller sections seldom operate at such a high drag lift ratio and when their coefficients do approach such a ratio it is at the thicker shank sections of the blade where the dependence of γ upon ϵ is very slight. Also, in actual cases, A will, in general, exceed 0.5. This diminishes the effect of ϵ percentagewise.

Therefore, the computational procedure will be to assume a reasonable range of values of ϵ along the blade, calculate the corresponding distribution of circulation, obtain the actual distribution of ϵ along the blade to fit the desired dimensions and force coefficients, and then to check for the necessity of correction due to this change. These statements are included in the outline of computational procedure in section 3.3f.

3.3e <u>To Calculate the Magnitude of the Blade Correction Factor K.</u>

Figure 3-5 gives the factor K as a function of the advance angle $(\phi + \alpha_i)$ with the blade stations as parameter. α_i is the induced angle corresponding to the induced axial and rotational velocities. Immediately it is seen that our calculations must include successive approximations. That is $\sin\phi$ is first calculated upon the basis of the forward velocity V and the rotational velocity Ωr. The proper values of K at each blade station are then selected and the computation of optimum circulation is carried out. This will then give a value of the axially and rotational induced velocities from which $(\phi + \alpha_i)$ may be determined. Having these, the corrected values of K are determined and the calculations repeated. It is sufficient to take the second computation as the final result since for lightly loaded propellers the convergence in the approximation procedure is very rapid. These steps are included in the outline of the computational procedure in section 3.3f.

3.3f Summary of Steps in the Calculation. (For determination of Optimum Circulation for Lightly Loaded Operation)

1. Determine the Mach number range along the blade corresponding to the desired operating conditon.
2. Assume a sweepback curve such that all blade stations are reduced to operation at a desired effective Mach number (say 0.8).
3. Assume a reasonable distribution of ϵ along the blade, keeping in mind the high Mach number operation. The range of $0.01 \leq \epsilon \leq 0.05$ should normally be sufficient.
4. Calculate K at each of the blade stations.
5. Determine C_P and J corresponding to the desired operating condition.
6. Calculate the coefficients in the cubic equation 3-(9) for representative values of A. (Two rather widely spaced will suffice).
7. Calculate the optimum μ from equation 3-(9) for each of the values of A.
8. The summation in the equation for C_P, equation 3-(10) may next be evaluated graphically and the proper A determined.
9. The distribution of μ corresponding to this A is then calculated and hence γ is determined.
10. Calculate $$C_L c = \frac{4\pi V^2}{B\Omega}\frac{\gamma}{W_e}$$
11. From $C_L c$ the proper values of ϵ along the blade can be chosen.

12. α_i is calculated using the relation,

$$u = \frac{V}{2}(\gamma + Z)$$

13. Proper values of K are chosen.

14. The solution is completed for this new value of K.

Note: The solution should finally be checked for the effect of variation of ϵ (step 11) and to see that the proper value of C_p has been maintained.

IV THE HYPOTHETICAL EXAMPLE

The following operating conditions were assumed:

$V = 550$ mph	$B = 4$
$h = 25,000$ ft.	RPM = 1260
$D = 13$ ft.	$P = 2800$ HP

Then

$$C_p = 0.422 \quad \text{and} \quad J = 2.96$$

1. Actual Mach number variation. The range of Mach number along the blade for the above operating condition is shown in Figure 1-4. Values of M at the desired blade stations are given in Table 4-1, row (2).

2. Calculation of sweepback angles. The effective Mach number is reduced to 0.8 at each station. (rows 3 and 4)

3. Assumed variation of ϵ along the blade. These figures (row 5) are based upon a thickness variation from 12% at $\xi = 0.3$ to 4% at $\xi = 0.95$. It is assumed that 16 series symmetrical airfoils are used throughout the length of the blade. See reference #6.

4. Calculation of the tip factor K (1st approximation). It is first assumed that the advance angle ϕ is determined uniquely by the forward velocity of flight and the rotational speed of the propeller. Values of K are taken from Figure 3-5. (see row 9)

5. Determination of the coefficients in the cubic. Several values of A are assumed and the coefficients evaluated for these values of A. Values of δ and $\cos\beta$ as determined

above are used. (see rows 10, 11, and 12)

6. <u>μ (the loading factor)</u> is now determined for those values of the coefficients as obtained in step 5. The corresponding γ is also obtained. (see rows 13, 14, and 15) Values of γ versus ξ (blade station) are plotted in Figure 4-1.

7. <u>Calculation of C_P.</u> The finite sum in equation 3-(10) is evaluated mechanically for each of the three values of A. Values of the integrand at each blade station and for each of the three values of A are given in row 16. The integrand is plotted versus blade station in Figure 4-2. $\frac{C_P D}{J^4}$ is plotted versus A, Figure 4-3.

8. <u>Determination of the proper A.</u> Our power condition gives a power coefficient of $C_P = 0.422$. The corresponding value of $\frac{C_P D}{J^4}$ is 0.0715. From the plot of $\frac{C_P D}{J^4}$ versus A it is found that we need an A = 4.1

9. <u>Calculation of γ for $C_P = 0.422$ (A = 4.1)</u> With A = 4.1 the coefficients in the cubic are calculated (row 17). γ is determined by solving for μ from the cubic. (row 18). Note: A check on C_P showed that the proper value has been attained.

10. <u>The blade element loading ($C_L c$)</u> is calculated from

$$C_L c = \frac{4\pi V^2}{B\Omega} \frac{\gamma}{W_e} = 19.14\,\gamma$$

Values are in row 19 and are plotted in Figure 4-4.

11. Check on tip correction factor K. The induced velocity is computed from

$$u = \frac{V}{2}(\gamma + 2)$$

or

$$u - V = V\frac{\gamma}{2} = 403.7\,\gamma$$

Its value, in feet per second, is given in row (20). The new $\sin\phi$ and K follow in rows (21) and (22). Note how little K has changed which is a direct result of the light loading of the propeller. (To the order of accuracy of which these calculations have been made it is not necessary to correct the solution for the change in K)

Table 4-1

Numerical Example

1.	Station	0.3	0.45	0.60	0.70	0.80	0.90	0.95
2.	M (operating)	.830	.878	.942	.993	1.045	1.100	1.132
3.	$\cos\beta$	.963	.910	.850	.806	.765	.727	.706
4.	β	15°22'	24°30'	31°47'	36°18'	40°6'	43°22'	45°15'
5.	ϵ	.05	.04	.03	.02	.01	.01	.01
6.	Ωr (ft/sec)	257	386	515	600	686	772	815
7.	$\tan\phi = V/\Omega r$	3.14	2.09	1.568	1.346	1.178	1.045	0.990
8.	$\sin\phi$	.953	.902	.843	.803	.762	.723	.704
9.	K	1.172	.917	.736	.625	.507	.358	.254
10.	Coeff., A = 1:							
	B	-14.21	-26.32	-47.16	-82.98	-191.4	-217.6	-230.8
	C	58.81	200.7	654.5	2041	10838	14301	16162
	D	-46.81	-185.2	-657.5	-2142	-11695	-15173	-16832
11.	Coeff., A = 3:							
	B	-14.16	-26.30	-47.21	-83.12	-191.7	-218.2	-231.4
	C	58.42	199.7	652.6	2038	10837	14311	16178
	D	-45.47	-176.8	-620.0	-2010	-10975	-14426	-16147
12.	Coeff., A = 5:							
	B	-14.14	-26.28	-47.23	-83.16	-191.8	-218.3	-231.6
	C	58.35	199.4	652.3	2038	10836	14311	16181
	D	-45.21	-175.1	-612.4	-1984	-10830	-14275	-16087
13.	Opt. Values, A=1:							
	μ	1.037	1.066	1.088	1.098	1.100	1.079	1.057
	γ	.074	.132	.176	.196	.200	.158	.114
14.	Opt. Values, A=3:							
	μ	1.006	1.016	1.024	1.029	1.0315	1.024	1.013
	γ	.012	.032	.048	.058	.063	.048	.026
15.	Opt. Values, A=5:							
	μ	1.0008	1.005	1.011	1.015	1.0178	1.013	1.009
	γ	.0016	.0090	.022	.030	.0356	.026	.018
16.	C_P integrand:							
	A = 1	.0236	.0613	.1040	.1292	.1434	.1184	.0861
	A = 3	.0037	.0142	.0267	.0358	.0424	.0342	.0189
	A = 5	.0005	.0039	.0121	.0183	.0235	.0184	.0130
17.	Coeff., A = 4.1:							
	B	-14.16	-26.29	-47.22	-83.17	-191.7	-218.2	-231.5
	C	58.37	199.6	652.3	2038	10836	14310	16180
	D	-45.29	-175.7	-614.9	-1993	-10877	-14324	-16128

Table 4-1 (continued)

1.	Station	0.3	0.45	0.60	0.70	0.80	0.90	0.95
18.	Opt. Values, A=4.1: μ γ	1.002 .004	1.009 .018	1.016 .032	1.020 .040	1.022 .044	1.017 .034	1.011 .022
19.	$C_L c$	.0765	.344	.612	.765	.842	.650	.421
20.	(u - V)	1.62	7.27	12.92	16.2	17.8	13.7	8.9
21.	$\sin(\phi + \alpha_i)$	.953	.904	.847	.808	.769	.729	.708
22.	K	1.172	.917	.735	.622	.504	.356	.253

V. CONCLUSIONS

In this paper we have developed a means for calculating the pitch distribution of a propeller of sweptback planform so as to give peak efficiency of operation. It would be advantageous to be able to calculate the characteristics of such a propeller by a strip analysis and thus to indicate that our purpose has been achieved. However, such a calculation would necessarily be based upon the same simplifying assumptions under which the pitch distribution was computed. Accordingly the most that could be expected from such a check would be the substantiation of the algebra. What is needed are the actual test data from a propeller which has been designed aerodynamically by this method and, of course, such data are not available. Substantiation of calculations such as the one put forward in this paper will have to await the appearance of test data for such configurations and the more complete understanding of the basic aerodynamic characteristics of swept-back airfoils.

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APPENDIX I. APPLICATION OF THE CALCULUS OF VARIATIONS TO PITCH DISTRIBUTION CALCULATIONS

(1) Description and Statement of the Purpose of the Calculus of Variations.

The following description may be found in many standard mathematical texts. What follows immediately here will be general in nature and will not apply directly to this particular problem. However, the calculus of variations lends itself so well to the calculation of propeller operating characteristics that it is worthwhile to include a brief, general description, in this paper.

Suppose we desire to find the form of a function, f, such that if y= f(x), then

$$\int_{X_0}^{X_1} \phi(X,y,y')\,dX$$

shall be a maximum or a minimum where

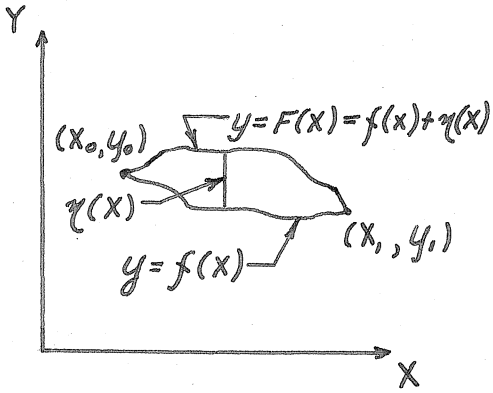
$$\phi(X,y,y')$$

is any desired function of these variables. $y' = \frac{dy}{dX}$, and x_0 and x_1 are any desired values of x. Thus we are looking for a definite integral

$$I = \int_{X_0}^{X_1} \phi(X,y,y')\,dX$$

which corresponds to the form of a curve giving an I which is greater than or less than any other curve which we may draw between the end points x_0 and x_1. Now let y = f(x) be the desired function, or curve, and y = F(x) be any other continuous curve joining the same end points. Then if,

$$\eta(x) = F(x) - f(x)$$



we will have

$$y = f(x) + \eta(x)$$

representing the same curve as

$$y = F(x)$$

Now consider the expression

$$y = f(x) + \alpha\eta(x)$$

where α is a parameter independent of x. This function then represents an entire family of curves in which

a) when $\alpha = 0$, $y = f(x)$

b) when $\alpha = 1$, $y = f(x) + \eta(x) = F(x)$

Then if we take α small enough, the function $\alpha\eta(x)$ for this α and for all smaller α's, and for all values of x between x_0 and x_1, will be less then any previously chosen ξ. For these values of α the curve

$$y = f(x) + \alpha\eta(x)$$

is said to be in the neighborhood of $y = f(x)$. Then, if $y = f(x)$ and $y = F(x)$ are given, and if by definition

$$I = \int_{x_0}^{x_1} \phi(x, y, y')\,dx$$

then I for any one of the α family of curves is given by,

$$I(\alpha) = \int_{x_0}^{x_1} \phi\left[x, y + \alpha\eta(x), y' + \alpha\eta'(x)\right] dx$$

where I (α) is a function of α alone. Then if $y = f(x)$ is the curve of the family for which I is to be a minimum it is readily apparent that I (α) must be a minimum when $\alpha = 0$. A necessary condition for this is that

$$\left[\frac{d}{d\alpha} I(\alpha)\right]_{\alpha=0} = 0$$

This condition is expressed by the equation

$$I'(0) = 0$$

(2) Notation

a) Variation of y (δy):

For any x between x_0 and x_1

$$\eta(x) = F(x) - f(x)$$

Thus $\eta(x)$ is an increment in y produced by changing f(x) into F(x) without change in x. $\eta(x)$ is the variation in y (represented by δy). δy is, in general, an arbitrary function of x.

b) Variation of y' ($\delta y'$):

The slope at any point of the curve $y = \eta(x)$ is the difference between the slopes of the curves $y = F(x)$ and f(x) for the corresponding x. Thus

$$\eta'(x) = \frac{d}{dx}(\delta y)$$

is the change produced in y' by changing f(x) into F(x).

c) Variation of y'' ($\delta y''$):

By extension of (b) we have that

$$\eta''(x) = \frac{d}{dx}(\delta y')$$

is the change produced in y'' by changing f(x) into F(x).

Now let us denote a given function of y as ϕ (y) and increase y by δ y. Then

$$\frac{d}{dy}\left[\phi(y)\right]\delta y$$

will be an approximation of the increment in ϕ(y), the order of the approximation depending upon the magnitude of δ y. If δ y is an infinitesimal then the approximation is in error by an infinitesimal of order higher then δ y. This approximate increment is called the variation of ϕ(y). Thus

$$\text{Variation of } \phi(y) = \delta\phi(y) = \frac{d}{dy}\left[\phi(y)\right]\delta y.$$

Also, if y and y' are treated as independent variables, the variation in a function ϕ(y, y') due to an increment δy' of y' and δ y of y will be

$$\text{Variation of } \delta\phi(y, y') = \frac{\partial\phi(y,y')}{\partial y}\delta y + \frac{\partial\phi(y,y')}{\partial y'}\delta y'$$

Similarly,

$$\text{Variation of } \delta\phi(y', y'') = \frac{\partial\phi(y',y'')}{\partial y'}\delta y' + \frac{\partial\phi(y',y'')}{\partial y''}\delta y''$$

All of the above variations are supposed to be caused by varying the f in y = f(x) without changing x. So,

$$\delta\phi(x,y,y') = \frac{\partial\phi(x,y,y')}{\partial y}\delta y + \frac{\partial\phi(x,y,y')}{\partial y'}\delta y'$$

Note that $d\phi$ is an approximate increment in ϕ caused by changing x without changing ϕ while $\delta\phi$ is an approximate increment in ϕ caused by changing ϕ without changing x. Hence, with this new notation, our condition that $I(\alpha)$ be a minimum when $\alpha =$ 0 is that

$$\int_{x_0}^{x_1} \delta\phi(x,y,y') = 0$$

And let it be defined that the variation of a definite integral is the integral of the variation of the integrand, i.e.,

$$\delta\int_{x_0}^{x_1} \phi\, dx = \int_{x_0}^{x} \delta\phi\, dx$$

So we may write our minimum condition as

$$\delta I = 0$$

(3) Optimum Pitch Distribution by Calculus of Variations

The acknowledgment here is to references #8 and #9. In applying the calculus of variations the British writers make use of the power loss and torque grading coefficients which are seldom used in this country. Therefore, rather then to outline the reference here we will merely consider the general ideas involved.

It is assumed that all factors are fixed except the blade setting θ ; that is, that determination of the optimum will involve investigating only a single variable. This restriction is not necessary and this entire procedure may be generalized to any desired number of variables. In general

θ will vary along the radius r in a fashion which it is the purpose of this analysis to determine. Then we let

$$\theta = f(r) + \alpha\eta(r)$$

where f(r) is a function to be determined, η(r) is an arbitrary function, and α is a parameter which will allow the determination of f(r) so that certain integrals involving θ are a minimum. These certain integrals may define the torque input and power losses of the propeller. By the reasoning of part (1) of this discussion it is seen that the integrals will be functions of the parameter α alone and when α is zero that the desired functions will be determined. To make α zero is to write the variation of the integrals equal to zero. One such minimum integral will exist for the torque input and one for the power loss. In order for both integrals to be a minimum the ratio of the two integrands will have to be related by a constant.

That is the general principle of this method and to check its utility we will apply it to the simpler cases of section III. Using the work output and power loss as the quantities of interest we apply the above principles and state that we want

$$\int \frac{\partial(dE/dr)}{\partial r}\,\delta r\,dr = 0$$

for all forms of δr which make

$$\int \frac{\partial(v\,dT/dr)}{\partial r}\,\delta r\,dr = 0$$

The only way that this can be true is for

$$\frac{\partial(dE/dr)}{\partial r} = A\,\frac{\partial(V\,dT/dr)}{\partial r}$$

Applying this criterion to a simple case, page 26, gives

$$\frac{\partial(V\,dT/dr)}{\partial r} = \frac{2\pi\rho V^4}{\Omega}\,x\,\frac{\partial\gamma}{\partial r}$$

and

$$\frac{\partial(dE/dr)}{\partial r} = \frac{2\pi\rho V^4}{\Omega}\,\frac{\partial\gamma}{\partial r}\left[\gamma\,\frac{1+x^2}{x} + \epsilon x^2\right]$$

so that the condition of optimum circulation satisfies the relation

$$\gamma\,\frac{1+x^2}{x} + \epsilon x^2 = Ax$$

The other cases treated in this paper could be checked in a similar fashion. This discussion makes it apparent that nothing has been added to our analysis method by utilizing the calculus of variations. However, it puts the entire procedure on a more formal basis and allows a great deal of algebraic simplification when we treat the more exact equations. It is also apparent that the great many similar problems which exist in propeller aerodynamics may very nicely be treated by the calculus of variations.

Table 3-3

Goldstein Factor (K) for 2, 3, and 4 Bladed Propellers

sin $(\phi+\alpha_i)$		.05	.10	.20	.30	.40	.60	.80	.90	1.00
x = .3	2	1.000	1.000	0.994	0.978	0.958	0.930	0.916	0.935	1.012
	3	1.000	1.000	0.997	0.992	0.984	0.973	1.010	1.061	1.185
	4	1.000	1.000	0.998	0.996	0.991	0.986	1.035	1.103	1.256
x = .45	2	1.000	0.998	0.991	0.959	0.906	0.784	0.677	0.657	0.633
	3	1.000	0.999	0.995	0.987	0.966	0.909	0.849	0.831	0.832
	4	1.000	1.000	0.997	0.994	0.984	0.954	0.924	0.917	0.923
x = .60	2	1.000	0.997	0.961	0.874	0.783	0.608	0.502	0.464	0.425
	3	1.000	0.999	0.998	0.955	0.902	0.771	0.661	0.622	0.588
	4	1.000	1.000	0.995	0.984	0.950	0.855	0.756	0.713	0.681
x = .70	2	0.999	0.988	0.901	0.774	0.663	0.492	0.398	0.360	0.325
	3	1.000	0.998	0.964	0.892	0.809	0.650	0.533	0.494	0.457
	4	1.000	1.000	0.989	0.945	0.883	0.745	0.627	0.581	0.543
x = .75	2	0.997	0.971	0.852	0.709	0.595	0.435	0.347	0.311	0.279
	3	0.999	0.994	0.935	0.843	0.746	0.582	0.470	0.429	0.393
	4	1.000	0.999	0.973	0.909	0.803	0.678	0.562	0.516	0.477
x = .80	2	0.994	0.937	0.784	0.634	0.520	0.376	0.297	0.265	0.238
	3	0.998	0.980	0.884	0.774	0.670	0.508	0.404	0.367	0.335
	4	0.999	0.993	0.940	0.852	0.759	0.601	0.490	0.447	0.412
x = .85	2	0.985	0.877	0.694	0.548	0.442	0.316	0.247	0.220	0.197
	3	0.990	0.948	0.810	0.684	0.581	0.429	0.337	0.304	0.278
	4	0.998	0.985	0.882	0.774	0.671	0.517	0.413	0.375	0.345
x = .90	2	0.950	0.773	0.578	0.444	0.351	0.249	0.193	0.172	0.154
	3	0.973	0.872	0.693	0.566	0.471	0.341	0.265	0.239	0.218
	4	0.995	0.943	0.777	0.651	0.554	0.414	0.329	0.298	0.272
x = .95	2	0.780	0.586	0.415	0.308	0.243	0.171	0.131	0.117	0.105
	3	0.863	0.692	0.512	0.406	0.331	0.236	0.182	0.164	0.149
	4	0.945	0.770	0.590	0.476	0.396	0.290	0.228	0.205	0.187

FIGURE 1-4

OPERATING MACH NUMBER

versus

BLADE STATION

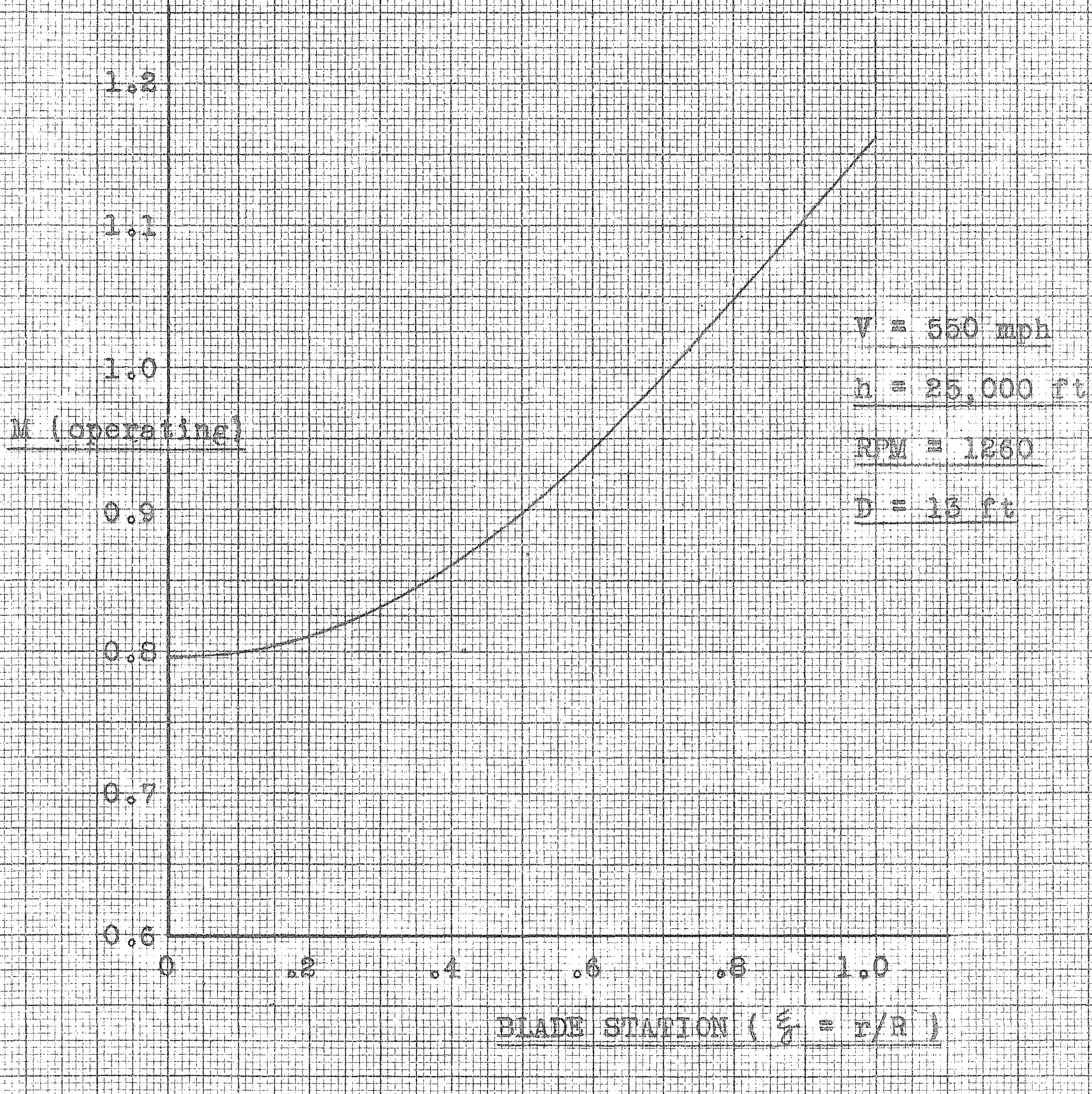


FIGURE 3-3

OPTIMUM DISTRIBUTION

OF CIRCULATION

$A = 0.5,\ \epsilon = 0.05,\ \beta = 60^{\circ}$

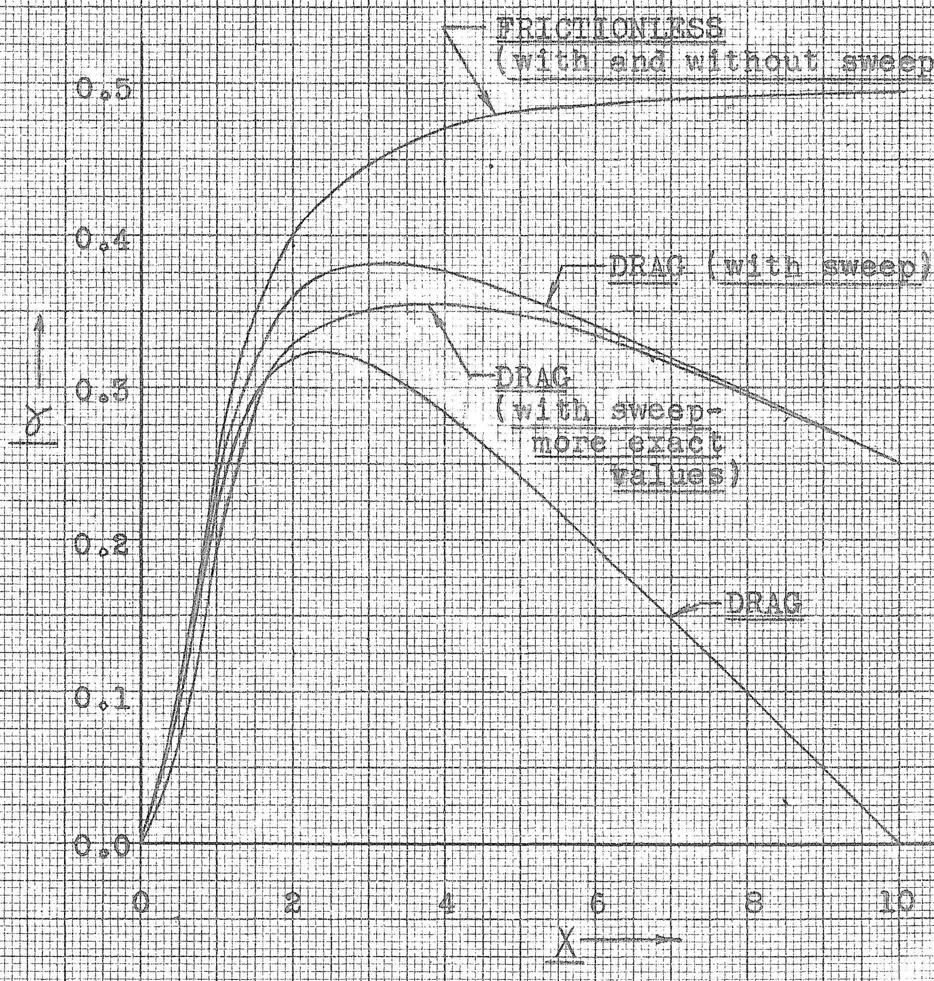


FIGURE 3-5

GOLDSTEIN FACTOR (K)

FOUR BLADED PROPELLER

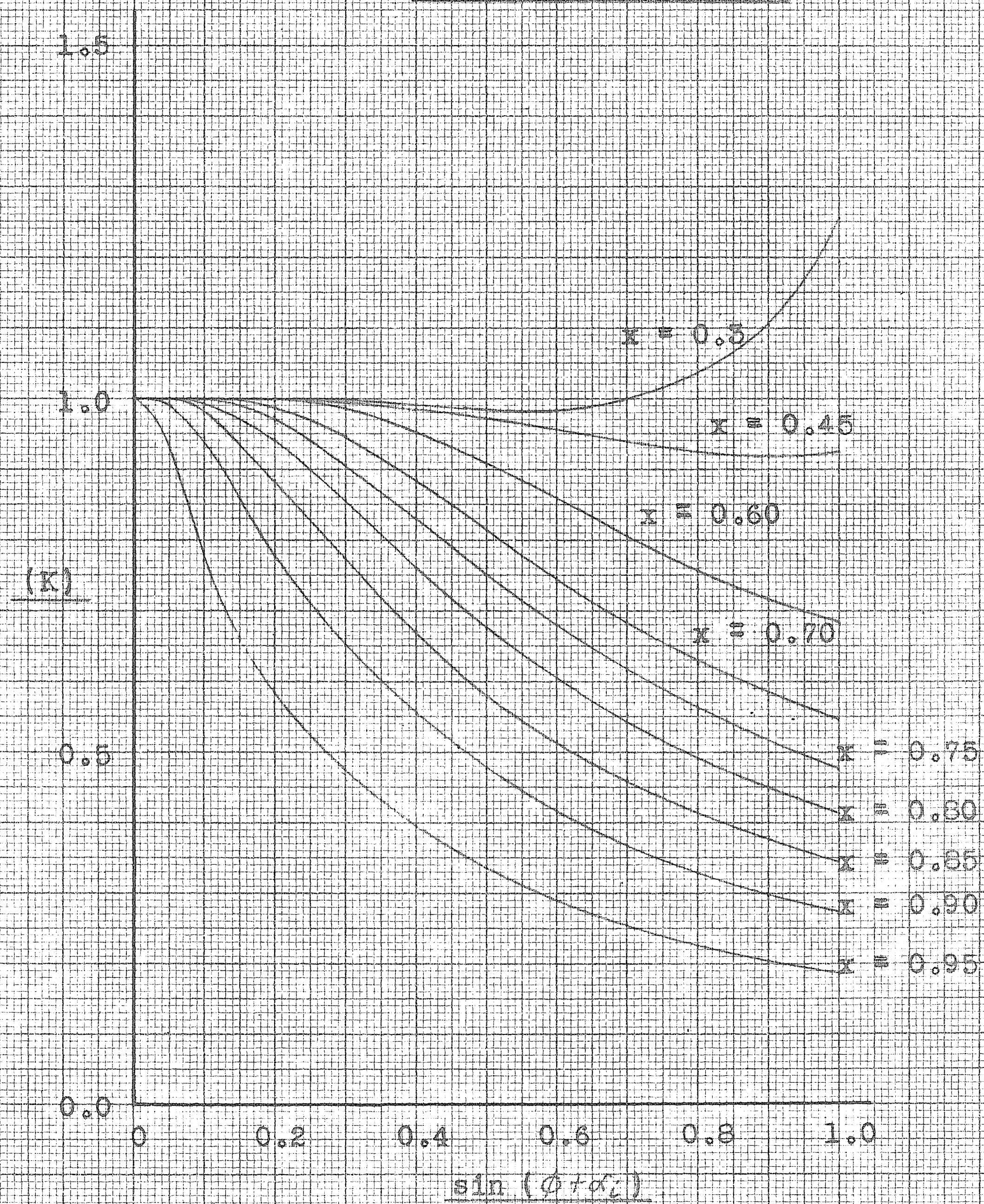


FIGURE 3-6

GOLDSTEIN FACTOR (K)

TWO BLADED PROPELLER

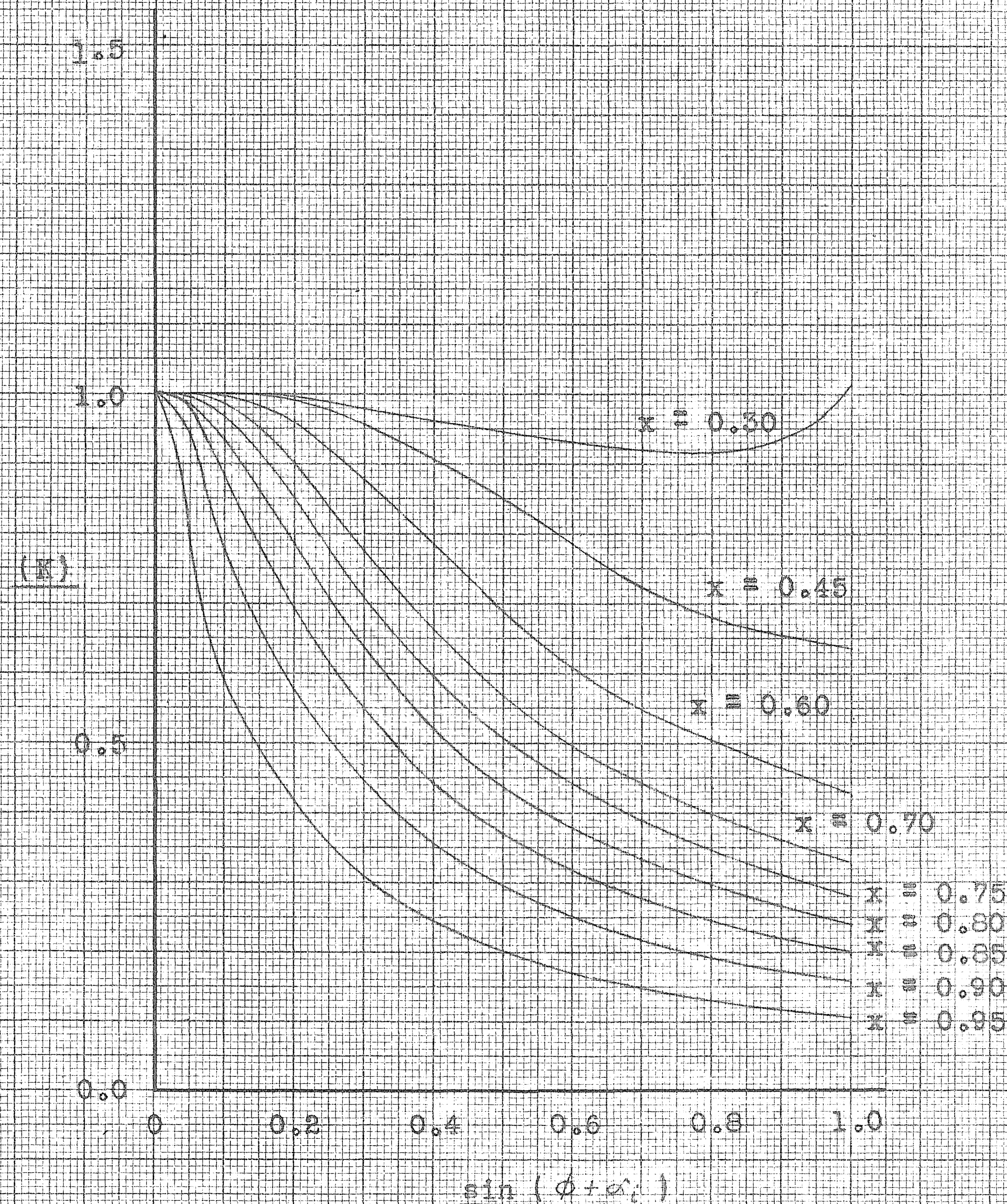
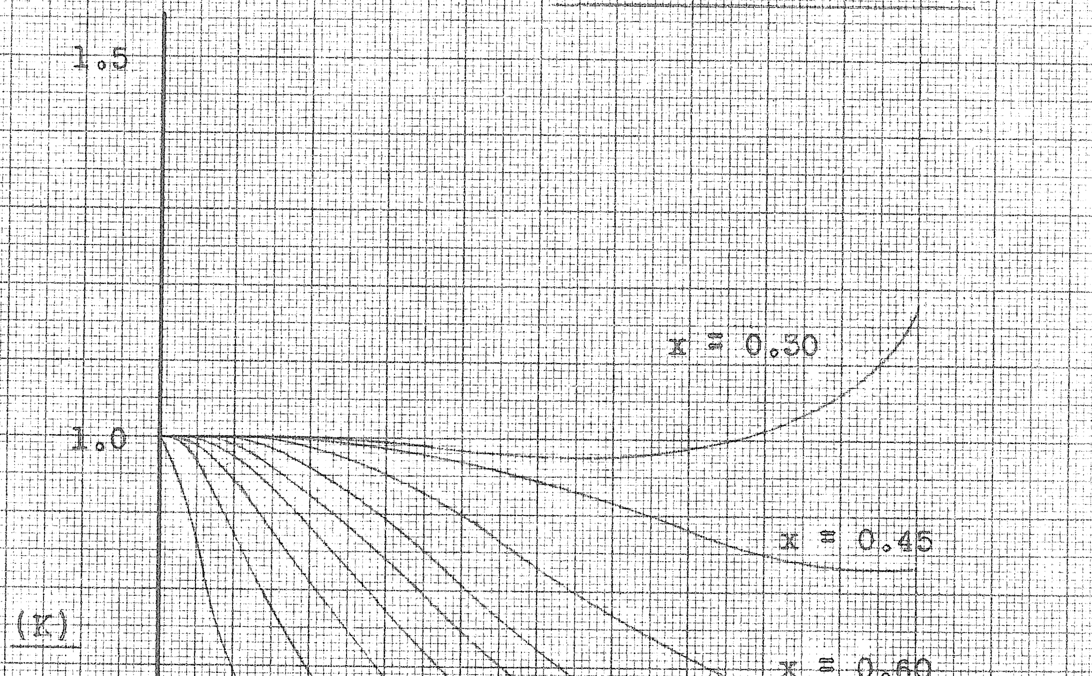


FIGURE 3-7

GOLDSTEIN FACTOR (K)

THREE BLADED PROPELLER



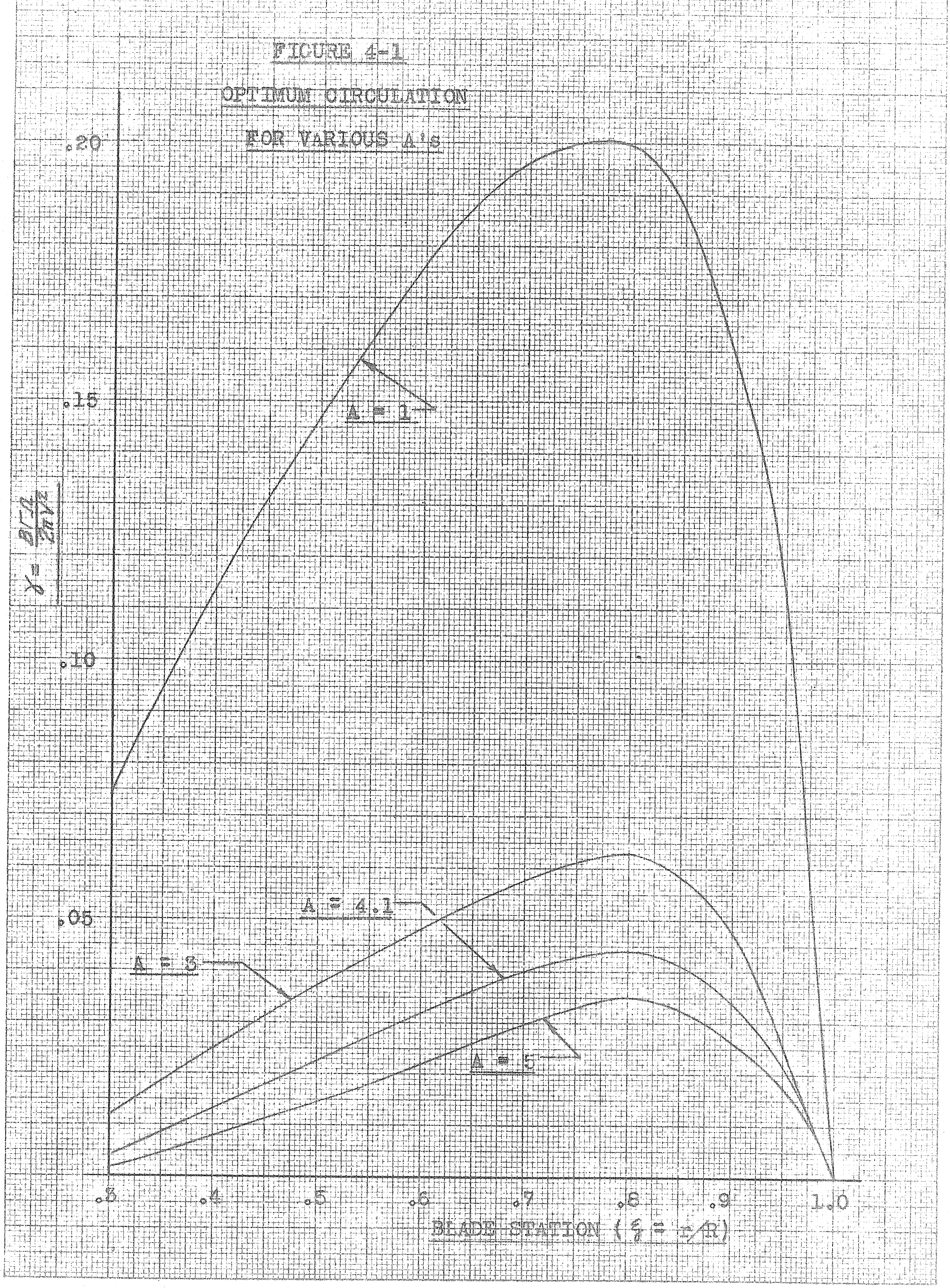


FIGURE 4-1
OPTIMUM CIRCULATION
FOR VARIOUS A's
.20
.15
.10
.05
$\gamma = \frac{B\Gamma\Omega}{2\pi V^2}$
A = 1
A = 4.1
A = 3
A = 5
.3
.4
.5
.6
.7
.8
.9
1.0
BLADE STATION (ξ = r/R)

FIGURE 4-2

POWER COEFFICIENT INTEGRAND

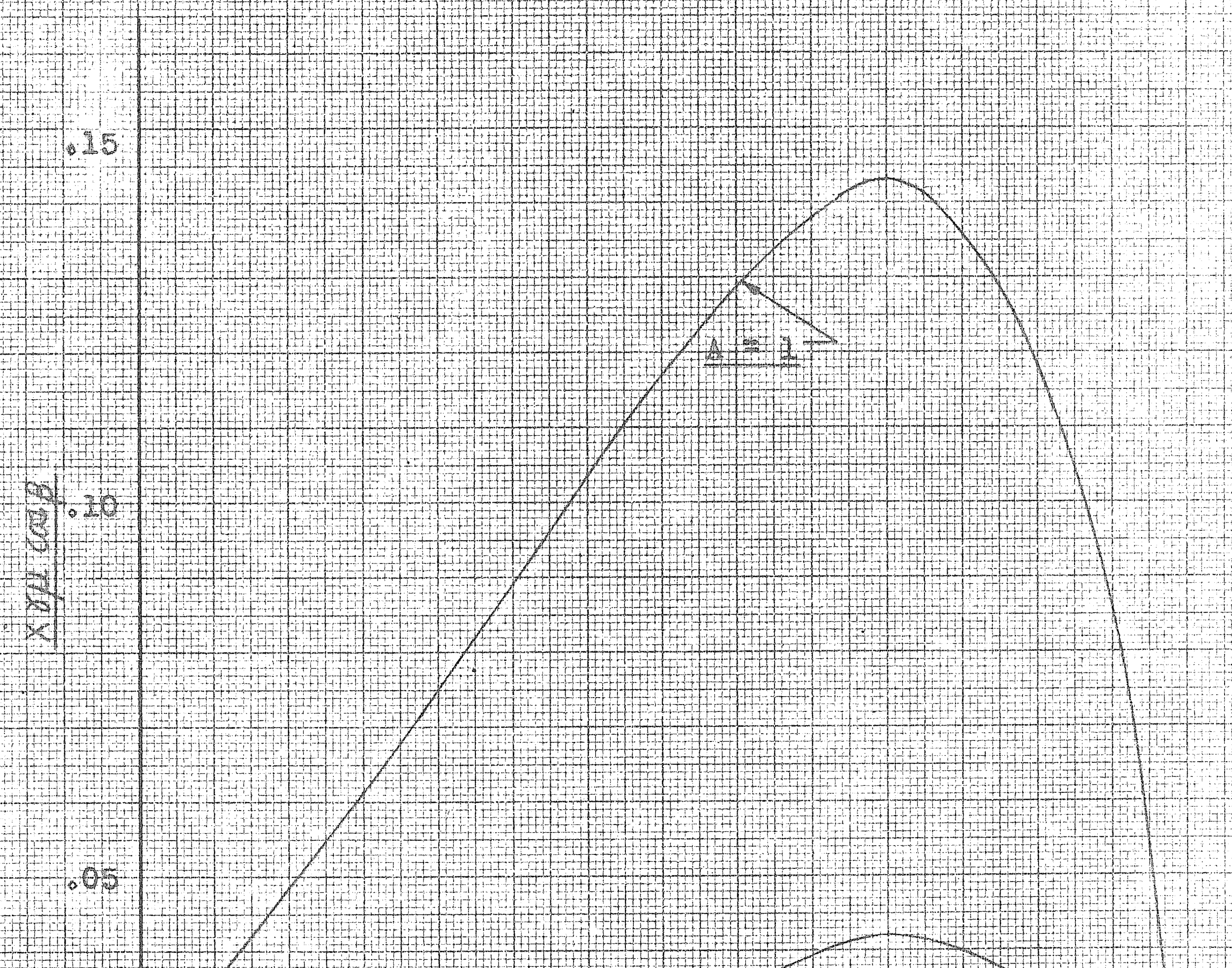
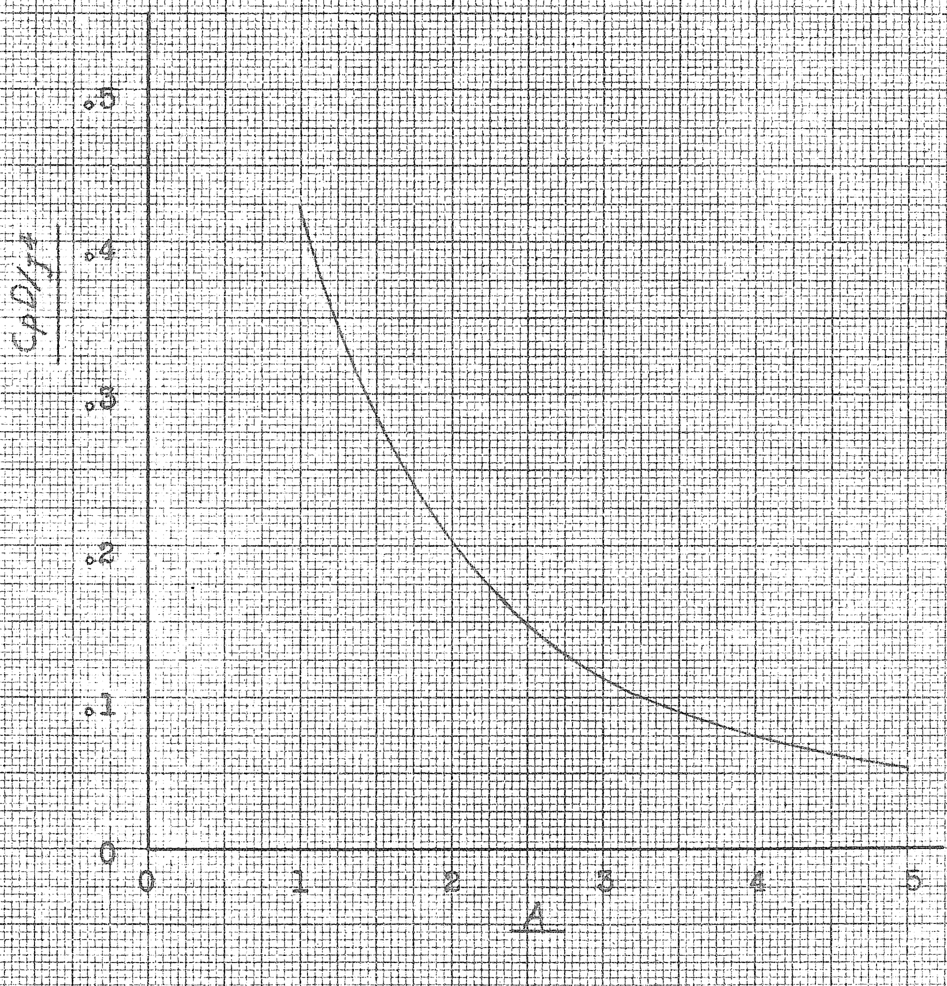


FIGURE 4-3

VARIATION OF C_p WITH A



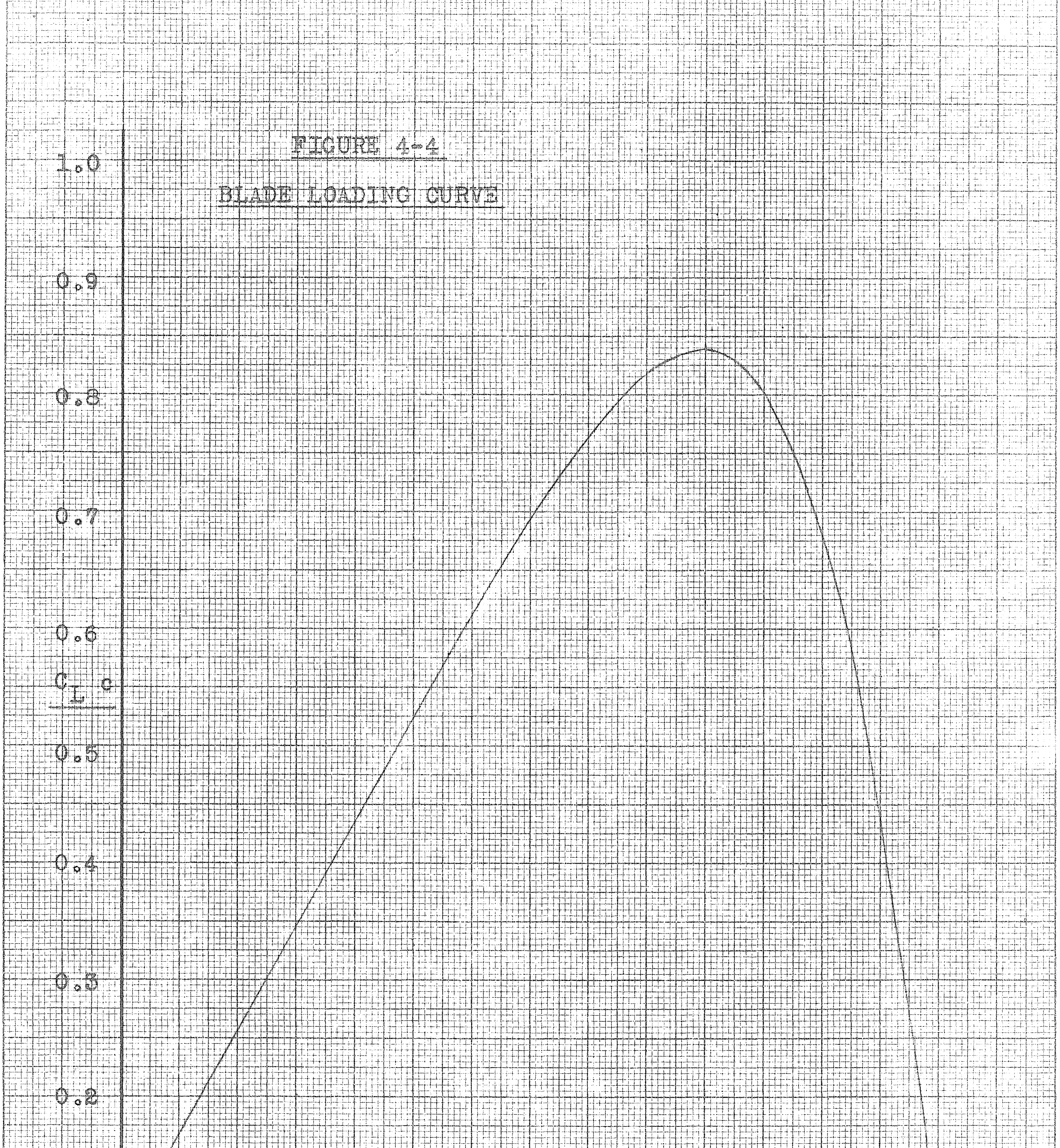


FIGURE 4-4
BLADE LOADING CURVE
1.0
0.9
0.8
0.7
0.6
$C_L c$
0.5
0.4
0.3
0.2
0.1
0.0
.3
.4
.5
.6
.7
.8
.9
1.0
BLADE STATION (ξ = r/R)