I PLASMA OSCILLATIONS AND RADIO NOISE FROM THE DISTURBED SUN

II A FIELD ANALYSIS OF THE M TYPE BACKWARD WAVE OSCILLATOR

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I PLASMA OSCILLATIONS AND RADIO NOISE FROM THE DISTURBED SUN

Many investigators have suggested that plasma oscillations in the solar corona may be the source of large bursts of radio noise in the meter wavelength region. Two aspects of this problem are considered in this report: (a) the excitation of plasma oscillations by directed beams of charged particles, and (b) the conversion of energy in the longitudinal plasma oscillations to transverse electromagnetic waves by means of random inhomogeneities in electron density.

It appears unlikely that charged particles whose velocity is much less than the r.m.s. thermal velocity of the coronal electrons will excite plasma oscillations. Charged particles whose velocity is much greater than the r.m.s. thermal velocity excite oscillations in a band of frequencies, including frequencies above the local plasma frequency. However, qualitative arguments indicate that the noise should be concentrated in a narrow band of frequencies slightly below the local plasma frequency. Thus it is impossible to explain the Type II (slow) bursts in the manner assumed and unlikely that the Type III (fast) bursts are explainable in this manner. The transfer of energy is studied in detail and it is shown that only waves whose phase velocity is less than the directed beam of charged particles receive energy from the beam.

It is shown that plasma oscillations radiate a small fraction of their energy if the electron density is not uniform. In particular, random fluctuations in density, of the amount expected in thermal equilibrium, cause about 10^{-5} of the plasma-oscillation energy to be radiated; the remainder is dissipated by short-range collisions. Larger fluctuations than this are likely, and hence more energy should be radiated.

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I. INTRODUCTION

Advances in high frequency radio receiving techniques during and following the war have made a valuable new tool available to the astronomers. Within the last decade radio observations have been made of the sun, the moon, planets in our solar system, and other sources of electromagnetic radiation within and without our galaxy . Long before these receiving techniques were available it was suggested that the sun should be a radiator of radio frequency electromagnetic energy just as it is a radiator of energy in other parts of the spectrum. Only within the last decade, however, has it been possible to study this radiation. Observations are limited to frequencies above a few tens of megacycles per second (10-20 meters) by the characteristics of the earth's ionosphere and to frequencies below several tens of thousands of megacycles, (.5 to 1 cm) by molecular absorbtion bands in the earth's atmosphere and by difficulties in receiver technique.

Solar radiation in the meter wavelength range is exceedingly complex. At a particular frequency the radiation consists of a steady or slowly varying component, on which are superimposed sudden increases of intensity, or bursts. These bursts may be many orders of magnitude greater than the steady component. The behavior at different frequencies is frequently very different. This complicated behavior has led to attempts to separate the radiation into distinct components, according to the duration, polarization, and spectrum of the radiation, and according to associated optical features and the apparent location of the source of the radiation. An excellent summary of components of radiation is to be found in "The Sun" (1) edited by G. P. Kuiper, and in a series of papers by J. P. Wild in the Australian Journal of Scientific Research (2).

Only the thermal component of solar radiation produced by the free-free transitions of electrons of the solar atmosphere can be said to be understood at the present time. The intensity of thermal radiation can be calculated from a knowledge of the physical state of solar atmosphere and the propagation characteristics of electromagnetic energy in the solar atmosphere (1),(3). Other components of solar radiation are generally associated with disturbances in the solar atmosphere and, because of the variety of characteristics which the bursts show, it seems likely that different components of the disturbed radiation are to be explained by different mechanisms.

To study the theoretical aspects of radio radiation from the disturbed sun requires a firm optimism. Observations are generally made at a single frequency; angular resolution of even the interferometer-type receivers is poor, and solar disturbances with which the radiation seems to be associated are not generally understood. It is not surprising to find many explanations proposed with little evidence to decide between them. One of the most interesting proposals, which has stimulated considerable theoretical investigation, is that plasma oscillations in the solar corona may be responsible for certain types of "excess" radiation. The basic problem is to account for the large intensities of this radiation and its relatively narrow spectrum. The attractive feature of the plasma oscillation theory is that the coherent motion of large numbers of electrons ought to radiate energy very freely and it should be easy to account for the large intensities. Unfortunately, however, it turns out that there is no radiation from the coherent motion of electrons in oscillation of a uniform plasma because the displacement current of the field and the convection current of the moving electrons exactly cancel, with no net current left to excite the electromagnetic field. Nevertheless, the plasma theory remains attractive because it is believed that only a very small fraction of the energy which might reside in the oscillations needs to be radiated in order to explain the observed intensities. The plasma oscillation problem naturally divides into two parts, (a) the excitation of the oscillations to large amplitudes, and (b) the radiation of a fraction of the energy in the form of electromagnetic waves. Both of these aspects of the problem are discussed in this paper.

The plasma oscillation theory probably originated with the discovery by Haeff (4) and others of the principle of the double stream amplifying mechanism. Haeff (5) suggested that a group of charged particles moving outward through the solar corona would interact with the coronal electrons in such a way as to amplify statistical fluctuations associated with either group of charged particles. It appears as though Haeff originally intended the theory to explain the fact that the continuous radiation in the meter wavelength region is much greater than expected from a 5000° K black body radiator. This is now known to be explainable by the $10^{6\circ}$ K temperature of the corona, from which meter wavelength thermal radiation originates. Haeff's original suggestion failed to include other pertinent characteristics of the solar atmosphere.

The general theory of plasma oscillations has since been developed considerably by Bohm (6) and collaborators. Feinstein and Sen (7) have extended Haeff's theory of the excitation mechanism. The principal extension of Feinstein and Sen was the inclusion of the effect of the thermal velocities of the coronal electrons in an approximate manner. They studied the growth of plasma disturbances caused by a group of monoenergetic charged particles passing through the corona and found that it was possible for the disturbances to grow larger when the monoenergetic beam was either slower or faster than the r.m.s. thermal velocity of the electrons. Bohm and Pines (6) find however, that when a single charged particle traverses a plasma it excites the plasma oscillation only if the electron travels faster than the r.m.s. thermal velocity of the plasma electrons. This discrepancy is examined in detail in Section IV and it shown that when the velocity of the monoenergetic beam of particles is less than the mean thermal velocity of coronal electrons, the growing waves found by Feinstein and Sen are really "evanescent" or "cutoff" waves, and do not give rise to amplification of small disturbances. Evanescent plasma waves, which occur in a bimodal distribution of electron velocities such as may exist in a shock front which moves through an ionized gas, have been misinterpreted as giving rise to growth of small disturbances (8). The conditions under which growth of small disturbances is possible is reexamined using the linear theory.

In Section V the nonlinear aspects of the problem are discussed qualitatively and an estimate is made of the amplitude which the plasma oscillations may attain. In Section VI the coupling of longitudinal plasma oscillations and transverse electromagnetic waves by inhomogeneities in electron density is evaluated and the conditions for escape of transverse waves from the solar atmosphere are discussed.

Characteristics of Type II and Type III Bursts. The plasma oscillation theory is a likely candidate for explaining the Type II (slow) and Type III (fast) bursts of radiation reported by Wild et al (9),(10), (11). Type II bursts last for 5 to 10 minutes and have a spectrum-time curve as shown in Figure 1a. Type III bursts last for 5 to 15 seconds and have a spectrum-time curve as shown in Figure 1b. At a particular instant of time, emission occurs over two relatively narrow bands which differ in frequency by about a factor of two. The striking similarity in the spectra in the two frequency bands suggest that both are due to the same source, a nonlinear oscillation in which both fundamental and second harmonic amplitudes are appreciable. Other characteristics of Wild's spectra indicate that part of the radiation in the low frequency band is attenuated because the radiation originates near the critical level for escape of radiation. Using a model of the undisturbed corona, the radial velocities of the disturbances can be determined.

Only four of the Type II (slow) bursts have been observed, but several hundred of the Type III (fast) bursts have been observed. Of the Type III bursts observed, second harmonic radiation was distinctly identifiable in twenty events and probable in about half of the events. Third harmonic radiation is less than one-tenth the intensity of fundamental and second harmonic radiation and has not been detected. Single frequency measurements (12),(13), indicate that bursts may be associated with solar flares. Radio fadeouts caused by intense ultraviolet radiation of the sun are nearly coincident with the onset of flares. Magnetic storms caused by corpuscular streams frequently follow the flare one day later (14). If the bursts of radiation are assumed to originate from a point in the corona where the frequency of the disturbance is equal to the local electron plasma frequency, the radial velocity of the disturbance through the corona can be computed. The velocities

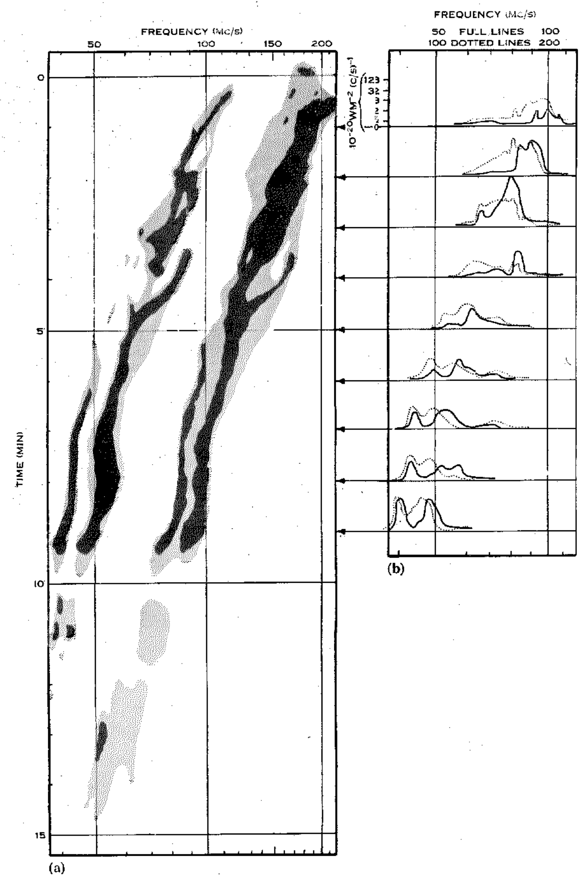

Fig. 1a. Spectrum-Time Curve of Type II Bursts (Reference 11)

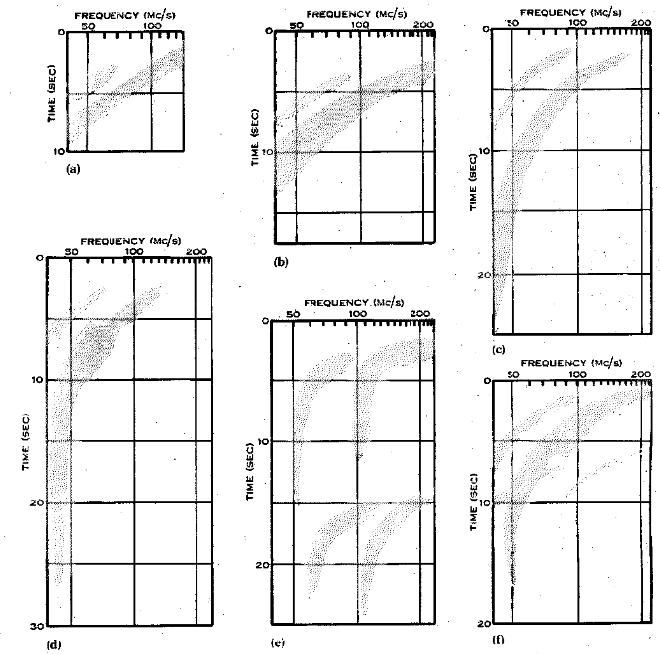

Fig. 1b. Spectrum-Time Curve of Type III Bursts (Reference 11)

obtained, using the electron density of the undisturbed corona, are about 500 km sec^{-1} for the slow bursts and between $.6 \times 10^5$ and 1.0×10^5 km sec^{-1} for the fast bursts. Optical observations give velocities somewhat less than 500 km sec^{-1} for material in an eruptive prominence. The velocities of corpuscular streams which reach the earth are about 1800 km sec^{-1}. Thus there is independent evidence for motion of the same general velocity as inferred from the Type II burst data. However, there is virtually no independent evidence for particles of the velocity implied by Type III burst data.

The observed radiation is randomly polarized. This is taken as an indication that steady magnetic fields do not play an important role in these phenomena. Measurements of the sun's magnetic field by Babcock (15) indicate fields of only a few gauss, except near a sunspot. The magnetic field effects will not be important in plasma oscillations if the square of the cyclotron frequency, $\omega_c = eB/m$ is much less than the square of the plasma frequency $\omega_p = \sqrt{n_o e^2/\epsilon_o m}$. When $B = 1$ gauss, ω_c is 2.8 mc. and the plasma frequency in the corona is typically 100 mc, hence the neglect of the magnetic field is valid. Several analyses (16),(17),(18),(19), have been made of plasma oscillations in a static magnetic field, in an attempt to explain the enhanced radiation (1) associated with sunspots. It cannot be said, however, that these theories explain the enhanced radiation.

Model for Type II and Type III Bursts. To be more specific about the process which might give rise to these bursts, assume that a volume V of quasi-neutral ionized gas of unknown density moves outward through the corona with a velocity u as shown in Figure 2. The

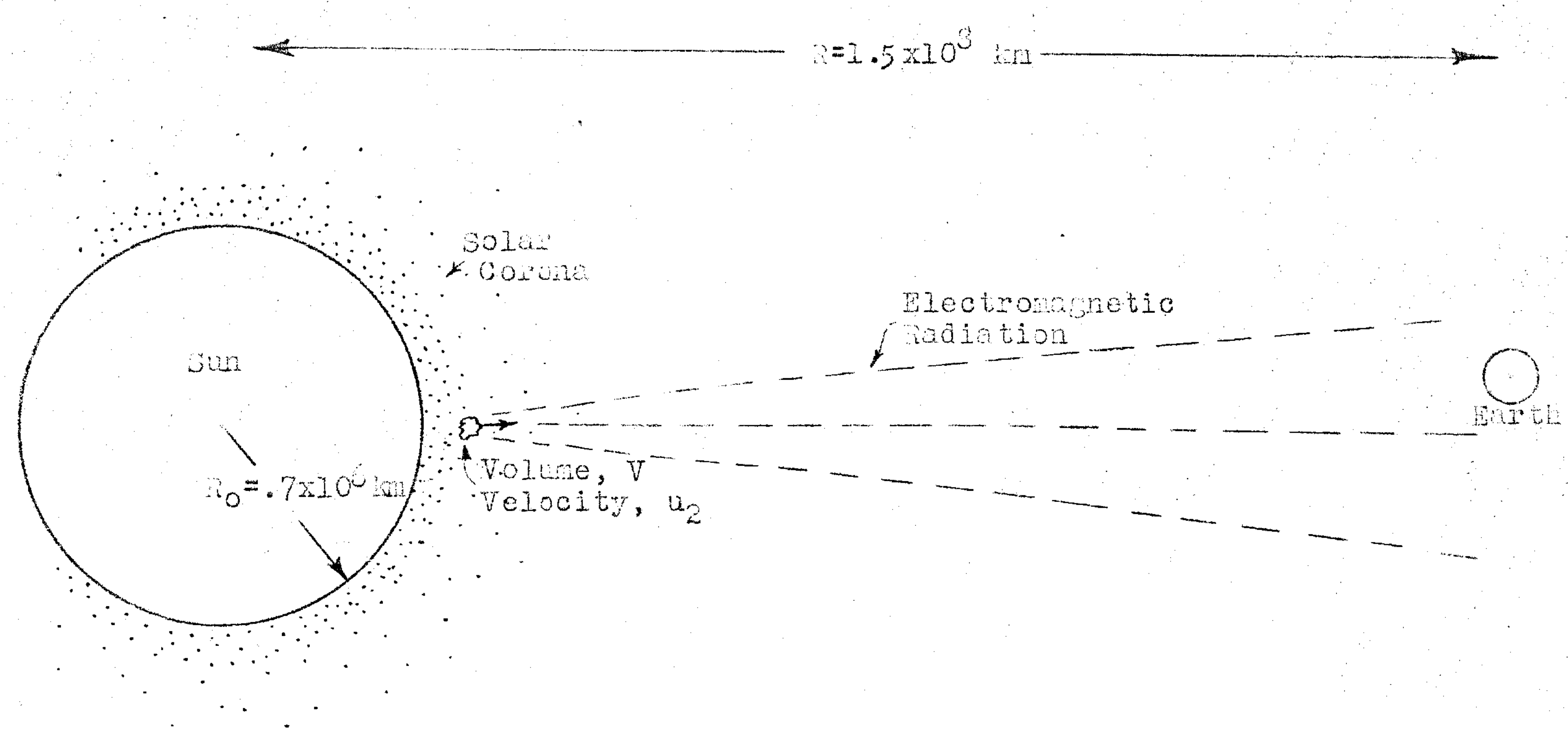

Figure 2. Model for Type II and Type III Bursts. A Quasi-neutral Group of Charged Particles Moves Outward through the Solar Corona with Velocity u_2.

velocity u is 500-1800 km sec^{-1} for Type II bursts and .6 to 1×10^5 km sec^{-1} for Type III bursts. Under certain conditions the outward moving gas cloud may excite the plasma oscillations of the coronal electrons. It will be the object of subsequent analysis to determine these conditions, the frequencies which are involved, the amplitude of oscillation which is attained, and the amount of electromagnetic energy which is radiated. The mechanism by which such a group of particles may attain this drift velocity will not be discussed.

II THEORY OF WAVES IN A PLASMA

For the purposes of this paper the term plasma will be used to denote a completely ionized gas which, in the absence of disturbances, is electrically neutral. Ionization and recombination will be neglected. The solar corona is a dilute plasma composed primarily of hydrogen ions, helium ions, and electrons in the ratio 5:1:7 . Due to the extremely high temperature of the corona only about one atom in 10^7 is not ionized. Short range collisions, i.e., collisions which deflect electrons through an appreciable angle, occur very infrequently. A measure of the average number of short range electron-ion collisions per second is obtained from the Chapman-Enskog method of solution of the Boltzmann equation (20),(3)

$$\nu_c = \frac{4}{3} \frac{n_o e^4}{m \varkappa T} \sqrt{\frac{\pi m}{2 \varkappa T}} \, \ln \left[1 + \left(\frac{4 d \varkappa T}{e^2} \right)^2 \right] \qquad \text{II.1}$$

where n_o is the number of electrons per cm^3 , e and m are the electronic charge and mass, respectively, $\varkappa$ is Boltzmann's constant, T is the temperature of the gas, and d is the maximum value of the impact parameter. d is usually taken to be the mean interparticle distance, $n_o^{-1/3}$, since binary collision theory does not apply for more distant encounters. Ion motion can be neglected, because of the large mass of the ions.

The effect of long range encounters, involving the simultaneous interaction of many electrons and ions with small momentum transfers may be taken into account by assuming that the electrons also move in the average field produced by many neighboring electrons. This average field is to be computed from the charge density averaged over a volume which is large enough to contain many electrons, but small enough so that macroscopic properties of the plasma do not change significantly in the volume. Collisions of intermediate range are not taken into account by either method, but their effect can be estimated by taking d_{max} to be the Debye wavelength. The value of ν_c obtained in this way differs very little from the value given by II.1 because of the logarithmic dependence on impact parameter. In the present problem, the short range collisions play a minor role and hence the effect of encounters of intermediate range may be neglected completely.

The Elementary Theory of Waves in a Plasma. The elementary theory of plasma waves neglects the thermal velocity of the electrons and short range collisions. The equations for the mean electron velocity, $\underline{v}$, and charge density, ρ , are

$$\frac{d\underline{v}}{dt} = -\frac{e}{m}(\underline{E} + \underline{v} \times \underline{B}) \qquad \text{II.2}$$

$$\frac{\partial \rho}{\partial t} + \nabla(\rho\, \underline{v}) = 0 \ . \qquad \text{II.3}$$

Small perturbations in the mean velocity and charge density from their steady values, $\underline{v} = 0$ and $\rho = \rho_o$, denoted by $\underline{v}_1$ and ρ_1 respectively, obey the linearized equations

$$\frac{\partial \underline{v}_1}{\partial t} = -\frac{e}{m}\underline{E}_1 \qquad \text{II.4}$$

$$\frac{\partial \rho_1}{\partial t} + \rho_o \nabla \cdot \underline{v}_1 = 0 \ . \qquad \text{II.5}$$

It is assumed that the steady magnetic field is zero. Maxwell's equations, with the above charge density ρ_1 and current density

$\underline{J}_1 = \rho_o \underline{v}_1$ as source terms, are

$$\nabla \cdot \underline{B}_1 = 0 \qquad \text{II.6}$$

$$\nabla \cdot \underline{D}_1 = \rho_1 + \rho^*_1 \qquad \text{II.7}$$

$$\nabla \times \underline{E}_1 = -\frac{\partial B_1}{\partial t} \qquad \text{II.8}$$

$$\nabla \times \underline{H} = \underline{J}_1 + \frac{\partial \underline{D}_1}{\partial t} + \underline{J}^*_1 \qquad \text{II.9}$$

$$\underline{D}_1 = \epsilon_o \underline{E}_1 \qquad \underline{B}_1 = \mu_o \underline{H}_1 \qquad \text{II.10}$$

where ρ^*_1 and $\underline{J}^*_1$ are charge and current densities other than those due to the plasma electrons. Because of II.6 and II.8, the magnetic and electric fields can be derived from vector and scalar potentials

$$\underline{B}_1 = \nabla \times \underline{A}_1 \qquad \nabla \cdot \underline{A}_1 = 0 \qquad \text{II.11}$$

$$\underline{E}_1 = -\frac{\partial \underline{A}_1}{\partial t} - \nabla \phi_1 \, . \qquad \text{II.12}$$

Substituting these expressions into II.7 and II.9 yields

$$\nabla^2 \phi = -\frac{1}{\epsilon_o} (\rho_1 + \rho^*_1) \qquad \text{II.13}$$

$$\nabla^2 \underline{A}_1 = -\mu_o \underline{J}_1 + \epsilon_o \mu_o \frac{\partial^2 \underline{A}_1}{\partial t^2} + \mu_o \epsilon_o \frac{\partial}{\partial t} \nabla \phi_1 - \mu_o \underline{J}^*_1 \, . \qquad \text{II.14}$$

Differentiating II.13 twice with respect to time and using II.5 and II.4 for $\frac{\partial \rho_1}{\partial t}$ and $\frac{\partial \underline{v}_1}{\partial t}$

$$\frac{\partial^2}{\partial t^2} \nabla^2 \phi = -\frac{1}{\epsilon_o} \frac{\partial^2 \rho^*_1}{\partial t^2} + \rho_o \frac{e}{m} \nabla \cdot \underline{E}_1$$

which may be rewritten,

$$\left[\frac{\partial^2}{\partial t^2} + \omega_p^2\right]\nabla^2 \phi_1 = -\frac{1}{\epsilon_o}\frac{\partial^2 \rho^*_1}{\partial t^2} \qquad \text{II.15}$$

where $$\omega_p^2 = -\frac{\rho_o}{\epsilon_o}\frac{e}{m} = \frac{n_o e^2}{\epsilon_o m} \quad . \qquad \text{II.16}$$

is the electron plasma frequency.

Differentiating II.14 with respect to time and using II.4 and II.12 yields

$$\frac{\partial}{\partial t}\left[\nabla^2 \underline{A}_1 - \mu_o \epsilon_o \frac{\partial^2 \underline{A}_1}{\partial t^2}\right] = \mu_o \frac{e}{m}\rho_o \left[-\frac{\partial \underline{A}_1}{\partial t} - \nabla\phi_1\right] + \mu_o \epsilon_o \frac{\partial^2}{\partial t^2}\nabla\phi_1 - \mu_o \frac{\partial \underline{J}^*_1}{\partial t}$$

or

$$\frac{\partial}{\partial t}\left[\nabla^2 \underline{A}_1 - \mu_o \epsilon_o \frac{\partial^2 \underline{A}_1}{\partial t^2} - \mu_o \epsilon_o \omega_p^2 \underline{A}_1\right] = \mu_o \epsilon_o \left[\frac{\partial^2}{\partial t^2} + \omega_p^2\right]\nabla\phi - \mu_o \frac{\partial \underline{J}^*_1}{\partial t} \quad . \qquad \text{II.17}$$

When free charge is not present $\rho^*_1 = \underline{J}^*_1 = 0$, and the solutions of II.15 are of two types

$$\nabla^2 \phi_1 = 0 \qquad \text{II.18}$$

$$\left[\frac{\partial^2}{\partial t^2} + \omega_p^2\right]\phi_1 = 0 \quad . \qquad \text{II.19}$$

Solutions of the first type, II.18 , are not required in charge-free regions (21). Solutions of the second type, II.19, describe the electron plasma oscillations. Note that ϕ_1 can be any arbitrary function of position and that II.19 prescribes only its time dependence. This means that any region of the plasma, if given an initial disturbance of this type, will continue to oscillate indefinitely with a frequency equal to the plasma frequency, without spreading outside the area of original disturbance. In these simple harmonic oscillations there is an interchange of the kinetic energy of the electron motion and the potential energy of the electric field. The sum of the convection current, $\underline{J}_1$, and the displacement current, $\frac{\partial \underline{D}_1}{\partial t}$, vanishes at every point, as does the magnetic field $\underline{B}_1$. If the electrons drift with an average velocity, $\underline{u}_o$ then it can be shown that energy of the plasma oscillations is transported with just this velocity.

Since ϕ_1 obeys II.19, the last term on the right side of II.17 is zero and the equation for $\underline{A}_1$ becomes

$$\nabla^2 \underline{A}_1 - \mu_o \epsilon_o \frac{\partial^2 \underline{A}_1}{\partial t^2} - \mu_o \epsilon_o \omega_p^2 \underline{A}_1 = 0 \qquad \text{II.20}$$

$$\frac{\partial \underline{A}_1}{\partial t} = 0 \qquad \text{II.21}$$

II.20 is sometimes called the Proca Equation. When the time dependence is sinusoidal, $\epsilon^{i\omega t}$, II.20 is the vector wave equation for a medium which has an index of refraction, n , given by

$$n = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \quad . \qquad \text{II.22}$$

When the frequency, ω , is less than the plasma frequency, ω_p , electromagnetic waves no longer propagate through the medium, but are totally reflected if incident from outside the plasma. Solutions of II.21 represent static magnetic fields, which are assumed to be absent in this analysis.

Figure 3 shows how the two characteristic frequencies ω_p and ν_c vary with height above the photosphere in the solar corona. Data is taken from references (3) and (22). It may be seen that the collision frequency is always many orders of magnitude less than the plasma frequency. In most aspects of this problem it will be permissible to

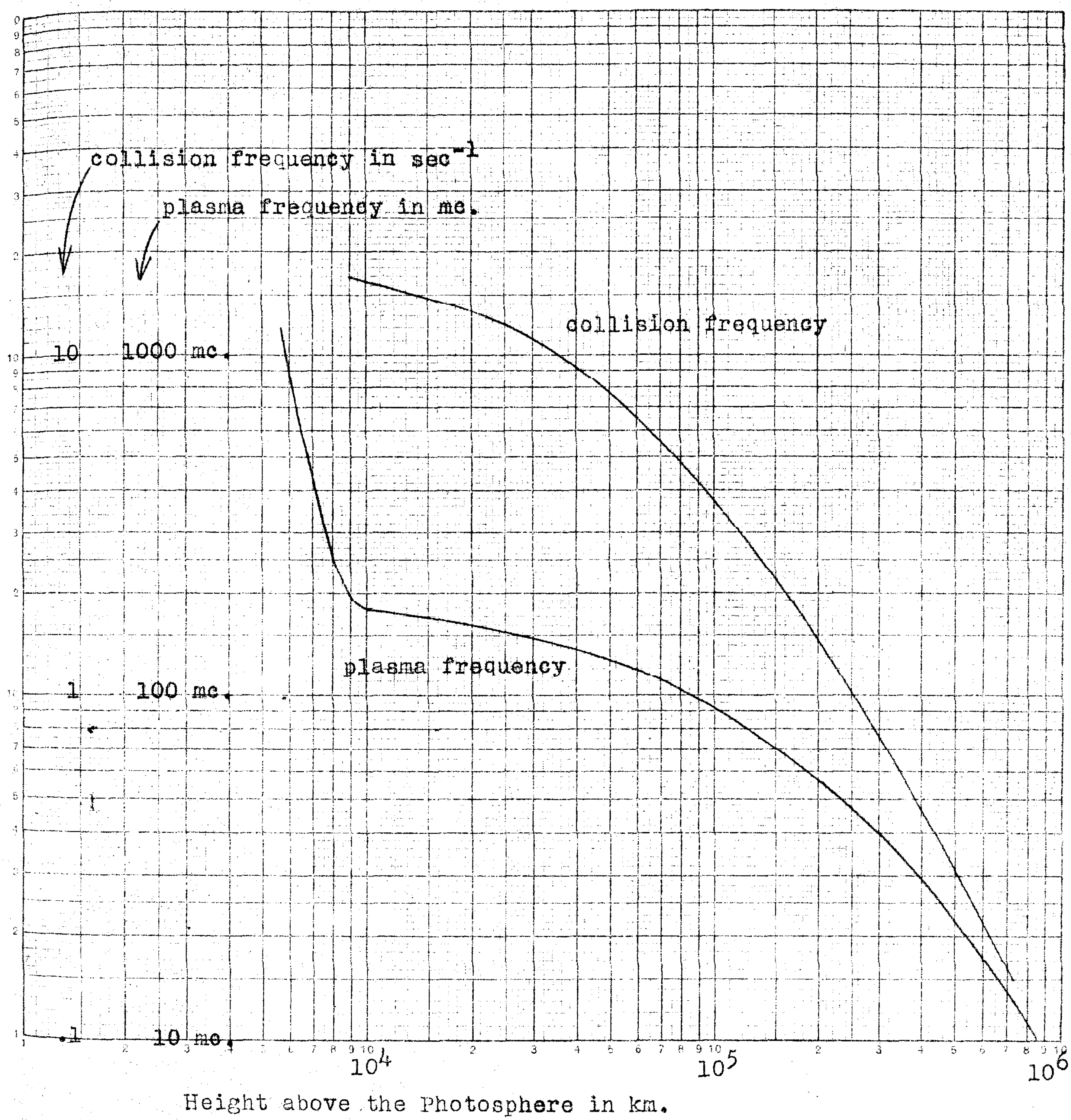

Figure 3. Electron Plasma Frequency and Collision Frequency versus Height in the Solar Corona

neglect short range collisions altogether.

The Effect of Thermal Velocities. When the random velocities of thermal agitation are taken into account, the plasma oscillations described by the scalar potential in the preceding paragraphs no longer remain confined to the region where the disturbance is originated. The thermal velocities allow the disturbance to spread at a maximum velocity which is of the order of the r.m.s. velocity of thermal agitation. The Boltzmann equation can be used to include the effect of thermal velocities of the coronal electrons. Long range interactions are included in the external force term, $(\underline{E} + \underline{u} \times \underline{B}) \cdot \nabla_{\underline{u}} f$, of the Boltzmann equation,

$$\frac{\partial f}{\partial t} + (\underline{u} \cdot \nabla) f - \frac{e}{m} (\underline{E} + \underline{u} \times \underline{B}) \cdot \nabla_u f = \left(\frac{\partial f}{\partial t}\right)_{\text{collisions}} . \qquad \text{II.23}$$

$f(\underline{r},\underline{u},t)$ is the number of electrons per unit volume in six-dimensional phase space in the vicinity of the point $(\underline{r},\underline{u})$ at time t , and $\underline{E}$ and $\underline{B}$ are the electric and magnetic fields averaged over a volume which contains many electrons. If the equilibrium distribution is denoted by f_o and small deviations from the equilibrium distribution are denoted by f_1, the linearized equation obeyed by f_1 is

$$\frac{\partial f_1}{\partial t} + (\underline{u} \cdot \nabla) f_1 - \frac{e}{m} (\underline{E}_1 + \underline{u} \times \underline{B}_1) \cdot \nabla_u f_o = \left(\frac{\partial f_1}{\partial t}\right)_{\text{collisions}} \qquad \text{II.24}$$

where $\underline{E}_1$ and $\underline{B}_1$ are the perturbations in the average electric and magnetic fields. The steady magnetic field is assumed to be zero throughout this paper. When the electron velocities are small compared with the velocity of light, the magnetic force $\underline{u} \times \underline{B}_1$ is small compared with the electric force $\underline{E}_1$. Although in some phases of the analysis which follows, a few electrons will have velocities which are a small fraction of the velocity of light, this magnetic force will nevertheless be neglected. A completely relativistic treatment would change the quantitative aspects of the results only slightly and the qualitative aspects not at all. In some problems, however, the qualitative nature of the results are changed by this approximation (19),(23).

The effect of short-range collisions may be taken into account approximately by writing

$$\left[\frac{\partial f_1}{\partial t}\right]_{\text{collisions}} = -\nu_c f_1 \quad . \qquad \text{II.25}$$

In the absence of other forces, this tends to return the distribution to the equilibrium distribution in a time $1/\nu_c$.

The electric and magnetic fields $\underline{E}_1$ and $\underline{B}_1$ satisfy the Maxwell equations II.7 through II.10 with ρ_1 and $\underline{J}_1$ given by

$$\rho_1 = -e \int f_1(\underline{u})\, d\underline{u} \qquad \text{II.26}$$

$$\underline{J}_1 = -e \int \underline{u}\, f_1(\underline{u})\, d\underline{u} \quad . \qquad \text{II.27}$$

These equations, together with the conditions which f_1, $\underline{E}_1$, $\underline{D}_1$, $\underline{B}_1$, $\underline{H}_1$ must satisfy at the boundaries, completely determine the fields and the electron distribution function in an arbitrary volume.

Some investigators have studied the propagation characteristics of plane waves with sinusoidal space dependence $(e^{-i\underline{k} \cdot \underline{r}})$. The former procedure is of value if a sinusoidal external driving force is present, while the latter is suited to the study of disturbances in the unbounded medium. Either procedure is legitimate since the linearity of the equations allows solutions obtained in this manner to be added in such a way as to satisfy the boundary conditions, or initial conditions of the problem. Landau (24) and Twiss (25) have demonstrated the usefulness of the Laplace transform in meeting boundary conditions and initial conditions.

If sinusoidal time dependence is assumed, waves whose amplitude increases without bound in some spatial direction are found to exist. The boundary conditions play a very important role in determining whether waves which increase or decrease in some spacial direction are excited or not, particularly in plasma problems where electrons travel in all directions. In most wave motion problems such waves are generally excluded, if the region of interest includes infinity, by the requirement that the fields be bounded everywhere or at least that the total energy is finite. Waves which become large at large distances are generally associated with sources at infinity and these are generally not of interest. In linearized plasma problems some of the waves which increase without bound are not associated with sources at infinity, but represent, instead, waves which derive their energy from the kinetic energy of the electrons. Of course, waves of this type cannot really increase without bound either, because only a finite amount of kinetic energy is available from the electrons. This ultimate limit on energy is absent in the linearized theory, however, and waves which increase without bound and which are excited by sources of finite energy arise in the linear theory. At some point non-linear effects prevent further growth of the wave amplitude.

The linear theory is sufficient to establish whether disturbances which are small enough to be described accurately by the linear theory, such as random fluctuations, can grow to very large amplitudes. For this reason it is essential to establish which of the increasing waves can be excited and represent a genuine tendency for small perturbations to be amplified. An analysis of the problem without linearization would be very difficult. Some of the qualitative aspects of non-linear operation are discussed in Section VI. It will be assumed in the following sections that the volume of the disturbed region is so large that the theory of waves in an unbounded medium is applicable. While it is true that some modification is expected near the boundaries, the fraction of the total volume which is occupied by the boundary regions is certain to be small compared with the total volume which is involved in the disturbance.

Plane Waves in a Plasma. One of the simplest problems which can be solved using this formalism is to determine the characteristics of plane waves in a plasma. Assuming that the time and spacial dependence is given by the factor $e^{i(\omega t - \underline{k} \cdot \underline{r})}$ the current in a plasma due to a field $\underline{E}_1$ is found from II.27, II.24, and II.25 to be

$$\underline{J}_1 = -\frac{e^2}{m}\int \frac{\underline{u}\,(\nabla_u f_o \cdot \underline{E}_1)}{i(\omega - \underline{k}\cdot\underline{u}) + \nu_c}\, d\,\underline{u} \quad . \qquad \text{II.28}$$

After integration by parts this expression can be written

$$\underline{J}_1 = i\omega\epsilon_o\left[\chi_t\,\underline{E}_1 + \underline{\underline{\chi}}_1 \cdot \underline{E}_1\right] \qquad \text{II.29}$$

where

$$\omega \chi_t = -\frac{e^2}{m\epsilon_o}\int \frac{f_o(\underline{u})\, d\underline{u}}{(\omega - \underline{k}\cdot\underline{u}) - i\nu_c} \qquad \text{II.30}$$

$$\omega \underline{\underline{\chi}}_1 = -\frac{e^2}{m\epsilon_o}\int \frac{\underline{u}\,\underline{k}\, f_o(\underline{u})\, d\underline{u}}{\left[(\omega - \underline{k}\cdot\underline{u}) - i\nu_c\right]^2} \quad . \qquad \text{II.31}$$

χ_t and $\underline{\underline{\chi}}_1$ are the electric susceptibilities of the plasma electrons; χ_o is a scalar and $\underline{\underline{\chi}}_1$ is a diadic. χ_t is the susceptibility for transverse waves, since for these waves, $\underline{k} \cdot \underline{E}_1 = 0$ and the second term of II.29 does not contribute to the current. When f_o is spherically symmetric in velocity space χ_1 is simply a constant times the diadic $(\underline{k}\,\underline{k})$,

$$\underline{\underline{\chi}}_1 = \frac{(\underline{k}\,\underline{k})}{k^2}\chi_1 \quad . \qquad \text{II.32}$$

For plane waves, the Maxwell equations II.6 through II.10 become (with $\rho^*_1 = \underline{J}^*_1 = 0$)

$$\underline{k} \cdot \underline{B}_1 = 0 \qquad \text{II.33}$$

$$\underline{k} \cdot \underline{D}_1 = \rho_1 \qquad \text{II.34}$$

$$\underline{k} \times \underline{E}_1 = \omega \underline{B}_1 \qquad \text{II.35}$$

$$\underline{k} \times \underline{H}_1 = -\omega\, \underline{D}_1 + i\, \underline{J}_1 \qquad \text{II.36}$$

$$\underline{D}_1 = \epsilon_o\, \underline{E}_1 \qquad \underline{B}_1 = \mu_o\, \underline{H}_1 \qquad \text{II.37}$$

Forming the vector triple product $\underline{k} \times (\underline{k} \times \underline{E}_1)$ and using II.36 and II.37

$$\underline{k} \times (\underline{k} \times \underline{E}) = \omega\mu_o(\underline{k} \times \underline{H}_1) = -\omega^2 \mu_o\epsilon_o\, \underline{E}_1 + i\omega\mu_o\, \underline{J}_1 \quad .$$

Using the vector identity, $\underline{k} \times (\underline{k} \times \underline{E}_1) = -k^2 \underline{E}_1 + \underline{k}(\underline{k} \cdot \underline{E}_1)$, and substituting II.29 for $\underline{J}_1$,

$$k^2 \underline{E}_1 + \underline{k}(\underline{k} \cdot \underline{E}_1) = -\omega^2 \mu_o \epsilon_o \underline{E}_1 - \omega^2 \mu_o \epsilon_o \chi_t \underline{E}_1 + \underline{\underline{\chi}}_1 \cdot \underline{E}_1 . \qquad \text{II.38}$$

When the velocity distribution is spherically symmetric in velocity space, the expression II.32 for $\underline{\underline{\chi}}_1$ can be used so that II.38 may be written

$$\left[k^2 + \omega^2 \mu_o \epsilon_o (1 + \chi_t) \right] \underline{E}_1 - \underline{k}(\underline{k} \cdot \underline{E}_1) + \omega^2 \mu_o \epsilon_o \chi_1 \frac{\underline{k}(\underline{k} \cdot \underline{E}_1)}{k^2} = 0 .$$

II.39

Any electric field $\underline{E}_1$ and wave vector $\underline{k}$ which satisfy this relation give permissible solutions to the above differential equations. Without loss of generality, it may be assumed that $\underline{k}$ is in the x direction. In component form II.39 becomes

$$k_x^2 + \omega^2 \mu_o \epsilon_o (1 + \chi_t) \quad E_{1x} - k_x^2 E_{1x} + \omega^2 \mu_o \epsilon_o \chi_1 E_{1x} = 0 \qquad \text{II.40a}$$

$$k_x^2 + \omega^2 \mu_o \epsilon_o (1 + \chi_t) \quad E_{1y} = 0 \qquad \text{II.40b}$$

$$k_x^2 + \omega^2 \mu_o \epsilon_o (1 + \chi_t) \quad E_{1z} = 0 \quad . \qquad \text{II.40c}$$

If E_{1y} or E_{1z} are not zero, II.40b and II.40 require that

$$k_x^2 + \omega^2 \mu_o \epsilon_o (1 + \chi_t) = 0 \quad . \qquad \text{II.41}$$

Using II.41 in II.40a shows that $E_{1x} = 0$ in this case. II.41 is called the transverse wave dispersion relation since $\underline{E}_1$ and $\underline{k}$ are perpendicular for this case. χ_t is the transverse wave susceptibility, There are two orientations for the electric vector, corresponding to the two possible polarizations.

When E_{1y} and E_{1z} are zero, and E_{1x} is not zero, II.40a becomes

$$\omega^2 \mu_o \epsilon_o (1 + \chi_t + \chi_1) = 0 \quad . \qquad \text{II.42}$$

This is called the longitudinal wave dispersion relation since $\underline{E}_1$ and $\underline{k}$ are parallel for this case. $\chi_t + \chi_1 \equiv \chi_\ell$ is the longitudinal wave susceptibility. Since $\omega^2 \mu_o \epsilon_o \neq 0$ the longitudinal wave behavior is obtained by setting χ_ℓ equal to -1 .

For a spherically symmetric velocity distribution

$$\chi_\ell = - \frac{i\,\omega + \nu_c}{i\,\omega} \; \frac{e^2}{m\,\epsilon_o} \int \frac{f_o\,(\underline{u})\,d\,\underline{u}}{\left[\omega - \underline{k} \cdot \underline{u} - i\,\nu_c\right]^2} \qquad \text{II.43}$$

To summarize, the transverse wave relationships are

$$\underline{k} \cdot \underline{B}_{1t} = 0$$

$$\underline{k} \cdot \underline{E}_{1t} = 0 \quad (\rho_1 = 0)$$

$$\underline{k} \times \underline{E}_{1t} = \omega\, \underline{B}_{1t}$$

$$\underline{k} \times \underline{H}_{1t} = \omega \epsilon_o (1 + \chi_t)\, \underline{E}_{1t}$$

and the longitudinal wave relationships are

$$\underline{k} \times \underline{E}_{1\ell} = 0 \qquad (\underline{B}_{1\ell} = 0)$$

$$(1 + \chi_\ell) = 0 \quad .$$

These relations are valid only when the velocity distribution is spherically symmetric. When this assumption is not valid, such as when a directed beam is present, the waves may no longer be purely longitudinal or purely transverse.

The diadic susceptibility of a single directed beam of velocity $\underline{u}$ is found from II.28 to be

$$\chi_t = \frac{1}{\omega}\frac{\omega_p^2}{(\omega - \underline{k}\cdot\underline{u})} \qquad \underline{\underline{\chi}}_1 = \frac{\omega_p^2}{(\omega - \underline{k}\cdot\underline{u})^2}(\underline{k}\,\underline{u}) \ .$$

Collisions have been neglected, and $\underline{e}\,\underline{e}$ is a unit diadic.

When collisions are neglected the expressions for the longitudinal and transverse susceptibilities of a spherically symmetric velocity distribution may be approximated by

$$\chi_\ell = -\frac{\omega_p^2}{\omega^2}\left[1 + \frac{k^2}{\omega^2}\ \overline{u^2} + \ldots\right] + i\,\chi_{\ell_i} \qquad \text{II.44}$$

$$\chi_t = -\frac{\omega_p^2}{\omega^2}\left[1 + \frac{1}{3}\frac{k^2}{\omega^2}\ \overline{u^2} + \ldots\right] + i\,\chi_{t_i} \qquad \text{II.45}$$

where $\overline{u^2} = \dfrac{\int u^2\, f_o(\underline{u})\, d\underline{u}}{\int f_o(\underline{u})\, d\underline{u}}$, and the properties of a spherically symmetric velocity distribution, $\bar{u} = 0$, $\overline{u^2_x} = \overline{u^2_y} = \overline{u^2_z} = \frac{1}{3}\ \overline{u^2}$, have been used. For the Maxwell-Boltzmann distribution

$$f_o = \left(\frac{m}{2\pi\kappa T}\right)^{3/2} e^{-\frac{mu^2}{2\kappa T}} \ , \quad \text{where} \quad \overline{u^2} = \frac{3\kappa T}{m} \ . \qquad \text{II.46}$$

The long wavelength approximation $\left(k^2 \ll \frac{\omega^2}{\overline{u^2}}\right)$ used by Sen and others consists of retaining only the first two terms in the expansion of the real part of χ and neglecting the imaginary part altogether. The imaginary parts of these expressions arise from integration around the contour being obtained by making ν_c small but not zero. This rule is also a result of the transform method of solution (24),(25). The imaginary part of the susceptibility signifies a damping originally noted by Landau (24) and is closely connected with particles which travel with nearly the wave velocity. Its exact significance is discussed later, in connection with a study of energy transfer in such waves. Since $\frac{\omega^2}{k^2} \gtrsim c^2$ for transverse waves, all terms but the first in the series II.40 are negligible so that II.40 becomes

$$\chi_t = -\frac{\omega_p^2}{\omega^2} .$$

Reinserting the effect of collisions changes this to

$$\chi_t = -\frac{\omega_p^2}{\omega(\omega - i\nu_c)} . \qquad \text{II.47}$$

For the Maxwell-Boltzmann distribution function, II.41, the longitudinal wave electric susceptibility can be shown to be

$$-\frac{\omega_p^2}{k^2}\frac{m}{\varkappa T}\left\{1 + \frac{\pi}{2}\, i\, \frac{\omega}{k}\sqrt{\frac{m}{\varkappa T}}\; e^{-\frac{1}{2}\left(\frac{\omega}{k}\right)^2 \frac{m}{\varkappa T}}\left[\operatorname{Erf}\left(i\,\frac{\omega}{k}\sqrt{\frac{m}{2\varkappa T}}\right) - 1\right]\right\} \qquad \text{II.48}$$

where $\operatorname{Erf} z$ is the error function of complex argument.

III ENERGY RELATIONS FOR PLASMA WAVES

In 1938 Tonks (26) gave an energy conservation theorem for a simple electron beam

$$\frac{\partial}{\partial t}\left[-\frac{m}{e}\ \frac{\rho \underline{v}^2}{2}+\frac{\epsilon_o\ \underline{E}^2}{2}+\ \frac{\mu_o\ \underline{H}^2}{2}\right]+\nabla\cdot\left[-\frac{m}{e}\ \frac{\rho \underline{v}^3}{2}+\underline{E}\times\underline{H}\right]=0\ . \qquad \text{III.1}$$

This follows simply from the equation of motion of the electrons,

$$\frac{\partial v}{\partial t}+(\underline{v}\cdot\nabla)\ \underline{v}\ =-\frac{e}{m}\ (\underline{E}+\underline{v}\times\underline{B}) \qquad \text{III.2}$$

the continuity equation,

$$\frac{\partial \rho}{\partial t}+\nabla\cdot\rho\underline{v}\ =\ 0\ , \qquad \text{III.3}$$

and Maxwell's equations for the electromagnetic field. $\underline{E}\times\underline{H}$ is the electromagnetic energy flux and $-\frac{1}{2}\frac{m}{e}\rho\underline{v}^3$ is the kinetic energy flux or power flow. $\frac{1}{2}\ \epsilon_o\ \underline{E}^2$, $\frac{1}{2}\mu_o\ \underline{H}^2$, and $-\frac{1}{2}\frac{m}{e}\rho\underline{v}^2$ are the electric, magnetic, and kinetic energy densities, respectively.

In nearly all electron beam problems it is necessary to linearize the equation of motion and continuity equation and hence restrict the discussion to small disturbances in order to solve the partial differential equations. The linearization process can be carried out in a consistent manner by expanding all field quantities in terms of an amplitude parameter δ by writing,

$$\underline{v}\ =\ \underline{u}_o+\delta\ \underline{v}_1+\delta^2\ \underline{v}_2+\cdots \qquad \text{III.4}$$

$$\rho\ =\ \rho_o+\delta\ \rho_1+\delta^2\ \rho_2+\cdots$$

etc. Symbols with the subscript zero denote the steady or undisturbed quantities, symbols with the subscript one denote the first order perturbations of these quantities and symbols with higher subscripts indicate higher order approximations which are usually neglected. Energy densities and fluxes of the disturbances with non-zero time averages will be of order δ^2 since they generally involve products of two time varying quantities. For example, the electric energy density of a disturbance would be $\delta^2(\frac{1}{2}\,\epsilon_o\,\underline{E}_1^2)$. If the definitions of kinetic power flow and kinetic energy density suggested by III.1 are used, it is necessary to know second order quantities ρ_2 , $\underline{u}_2$ as well as zero and first order quantities, provided $\underline{u}_o$ is not zero. In this section the electrons are assumed to have a non-zero drift velocity $\underline{u}_o$. A thermal plasma may be regarded as a collection of electrons with a distribution of drift velocities, $\underline{u}_o$, as shown at the end of this section. The kinetic power flow is given by

$$\rho\underline{v}^3 = \rho v^2\underline{v} = \rho_o\, u^2_o\underline{u}_o + \delta(\rho_o\, u^2_o\underline{v}_1 + 2\rho_o\,\underline{u}_o\,\underline{u}_o\cdot\,\underline{v}_1 + \rho_o\,\underline{u}_o\,u^2_o)$$
$$+\,\delta^2\,(\rho_o\,u^2_o\,\underline{v}_2 + \rho_2\,u^2_o\,\underline{u}_o + 2\rho_o\,\underline{u}_o\cdot\underline{v}_2\,\underline{u}_o + \qquad \text{III.5}$$
$$2\underline{u}_o\cdot\underline{v}_1\,\rho_o\,\underline{v}_1 + 2\underline{u}_o\cdot\underline{v}_1\,\rho_1\,\underline{u}_o + \rho_1\,\underline{v}_1\,u^2_o + \rho_o\,\underline{u}_o\,v^2_1)\,.$$

To compute this quantity correctly to order δ^2 , $\underline{u}_2$ and ρ_2 must be found. This requires the solution of an inhomogeneous partial differential equation for the second order quantities in which products of first order quantities appear as source terms. A special problem has been treated in this manner by Walker (27).

Energy Relations from the Linearized Equations. To avoid having to compute second order quantities and to work completely within the framework of the linear theory it will be postulated that the linearized equations which are of first order in the expansion parameter, δ , are the rigorously correct equations and a conservation theorem will be derived by manipulating these equations. This is a generalization of an idea due to Chu (28). The following linearized equations are used

$$\nabla \times \underline{E}_1 = -\frac{\partial \underline{B}_1}{\partial t} \qquad \text{III.6}$$

$$\nabla \times \underline{H}_1 = \rho_o \underline{v}_1 + \rho_1 \underline{u}_o + \epsilon_o \frac{\partial \underline{E}_1}{\partial t} \qquad \text{III.7}$$

$$\frac{\partial \underline{v}_1}{\partial t} + (\underline{u}_o \cdot \nabla)\, \underline{v}_1 = -\frac{e}{m} \underline{E}_1 \qquad \text{III.8}$$

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_o \underline{v}_1 + \rho_1 \underline{u}_o) = 0 \ . \qquad \text{III.9}$$

If the solutions are irrotational, $\nabla \times \underline{E}_1 = \nabla \times \underline{v}_1 = 0$, the following relation can be shown to hold

$$\frac{\partial}{\partial t}\left[-\frac{m}{e}\left(\rho_o \frac{\underline{v}_1^2}{2} + \rho_1\, \underline{u}_o \cdot \underline{v}_1\right) + \frac{1}{2}\epsilon_o \underline{E}_1^2\right] + \nabla \cdot \left[-\frac{m}{e}(\rho_o \underline{v}_1 + \underline{u}_o \rho_1)(\underline{u}_o \cdot \underline{v}_1)\right] = 0$$

III.10

The irrotational fields correspond to the longitudinal plane waves of Section II. If the solutions are solenoidal, $\nabla \cdot \underline{E}_1 = \nabla \cdot \underline{v}_1 = 0$, the following relation can be shown to hold,

$$\frac{\partial}{\partial t}\left[-\frac{m}{e}\rho_o\frac{\underline{v}_1^2}{2}+\epsilon_o\frac{\underline{E}_1^2}{2}+\mu_o\frac{\underline{H}_1^2}{2}\right]+\nabla\cdot\left[-\frac{m}{e}\underline{u}_o\frac{\underline{v}_1^2}{2}+\underline{E}_1\times\underline{H}_1\right]=0\,. \qquad \text{III.11}$$

The solenoidal fields correspond to the transverse plane waves of Section II. In the expression for the irrotational waves, the first two terms can be interpreted as the kinetic energy density of the electrons as a result of the small disturbance, the third term as the energy density of the electric field, and the last term as the kinetic power flow in the disturbance. In the expression for the solenoidal waves, the first term can be interpreted as the kinetic energy density of the electrons, the second and third terms as the electric and magnetic energy densities, respectively, and the fourth and fifth terms as the kinetic and electromagnetic power flow, respectively.

The first relation, III.10, is obtained in the following way. The usual conservation theorem for the electromagnetic field may be written

$$\frac{\partial}{\partial t}\left[\epsilon_o\frac{\underline{E}_1^2}{2}+\mu_o\frac{\underline{H}_1^2}{2}\right]+\nabla\cdot\left[\underline{E}_1\times\underline{H}_1\right]=-\underline{E}_1\cdot\underline{J}_1=-\underline{E}_1\cdot(\rho_1\underline{u}_o+\rho_o\underline{v}_1). \qquad \text{III.12}$$

The term $\underline{E}_1\cdot(\rho_1\underline{u}_o+\rho_o\underline{v}_1)$ gives the rate at which the electron beam takes energy from the field, in the linear approximation. To express this term as the time derivative of an energy density plus the divergence of a power flow, multiply III.6 by $(\rho_1\underline{u}_o+\rho_o\underline{v}_1)=\underline{J}_1$

$$\underline{J}_1 \cdot \underline{E}_1 = -\frac{m}{e}\left\{(\rho_1\,\underline{u}_o + \rho_o\,\underline{v}_1)\cdot\frac{\partial \underline{v}_1}{\partial t} + \underline{J}_1\cdot\left[(\underline{u}_o\cdot\nabla)\,\underline{v}_1\right]\right\}$$

$$= -\frac{m}{e}\left\{\frac{\partial}{\partial t}\left(\rho_o\,\frac{\underline{v}_1^2}{2} + \rho_1\,\underline{u}_o\cdot\underline{v}_1\right) - \underline{u}_o\cdot\underline{v}_1\,\frac{\partial\rho_1}{\partial t} + \underline{J}_1\cdot\left[(\underline{u}_o\cdot\nabla)\underline{v}_1\right]\right\}.$$

III.13

Using $\frac{\partial\rho_1}{\partial t} = -\nabla\cdot\underline{J}_1$ in this expression

$$\underline{J}_1\cdot\underline{E}_1 = -\frac{m}{e}\left\{\frac{\partial}{\partial t}\left(\rho_o\,\frac{\underline{v}_1^2}{2} + \rho_1\,\underline{u}_o\cdot\underline{v}_1\right) + (\underline{u}_o\cdot\underline{v}_1)\,\nabla\cdot\underline{J}_1 + \underline{J}_1\cdot\left[(\underline{u}_o\cdot\nabla)\underline{v}_1\right]\right\}$$

III.14

Since $\underline{u}_o$ is a constant, and it is assumed that $\nabla \times \underline{v}_1 = 0$, $-\frac{m}{e}\,\underline{J}_1\cdot\left[(\underline{v}_1\cdot\nabla)\,\underline{u}_o + \underline{v}_1\times(\nabla\times\underline{u}_o) + \underline{u}_o\times(\nabla\times\underline{v}_1)\right]$ can be added to the right side of III.14 without changing the result. These terms, together with the last two terms on the right side of III.14, continue to produce $\nabla\cdot\left[(\underline{u}_o\cdot\underline{v}_1)\,\underline{J}_1\right]$. Upon substituting the value of $\underline{J}_1\cdot\underline{E}_1$ obtained in this manner, III.10 is obtained. III.11 may be proved in a similar manner.

If the plasma consists of electrons with more than one steady velocity such as in a thermal plasma, the above relations can be generalized. Since the magnitudes of $\underline{v}_1$ and ρ_1 depend on the steady velocity, these quantities will be denoted by $\underline{v}_1(\underline{u}_o)$ and $\rho_1(\underline{u}_o)$. The energy relation for irrotational waves becomes

$$\frac{\partial}{\partial t}\left\{-\frac{m}{e}\sum_{\underline{u}_o}\left[\frac{\rho_o(\underline{u}_o)\underline{v}_1^2(\underline{u}_o)}{2} + \underline{u}_o\,\rho_1(\underline{u}_o)\,\underline{v}\,(\underline{u}_o)\right] + \epsilon_o\,\frac{E_1^2}{2}\right\}$$

$$+\nabla\cdot\left\{-\frac{m}{e}\sum_{\underline{u}_o}\left[\rho_o(\underline{u}_o)\,\underline{v}_1(\underline{u}_o) + \rho_1(\underline{u}_o)\,\underline{u}_o\,\underline{u}_o\cdot\underline{v}_1(\underline{u}_o)\right]\right\} = 0\,.$$

III.15

The energy relation for solenoidal waves becomes

$$\frac{\partial}{\partial t}\left\{-\frac{m}{e}\sum_{\underline{u}_o}\frac{\rho_o(\underline{u}_o)\,\underline{v}_1^2\,(\underline{u}_o)}{2}+\epsilon_o\frac{\underline{E}_1^2}{2}+\mu_o\frac{\underline{H}_1^2}{2}\right\}$$

$$+\nabla\cdot\left\{-\frac{m}{e}\sum_{\underline{u}_o}\frac{\rho_o(\underline{u}_o)\,\underline{u}_o\,v^2(\underline{u}_o)}{2}+\underline{E}_1\times\underline{H}_1\right\}=0\,. \qquad \text{III.16}$$

The power carried by the various components of the plasma is additive since there are no cross product terms in this expression.

Power Flow and Energy Transfer in Plane Wave Disturbances. The expressions just developed will now be used to calculate the kinetic power flow of one group of electrons in the plasma, and the exchange of energy between this group of electrons and the electric field. Two cases are of interest, (a) frequency real and propagation vector complex, and (b) propagation vector real and frequency complex. Both cases are treated simultaneously by assuming the space and time dependence to be given by the factors $e^{(\nu+i\omega t)}\,e^{-(\underline{\ell}+i\underline{k})\cdot\underline{r}}$. Using the equation of motion and the continuity equation it is easily shown that the perturbations in velocity, charge density, and current of a group of electrons whose steady velocity is $\underline{u}_o$ is given by

$$\underline{v}_1(\underline{u}_o)=\frac{-\frac{e}{m}\,\underline{E}_1}{\left[\nu+i\omega-(\underline{\ell}+i\underline{k})\cdot\underline{u}_o\right]} \qquad \text{III.17}$$

$$\rho_1(\underline{u}_o)=\frac{-\frac{e}{m}\,\rho_o(\underline{u}_o)\left[(\underline{\ell}+i\underline{k})\cdot\underline{E}_1\right]}{\left[\nu+i\omega-(\underline{\ell}+i\underline{k})\cdot\underline{u}_o\right]^2} \qquad \text{III.18}$$

$$\underline{J}_1(\underline{u}_o) = \frac{-\frac{e}{m}\rho_o(\underline{u}_o)\left[\underline{u}_o(\underline{\ell} + i\underline{k})\cdot\underline{E}_1 + (\nu + i\omega)\,\underline{E}_1 - (\underline{\ell} + i\underline{k})\cdot\underline{u}_o\,\underline{E}_1\right]}{\left[\nu + i\omega - (\underline{\ell} + i\underline{k})\cdot\underline{u}_o\right]^2}. \qquad \text{III.19}$$

Note that $\omega' - i\nu' = (\omega - \underline{k}\cdot\underline{u}_o) - i(\nu - \underline{\ell}\cdot\underline{u}_o)$ is the complex frequency of disturbance as seen by these electrons. After some algebraic manipulation the average† rate of transfer of energy from the field to the electrons, Re $(\underline{J}_1\cdot\underline{E}_1^*) = -\frac{\partial\overline{W}_E}{\partial t}$, may be written,

$$\text{Re}\,(\underline{J}_1\cdot\underline{E}_1^*) = \epsilon_o\,\omega_p^2(\underline{u}_o)\frac{\nu(\nu'^2 - \omega'^2) + 2\omega\,\omega'\nu'}{\left[\nu'^2 + \omega'^2\right]^2}\,\underline{E}_1\cdot\underline{E}_1^*\,e^{2(\nu t - \underline{\ell}\cdot\underline{r})}. \qquad \text{III.20}$$

The average kinetic power flow, $\frac{1}{2}$ Re $\underline{J}_1\cdot(\underline{u}_o\cdot\underline{v}_1)$ may be written,

$$\overline{P}_k = \frac{1}{2}\,\omega_p^2(\underline{u}_o)\;\frac{(\nu'^2 + \omega'^2)\underline{E}_1 + (\nu'\underline{\ell} + \omega'\underline{k})\,E_1\,\underline{u}_o}{\left[\nu'^2 + \omega'^2\right]^2}\;\underline{u}_o\cdot\underline{E}_1^*\,e^{2(\nu t - \underline{\ell}\cdot\underline{r})}. \qquad \text{III.21}$$

The average† kinetic energy density may be written

$$\overline{W}_k = \frac{1}{4}\,\epsilon_o\,\omega_p^2(\underline{u}_o)\left[\frac{\frac{1}{2}\,\underline{E}_1\cdot\underline{E}_1^*}{(\nu'^2 + \omega'^2)} + \frac{(\nu'\underline{\ell} + \omega'k)\,E_1\,\underline{u}_o\cdot\underline{E}_1^*}{(\nu'^2 + \omega'^2)^2}\right]e^{2(\nu t - \underline{\ell}\cdot\underline{r})}, \qquad \text{III.22}$$

where $\omega_p^2(\underline{u}_o) = -\rho_o(\underline{u}_o)\,\frac{e}{m\epsilon_o}$ is the plasma frequency of the group of electrons with velocity $\underline{u}_o$. The features of these expressions are best examined by applying them to the special cases.

When the frequency is real and the propagation vector is complex, the amplitude of the disturbance increases or decreases as it propagates. The average rate at which electrons receive energy from the field is given by

$$\frac{-\omega\,(\omega - \underline{k}\cdot\underline{u}_o)\,\underline{\ell}\cdot\underline{u}_o}{\left[(\underline{\ell}\cdot\underline{u}_o)^2 + (\omega - \underline{k}\cdot\underline{u}_o)^2\right]^2}\;\epsilon_o\,\omega_p^2(\underline{u}_o)\,\underline{E}_1\cdot\underline{E}_1^*\,e^{-2\underline{\ell}\cdot\underline{r}}. \qquad \text{III.23}$$

† Average over one period of the disturbance.

The denominator of this expression is always positive, the frequency is positive, hence the electrons receive energy from the field if

$$(\omega - \underline{k} \cdot \underline{u}_o)\, \underline{\ell} \cdot \underline{u}_o < 0 \quad .$$

Conversely, the electrons give energy to the field if

$$(\omega - \underline{k} \cdot \underline{u}_o)\, \underline{\ell} \cdot \underline{u}_o > 0 \quad .$$

For example, if the electrons move so as to see a field which increases with time, $\underline{\ell} \cdot \underline{u}_o$ is negative and the electrons give energy to the field only when $(\omega - \underline{k} \cdot \underline{u}_o)$ is negative. This will occur if the component of electron velocity in the direction of the wave front normal is greater than the phase velocity of the wave. The electrons also give up energy to the field when they move in the direction in which the field decreases, if their component of velocity in the direction of the wave vector is less than the phase velocity of the wave.

It is interesting to examine the kinetic power flow under these circumstances. For convenience, assume that the electrons move in the direction of propagation, call it the x direction, then

$$\bar{P}_k = \frac{\omega(\omega - k_x u_{ox})}{\left[(\ell_x u_{ox})^2 + (\omega - k_x u_{ox})^2\right]^2} \; \omega_p^2(u_{ox})\, u_{ox} \frac{\underline{E}_1 \cdot \underline{E}_1^*}{2} e^{-2 \ell x \cdot x} \quad . \tag{III.24}$$

Thus the kinetic power flow is in the direction of the electron velocity only if $(\omega - k_x u_{ox}) > 0$ and in the opposite direction if $(\omega - k_x u_{ox}) < 0$. The latter curious situation, in which the power flow is in the opposite direction from the direction of the carriers of the energy, comes about because the kinetic energy of the disturbance is negative. This simply means that the total kinetic energy of the electrons is less when a disturbance is present than when no disturbance is present.

The physical explanation of this result is as follows. Assume that d.c. electron velocity and propagation vector are both in the x direction and that the electrons travel slightly faster than the constant phase planes of the wave. In a coordinate system which moves with the wave, the electric field of the wave is a sinusoidal static field and the electrons move slowly through this field in the +x direction. The electrons tend to bunch in the regions where they travel more slowly, i.e., where the potential is low, and spread out where they travel more rapidly. In the stationary coordinate system the electrons also appear bunched in the regions of lowest velocity, and since there are more electrons with reduced velocity than there are with increased velocity, the total kinetic energy is less than in the undisturbed case. If, however, the d.c. electron velocity is less than wave velocity in the stationary system, in the coordinate system which moves with the wave, the electrons move slowly through the potential of the wave in the -x' direction. They are strongly bunched in the regions of low velocity (in the -x' direction) in this coordinate system but in the stationary coordinate system the electrons appear bunched in the regions of highest velocity. The total kinetic energy is then greater than in the undisturbed situation.

Thus under some circumstances it is possible for a disturbance to increase as it propagates along the electron beam or beams with the energy in the field increasing at the expense of the kinetic energy of the electrons which travel faster than the wave velocity.

A similar situation occurs when the propagation vector is assumed to be real and the frequency is complex. A disturbance of this kind increases or decreases exponentially with time throughout all space. In such a disturbance the average† rate at which electrons of velocity $\underline{u}_o$ absorb energy from the electric field is

$$-\frac{\partial \overline{W}_E}{\partial t} = \frac{\nu\left[\nu^2 + \omega^2 - (\underline{k}\cdot\underline{u}_o)^2\right]}{\left[\nu^2 + (\omega - \underline{k}\cdot\underline{u}_o)^2\right]^2}\ \omega_p^2(\underline{u}_o)\ \frac{\epsilon_o\, \underline{E}_1\cdot \underline{E}_1^*}{2}\ e^{2\nu t} \quad . \qquad \text{III.25}$$

The average† kinetic energy density of these electrons is

$$\overline{W}_k = \frac{1}{2}\ \frac{\nu^2 + \omega^2 - (\underline{k}\cdot\underline{u}_o)^2}{\left[\nu^2 + (\omega - \underline{k}\cdot\underline{u}_o)^2\right]^2}\ \omega_p^2(\underline{u}_o)\ \frac{\epsilon_o\, \underline{E}_1\cdot \underline{E}_1^*}{2}\ e^{2\nu t} \quad . \qquad \text{III.26}$$

From the first of these two expressions it may be seen that electrons receive energy from the field if

$$\nu^2 + \omega^2 - (\underline{k}\cdot\underline{u}_o)^2 > 0$$

whereas they give energy to the field if

$$\nu^2 + \omega^2 - (k \cdot u_o)^2 < 0 \quad .$$

When the amplitude of the disturbance increases with time, ν is positive, hence electrons give energy to the field if

$$(\underline{k}\cdot\underline{u}_o)^2 > \nu^2 + \omega^2$$

or when the component of the electron velocity in the direction of the propagation vector is a little larger than the phase velocity of the wave.

From the second of these two expressions, it may be seen that the kinetic energy of the disturbance is negative if

---

† averaged over a spatial period.

$$(\underline{k} \cdot \underline{u}_o)^2 > \nu^2 + \omega^2 \quad .$$

Thus under some circumstances it will be possible for a disturbance to increase exponentially with time, the energy in the field increasing at the expense of the kinetic energy of electrons which travel, roughly speaking, faster than the wave.

It should be noted, however, that the net or total time-average power flow of a wave which increases in some spacial direction is zero. If this were not so, the power flow would be proportional to the square of the wave amplitude and hence would increase in the particular spacial direction. This would imply a source of energy which, according to the conservation laws, does not exist. When a wave does increase in some spacial direction, some of the electrons transmit energy in the direction in which the waves increase and others transmit an equal amount of energy in the opposite direction. Electrons which travel slower than the wave may transmit energy in a direction opposite to their steady velocity, $\underline{u}_o$.

Similarly, when an oscillation increases with time, the space-average of net or total energy density does not increase, but remains constant. The kinetic energy density of the electrons which travel faster than the wave have their kinetic energy reduced at a rate which is just equal to the rate at which the energy in the field and the kinetic energy of other electrons increases.

The Electric Susceptibilities. Equations III.6 and III.7 can also be used to determine the current which is induced in the plasma by the electric field $\underline{E}_1$. Assuming sinusoidal time and spatial dependence,

$e^{i(\omega t - \underline{k} \cdot \underline{r})}$ these equations become

$$i(\omega - \underline{k} \cdot \underline{u}_o)\, \underline{v}_1 = -\frac{e}{m} \underline{E}_1$$

$$i(\omega - \underline{k} \cdot \underline{u}_o)\, \rho_1 - i\, \rho_o(\underline{u}_o)\, \underline{k} \cdot \underline{v}_1 = 0 \ .$$

Solving these for ρ_1 and $\underline{v}_1$ and forming $\underline{J}_1 = \rho_o\, \underline{v}_1 + \underline{u}_o\, \rho_1$ gives

$$\underline{J}_1 = i \sum_{\underline{u}_o} \frac{e}{m} \rho_o(u_o) \quad \frac{(\omega - \underline{k} \cdot \underline{u}_o)\, \underline{E}_1 + \underline{u}_o\, \underline{k} \cdot \underline{E}}{(\omega - \underline{k} \cdot \underline{u}_o)^2} \ .$$

The electric susceptibilities, as defined by II.29 are,

$$\chi_t = -\sum_{\underline{u}_o} \frac{\omega^2{}_p\, (\underline{u}_o)}{\omega\, (\omega - \underline{k} \cdot \underline{u}_o)}$$

$$\underline{\underline{\chi}}_1 = -\sum_{\underline{u}_o} \frac{\underline{k} \cdot \underline{u}_o\, \omega_p^2\, (\underline{u}_o)}{\omega(\omega - \underline{k} \cdot \underline{u}_o)^2} \ .$$

These expressions are completely equivalent to II.30 and II.31 if short range collisions are neglected.

IV CHARACTERISTICS OF LONGITUDINAL WAVES IN A PLASMA

Energy Loss Due to Short-Range Collisions. In this section it will be assumed that a quasi-neutral group of charged particles traverses the same space as the coronal electrons and ions. The electrons interact more strongly with the field than do the ions because of their smaller mass. To determine whether high speed particles can travel an appreciable distance in the solar corona, the energy loss is estimated from the formula (29)

$$\frac{dE}{dx} = - \frac{n_o e^4}{4\pi \epsilon_o^2 m V^2} \ln \frac{d_{max}}{d_{min}} \quad . \qquad \text{IV.1}$$

Although this formula is not accurate for electron-electron collisions involving large momentum transfers, it will suffice for an estimate. The minimum value of the impact parameter, d_{min}, is determined by the maximum energy which may be transferred in a single collision

$$d_{min} = \frac{e^2}{4\pi \epsilon_o m V^2} \qquad \text{IV.2}$$

The maximum value of the impact parameter, d_{max}, may be taken as the mean interparticle distance, $n_o^{-1/3}$. For electrons whose velocity is about one-third the velocity of light,

$$- E \frac{dE}{dx} \approx \frac{1}{1.6 \times 10^7 \text{ km}} \qquad \text{IV.3}$$

in the solar corona. Since the distance these electrons must travel to escape from the corona is about 10^6 km, the energy loss due to short-range collisions will not be appreciable. For much slower electrons, however, this energy loss will be very important.

Waves Associated with a Square Distribution Function. Before discussing waves in non-thermal plasmas, several aspects of waves in thermal plasmas will be discussed. When the distribution function can be written

$$f_o(\underline{u}) = f_{ox}(u_x)\, f_{oy}(u_y)\, f_{oz}(u_z) \quad , \qquad \text{IV.4}$$

the longitudinal wave dispersion relation II.48 can be written

$$1 = \omega_p^2 \int_{-\infty}^{\infty} \frac{f_{ox}(u_x)\, du_x}{(\omega - k_x u_x)^2} \quad . \qquad \text{IV.5}$$

The x direction is taken as the direction of the wave vector $\underline{k}$. Short range collisions have been neglected. If the Maxwell Boltzmann distribution is approximated by assuming that velocities up to a value U_o are equally probable and that no electrons have a velocity greater than U_o , integration of IV.5 leads to

$$\omega^2 = \omega_p^2 + k_x^2\, U_o^2 \quad . \qquad \text{IV.6}$$

The distribution function is shown in Figure 4a, and the relationship IV.6 is shown in Figure 4b. Interpreted as a relationship which gives the frequency of oscillation of a disturbance whose wavelength is $2\pi/k_x$, it is seen that all disturbances have a natural oscillation frequency greater than the plasma frequency and that long wavelength disturbances $(k_x^2 \ll \omega_p^2/U_o^2)$ oscillate with a frequency nearly equal to the plasma frequency. Written as a relationship which gives the propagation characteristics of waves with frequency ω , it becomes

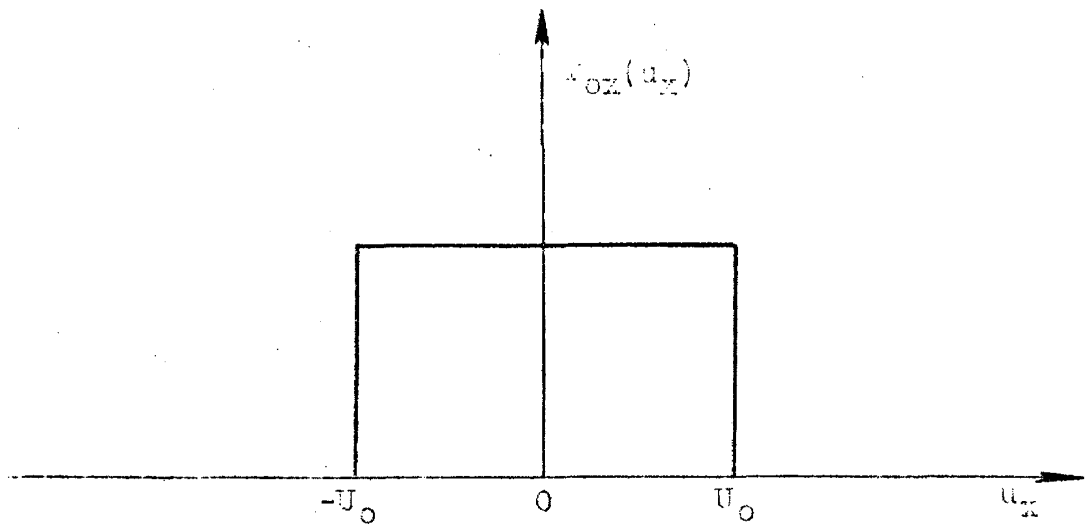

Figure 4a. The Distribution Function

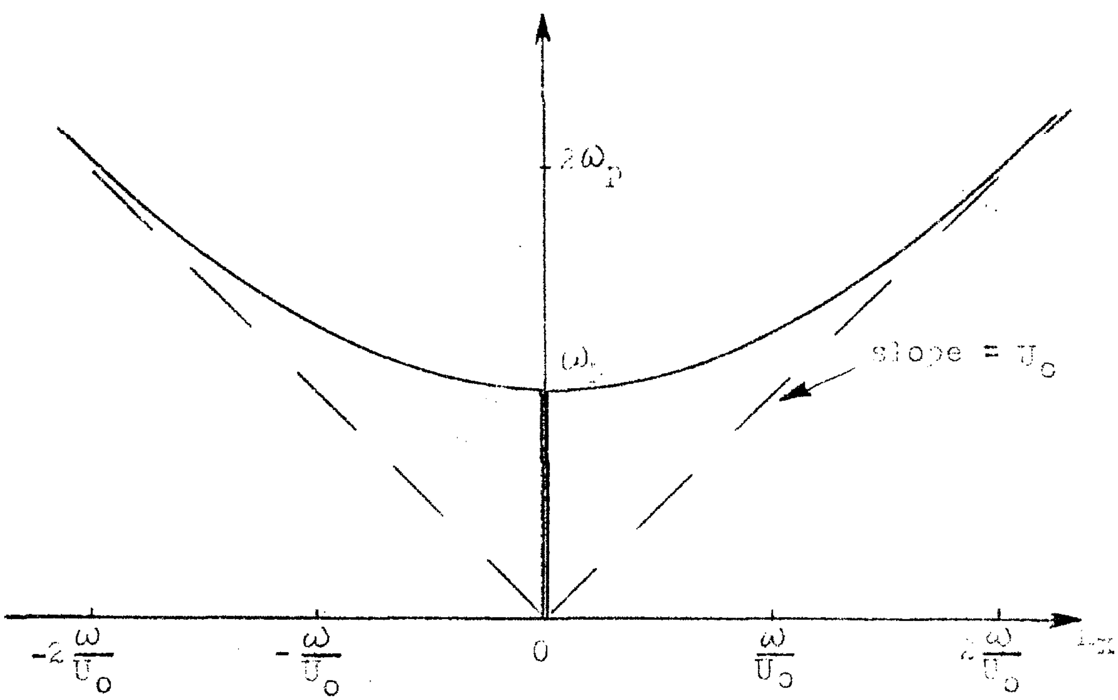

Figure 4b. The Dispersion Relation

$$k_x = \pm \frac{\sqrt{\omega^2 - \omega_p^2}}{U_o} . \qquad \text{IV.7}$$

When the frequency is greater than the plasma frequency the phase velocity of the waves is

$$v_p = \frac{\omega}{k_x} = \frac{U_o}{\sqrt{1 - (\frac{\omega_p}{\omega})^2}} \qquad \text{IV.8}$$

and the group velocity is

$$v_g = \frac{\partial\omega}{\partial k_x} = U_o \sqrt{1 - (\omega_p^2/\omega)^2} . \qquad \text{IV.9}$$

Note that $v_p \, v_g = U_o^2$, and that these formulas are similar to waveguide formulas where the cutoff frequency is ω_p and intrinsic velocity of propagation is U_o instead of the velocity of light.

The energy transferred by these waves can be computed from the expressions of Section III. III.23 shows that, on the average, the electrons do not exchange energy with the field, since $\ell = 0$. III.24 gives the average kinetic power flow of electrons whose velocity is u_x

$$\bar{P}_k = \frac{\omega \, \omega_p^2 (u_x)}{(\omega - k_x \, u_x)^3} \; u_x \; \frac{\epsilon_o \, E_1 \, E^*_1}{2} . \qquad \text{IV.10}$$

Since $\omega > k_x \, u_x$, the denominator of this expression is always positive. Thus electrons traveling in the positive x direction transport a.c. energy in the positive x directions and electrons traveling in the negative x direction transport a.c. energy in the negative x direction. Because of the way in which the denominator depends on u_x , electrons traveling in the positive x direction transport more energy in that direction than electrons traveling in the negative direction, provided k_x is positive. The total kinetic power flow is obtained by integrating II.9 over all velocities,

$$(\bar{P}_k)_{total} = \left(\frac{\omega^2}{\omega_p^2}\right) U_o \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \; \frac{\epsilon_o E_{1x} E^*_{1x}}{2} \qquad \text{IV.11}$$

$$= \left(\frac{\omega^2}{\omega_p^2}\right) v_g \frac{\epsilon_o E_{1x} E^*_{1x}}{2} \quad .$$

The average stored electric energy is given by III.16 and the average kinetic energy may be obtained by taking $\nu = 0$ in III.25 and integrating over all velocities. The result is easily shown to be

$$(\bar{W}_k)_{total} = \left(\frac{2\omega^2}{\omega_p^2} - 1\right) \frac{\epsilon_o E_{1x} E^*_{1x}}{4} \quad . \qquad \text{IV.12}$$

These relationships may be combined to give

$$(\bar{P}_k)_{total} = v_g \left[(\bar{W}_k)_{total} + \bar{W}_E \right] \qquad \text{IV.13}$$

so that the total a.c. energy of the disturbance can be regarded as being transported at a velocity equal to the group velocity of the wave. This type of relationship has been proved for many other wave-like disturbances. Since the group velocity is very small for long wavelength disturbances, i.e., when the frequency is equal to or slightly greater than the plasma frequency, little energy is transported in this region. The maximum velocity of energy transport is U_o , and this is achieved at frequencies which are much higher than the plasma frequency. The disturbances then have a very short wavelength compared with the Debye wavelength and the rectangular velocity distribution used in this discussion is not a valid approximation to a true thermal distribution (6),(24).

When the frequency is less than the plasma frequency (this situation could exist if the medium were excited by an external source of this frequency) the propagation constant becomes pure imaginary ($k_x = 0$, $\ell_x \neq 0$ in the notation of Section III) and waves increase or decrease with distance. The nature of these waves may be determined by examining the energy exchange. III.24, with $k_x = 0$, gives the average kinetic power flow of electrons with velocity u_o

$$\bar{P}_k = \frac{\omega^2 \omega_p^2 (u_x)}{\left[\omega^2 + \ell_x^2 u_x^2\right]^2} u_x \frac{\epsilon_o E_{1x} E^*_{1x}}{2} e^{-2\ell_x x} . \qquad \text{IV.14}$$

Thus, when $\ell_x > 0$, electrons traveling in the positive x direction transport a.c. energy in the positive x direction, but the amount of energy which they transport decreases as they travel because of the factor $e^{-2\ell_x x}$). Conversely, electrons traveling in the $-x$ direction transport energy in that direction but the amount of energy transported increases as they travel. Examination of the rate of energy transfer from the field to the electrons,

$$\frac{-\omega^2 \omega_p^2 \ell_x u_x}{\left[\omega^2 + \ell_x^2 u_x^2\right]^2} \epsilon_o E_{1x} E^*_{1x} e^{-2\ell_x x} , \qquad \text{IV.15}$$

shows that the electrons which transport less energy as they travel ($u_x > 0$) are giving their energy to the electric field, while the electrons which transport more and more energy as they travel ($u_x < 0$) are gaining this energy from the field. This is as expected.

If the observer is imagined to be at $x = 0$, the source of the disturbance must be at some negative x coordinate. A source at $x = L$, for example, can be imagined as sending out a.c. energy in the positive x direction via electrons which travel in that direction. As these electrons travel they give up their a.c. energy to the field. Electrons traveling in the -x direction, in turn, receive this energy from the field and carry it back to the source. There is no net kinetic power flow in such a wave, for at each point, on the average, as much energy flows in the -x direction as in the +x direction. All energy sent out by the source is returned to it. If ℓ_x is negative, the disturbance increases in the -x direction, and the source is located at some +x coordinate.

From this result it may be concluded that when the frequency is less than the plasma frequency, only waves which decrease away from the source may be excited. The situation is analogous to that of a waveguide operated below its lowest cutoff frequency; waves always decay away from their source.

The Double Stream Amplification Process. In Section III it was shown that electrons can give up energy to the field and, at the same time, acquire a greater velocity modulation if the electron velocity is greater than the wave velocity. A simple example of this is the double-stream amplification process discovered by Haeff (4) and others. Assume that the plasma consists of two groups of particles, one which travels with velocity u_1 and the other which travels with velocity u_2. The distribution function may be represented by

$$f_{ox}(u_x) = n_1\,\delta(u_x - u_1) + n_2\,\delta(u_x - u_2) \qquad \text{IV.16}$$

and the resulting longitudinal wave dispersion relationship is

$$1 = \frac{\omega^2_{p1}}{(\omega + i\ell_x u_1 - k_x u_1)^2} + \frac{\omega^2_{p2}}{(\omega + i\ell_x u_2 - k_x u_2)^2} \qquad \text{IV.17}$$

where $\omega^2_{p1} = \frac{n_1 e^2}{\epsilon_o m}$ and $\omega^2_{p2} = \frac{n_2 e^2}{\epsilon_o m}$. Following Haeff, only the special case $\frac{n_1}{n_2} = (\frac{u_1}{u_2})^2$ will be discussed and the "mean" velocity u_m and "mean" plasma frequency ω_p may be defined as

$$\frac{1}{u_m} = \frac{1}{2}\left(\frac{1}{u_1} + \frac{1}{u_2}\right)$$

$$\frac{\omega^2_p}{u^2_m} = \frac{\omega^2_{p1}}{u^2_1} = \frac{\omega^2_{p2}}{u^2_2} \quad . \qquad \text{IV.18}$$

Using these definitions in IV.17 and solving for $k_x - i\ell_x$

$$(k_x - i\ell_x - \frac{\omega}{u_m})^2 = (\frac{\omega}{u_m} - \frac{\omega}{u_1})^2 + \frac{\omega^2_p}{u^2_m} \pm \sqrt{\frac{\omega^4_p}{u^4_m} + 4(\frac{\omega}{u_m} - \frac{\omega}{u_1})^2 \frac{\omega^2_p}{u^2_m}} \, . \qquad \text{IV.19}$$

The lower sign yields one wave which increases in the +x direction, $\ell_x < 0$, and one wave which decreases in the +x direction, $\ell_x > 0$, if

$$(\frac{\omega}{\omega_p})^2 < 2\,\frac{u_1 + u_2}{u_1 - u_2} \quad . \qquad \text{IV.20}$$

For both waves $k_x = \omega/u_o$, so that the phase velocity of the wave is u_o and thus lies between u_1 and u_2 .

With this result in mind, the power flow in the two waves can be discussed. Consider the increasing wave, $\ell_x < 0$, first. $\omega - k_x u_1$ is positive and $\omega - k_x u_2$ is negative, hence by III.23 the first group of electrons receives energy from the field and the second group of electrons gives energy to the field. From III.24 it may be established that the first group of electrons transports energy in the $+x$ direction and the second group of electrons transports energy in the $-x$ direction. Power flow in a direction opposite to the direction of motion comes about, as explained in Section II, because the second group of electrons loses kinetic energy by becoming bunched. Thus both groups of electrons become more strongly bunched as they travel, the faster electrons giving energy to the field and to the slower electrons.

For the decreasing wave, ℓ_x is positive, and the direction of kinetic power flow of the two groups of electrons is the same as in the increasing wave. But now the fast electrons receive energy from the field in order to increase their total energy to that of the unperturbed state which they ultimately reach. The slow electrons give up their excess energy to the field.

Pierce has recently given an explanation of the double stream amplification process in terms of a coupling of modes of propagation (30). For stream number one, in the absence of stream number two, the characteristic waves are obtained from III.16 by setting ω_{p2} equal to zero

$$k_x = \frac{\omega \pm \omega_{p1}}{u_1} \quad \ell_x = 0 \qquad \text{IV.21}$$

and the characteristic waves of the second stream in the absence of the first are

$$k_x = \frac{\omega \pm \omega_{p2}}{u_2} \quad \ell_x = 0 \quad . \qquad \text{IV.22}$$

These solutions are shown in Figure 5 by the solid lines. The solutions

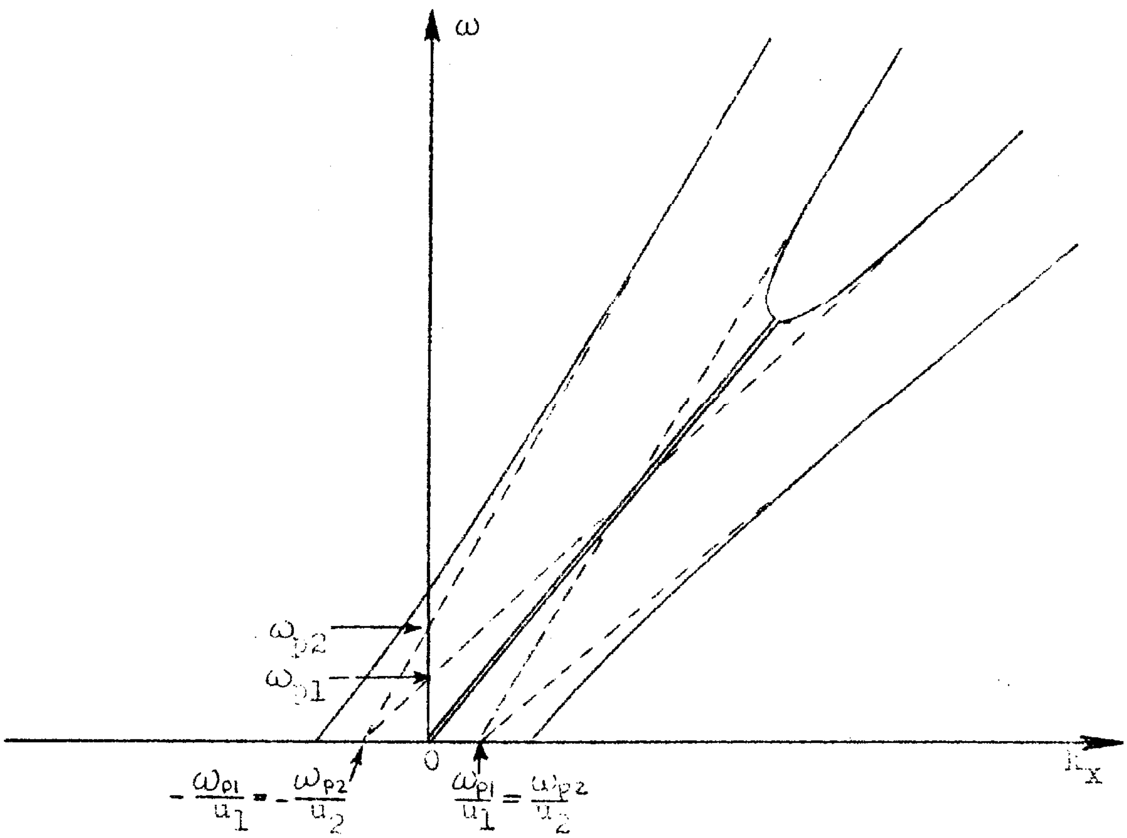

Figure 5. Characteristics of Waves on Two Electron Beams - ω versus k_x

dashed lines: electron beams separately (III.20, III.21)
solid lines: coupled electron beams (III.13)
double lines: waves which increase and decrease with distance ($\ell_x \neq 0$)

with the plus signs are called the slow space charge waves and the solutions with the minus signs are called the fast space charge waves because the waves have phase velocities which are less than or greater than the electron velocities of the appropriate electrons. Pierce's theory of coupling of modes of propagation shows that if two modes whose phase velocities are approximately equal and whose power flow is in opposite directions, are coupled together, one of the modified waves will increase with distance and the other will decrease with distance. When the power flow of the two modes are in the same direction, the modified waves are constant amplitude waves. From III.24 it may be shown that the power flow in a slow space charge wave of a single stream $(\omega - k_x u < 0)$ is negative and the power flow in a fast space charge wave of a single stream $(\omega - k_x u > 0)$ is positive. In the double stream process there is a frequency band in which the slow space charge wave of the fast stream couples with the fast space charge wave of the slow stream to produce a pair of new waves, one of which increases with distance. The energy which appears in the electric field in this increasing wave comes from the kinetic energy of the faster of the two groups of electrons. Furthermore, since the group velocity of each of the two original waves is positive, both the increasing wave and the decreasing wave can be established in the region $x > 0$ by a source at $x = 0$. At large x, of course, the increasing wave becomes much larger than the decreasing or constant amplitude waves.

The two-stream theory is not applicable to the solar corona because the coronal electrons do not all have the same velocity. The electrons of the undisturbed corona are approximately in thermal equilibrium and thus their velocities are distributed according to the

Maxwell-Boltzmann law, II. . Assuming a temperature of $10^{6\circ}$K, their r.m.s. x-directed velocity is given by

$$\sqrt{\overline{u_x^2}} = \sqrt{\frac{\kappa T}{m}} \approx 3900 \text{ km sec}^{-1} \quad . \qquad \text{IV.23}$$

According to the measurements of Wild, cited in the introduction, the Type II (slow) bursts appear to travel outward through the corona at a velocity which is about one-tenth this value, and the Type III (fast) bursts appear to travel outward at a velocity which is about ten times this value. Thus if the slow bursts are to be explained as an excitation of the plasma oscillations of the coronal electrons by a directed beam of charged particles, it is essential to include the effect of the distribution of coronal electron velocities.

The Analysis of Feinstein and Sen. Feinstein and Sen (7) approximated the coronal electron velocity distribution by the "square" distribution discussed earlier,

$$\begin{aligned} f_o(u) &= \frac{n_1}{2U_o} \quad |u| < U_o \\ &= 0 \qquad |u| > U_o \end{aligned} \qquad \text{IV.24}$$

and took the directed beam to be monoenergetic

$$f_o(u) = n_2\, \delta(u - u_o) \quad . \qquad \text{IV.25}$$

The resulting dispersion relation is

$$1 = \frac{\omega_{p1}^2}{\omega^2 - (k_x - i\ell_x)^2 U_o^2} + \frac{\omega_{p2}^2}{(\omega - k_x u_o + i\ell_x u_o)^2} \qquad \text{IV.26}$$

where $\omega_{p1}^2 = \frac{n_1 e^2}{\epsilon_o m}$ is the plasma frequency of the coronal electrons and $\omega_{p2}^2 = \frac{n_1 e^2}{\epsilon_o m}$ is the plasma frequency of the electrons in the directed beam. Feinstein and Sen have solved this relation for $k_x - i\ell_x$ assuming the frequency to be real, for a wide variety of electron densities and velocities. In particular, they looked for frequencies, ω, for which the wave amplitude increased in the direction of the travel of the directed beam, i.e., for which $\ell_x u_o < 0$. It was found that such increasing waves were possible even when the velocity of the directed beam, u_o, was less than the velocity of the coronal electrons, U_o. Sen (31) later applied these results to estimate the density and motion of solar material associated with the Type II bursts.

The idea that slow electrons can excite the plasma waves appears to be in contradiction with the result of Bohm and Pines who have shown that a single charged particle moving through a thermal plasma loses energy to the plasma oscillations and hence excites them, only when its velocity is greater than the r.m.s. thermal velocity of the plasma electrons.

The energy concepts discussed in Section II can be applied to the increasing waves found by Feinstein and Sen to determine their nature. Since Feinstein and Sen do not give k_x and ℓ_x as a function of ω, but give only the values of ω for which $\ell_x u_o < 0$, it is necessary to solve IV.26 again. The qualitative nature of the solutions may be obtained by using Pierce's theory of coupling at modes of propagation. The waves associated with the distribution IV.24 have already been discussed. The waves associated with distribution IV.25 are described by the relation

$$k_x = \frac{\omega \pm \omega_{p2}}{u_o} \qquad \ell_x = 0 \ . \qquad \text{IV.27}$$

The characteristics of the waves of the two separate distributions are shown in Figures 6a and 6b by the solid lines. The characteristics of waves of the composite distribution are depicted by the dashed lines. A pair of increasing and decreasing waves are present, for which $k_x u_o < 0$, over a band of frequencies which is centered about ω_{p1} when $\omega_{p1} = \omega_{p2}$. The center frequency is higher when ω_{p2} is greater than ω_{p2} . The essential point to note is that $k_x u_o$ is negative and hence

$$\omega - k_x u_o > 0$$

for both the increasing and decreasing wave. Thus the directed beam of electrons receives energy from the field when $\ell_x u_o < 0$, rather than giving energy to the field, and the kinetic power flow for this group of electrons is in the +x direction. These waves are really modified evanescent waves, similar to those exhibited by the square distribution.

From the point of view of the coupling of modes of propagation, the increasing and decreasing waves can be thought of as arising from a coupling of the fast space charge wave of the directed beam with the wave of the thermal plasma which has a phase velocity in the negative x direction. The power flow of these two waves is in opposite directions and hence increasing and decreasing waves are expected when they are coupled. Furthermore, since the group velocities of the two waves are also in opposite directions, it is not possible to say that both increasing and decreasing waves are established in the region $0 < x < L$ by

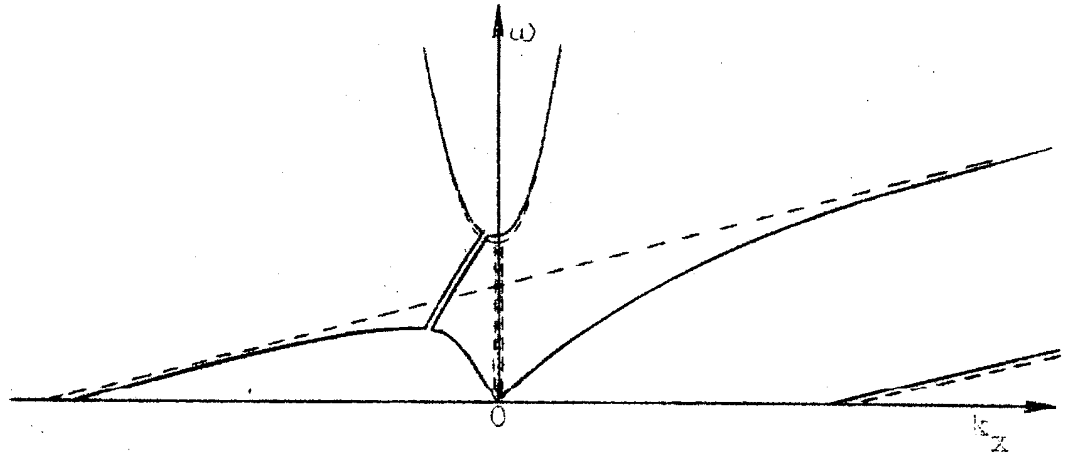

Figure 6a. ω_{p2}/ω_{p1} = .5, u_o/U_o = .2

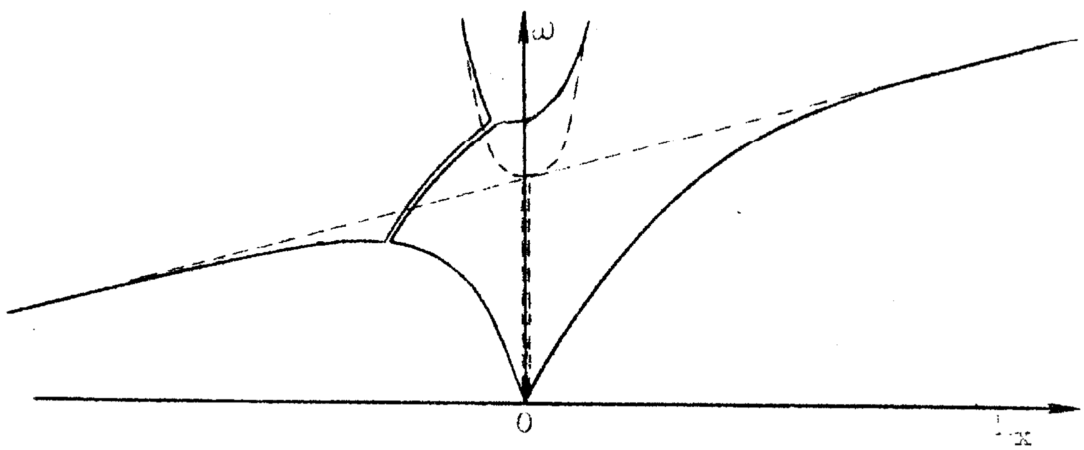

Figure 6b. ω_{p2}/ω_{p1} = 1, u_o/U_o = .2

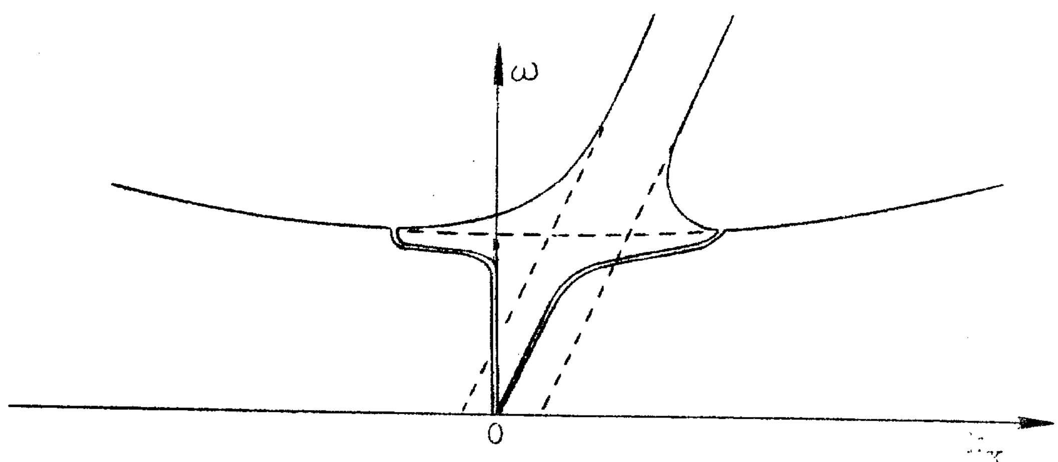

Figure 6c. ω_{p2}/ω_{p1} = .3, u_o/U_o = 5.

a source at $x = 0$. A study of the excitation of two waves of this type (32) shows that the excitation of the wave which increases in the $+x$ direction is determined primarily by the boundary condition at $x = 0$ whereas the excitation of the wave which decreases in the $+x$ direction is determined primarily by the boundary condition at $x = L$.

When the velocity of the directed beam is greater than the velocity of the coronal electrons, increasing and decreasing waves are found in two different regions as shown in Figure 6c . One set of increasing and decreasing waves resemble the evanescent waves of the thermal electrons alone and the previous description of the evanescent waves of the thermal plasma applies with one exception: k_x is no longer zero but slightly negative. This causes the thermal electrons to transport a small emount of energy in the negative x direction as may be seen by integrating III.26 over all velocities. This energy flow just compensates for the extra energy which the directed beam transports in the $+x$ direction.

As a final argument against the excitation of plasma oscillations by slow electrons, it may be shown that the dispersion relation IV.26 if interpreted as a relation which gives the oscillation frequency of disturbance of wave number $k_x(\ell_x = 0)$, yields no solutions for which ν is positive if $u_o < U_o$. The proof of this statement will now be outlined. Rewriting III.26

$$\frac{k_x^2}{\omega_{pl}^2} = \frac{1}{u^2 - U_o^2} + \frac{r}{(u - u_o)^2}$$

where $r = \frac{\omega_{p2}^2}{\omega_{p1}^2}$ and $u = \frac{\omega - i\nu}{k_x}$ is the complex wave velocity. Putting the right side over a common denominator

$$\frac{k_x^2}{\omega_{p1}^2} = \frac{(1+r)\,(u-u_1)\,(u-u_2)}{(u^2 - U_o^2)\,(u-u_o)^2} \qquad \text{IV.28}$$

where
$$u_1 = \frac{u_o - \sqrt{r\,(1+r)\,U_o^2 - u_o^2}}{1+r}$$

$$u_2 = \frac{u_o + \sqrt{r\,(1+r)\,U_o^2 - u_o^2}}{1+r}$$

If it is assumed that $0 < u_o < U_o$ it may easily be established that

$$-U_o < u_1 < u_o$$

$$u_o < u_2 < U_o \quad .$$

The right hand side of IV.28 is easily sketched for real u from this knowledge of the pole and the zero locations. Such a sketch is shown in Figure 7. For every value of k_x, i.e. for all wavelengths of disturbance, the left side of IV.28 is real and positive. Figure 7 shows that for each value of k_x there are four real values of the wave velocity which satisfy IV.28. Since IV.28 has only four solutions, all solutions are real. From the definition of the wave velocity, it is seen that the four frequencies of vibration are real $(\nu = 0)$. However, when the velocity of the directed beam is greater than the velocity of the coronal electrons $u_o > U_o$, one of the characteristic frequencies of vibration has a negative imaginary part, $(\nu > 0)$, indicating that disturbances may grow larger with time.

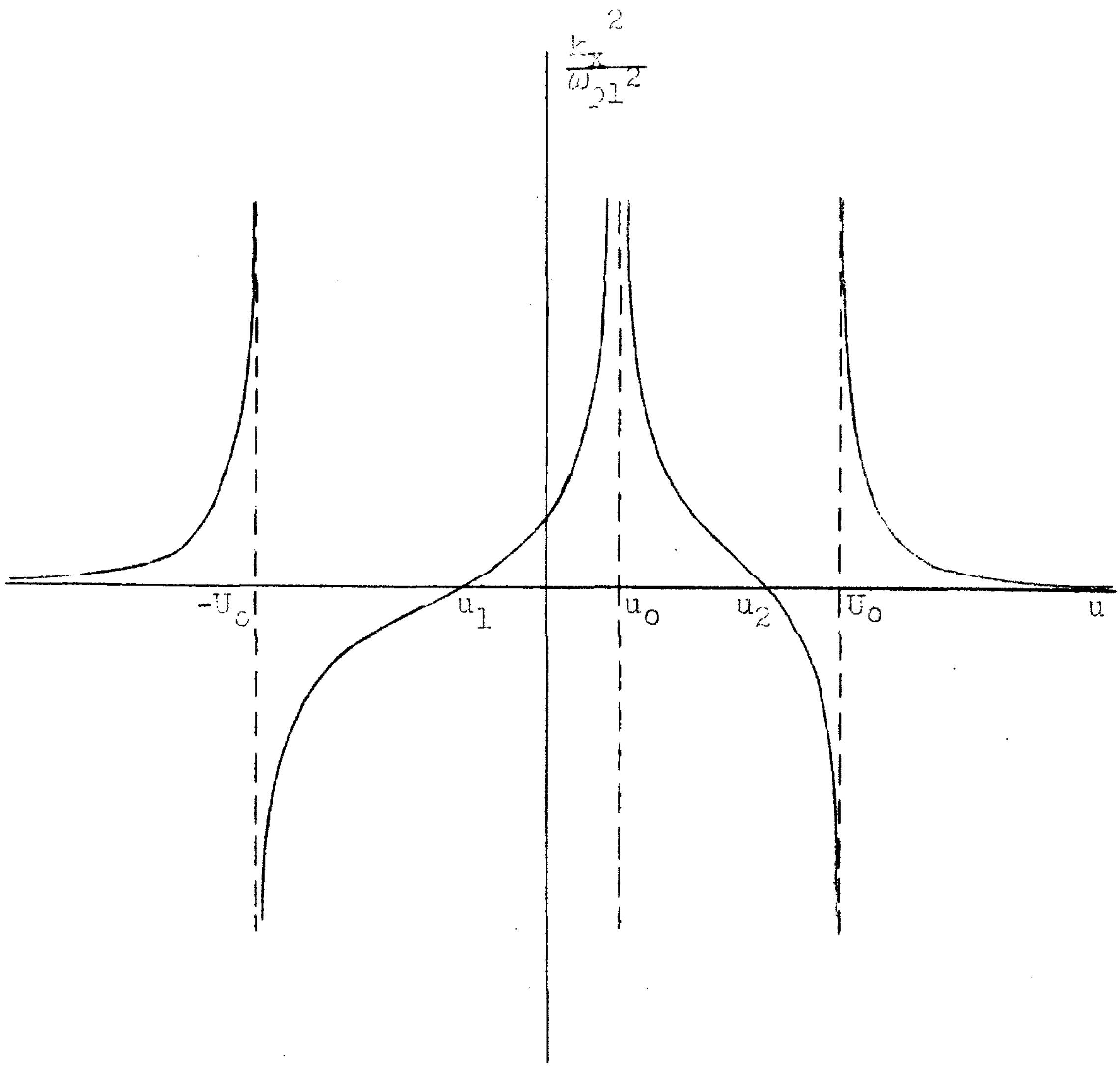

Figure 7. Sketch of $(k_x/\omega_{p1})^2$ versus the Wave Velocity, u.

Sumi (33) has also studied the excitation of longitudinal oscillations of a thermal plasma by a directed beam of electrons, and reached the conclusion that the phase velocity of the wave must be greater than the velocity of the directed beam for excitation to take place. This conclusion is incorrect as shown in Section III by physical, as well as mathematical arguments. Sumi's error can be traced to the rule used in indenting around the singularity at the wave velocity $(\omega - \underline{k} \cdot \underline{u} = 0)$ in the integration over all electron velocities. The correct rule has been given in Section II. The results of Ahiezer and Feinberg (34) suffer from a similar difficulty.

Sen has also studied the characteristics of long wavelength plasma waves in a bimodal velocity distribution (8) such as is thought to exist in the transition region of a shock wave as it propagates through an ionized gas. However, the long wavelength approximation is not generally applicable when looking for amplifying waves, because the phase velocities of the waves must be much larger than the velocities of any particles. Thus there can be no extraction of energy from fast particles. All increasing and decreasing waves found under these circumstances must be evanescent waves.

There have been other instances in which solutions of the plasma equations were incorrectly interpreted as representing amplification of small disturbances. On closer examination many of these solutions have been found to represent evanescent waves. Bailey (19), for example, suggested that two electron beams in relative motion in the presence of a constant magnetic field can make possible the amplification of transverse electromagnetic waves. If this were true it would be very fortunate for it might end the search for schemes which convert longitudinal plasma oscillation energy to transverse electromagnetic wave energy. However, Twiss (25) has been able to show, by means of a transient solution of a boundary value problem in Bailey's medium, that the increasing waves are excited only by reflection and these waves decrease away from the plane of reflection. Roberts (35) suggested that amplification is possible in a dominant mode waveguide when an electron beam is present, but Walker (23) has pointed out that the wave discussed by Roberts is simply the evanescent or cutoff waveguide mode, modified very slightly by the presence of the electron beam.

Most of these errors have been errors in interpretation. There has been a great tendency for investigators to regard all waves whose amplitude increases in the direction of flow of an electron beam as amplifying waves which derive their energy from the kinetic energy of the electron beam.

Landau Damping. While the imaginary part of the transverse wave susceptibility given by II.45 is always negligible, the longitudinal wave susceptibility given by II.44 can have an appreciable imaginary part when the wave velocity is small enough for a significant number of electrons to be traveling faster than the wave. This imaginary part of the susceptibility gives rise to a damping of the plasma oscillations as first noted by Landau (24). According to the arguments of Section III, which neglects the effects of short range collisions, the total a.c. energy in a damped oscillation must be zero since there is no place for the energy to go when the oscillation is damped out. It is possible for the total a.c. energy to be zero because the a.c. kinetic energy of the electrons which travel faster than the wave is negative and equal in magnitude to the positive a.c. kinetic energy of electrons which are slower than the wave and the energy in the electric field.

Excitation of Plasma Oscillations by a Directed Beam. Previous calculations have either been incorrect or incomplete, so it has been necessary to find the solutions of the longitudinal wave dispersion relation for a variety of parameters. The formula II.42 for the susceptibility of a Maxwell-Boltzmann distribution of electrons is not a convenient one for computation, so the one-dimensional form of the distribution function has been approximated by

$$f_{o1}(u_x) = \frac{\alpha_1/\pi}{u_x^2 + \alpha_1^2} \qquad \text{IV.28}$$

The width at half-maximum of this distribution is $2\alpha_1$. The moments of the distribution do not exist and it has the disadvantage that there are more high speed particles than in the Maxwell-Boltzmann distribution. These disadvantages are not particularly serious. The directed beam of electrons is approximated by the distribution

$$f_{o2}(u_x) = \frac{\alpha_2/\pi}{(u_x - u_2)^2 + \alpha_2^2} \quad . \qquad \text{IV.29}$$

u_2 is the mean value of the x component of the velocity of the directed beam and $2\alpha_2$ is the width of its distribution. Letting n_1 be the number density of electrons in the thermal plasma and n_2 the number density of electrons of the directed beam, the resulting longitudinal wave electric susceptibility may be written

$$= - \frac{\omega_{p1}^2}{\left[\omega - i\nu - k_x\, i\alpha_1\right]^2} - \frac{\omega_{p2}^2}{\left[\omega - i\nu - k_x\,(u_2 + i\alpha_2)\right]^2} \,. \qquad \text{IV.30}$$

This result is obtained by substituting IV.28 and IV.29 into II.32 and neglecting collisions. The first term is the susceptibility of the coronal electrons and the second term is the susceptibility of the directed beam. The longitudinal wave dispersion relation is obtained by setting this equal to -1 . This operation has the following significance. When IV.30 is equal to -1 , the imaginary part of the first term of the expression must be the negative of the imaginary part of the second term of the expression. The imaginary part of the susceptibility is proportional to the conductance of the electrons, hence is positive if the electrons take energy from the field and negative if the electrons give energy to the field. Thus one group of electrons gives energy to the field and the other receives energy. The case where the directed beam gives energy to the field is of primary interest in this problem. The real part of the susceptibility must be equal to -1 in order to cancel the susceptibility of free space. Since free space is capacitive, the electrons supply an inductive susceptance. Energy oscillates back and forth between the electric field and the kinetic energy of the electrons, much as in a resonant electric circuit.

The dispersion relation IV.30 may be solved for $\omega - i\nu$ by means of the root locus method (36) . Let $(\omega - i\nu)/k_x$ be defined as the complex wave velocity, u, and let $\bar{u}_1$ and $\bar{u}_2$ denote $i\alpha_1$ and $u_2 + i\alpha_2$ respectively. IV.30 can be written

$$\frac{k_x^2}{\omega_{p1}^2} = \frac{1}{(u-\bar{u}_1)^2} + \frac{r}{(u-\bar{u}_2)^2} \qquad \text{IV.31}$$

or

$$\frac{k_x^2}{(1+r)\omega_{p1}^2} = \frac{(u-\bar{u}_3)\ (u-\bar{u}_4)}{(u-\bar{u}_1)^2\ (u-\bar{u}_2)^2} \qquad \text{IV.32}$$

where

$$\bar{u}_3 = \frac{\bar{u}_2 + r\,\bar{u}_1 + i\sqrt{r}\,(\bar{u}_2 - \bar{u}_1)}{1+r}$$

$$\bar{u}_4 = \frac{\bar{u}_2 + r\,\bar{u}_1 - i\sqrt{r}\,(\bar{u}_2 - \bar{u}_1)}{1+r}$$

$$r = \left(\frac{\omega_{p2}}{\omega_{p1}}\right)^2 \quad .$$

Since k_x is assumed to be real, the left side of this expression is real and positive. Hence the right side must be real and positive. Figure 8 shows the points $\bar{u}_1$, $\bar{u}_2$, $\bar{u}_3$, and $\bar{u}_4$ plotted in the complex wave velocity plane. In order for the point u to be a solution of IV.50, the angles shown in Figure 8a,b must total an integral number times 2π,

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = 2\pi n \quad . \qquad \text{IV.33}$$

If the point u satisfies IV.33, $k_x^2/(1+r)\,\omega_{p1}^2$ may be found from

$$\frac{k_x^2}{\omega_{p1}^2} = (1+r)\ \frac{\ell_3\,\ell_4}{\ell_1^2\,\ell_2^2} \quad . \qquad \text{IV.34}$$

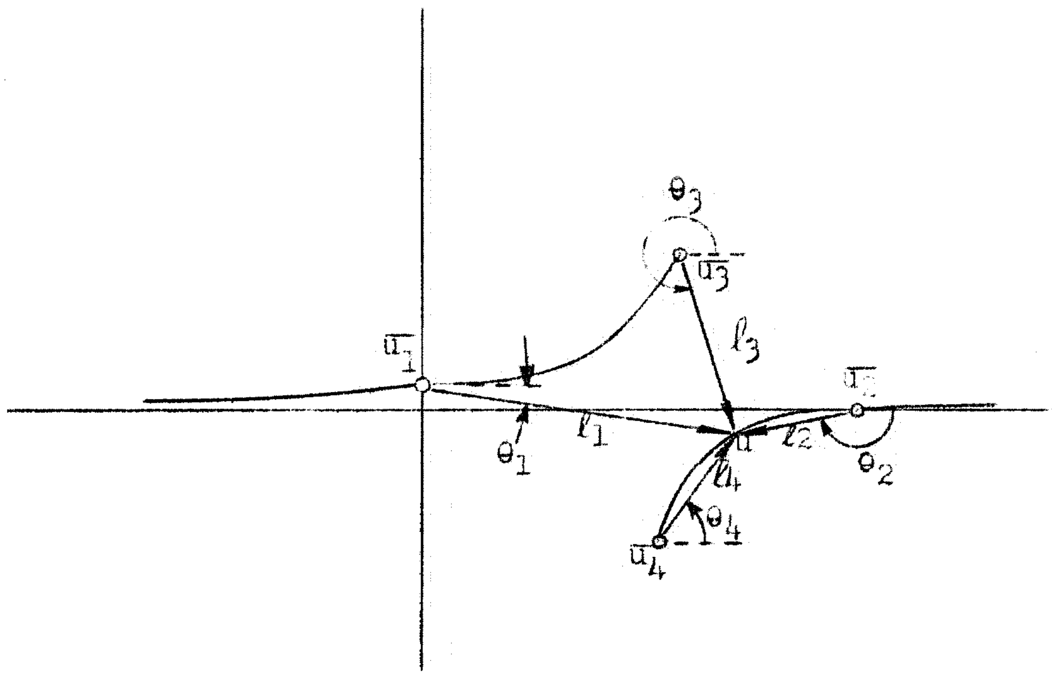

Figure 3a. Locus of Roots in the Complex Wave Velocity Plane for Real Values of the Wave Number

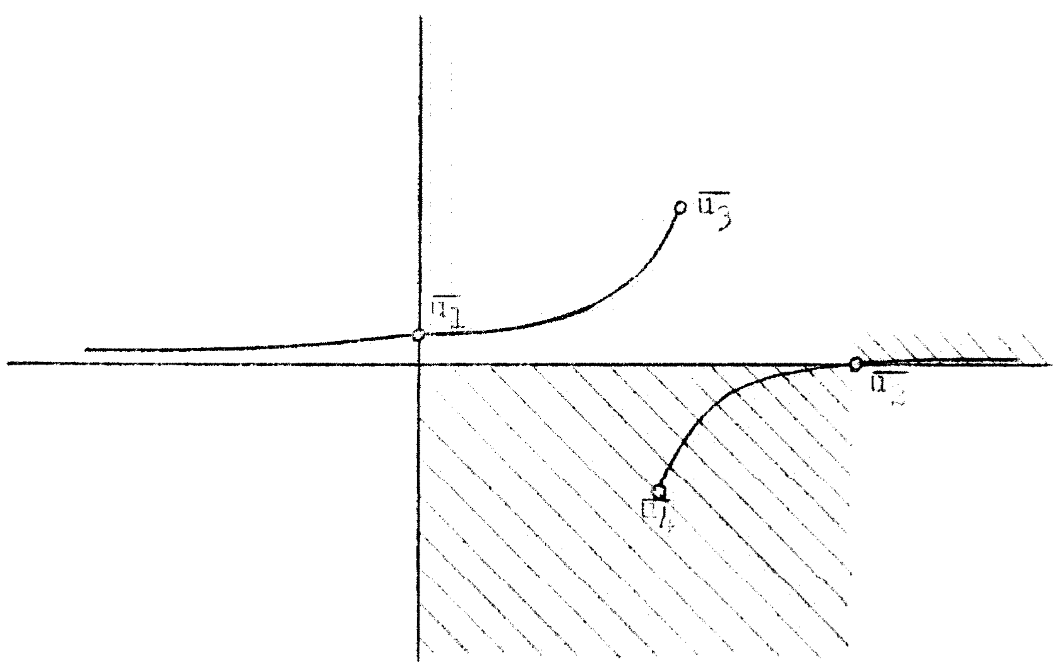

Figure 3b. Shaded Areas Denote Complex Wave Velocities for Which the Second Group of Particles Gives Energy to the First Group of Particles.

Oscillations which increase with time correspond to points in the lower half of the wave velocity plane, since in that region ν is positive. The shaded area of Figure 8c shows the region of wave velocities for which the directed beam gives energy to the field and the coronal electrons receive energy from the field. Only solutions in this region are of interest in the present study.

Although the root locus method was useful in a preliminary study of the solutions of IV.31, more extensive calculations have been made with the Electrodata Datatron computer using an iterative method. Some of the results of this study are shown in Figures 9 through 14. The remainder of the results are tabulated in Appendix I. Figures 9, 10, and 11 show the loci traced out by the complex wave velocity, u , as k_x varies from zero to infinity with α_1 equal to .1, 1.0, and 3.0, respectively. In these three cases α_2 is assumed to be zero. The different loci correspond to different values of the parameter $r = (\frac{\omega_{p2}}{\omega_{p1}})^2$. The loci start at $\bar{u}_4$ $(k_x = 0)$ and end at $\bar{u}_2$ $(k_x = \infty)$.

Figures 12, 13, and 14 show the loci traced out by the complex frequency as k_x varies from zero to infinity for the three sets of parameters of Figures 9, 10, and 11. The complex frequency always has a negative imaginary part $(\nu > 0)$, indicating that such disturbances increase with time. When the wavelength of the disturbance is infinite $(k_x = 0)$, the frequency is zero. It can be seen from the figures that the rate of growth is largest when $k_x u_2/\omega_{p1}$ is of the order of unity.

The results given in Figures 9, 10, and 11 are somewhat more general than might first appear. As may be seen from IV.32, the addition of a constant, $(u' + i\alpha)$, to u , $\bar{u}_1$, $\bar{u}_2$, $\bar{u}_3$, and $\bar{u}_4$ does not

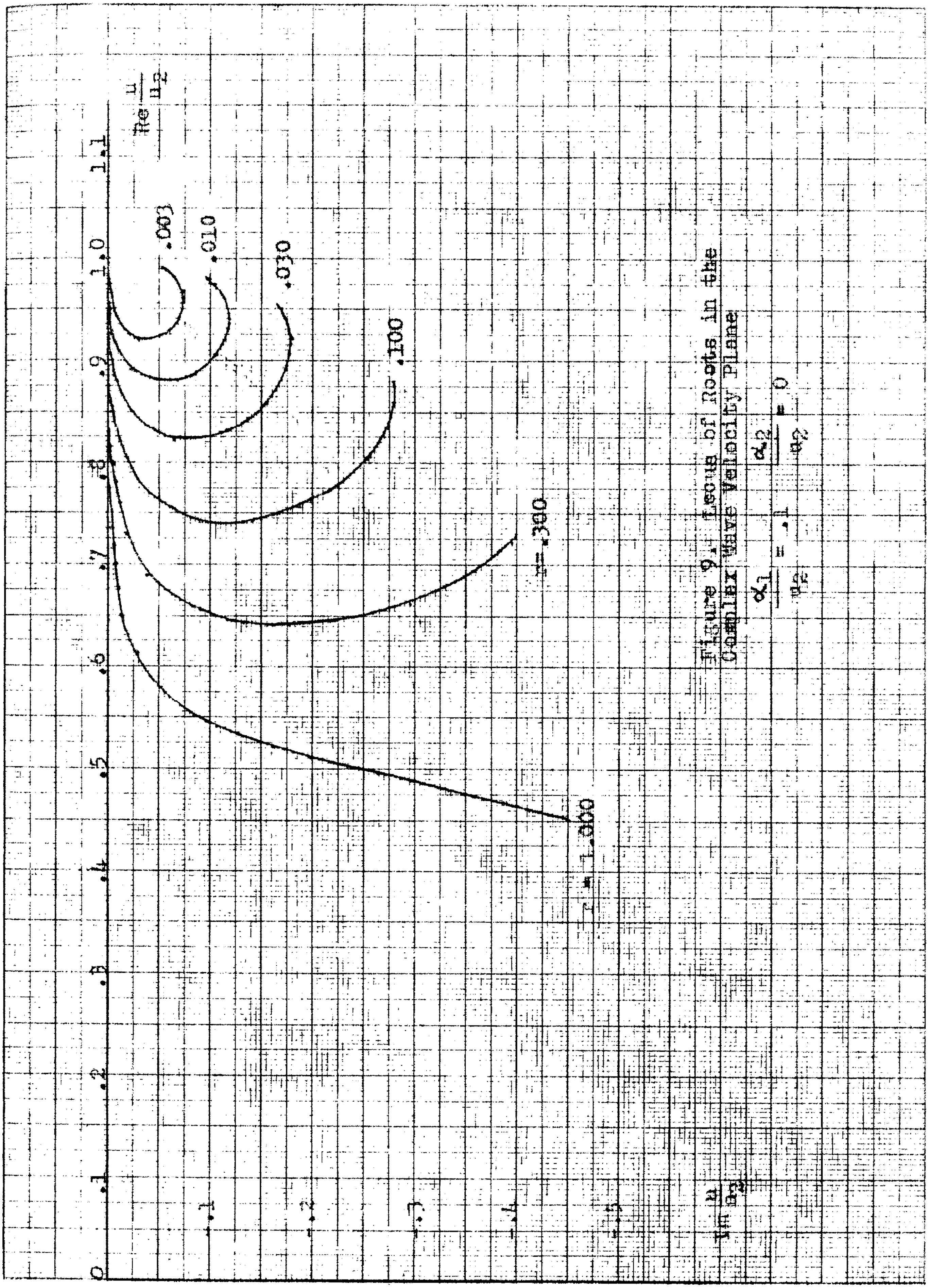

Figure 9. Locus of Roots in the Complex Wave Velocity Plane

$$\frac{\alpha_1}{u_2} = .1 \qquad \frac{\alpha_2}{u_2} = 0$$

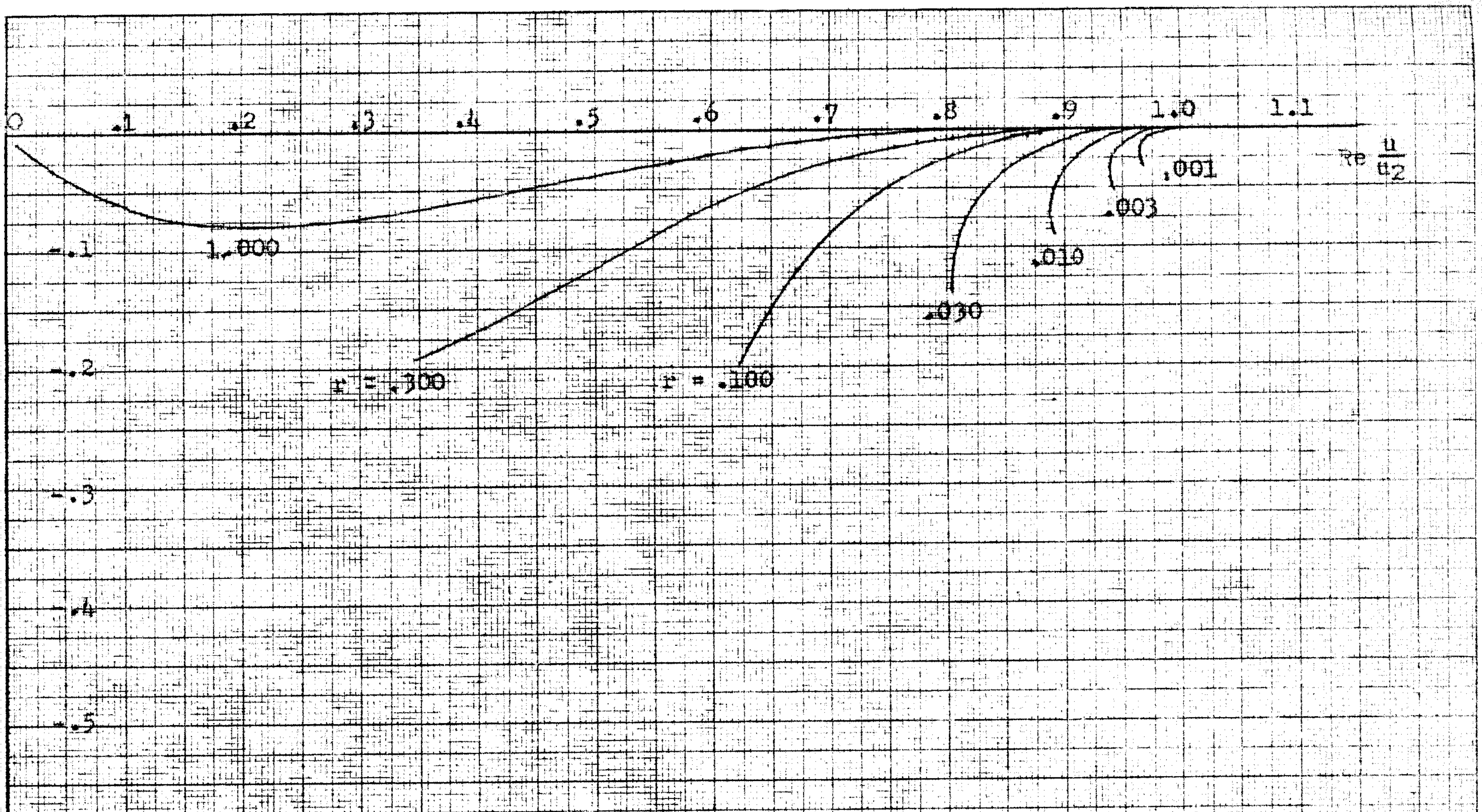

Figure 10. Locus of Roots in the Complex Wave Velocity Plane

$$\frac{\alpha_1}{u_2} = 1 \qquad \frac{\alpha_2}{u_2} = 0$$

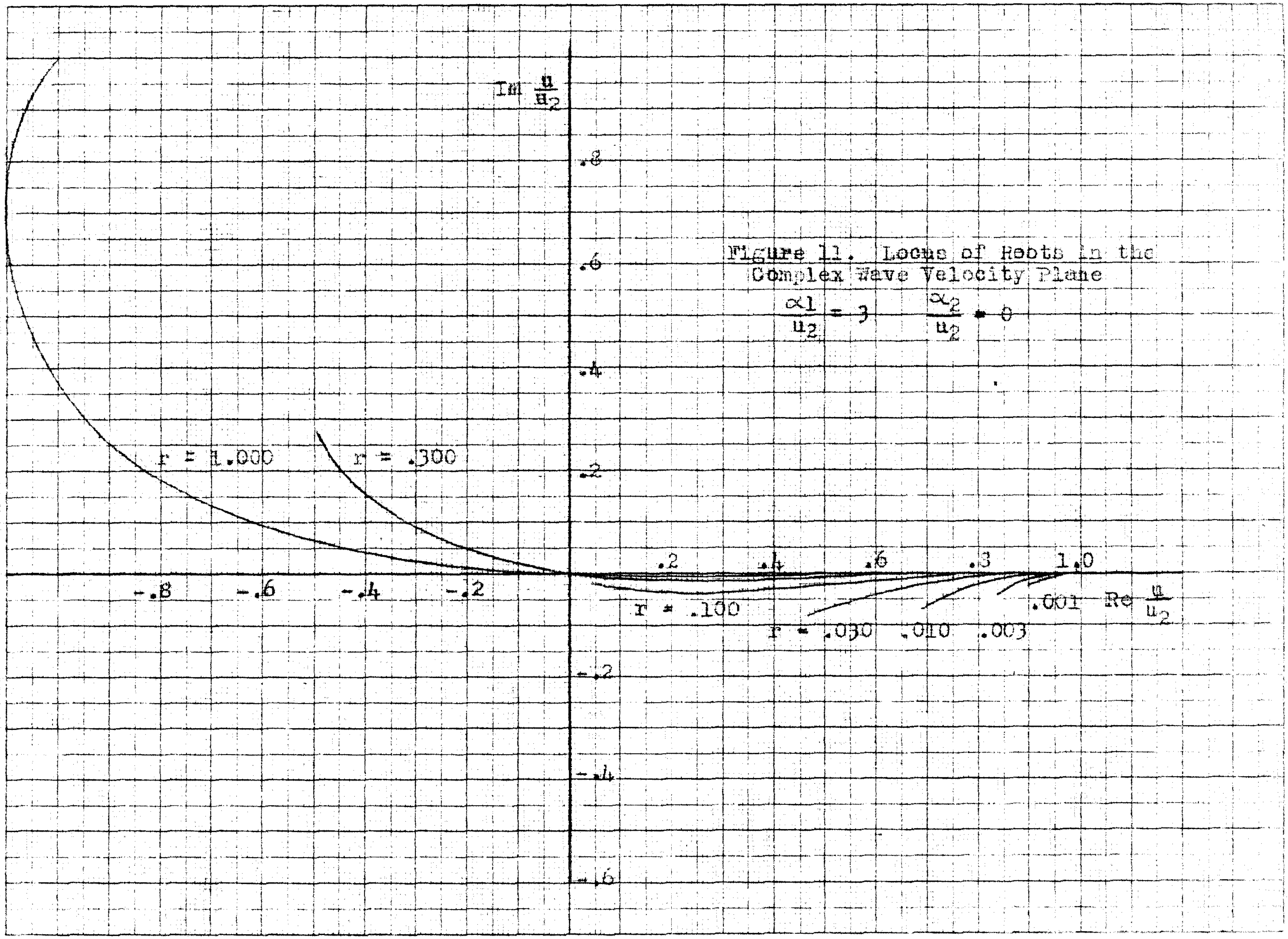

Figure 11. Locus of Roots in the Complex Wave Velocity Plane

$\frac{\alpha_1}{u_2} = 3 \qquad \frac{\alpha_2}{u_2} = 0$

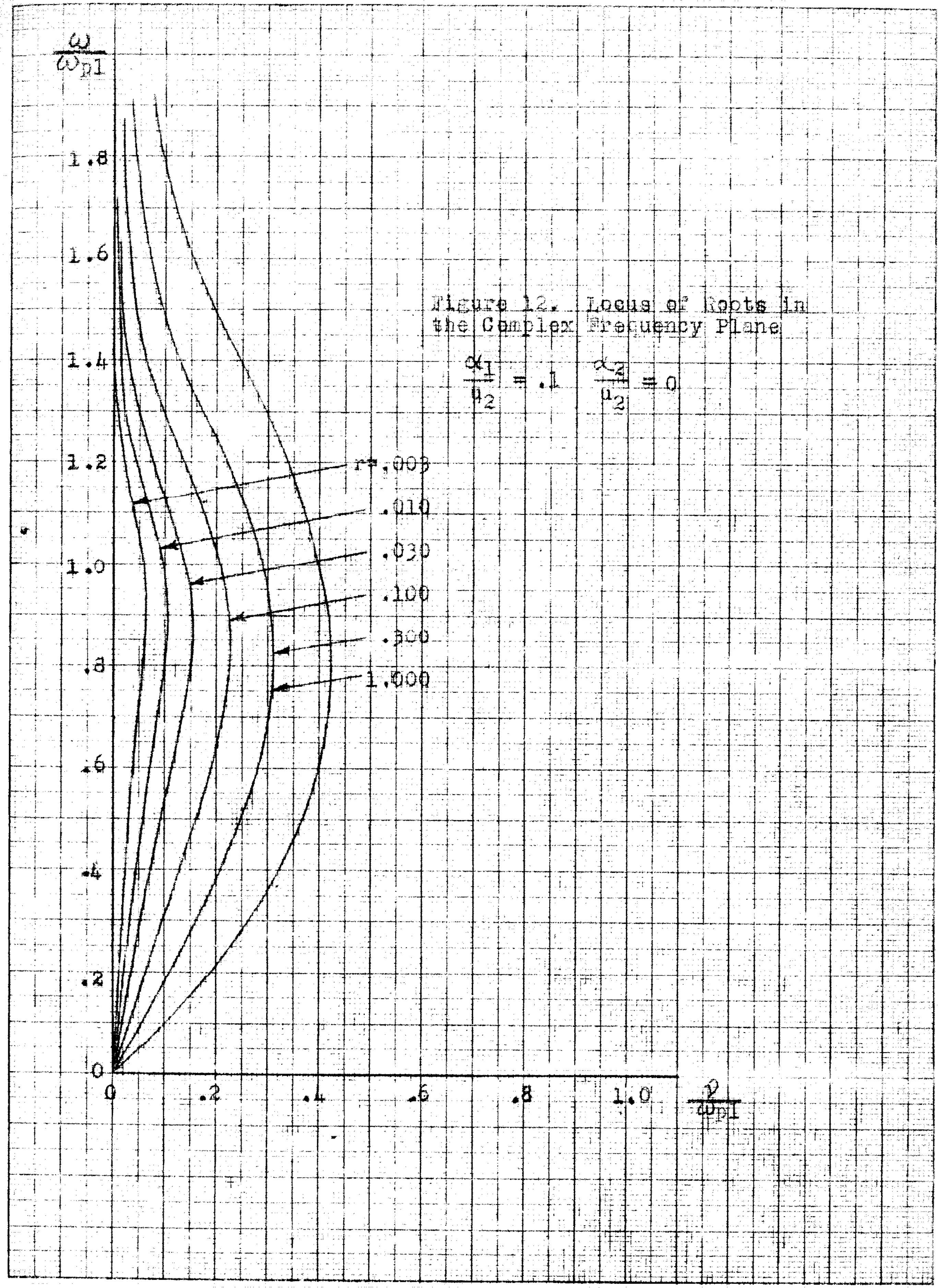
Figure 12. Locus of Roots in the Complex Frequency Plane
$\frac{\alpha_1}{u_2} = .1 \quad \frac{\alpha_2}{u_2} = 0$
r = .003
.010
.030
.100
.300
1.000
$\frac{\omega}{\omega_{p1}}$
1.8
1.6
1.4
1.2
1.0
.8
.6
.4
.2
0
0
.2
.4
.6
.8
1.0
$\frac{\nu}{\omega_{p1}}$

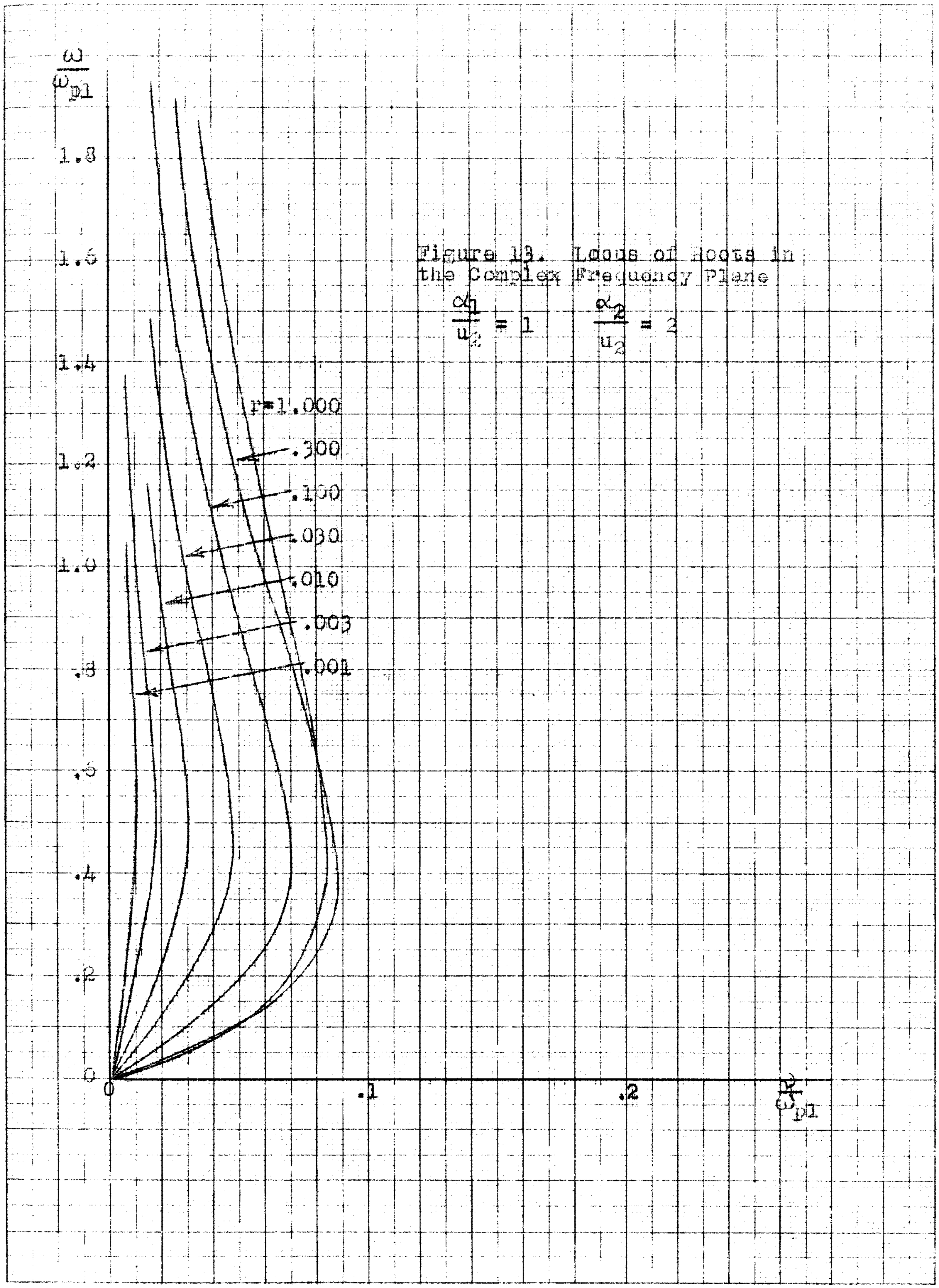

Figure 13. Locus of Roots in the Complex Frequency Plane

$\frac{\alpha_1}{u_2} = 1 \qquad \frac{\alpha_2}{u_2} = 2$

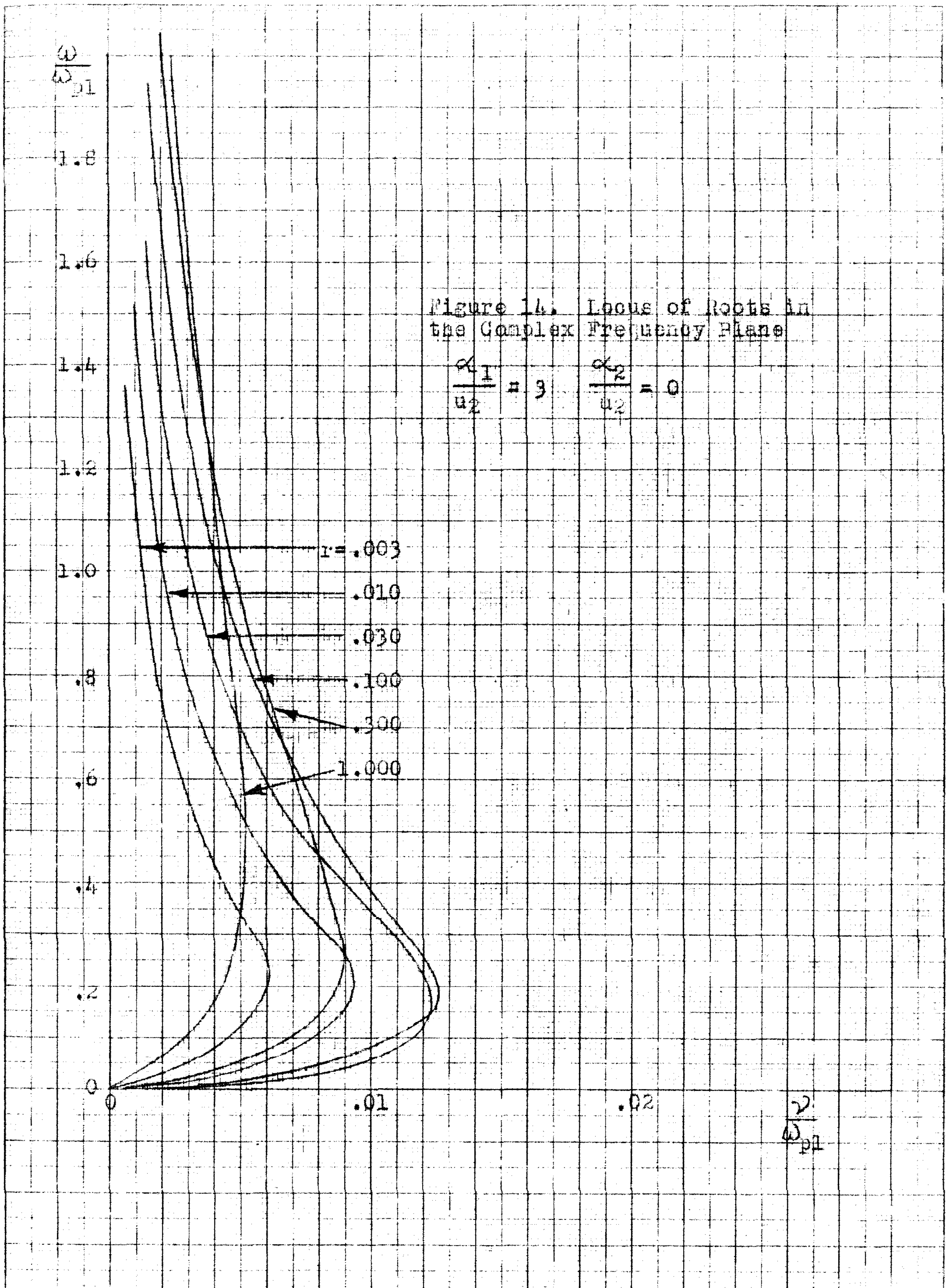

Figure 14. Locus of Roots in the Complex Frequency Plane

$\frac{\alpha_1}{u_2} = 3 \quad \frac{\alpha_2}{u_2} = 0$

change k_x. This is equivalent to a translation in the complex wave velocity plane. The addition of a real constant $(\alpha' = 0)$ is equivalent to transforming to a coordinate system which moves with the velocity u'. The real part of frequency of oscillation in the new coordinate system is given by the non-relativistic Doppler shift formula,

$$\omega' = \omega + k_x u' \qquad \text{IV.35}$$

and the imaginary part of the frequency is unchanged. The addition of an imaginary constant $(u' = 0)$ is equivalent to a broadening of the distributions

$$\begin{aligned} \alpha_1' &= \alpha_1 + \alpha' \\ \alpha_2' &= \alpha_2 + \alpha' \quad . \end{aligned} \qquad \text{IV.36}$$

The real part of the frequency is unchanged but the imaginary part of the frequency $(-\nu)$ is increased by α'

$$-\nu' = -\nu + \alpha' \qquad \text{IV.37}$$

and hence the rate of growth of the oscillation is <u>decreased</u>.

A number of features of these results have been noted: (a) If the directed beam is monoenergetic $(\alpha_2 = 0)$ and the velocity spread of the coronal electrons is not neglected $(\alpha_1 \neq 0)$ disturbances of <u>all</u> wavelengths increase with time. However, disturbances whose wavelength is greater than two or three times ω_{p1}/u_2 increase at a much slower rate than disturbances of other wavelengths and after a short time are unimportant. If the directed beam has a slight distribution of velocities $(\alpha_2 \neq 0)$ only the longer wavelength disturbances increase with time. This result is obtained by translating the wave velocity loci in the imaginary u direction and noting that the parts of the root loci corresponding to disturbances of short wavelengths are shifted into the upper half plane. Complex wave velocities in the upper half plane represent damped disturbances. (b) Long wavelength disturbances increase with time even if the directed beam has a velocity which is less than the width of the thermal velocity distribution of the coronal electrons $(\frac{\alpha_1}{u_2} = 3)$. A small velocity spread in the directed beam $(\frac{\alpha_2}{u_2} \approx .05)$ completely eliminates this effect, however. (c) For the disturbances which increase most rapidly, the real part of the frequency is generally just a little less than the plasma frequency of the coronal electron, ω_{p1} . In none of the cases is this frequency greater than the plasma frequency of the coronal electrons, although in many it is just a few percent less. Appreciable growth occurs for disturbances whose frequency is greater than plasma frequency ω_{p1} or the critical frequency for escape, $\sqrt{\omega_{p1}^2 + \omega_{p2}^2} = \omega_{p1}\sqrt{1 + r}$. (d) The real part of the frequency in a coordinate system which moves with the directed beam is $\omega' = \omega - k_x u_2$, which is approximately equal to the plasma frequency of the directed beam. Thus each group of particles oscillates at a frequency which is nearly equal to its natural oscillation frequency.

Figure 15 summarizes one aspect of these results; the conditions under which a plasma may be unstable due to the presence of a directed beam of electrons. The lines r = constant denote the stability boundaries for the various values of r . Regions to the left and

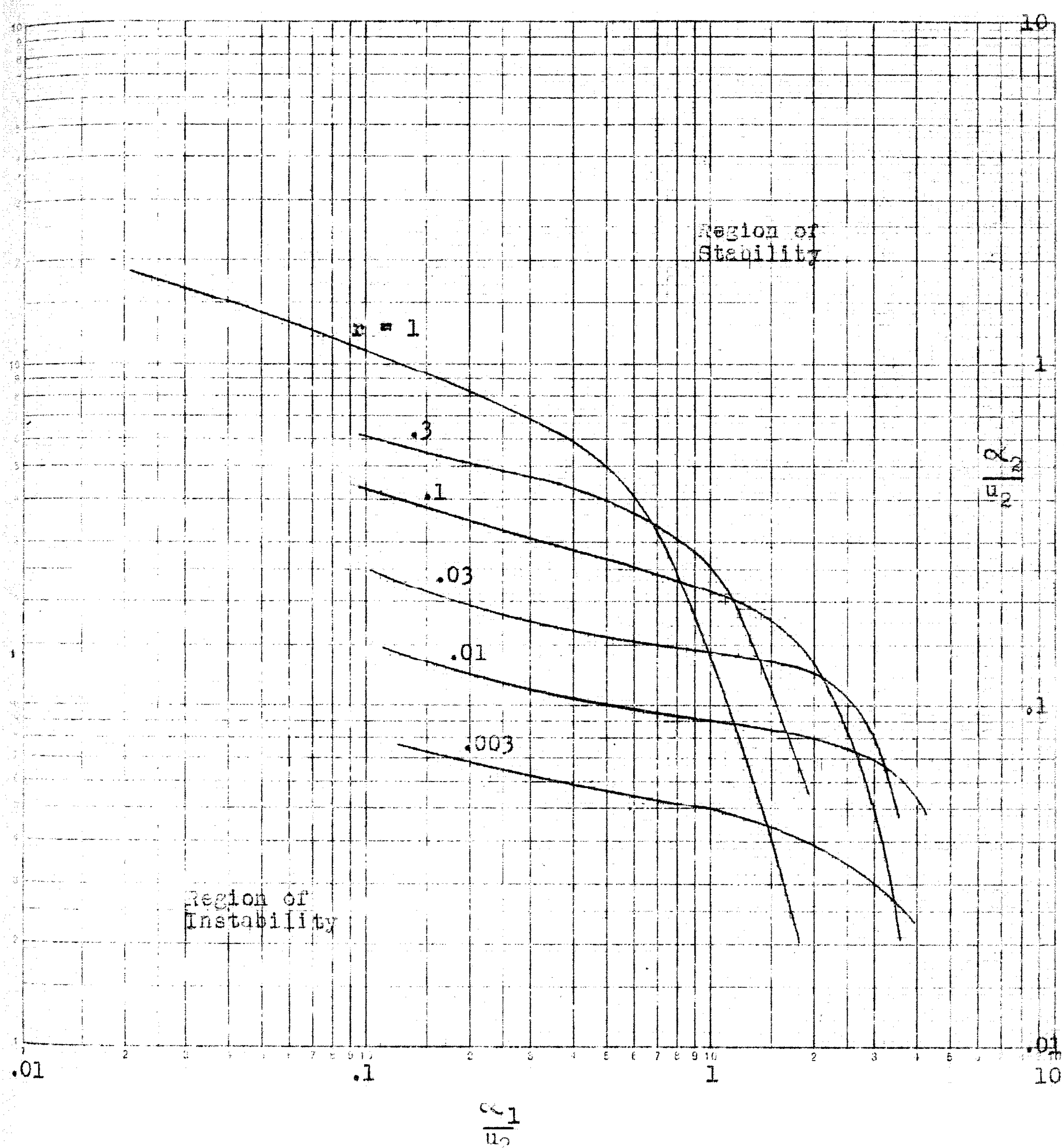

Figure 15. Stability Boundary Diagram

below these lines represent values of α_1 and α_2 for which some disturbances increase with time. Regions to the right and above these lines represent values of α_1 and α_2 for which no disturbances increase with time. From this curve it appears unlikely that any disturbances increase with time when $\frac{\alpha_2}{u_1} \approx 10$. If a particular set of parameters α_1, α_2, and r lie just within the unstable region, only the very long wavelength disturbances increase with time and the frequency of these disturbances are much less than the plasma frequency.

V REMARKS ON THE NON-LINEAR ASPECT OF THE PROBLEM

In the preceding section it was found that disturbances of certain wavelengths increase with time very rapidly. Even if the initial disturbances are the small random fluctuations which are always present, it takes very little time for these disturbances to grow to extremely large amplitudes. For example, the amplitude of the fastest growing disturbances $(\nu/\omega_{pl} \approx .3)$ increases by a factor of 10^{10} in only 16 plasma periods or about 10^{-7} second. Other disturbances may grow at a slower rate but it is clear that, as far as observations are concerned, the non-linear behavior of the plasma process is the most important. Very little progress has been made in this important aspect of the problem. In this section an estimate of the final amplitude will be made using the results of a simple non-linear calculation.

Non-Linear Oscillations of a Simple Plasma. If all electrons have the same velocity, zero velocity for convenience, a traveling-wave solution of the non-linear equations can be obtained (37),(38),(39). Assume that all field quantities are functions only of the quantity $x' = x - Vt$, where V is the wave velocity. In a coordinate system which moves with the wave velocity, all field quantities are independent of time. The equation of motion becomes

$$\frac{\partial v}{\partial t'} + v\,\frac{\partial v}{\partial x'} = \frac{\partial}{\partial x'}\left(\frac{v^2}{2}\right) = +\,\frac{e}{m}\,\frac{\partial \phi}{\partial x'} \quad . \qquad \text{V.1}$$

Integration of V.1 yields

$$\frac{v^2}{2} = \frac{e}{m}\,\phi + \frac{V^2}{2} \quad , \qquad \text{V.2}$$

where the constant of integration has been chosen to make $v = V$ in the absence of a disturbance $(\phi = 0)$. The continuity equation becomes

$$\frac{\partial \rho}{\partial t'} + \frac{\partial}{\partial x'}(\rho v) = \frac{\partial}{\partial x'}(\rho v) = 0 \quad . \qquad \text{V.3}$$

Integration of V.3 yields

$$\rho v = \rho_o V \qquad \text{V.4}$$

where the constant of integration has been chosen to give $\rho = \rho_o$ in the absence of a disturbance $(v = V)$. Poisson's equation, from which the field is determined, can be written

$$\frac{\partial^2 \phi}{\partial x'^2} = \frac{\rho - \rho_o}{\epsilon_o} = \frac{\rho_o}{\epsilon_o}\left[\frac{V}{v} - 1\right] = \frac{\rho_o}{\epsilon_o}\left[\frac{1}{\sqrt{1 + \frac{2e\,\phi}{m\,V^2}}} - 1\right] \qquad \text{V.5}$$

where V.4 and V.2 have been used. V.5 is conveniently expressed in terms of the dimensionless variables

$$\xi = \frac{\omega_p x'}{V} \qquad \Phi = \frac{2e\,\phi}{mV^2} \quad , \qquad \text{V.6}$$

as

$$\frac{d^2\Phi}{d\xi^2} = 2\left[\frac{1}{\sqrt{1+\Phi}} - 1\right] \quad . \qquad \text{V.7}$$

When Φ is small compared to unity, the right side of V.7 can be approximated by simply $-\Phi$. Evidently this yields the linearized, or small amplitude, result of simple harmonic equation. V.7 can be integrated by multiplying by $2\,\frac{d\Phi}{d\xi}\,d\xi$

$$\left(\frac{d\Phi}{d\xi}\right)^2 = 4\int\left[\frac{1}{1+\Phi} - 1\right] d\Phi$$

$$= 4\left[2 \quad 1+\Phi - \quad + A^2 - 2\right] \qquad \text{V.8}$$

where the constant of integration has been taken as $4(A^2 - 2)$. Letting $\sqrt{1+\Phi} = 1 + \eta$ and solving for $\frac{d\eta}{d\xi}$

$$\frac{d\eta}{d\xi} = \frac{\sqrt{A^2 - \eta^2}}{1+\eta} \quad . \qquad \text{V.9}$$

Using Dwight (40) 320.01 and 321.01, yields

$$\xi = \sin^{-1}\frac{\eta}{A} - \sqrt{A^2 - \eta^2} + \frac{\pi}{2} \quad . \qquad \text{V.10}$$

The constant of integration was taken to be $\frac{\pi}{2}$ so as to make Φ symmetric about $\xi = 0$. It is easily shown that the normalized electric field and charge density is

$$\frac{d\Phi}{d\xi} = 2\sqrt{A^2 - \eta^2} = \qquad \text{V.11}$$

$$\frac{d^2\Phi}{d\xi^2} = \frac{-2\eta}{1+\eta} = \frac{1}{2}\frac{\rho - \rho_0}{\rho_0} \qquad \text{V.12}$$

Figure 16 shows Φ, $\frac{d\Phi}{d\xi}$, and $1 + 2\frac{d^2\Phi}{d\xi^2}$ versus ξ for $A = .1, .5$ and 1.0. The small amplitude approximation $(A \ll 1)$ of V.10 is simply

$$\eta \cong \frac{\Phi}{2} = A \sin\xi \qquad \text{V.13}$$

hence A is an amplitude parameter. From V.12 and Figure 16 it is

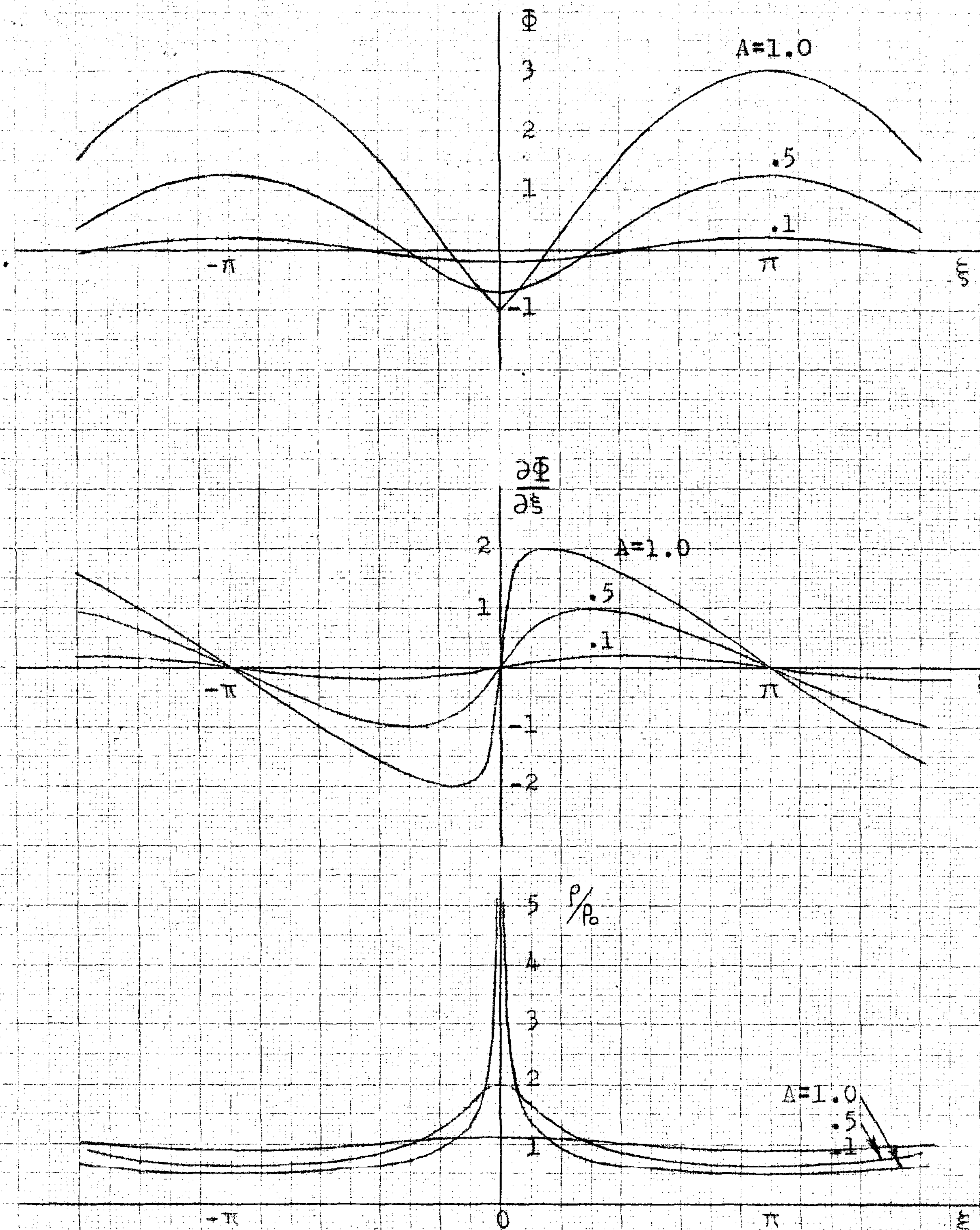

Figure 16. Normalized Potential, Electric Field, and Charge Density Waveforms of a Non-linear Plasma Oscillation

seen that the charge density becomes infinite when $\eta = -1$ $(A = 1)$. At this amplitude electrons are just beginning to overtake their neighbors. At larger amplitudes the velocity would be a multivalued function of position x'. The normalized charge density, electric field, and potential wave forms may be expanded in a Fourier series,

$$\frac{\rho}{\rho_o} = 2\frac{d^2\Phi}{d\xi^2} + 1 = 1 + a_1 A \cos\xi + a_2 A^2 \cos 2\xi + a_3 A^3 \cos 3\xi + \cdots \qquad \text{V.14}$$

$$\frac{d\Phi}{d\xi} = \frac{1}{2}\left[a_1 A \sin\xi + \frac{a_2 A^2}{2} \sin 2\xi + \frac{a_3 A^3}{3} \sin 3\xi + \cdots\right] \qquad \text{V.15}$$

$$\Phi = -\frac{1}{2}\left[a_1 A \cos\xi + \frac{a_2 A^2}{4} \cos 2\xi + \frac{a_3 A^3}{9} \cos 3\xi + \cdots\right]. \qquad \text{V.16}$$

The coefficients a_1, a_2, and a_3 are approximately equal to unity but depend somewhat on A. They are plotted as a function of A in Figure 17. Thus these simple non-linear plasma oscillations are very rich in harmonics.

Estimate of the Maximum Limiting Amplitude of Plasma Oscillations. The linear theory is useful in predicting whether a particular distribution of velocities is unstable, i.e., whether small disturbance may grow larger. In Section III the source of energy for these increasing disturbances was shown to be the kinetic energy of the directed beam. Clearly, only a limited amount of energy is available and disturbances must cease to grow when non-linear effects become important. The previous non-linear calculation shows that these effects are quite important when $\frac{2e\phi}{mV^2}$ becomes of the order unity, V is the velocity of the electrons relative to the wave.

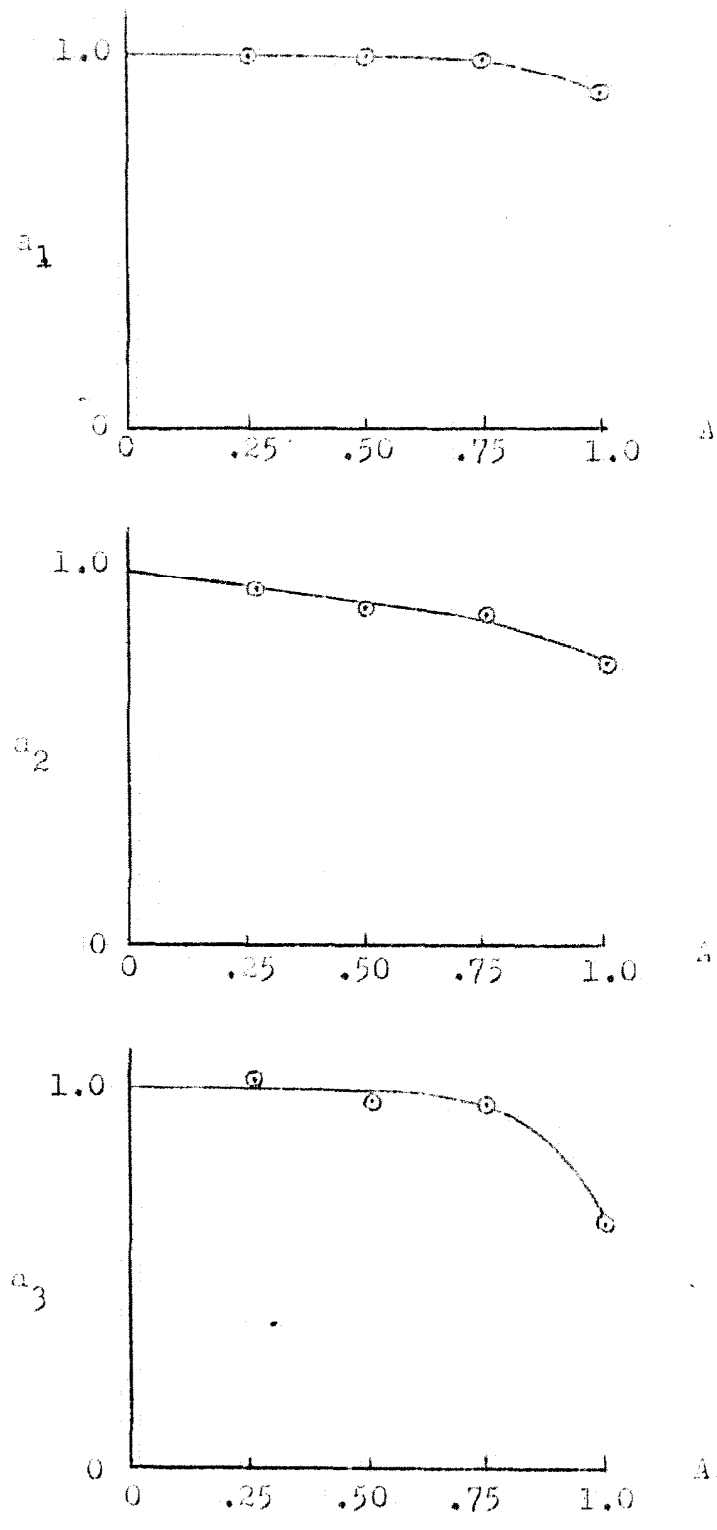

Figure 17. The Coefficients a_1, a_2, and a_3 Versus the Amplitude Parameter, A.

Using the relations of the linear theory,

$$\phi = \frac{\underline{k} \cdot \underline{E}_1}{i\,k^2} \quad , \qquad V^2 = \frac{\left|\omega - i\nu - \underline{k} \cdot \underline{u}_2\right|^2}{k^2} , \qquad \text{V.17}$$

the condition for the validity of the linear approximation can be expressed as

$$2\,\frac{e}{m}\;\frac{\underline{k} \cdot \underline{E}_1}{\left|\omega - i\nu - \underline{k} \cdot \underline{u}_2\right|^2} \ll 1 \quad . \qquad \text{V.18}$$

The growth of the disturbances may not be limited by non-linear effects until this quantity becomes of the order of unity, however. It is easily shown from the equation of motion, III.17, that the above quantity, $\frac{2e\phi}{mV^2}$, is simply twice the ratio of the a.c. velocity of the electrons, $\underline{v}_1$, to the difference between the wave velocity and the component of d.c. electron velocity in the direction of the wave. It is also equal to twice the ratio of the a.c. charge density to the d.c. charge density of the electron beam.

It will be assumed that growth of the disturbance is limited by non-linear effects when this quantity, evaluated for electrons which travel faster than the wave and hence the ones that supply energy to the disturbance, is equal to unity. It is interesting to note that limiting amplitudes of increasing waves in traveling wave tubes may be predicted with a reasonable accuracy in this manner. Because of the similarity of the situations (in one case the beam of fast electrons interacts with a slowly traveling electromagnetic wave of a helix and in the other the fast electrons interact with a traveling plasma wave of the coronal electrons) it is reasonable to apply the same considerations to determine the limiting amplitude of the increasing waves.

The space-average of the energy in the electric field of the plasma oscillation at this limiting amplitude is given by

$$\overline{W}_E = \frac{1}{4} \epsilon_o \underline{E}_1 \cdot \underline{E}_1^*$$
$$= \frac{1}{4} \frac{\omega^2}{\omega_p^2} \frac{n_2 m u_2^2}{2} \left[(1 - \frac{\underline{k} \cdot \underline{u}_2}{\omega})^2 + (\frac{\nu}{\omega})^2 \right] \frac{\underline{v}_1 \cdot \underline{v}_1^*}{u_2^2}$$
$$\approx \frac{1}{8} (\frac{\omega^2}{\omega_p^2}) \frac{n_2 m u_2^2}{2} \left| \frac{(u - u_2)^2}{u\, u_2} \right|^2 . \qquad V.19$$

$\frac{1}{2} n_2 m u_2^2$ is the kinetic energy of the directed motion of the fast electrons, $\frac{\omega^2}{\omega_p^2}$ is of the order unity. The last factor is much less than one, and can be evaluated using the results of linear theory given in Section IV. This factor is roughly fourth power difference between the complex wave velocity and the fast electron velocity divided by the electron velocity. This factor is tabulated in the Table I for $\frac{\alpha_1}{u_2} = .1$. The wavelength of the disturbance is taken as the one which increases most rapidly with time.

Table I

r	.003	.010	.030	.100	.300	1.000
$\left\lvert \frac{(u - u_2)^2}{u - u_2} \right\rvert^2$	.000074	.00046	.0026	.016	.072	.37

An approximate analytical expression for the factor given in Table I is $.4\, r^{3/2}$. Thus less than a few percent of the kinetic energy of the directed beam is transferred to energy of plasma oscillation.

Spectral Distribution of Energy. The spectral distribution of the noise energy at a time t such that the linear theory is still applicable can, in principle, be computed by assuming that initially $(t = 0)$

the disturbances which are present are simply the random fluctuations of thermal equilibrium. As shown in the preceding section, certain wavelength components of these disturbances increase very rapidly with time. At the time when the limiting amplitude is reached, the oscillation energy will be confined to a narrow range of wavelengths, and hence to a narrow range of frequencies. This can be seen as follows. Suppose there are $N(\omega)$ modes of oscillation per unit volume whose frequency (real part) is between ω and $\omega + d\omega$ and that at $t = 0$ the energy of these modes are $\epsilon(\omega)$. At a time t later the energy density spectrum is

$$W(\omega) = N(\omega)\,\epsilon(\omega)\,e^{2\nu(\omega)t} \quad . \qquad \text{V.20}$$

If ω_m is the frequency whose rate of growth is largest, $\nu(\omega)$ can be approximated by $\nu(\omega_m) + \frac{1}{2}\nu''(\omega_m)(\omega - \omega_m)^2$, where $\nu''(\omega_m)$ is negative. For large $\nu(\omega_m)t$ the energy density spectrum can be written

$$W(\omega) \approx N(\omega_m)\,\epsilon(\omega_m)e^{2\nu(\omega_m)t}\,e^{\nu''(\omega_m)(\omega-\omega_m)^2 t} \quad , \qquad \text{V.21}$$

assuming that $N(\omega)$ and $\epsilon(\omega)$ are slowly-varying functions of ω. The fractional width of the distribution

$$\frac{\Delta\omega}{\omega_m} \approx \frac{1}{\omega_m\sqrt{\nu''(\omega_m)t}} = \frac{1}{\sqrt{\nu''(\omega_m)\omega_m\,(\omega_m t)}} \qquad \text{V.22}$$

decreases with time. From Figures 12, 13 and 14 it is found that $\nu''(\omega_m)\,\omega_m$ is of the order of unity. Since the exact initial conditions are not known, the time t at which limiting amplitude is reached cannot be determined. But the limiting amplitude is so much larger than the initial amplitude, that $\omega_m t$ will be a large number, and hence the spectrum will be narrow with the maximum energy at ω_m the frequency whose growth rate is greatest.

VI COUPLING OF LONGITUDINAL PLASMA OSCILLATIONS AND TRANSVERSE ELECTROMAGNETIC WAVES

The previous sections have dealt with special aspects of longitudinal plasma oscillations. These waves represent the collective motions of the electrons and the Poynting vector vanishes identically. The disturbances which reach the earth, on the other hand, are transverse electromagnetic waves. In a uniform unbounded plasma these two wave types are completely independent, or uncoupled. The solar atmosphere is not unbounded, nor uniform. The principal deviation from uniformity is the slow steady decrease in electron density between the inner corona and interplanetary space, where the electron density can be considered to be negligible. In addition, the corona is not simply a quiet atmosphere at a high temperature, but there are many irregularities of varying size (1). There is undoubtedly a small amount of coupling between transverse and longitudinal waves which arises from the steady decline in electron density with radius in the undisturbed corona. More abrupt discontinuities in density, associated with departures from the quiet or undisturbed coronal conditions, should give rise to an even greater amount of radiation. G. B. Field (41) has, for example, computed the radiation which is generated when a plasma wave impinges on a plane interface at which the electron density drops abruptly to zero. This model is a rather drastic one to apply to the situation in the corona, but it is at least indicative of a phenomenon which is likely to be significant in accounting for the observed radiation.

In this section, the problem of radiation and coupling between wave types will be approached from a different point of view. To a certain extent it is complementary to Field's. The small density gradient which extends over great distances will be neglected, and it will be assumed that there are many small regions where the electron density is slightly higher, or slightly lower than the average. The dynamics of the density fluctuations will not be treated; they are assumed to be independent of time. Actually, the fluctuations are not steady in time but have their own dynamic character. However, when their characteristic frequencies are very much lower than electron plasma frequency, they may be considered independent of time for the analysis which follows. It is assumed that the plasma electrons have been left in a state of large amplitude oscillation by the passage of a directed beam. The presence of the directed beam is neglected in this part of the analysis, although its presence is necessary to maintain the plasma oscillations at a high level because of damping by short range collisions. The principal result of this analysis is to show that the fluctuations in density give rise to a coupling between the longitudinal plasma oscillations and the transverse electromagnetic waves. A plasma wave is scattered by an irregularity in density and some of its energy is transferred to the electromagnetic wave. The amount of scattering is evaluated when the fluctuations are random and can be characterized by a mean square fluctuation and a correlation distance.

Coupling by Density Fluctuations. The analysis begins with Maxwell's equations and what might be termed the macroscopic equations of a plasma:

$$\nabla \times \underline{E} = -\frac{\partial B}{\partial t} \qquad \text{VI.1}$$

$$\nabla \times \underline{H} = \frac{\partial D}{\partial t} + \underline{J}_t \qquad \text{VI.2}$$

$$\nabla \cdot \underline{D} = \rho_t \qquad \text{VI.3}$$

$$\nabla \cdot \underline{B} = 0 \qquad \text{VI.4}$$

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot \rho_e \underline{v} = 0 \qquad \text{VI.5}$$

$$\rho_e \frac{\partial \underline{v}}{\partial t} = -\frac{e}{m} \rho_e \underline{E} - w^2 \nabla \rho_e \qquad \text{VI.6}$$

where ρ_e is the electron charge density, ρ_t is the total charge density (electrons and ions), $\underline{J}_t$ is the total current, and $\underline{v}$ is the electron mass velocity. The second term on the right of VI.6 is a pressure term which takes into account, in an approximate manner, the convection of momentum by the electron thermal velocities. This term causes the frequency of oscillation to depend on the wavelength, and choosing w^2 equal to $\overline{u^2} = 3\kappa T/m$ gives a long wavelength dispersion relation in agreement with II.39

$$\omega^2 = \omega_p^2 + k^2 \overline{u^2} \quad . \qquad \text{VI.7}$$

This result can also be obtained from the equations of change of Chapman and Cowling (20) by assuming the electron gas to be adiabatic and noting that in a plane plasma wave only one degree of freedom of the electrons is involved, motion in the direction of the electric field, or wave vector.

The effect of convected momentum is to couple adjacent regions of the medium. Without this effect the longitudinal oscillations in a plasma of varying density would simply occur at the frequency which is characteristic of the medium at each point. If the density varied, each region would oscillate at a different frequency. The effect of the pressure term is to couple adjacent regions of space so that the oscillation has the same frequency everywhere. In this case when the density varies from point to point the wavelength can be considered to vary from point to point according to Eq. VI.7 .

Rather than attempting to obtain exact solutions of (1) - (6) for specified discontinuities, the irregularities will be considered as a perturbation from the uniform case. The irregularities in density will be characterized by an expansion parameter, δ , and only the first order corrections will be obtained. All conditions and solutions for the undisturbed conditions are denoted by a superscript zero. First order perturbations in this procedure are denoted by a superscript one. Quantities which are independent of time are denoted by a subscript zero, and a.c. or time dependent quantities are denoted by a subscript one. The need for such a notation will become clear.

The irregularities in ion density are assumed to be known. Due to their large mass, the characteristic frequencies of the ions are much lower than the frequencies of the disturbances, which are of the order of the electron plasma frequency. The electron density tends to follow the ion density and maintain approximate neutrality. The electron and ion densities may be written

$$\rho_e = \rho^o_o + \rho^o_1 + \delta\ (\rho^1_o + \rho^1_1)$$

$$\rho_i = -\rho^o_o - \delta\ \rho^1_i$$

where the ion density is taken to be independent of time. Since the electrons have no steady or d.c. velocity, the electron velocity may be written

$$\underline{v}_e = \underline{v}^o_1 + \delta\ \underline{v}^1_1\ .$$

Similarly there is no d.c. electric field when the perturbation is absent $(\delta = 0)$ but there is, in general, a d.c. field when the perturbation is present, so that

$$\underline{E} = \underline{E}^o_1 + \delta\ (\underline{E}^1_o + \underline{E}^1_1)\ .$$

When the perturbation is absent and there is no a.c. disturbance, the electron and ion densities are uniform and there are no steady electric or magnetic fields. The velocity is zero.

<u>Plasma Conditions when Perturbation is Present but with A.C. Disturbances Absent.</u> Terms of VI.3 and VI.6 of first order in δ and independent of time are,

$$\epsilon_o\ \nabla \cdot \underline{E}^1_o = -\rho^1_i + \rho^1_o \qquad \text{VI.8}$$

$$\frac{e}{m}\underline{E}^1_o\ \rho^o_o + w^2\nabla\rho^1_o = 0\ . \qquad \text{VI.9}$$

To solve VI.8 and VI.9 in terms of ρ^1_i which is assumed to be known, let $\underline{E}^1_o = -\nabla\phi^1_o$ (since $\nabla \times \underline{E}^1_o = 0$). Then VI.8 and VI.9 become

$$- \epsilon_o \nabla^2 \phi^1_o = - \rho^1_i + \rho^1_o \quad , \qquad \text{VI.10}$$

$$- \frac{e}{m} \rho^o_o \nabla \phi^1_o + w^2 \nabla \rho^1_o = 0 \quad . \qquad \text{VI.11}$$

The latter equation implies that $- \frac{e}{m} \rho^o_o \phi^1_o + w^2 \rho^1_o$ is a constant which is independent of position and which can be taken to be zero,

$$- \frac{e}{m} \rho_o \phi^1_o + w^2 \rho^1_o = 0 \ . \qquad \text{VI.12}$$

Combining VI.10 and VI.12 the following equation for ϕ^1_o is obtained:

$$\nabla^2 \phi^1_o - \frac{\omega_{po}^2}{w^2} \phi^1_o = \frac{\rho^1_i}{\epsilon_o} \qquad \text{VI.13}$$

where $\omega^2_{po} = \frac{-\rho^o_o}{\epsilon_o} \frac{e}{m}$.

This equation gives rise to a shielded coulomb field,

$$\phi^1_o = - \frac{1}{4\pi\epsilon_o} \int \frac{\rho^1_i (\underline{r}') \, e^{- \frac{\omega_{po}}{w} |\underline{r} - \underline{r}'|}}{|\underline{r} - \underline{r}'|} \, dV' \qquad \text{VI.14}$$

with a shielding length $\frac{w}{\omega_{po}} = \ell_D$, a result which is already known from the Debye-Huckle theory of electrolytes.

Thus when the perturbation in ion charge density ρ^1_i is known the potential ϕ^1_o, the electric field $\underline{E}^1_o$, and the electron charge density ρ^1_o, may be easily obtained. One of the consequences of the "shielding" effect is that, although ρ^1_i may have rapid fluctuations in space, ϕ^1_o cannot. ϕ^1_o has virtually no fluctuations of wavelength less than the shielding length ℓ_D. Furthermore, it is easily shown that ρ^1_o is related to ϕ^1_o in the following manner:

$$\rho^1_o = \frac{\epsilon_o}{\ell^2_D} \phi^1_o \quad . \qquad \text{VI.15}$$

Thus ρ^1_o also has virtually no irregularities of wavelength less than ℓ_D.

A.C. Disturbances in the Absence of Irregularities in Density.

The parts of Eqs (1)-(7) which are time dependent but of zero order in δ are

$$\nabla \times \underline{E}^o_1 = - \frac{\partial \underline{B}^o_1}{\partial t} \qquad \text{VI.16}$$

$$\nabla \times \underline{H}^o_1 = \underline{J}^o_1 + \frac{\partial \underline{D}^o_1}{\partial t} \qquad \text{VI.17}$$

$$\rho^o_o \frac{\partial \underline{v}^o_1}{\partial t} = - \frac{e}{m} \rho^o_o \underline{E}^o_1 - w^2 \nabla \rho^o_1 \qquad \text{VI.18}$$

$$\nabla \cdot \underline{B}^o_1 = 0 \qquad \text{VI.19}$$

$$\nabla \cdot \underline{D}^o_1 = \rho^o_1 \qquad \text{VI.20}$$

$$\frac{\partial \rho^o_1}{\partial t} + \nabla \cdot (\rho^o_o \underline{v}^o_1) = 0 \quad . \qquad \text{VI.21}$$

In addition, the current is given by $\underline{J}^o_1 = \rho^o_o \underline{v}^o_1$. The solution of equations VI.16 through VI.21 are conveniently discussed in terms of the vector and scalar potentials $\underline{A}^o_1$ and ϕ^o_1. In view of VI.19 the magnetic field can be derived from a vector potential whose divergence is zero

$$\underline{B}^o_1 = \nabla \times \underline{A}^o_1 \qquad \nabla \cdot \underline{A}^o_1 = 0 \qquad \text{VI.22}$$

and in view of VI.16 the electric field differs from $- \frac{\partial \underline{A}^o_1}{\partial t}$ by the gradient of a scalar

$$\underline{E}^o_1 = - \frac{\partial \underline{A}^o_1}{\partial t} - \nabla \phi^o_1 \quad . \tag{VI.23}$$

Substituting VI.23 into VI.20 gives

$$- \epsilon_o \nabla^2 \phi^o_1 = \rho^o_1 \quad .$$

Differentiating with respect to time and using VI.21 results in

$$- \epsilon_o \frac{\partial}{\partial t} \nabla^2 \phi^o_1 = \frac{\partial \rho^o_1}{\partial t} = - \nabla \cdot (\rho^o_o \underline{v}^o_1) \quad .$$

Differentiating with respect again to time and using VI.18 and VI.20 results in

$$- \epsilon_o \frac{\partial^2}{\partial t^2} \nabla^2 \phi^o_1 = - \nabla \cdot (\rho^o_o \frac{\partial \underline{v}^o_1}{\partial t}) = - \nabla \cdot \left[- \frac{e}{m} \rho^o_o \underline{E}^o_1 - w^2 \nabla \rho^o_1 \right]$$

$$= \rho_o \frac{e}{m} \nabla^2 \phi^o_1 + w^2 \nabla^2 \rho^o_1$$

$$= \rho_o \frac{e}{m} \nabla^2 \phi^o_1 + w^2 \nabla^2 (- \epsilon_o \nabla^2 \phi^o_1) \quad .$$

Thus ϕ^o_1 must satisfy the following equation

$$(\frac{\partial^2}{\partial t^2} + \omega^2_{po} - w^2 \nabla^2) \nabla^2 \phi^o_1 = 0 \quad . \tag{VI.24}$$

VI.24 is satisfied if

$$\nabla^2 \phi^o_1 = 0 \tag{VI.25}$$

or if

$$(\frac{\partial^2}{\partial t^2} + \omega^2_{po} - w^2 \nabla^2) \phi^o_1 = 0 \quad . \tag{VI.26}$$

The equation satisfied by $\underline{A}^o_1$ may be derived in a similar manner.

VI.17 may be written

$$\nabla \times (\nabla \times \underline{A}_1^o) = \mu_o \rho_o^o \underline{v}_1^o + \mu_o \epsilon_o \frac{\partial \underline{E}_1^o}{\partial t} .$$

Differentiating with respect to time and using VI.18

$$\frac{\partial}{\partial t} \nabla \times (\nabla \times \underline{A}_1^o) = \mu_o \left[-\frac{e}{m} \rho_o^o \underline{E}_1^o - w^2 \nabla \rho_1^o \right] + \mu_o \epsilon_o \frac{\partial^2 \underline{E}_1^o}{\partial t^2} .$$

Using VI.20 and VI.23, this may be written in the form

$$\frac{\partial}{\partial t} \left[-\nabla^2 \underline{A}_1^o + \mu_o \epsilon_o \left(\frac{\partial^2}{\partial t^2} + \omega_{po}^2\right) \underline{A}_1^o \right] = -\mu_o \epsilon_o \left[\frac{\partial^2 \phi_1^o}{\partial t^2} + \omega_{po}^2 \phi_1^o - w^2 \nabla^2 \phi_1^o \right] . \qquad \text{VI.27}$$

Solutions of VI.25 are not required when there is no free charge present in the plasma. An analogous situation occurs in the electrodynamics of free space,(21). Hence only solutions to VI.26 and VI.27 are required. Notice that these solutions are independent, i.e., $\underline{A}_1^o$ does not enter into the equation for ϕ_1^o, and any ϕ_1^o which is a solution of VI.26 makes the right side of the equation for $\underline{A}_1^o$ equal to zero. VI.27 may therefore be written

$$\nabla^2 \underline{A}_1^o - \mu_o \epsilon_o \left(\frac{\partial^2}{\partial t^2} + \omega_{po}^2\right) \underline{A}_1^o = 0 . \qquad \text{VI.28}$$

The solutions of VI.26 are the longitudinal plasma waves or oscillations of the medium and the solutions of VI.28 are the transverse electromagnetic waves of the medium. The properties of these waves have been discussed in some detail by others. It is sufficient to remark here that VI.28 gives propagating electromagnetic waves only when the frequency (of a sinusoidal wave) is greater than ω_{po} . The sinusoidal solutions of VI.28 are the same as for the ordinary electromagnetic wave equation in a medium of dielectric constant $\epsilon_o(1 - \frac{\omega^2_{po}}{\omega^2})$.

VI.26 is valid only for wavelengths which are long compared to the screening wavelength (long wavelength approximation) which occurs when the frequency is close to ω_{po} . For plane waves of sinusoidal time dependence, $e^{i(\omega t - \underline{k} \cdot \underline{r})}$, one obtains the frequency-wave number relationship

$$\omega^2 = \omega^2_{po} + w^2 k^2 \quad . \qquad \text{VI.29}$$

Only the frequency range $.8\,\omega_{po} < \omega < 1.2\,\omega_{po}$, where the long wavelength approximation is valid, is of interest.

Effect of Density Irregularities on the Sinusoidal Fields. The effect of the perturbation on the a.c. fields will now be considered. The a.c. parts of VI.1 through VI.6 which are first order in δ are,

$$\nabla \times \underline{E}^1_1 = -\frac{\partial \underline{B}^1_1}{\partial t} \qquad \text{VI.30}$$

$$\nabla \times \underline{H}^1_1 = \underline{J}^1_1 + \frac{\partial \underline{D}^1_1}{\partial t} \qquad \text{VI.31}$$

$$\rho^o_o \frac{\partial \underline{v}^1_1}{\partial t} + \rho^1_o \frac{\partial \underline{v}^o_1}{\partial t} = \frac{e}{m}\left[-\rho^o_o \underline{E}^1_1 - \rho^1_o \underline{E}^o_1 - \rho^o_1 \underline{E}^1_o\right] + w^2 \nabla \rho^1_1 \qquad \text{VI.32}$$

$$\nabla \cdot \underline{B}^1_1 = 0 \qquad \text{VI.33}$$

$$\nabla \cdot \underline{D}^1_1 = \rho^1_1 \qquad \text{VI.34}$$

$$\frac{\partial \rho^1_1}{\partial t} + \nabla \cdot \left[\rho^o_o \underline{v}^1_1 + \rho^1_o \underline{v}^o_1\right] = 0 \quad . \qquad \text{VI.35}$$

These equations are very similar to equations VI.16 through VI.21 except that several new terms appear, each of which is a product of a time independent factor associated with the perturbation and a time dependent factor associated with the free wave solutions of VI.26 or VI.28. These new terms will act as <u>source terms</u> in the equations for ϕ^1_1 and $\underline{A}^1_1$. Proceeding as before and letting $\underline{B}^1_1 = \nabla \times \underline{A}^1_1$, $\underline{E}^1_1 = -\frac{\partial \underline{A}^1_1}{\partial t} - \nabla \phi^1_1$ and

$$(1 - w^2 \mu_o \epsilon_o) \frac{\partial}{\partial t} \nabla \cdot A^1_1 + \mu_o \epsilon_o \left[\frac{\partial^2}{\partial t^2} + \omega^2_{po} - w^2 \nabla^2\right] \phi^1_1 = 0^{\dagger} \qquad \text{VI.36}$$

the following equations for $\underline{A}^1_1$ and ϕ_1 are obtained:

$$\frac{\partial}{\partial t}\left[-\nabla^2 \underline{A}^1_1 + \mu_o \epsilon_o \left(\frac{\partial^2 \underline{A}^1_1}{\partial t^2} + \omega^2_{po} \underline{A}^1_1\right)\right] = -\mu_o \frac{e}{m}\left[\rho^1_o \underline{E}^o_1 + \rho^o_1 \underline{E}^1_o\right] \qquad \text{VI.37}$$

$$\left[\frac{\partial^2}{\partial t^2} + \omega^2_{po} - w^2 \nabla^2\right]\left[\frac{\nabla^2 - \mu_o \epsilon_o \left(\frac{\partial^2}{\partial t^2} + \omega^2_{po}\right)}{1 - w^2 \mu_o \epsilon_o}\right] \phi^1_1 = \omega^2_{po} \nabla \cdot \left[\frac{\rho^1_o}{\rho^o_o} \underline{E}^o_1 + \frac{\rho^o_1}{\rho^o_o} \underline{E}^1_o\right]. \qquad \text{VI.38}$$

Since $w^2 \mu_o \epsilon_o \sim 10^{-3}$ in the solar corona, this quantity can be neglected in comparison with unity.

Thus it is seen that the effect of the discontinuities is to produce source terms which are equivalent to a charge density, ρ_1,

$$\frac{\partial^2}{\partial t^2} \rho_1 = \frac{e}{m} \nabla \cdot \left[\rho^1_o \underline{E}^o_1 + \rho^o_1 \underline{E}^1_o\right] \qquad \text{VI.39}$$

and a current density, $\underline{J}^*_1$,

$$\frac{\partial \underline{J}^*_1}{\partial t} = -\frac{e}{m}\left[\rho^1_o \underline{E}^o_1 + \rho^o_1 \underline{E}^1_o\right] \qquad \text{VI.40}$$

---

† It is assumed that $\nabla \cdot \underline{A}^1_1$ be related to ϕ^1_1 in this manner rather than taking $\nabla \cdot \underline{A}^1_1$ to be zero. This amounts to a choice of the gauge.

From VI.39 and VI.40 it is seen that

$$\frac{\partial}{\partial t}\left(\frac{\partial \rho_1^*}{\partial t} + \nabla \cdot J_1^*\right) = 0$$

i.e., that the equivalent source current density and charge density satisfy the continuity equation.

When the time dependence is sinusoidal, Eq. VI.38 may be solved with the aid of the scalar Green's function

$$G_I(\underline{r},\underline{r}') = \frac{\omega^2}{\omega^2 - \omega_{po}^2} \; \frac{e^{-i\sqrt{\frac{\omega^2-\omega_{po}^2}{c}}|\underline{r}-\underline{r}'|} - e^{-i\sqrt{\frac{\omega^2-\omega_{po}^2}{W}}|\underline{r}-\underline{r}'|}\, e^{i\omega t}}{4\pi\,|\underline{r}-\underline{r}'|} \quad \text{VI.41}$$

The first term gives the modified longitudinal field and the second term gives an outward traveling spherical plasma wave. VI.37 may be solved with the aid of the diadic Green's function

$$G_{II}(\underline{r},\underline{r}') = (\underline{ee})\,\frac{1}{i\omega}\; \frac{e^{-i\sqrt{\frac{\omega^2-\omega_{po}^2}{c}}\,|\underline{r}-\underline{r}'|}}{4\,\pi\,|\underline{r}-\underline{r}'|}\; e^{i\omega t} \quad \text{VI.42}$$

where $(\underline{ee})$ is the unit diadic. This part of the solution gives outward traveling electromagnetic waves.

<u>Scattering of Plasma Waves and the Generation of Electromagnetic Waves by Irregularities in Density.</u> The preceding theory will now be used to calculate the energy scattered out of a plane plasma wave into electromagnetic radiation. The plasma wave is assumed to propagate in the z direction, as shown in Figure 18. The electric field is given by

$$\underline{E}_1^o = E_1^o\,\underline{e}_z\, e^{i(\omega t - \underline{k}\cdot\underline{r})} \quad \text{VI.43}$$

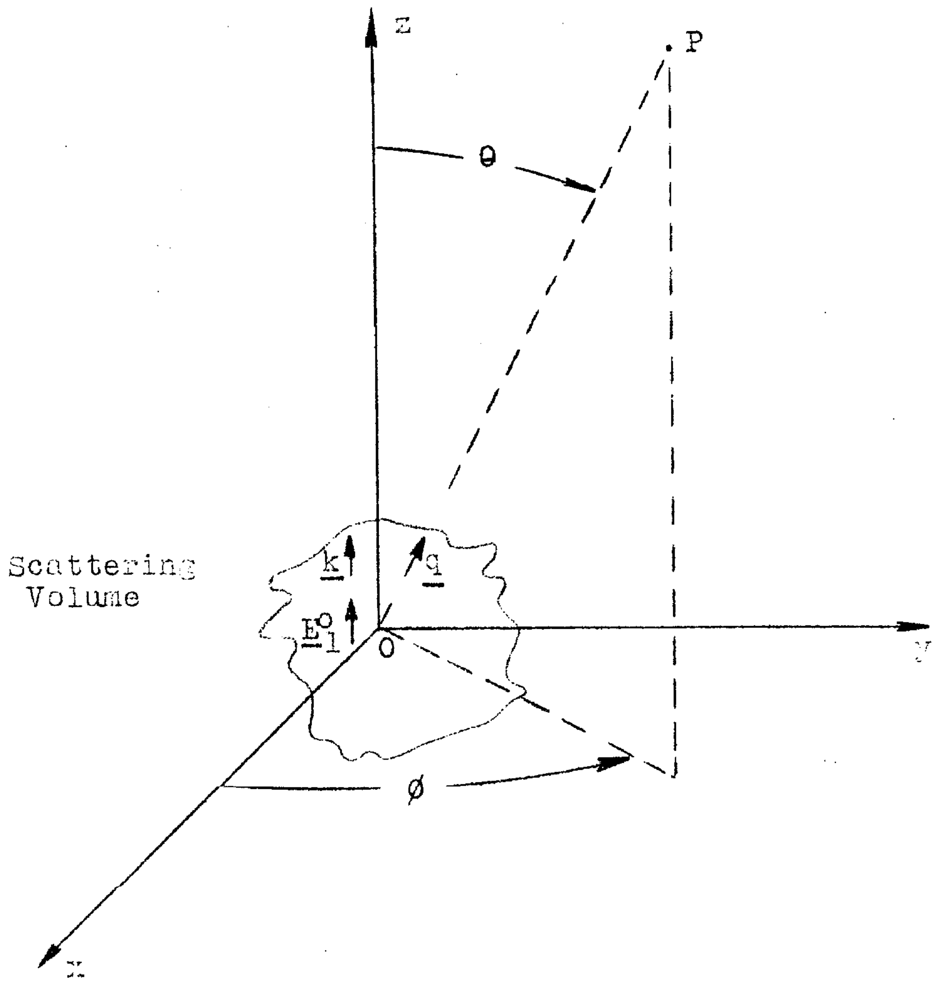

Figure 13. Radiation from Plasma Irregularities

where $k_z^2 = \frac{\omega^2 - \omega_{po}^2}{W^2}$, $k_x = k_y = 0$.

The point of observation, P, is assumed to be a large distance away from the scattering region in the vicinity of the origin. The vector potential $\underline{A}_1^1$ may be written

$$\underline{A}_1^1 = \frac{\mu_o e}{i\omega m} e^{i\omega t} \int_V (\underline{e}\underline{e}) \cdot \left[\rho_o^1 \underline{E}_1^o + \rho_1^o \underline{E}_o^1\right] \frac{e^{-i\frac{\sqrt{\omega^2 - \omega_{po}^2}}{c}|\underline{r} - \underline{r}'|}}{4\pi |\underline{r} - \underline{r}'|} dx'dy'dz'. \qquad \text{VI.44}$$

Letting $\underline{q}$ be a vector in the direction OP with magnitude $\omega\sqrt{\mu_o \epsilon} = \frac{\sqrt{\omega^2 - \omega_{po}^2}}{c}$ and introducing the vector to the source-point $\underline{r}'$, the above equation can be written

$$\underline{A}_1^1 = \frac{\mu_o e}{i\omega m} \frac{e^{i(\omega t - \underline{q} \cdot \underline{r})}}{4\pi r} \int_V (ee) \cdot \left[\rho_o^1 \underline{E}_1^o + \rho_1^o \underline{E}_o^1\right] e^{-i\underline{q} \cdot \underline{r}'} dx'dy'dz' \qquad \text{VI.45}$$

at large distances from the scattering volume. The contribution from the second term in the square brackets can be neglected in comparison with the first when the plasma wavelength is long compared to the screening length. Under these conditions, the integral is a vector in the z direction since $\underline{E}_1^o$ is in the z direction. Thus $\underline{A}_1^1$ is in the z direction.

The time average of the radial component of the Poynting vector is

$$\overline{\Pi} = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu_o}} \sin^2\theta \, \omega^2 \underline{A}_1^1 \cdot \underline{A}_1^{1*} \qquad \text{VI.46}$$

(ϕ_1^1 does not contribute to the radial Poynting vector.) Substituting VI.45 into VI.46 with the indicated approximation, VI.46 becomes

$$\overline{II} = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu_o}} \sin^2\theta \frac{\omega_{po}^4 \mu_o^2 \epsilon_o^2}{16\pi^2 R^2} (\underline{E}_1^o \cdot \underline{E}_1^{o*})$$

$$\iint \frac{\rho'_o(\underline{r}')}{\rho_o^o} e^{-i(\underline{k}+\underline{q})\cdot(\underline{r}'-\underline{r}'')} \frac{\rho'_o(\underline{r}'')}{\rho_o^o} d\,\underline{r}'\, d\,\underline{r}'' . \qquad \text{VI.47}$$

The expected value of II can be evaluated using the correlation function of the density fluctuations, if the correlation function becomes small in a distance which is small compared to a dimension of the region of integration, V . The volume V is assumed to be finite either because the density fluctuations are limited in extent or as is more likely, because the plasma wave itself is present only in a limited region of space. The correlation function $\Psi(\tau)$ is defined by

$$\Psi(\tau) = \frac{1}{(\rho_o^o)^2}\langle\rho(\underline{r})\ \rho(\underline{r}+\underline{\tau})\rangle = \lim_{V\to\infty} \frac{1}{V}\int_V \frac{\rho(\underline{r})\ \rho(\underline{r}+\underline{\tau})}{(\rho_o^o)^2}\, d\,\underline{r} . \qquad \text{VI.48}$$

VI.47 may then be written

$$\langle II\rangle = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu_o}} \sin^2\theta \frac{\omega_{po}^4 \mu_o^2 \epsilon_o^2}{16\pi R^2} E_1^o\ E_1^{o*}\ V\int_V \Psi(\tau)\, e^{-i\underline{k}\cdot\underline{r}}\, e^{-i\underline{q}\cdot\underline{r}}\, d\,\underline{\tau} \qquad \text{VI.49}$$

where $\tau^2 = x^2 + y^2 + z^2$, and Ψ is assumed to depend only on the magnitude of $\underline{\tau}$.

Assuming that $\Psi(\tau)$ vanishes sufficiently rapidly at infinity, the integration can be extended to the infinite domain. Converting to spherical coordinates, in which the polar axis is parallel to the vector $\underline{q} + \underline{k}$, the integral appearing in VI.49 can be written

$$I = \int_0^\infty \int_0^\pi \Psi(\tau)\, e^{-i|\underline{q}+\underline{k}|\tau\cos\theta}\, 2\pi\tau^2\, d\tau \sin\theta\, d\theta \qquad \text{VI.50}$$

$$= \int_0^\infty \Psi(\tau)\, \frac{2\pi}{|\underline{k}+\underline{q}|}\, \tau \sin(|\underline{k}+\underline{q}|\tau)\, d\tau .$$

To evaluate this integral the density correlation function must be known. As a <u>lower</u> limit, the density fluctuations in thermal equilibrium may be used. In Appendix II it is shown that the density correlation function is

$$\Psi(\tau) = \frac{1}{8\pi n_o \ell_D^3}\, e^{-\tau/\ell_D} . \qquad \text{VI.51}$$

It is probably not a very good approximation to assume that the ions are in equilibrium after the passage of a directed beam of charged particles since low frequency ion oscillations may be excited and the density fluctuations may be greatly enhanced. At least one feature of VI.51 is qualitatively correct, however, the maximum distance over which correlation exists is roughly the Debye wavelength ℓ_D. The amplitude of the fluctuations may be much greater. It should be noted from VI.50 that only density fluctuations whose mean spacing is greater than about $\frac{\pi}{|\underline{k}+\underline{q}|} \approx \frac{\pi}{|\underline{k}|}$ contribute appreciably to radiation.

Substituting correlation function VI.51 into VI.50 results in

$$I = \frac{1}{2n_o}\, \frac{1}{\left[1 + |k_z\, \underline{e}_z + \underline{q}|^2 \ell_d^2\right]^2} . \qquad \text{VI.52}$$

The magnitude of $\underline{q}$ is small compared with the magnitude of $\underline{k}$ and $(k_z \ell_d)^2$ is small compared with unity. Hence the second factor of VI.52

is approximately unity.

The expected value of the Poynting vector at a large distance is given by

$$\langle \Pi \rangle = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu_o}} \sin^2\theta \; \frac{\omega^4_{po}\,\mu^4_o \epsilon^2_o}{16\pi R^2} \, \underline{E}^o_1 \cdot \underline{E}^{o*}_1 \, V \frac{1}{2n_o} \quad . \qquad \text{VI.53}$$

Introducing the following definitions,

$$\omega_{po}\sqrt{\mu_o \epsilon_o} = \frac{1}{\lambda_o} \qquad\qquad W = \epsilon_o \frac{\underline{E}^o_1 \cdot \underline{E}^{o*}_1}{4} V$$

and integrating the Poynting vector over a large sphere, the total radiated power is found to be

$$P = \frac{1}{6}\sqrt{\frac{\epsilon}{\epsilon_o}} \; \frac{1}{\lambda^4_o} \; \frac{c}{n_o} \; W \quad . \qquad \text{VI.54}$$

Noting that W is the total energy stored in the electric field of the plasma oscillation, and very nearly equal to the total energy of the oscillation†, it is seen that P/W is just the reciprocal of the time required for the oscillation energy to decrease by a factor $1/e$ due to the radiated energy when other sources of damping are absent. At a point in the corona where the plasma frequency is 100 mc, λ_o is about 50 cm and n_o is about $.4 \times 10^8$ cm^{-3} . If the oscillation frequency is $1.10\omega_{po}$, for which $\sqrt{\frac{\epsilon}{\epsilon_o}} \approx .42$, it is found that

† When the frequency of the plasma oscillation is ω_{po} , the potential energy is $1/4 \; \epsilon_o \underline{E}^o_1 \underline{E}^{o*}_1$ which is also equal to the kinetic energy of the oscillation. When the oscillation takes place at a slightly higher frequency, there is an additional potential energy term $\frac{1}{4}\frac{m}{e} w^2 \frac{1}{\rho^o_o} \rho^o_1 \rho^{o*}_1$, which may be regarded as the internal energy of the electron gas. It is $(k_z \ell_D)^2$ times the electric energy and hence praccally negligible.

$$\frac{1}{T} = \frac{P}{W} \approx 10^{-5}\ \text{sec}^{-1} \qquad \text{VI.55}$$

Damping of the plasma oscillations by short range collisions has been neglected. Since the collision damping time is of the order of a second, most of the energy in plasma oscillations will actually be dissipated by this mechanism but the fraction, $\frac{\nu_c}{T} \approx 10^{-5}$, of the energy is radiated in the form of transverse electromagnetic waves.

Propagation of Electromagnetic Waves in the Solar Atmosphere and their Escape into Space. Propagation of electromagnetic waves in the solar atmosphere has been adequately treated in the literature and only a few essentials are reviewed here in order that they may be applied to the present problem. The electric susceptibility for transverse waves is given by II.17. The corresponding complex index of refraction is

$$\sqrt{1 + \chi_t} = \sqrt{1 - \frac{\omega_p^{\,2}}{\omega(\omega - i\nu_c)}} \qquad \text{VI.56}$$

Since the change in the refractive index in an electromagnetic wavelength is generally very small the ray concept of electromagnetic disturbances is useful in studying the propagation characteristics of the solar atmosphere. Actual ray paths and absorbtion characteristics of transverse electromagnetic waves in the solar atmosphere have been computed by Young (3) and by Jaeger and Westfold (42). To get a qualitative idea of the ray paths, consider the solar atmosphere to be a plane stratified medium of varying index of refraction and neglect the absorbtion due to short-range collisions. The angle, θ, which a ray makes with a vector which is perpendicular to the planes of equal refractive index can be found from Snell's law

$$n \sin \theta = n_o \sin \theta_o , \qquad \text{VI.57}$$

where n is the refractive index at the point of interest and n_o and θ_o are the index of refraction and the angle which the ray makes with the normal to planes of constant refractive index at a reference point along the ray path.

Since the refractive index is less than one in the solar atmosphere and it increases monotonically to unity with increasing distance from the center of the sun, rays are bent toward the normal in traveling outward. If the reference point is taken to be the point of origin of the electromagnetic wave and the point of interest is far enough from the sun that the refractive index is substantially unity, the angle θ is given by

$$\sin \theta = n_o \sin \theta_o . \qquad \text{VI.58}$$

Typical paths are illustrated in Figure 19. The maximum value of θ is obtained when $\theta_o = \frac{\pi}{2}$,

$$\sin \theta_m = n_o < 1 . \qquad \text{VI.59}$$

When the disturbance originates just above the critical layer for the frequency of the disturbance (the critical layer is the layer at which the plasma frequency is equal to the frequency of the disturbance) n_o is small and hence θ_m is small. At large distances from the sun the disturbance is confined to cone of half angle, θ_m, whose axis of symmetry is a solar radius passing through the point of origin of the disturbance. More accurate calculations, taking into account the spherical nature of surfaces of constant refractive index by Smerd, cited by Wild et al (11), give

$$\sin .87\,\theta_m = n_o \quad . \qquad \text{VI.60}$$

Thus the half angle of the escape cone is a little larger than given by the simple theory. Figure 20 shows the half angle of the cone into which radiation escapes as a function of the ratio of the frequency of the radiation to the critical frequency at the point where the radiation originates.

VI.54 shows that the energy radiated is proportional to $\sqrt{\epsilon/\epsilon_o}$ or to the refractive index, n_o, at the point where the radiation originates. Thus the radiation efficiency is small when the frequency of radiation is only slightly greater than the critical frequency of the point where the radiation originates. This is more than offset by the focusing effect of the varying refractive index, however. The radiation escapes only into a solid angle,

$$= 2\pi(1 - \cos\theta_m)$$

$$\cong 1.32\pi\, n_o^2 \qquad n_o^2 \ll 1 \quad .$$

The ratio of the energy flux of a distant point in the escape cone to the flux which would exist at that point if the same currents (VI.40) were to radiate in free space is approximately

$$n_o \cdot \frac{4\pi}{1.32\pi\, n_o^2} = \frac{3.03}{n_o} \qquad (n_o^2 \ll 1) \quad .$$

Thus, although the total energy radiated is smaller when the frequency is only slightly greater than the local critical frequency, this energy is radiated into a small solid angle and the energy radiated per unit

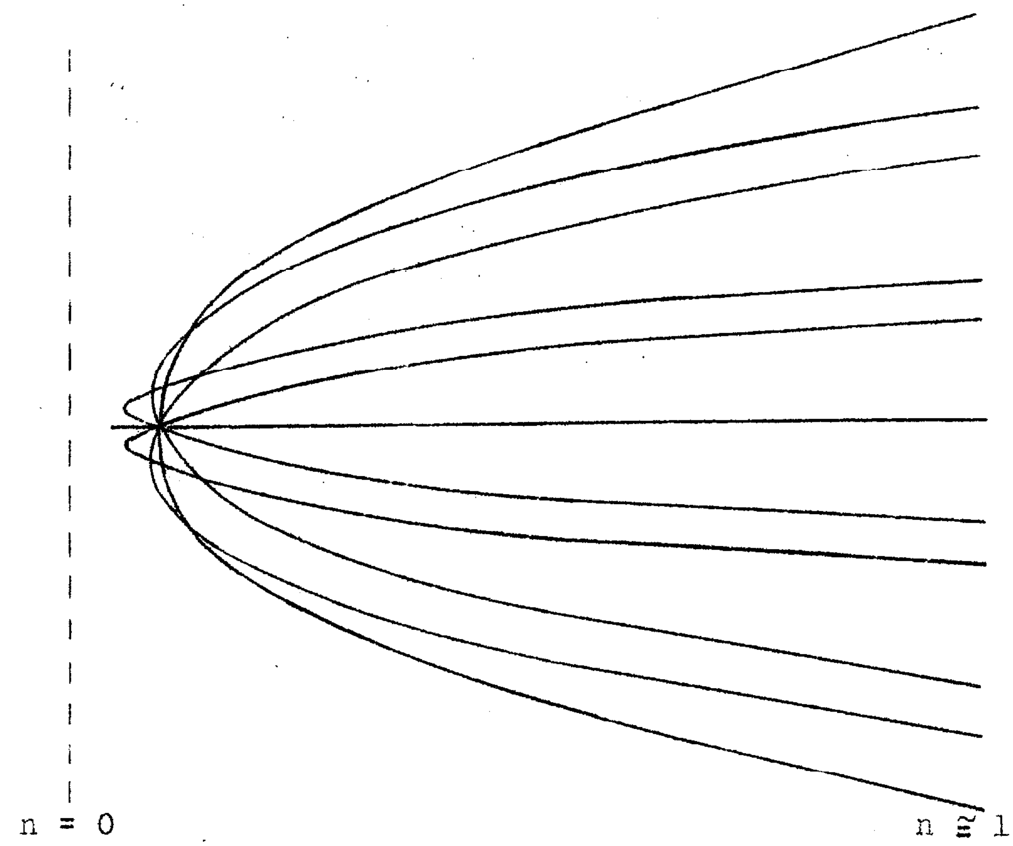

Figure 19. Typical Ray Paths in a Plane Stratified Medium whose Refractive Index Increases from Zero to Unity

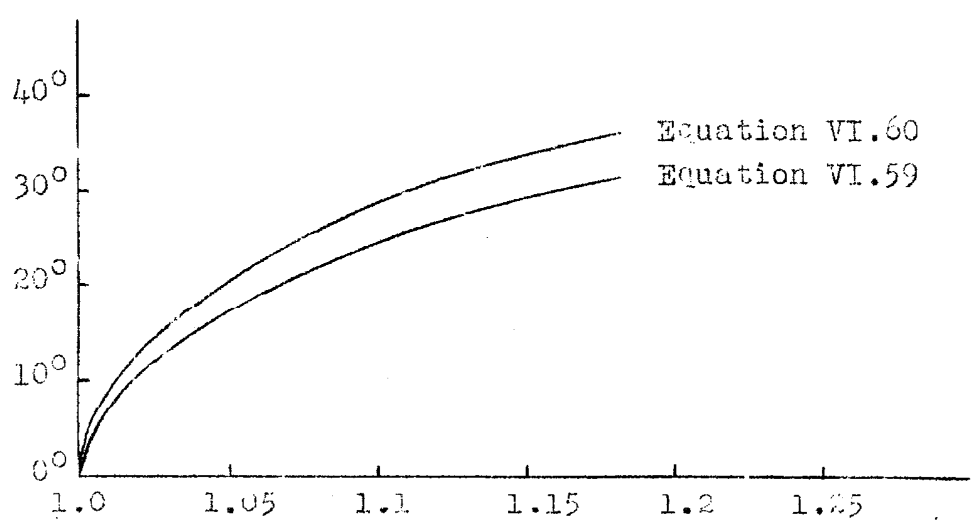

Figure 20. Half Angle of the Escape Cone. Abscissa is the Ratio of the Frequency of the Disturbance to the Plasma Frequency at the Source.

solid angle is actually increased.

Absorption of electromagnetic energy along the ray path must also be taken into account. The fraction of the energy which escapes is given by $e^{-\tau}$ where τ is the "optical depth" at the point where the radiation originates and is related to the imaginary part of the refractive index, n_i, through the relation

$$\tau = 2 \int_{s_o}^{\infty} n_i \frac{\omega}{c} \, ds \quad . \qquad \text{VI.61}$$

The integration is along a ray, and s_o is the origin of the radiation. The optical depth of the point where the real part of the refractive index vanishes given by Young is approximately .4 , 2.0 , 1.0 , and 20 at a frequency of 30, 100, 300, and 1000 megacycles per second respectively. Thus the absorption of electromagnetic energy is not very great. However, the optical depths of the same points are much smaller at twice the local plasma frequency so that second harmonic radiation would hardly be absorbed at all. It has been suggested (11), (38) that this differential absorption may account for the fact that the second harmonic radiation is about as intense as the fundamental radiation.

VII SUMMARY AND CONCLUSIONS

In the preceding sections several special aspects of the theory of plasma oscillations in the solar corona have been explored. It has been shown that the longitudinal plasma oscillations are independent of the transverse electromagnetic waves in a uniform plasma, but that the two wave types can be coupled by deviations from the uniform conditions. The conditions under which a directed beam of charged particles can excite the longitudinal plasma oscillations has been examined in detail by considering the energy exchange process. It is shown that there are evanescent plasma waves, which do not represent an excitation of plasma oscillations, but which have previously been interpreted in this way. Qualitative non-linear arguments are used to estimate the maximum amplitude which the plasma oscillations can attain. Prior to the attainment of this limiting amplitude it is shown that the frequency spectrum of the oscillations is nearly Gaussian, but it has not been possible to estimate the actual width of the spectrum. In the remaining few paragraphs some of the observations of Type II and Type III bursts will be considered in the light of these results.

Type II (slow) Bursts. The results of Section IV indicate quite clearly that slow charged particles do not excite coronal plasma oscillations appreciably. While this does not rule out the possibility that the Type II bursts involve a plasma oscillation phenomenon, the excitation mechanism is not the double stream process. If there are other, as yet undiscovered, excitation mechanisms it is possible that the radiation process suggested in Section VI may be important in accounting for the observed electromagnetic radiation.

Type III (fast) Bursts. In Section VI it was noted that only electromagnetic radiation of frequency greater than the local electron plasma frequency can escape from the solar atmosphere. Thus in applying the results of Section IV it is important to find conditions where appreciable noise energy of frequency greater than the local plasma frequency is produced. It was noted that in all cases the disturbance which increases most rapidly always has a frequency (real part) which is less than the plasma frequency. However, for a wide variety of densities of the fast particles disturbances whose frequency is above the local plasma frequency increase almost as rapidly as those which increase most rapidly. According to the argument of Section V, when the limiting amplitude is reached in a relatively short time, the spectrum may still be rather broad. Assuming that at later times the spectrum does not differ significantly from the spectrum which exists at the time limiting amplitude is reached, except for the presence of second and higher harmonics, there may be appreciable energy above the critical frequency. The time required to reach limiting amplitude, and hence the spectrum width, has not been determined. One of the principal factors which makes this difficult to estimate is the lack of knowledge of the initial conditions and boundary conditions. It should be possible, however, to make such an estimate by devising a very specific model of process in which the boundary conditions are known and the initial disturbances are the random fluctuations present in thermal equilibrium. This would seem to be worthy of further investigation.

To see that an explanation of Type III bursts in terms of the excitation of plasma oscillations by fast electrons is plausible from the energy standpoint, the total electromagnetic energy radiated may be estimated. At the present status of the theory the results of computations on the dispersion relation given in Section IV and Appendix I suggest that the noise amplification can take place for a variety of densities of fast particles and does not suggest that a particular density might be required to explain the observed bursts. Hence the electron density of the directed beam is not known but it may be written in terms of the dimensionless variable r as $10^8\ r\ \mathrm{cm}^{-3}$ and its mean velocity is about $10^{10}\ \mathrm{cm\ sec}^{-1}$. Assuming that the volume of the disturbed region is $10^{27}\ \mathrm{cm}^3$ (1), the kinetic energy of the electrons is

$$n_2 V(\frac{1}{2} m u_2^2) = 4.6 \times 10^{27}\ r\ \text{erg} .$$

In Section V it was shown that only a fraction, $.05\ r^{3/2}$, of this energy may be converted to energy of plasma oscillations and in Section VI a radiation process which radiates about 10^{-5} of the energy of plasma oscillations was described. Hence the total radiated energy is about

$$W = .2 \times 10^{22}\ r^{5/2}\ \text{erg} = .2 \times 10^{15}\ r^{5/2}\ \text{joule} .$$

The above radiation efficiency is actually an optimistic estimate since frequencies below the local plasma frequency produce no radiation at all. To estimate the energy flux at the earth it may be assumed that the electromagnetic energy is radiated uniformly into a small solid angle, of the order of a half a steradian. The total energy falling on a square meter at the earth, assuming the earth to be within the escape cone, is approximately

$$\frac{W}{\Omega R^2_{\substack{sun-\\earth}}} \approx \frac{.2 \times 10^{15}\, r^{5/2}}{.5 \times 2 \times 10^{22}} \approx .2 \times 10^{-7}\, r^{5/2} \text{ joule } m^{-2} \ .$$

Wild et al (11) observe a flux of about 10^{-20} watts m^{-2} $(cps)^{-1}$ over a bandwidth of about 3×10^{7} cps for a period of about 15 seconds. The total energy received per square meter is therefore about $.45 \times 10^{-11}$ joule m^{-2}. Taking r to be about .03 produces about the right amount of energy.

One important aspect of the problem which was not treated in Section IV where the oscillations of a uniform unbounded plasma were studied, may be of importance here. The electron density is a slowly varying function of solar radius and as the electrons in the directed beam move out through the corona they encounter regions of lower and lower plasma frequency. At some point in the corona these electrons become modulated to the maximum amplitude with narrow band noise whose center frequency is just below the local plasma frequency at that particular point. As these electrons continue to move outward they eventually find themselves in a medium whose plasma frequency is below the noise frequencies. Since the plasma frequency of the coronal electrons is a relatively slowly varying function of solar radius, the change in plasma frequency seen by the electrons of the directed beam as they move outward changes appreciably only in many plasma periods and it may be possible to have a continual readjustment of spectral distribution in such a way that the noise frequency always remains <u>below</u> the local plasma frequency. If, however, there is a slight delay in the readjustment it may be possible for the noise frequency to be above the local plasma frequency. This entire phase of the process takes place at limiting amplitudes where the nonlinearities are of primary importance. This is a difficult nonlinear noise problem and it has not been possible to deal with it at all. This effect should be considered in more detail.

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APPENDIX I

The following tables give the solution of IV.32 for a variety of the parameters α_1/u_2 , α_2/u_2 , r , and k_x/ω_{p1} .

$\frac{\alpha_1}{u_2} = 0 \quad \frac{\alpha_2}{u_2} = 0 \quad r = 1.000$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.500	-.500	.000	.000
.2	.500	-.495	.100	.099
.4	.500	-.481	.200	.192
.6	.500	-.460	.300	.276
.8	.500	-.434	.400	.347
1.0	.500	-.405	.500	.405
1.2	.500	-.375	.600	.449
1.4	.500	-.343	.700	.480
1.6	.500	-.310	.800	.497
1.8	.500	-.277	.900	.499
2.0	.500	-.243	1.000	.486
2.2	.500	-.207	1.100	.455
2.4	.500	-.017	1.200	.400
2.6	.500	-.012	1.300	.309
2.8	.500	-.004	1.400	.115

$\frac{\alpha_1}{u_2} = 0 \quad \frac{\alpha_2}{u_2} = 0 \quad r = .300$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.769	-.421	.000	.000
.2	.763	-.419	.153	.084
.4	.748	-.409	.299	.164
.6	.725	-.392	.435	.235
.8	.701	-.367	.560	.294
1.0	.678	-.336	.678	.336
1.2	.659	-.300	.790	.360
1.4	.642	-.261	.899	.365
1.6	.628	-.218	1.004	.350
1.8	.616	-.170	1.109	.307
2.0	.606	-.110	1.212	.221
2.2	.541	-.000	1.190	.000
2.4	.460	-.000	1.103	.000
2.6	.412	-.000	1.071	.000

$\frac{\alpha_1}{u_2} = 0 \quad \frac{\alpha_2}{u_2} = 0 \quad r = .100$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.909	-.288	.000	.000
.2	.904	-.289	.181	.058
.4	.888	-.293	.355	.117
.6	.861	-.293	.516	.176
.8	.824	-.283	.659	.226
1.0	.787	-.260	.787	.260
1.2	.753	-.225	.903	.270
1.4	.724	-.180	1.013	.252
1.6	.700	-.120	1.120	.192
1.8	.633	-.000	1.139	.000
2.0	.531	-.000	1.062	.000
2.2	.472	-.000	1.039	.000

$\frac{\alpha_1}{u_2} = 0 \quad \frac{\alpha_2}{u_2} = 0 \quad r = .030$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.971	-.168	.000	.000
.2	.969	-.171	.194	.034
.4	.961	-.178	.384	.071
.6	.943	-.190	.566	.114
.8	.909	-.198	.727	.158
1.0	.863	-.187	.863	.187
1.2	.818	-.153	.981	.183
1.4	.780	-.090	1.091	.126
1.6	.659	-.000	1.055	.000
1.8	.570	-.000	1.026	.000
2.0	.508	-.000	1.016	.000
2.2	.459	-.000	1.011	.000

$\frac{\alpha_1}{u_2} = 0 \quad \frac{\alpha_2}{u_2} = 0 \quad r = .010$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.990	-.099	.000	.000
.2	.989	-.101	.198	.020
.4	.986	-.107	.395	.043
.6	.978	-.119	.587	.071
.8	.955	-.135	.764	.108
1.0	.908	-.135	.908	.135
1.2	.855	-.097	1.026	.117
1.4	.744	-.000	1.041	.000
1.6	.634	-.000	1.015	.000
1.8	.560	-.000	1.008	.000
2.0	.503	-.000	1.005	.000

$\frac{\alpha_1}{u_2} = 0 \quad \frac{\alpha_2}{u_2} = 0 \quad r = .003$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.997	-.055	.000	.000
.2	.997	-.056	.199	.011
.4	.996	-.059	.398	.024
.6	.993	-.067	.596	.040
.8	.982	-.083	.785	.066
1.0	.940	-.093	.940	.093
1.2	.872	-.042	1.047	.050
1.4	.721	-.000	1.010	.000
1.6	.628	-.000	1.004	.000
1.8	.557	-.000	1.002	.000

$\frac{\alpha_1}{u_2} = 0 \quad \frac{\alpha_2}{u_2} = 0 \quad r = .001$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.999	-.032	.000	.000
.2	.999	-.032	.200	.006
.4	.999	-.035	.399	.014
.6	.998	-.039	.599	.024
.8	.993	-.051	.794	.041
1.0	.959	-.066	.959	.066
1.2	.846	-.000	1.015	.000
1.4	.716	-.000	1.003	.000
1.6	.626	-.000	1.001	.000
1.8	.556	-.000	1.001	.000
2.0	.500	-.000	1.001	.000

$\frac{\alpha_1}{u_2} = .1 \quad \frac{\alpha_2}{u_2} = 0 \quad r = 1.000$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.450	-.450	.000	.000
.2	.451	-.445	.090	.089
.4	.455	-.432	.182	.173
.6	.461	-.411	.277	.247
.8	.468	-.385	.374	.308
1.0	.474	-.357	.474	.357
1.2	.481	-.327	.578	.392
1.4	.488	-.295	.683	.413
1.6	.495	-.263	.792	.421
1.8	.502	-.231	.904	.415
2.0	.511	-.197	1.021	.395
2.2	.520	-.163	1.144	.358
2.4	.533	-.126	1.278	.303
2.6	.551	-.088	1.433	.229
2.8	.580	-.052	1.625	.146
3.0	.617	-.029	1.850	.086
3.2	.650	-.017	2.079	.055
3.4	.677	-.012	2.302	.039
3.6	.700	-.008	2.519	.029
3.8	.719	-.006	2.731	.023

$\frac{\alpha_1}{u_2} = .1 \quad \frac{\alpha_2}{u_2} = 0 \quad r = .300$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.727	-.398	.000	.000
.2	.722	-.394	.144	.079
.4	.710	-.380	.284	.152
.6	.693	-.358	.416	.215
.8	.676	-.329	.541	.263
1.0	.662	-.295	.662	.295
1.2	.652	-.258	.782	.309
1.4	.644	-.218	.902	.305
1.6	.642	-.176	1.026	.282
1.8	.644	-.131	1.160	.236
2.0	.658	-.082	1.316	.164
2.2	.691	-.040	1.520	.088
2.4	.729	-.020	1.750	.047
2.6	.759	-.011	1.974	.030
2.8	.782	-.007	2.190	.021
3.0	.800	-.005	2.401	.016

$\frac{\alpha_1}{u_2} = .1 \quad \frac{\alpha_2}{u_2} = 0 \quad r = .100$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.880	-.278	.000	.000
.2	.875	-.279	.175	.056
.4	.859	-.277	.344	.111
.6	.834	-.270	.500	.162
.8	.804	-.252	.643	.201
1.0	.777	-.223	.777	.223
1.2	.755	-.186	.906	.223
1.4	.741	-.141	1.038	.197
1.6	.741	-.088	1.186	.141
1.8	.767	-.038	1.381	.069
2.0	.804	-.016	1.608	.033
2.2	.831	-.009	1.828	.020
2.4	.850	-.006	2.041	.013
2.6	.865	-.004	2.249	.010
2.8	.877	-.003	2.456	.008
3.0	.887	-.002	2.660	.006

$\frac{\alpha_1}{u_2} = .1 \quad \frac{\alpha_2}{u_2} = 0 \quad r = .030$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.954	-.165	.000	.000
.2	.951	-.167	.190	.033
.4	.941	-.172	.377	.069
.6	.922	-.177	.553	.106
.8	.891	-.173	.713	.139
1.0	.857	-.154	.857	.154
1.2	.831	-.116	.997	.140
1.4	.823	-.064	1.153	.090
1.6	.851	-.022	1.361	.035
1.8	.879	-.009	1.582	.017
2.0	.897	-.005	1.794	.010

$\frac{\alpha_1}{u_2} = .1 \quad \frac{\alpha_2}{u_2} = 0 \quad r = .010$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.980	-.098	.000	.000
.2	.979	-.100	.196	.020
.4	.974	-.104	.390	.042
.6	.963	-.112	.578	.067
.8	.939	-.117	.751	.094
1.0	.905	-.105	.905	.105
1.2	.881	-.067	1.057	.080
1.4	.893	-.022	1.250	.030
1.6	.917	-.008	1.468	.013
1.8	.932	-.004	1.677	.007

$\frac{\alpha_1}{u_2} = .1 \quad \frac{\alpha_2}{u_2} = 0 \quad r = .003$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.992	-.054	.000	.000
.2	.991	-.055	.198	.011
.4	.989	-.059	.396	.023
.6	.983	-.065	.590	.039
.8	.969	-.072	.775	.058
1.0	.940	-.066	.940	.066
1.2	.925	-.030	1.110	.036
1.4	.943	-.008	1.321	.011

$\frac{\alpha_1}{u_2} = .2 \quad \frac{\alpha_2}{u_2} = 0 \quad r = 1.000$

$\frac{k_x}{\omega_{p1}}$	$\mathrm{Re}(\frac{u}{u_2})$	$\mathrm{Im}(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.400	-.400	.000	.000
.2	.403	-.396	.081	.079
.4	.411	-.383	.164	.153
.6	.422	-.364	.253	.218
.8	.435	-.340	.348	.271
1.0	.449	-.312	.449	.312
1.2	.463	-.283	.555	.339
1.4	.476	-.253	.667	.354
1.6	.490	-.222	.784	.355
1.8	.504	-.191	.908	.344
2.0	.520	-.160	1.040	.320
2.2	.537	-.130	1.181	.285
2.4	.557	-.100	1.337	.240
2.6	.581	-.073	1.511	.189
2.8	.608	-.051	1.703	.142
3.0	.636	-.035	1.908	.104
3.2	.662	-.024	2.118	.078
3.4	.685	-.018	2.329	.060
3.6	.705	-.013	2.538	.047
3.8	.723	-.010	2.746	.038
4.0	.738	-.008	2.953	.032

$\frac{\alpha_1}{u_2} = .2 \quad \frac{\alpha_2}{u_2} = 0 \quad r = .300$

$\frac{k_x}{\omega_{p1}}$	$\mathrm{Re}(\frac{u}{u_2})$	$\mathrm{Im}(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.685	-.375	.000	.000
.2	.681	-.369	.136	.074
.4	.673	-.352	.269	.141
.6	.663	-.327	.398	.196
.8	.654	-.294	.523	.235
1.0	.647	-.259	.648	.258
1.2	.645	-.221	.774	.265
1.4	.647	-.182	.906	.255
1.6	.654	-.143	1.046	.229
1.8	.667	-.104	1.200	.188
2.0	.688	-.070	1.376	.139
2.2	.715	-.043	1.573	.095
2.4	.743	-.027	1.782	.065
2.6	.767	-.018	1.994	.046
2.8	.787	-.012	2.203	.034
3.0	.804	-.009	2.411	.027
3.2	.818	-.007	2.617	.022
3.4	.830	-.005	2.822	.018
3.6	.841	-.004	3.026	.015
3.8	.850	-.003	3.229	.013

$\frac{\alpha_1}{u_2} = .2 \quad \frac{\alpha_2}{u_2} = 0 \quad r = .100$

$\frac{k_x}{\omega_{p1}}$	$\mathrm{Re}(\frac{u}{u_2})$	$\mathrm{Im}(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.852	-.269	.000	.000
.2	.846	-.268	.169	.054
.4	.830	-.263	.332	.105
.6	.809	-.248	.486	.149
.8	.787	-.224	.629	.179
1.0	.768	-.191	.768	.191
1.2	.757	-.153	.909	.184
1.4	.756	-.112	1.059	.157
1.6	.767	-.072	1.227	.115
1.8	.790	-.041	1.422	.074
2.0	.815	-.023	1.631	.046
2.2	.837	-.014	1.841	.031
2.4	.854	-.009	2.049	.023
2.6	.867	-.007	2.255	.017
2.8	.878	-.005	2.460	.014
3.0	.888	-.004	2.663	.011

$\frac{\alpha_1}{u_2} = .2 \quad \frac{\alpha_2}{u_2} = 0 \quad r = .030$

$\frac{k_x}{\omega_{p1}}$	$\mathrm{Re}(\frac{u}{u_2})$	$\mathrm{Im}(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.937	-.162	.000	.000
.2	.934	-.163	.187	.033
.4	.923	-.165	.369	.066
.6	.903	-.163	.542	.098
.8	.878	-.152	.702	.121
1.0	.855	-.127	.855	.127
1.2	.842	-.091	1.010	.110
1.4	.846	-.053	1.185	.075
1.6	.865	-.027	1.385	.043
1.8	.885	-.014	1.593	.026
2.0	.900	-.009	1.800	.017
2.2	.912	-.006	2.006	.012

$\frac{\alpha_1}{u_2} = .2 \quad \frac{\alpha_2}{u_2} = 0 \quad r = .010$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.970	-.097	.000	.000
.2	.969	-.098	.194	.020
.4	.962	-.101	.385	.040
.6	.949	-.104	.570	.063
.8	.928	-.101	.742	.081
1.0	.906	-.082	.906	.082
1.2	.897	-.051	1.076	.061
1.4	.907	-.024	1.270	.034
1.6	.923	-.012	1.476	.019
1.8	.934	-.007	1.682	.012
2.0	.943	-.004	1.886	.008

$\frac{\alpha_1}{u_2} = .2 \quad \frac{\alpha_2}{u_2} = 0 \quad r = .003$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.986	-.054	.000	.000
.2	.985	-.055	.197	.011
.4	.982	-.057	.393	.023
.6	.975	-.061	.585	.037
.8	.960	-.061	.768	.049
1.0	.942	-.049	.942	.049
1.2	.939	-.025	1.127	.030

$\frac{\alpha_1}{u_2} = .2 \quad \frac{\alpha_2}{u_2} = 0 \quad r = .001$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.993	-.031	.000	.000
.2	.992	-.032	.198	.006
.4	.991	-.033	.396	.013
.6	.986	-.036	.592	.022
.8	.977	-.038	.782	.030
1.0	.964	-.029	.964	.029
1.2	.964	-.013	1.157	.016
1.4	.971	-.057	1.359	.008

$\frac{\alpha_1}{u_2} = .3 \quad \frac{\alpha_2}{u_2} = 0 \quad r = 1.000$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.350	-.350	.000	.000
.2	.354	-.346	.071	.069
.4	.366	-.336	.146	.134
.6	.383	-.318	.230	.191
.8	.403	-.296	.322	.237
1.0	.423	-.271	.423	.271
1.2	.444	-.243	.533	.292
1.4	.464	-.021	.650	.300
1.6	.485	-.186	.775	.297
1.8	.505	-.157	.909	.283
2.0	.527	-.130	1.053	.260
2.2	.549	-.104	1.209	.229
2.4	.574	-.081	1.377	.194
2.6	.599	-.060	1.557	.158
2.8	.625	-.045	1.749	.127
3.0	.649	-.033	1.948	.101
3.2	.672	-.025	2.150	.080
3.4	.692	-.019	2.354	.065
3.6	.711	-.015	2.558	.054
3.8	.727	-.012	2.762	.045
4.0	.742	-.010	2.966	.038

$\frac{\alpha_1}{u_2} = .3$ $\frac{\alpha_2}{u_2} = 0$ $r = .300$

$\frac{k_x}{\omega_{pl}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{pl}}$	$\frac{\nu}{\omega_{pl}}$
0	.643	-.352	.000	.000
.2	.641	-.345	.128	.069
.4	.637	-.325	.255	.130
.6	.633	-.297	.380	.178
.8	.632	-.262	.506	.210
1.0	.634	-.226	.634	.226
1.2	.640	-.189	.767	.226
1.4	.649	-.152	.909	.213
1.6	.664	-.117	1.062	.187
1.8	.683	-.086	1.229	.154
2.0	.705	-.060	1.411	.119
2.2	.730	-.040	1.606	.089
2.4	.753	-.277	1.808	.066
2.6	.774	-.196	2.013	.051
2.8	.792	-.143	2.218	.040
3.0	.807	-.107	2.422	.032
3.2	.821	-.083	2.626	.026

$\frac{\alpha_1}{u_2} = .3$ $\frac{\alpha_2}{u_2} = 0$ $r = .100$

$\frac{k_x}{\omega_{pl}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{pl}}$	$\frac{\nu}{\omega_{pl}}$
0	.823	-.260	.000	.000
.2	.818	-.257	.164	.051
.4	.804	-.247	.322	.099
.6	.787	-.227	.472	.136
.8	.772	-.199	.617	.159
1.0	.762	-.164	.762	.164
1.2	.760	-.128	.912	.153
1.4	.767	-.092	1.074	.128
1.6	.783	-.061	1.253	.097
1.8	.804	-.038	1.447	.069
2.0	.825	-.024	1.649	.049
2.2	.843	-.016	1.854	.036
2.4	.858	-.011	2.058	.027
2.6	.870	-.008	2.262	.021
2.8	.880	-.006	2.465	.017

$\frac{\alpha_1}{u_2} = .3$ $\frac{\alpha_2}{u_2} = 0$ $r = .030$

$\frac{k_x}{\omega_{pl}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{pl}}$	$\frac{\nu}{\omega_{pl}}$
0	.920	-.159	.000	.000
.2	.917	-.159	.183	.032
.4	.905	-.158	.362	.063
.6	.887	-.150	.532	.090
.8	.867	-.133	.694	.106
1.0	.853	-.105	.853	.105
1.2	.850	-.074	1.020	.089
1.4	.860	-.045	1.203	.063
1.6	.875	-.026	1.401	.042
1.8	.891	-.016	1.604	.029

$\frac{\alpha_1}{u_2} = .3$ $\frac{\alpha_2}{u_2} = 0$ $r = .010$

$\frac{k_x}{\omega_{pl}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{pl}}$	$\frac{\nu}{\omega_{pl}}$
0	.960	-.096	.000	.000
.2	.958	-.096	.192	.019
.4	.951	-.098	.380	.039
.6	.938	-.096	.563	.058
.8	.920	-.087	.736	.070
1.0	.907	-.066	.907	.066
1.2	.906	-.042	1.088	.050
1.4	.916	-.023	1.282	.032
1.6	.927	-.013	1.484	.021
1.8	.937	-.008	1.687	.001
2.0	.945	-.005	1.889	.001

$\alpha_1/u_2 = .3$, $\alpha_2/u_2 = 0$, $r = .003$

k_x/ω_{p1}	$Re(u/u_2)$	$Im(u/u_2)$	ω/ω_{p1}	ν/ω_{p1}
0	.981	-.054	.000	.000
.2	.980	-.054	.196	.011
.4	.976	-.056	.390	.022
.6	.967	-.057	.580	.034

$\alpha_1/u_2 = .3$, $\alpha_2/u_2 = 0$, $r = .001$

k_x/ω_{p1}	$Re(u/u_2)$	$Im(u/u_2)$	ω/ω_{p1}	ν/ω_{p1}
0	.990	-.031	.000	.000
.2	.989	-.032	.198	.006
.4	.987	-.033	.395	.013
.6	.982	-.034	.589	.020
.8	.974	-.032	.779	.025
1.0	.967	-.023	.967	.023
1.2	.968	-.012	1.162	.014
1.4	.973	-.006	1.362	.008

$\alpha_1/u_2 = 1.0$, $\alpha_2/u_2 = 0$, $r = 1.000$

k_x/ω_{p1}	$Re(u/u_2)$	$Im(u/u_2)$	ω/ω_{p1}	ν/ω_{p1}
0	.000	-.000	.000	.000
.2	.010	-.010	.002	.002
.4	.044	-.035	.018	.014
.6	.102	-.061	.061	.037
.8	.176	-.076	.141	.061
1.0	.252	-.077	.252	.077
1.2	.322	-.070	.387	.084
1.4	.384	-.060	.538	.084
1.6	.439	-.049	.702	.079
1.8	.487	-.040	.876	.072
2.0	.528	-.032	1.057	.064
2.2	.565	-.026	1.243	.056
2.4	.597	-.021	1.433	.049
2.6	.625	-.017	1.625	.043
2.8	.650	-.013	1.820	.038
3.0	.672	-.011	2.016	.033
3.2	.692	-.009	2.213	.029
3.4	.071	-.008	2.411	.026
3.6	.072	-.006	2.609	.023

$\alpha_1/u_2 = 1.0$, $\alpha_2/u_2 = 0$, $r = .300$

k_x/ω_{p1}	$Re(u/u_2)$	$Im(u/u_2)$	ω/ω_{p1}	ν/ω_{p1}
0	.348	-.191	.000	.000
.2	.364	-.184	.073	.037
.4	.406	-.166	.162	.066
.6	.458	-.139	.275	.083
.8	.510	-.110	.408	.088
1.0	.560	-.084	.560	.084
1.2	.605	-.062	.726	.075
1.4	.644	-.046	.902	.064
1.6	.679	-.034	1.087	.054
1.8	.709	-.025	1.276	.045
2.0	.735	-.019	1.470	.038
2.2	.757	-.014	1.665	.032
2.4	.776	-.011	1.862	.027
2.6	.792	-.009	2.060	.023
2.8	.806	-.007	2.258	.020
3.0	.819	-.006	2.457	.017

$\frac{\alpha_1}{u_2} = 1.0 \quad \frac{\alpha_2}{u_2} = 0 \quad r = .100$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.622	-.197	.000	.000
.2	.628	-.184	.126	.037
.4	.645	-.153	.258	.061
.6	.671	-.117	.403	.070
.8	.702	-.085	.561	.068
1.0	.733	-.059	.733	.059
1.2	.762	-.041	.915	.049
1.4	.788	-.028	1.104	.040
1.6	.810	-.020	1.297	.032
1.8	.829	-.014	1.493	.026
2.0	.845	-.011	1.690	.021
2.2	.858	-.008	1.888	.018
2.4	.870	-.006	2.087	.015

$\frac{\alpha_1}{u_2} = 1.0 \quad \frac{\alpha_2}{u_2} = 0 \quad r = .030$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.803	-.139	.000	.000
.2	.802	-.130	.160	.026
.4	.804	-.107	.322	.043
.6	.814	-.079	.482	.047
.8	.830	-.054	.664	.043
1.0	.848	-.035	.848	.035
1.2	.866	-.023	1.039	.028
1.4	.882	-.016	1.234	.022
1.6	.895	-.011	1.432	.017

$\frac{\alpha_1}{u_2} = 1.0 \quad \frac{\alpha_2}{u_2} = 0 \quad r = .010$

0	.891	-.089	.000	.000
.2	.889	-.084	.178	.017
.4	.887	-.070	.355	.028
.6	.890	-.050	.534	.030
.8	.899	-.033	.719	.027
1.0	.911	-.021	.911	.021
1.2	.922	-.014	1.106	.016

$\frac{\alpha_1}{u_2} = 1.0 \quad \frac{\alpha_2}{u_2} = 0 \quad r = .003$

0	.942	-.052	.000	.000
.2	.941	-.049	.188	.010
.4	.938	-.041	.375	.016
.6	.939	-.029	.563	.018
.8	.944	-.019	.755	.015
1.0	.951	-.012	.951	.012
1.2	.957	-.007	1.148	.009
1.4	.962	-.005	1.347	.007

$\frac{\alpha_1}{u_2} = 1.0 \quad \frac{\alpha_2}{u_2} = 0 \quad r = .001$

0	.967	-.031	.000	.000
.2	.966	-.029	.193	.006
.4	.965	-.025	.386	.010
.6	.965	-.017	.579	.010
.8	.967	-.011	.774	.009
1.0	.971	-.007	.971	.007

$\frac{\alpha_1}{u_2} = 3.0$ $\frac{\alpha_2}{u_2} = 0$ $r = 1.000$

$\frac{k_x}{\omega_{pl}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{pl}}$	$\frac{\nu}{\omega_{pl}}$
0	-1.000	1.000	-.000	-.000
.2	-1.084	.842	-.217	-.168
.4	-.919	.268	-.368	-.107
.6	-.461	.053	-.276	-.032
.8	-.154	.009	-.123	-.007
1.0	.051	-.002	.051	.002
1.2	.197	-.004	.236	.004
1.4	.305	-.004	.426	.005
1.6	.388	-.003	.620	.005
1.8	.453	-.003	.816	.005
2.0	.506	-.002	1.013	.004
2.2	.550	-.002	1.210	.004
2.4	.587	-.001	1.408	.003
2.6	.618	-.001	1.607	.003
2.8	.645	-.001	1.806	.003
3.0	.668	-.001	2.005	.002
3.2	.689	-.001	2.205	.002
3.4	.707	-.001	2.404	.002
3.6	.723	-.001	2.604	.002
3.8	.738	-.000	2.803	.002
4.0	.751	-.000	3.003	.002

$\frac{\alpha_1}{u_2} = 3.0$ $\frac{\alpha_2}{u_2} = 0$ $r = .300$

$\frac{k_x}{\omega_{pl}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{pl}}$	$\frac{\nu}{\omega_{pl}}$
0	-.495	.271	-.000	-.000
.2	-.370	.134	-.074	-.027
.4	-.051	.007	-.020	-.003
.6	.200	-.012	.120	.007
.8	.366	-.011	.293	.009
1.0	.479	-.008	.479	.008
1.2	.559	-.006	.670	.007
1.4	.618	-.004	.866	.005
1.6	.664	-.003	1.062	.005
1.8	.700	-.002	1.260	.004
2.0	.729	-.002	1.459	.003
2.2	.753	-.001	1.657	.003
2.4	.774	-.001	1.857	.002
2.6	.791	-.001	2.056	.002
2.8	.806	-.001	2.255	.002
3.0	.818	-.001	2.455	.002

$\frac{\alpha_1}{u_2} = 3.0$ $\frac{\alpha_2}{u_2} = 0$ $r = .100$

$\frac{k_x}{\omega_{pl}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{pl}}$	$\frac{\nu}{\omega_{pl}}$
0	.047	-.015	.000	.000
.2	.179	-.038	.036	.008
.4	.387	-.031	.155	.012
.6	.535	-.018	.321	.011
.8	.632	-.011	.506	.008
1.0	.698	-.006	.698	.006
1.2	.745	-.004	.894	.005
1.4	.779	-.003	1.091	.004
1.6	.806	-.002	1.289	.003
1.8	.827	-.001	1.488	.002
2.0	.844	-.001	1.687	.002
2.2	.858	-.001	1.887	.002

$\frac{\alpha_1}{u_2} = 3.0$ $\frac{\alpha_2}{u_2} = 0$ $r = .030$

$\frac{k_x}{\omega_{pl}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{pl}}$	$\frac{\nu}{\omega_{pl}}$
0	.466	-.081	.000	.000
.2	.544	-.058	.109	.012
.4	.660	-.029	.264	.011
.6	.743	-.014	.446	.008
.8	.798	-.007	.638	.006
1.0	.834	-.004	.834	.004
1.2	.860	-.002	1.032	.003
1.4	.879	-.002	1.231	.002
1.6	.894	-.001	1.430	.002
1.8	.905	-.001	1.629	.001

$\frac{\alpha_1}{u_2} = 3.0 \quad \frac{\alpha_2}{u_2} = 0 \quad r = .010$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.693	-.069	.000	.000
.2	.736	-.045	.147	.009
.4	.803	-.020	.321	.008
.6	.851	-.009	.511	.005
.8	.883	-.004	.706	.004
1.0	.904	-.002	.904	.002
1.2	.919	-.002	1.103	.002
1.4	.930	-.001	1.303	.001
1.6	.939	-.001	1.502	.001

$\frac{\alpha_1}{u_2} = 3.0 \quad \frac{\alpha_2}{u_2} = 0 \quad r = .003$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.833	-.046	.000	.000
.2	.855	-.029	.171	.006
.4	.891	-.012	.357	.005
.6	.918	-.005	.551	.003
.8	.936	-.003	.749	.002
1.0	.947	-.001	.947	.001
1.2	.956	-.001	1.147	.001
1.4	.962	-.001	1.346	.001

$\frac{\alpha_1}{u_2} = 3.0 \quad \frac{\alpha_2}{u_2} = 0 \quad r = .001$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.904	-.029	.000	.000
.2	.916	-.002	.183	.000
.4	.937	-.007	.375	.000
.6	.953	-.003	.572	.000
.8	.963	-.002	.770	.000
1.0	.970	-.001	.970	.000

$\frac{\alpha_1}{u_2} = 0 \quad \frac{\alpha_2}{u_2} = .1 \quad r = 1.000$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.550	-.450	.000	.000
.2	.549	-.445	.110	.089
.4	.545	-.432	.218	.173
.6	.539	-.411	.323	.247
.8	.532	-.385	.426	.308
1.0	.526	-.357	.526	.357
1.2	.519	-.327	.622	.392
1.4	.512	-.295	.717	.413
1.6	.505	-.263	.808	.421
1.8	.498	-.231	.896	.415
2.0	.489	-.197	.979	.395
2.2	.480	-.163	1.056	.358
2.4	.467	-.126	1.122	.303
2.6	.449	-.088	1.167	.229
2.8	.420	-.052	1.175	.146
3.0	.383	-.029	1.150	.086
3.2	.350	-.017	1.121	.055
3.4	.323	-.012	1.098	.039
3.6	.300	-.008	1.081	.029
3.8	.281	-.006	1.069	.023
4.0	.265	-.005	1.059	.019

$\frac{\alpha_1}{u_2} = 0 \quad \frac{\alpha_2}{u_2} = .1 \quad r = .300$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.811	-.344	.000	.000
.2	.805	-.344	.161	.069
.4	.786	-.339	.315	.136
.6	.759	-.328	.455	.197
.8	.727	-.308	.582	.246
1.0	.696	-.281	.696	.281
1.2	.667	-.248	.800	.298
1.4	.640	-.211	.896	.296
1.6	.614	-.171	.983	.274
1.8	.588	-.127	1.058	.229
2.0	.554	-.080	1.108	.159
2.2	.504	-.038	1.109	.083
2.4	.451	-.018	1.082	.043
2.6	.408	-.010	1.062	.026
2.8	.374	-.006	1.048	.018
3.0	.346	-.004	1.038	.013
3.2	.322	-.003	1.031	.010
3.4	.302	-.002	1.026	.008
3.6	.284	-.002	1.022	.007
3.8	.268	-.001	1.019	.005
4.0	.254	-.001	1.016	.005

$\frac{\alpha_1}{u_2} = 0 \quad \frac{\alpha_2}{u_2} = .1 \quad r = .100$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.938	-.197	.000	.000
.2	.934	-.200	.187	.040
.4	.919	-.208	.368	.083
.6	.890	-.217	.534	.130
.8	.848	-.218	.678	.174
1.0	.799	-.202	.799	.202
1.2	.752	-.173	.903	.207
1.4	.708	-.133	.991	.186
1.6	.660	-.083	1.056	.133
1.8	.594	-.035	1.070	.062
2.0	.525	-.014	1.049	.027
2.2	.470	-.007	1.034	.015
2.4	.427	-.004	1.025	.010
2.6	.392	-.003	1.019	.007
2.8	.362	-.002	1.015	.005
3.0	.337	-.001	1.012	.004
3.2	.316	-.001	1.010	.003
3.4	.297	-.001	1.008	.002

$\frac{\alpha_1}{u_2} = 0 \quad \frac{\alpha_2}{u_2} = .1 \quad r = .030$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.988	-.071	.000	.000
.2	.986	-.074	.197	.015
.4	.981	-.084	.393	.034
.6	.967	-.103	.580	.062
.8	.932	-.126	.746	.100
1.0	.873	-.129	.873	.129
1.2	.807	-.104	.969	.124
1.4	.737	-.057	1.032	.080
1.6	.647	-.018	1.035	.028
1.8	.567	-.006	1.021	.012
2.0	.507	-.003	1.014	.006

$\frac{\alpha_1}{u_2} = 0 \quad \frac{\alpha_2}{u_2} = .1 \quad r = .010$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	1.000	-.000	.000	.000
.2	1.000	-.002	.200	.000
.4	.999	-.009	.400	.004
.6	.995	-.024	.597	.014
.8	.978	-.054	.782	.043
1.0	.916	-.078	.916	.078
1.2	.832	-.055	.999	.066
1.4	.729	-.016	1.020	.023
1.6	.632	-.005	1.011	.007
1.8	.559	-.002	1.007	.003
2.0	.502	-.001	1.005	.002

$\frac{\alpha_1}{u_2} = 0 \quad \frac{\alpha_2}{u_2} = .2 \quad r - 1.000$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.600	-.400	.000	.000
.2	.597	-.396	.119	.079
.4	.589	-.383	.236	.153
.6	.578	-.364	.347	.218
.8	.565	-.340	.452	.272
1.0	.551	-.312	.551	.312
1.2	.537	-.283	.645	.339
1.4	.524	-.253	.733	.354
1.6	.510	-.222	.816	.355
1.8	.496	-.191	.892	.344
2.0	.480	-.160	.960	.320
2.2	.463	-.130	1.019	.285
2.4	.443	-.100	1.063	.240
2.6	.419	-.073	1.089	.189
2.8	.392	-.051	1.097	.142
3.0	.364	-.035	1.092	.104
3.2	.338	-.024	1.082	.078
3.4	.315	-.018	1.071	.060
3.6	.295	-.013	1.062	.047
3.8	.277	-.010	1.054	.038
4.0	.262	-.080	1.047	.032

$\frac{\alpha_1}{u_2} = 0 \quad \frac{\alpha_2}{u_2} = .2 \quad r = .300$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.854	-.268	.000	.000
.2	.847	-.269	.169	.054
.4	.826	-.271	.331	.108
.6	.794	-.265	.477	.159
.8	.755	-.253	.604	.202
1.0	.715	-.231	.715	.231
1.2	.676	-.202	.811	.242
1.4	.640	-.169	.896	.236
1.6	.604	-.134	.966	.214
1.8	.567	-.098	1.020	.176
2.0	.525	-.064	1.051	.129
2.2	.481	-.039	1.057	.086
2.4	.438	-.024	1.052	.057
2.6	.401	-.015	1.043	.040
2.8	.370	-.010	1.035	.029
3.0	.343	-.007	1.029	.022
3.2	.320	-.005	1.025	.017
3.4	.300	-.004	1.021	.014
3.6	.283	-.003	1.018	.011
3.8	.267	-.003	1.016	.010
4.0	.253	-.002	1.014	.008

$\frac{\alpha_1}{u_2} = 0 \quad \frac{\alpha_2}{u_2} = .2 \quad r = .100$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.967	-.106	.000	.000
.2	.963	-.110	.193	.022
.4	.951	-.123	.380	.049
.6	.923	-.143	.554	.086
.8	.875	-.156	.700	.125
1.0	.815	-.151	.815	.151
1.2	.755	-.129	.906	.155
1.4	.696	-.097	.975	.136
1.6	.636	-.062	1.018	.099
1.8	.573	-.034	1.032	.061
2.0	.514	-.018	1.029	.035
2.2	.465	-.010	1.023	.023
2.4	.424	-.006	1.018	.016
2.6	.390	-.004	1.014	.011
2.8	.361	-.003	1.012	.008
3.0	.337	-.002	1.010	.007

$\frac{\alpha_1}{u_2} = 0 \quad \frac{\alpha_2}{u_2} = .2 \quad r = .030$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	1.005	-.026	.000	.000
.2	1.004	-.022	.201	.004
.4	1.002	-.011	.401	.004
.6	.995	-.015	.597	.009
.8	.963	-.057	.770	.045
1.0	.888	-.081	.888	.081
1.2	.802	-.068	.963	.081
1.4	.717	-.040	1.004	.056
1.6	.634	-.018	1.014	.029
1.8	.562	-.009	1.012	.016
2.0	.505	-.005	1.009	.009
2.2	.458	-.003	1.007	.006
2.4	.419	-.002	1.005	.004
2.6	.386	-.001	1.004	.003
2.8	.358	-.001	1.003	.002

$\frac{\alpha_1}{u_2} = 0 \quad \frac{\alpha_2}{u_2} = .3 \quad r = 1.000$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.650	-.350	.000	.000
.2	.646	-.346	.129	.069
.4	.634	-.336	.254	.134
.6	.617	-.318	.370	.191
.8	.597	-.296	.478	.237
1.0	.577	-.271	.577	.271
1.2	.556	-.243	.667	.292
1.4	.536	-.214	.750	.300
1.6	.515	-.186	.825	.297
1.8	.495	-.157	.891	.283
2.0	.473	-.130	.947	.260
2.2	.451	-.104	.991	.229
2.4	.426	-.081	1.023	.194
2.6	.401	-.061	1.043	.158
2.8	.375	-.045	1.051	.127
3.0	.351	-.034	1.052	.101
3.2	.328	-.025	1.050	.080
3.4	.308	-.019	1.046	.065
3.6	.289	-.015	1.042	.054
3.8	.273	-.012	1.038	.045
4.0	.258	-.010	1.034	.038

$\frac{\alpha_1}{u_2} = 0 \quad \frac{\alpha_2}{u_2} = .3 \quad r = .300$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.896	-.191	.000	.000
.2	.889	-.194	.178	.039
.4	.868	-.201	.351	.080
.6	.832	-.205	.499	.123
.8	.785	-.201	.628	.161
1.0	.735	-.186	.735	.186
1.2	.687	-.162	.824	.194
1.4	.641	-.134	.898	.187
1.6	.597	-.104	.955	.166
1.8	.553	-.076	.995	.137
2.0	.509	-.052	1.018	.105
2.2	.467	-.035	1.027	.077
2.4	.428	-.023	1.028	.056
2.6	.395	-.016	1.026	.042
2.8	.365	-.012	1.023	.032
3.0	.340	-.009	1.020	.026
3.2	.318	-.006	1.172	.021
3.4	.299	-.005	1.015	.017

$\frac{\alpha_1}{u_2} = 0 \quad \frac{\alpha_2}{u_2} = .3 \quad r = .100$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	.995	-.015	.000	.000
.2	.993	-.020	.199	.004
.4	.984	-.038	.394	.015
.6	.960	-.069	.576	.042
.8	.907	-.100	.726	.080
1.0	.834	-.108	.834	.108
1.2	.760	-.094	.912	.113
1.4	.690	-.071	.965	.099
1.6	.624	-.047	.998	.075
1.8	.562	-.028	1.011	.051
2.0	.507	-.017	1.014	.035
2.2	.460	-.011	1.013	.024
2.4	.421	-.007	1.011	.017
2.6	.388	-.005	1.009	.013
2.8	.360	-.004	1.008	.010
3.0	.336	-.003	1.007	.008
3.2	.314	-.002	1.006	.007
3.4	.296	-.002	1.005	.005
3.6	.279	-.001	1.004	.005

$\frac{\alpha_1}{u_2} = 0 \quad \frac{\alpha_2}{u_2} = .3 \quad r = .030$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	1.021	-.123	.000	.000
.2	1.022	-.119	.204	.024
.4	1.024	-.106	.410	.043
.6	1.027	-.076	.616	.046
.8	1.006	-.007	.804	.006
1.0	.907	-.037	.907	.037
1.2	.803	-.043	.964	.052
1.4	.709	-.028	.993	.039
1.6	.627	-.015	1.003	.024
1.8	.558	-.008	1.005	.015
2.0	.502	-.005	1.005	.010
2.2	.456	-.003	1.004	.007
2.4	.418	-.002	1.003	.005
2.6	.386	-.001	1.003	.004

$\frac{\alpha_1}{u_2} = 0 \quad \frac{\alpha_2}{u_2} = .3 \quad r = .010$

$\frac{k_x}{\omega_{p1}}$	$Re(\frac{u}{u_2})$	$Im(\frac{u}{u_2})$	$\frac{\omega}{\omega_{p1}}$	$\frac{\nu}{\omega_{p1}}$
0	1.020	-.198	.000	.000
.2	1.021	-.196	.204	.039
.4	1.025	-.189	.410	.076
.6	1.035	-.173	.621	.104
.8	1.073	-.124	.859	.099
1.0	.955	-.012	.955	.012
1.2	.822	-.021	.986	.025
1.4	.713	-.010	.999	.014
1.6	.626	-.005	1.002	.008
1.8	.557	-.003	1.002	.005
2.0	.501	-.002	1.002	.003
2.2	.455	-.001	1.001	.002
2.4	.417	-.001	1.001	.002

APPENDIX II

To calculate the correlation function of the electron density in thermal equilibrium consider a box of unit volume and let the ion charge density in this box be described by giving the coefficients of Fourier expansion of the density, η_k. The expansion coefficient of the electron density of the same mode is

$$\rho_k = \frac{\eta_k}{1 + k^2 \ell_D^2} \quad .$$

In thermal equilibrium,

$$\left\langle \frac{\eta_k \ \eta_{k'}}{\eta_o^2} \right\rangle = \frac{\delta_{kk'}}{n_o} \qquad \text{AII.1}$$

where n_o is the number of ions per unit volume. This states that the amplitudes of the k and k' modes are independent. The above value of $\left\langle \frac{\eta_k}{\eta_o} \right\rangle^2$ may be obtained by assuming that the gas propagates sound waves. The energy in the k^{th} mode is $\frac{m}{e} \ \rho_o \ c'^2 \ (\frac{\eta_k^2}{\eta_o})$ and in thermal equilibrium its expected value is just kT. Using the isothermal sound speed $c'^2 = \kappa T/M$, the above result is obtained.

The density correlation function is

$$\Psi(\underline{\tau}) = \left\langle \frac{\rho(\underline{r}) \ \rho(\underline{r} + \underline{\tau})}{\rho_o^{o\,2}} \right\rangle = \left\langle \sum_{k,k'} \frac{\rho_k}{\rho_o^o} \frac{\rho_{k'}}{\rho_o} e^{-i\underline{k} \cdot \underline{r}} e^{i\underline{k}' \cdot (\underline{r} + \underline{\tau})} \right\rangle$$

$$= \sum_k \left\langle \frac{\rho_k \ \rho_k}{\rho_o^2} \right\rangle e^{-i\underline{k} \cdot \underline{\tau}} \quad .$$

Converting the sum to an integral and using AII.1,

$$\Psi(\tau) = \frac{1}{n_o} \int \frac{e^{-i\underline{k} \cdot \underline{\tau}}}{(1 + k^2 \ell_D^2)^2} \frac{d\,\underline{k}}{(2\pi)^3} \quad .$$

There is an upper limit on the magnitude of $\underline{k}$, of the order of the reciprocal of the interparticle distance, which is determined by equating the number of modes so obtained to the total number of degrees of freedom of the particles. However, negligible error is committed by extending the above limits of integration to $\pm\infty$. The integral may be evaluated by converting to spherical k-coordinates and the result is found to be

$$\Psi(r) = \frac{1}{8\pi\, n_o\, \ell_D^3}\, e^{-r/\ell_D} \quad . \qquad \text{AII.2}$$

II A FIELD ANALYSIS OF THE M TYPE BACKWARD WAVE OSCILLATOR

II A FIELD ANALYSIS OF THE M TYPE BACKWARD WAVE OSCILLATOR

ABSTRACT

A field theory of electron beams focused by crossed electric and magnetic fields is given. The theory is basic to the understanding of the small signal behavior of crossed field electron devices. It is applied to explain the slipping stream, or diocotron, effect as a coupling of two surface waves of the electron beam, and to derive the start oscillation conditions of the M-type backward wave oscillator. It is found that the slipping stream effect can reduce the starting current by an appreciable factor. The results are compared with the thin beam theory which neglects space charge effects.

An analysis of a loaded strip transmission line is given, from which a method of representing space harmonic slow wave circuits by a surface admittance boundary condition is obtained. Forward and backward space harmonic interaction may be treated equally well.

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I. INTRODUCTION

The last few years have seen the invention of a host of new microwave amplifier and oscillator tubes. The small signal theory of tubes using electron beams focused by axial or longitudinal magnetic fields is now rather well developed (1),(2),(3),(4),(5). The theory of tubes using electron beams focused by crossed electric and magnetic fields is not so well developed, perhaps because this type of tube has not been so important until recently. The M-type (M for magnetron, because of the similar steady flow conditions) backward wave oscillator is likely to be very important because of its higher efficiency of conversion of d.c. energy to a.c. energy and its greater tuning rate than the longitudinally focused, or O-type (O for ordinary) backward wave oscillator. The operating characteristics and the theory of this new type of tube are summarized in reference (6).

The major contribution of this paper is to present a field analysis of M-type tubes which makes it possible to take into account space charge effects, that is, the effect on the motion of the charge of fields generated by the space charge. It is not possible to do this without a number of assumptions, to be discussed later, the principal one of which is that the unperturbed condition in the beam is a generalization of the planar Brillouin(7) state. While, in principle, this state can be realized in beam type tubes, it is doubtful whether most tubes fulfill this condition very closely.

Although this paper concentrates on the application of the theory to the M-type backward wave oscillator, the theory developed here is fundamental to all M-type beam tubes. The diocotron, or slipping stream amplifier, is discussed briefly, inasmuch as some of the effects which it exhibits have a bearing on the backward wave oscillator discussion.

A schematic diagram of the M-type backward wave oscillator is shown in Figure 1. Electrons emitted from the cathode, C, are focused into a beam through the combined action of the magnetic field B_{ox}, and the electric field produced by the plate, P, the sole, S, and the slow wave circuit, or delay line, L. When the electrons travel to the right with a velocity approximately equal to the phase velocity of one of the space harmonics of a wave of the slow wave circuit, a strong modulation of the electrons occurs and they may give energy to the field, much as in a magnetron. If the energy flow of the circuit wave is to the left, this energy, reinforcing the modulation of the beam as it goes, is delivered to the transmission line connected to the circuit on the left, and ultimately delivered to a load. The circuit is terminated on the right, by T, so that there is no reflection of electromagnetic energy at this end of the circuit. The electrons are collected by K, after their passage through the interaction region.

In order for the modulation of the beam to reinforce, and for oscillations to increase with time until non-linearities limit the amplitude, the circuit wave and the electrons must have a certain relative velocity and the tube must be greater than a certain minimum length. The latter condition can also be interpreted to mean that, for a given length, a certain minimum current in the electron beam is required. These "start oscillation" conditions will be determined in this analysis.

The elementary theory neglects space charge effects (8), (9). This

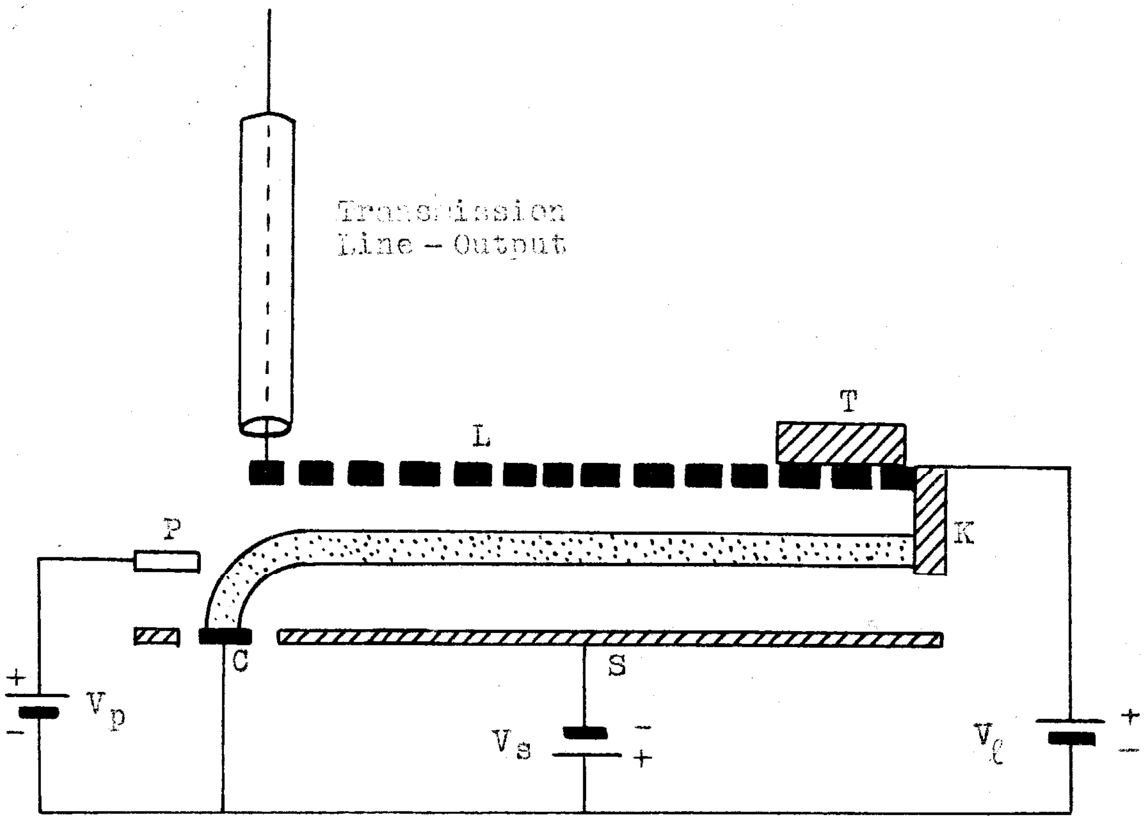

Figure 1. Schematic Structure of the Linear
M-Type Backward Wave Oscillator

analysis studies the effects of space charge (particularly the slipping stream effect) on the start oscillation conditions. French workers report (6) that the current required to start oscillations is frequently only one-half or one-third the value predicted by the elementary theory and suggest that space charge effects are responsible. An approximate theory which they have devised (6),(10) to explain this result is discussed at the conclusion of this analysis.

In order to carry out the analysis it is necessary to make a number of assumptions. These are listed here for reference, although in many cases a more detailed discussion will be found at the point at which they are introduced.

1. All quantities are assumed to be independent of the x coordinate over the width of the tube w. Fringing fields are neglected, and it is assumed that the beam does not spread in this direction.

2. A self-consistent field method is used. The particle aspect of the electron is ignored by considering the motion of an equivalent charged fluid.

3. The analysis is non-relativistic. Non-relativistic equations of motion are used. Magnetic fields are neglected, except in Section V where the slow wave circuit is discussed. The electric field is assumed irrotational.

4. The analysis is restricted to small signals. All equations are linearized by neglecting the products of time-varying quantities. The t and z dependences are assumed to be $e^{j(\omega t - \beta z)}$, and superposition applies.

5. An equivalent surface charge density and surface current

density is used to take into account the deformation of the boundary of the electron beam.

6. Only a finite number of modes of propagation are used, so that it is not possible to meet all the boundary conditions on the motion of the electron beam at the point at which it enters the interaction region.

7. It is assumed that the electron beam affects only one of the circuit space harmonics. Thus the "rising sun" effect and operation near a circuit cutoff frequency is not analyzed, although the manner in which these two effects may be studied is outlined in Section V .

8. It is assumed that the steady state of the electron beam is the planar Brillouin state, or a modification of it, so that the steady or d.c. velocity of the beam is in the z direction only.

9. Numerical computations are carried out for small $\frac{\omega_p^2}{\omega_c^2}$ only, although there is reason to believe that the results would not be significantly different if $\frac{\omega_p^2}{\omega_c^2}$ were as large as unity.

10. The sole is assumed to be far from the beam in the numerical position of the analysis. This is not an essential assumption. Other cases may be calculated with no additional difficulty. Actually, it is desirable to have the beam close to sole and far from the circuit for high efficiency operation.

II. THE ELECTRONIC EQUATIONS

The equations obeyed by the field quantities in the interior of the electron beam will be derived in this section. The motion of the charge will be discussed from the Euler point of view, as well as from the Lagrange point of view, since some confusion exists in the literature where these methods have been applied to electron beam problems. Figure 2, shows the configuration to be analyzed.

Steady State of the Beam. It is assumed that the electric and magnetic forces balance at every point within the beam and that the flow is rectilinear. Thus

$$E_{oy} + u(y)\, B_{ox} = 0 \qquad \text{II.1}$$

where the velocity, u, may depend on y. The electric field, E_{oy}, varies with y because of the charge in the beam,

$$\frac{\partial E_o}{\partial y} = \frac{\rho_o}{\epsilon_o}\, s \qquad \text{II.2}$$

where the factor s has been introduced to account for the possibility of neutralization of the electron charge by ions. s is 1 when there are no ions, and 0 when the electron charge is completely neutralized. s might be termed the "slip" parameter since when $s = 0$, all electrons move with the same velocity, while when $0 < s \leq 1$, the upper electrons slip past the lower electrons. Differentiating II.1 with respect to y and combining with II.2, the gradient of the steady velocity is found to be

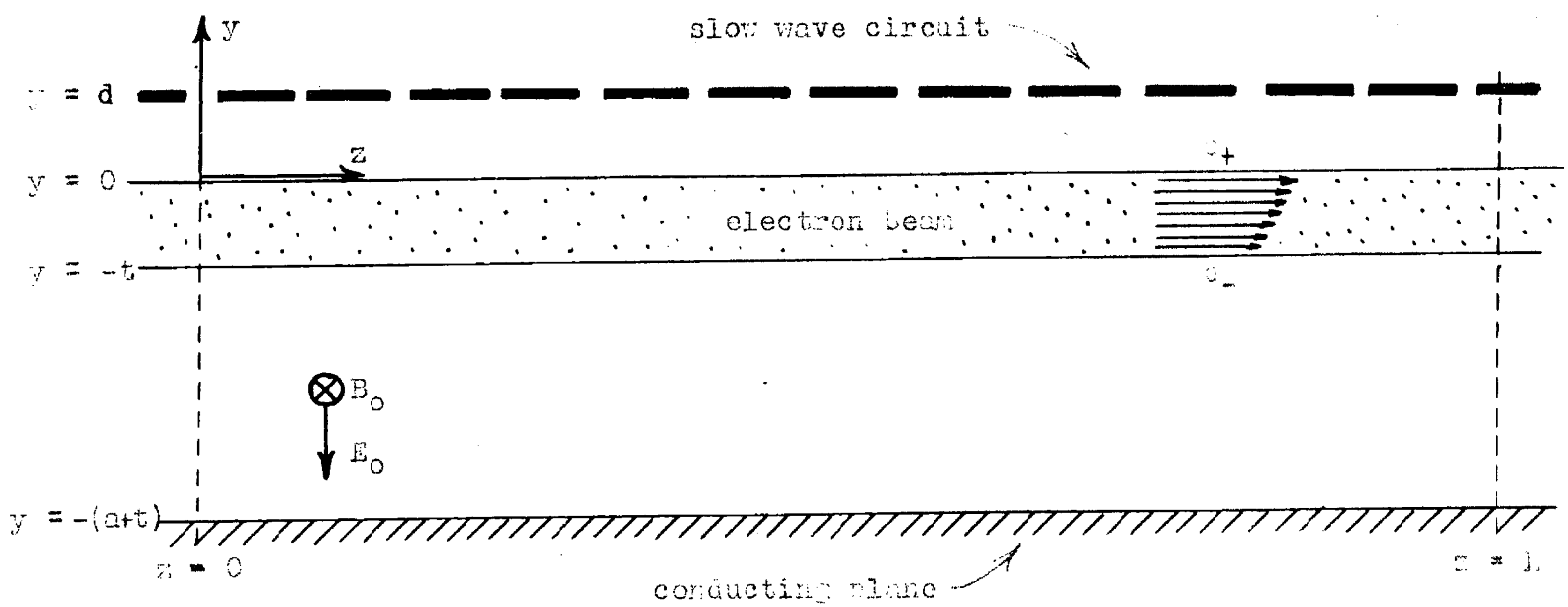

Figure 2. Configuration of the Model which is Analyzed

$$\frac{\partial u}{\partial y} = -\frac{1}{B_{ox}} \frac{\rho_o}{\epsilon_o} s = \frac{\omega_p^2}{\omega_c} s \qquad \text{II.3}$$

In the remainder of the analysis the velocity gradient will be denoted by

$$\Delta = \frac{\partial u}{\partial y} = \frac{\omega_p^2}{\omega_c} s \qquad \text{II.4}$$

It will be assumed that the sense of the steady magnetic field is as shown in Figure 2. However, reversing the magnetic field and the electric field simply changes the sign of ω_c .

It should be pointed out that the steady flow condition just described is somewhat more general than can be obtained if the electrons are emitted from a unipotential cathode, for in this case there is the additional restriction

$$\frac{1}{2} u^2 = \eta \phi_o \qquad \text{II.5}$$

where ϕ_o is the potential from which E_o is derived, measured from the cathode. It is easily shown that this additional restriction is compatible with II.3 only if $(\frac{\omega_p}{\omega_c})^2 s = 1$. Differentiating II.5 with respect to y and using II.1 ,

$$u \frac{\partial u}{\partial y} = \eta \frac{\partial \phi_o}{\partial y} = -\eta E_{oy} = \eta u B_{ox}$$

which may be written

$$\frac{\partial u}{\partial y} = \eta B_{ox} = \omega_c \; . \qquad \text{II.6}$$

This restriction means that in the absence of ions, the electron plasma frequency and the electron cyclotron frequency must be the same. This is the planar Brillouin condition (7). The more general condition II.3

will be assumed unless otherwise noted.

In practice, this type of steady flow is seldom realized, although in principle it can be obtained. The use of these steady flow conditions may be regarded as a working hypothesis, whose usefulness is to be determined by a comparison with the experimental results. It is true that in many physical problems some of the simpler results of a calculation do not depend particularly on the model chosen for the analysis, so long as the model is internally consistent. The field point of view makes possible an internally consistent analysis in which most of the approximations which must be made to solve the problem are mathematical approximations which have previously received careful scrutiny. The major physical assumption in this analysis is that the above steady flow conditions are realized.

In practical beams of this type the indication seems to be that the current densities are such as to make ω_p^2 considerably less than ω_c^2 and electrons do not follow linear trajectories. Perhaps some of the features of such a beam may be described satisfactorily by replacing it by a Brillouin flow beam with the same value of ω_p^2.

<u>Perturbations from the Steady State</u>. The linearized equations, in Eulerian form, for small perturbations from the steady state are

$$\frac{\partial \underline{v}_1}{\partial t} + (\underline{u} \cdot \nabla)\, \underline{v}_1 + (\underline{v}_1 \cdot \nabla)\, \underline{u} = -\eta\,(\underline{E}_1 + \underline{v}_1 \times \underline{B}_o) \qquad \text{II.7}$$

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_o\, \underline{v}_1 + \underline{u}\, \rho_1) = 0 \qquad \text{II.8}$$

$$\epsilon_o \nabla \cdot \underline{E}_1 = \rho_1 \qquad \text{II.9}$$

$$\nabla \times \underline{E}_1 = 0 \quad . \qquad \text{II.10}$$

The subscript 1 denotes the perturbation of a quantity, and the subscript 0 denotes the unperturbed or steady value of the quantity. The term $\underline{u} \times \underline{B}_1$ has been neglected in II.7 because it is of order $(u/c)^2$ smaller than $\underline{E}_1$. Equation II.10 is one of the Maxwell equations with the magnetic field, $\underline{B}_1$, neglected, and expresses the static approximation.

It will be convenient to assume that the t and z dependence of all quantities is given by the factor

$$e^{j(\omega t - \beta z)} \quad . \qquad \text{II.11}$$

In the remainder of the analysis, this dependence will be understood and the above factor omitted, except where required for clarity. The symbols for the field quantities are subsequently to be considered as phasors, denoting the amplitude and phase of the sinusoidal, or a.c., perturbation of the quantity in question.

Since $\nabla \times \underline{E}_1 = 0$, the electric field is derivable from a potential. Because of II.11 the z component of the electric field is just a constant times the potential, and it is possible to omit using a potential and express all quantities in terms of E_{1z} . For example, the x component of II.10 becomes

$$\frac{\partial E_{1z}}{\partial y} + j\beta\, E_{1y} = 0 \qquad \text{or} \qquad E_{1y} = -\frac{1}{j\beta}\,\frac{\partial E_{1z}}{\partial y} \quad . \qquad \text{II.12}$$

II.9 becomes

$$\frac{\partial E_{1y}}{\partial y} - j\beta\, E_{1z} = \frac{\rho_1}{\epsilon_0} \quad . \qquad \text{II.13}$$

Eliminating E_{1y} between these equations gives

$$\frac{\partial^2 E_{1z}}{\partial y^2} - \beta^2 E_{1z} = - j\beta \frac{\rho_1}{\epsilon_o} \quad , \qquad \text{II.14}$$

and II.8 may be solved for ρ_1

$$\rho_1 = - \frac{\rho_o \left(\frac{\partial v_{1y}}{\partial y} - j\beta v_{1z}\right)}{j(\omega - \beta u)} \quad . \qquad \text{II.15}$$

In component form II.7 is

$$j(\omega - \beta u)\, v_{1y} = - \eta(E_{1y} + v_{1z}\, B_{ox}) \qquad \text{II.16}$$

$$j(\omega - \beta u)\, v_{1z} + \Delta v_{1y} = -\eta(E_{1z} - v_{1y}\, B_{ox}) \quad . \qquad \text{II.17}$$

The x equation is not required since the problem is assumed to be two-dimensional; $v_{1x} = 0$. Solving for v_{1y} and v_{1z} yields

$$v_{1y} = \frac{j(\omega - \beta u)\, E_{1y} - \omega_c\, E_{1z}}{\Omega^2} \qquad \text{II.18}$$

$$v_{1z} = \frac{j(\omega - \beta u)\, E_{1z} + (\omega_c - \Delta)\, E_{1y}}{\Omega^2} \quad , \qquad \text{II.19}$$

where $\Omega^2 = (\omega - \beta u)^2 - \omega_c(\omega_c - \Delta)$.

To obtain the differential equation obeyed by E_{1z} , substitute ρ_1 from II.15 into II.14, obtaining

$$\frac{\partial^2 E_{1z}}{\partial y^2} - \beta^2 E_{1z} = - \frac{\rho_o}{\epsilon_o} \; \frac{\beta\left(\frac{\partial v_{1y}}{\partial y} - j\beta\, v_{1z}\right)}{(\omega - \beta u)} \quad . \qquad \text{II.20}$$

Differentiating II.18 with respect to y , bearing in mind that u as well as E_{1y} and E_{1z} are functions of y , and using II.19 ,

$$\frac{\partial^2 E_{1z}}{\partial y^2} - \beta^2 E_{1z} = \frac{\omega_p^2}{\Omega^2}\left[\frac{\partial^2 E_{1z}}{\partial y^2} - \beta^2 E_{1z} + \frac{2\beta\Delta\left\{(\omega-\beta u)\frac{\partial E_{1z}}{\partial y} + \beta\omega_c E_{1z}\right\}}{\Omega^2}\right] , \qquad \text{II.21}$$

which may be rewritten,

$$\left[1 - \frac{\omega_p^2}{\Omega^2}\right]\frac{\partial^2 E_{1z}}{\partial y^2} - 2\frac{\omega_p^2}{\Omega^4}\beta(\omega-\beta u)\Delta\frac{\partial E_{1z}}{\partial y} - \beta^2\left[1 - \frac{\omega_p^2}{\Omega^2} + \frac{2\omega_p^2\omega_c\Delta}{\Omega^4}\right]E_{1z} = 0 . \qquad \text{II.22}$$

This result may be obtained by another method, analogous to the Lagrangian description in hydrodynamics (11), in which the motion of an individual particle is described. The position of the electron is given by

$$\underline{r} = \underline{r}_o(a,b,c,t) + \underline{r}_1(a,b,c,t) \qquad \text{II.23}$$

where $\underline{r}_o(a,b,c,t)$ is the position of the electron in the absence of the small a.c. disturbances; a,b,c, are parameters which tell which electron is being described (for example, a,b,c, might be the xyz coordinates of the electron at $t = 0$); $\underline{r}_1$ is the a.c. perturbation of the electron position as a function of time.

The equation of motion of the electron is

$$\frac{d^2\underline{r}}{dt^2} = -\eta\left[\underline{E}(r) + \frac{d\underline{r}}{dt} \times \underline{B}(r)\right] \qquad \text{II.24}$$

where $\underline{E}(r) = \underline{E}_o(r) + \underline{E}_1(r)$

$\underline{B}(r) = \underline{B}_o(r) + \underline{B}_1(r)$ ($\underline{B}_1(r)$ may be neglected, however.)

$\underline{E}_o$ and $\underline{B}_o$ are the steady parts of the electric and magnetic fields and it is assumed that $\underline{E}_o(r_o) + \underline{u} \times \underline{B}_o(\underline{r}_o) = 0$ for all electrons so that the unperturbed trajectories are straight lines. The linear, time-dependent part of the above equation is

$$\frac{d^2\underline{r}_1}{dt^2} = -\eta\left[\underline{E}_1(r_o) + (\underline{r}_1 \cdot \nabla)\, \underline{E}_o(\underline{r}_o) + \frac{d\underline{r}_1}{dt} \times \underline{B}_o\right] \quad . \qquad \text{II.25}$$

The first term arises from the change in $\underline{E}$ along the unperturbed trajectory, while the second term takes into account the fact that motion of the electron into a region where the d.c. field is different appears as an a.c. force on the electron since the motion is time varying. In the linear approximation $\underline{E}_1$ may be evaluated at its unperturbed position r_o . If a,b,c, are the unperturbed electron coordinates at $t = 0$, the unperturbed coordinates at time t later are,

$$\begin{aligned} x_o &= a \\ y_o &= b \\ z_o &= c + u_o(b)t \quad . \end{aligned} \qquad \text{II.26}$$

Assuming the fields to vary as $e^{j\omega t}\, e^{-j\beta z}$ it is easily seen that in the linear approximation the time dependence of the force acting on the electron is $e^{j(\omega - \beta u)t}$ hence differentiation with respect to time, $\frac{d}{dt}$, is equivalent to multiplication by $j(\omega - \beta u)$. In component form II.25 becomes,

$$-(\omega - \beta u)^2\, y_1 = -\eta\left[E_{1y} + \frac{\partial E_{oy}}{\partial y}\, y_1 + j(\omega - \beta u)\, z_1\, B_{ox}\right] \qquad \text{II.27}$$

$$-(\omega - \beta u)^2\, z_1 = -\eta\left[E_{1z} - j(\omega - \beta u)\, y_1\, B_{ox}\right] \qquad \text{II.28}$$

($e^{j(\omega - \beta u)t}$ is understood.)

Solving for y_1 and z_1 ,

$$j(\omega - \beta u)\, y_1 = \frac{j(\omega - \beta u)\, E_{1y} - \omega_c\, E_{1z}}{\Omega^2} \qquad \text{II.29}$$

$$j(\omega - \beta u)\, z_1 = \frac{j(\omega - \beta u)\, \frac{\omega_c \quad \Delta}{j(\omega - \beta u)}\, E_{1z} + \omega_c\, E_{1y}}{\Omega^2} \qquad \text{II.30}$$

where $\eta \frac{\partial E_o}{\partial y}$ has been replaced by the value obtained from II.2 and II.4,

$$\eta \frac{\partial E_o}{\partial y} = \eta \frac{\rho_o}{\epsilon_o}\, s = -\omega_p^2\, s = -\omega_c\, \Delta\, .$$

Notice that II.30 differs from II.19 although II.29 and II.18 are the same. II.18 and II.19 describe the velocity field at a particular point in space, while II.29 and II.30 describe the a.c. velocity of a particle whose unperturbed trajectory passes through this point, but whose perturbed trajectory does not.

The charge density in the vicinity of a particular particle is calculated from the Lagrange continuity equation

$$\rho = \frac{\rho_o}{\begin{vmatrix} \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{vmatrix}}$$

using the particle positions,

$$y = b + y_1(u,\, E_{1y} . E_{1z})\, e^{j(\omega - \beta u)t}\, e^{-j\beta c}$$

$$z = c + ut + z_1(u, E_{1y}\, E_{1z})\, e^{j(\omega - \beta u)t}\, e^{-j\beta c} \quad .$$

Performing the indicated computations, and using $\frac{\partial u}{\partial b} = \Delta$,

$$\frac{\partial y}{\partial b} = 1 + \left[\frac{\partial y_1}{\partial u}\Delta + \frac{\partial y_1}{\partial E_{1y}}\frac{\partial E_{1y}}{\partial b} + \frac{\partial y_1}{\partial E_{1z}}\frac{\partial E_{1z}}{\partial b} - j\beta t\Delta y\right] e^{j(\omega-\beta u)t}\, e^{-j\beta c}$$

$$\frac{\partial y}{\partial c} = -j\beta y_1\, e^{j(\omega-\beta u)t}\, e^{-j\beta c}$$

$$\frac{\partial z}{\partial b} = \Delta\, t + \left[\frac{\partial z_1}{\partial u}\frac{\partial E_{1y}}{\partial b} + \frac{\partial z_1}{\partial E_{1z}}\frac{\partial E_{1z}}{\partial b}\right] e^{j(\omega-\beta u)t}\, e^{-j\beta c}$$

$$\frac{\partial z}{\partial c} = 1 - j\beta\, z_1\, e^{j(\omega-\beta u)t}\, e^{-j\beta c}$$

$$J = \frac{\partial y}{\partial b}\frac{\partial z}{\partial c} - \frac{\partial z}{\partial b}\frac{\partial y}{\partial c} = 1 + \left[\frac{\partial y_1}{\partial u}\Delta + \frac{\partial y_1}{\partial E_{1y}}\frac{\partial E_{1y}}{\partial b} + \frac{\partial y_1}{\partial E_{1z}}\frac{\partial E_{1z}}{\partial b} - j\beta t\Delta y_1 - j\beta z_1\right]$$

$$e^{j(\omega-\beta u)t}\, e^{-j\beta c} + j\beta\Delta t y_1\, e^{j(\omega-\beta u)t}\, e^{-j\beta c} + \text{second order terms.}$$

Thus

$$\rho \approx \frac{1}{1 + \left[\frac{\partial y_1}{\partial u}\Delta + \frac{\partial y_1}{\partial E_{1y}}\frac{\partial E_{1y}}{\partial b} + \frac{\partial y_1}{\partial E_{1z}}\frac{\partial E_{1z}}{\partial b} - j\beta z_1\right] e^{j(\omega-\beta u)t}\, e^{-j\beta c}}$$

$$\rho_1 = (\rho - \rho_o) = \rho_o\left[\frac{\partial y_1}{\partial u}\Delta + \frac{\partial y_1}{\partial E_{1y}}\frac{\partial E_{1y}}{\partial b} + \frac{\partial y_1}{\partial E_{1z}}\frac{\partial E_{1z}}{\partial b} - j\beta z_1\right] e^{j(\omega-\beta u)t}\, e^{-j\beta c} \qquad \text{II.31}$$

From II.29 and II.30

$$y_1 = \eta\left[\frac{E_{1y}}{\Omega^2} - \frac{\omega_c}{j(\omega-\beta u)}\frac{E_{1z}}{\Omega^2}\right]$$

$$z_1 = \eta\left[1 + \frac{\omega_c\,\Delta}{(\omega-\beta u)^2}\right]\frac{E_{1z}}{\Omega^2} + \eta\,\frac{\omega_c}{j(\omega-\beta u)}\frac{E_{1y}}{\Omega^2} \quad .$$

Thus

$$\frac{\partial y_1}{\partial u} = \eta\left\{\frac{+2\beta(\omega-\beta u)}{\Omega^4}\left[E_{1y} - \frac{\omega_c}{j(\omega-\beta u)}E_{1z}\right] - \frac{\omega_c\,\beta}{j(\omega-\beta u)^2}\frac{E_{1z}}{\Omega^2}\right\}$$

$$\frac{\partial y_1}{\partial E_{1y}} = \eta\,\frac{1}{\Omega^2} \qquad\qquad \frac{\partial y_1}{\partial E_{1z}} = -\eta\,\frac{\omega_c}{j(\omega-\beta u)}\frac{1}{\Omega^2} \quad .$$

Substituting into II.31 for ρ_1 ,

$$\rho_1 = -\rho_o \left\{ \begin{array}{l} \frac{2\beta\,\Delta(\omega-\beta u)}{\Omega^4}\left[E_{1y} - \frac{\omega_c}{j(\omega-\beta u)}E_{1z}\right] - \frac{\Delta\,\omega_c\,\beta}{j(\omega-\beta u)^2}\frac{E_{1z}}{\Omega^2} \\ + \frac{1}{\Omega^2}\frac{\partial E_{1y}}{\partial b} - \frac{\omega_c}{j(\omega-\beta u)}\frac{1}{\Omega^2}\frac{\partial E_{1z}}{\partial b} - j\beta\left[1+\frac{\omega_c\,\Delta}{(\omega-\beta u)^2}\right]\frac{E_{1z}}{\Omega^2} \\ - j\beta\frac{\omega_c}{j(\omega-\beta u)}\frac{E_{1y}}{\Omega^2} \end{array} \right\} x\; e^{j(\omega-\beta u)t}\, e^{-j\beta c}$$

or

$$-\frac{j\beta}{\epsilon_o}\rho_1 = \frac{\omega_p^2}{\Omega^2}\left[\frac{\partial^2 E_{1z}}{\partial b^2} - \beta^2 E_{1z} + \frac{2\beta\,\Delta\,(\omega-\beta u)\frac{\partial E_{1z}}{\partial b} + \beta\omega_c E_{1z}}{\Omega^2}\right] \qquad \text{II.32}$$

where $E_{1y} = -\frac{1}{j\beta}\frac{\partial E_{1z}}{\partial b}$ has been used. This is an expression for the charge density in the vicinity of a given particle. Taking $c + u_o t = z$ to be a constant in this expression rather than c = constant, and $b = y$, the a.c. charge density at the point $(x, y + y_1\,e^{j(\omega t-\beta z)}, z)$ is obtained. To a first approximation this is the a.c. charge density at the point x,y,z . II.32 is precisely the same as the right side of II.21. Thus the two methods give identical results, as they must, although they differ in detail.

This will now be compared with the method of Warnecke, Doehler and Bobot (12), which is in error. To compute the a.c. charge density they use $j(\omega-\beta u)y_1$ and $j(\omega-\beta u)z_1$ (II.29 and II.30), as the y and z components of a.c. velocity in the Eulerian continuity equation II.15, whereas the Lagrange continuity equation should be used. Another way of stating the difficulty is to note that II.29 and II.30 do not give the velocity at a fixed point in space (x,y,z) but rather the velocity at the point $(y + y_1, z + z_1)$. The velocity at (y,z) can be computed from this however, if it is remembered that the particles which are at (y,z) came from an unperturbed position $(y - y_1, z - z_1)$ where the steady part of the z velocity is

$$u(y) = \frac{\partial u}{\partial y} y_1 = u(y) - \Delta y_1 \quad .$$

This contributes to the a.c. velocity at the fixed point giving

$$v_{1y} = j(\omega - \beta u)\, y_1 \qquad \text{II.33}$$

$$v_{1z} = j(\omega - \beta u)\, z_1 - \Delta\, y_1 \quad . \qquad \text{II.34}$$

Using II.29 and II.30 in II.34 gives exactly II.19 . The extra term in II.34 subtracts from the z component of velocity. This can be understood as follows: If the particles move upward, $y_1 > 0$, the z velocity of a given point will be less because the particles which are at this point have come from a point below where the steady velocity in the z direction is less (if $\Delta > 0$). This effect contributes to the a.c. velocity of the point since y_1 , the vertical displacement is an a.c. effect.

Discussion of the Differential Equation for E_{1z}. II.21 may be simplified in three special cases. First, if $s = 0$ all electrons have the same velocity, u . Because of the constant electric field, however, different electrons are at different electrostatic potentials, depending on their position in the beam. Since different electrons are at different potentials but all have the same velocity, they cannot have been emitted from a unipotential cathode with zero initial velocities. Thus the following analysis does not apply to a situation which is easily realized in practice. Nevertheless, it is instructive to consider this case in some detail because of its simplicity. This relatively simple case forms the basis for discussion of the more complicated slipping stream case $(s \neq 0)$.

When $s = \Delta = 0$, II.21 reduces to

$$\left[1 - \frac{\omega_p^2}{(\omega - \beta u)^2 - \omega_c^2}\right]\left[\frac{d^2 E_{1z}}{dy^2} - \beta^2 E_{1z}\right] = 0 \quad . \qquad \text{II.35}$$

The solutions of this equation are of two types:

$$\text{(a)} \quad (\omega - \beta u)^2 = \omega_p^2 + \omega_c^2 \qquad \text{II.36}$$

$$\text{(b)} \quad \frac{d^2 E_{1z}}{dy^2} - \beta^2 E_{1z} = 0 \quad . \qquad \text{II.37}$$

Solutions of type (a) have a charge density in the interior of the beam associated with them. In a coordinate system which moves with the electrons all disturbances of this type oscillate with the frequency $\sqrt{\omega_p^2 + \omega_c^2}$. These disturbances are the plasma oscillations of the beam, modified by the magnetic field (when ω_c is zero, the frequency in the moving coordinate system is simply ω_p). The frequency of oscillation does not depend on the variation of the disturbance with the transverse, or y, coordinate. Fields which go with this type of solution are localized within the beam, and are not coupled to external electromagnetic circuits (gridded cavities excepted). This type of solution is similar to the solution $(\omega - \beta u)^2 = \omega_p^2$ found by Rigrod and Lewis[4] in their study of wave propagation along a magnetically focused cylindrical electron beam.

The charge density modulation in the interior of the beam is zero for solutions of the second type, and the differential equation for E_{1z}

is especially simple. The nature of these solutions is discussed in Section IV, following the discussion of boundary conditions in Section III.

When s is unity (intermediate values of s will not be discussed), it is convenient to express II.21 in terms of dimensionless variables,

$$r = \frac{\omega_p}{\omega_c} \qquad \text{II.38}$$

$$\xi = -\frac{\omega - \beta u}{\Delta} \qquad \text{II.39}$$

or

$$\omega - \beta u = -\Delta\xi = -\frac{\omega_p^2}{\omega_c} \quad .$$

The dependence of ξ on y is through the dependence of u on y. With the aid of II.4, it is easily shown that

$$d\xi = \beta\, dy \; . \qquad \text{II.40}$$

With these substitutions the differential equation for E_{1z} becomes

$$\left[1 - \frac{r^2}{r^4\xi^2 - 1 + r^2}\right]\frac{d^2E_{1z}}{d\xi^2} + \frac{2r^6}{(r^4\xi^2 - 1 + r^2)^2}\frac{dE_{1z}}{d\xi} - \left[1 - \frac{r^2}{r^4\xi^2 - 1 + r^2} + \frac{2r^4}{(r^4\xi^2 - 1)}\right]E_{1z} = 0 \qquad \text{II.41}$$

which may be rewritten

$$\frac{d^2E_{1z}}{d\xi^2} + \frac{2r^6}{(r^4\xi^2 - 1 + r^2)(r^4\xi^2 - 1)}\frac{dE_{1z}}{d\xi} - \left[1 + \frac{2r^4}{(r^4\xi^2 - 1 + r^2)(r^4\xi^2 - 1)}\right]E_{1z} = 0 \quad . \qquad \text{II.42}$$

The second instance in which a simplification is obtained is when ω_p and ω_c are equal, $(r = 1)$. When $r = 1$ II.42 becomes

$$\frac{d^2E_{1z}}{d\xi^2} + \frac{2}{\xi(\xi^2 - 1)} \frac{dE_{1z}}{d\xi} - \left[1 + \frac{2}{\xi^2(\xi^2 - 1)}\right]E_{1z} = 0 \quad . \qquad \text{II.43}$$

The substitution, $E_{1z} = \xi\Psi$, further simplifies this to

$$\frac{d^2\Psi}{d\xi^2} + \frac{2}{\xi^2 - 1} \frac{d\Psi}{d\xi} - \Psi = 0 \quad . \qquad \text{II.44}$$

II.44 has also been obtained by Macfarlane and Hay (13) in their analysis of wave propagation along a slipping stream of electrons using an action function Ψ. Their results apply only to the case $r = s = 1$ and to forward wave electromagnetic circuits. Although the case $r = s = 1$ is of interest in the analysis of the backward wave oscillator the use of the functions defined by II.44 complicates the analysis considerably. Since the numerical work of this report deals only with the simpler cases, a detailed discussion of the properties of these functions is not given here. It will suffice to note that $\xi = -1$ and $\xi = +1$ are regular singular points of the equation, and $\xi = \infty$ is an irregular singular point. Two linearly independent solutions of equation II.43 in the range $-1 < \xi < 1$, together with their derivatives have been obtained by numerical integration of II.44 and are shown in Figure 3. One of these solutions has been chosen to be an even function of ξ and the other to be odd. Both are singular at $\xi = 1, -1$, although there exists a linear combination which is not.

The third instance in which II.42 simplifies is when r^4 is small compared with unity, and only waves whose phase velocity is approximately

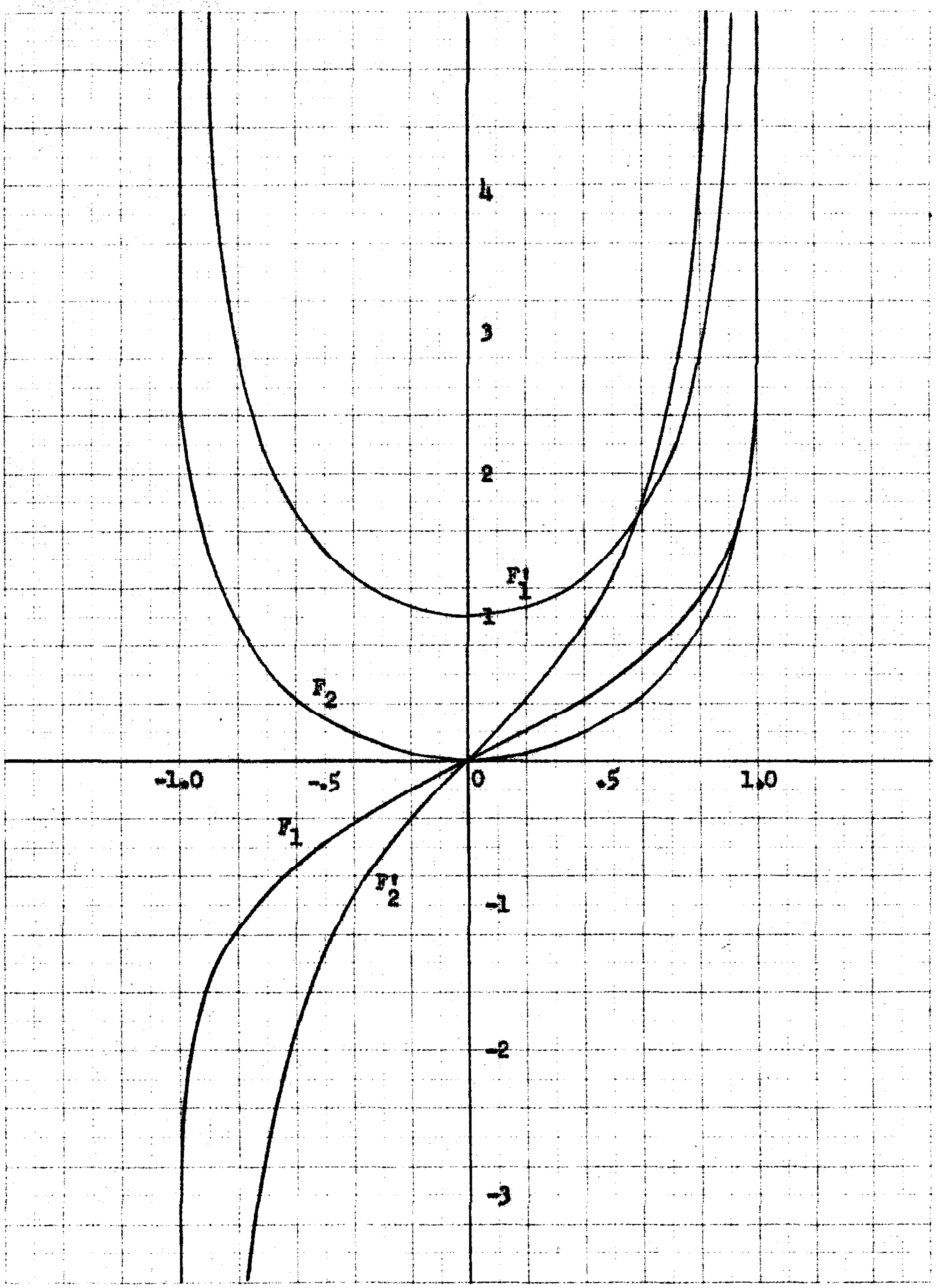

Figure 3. TWO SOLUTIONS OF THE DIFFERENTIAL EQUATION FOR E_{1z} AND THEIR DERIVATIVES, FOR THE CASE $r = s = 1$.

equal to the electron velocity are considered. The latter restriction may be expressed more precisely by $|\xi| \lesssim 1$. When these two conditions are satisfied, the coefficient of the first derivative term in II.42 is small, and the coefficient of E_{1z} is approximately equal to one so that the following approximate differential equation is obtained,

$$\frac{d^2E_{1z}}{d\xi^2} - E_{1z} = 0 . \qquad \text{II.45}$$

With the aid of the relation $d\xi = -\beta dy$, this becomes

$$\frac{d^2E_{1z}}{dy^2} - \beta^2 E_{1z} = 0 \qquad \text{II.46}$$

which is the same equation as obeyed by E_{1z} outside the beam. In later sections it is primarily this third special case that will be of interest.

It should be noted that in this special case, as in the first special case, the a.c. charge density in the beam vanishes. The previously cited error in the French work, together with an error in the sign of the force given by the second term of II.27, prevented them from obtaining this simple equation for the field in the interior of the electron beam.

Conditions at the Surfaces of the Beam. The equations applying to the interior of the beam have been derived. In the free space between the beam and the sole, and between the beam and the slow-wave circuit, E_{1z} obeys the free space equation

$$\frac{d^2E_{1z}}{dy^2} - \beta^2E_{1z} = 0 \qquad \text{II.47}$$

which is obtained from II.14 by setting ρ_1 equal to zero. The joining of the solutions in three principal regions is accomplished by means of the equivalent surface charge method used by Hahn (1) and Feenberg (3) and others. The motion of edge electrons may be rather complicated, but the effect is to produce a rippling of the boundary as shown in Figure 4a. As far as an interior electron is concerned, the difference between this situation and the unmodulated situation can be represented by a charge distribution as shown in Figure 4b or 4c. While it may be difficult to calculate the exact motion of the edge electrons, only the total excess charge which accumulates on the boundary influences motion of the interior electrons, and for this purppse it is sufficiently accurate to use

$$\sigma_{\pm} = \pm \rho_o y_1 = \pm \rho_o \frac{v_{1y}}{j(\omega - \beta u_{\pm})} \qquad \text{II.48}$$

for the surface charge density. The upper signs apply at the upper surface of the beam while the lower signs apply at the lower surface of the beam. u_+ is the velocity of the upper edge electrons and u_- is the velocity of the lower edge electrons. At a boundary E_{1y} is discontinuous by an amount $\sigma_{\pm}/\epsilon_o$ and E_{1z} is continuous.

Solutions in the three regions may be written

$$\begin{aligned} E_{1z} &= C_1 \sinh \beta(y + t + a) && -(a+t) < y < -t \\ &= C_2 F_1(\xi) + C_3 F_2(\xi) && -t < y < 0 \qquad \text{II.49 abc} \\ &= C_4 \cosh \beta y + C_5 \sinh \beta y && 0 < y < d \end{aligned}$$

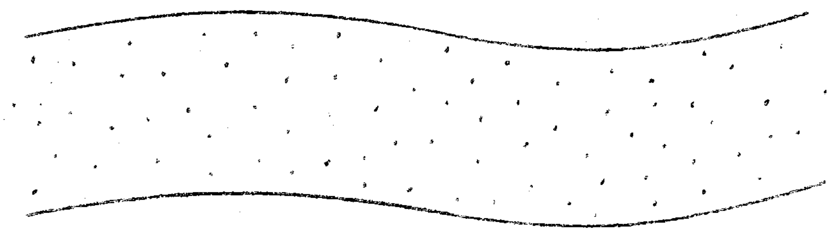

Figure 4a. Deformation of Beam when Modulated

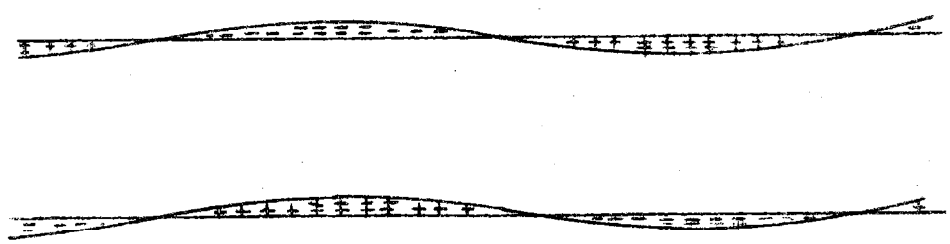

Figure 4b. Time-varying Part of Charge Density When Beam is Modulated

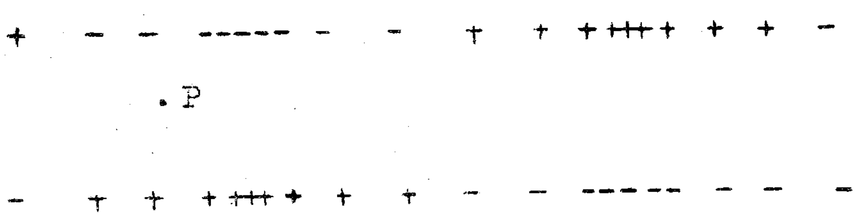

Figure 4c. Equivalent Surface Charge Density which is used to Represent the Charge Density of 4b in Computing the Field at an Interior Point, P.

$$E_{1y} = jC_1 \cosh \beta(y+t+a) \qquad -(a+t)< y< -t$$
$$= j\left[C_2 F_1'(\xi) + C_3 F_2'(\xi)\right] \qquad -t<y<0 \qquad \text{II.50 abc}$$
$$= j\left[C_4 \sinh \beta y + C_5 \cosh \beta y\right] \qquad 0< y< d$$

where $F_1(\xi)$ and $F_2(\xi)$ denote the two fundamental solutions of II.37, II.43, or II.46. In the first and third cases $F_1(\xi)$ and $F_2(\xi)$ may be regarded as standing for cosh βy and sinh βy, respectively. In all three cases, $\frac{dF(\xi)}{dy} = \beta F'(\xi)$.

Requiring E_{1z} to be continuous at $y = -t$ and $y = 0$ ($\xi = \xi_-$ and ξ_+ respectively) gives two relationships among the five constants,

$$C_1 \sinh \beta a = C_2 F_1(\xi_-) + C_3 F_2(\xi_-) \qquad \text{II.51}$$

$$C_2 F_1(\xi_+) + C_3 F_2(\xi_+) = C_4 \quad . \qquad \text{II.52}$$

Requiring E_{1y} to be discontinuous by an amount $\sigma_{\pm}/\epsilon_o$ gives two more relationships among the five constants,

$$C_2 F_1'(\xi_-) + C_3 F_2'(\xi_-) = C_1 \cosh \beta a + \frac{\omega_p^2}{\Omega_-^2}\left[C_2 F_1'(\xi_-) + C_3 F_2'(\xi_-) + \frac{\omega_c}{\omega - \beta u_-} C_1 \sinh \beta a\right] \qquad \text{II.53}$$

$$C_5 = C_2 F_1'(\xi_+) + C_3 F_2'(\xi_+) - \frac{\omega_p^2}{\Omega_+^2}\left[C_2 F_1'(\xi_+) + C_3 F_2'(\xi_+) + \frac{\omega_c}{\omega - \beta u_+} C_5\right] \qquad \text{II.54}$$

Two boundary conditions at $y = d$ must still be applied; E_{1y} and E_{1z} may be specified. Actually, only the ratio of E_{1y} to E_{1z} is significant in determining the allowed values of β , since specifying either E_{1y} or E_{1z} in addition to the ratio only determines the amplitude of the wave. With the four equations II.51, II.52, II.53 and

II.54, and the ratio of E_{1y} to E_{1z} at $y = d$, it is possible to eliminate all five constants and obtain a transcendental equation which determines the allowed or characteristic values of β and hence the characteristic waves of the system.

The characteristic waves when the circuit is absent will be discussed in the next section.

III. THE ADMITTANCE METHOD AND SPACE CHARGE WAVES

The Admittance Method. It is sometimes more convenient to use the admittance method (14),(15), to satisfy the boundary conditions at beam edges. The normalized admittance will be defined as

$$P + jQ \equiv \frac{E_{1y}}{E_{1z}} \qquad \text{III.1}$$

where P is the real part of the admittance, or conductance, and Q is the imaginary part, or susceptance. This normalized admittance is related to the usual E-mode admittance (14),(15),

$$Y_E = -\frac{H_{1x}}{E_{1z}} \qquad \text{III.2}$$

through the equation

$$Y_E = +\frac{k}{\beta}\sqrt{\frac{\epsilon_o}{\mu_o}}\ (P + jQ) \qquad \text{III.3}$$

where $\frac{k}{\beta}$ is the phase velocity of the wave divided by the velocity of light (which is small) and $\sqrt{\frac{\epsilon_o}{\mu_o}}$ is the characteristic admittance of free space. $P + jQ$ is normalized in the sense that the surface admittance of free space is given by

$$P + jQ = \pm j \qquad \text{III.4}$$

This is easily seen from the following consideration:

For $y<0$ an appropriate solution of II.47 which is bounded at $-\infty$ is

$$E_{1z} = C_1 e^{\beta y}$$

Using II.12

$$E_{1y} = jC_1 e^{\beta y}$$

Consequently

$$P + jQ = \frac{E_{1y}}{E_{1z}} = j \ .$$

Similarly, for $y > 0$,

$$E_{1z} = C_1 e^{-\beta y} \qquad E_{1y} = jC_1 e^{-\beta y} \ .$$

Hence $P + jQ = -j$.

The normalized admittance of a conducting plane at a distance d is $\pm j \coth \beta d$. It is easily shown that if the admittance at a particular plane is $(P + jQ)_1$, the admittance at distance d above this plane is

$$(P + jQ)_2 = \frac{(P + jQ)_1 + j \tanh \beta d}{1 - j \tanh \beta d \ (P + jQ)_1} \qquad \text{III.5}$$

or when P is zero simply

$$Q_2 = \frac{Q_1 + \tanh \beta d}{1 + Q_1 \tanh \beta d} \qquad \text{III.6}$$

The usefulness of the normalized admittance lies in the fact that only the ratio E_{1y}/E_{1z} is important in determining the characteristic waves of the system. This method is used in this section to obtain the propagation constants of the space charge waves which propagate on a beam between two conducting planes. Throughout the remainder of this section it will be assumed that the fields E_{1z} and E_{1y} obey the free space equation II.47 inside, as well as outside the beam, and that the slow wave circuit at $y = d$ in Figure 2 is replaced by a conducting plane.

The boundary condition at the lower edge of the beam may be written

$$\left.\frac{E_{1y}}{E_{1z}}\right|_{y=-t^+} = \left.\frac{E_{1y}}{E_{1z}}\right|_{y=-t^-} + \frac{\sigma_-}{\epsilon_o E_{1z}} \qquad \text{III.7}$$

The first term is the admittance just above the lower surface, the second term is the admittance just below the lower surface (and equal to $j \coth \beta a$), and the third term is the discontinuity in surface admittance provided by the equivalent surface charge. The latter term may be expressed in terms of the admittances with the aid of II.48, II.18, and the definition of ω_p^2,

$$\frac{\sigma_-}{\epsilon_o E_{1z}} = \frac{\omega_p^2}{\Omega_-^2}\left[\left.\frac{E_{1y}}{E_{1z}}\right|_{y=t^+} - \frac{\omega_c}{j(\omega-\beta u_-)}\right]. \qquad \text{III.8}$$

In terms of the susceptance Q,

$$Q_{t^+}\left[1-\frac{\omega_p^2}{\Omega_-^2}\right] = Q_{t^-} + \frac{\omega_p^2\ \omega_c}{\Omega_-^2(\omega-\beta u_-)}$$

or

$$Q_{t^+} = \frac{Q_{t^-}}{1-\dfrac{\omega_p^2}{\Omega_-^2}} + \frac{\omega_p^2\ \omega_c}{(\Omega_-^2-\omega_p^2)(\omega-\beta u_-)} \qquad \text{III.9}$$

Since $Q_{-t^-} = \coth a$, III.9 may be written

$$Q_{-t^+} = \frac{\Omega_-^2(\omega-\beta u_-)\coth\beta a + \omega_p^2\ \omega_c}{(\Omega_-^2-\omega_p^2)(\omega-\beta u_-)} \qquad \text{III.10}$$

Using the transformation formula III.6, since the fields obey the free space equation in the beam, this admittance appears as

$$Q_{o^-} = \frac{\left[\Omega_-^2(\omega-\beta u_-)\coth\beta a+\omega_p^2\omega_c\right] + \left[(\Omega_-^2-\omega_p^2)(\omega-\beta u_-)\right]\tanh\beta t}{\left[\Omega_-^2(\omega-\beta u_-)\coth\beta\beta a+\omega_p^2\omega_c\right]\tanh\beta t+\left[(\Omega_-^2-\omega_p^2)(\omega-\beta u_-)\right]} \qquad \text{III.11}$$

just below the upper edge of the beam. The boundary condition at the upper edge of the beam can be written,

$$\left.\frac{E_{1y}}{E_{1z}}\right|_{y=0^+} = \left.\frac{E_{1y}}{E_{1z}}\right|_{y=0^-} + \frac{\sigma_+}{\epsilon_o E_{1z}} \qquad \text{III.12}$$

which may be expressed in terms of the admittances with the aid of II.48, II.18, and the definition of ω_p^2 ,

$$\frac{\sigma_+}{\epsilon_o E_{1z}} = -\frac{\omega_p^2}{\Omega_+^2}\left[\left.\frac{E_{1y}}{E_{1z}}\right|_{y=0^-} - \frac{\omega_c}{j(\omega-\beta u_+)}\right] . \qquad \text{III.13}$$

In terms of the susceptance Q ,

$$Q_{o^+} = Q_{o^-}\left[1-\frac{\omega_p^2}{\Omega_+^2}\right] - \frac{\omega_p^2\,\omega_c}{\Omega_+^2(\omega-\beta u_+)} \qquad \text{III.14}$$

Thus the normalized susceptance just above the upper edge of the beam is

$$Q_{o^+} = \left\{\frac{\left[\Omega_-^2(\omega-\beta u_-)\coth\beta a+\omega_p^2\omega_c\right] + \left[(\Omega_-^2-\omega_p^2)(\omega-\beta u_-)\right]\tanh\beta t}{\left[\Omega_-^2(\omega-\beta u_-)\coth\beta a+\omega_p^2\omega_c\right]\tanh\beta t+\left[(\Omega_-^2-\omega_p^2)(\omega-\beta u_-)\right]}\right\} \cdot \left\{1-\frac{\omega_p^2}{\Omega_+^2}\right\} - \left\{\frac{\omega_p^2\,\omega_c}{\Omega_+^2(\omega-\beta u_+)}\right\} . \qquad \text{III.15}$$

Finally, this admittance must be transformed to $y = d$ by means of

$$Q_d = \frac{Q_{o^+} + \tanh\beta d}{1+Q_{o^+}\tanh\beta d} . \qquad \text{III.16}$$

The resulting susceptance is a function of the propagation constant β .

The propagation constants of the characteristic waves of the system are obtained by equating this expression to the susceptance of the circuit at $y = d$. Since in this section the circuit is replaced by a conducting plane, the susceptance Q_d must be infinite. From III.16 this is easily seen to occur when

$$Q_{o^+} = -\coth \beta d \qquad \text{III.17}$$

Space Charge Waves of the Non-slipping Beam. When $s = 0$, all electrons have the same velocity, $u_+ = u_- = u_o$. If the beam is very thick, $\beta t \gg 1$, the fields generated by the equivalent surface charge on the upper edge of the beam will be negligible at the lower edge of the beam and the fields generated by the equivalent surface charge on the lower edge of the beam will be negligible at the upper edge, and the waves associated with the upper and lower boundaries of the beam may be analyzed separately. The waves associated with the upper boundary may be found by setting $\tanh \beta t = 1$ in III.15 and equating this to III.17,

$$1 - \frac{\omega_p^2}{\Omega^2} - \frac{\omega_p^2 \ \omega_c}{\Omega^2 (\omega - \beta u)} = -\coth \beta d \ . \qquad \text{III.18}$$

Using the definition of Ω^2 and rearranging slightly, this becomes,

$$(\omega - \beta u - \omega_c)\ (\omega - \beta u) = \frac{\omega_p^2}{1 + \coth \beta d} \ . \qquad \text{III.19}$$

Solving for β

$$\beta_1 = \beta_e - \frac{\beta_c}{2} + \sqrt{\left(\frac{\beta_c}{2}\right)^2 + \frac{\beta_p^2}{1 + \coth \beta d}} \qquad \text{III.20}$$

$$\beta_2 = \beta_e - \frac{\beta_c}{2} - \sqrt{\left(\frac{\beta_c}{2}\right)^2 + \frac{\beta_p^2}{1 + \coth \beta d}} \qquad \text{III.21}$$

where the usual traveling wave tube notation is employed:

$$\beta_e = \frac{\omega}{u} \qquad \beta_c = \frac{\omega_c}{u} \qquad \beta_p = \frac{\omega_p}{u} \quad .$$

The first wave has a phase velocity which is less than the electron velocity, while the second wave has a phase velocity which is greater than the electron velocity. The electron velocity field corresponding to these waves can be found from II.18 and II.19. Rewriting them as

$$\frac{u\, v_{1y}}{\eta} = - \frac{j(\beta - \beta_e)\, E_{1y} + \beta_c\, E_{1z}}{(\beta - \beta_e)^2 - \beta_c^2} \qquad \text{III.22}$$

$$\frac{u\, v_{1z}}{\eta} = - \frac{j(\beta - \beta_e)\, E_{1z} - \beta_c\, E_{1y}}{(\beta - \beta_e)^2 - \beta_c^2} \qquad \text{III.23}$$

and using the ratio of E_{1y} to E_{1z} in the interior of the beam $(+j)$

$$\frac{u\, v_{1y}}{\eta} = \frac{E_{1z}}{\beta - \beta_e + \beta_c} \qquad \text{III.24}$$

$$\frac{u\, v_{1z}}{\eta} = \frac{-jE_{1z}}{\beta - \beta_e + \beta_c} \quad . \qquad \text{III.25}$$

Since the electric field decays exponentially $(e^{\beta y})$ in the interior of the beam, electrons near the surface of the beam deviate further from their equilibrium paths than do electrons in the interior of the beam. Because these waves are, in a sense, supported by the electrons near the surface of the beam, they are called surface waves. Comparison of III.25 with III.24 shows that the phase of the z velocity differs from the phase of the y velocity by $\frac{\pi}{2}$, but the amplitude of the two components of velocity are equal. Thus the motion of each electron is the combination of a drift at the electron is the combination of a drift at the velocity u and a circular motion. A given amplitude of the electric field, E_{1z}, produces a greater velocity modulation of the electrons for wave 2 than for wave 1 since the denominators of III.24 and III.25 are smaller for wave 2 than for wave 1.

In a coordinate system which moves with the electrons the frequency of oscillation is higher for wave 2

$$\omega'_2 = \omega - \beta_2 u = \frac{\omega_c}{2} + \sqrt{\left(\frac{\omega_c}{2}\right)^2 + \frac{\omega_p^2}{1 + \coth \beta d}} \qquad \text{III.26}$$

than for wave 1

$$\omega'_1 = \omega - \beta_1 u = \frac{\omega_c}{2} - \sqrt{\left(\frac{\omega_c}{2}\right)^2 + \frac{\omega_p^2}{1 + \coth \beta d}} \quad . \qquad \text{III.27}$$

Electrons execute many more circles in drifting a fixed distance when wave 2 is excited than when wave 1 is excited. In addition ω'_2 is positive and the electrons execute counter-clockwise circular motion, when wave 1 is present, while ω'_1 is negative and the electrons execute clockwise motion.

The waves associated with the lower boundary may be found by setting normalized susceptance, III.10, equal to -1 , since the fields in the beam may be assumed to be proportional to $e^{-\beta y}$. The resulting equation can be written,

$$(\omega - \beta u + \omega_c)\,(\omega - \beta u) = \frac{\omega_p^2}{1 + \coth \beta u} \qquad \text{III.28}$$

and its solutions are

$$\beta_3 = \beta_e + \frac{\beta_c}{2} - \sqrt{\left(\frac{\beta_c}{2}\right)^2 + \frac{\beta_p^2}{1 + \coth \beta d}} \qquad \text{III.29}$$

$$\beta_4 = \beta_e + \frac{\beta_c}{2} + \sqrt{\left(\frac{\beta_c}{2}\right)^2 + \frac{\beta_p^2}{1 + \coth \beta d}} \quad . \qquad \text{III.30}$$

The electron velocity field is given by

$$\frac{u\, v_{1y}}{\eta} = \frac{-E_{1z}}{\beta - \beta_e - \beta_c} \qquad \text{III.31}$$

$$\frac{u\, v_{1z}}{\eta} = \frac{-j\, E_{1z}}{\beta - \beta_e - \beta_c} \quad . \qquad \text{III.32}$$

As before, the velocities are equal in magnitude but differ in phase by $\frac{\pi}{2}$. For wave 3

$$\omega'_3 = \omega - \beta_3 u = \frac{\omega_c}{2} - \sqrt{\left(\frac{\omega_c}{2}\right)^2 + \frac{\omega_p^2}{1 + \coth \beta a}} \qquad \text{III.33}$$

the frequency is negative and the circular motion is clockwise; while for wave 4

$$\omega'_4 = \omega - \beta_4 u = \frac{\omega_c}{2} + \sqrt{\left(\frac{\omega_c}{2}\right)^2 + \frac{\omega_p^2}{1 + \coth \beta a}} \qquad \text{III.34}$$

the frequency is positive and the motion is counterclockwise. The circular motion is greatest for edge electrons, less for electrons in the interior of the beam, being practically negligible for electrons which are more than a small fraction of an electronic wavelength (distance measured in units $\lambda_e = \frac{2\pi}{\beta_e}$) from the boundary of the beam.

It is interesting to note that when $\omega_p^2 \ll \omega_c^2$ the propagation constants of the four waves are given by

$$\beta_1 = \beta_e + \frac{\beta_p^2/\beta_c}{1 + \coth \beta d} \qquad \text{III.35}$$

$$\beta_2 = \beta_e - \beta_c - \frac{\beta_p^2/\beta_c}{1 + \coth \beta d} \qquad \text{III.36}$$

$$\beta_3 = \beta_e - \frac{\beta_p^2/\beta_c}{1 + \coth \beta a} \qquad \text{III.37}$$

$$\beta_4 = \beta_e + \beta_c + \frac{\beta_p^2/\beta_c}{1 + \coth \beta a} \qquad \text{III.38}$$

Waves 1 and 3 have phase velocities approximately equal to the electron velocity and are similar to those to be discussed later in the slipping stream analysis. Waves 2 and 4 are sometimes referred to as the fast and slow cyclotron waves, respectively, since their phase velocities are less than and greater than the electron velocity and the frequency in the moving coordinate system is approximately equal to the cyclotron frequency. As the charge density of the beam approaches zero, the frequency of these waves in a system moving with the average velocity of the electrons approaches ω_c and these waves describe the natural circular motion which an electron executes in a magnetic field, which is counterclockwise for the sense of magnetic field assumed here. Similarly, as the charge density of the beam decreases the frequency of waves 1 and 3 tends to zero and these waves describe the natural drift motion which an electron executes in crossed electric and magnetic fields.

To discuss the waves it is convenient to introduce a new variable, $\delta = \beta/\beta_e$. The susceptance just above the beam III.15 can be written

$$Q_{o^+} = \frac{\left\{\left[(\delta-1)^2 - m^2\right]\left[\delta-1\right]\coth\beta_e a\delta - r^2m^3\right\} + \left\{\left[\delta-1\right]\left[(\delta-1)^2 - m^2 - m^2r^2\right]\right\}\tanh\beta_e t\delta}{\left\{\left[(\delta-1)^2 - m^2\right]\left[\delta-1\right]\coth\beta_e a\delta - r^2m^3\right\}\tanh\beta_e t\delta + \left\{\left[\delta-1\right]\left[(\delta-1)^2 - m^2 - m^2r^2\right]\right\}}$$

$$\cdot\left\{\frac{(\delta-1)^2 - m^2 - m^2r^2}{(\delta-1)^2 - m^2}\right\} + \frac{r^2m^3}{\left[\delta-1\right]\left[(\delta-1)^2 - m^2\right]} \quad . \qquad \text{III.39}$$

Upon putting this over a common denominator, a common factor $(\delta-1)^2 - m^2$ may be cancelled from numerator and denominator, so that Q may be written

$$Q_{o^+} = \frac{\left\{\delta-1\right\}\left\{\left[\delta-1\right]\left[(\delta-1)^2 - m^2 - m^2r^2\right]\left[1+\tanh(\beta_e t\delta)\right] - r^2m^2(\delta-1)\tanh(\beta_e t\delta) + r^2m^3\tanh(\beta_e t\delta)\right\} + m^4r^4\tanh\beta_e t\delta}{\delta-1\left[\left\{\left[(\delta-1)^2 - m^2\right]\left[\delta-1\right] - r^2m^3\right\}\tanh\beta_e t\delta + \left[\delta-1\right]\left[(\delta-1)^2 - m^2 - m^2r^2\right]\right]}$$

where $\coth\beta_e a\delta$ has been taken equal to 1 $(a = \infty)$ for simplicity. To find the modes of propagation when there is a conducting plane at a distance d above this beam it is necessary to equate this expression to $-\coth\beta_e d\delta$ and find the values of δ which satisfy the resulting relation. The solution may be effected by plotting the susceptance of the beam and $-\coth\beta_e d\delta$ as a function of δ and locating the intersections of the two curves. This procedure is illustrated in Figure 5, where it has been assumed that $\coth\beta_e a\delta = 1$, $\beta_e t = .50$, $m = 1/2$, and $r = 1$. The susceptance at a distance $d = \frac{.50}{\beta_e}$ above the beam is also shown. Intersections of the beam susceptance curve with the free space curve, $Q = -1$ occur at

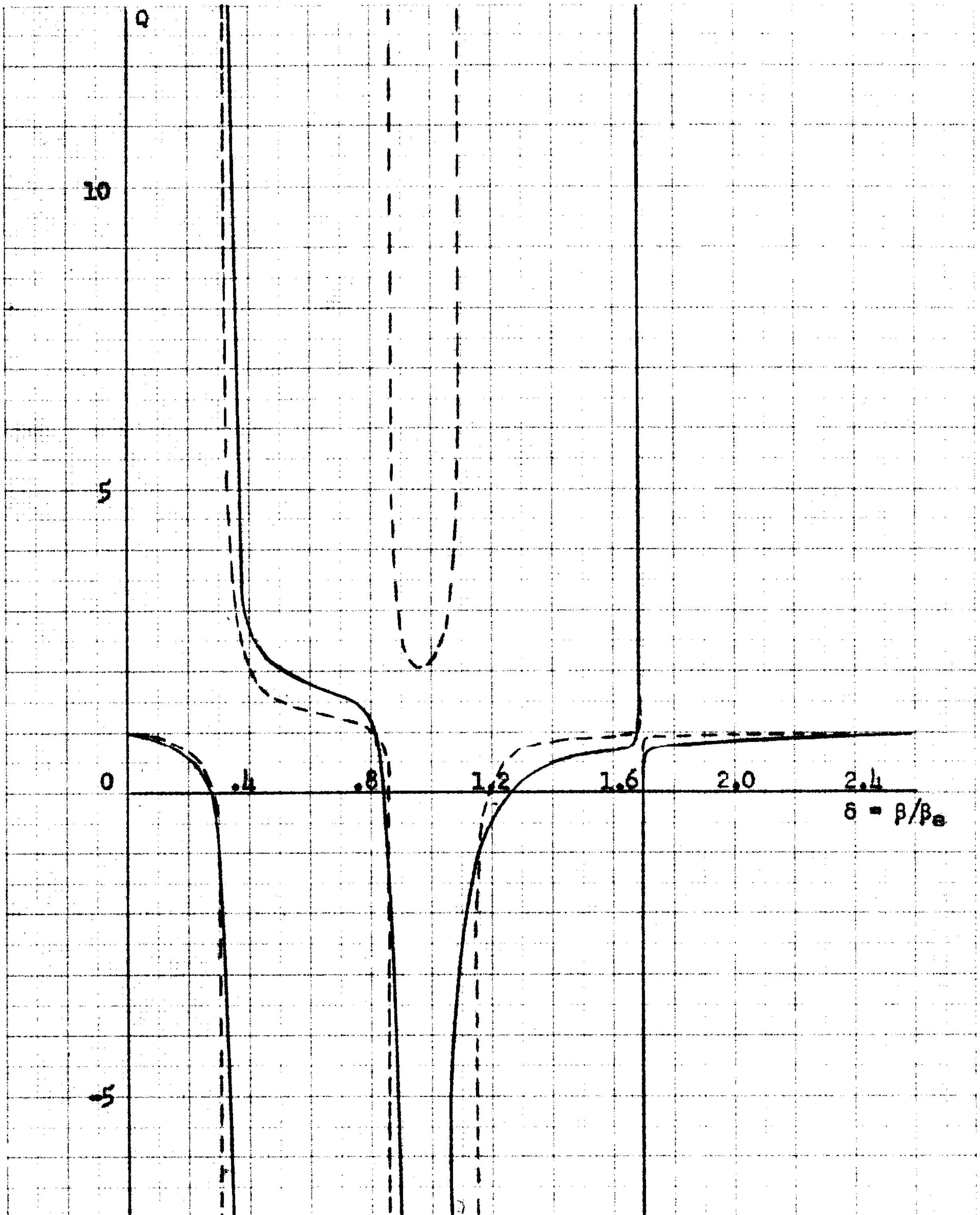

Figure 5. NORMALIZED SUSCEPTANCE OF A NON-SLIPPING BEAM OF FINITE THICKNESS, $\omega_p = \omega_c = \frac{\omega}{2}$.

$$\delta = \frac{\beta}{\beta_e} = 0.30,\ 0.86,\ 1.16,\ 1.69 \ .$$

The waves for which $\delta = .30,\ 1.69$ may be termed the fast and slow cyclotron waves respectively while the waves for which $\delta = .86,\ 1.16$ are similar to the waves whose phase velocities were approximately equal to electron velocity which were found in the thick beam case.

Figure 6 shows the distribution of E_{1z} across the beam for each of the four modes. The values calculated from the thick beam formulas III.20, III.21, III.29, and III.30 are

$$\delta = \frac{\beta}{\beta_e} = 0.134,\ 0.634,\ 1.366,\ 1.866 \ ,$$

and are in qualitative agreement with above results. In both cases there is one fast cyclotron wave, one slow cyclotron wave, and two waves with a phase velocity near the electron velocity, one a little faster and one a little slower than the electrons. Previous investigators have found only one wave (8),(9), or claim that only one of the two waves near the electron velocity can couple to external circuits (6). The field analysis shows that there are two waves near the electron velocity which can couple to external circuits. In many other respects the curves of Figure 5 are similar to those obtained from an equivalent circuit theory (8),(9).

<u>Connection between Energy Transfer and $\frac{\partial Q}{\partial \beta}$</u>. The power extracted from the electron beam can be obtained by integrating the Poynting vector over a surface just above the beam. The time average of the Poynting vector is

$$\Pi = \frac{1}{2} \operatorname{Re} (E^*_z \ H_x) \qquad \text{III.41}$$

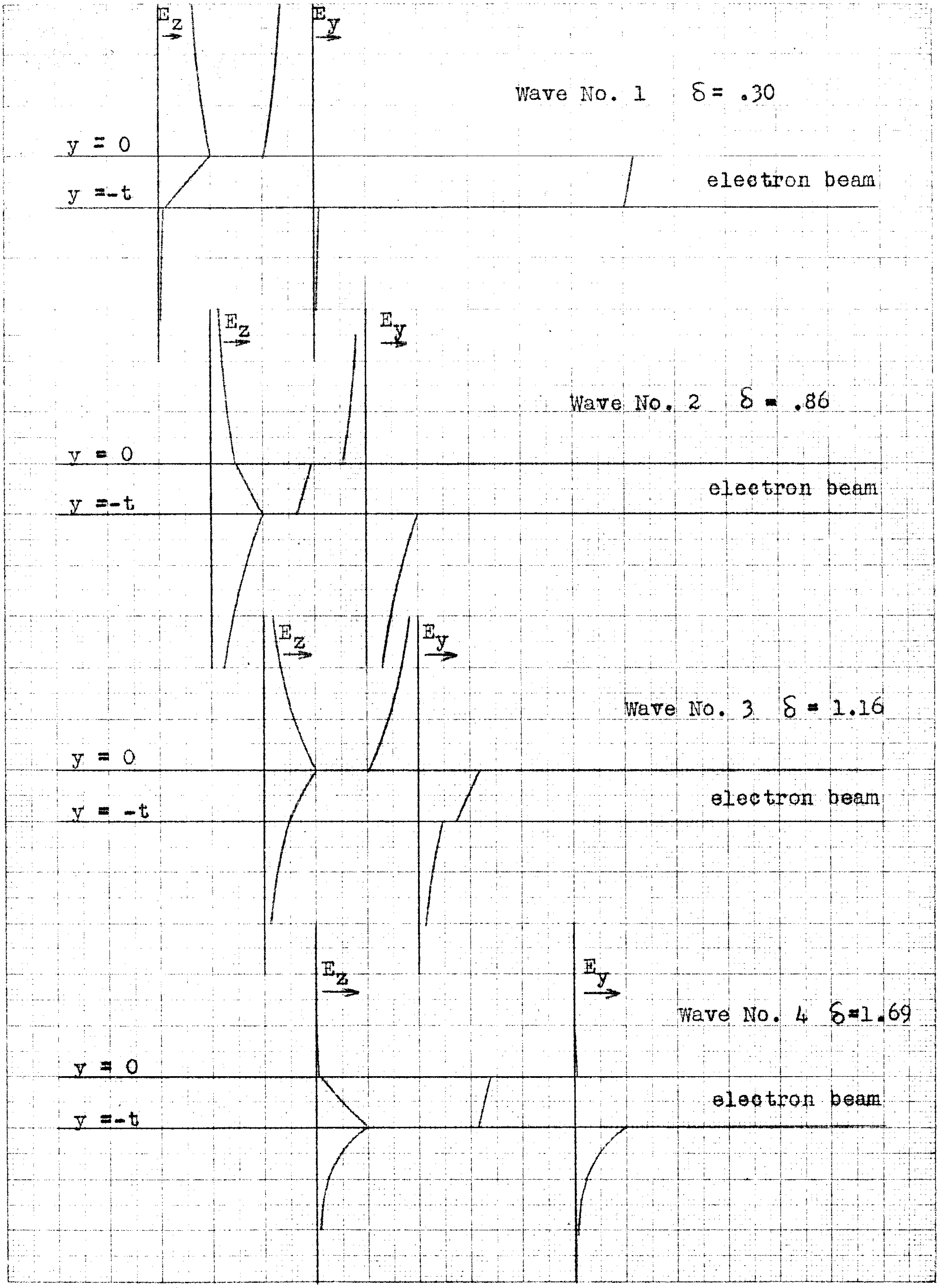

Figure 6. Distribution of Electric Field for Waves on a Non-slipping Beam of Finite Thickness

If a single wave whose space and time dependence $e^{j(\omega t - \beta z)}$ is present this may be rewritten

$$\overline{\Pi} = \frac{1}{2} E_z E^*_z \text{ Re } \left(\frac{H_x}{E_z}\right) = -\frac{1}{2} E_z E^*_z \text{ Re}(Y_E) \qquad \text{III.42}$$

According to III.3 Y_E is proportional to $P + jQ$ with a real constant of proportionality. In the preceding section it was found that P is zero for real values of the propagation constant. Thus the net energy transfer is zero since the time average of the Poynting vector is zero. Energy may be extracted from the beam if more than one wave is present or if the amplitude of the wave increases or decreases with z.

Increasing and decreasing waves can be represented by assuming the z dependence to be $e^{-(\alpha + j\beta)z}$; positive α represents a decreasing wave and negative α an increasing wave. III.42 is still valid. Since Y_E is analytic along the β axis, except at four points (see III.40 and Figure 5), the real part of the admittance can be found from the imaginary part in the following way:

In a neighborhood where Y_E is analytic,

$$Y_E\,(\beta - j\alpha) \approx Y_E(\beta) + (-j\alpha)\frac{\partial Y_E}{\partial \beta} \qquad \alpha \text{ small} \qquad \text{III.43}$$

Since $Y_E(\beta)$ and $\frac{\partial Y_E}{\partial \beta}$ are pure imaginary, the real part of $Y_E(\beta - j\alpha)$ is given by

$$\text{Re } Y_E\,(\beta - j\alpha) \approx -j\alpha\,\frac{\partial Y_E}{\partial \beta}$$

or in terms of the normalized admittance

$$P(\beta - j\alpha) \approx \frac{\partial Q}{\partial \beta}\,\alpha\ . \qquad \text{III.44}$$

It may be concluded from this result that energy is extracted from the beam if

$$\frac{\partial Q}{\partial \beta}\,\alpha < 0^{*} \qquad . \qquad \text{III.45}$$

The most interesting case is when $\alpha < 0$, for in this case the wave increases with distance and the additional energy which appears in the field as the wave increases is obtained from the kinetic energy of the electrons. Reference to Figure 5 shows that $\frac{\partial Q}{\partial \beta}$ is positive only if β is greater than β_e, that is, if the wave velocity is slower than the electron velocity.

When two constant amplitude waves are present,

$$E_z = E_{z1}\, e^{-j\beta_1 z} + E_{z2}\, e^{-j\beta_2 z}$$

$$H_x = -Y_{E1}\, E_{z1}\, e^{-j\beta_1 z} - Y_{E2}\, E_{z2}\, e^{-j\beta_2 z}$$

and the time average of the Poynting vector is given by

$$\Pi = -\frac{1}{2}\,\mathrm{Re}\Big[Y_{E1}\, E_{z1}\, E_{z1}^{*} + Y_{E2}\, E_{z2}\, E_{z2}^{*} + Y_{E1}\, E_{z1}\, E_{z2}^{*}\, e^{-j(\beta_1 - \beta_2)z} + Y_{E2}\, E_{z2}\, E_{z1}^{*}\, e^{+j(\beta_1 - \beta_2)z} \Big] .$$

---

* In some respects this condition is analogous to a theorem for electrical networks: the z coordinate replaces the time variable and $-(\alpha + j\beta)$ replaces the complex frequency variable $p = \sigma + j\omega$ (the frequency is a constant in the electron beam problem). In the electron beam problem the admittance is pure imaginary along the β axis $(\alpha = 0)$; this corresponds to a reactance network. The susceptance slope $\frac{\partial \beta}{\partial \omega}$ of a passive reactance network is positive. If the susceptance slope is negative over any part of the frequency range, energy can be extracted from the network. Energy can be extracted from the electron beam if $\frac{\partial Q}{\partial \beta}$ is positive; the difference in sign arises from the negative sign preceding β in the exponential z dependence.

Writing $Y_{E1} = jB_{E1}$ and $Y_{E2} = jB_{E2}$ where B_{E1} and B_{E2} are pure real,

$$\Pi = -\frac{1}{2} \operatorname{Re}\left[jB_{E1} E_{z1} E^*_{z2} e^{-j(\beta_1 - \beta_2)z} + jB_{E2} E_{z2} E^*_{z1} e^{j(\beta_1 - \beta_2)z} \right] \qquad \text{III.46}$$

This expression has oscillatory z dependence and energy may be imagined to flow out of the electron beam at one point and into the beam at another. Since the length of the beam is finite, it is possible to obtain a net extraction of energy.

<u>Space Charge Waves of the Slipping Beam: ω_p^2 Small Compared with ω_c^2, Waves Near the Electron Velocity.</u> The restriction to waves near the electron velocity can be stated more precisely,

$$(\omega - \beta u)^2 << \omega_c^2 \quad . \qquad \text{III.47}$$

Since for the waves discussed in this and later sections

$$(\omega - \beta u)^2 \approx \frac{\omega_p^4}{\omega_c^2} \quad .$$

III.47 is satisfied if ω_p^2 is small compared with ω_c^2. Under these circumstances,

$$\Omega^2 = (\omega - \beta u)^2 - \omega_c(\omega_c - \Delta) \approx -\omega_c^2 \qquad \text{III.48}$$

and the surface charge density on the lower edge of the beam is

$$\frac{\sigma_-}{\epsilon_o E_{1z}} = \frac{\omega_p^2}{\omega_c} \frac{1}{j(\omega - \beta u_-)} \qquad \text{III.49}$$

while the surface charge density on the upper edge of the beam is

$$\frac{\sigma_+}{\epsilon_o E_{1z}} = -\frac{\omega_p^2}{\omega_c} \frac{1}{j(\omega - \beta u_+)} \qquad \text{III.50}$$

The last two equations are obtained from III.8 and III.13 by neglecting E_{1y}/E_{1z} in comparison with $\omega_c/(\omega - \beta u)$ since the former is of order of magnitude unity and the latter will be of order of magnitude $(\frac{\omega_c}{\omega_p})^2$.

The normalized susceptance just above the beam is

$$Q_{o^+} = \frac{(\omega - \beta u_-)(\tanh \beta t + \coth \beta a) - \frac{\omega_p^2}{\omega_c}}{(\omega - \beta u_-)(1 + \tanh \beta t \coth \beta a) - \frac{\omega_p^2}{\omega_c} \tanh \beta t} + \frac{\omega_p^2}{\omega_c(\omega - \beta u_+)} \qquad \text{III.51}$$

which may also be written

$$Q_{o^+} = \frac{(\beta - \beta_-)(\tanh \beta t + \coth \beta a) - r^2 m \beta_-}{(\beta - \beta_-)(1 + \tanh \beta t \coth \beta a) - r^2 m \beta_- \tanh \beta t} - \frac{r^2 m \beta_+}{(\beta - \beta_+)} \qquad \text{III.52}$$

where $\beta_- = \frac{\omega}{u_-}$ $\quad \beta_+ = \frac{\omega}{u_+}$.

To simplify the writing of this expression, a new variable ν , is defined by

$$\beta = \beta_+ (1 + r^2 m \nu) \qquad \text{III.53}$$

and the following additional definitions are made,

$$\tanh \beta t \equiv T \qquad \coth \beta a \equiv C$$

$$\epsilon = \frac{u_+ - u_-}{u_-} = \frac{\Delta t}{u_-} = s\, m\, r^2 \frac{\omega t}{u_-}$$

ϵ is the fractional velocity spread and is assumed to be small compared to unity. The slip parameter, s, will be retained in order that subsequent expressions will apply to both the slipping or non-slipping cases. Since T and C are slowly varying functions of β , they may be taken as constants, evaluated at

$$\beta = \frac{\omega}{\sqrt{u_+ u_-}} \approx \frac{\omega}{u_+} \approx \frac{\omega}{u_-} \quad .$$

The expression for the normalized susceptance, III.52 becomes

$$Q_{\overset{+}{o}} = \frac{(\nu - s\beta_- t)(T + C) + 1}{(\nu - s\beta_- t)(1 + TC) + T} - \frac{1}{\nu}$$

$$= \frac{T + C}{1 + TC} \; \frac{\nu^2 - \nu\left[s\beta_- t + \frac{TC}{1 + TC}\right] + \frac{s\beta_- t(1 + TC) - T}{T + C}}{\nu\left[\nu - s\beta_- t + \frac{T}{1 + TC}\right]} \quad . \qquad \text{III.55}$$

III.55 is subject to obvious simplification if the beam does not slip ($s = 0$ or if the conducting plane below the beam is far removed ($C = 1$). When $C = 1$

$$Q_{\overset{+}{o}} = \frac{\nu^2 - \nu\left[s\beta_- t + \frac{T}{1 + T}\right] + \quad s\beta_- t - \frac{T}{1 + T}}{\nu\left[\nu - s\beta_- t + \frac{T}{1 + T}\right]} \quad . \qquad \text{III.56}$$

Figure 7 shows a plot of III.56. The slipping and non-slipping cases are shown separately. Two features which are common to both curves should be noted: (a) the normalized susceptance is very nearly equal to $+1$, the free space value if $\nu \gtrsim 1$, and (b) the slope of the susceptance curve is positive in the vicinity of $\nu = 0$, indicating a range of wave velocities in which it is possible to extract energy from the beam. Intersections of these curves with line $Q = -1$, the normalized susceptance of the free space above the beam, determines the propagation constants of the waves when the upper conductor is also far removed. It can be seen that when $s = 1$, no intersections are obtained. An analytical solution may be

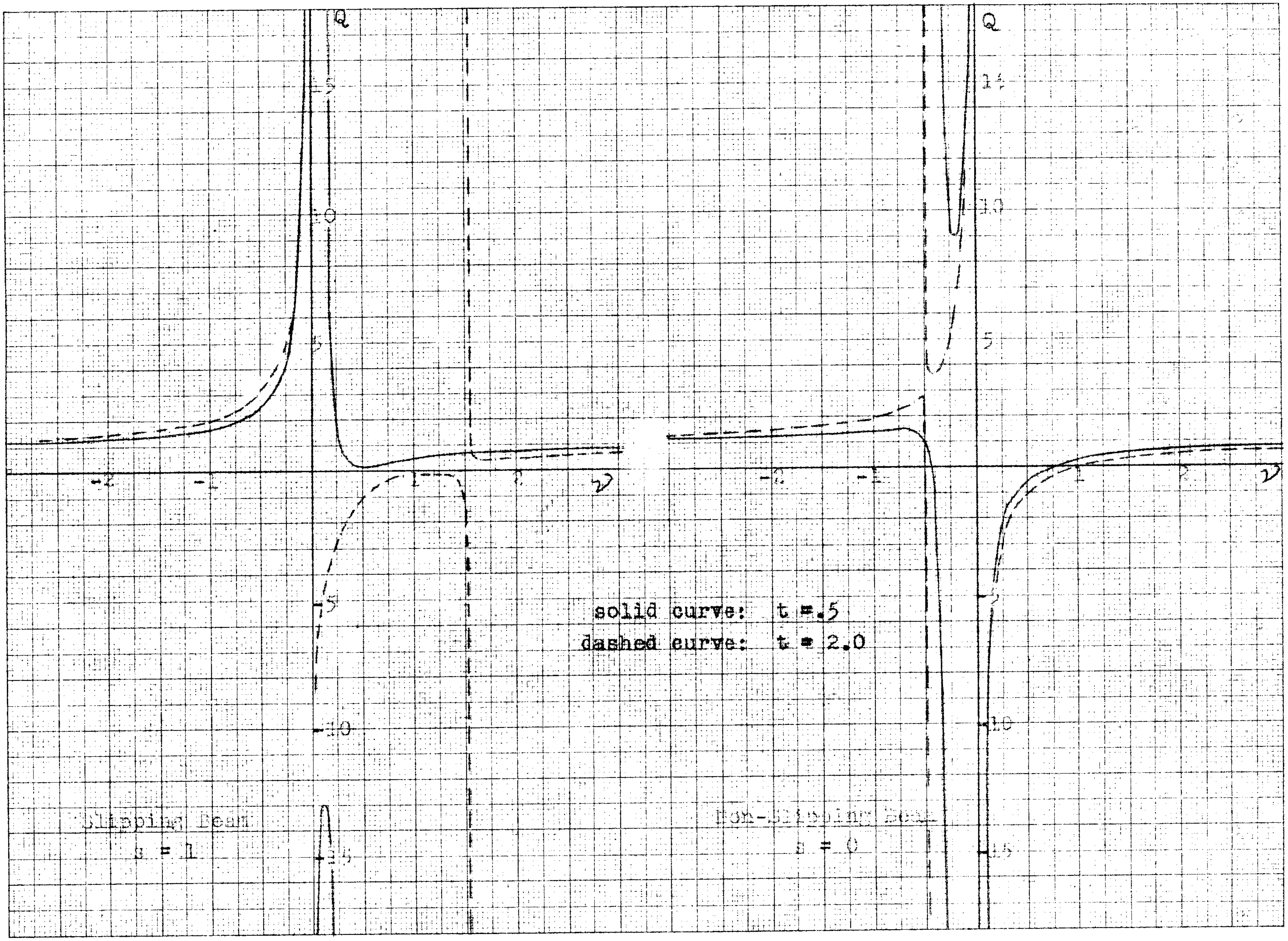

FIGURE 7 NORMALIZED SUSCEPTANCE OF THE SLIPPING AND NON-SLIPPING BEAMS

obtained by setting III.56 equal to -1 . After a slight amount of algebraic manipulation the resulting equation can be written

$$\nu^2 - (s\,\beta_-t)\,\nu + \frac{1}{2}\left(s\,\beta_-t - \frac{T}{1+T}\right) = 0 \;. \qquad \text{III.57}$$

The solutions of this equation are

$$\nu = \frac{s\beta_-t}{2} \pm \sqrt{\left(\frac{s\beta_-t}{2}\right)^2 - \frac{s\beta_-t}{2} + \frac{T}{2(1+T)}} \;. \qquad \text{III.58}$$

When the beam does not slip (s = 0) this reduces to

$$\nu = \pm\sqrt{\frac{T}{2(1+T)}} \qquad \text{III.59}$$

and upon substituting into III.53

$$\beta = \beta_+ \left(1 \pm r^2 m \sqrt{\frac{T}{2(1+T)}}\right) \qquad \text{III.60}$$

(note $\beta_+ = \beta_-$ when $s = 0$)

One wave has a phase velocity which is slightly greater than the electron velocity (lower sign) and the wave has a phase velocity which is slightly less than the electron velocity. When the beam is thick ($T \approx 1$) III.59 gives the same result as III.35 and III.39 derived earlier,

$$\beta = \frac{\omega}{u} \pm \frac{\omega_p^2}{2\omega_c u} \qquad \text{III.61}$$

When the beam slips (s = 1) III.58 becomes

$$\nu = \frac{\beta_-t}{2} \pm \sqrt{\left(\frac{\beta_-t}{2}\right)^2 - \frac{\beta_-t}{2} - \frac{T}{2(T+1)}} \;. \qquad \text{III.62}$$

Furthermore when the beam is thick ($T \approx 1$) the solution of this equation is

$$\nu_1 = \frac{1}{2}$$

$$\nu_2 = \beta_- t - \frac{1}{2} \quad . \qquad \text{III.63}$$

Using III.53 and III.54 the corresponding propagation constants are found to be

$$\beta_1 = \frac{\omega}{u_+} + \frac{\omega_p^2}{2\omega_c u_+} \qquad \text{III.64}$$

$$\beta_2 = \frac{\omega}{u_-} - \frac{\omega_p^2}{2\omega_c u_-} \quad . \qquad \text{III.65}$$

The first wave is a surface wave associated with the upper surface and has a phase velocity a little less than the velocity of the upper edge electrons. The second wave is also a surface wave associated with the lower surface and it has a phase velocity which is a little greater than the velocity of the lower edge electrons. These waves are similar to the waves described by III.61 except that the velocity of the appropriate edge electrons is different.

It is interesting to note that most of the longitudinal current in the electron beam is the surface current:

$$i_\pm = \sigma_\pm u_\pm = \pm j \frac{\epsilon_0 E_{1z_\pm}}{2} u_\pm \qquad \text{III.66}$$

where $E_{1z_\pm}$ is the longitudinal field at appropriate edge of the beam. III.66 is obtained by substituting III.64 or III.65 into III.50 or III.49. The current in the body of the beam is obtained by integrating $\rho_0 v_{1z}$

$$i = \int \rho_0 v_{1z} \, dy$$

$$= \pm j \frac{\omega_p^2}{\omega_c \omega} \epsilon_0 E_{1z} u_\pm \quad .$$

Thus the body current is $2\frac{\omega_p^2}{\omega\omega_c}$ times the surface current in the thick beam.

When the beam thickness is less than an amount given by $\beta t \approx 1.3$, III.62 gives complex conjugate values of γ. Consequently one wave increases with distance and the other decreases with distance, and small perturbations may grow as they are propagated along the electron beam. It is important to note that a source of disturbances at $z = 0$ sets up both the increasing wave and decreasing wave in the region $z > 0$. Small disturbances will grow large until the growth is limited by non-linear effects or until the electrons are collected. Figure 8 shows the rate of growth of the disturbance with distance as a function of the beam thickness:

$$\alpha = \frac{\omega_p^2}{\omega_c u_+} \sqrt{\left(\frac{\beta_- t}{2}\right) - \left(\frac{\beta_- t}{2}\right)^2 - \frac{1}{2}\frac{T}{1+T}} \,. \qquad \text{III.67}$$

This result can be understood in terms of Pierce's theory of coupling of modes of propagation (16). The increasing and decreasing waves come about through the coupling at the upper surface wave with the lower surface wave. When the beam is not very thick, the fields of the upper and lower surface waves overlap and the waves become coupled. The arguments of the preceding section indicate that the electron beam gives up energy to the field by being modulated with the upper surface wave and receives energy from the field by being modulated with the lower surface wave. Thus in the wave which increases with distance, the source of energy which appears in the field comes primarily from the upper edge electrons. The energy exchange takes place in the same way as in a magnetron: the electrons move into a region of higher d.c.

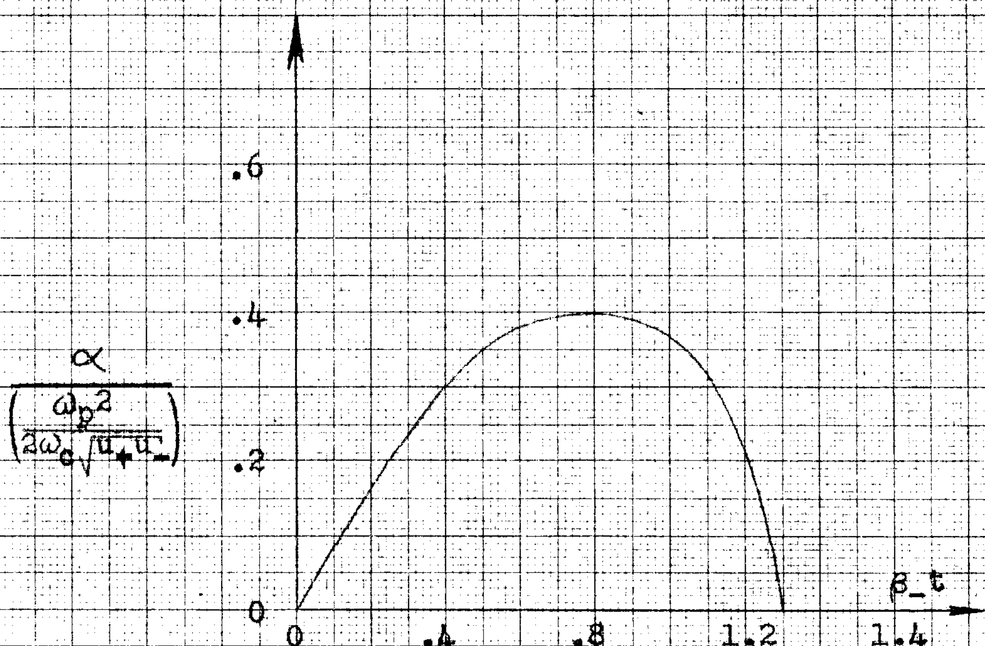

Figure 8. Slipping Beam Gain versus Beam Thickness and Frequency

potential without appreciable change in velocity by interaction with the high frequency field.

The critical thickness for growing waves is independent of the plasma frequency and hence of the current density in the beam. This result can be explained in the following way. Ordinarily, an increase in current increases the coupling between space charge waves but, in the slipping beam type of flow, an increase in current also increases the difference in velocity of the upper and lower edge electrons and hence increases the difference in velocity between the two surface waves. These two effects change the coupling in opposite directions with the net result that the coupling is independent of current.

Similarly, the beam thickness enters in two ways: the thicker the beam the greater the velocity separation $u_+ - u_-$ and, the less the fields of the two surface waves overlap. Both effects tend to reduce the coupling between the two surface waves.

The tendency for small perturbations to grow larger as they propagate along the electron beam has been termed the "diocotron" effect by French workers. A fundamental error in their analysis (12) of this effect has already been pointed out. This led to a prediction that the rate of growth of perturbations is proportional to the square root of the beam thickness for thin beams,

$$\alpha = \frac{\omega_p^2}{\omega_c u_+} \sqrt{\left(\frac{\beta_- t}{2}\right)} .$$

This result is incorrect. The analysis of the present paper furthermore shows that there exists a maximum thickness, beyond which growth does not occur.

Two of the results derived here are easily compared with the results of Macfarlane and Hay's(13) analysis which applies when $\omega_p = \omega_c$. The latter predicts a maximum rate of growth and a maximum frequency at which amplification can occur

$$\alpha_{max} \approx .24 \frac{\omega_p}{\left(\frac{u_+ + u_-}{2}\right)} \qquad \frac{u_+ - u_-}{u_+ + u_-} < .42$$

$$\omega_{max} \approx .707\, \omega_p \frac{u_+ + u_-}{u_+ - u_-} \; .$$

while the analysis of this paper predicts

$$\alpha_{max} \approx .20 \frac{\omega_p^2}{\omega_c \left(\frac{u_+ + u_-}{2}\right)}$$

$$\omega_{max} \approx .65 \frac{\omega_p^2}{\omega_c} \frac{u_+ + u_-}{u_+ - u_-} \; .$$

It is seen that the theory presented here would actually give fairly accurate results when applied to the extreme case, $\omega_p = \omega_c$, even though the principal assumption made in the derivation $(\omega_p^2 \ll \omega_c^2)$ does not apply.

Susceptance of the Beam at the Circuit. In the subsequent analysis it will be assumed that the lower plane is far removed. It will be necessary to know the normalized susceptance of the beam at a distance d above its upper surface, minus the normalized susceptance of free space at this same point. Using the susceptance transformation formula III.6 on III.56 gives

$$Q_d = \frac{\nu^2 + \nu\left[\frac{T}{1+T}\frac{T'-1}{T'+1} - s\,\beta_-t\right] + \frac{1}{T'+1}\left[s\,\beta_-t - \frac{T}{1+T}\right]}{\nu^2 + \nu\left[\frac{T}{1+T}\frac{1-T'}{T'+1} - s\beta_-t\right] + \frac{T'}{T'+1}\left[s\,\beta_-t - \frac{T}{1+T}\right]} \qquad \text{III.68}$$

where $T' \equiv \tanh \beta d$

$$Q_b - 1 = \frac{-\nu\left[\frac{2T}{1+T}\frac{1-T'}{1+T'}\right] + \frac{1-T'}{1+T'}\left[s\,\beta_-t - \frac{T}{1+T}\right]}{\nu^2 + \nu\left[\frac{T}{1+T}\frac{1-T'}{1+T'} - s\,\beta_-t\right] + \frac{T'}{1+T'}\left[s\,\beta_-t - \frac{T}{1+T}\right]} \; . \qquad \text{III.69}$$

To determine the characteristic waves of the system when a slow wave circuit is present at $z = d$, it is necessary to match this susceptance to a similar susceptance for the circuit. This concludes the discussion of the electron beam characteristics. In the following section the circuit characteristics are discussed.

IV THE SURFACE ADMITTANCE OF THE SLOW WAVE CIRCUIT

Characteristics of a Periodic Circuit. The surface admittance of the electron beam which is a function of the propagation constant of the wave and other parameters, was derived in the preceding section. This section will deal with the characteristics of the slow wave circuit. The structure shown in Figure 9 is analyzed and these results are then generalized to include other types of slow wave circuits. The principal result of this analysis is the determination of the surface admittance which such a slow wave circuit presents to the electron beam, particularly when the electrons have a velocity nearly equal to phase velocity of one of the space harmonics of the circuit. Only the transverse magnetic modes will be discussed.

In some respects this analysis is similar to one made by Parzen (17) of the same type of circuit. His analysis applies only to very low current electron beams focused by a very large axial magnetic field, while in the treatment presented here it is not necessary to specify the beam conditions specifically when deriving the circuit properties. Parzen's treatment can be shown to be equivalent to a presentation of the circuit by a fixed admittance wall (15), and thus backward wave circuits are not treated correctly. This is equivalent to the neglect of the sum in IV.39. The sum turns out to be very important, even though it is zero for waves which propagate at exactly the circuit velocity. The analysis of this section differs from a previous analysis of the same type of circuit by the author (18).

While it is possible to write down the complete set of equations which determine the fields and propagation constants of this structure,

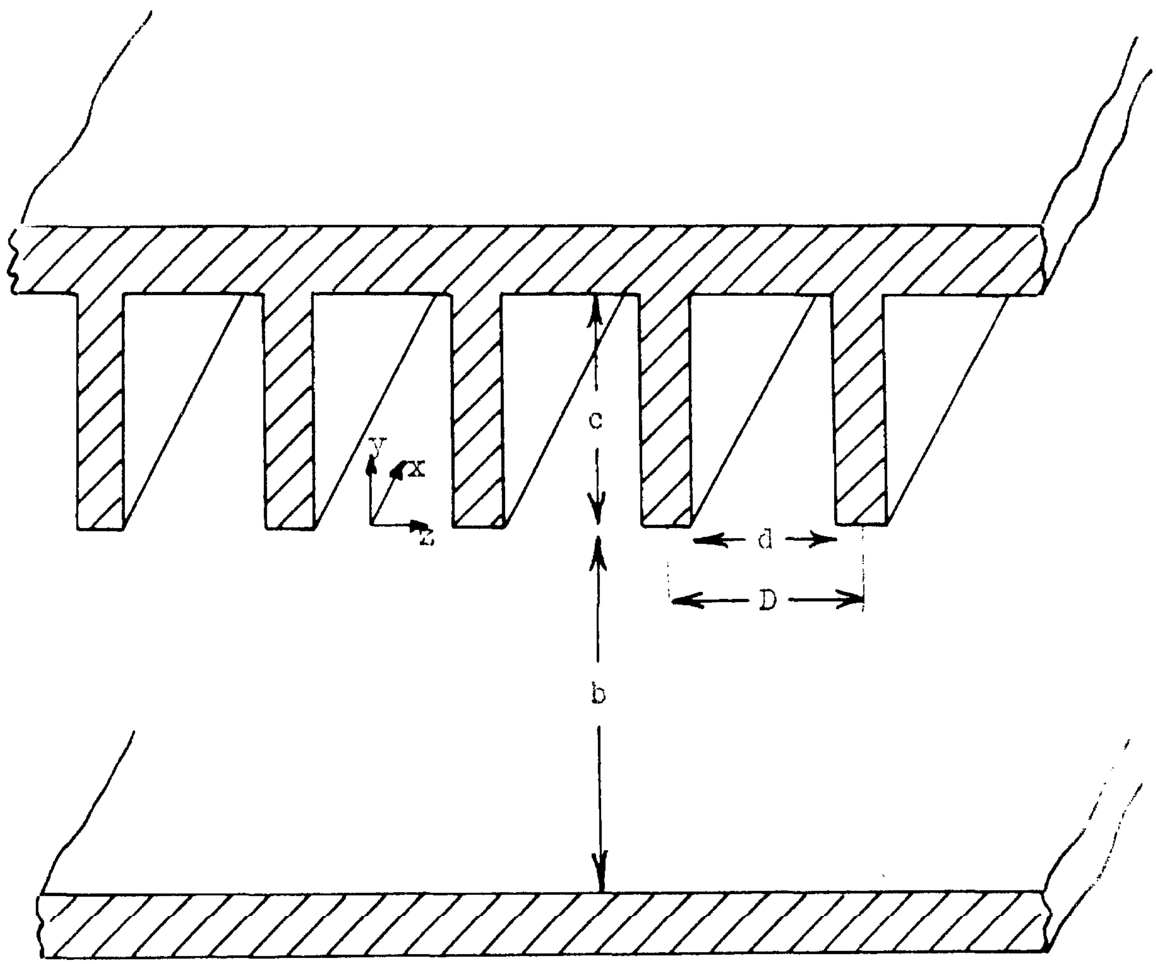

Figure 9. Cross-Section of the Slow Wave Circuit

with or without the electron beam, the boundary conditions are mixed and the solution of the set of equations requires the solution of an infinite determinant (19). Instead of following this procedure a variational method will be used in which the longitudinal electric field at $y = 0$ is the trial function. First, an integral equation for this field will be derived.

According to Bloch's (20) theorem the true fields of this structure satisfy the periodicity requirement

$$\begin{aligned} E_z(y, z + D) &= e^{-j\beta_o D} E_z(y, z) \\ E_y(y, z + D) &= e^{-j\beta_o D} E_y(y, z) \\ H_x(y, z + D) &= e^{-j\beta_o D} H_x(y, z) \end{aligned} \qquad \text{IV.1}$$

where the phase factor $e^{-j\beta_o D}$ is determined by the boundary conditions of the problem. E_z, E_y, H_x are the only field components present in the transverse magnetic modes when the fields are independent of the x coordinate. An expression for the electric field at $y = 0$ which satisfies periodicity condition IV.1 may be written

$$\mathcal{E}(z) = E_z(0, z) = e^{-j\beta_o z} \sum_{n=-\infty}^{+\infty} M_n \, e^{-j\frac{2\pi n z}{D}} \quad . \qquad \text{IV.2}$$

The inverse of IV.2 is obtained by multiplying IV.2 by $\frac{1}{D} e^{j\beta_o z} e^{+j\frac{2\pi m z}{D}}$ and integrating from $-\frac{D}{2}$ to $\frac{D}{2}$:

$$M_m = \frac{1}{D} \int_{-D/2}^{D/2} \mathcal{E}(z) \, e^{j\beta_o z} \, e^{j\frac{2\pi m z}{D}} \, dz \quad . \qquad \text{IV.3}$$

In the absence of the electron beam the electric and magnetic fields in the region $-b < y < 0$ are given by

$$E_z = e^{-j\beta_o z} \sum_{n=-\infty}^{+\infty} M_n \frac{\sinh \gamma_n (y+b)}{\sinh \gamma_n b} e^{-j \frac{2\pi n z}{D}} \qquad \text{IV.4}$$

$$E_y = e^{-j\beta_o z} \sum_{n=-\infty}^{+\infty} M_n \left(\frac{-j\beta_n}{\gamma_n}\right) \frac{\cosh \gamma_n (y+b)}{\sinh \gamma_n b} e^{-j \frac{2\pi n z}{D}} \qquad \text{IV.5}$$

$$H_x = e^{-j\beta_o z} \sum_{n=-\infty}^{\infty} M_n \left(\frac{-j\omega \epsilon_o}{\gamma n}\right) \frac{\cosh \gamma_n (y+b)}{\sinh \gamma_n b} e^{-j \frac{2\pi n z}{D}} \qquad \text{IV.6}$$

where $\gamma_n^2 = \beta_n^2 - k^2$ IV.7

$$\beta_n = \beta_o + \frac{2\pi n}{D} \qquad \text{IV.8}$$

$$k^2 = \omega^2 \mu_o \epsilon_o \quad . \qquad \text{IV.9}$$

These fields satisfy Maxwell's equations and the boundary condition at $y = -b$,

$$E_z = 0 .$$

The magnetic field at $y = 0$ is

$$H_x(0,z) = -e^{-j\beta_o z} \sum_{n=-\infty}^{\infty} Y_{no} M_n e^{-j\frac{2\pi n z}{D}} \qquad \text{IV.10}$$

where $Y_{no} = \frac{j\omega \epsilon_o}{\gamma n} \coth \gamma_n b$ is the E-mode (transverse magnetic) surface admittance of free space for the n^{th} space harmonic.

When the electron beam is present IV.10 may be written

$$H_x(0,z) = -e^{-j\beta_o z} \sum_{n=-\infty}^{\infty} Y_n M_n e^{-j\frac{2\pi n z}{D}} \qquad \text{IV.11}$$

where Y_n is the E mode surface admittance at $y = 0$ with the electron beam present, and is determined by the methods of the previous sections. In this section, the total field in the region $-b < y < 0$ is the superposition of an infinite number of partial waves or space harmonics as given by the summations in IV.4, IV.5 and IV.6 . Since the propagation constant of each space harmonic is different there is a different E mode admittance, Y_n , for each of these space harmonics. However, in the preceding section it was shown that when $s = 0$ the surface admittance with the beam present is practically equal to that of free space except when

$$\beta_n \approx \beta_e ,$$

$$\beta_n \approx \beta_e \pm \beta_c .$$

Thus most Y_n will not differ appreciably from Y_{no} . It is assumed that this is also the case when $s = 1$.

The fields in the slot which lies between $-\frac{d}{2}$ and $\frac{d}{2}$ may be written

$$E_z = \sum_{n=0}^{\infty} (2 - \delta_{no}) B_n \frac{\sin q_n(c-y)}{\sin q_n c} \cos \frac{n\pi(z+\frac{d}{2})}{d} \qquad \text{IV.12}$$

$$E_y = \sum_{n=0}^{\infty} (2 - \delta_{no}) B_n \left(\frac{n\pi}{q_n d}\right) \frac{\cos q_n(c-y)}{\sin q_n c} \sin \frac{n\pi(z+\frac{d}{2})}{d} \qquad \text{IV.13}$$

$$H = \sum_{n=0}^{\infty} (2 - \delta_{no}) B_n \left(-\frac{j\omega\epsilon_o}{q_n}\right) \frac{\cos q_n(c-y)}{\sin q_n c} \cos \frac{n\pi(z+\frac{d}{2})}{d} \qquad \text{IV.14}$$

where $q_n^2 = k^2 - (\frac{n\pi}{d})^2$ $\qquad \delta_{no} = 1 \quad n = 0$

$$\qquad\qquad = 0 \quad n \neq 0$$

Only the fields in one slot are specified, fields in adjacent slots differ only by the phase factor $e^{-j\beta_o D}$.

These fields have been chosen to satisfy the appropriate boundary conditions at the conducting surfaces:

$$E_z = 0 \quad \text{at} \quad y = c$$

$$E_y = 0 \quad \text{at} \quad z = \pm \frac{d}{2} \quad .$$

At $y = 0$ IV.12 and IV.14 become,

$$E_z(0,z) = \sum_{n=0}^{\infty} (2 - \delta_{no}) \, B_n \cos \frac{n\pi(z + \frac{d}{2})}{d} \qquad \text{IV.15}$$

$$H_x(0,z) = \sum_{n=0}^{\infty} (2 - \delta_{no}) \, (-Y'_n) B_n \sin \frac{n\pi(z + \frac{d}{2})}{d} \qquad \text{IV.16}$$

where $Y'_n = \frac{j\omega\epsilon_o}{q_n} \cot q_n c$ is the admittance of the n^{th} harmonic in the slot.

The following boundary conditions must still be satisfied:

(a) $E_z(0,z)$ as given by IV.15 must be equal to $E_z(0,z)$ as given by IV.2 ,

(b) $H_x(0,z)$ as given by IV.16 must be equal to $H_x(0,z)$ as given by IV.11 ,

(c) $E_z(0,z)$ must vanish in the interval $\frac{d}{2} < |z| < \frac{D}{2}$.

The first of these conditions will be satisfied if

$$(z) = \sum_{n=0}^{\infty} (2 - \delta_{no}) \, B_n \cos \frac{n\pi(z + \frac{d}{2})}{d} \, . \qquad \text{IV.17}$$

Multiplying IV.17 by $\frac{1}{d} \cos \frac{m\pi(z + \frac{d}{2})}{d}$ and integrating from $-\frac{d}{2}$ to $\frac{d}{2}$ yields

$$B_m = \frac{1}{d} \int_{-d/2}^{d/2} \mathcal{E}(z) \cos \frac{m\pi(z + \frac{d}{2})}{d} \, dz \, . \qquad \text{IV.18}$$

The second condition will be satisfied if

$$e^{-j\beta_o z}\sum_{n=-\infty}^{+\infty} Y_n M_n e^{-j\frac{2\pi n z}{D}} = \sum_{n=0}^{\infty} (2-\delta_{no}) + Y'_n B_n \cos\frac{n\pi(z+\frac{d}{2})}{d} \quad . \qquad \text{IV.19}$$

Upon substituting IV.3 and IV.18 for M_n and B_n , respectively, and interchanging the order of integration and summation, IV.19 becomes

$$\frac{1}{D}\int_{-D/2}^{D/2} \mathcal{E}(z') \sum_{n=-\infty}^{\infty} Y_n e^{-j(\beta_o+\frac{2\pi n}{D})(z-z')} dz' =$$

$$\frac{1}{d}\int_{-d/2}^{d/2} \mathcal{E}(z') \sum_{n=0}^{\infty} (2-\delta_{no}) Y'_n \cos\frac{n\pi(z'+\frac{d}{2})}{d} \cos\frac{n\pi(z'+\frac{d}{2})}{d} dz' \, . \qquad \text{IV.20}$$

This is an integral equation for $\mathcal{E}(z)$. $\mathcal{E}(z)$ must satisfy this integral equation in the region $-\frac{d}{2} < z < \frac{d}{2}$ and must vanish in the intervals $\frac{d}{2} < |z| < \frac{D}{2}$. Letting

$$G_I(z,z') = \frac{1}{D}\sum_{n=-\infty}^{\infty} Y_n e^{-j(\beta_o+\frac{2\pi n}{D})(z-z')} \qquad \text{IV.21}$$

$$G_{II}(z,z') = \frac{1}{d}\sum_{n=0}^{\infty} (2-\delta_{no}) Y'_n \cos\frac{n\pi(z+\frac{d}{2})}{d} \cos\frac{n\pi(z'+\frac{d}{2})}{d} \qquad \text{IV.22}$$

denote the Green's functions for regions I and II, the equation which $\mathcal{E}(z')$ must satisfy is conveniently written as,

$$\int_{-d/2}^{d/2} \left[G_I(z,z') - G_{II}(z,z')\right] \mathcal{E}(z')dz' = 0 \qquad |z| < \frac{d}{2}$$

$$\mathcal{E}(z') = 0 \qquad \frac{d}{2} < |z'| < \frac{D}{2} \quad . \qquad \text{IV.23}$$

In IV.23 $G_I(z,z')$ is a function of the propagation constant, β_o . Both $\mathcal{E}(z')$ and β_o are unknown. From the general theory of periodic structures it is known that for a given set of dimensions and for a particular frequency the slow wave circuit of Figure 9 supports waves only for special values of β_o. Consequently, it is expected that IV-23 can be satisfied only for these special values of β_o. In other words, IV.23 determines the propagation constant β_o as well as the field $\mathcal{E}(z')$.

IV.23 may be converted to a stationary form for the determination of the propagation constant, β_o, by multiplying by $\mathcal{E}(-z)$ and integrating from $-\frac{d}{2}$ to $+\frac{d}{2}$,

$$\int_{-d/2}^{d/2}\int_{-d/2}^{d/2} \mathcal{E}(-z)\left[G_I(z,z') - G_{II}(z,z')\right]\mathcal{E}(z')\,dz'\,dz = 0 \quad \text{IV.24}$$

For the true field $\mathcal{E}(z)$ and true propagation constant β_o this is a trivial operation since the integral IV.23 is identically zero for all z between $-\frac{d}{2}$ and $\frac{d}{2}$. However, if $\mathcal{E}(z)$ differs from the true field it will not, in general, be possible to satisfy IV.24, whereas it may be possible to satisfy IV.24 with a value of β_o which differs slightly from the true value. IV.24 is a weaker condition than IV.23; only a weighted average of the difference in tangential magnetic fields is required to vanish, the weighting function being $\mathcal{E}(-z)$.

By substituting various trial functions $\mathcal{E}(z)$ into IV.24, an implicit relation for the propagation constant β_o is obtained. It will now be shown that the value of β_o thus obtained is insensitive to small deviations in the trial function from the true field. Denoting the true field by $\mathcal{E}^o(z)$ and the true propagation constant by β_o^o, the first variation of IV.24 is

$$\iint_{-d/2}^{d/2} \delta\mathcal{E}(-z) \left[G_I(z,z') - G_{II}(z,z') \right]_{\beta_o=\beta_o^o} \mathcal{E}^o(z')\, dzdz'$$

$$+ \delta\beta_o \iint_{-d/2}^{d/2} \mathcal{E}^o(-z) \frac{\partial}{\partial\beta_o} \left[G_I(z,z') - G_{II}(z,z') \right]_{\beta_o=\beta_o^o} \mathcal{E}^o(z')\, dzdz'$$

$$+ \iint_{-d/2}^{d/2} \mathcal{E}^o(-z) \left[G_I(z,z') - G_{II}(z,z') \right]_{\beta_o=\beta_o^o} \delta\mathcal{E}(z')\, dzdz' = 0 . \qquad \text{IV.25}$$

The first term vanishes because of the integral equation

$$\int_{-d/2}^{d/2} \left[G_I(z,z') - G_{II}(z,z') \right]_{\beta_o=\beta_o^o} \mathcal{E}^o(z')\, dz' = 0 . \qquad \text{IV.23}$$

The third term vanishes because

$$\int_{-d/2}^{d/2} \mathcal{E}^o(-z) \left[G_I(z,z') - G_{II}(z,z') \right]_{\beta=\beta_o^o} dz = 0 \qquad \text{IV.26}$$

vanishes for all z'. This follows from the symmetry properties of the Green's functions,

$$G_I(z,z') = G_I(-z',-z)$$
$$G_{II}(z,z') = G_{II}(-z',-z) \qquad \text{IV.27}$$

and the original integral equation IV.23 . IV.26 is proved by substituting IV.27 into IV.23, replacing $-z'$ by z and $-z$ by z', and interchanging the limits of integration.

Thus

$$\delta\,\beta_o \iint_{-d/2}^{d/2} \mathcal{E}^o(-z)\, \frac{\partial}{\partial\beta_o}\left[G_I(z,z') - G_{II}(z,z')\right]_{\beta_o=\beta_o^o} \mathcal{E}^o(z')\,dz\,dz' = 0 \qquad \text{IV.28}$$

or $\delta\,\beta_o = 0$, provided the integral which multiplies $\delta\,\beta_o$ does not vanish. It seems unlikely that this integral vanishes and numerical calculation for the example discussed later bears out this conjecture. As a result, when IV.24 is used to determine the propagation constant, small errors in the trial function $\mathcal{E}(z)$ produce no error in the propagation constant to first order.

Substituting the series expansions for $\mathcal{E}(z)$ (IV.2 and IV.15) and the series expansion for the Green's functions (IV.21 and IV.22) into IV.24 and performing the indicated integrations yields

$$D \sum_{n=-\infty}^{\infty} Y_n M_n^2 + d \sum_{n=0}^{\infty} (2 - \delta_{no})\, Y'_n B_n^2 = 0 . \qquad \text{IV.29}$$

If the field is assumed to be uniform in the slot, $\mathcal{E}(z) = 1$,

$$M_n = \frac{1}{D} \int_{-d/2}^{d/2} e^{j\beta_o z}\, e^{j\frac{2\pi n z}{D}}\, dz = \frac{d}{D}\, \frac{\sin(\beta_o + \frac{2\pi n}{D})\frac{d}{2}}{(\beta_o - \frac{2\pi n}{D})\frac{d}{2}} \qquad \text{IV.30}$$

$$B_n = \frac{1}{d} \int_{-d/2}^{d/2} \cos\frac{n\pi(z+\frac{d}{2})}{d}\, dz = \delta_{no} \qquad \text{IV.31}$$

and IV.29 becomes

$$\sum_{n=-\infty}^{\infty} Y_n \left[\frac{d}{D}\; \frac{\sin(\beta_o + \frac{2\pi n}{D})\frac{d}{2}}{(\beta_o + \frac{2\pi n}{D})\frac{d}{2}}\right]^2 + \frac{d}{D}\, Y'_o = 0 . \qquad \text{IV.32}$$

A better trial function, which takes into account the singular nature of the fields at corners of the fins when $d \approx D$, is

$$\mathcal{E}(z) = \frac{\pi/2}{\sqrt{1-\left(\frac{2z}{d}\right)^2}} \quad . \qquad \text{IV.33}$$

In this case $M_n = \frac{d}{D} J_o\left(\frac{\beta_o d}{2} + \frac{\pi n d}{D}\right)$ IV.34

$$B_n = (-1)^{n/2} J_o\left(\frac{n\pi}{2}\right) \qquad n \text{ even}$$

$$= 0 \qquad n \text{ odd} \qquad \text{IV.35}$$

and IV.29 becomes

$$\sum_{n=-\infty}^{\infty} Y_n \frac{d}{D}\left[J_o\left(\frac{\beta_n d}{2} + \frac{\pi n d}{D}\right)\right]^2 + \frac{d}{D}\sum_{\substack{n=0\\ \text{odd}}}^{\infty} (2-\delta_{no})Y'_n J_o^2\left(\frac{n\pi}{2}\right) = 0 \quad . \qquad \text{IV.36}$$

Either IV.32 or IV.36 with the proper values of Y_n and Y'_n can be used to determine the propagation constant β_o. These two trial functions are not good approximations to the true field at very low frequencies, where the field, $\mathcal{E}(z)$, is nearly antisymmetric. In this region it would be better to employ a linear combination of a symmetric function and an antisymmetric function, determining the relative proportions by using the stationary property of IV.29. The solution of IV.32 for a particular set of dimensions is shown in Fig. 10. Computation is facilitated by converting IV.32 to a more rapidly converging series as discussed in ref.(17). The electron beam is assumed to be absent and only the propagation constant for the lowest mode has been determined. Curves of this type have been discussed by others, (21),(22),(23). A brief resume is given here.

Without loss of generality $\beta_o D$ may be assumed to lie between $-\pi$ and π. Since $\beta_n D = \beta_o D + 2\pi n$, the phase constants of the other space harmonics are obtained by simply displacing the fundamental curve $(n=0)$ by multiples of 2π. The velocity of energy propagation can be shown to be equal to the group velocity (24), which is given by

$$v_g = c\,\frac{\partial(kD)}{\partial(\beta_o D)} \qquad \text{IV.37}$$

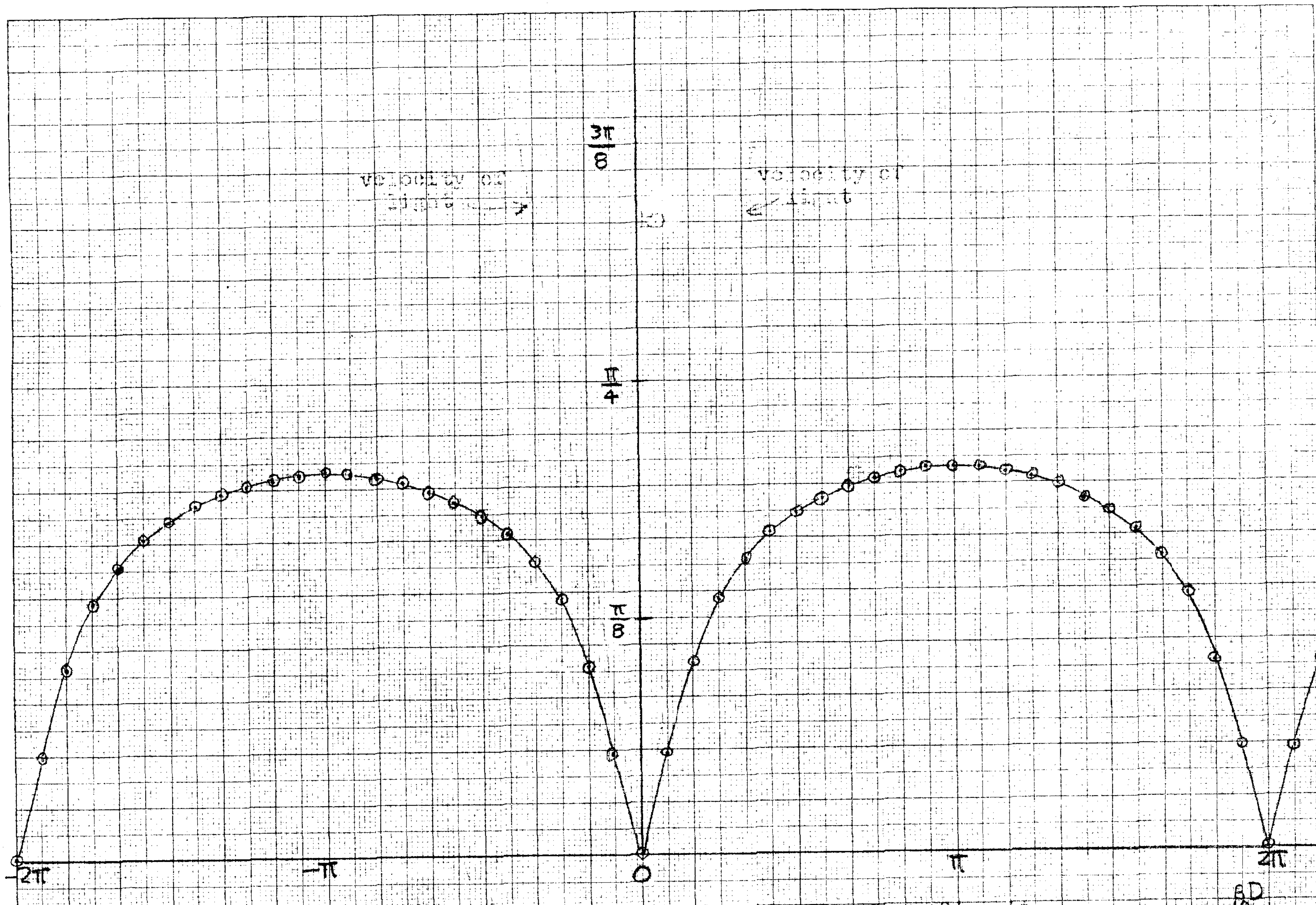

Figure 10 Phase Characteristics of the Loaded Guide Slow Wave Circuit
βD versus $\beta_0 D$ for $d/h = .8$, $c/b = 1.0$, $c/h = 2.0$

Thus when $\beta_o D$ is between 0 and π energy flows in the +z direction, while if $\beta_o D$ is between $-\pi$ and 0 energy flows in the -z direction. It should be noted that the phase velocity of space harmonics $\frac{\omega}{\beta_n}$ is not always of the same sign as the group velocity. For example, when $-\pi < \beta_o D < 0$ the group velocity is negative and the phase velocity of all space harmonics for which $n > 0$ is positive. A space harmonic whose phase velocity is of opposite sign from the group velocity at the wave is commonly called a backward space harmonic and sometimes, less precisely, a backward wave. When the phase velocity of a space harmonic has the same sign as the group velocity of the wave it is called a forward space harmonic or, sometimes, a forward wave. Backward space harmonics are of importance in oscillator tubes while forward space harmonics are of importance in amplifier tubes. At low frequencies (small kD, $kD = \frac{\omega D}{c}$), the behavior of this slow wave circuit is much like a strip transmission line, which propagated a transverse magnetic wave at the velocity of light. The effect of the fins is to capacitively load the line so as to decrease the phase velocity of the wave to about .65 times the velocity of light. As the frequency is increased, the phase shift per section increases until $\beta_o D = \pi$, where reflections from successive fins (or slots) reinforce and total reflection of the wave occurs. This is analogous to Bragg reflection of X-rays in crystal lattices. For a range of frequencies above the frequency at which total reflection occurs, no transmission occurs. Other transmission bends occur still higher in frequency but these are not of interest in the present discussion.

<u>Solution with the Electron Beam Present</u>. Solution of IV.29 is carried out in a similar manner when the electron beam is present, except

that the admittances Y_n are no longer the simple Y_{no} . In the preceding section it was found that the admittance with the beam present differs appreciably from that of free space only for certain rather narrow ranges of the propagation constant, β . The case of most interest in electron tubes is when the electron velocity is approximately equal to the phase velocity of one of the space harmonics, so that the electrons interact strongly with the circuit field. Under these circumstances the admittance for this particular space harmonic, n = m for example, will differ drastically from the free space value, while the other space harmonic admittances will be practically equal to their free space values. An important exception to this situation occurs when other space harmonics have approximately the same phase velocity as the cyclotron waves of the beam. This effect will be discussed later in this section, and for the present it will be assumed that the space harmonic admittances Y_n are equal to their free space values Y_{no} except for $n = m$. IV.29 can then be written

$$(Y_m - Y_{mo}) M_m^2 + \sum_{n=-\infty}^{\infty} Y_{no} M_n^2 + \frac{d}{D} \sum_{n=0}^{\infty} (2 - \delta_{no}) Y'_n B_n^2 = 0 \qquad \text{IV.38}$$

by adding and subtracting a term $Y_{mo} M_m^2$. Solving IV.38 for Y_m

$$Y_m = Y_{mo} + \frac{1}{M_m^2} \left[\sum_{n=-\infty}^{\infty} Y_{no} M_n^2 + \frac{d}{D} \sum_{n=0}^{\infty} (2 - \delta_{no}) Y'_n B_n^2 \right] . \qquad \text{IV.39}$$

This equation may be interpreted as follows. On the left stands the admittance of the electron beam at a plane which just grazes the circuit. The right side also has the dimensions of an admittance, and since the propagation constants of the system are found by equating this admittance to the beam admittance, this expression is just the admittance which the circuit presents to the electron beam. In the remainder of the analysis it will be convenient to subtract Y_{mo} from both sides of IV.39 and denote the quantity in brackets by jS. The j has been used in the definition in order that S be real for real β. S is a susceptance.

$$Y_m - Y_{mo} = \frac{jS}{M_m^2} \quad . \qquad \text{IV.40}$$

The quantity

$$S = \frac{1}{j}\left[\sum_{n=-\infty}^{\infty} Y_{no} M_n^2 + \frac{d}{D} \sum_{n=0}^{\infty} (2 - \delta_{no}) Y'_n B_n^2 \right] \qquad \text{IV.41}$$

is closely connected with the problem of finding the propagation constants of the circuit in the absence of the electron beam; when β_o is equal to the propagation constant of the circuit in the absence of the beam, S is zero. If the propagation constant of the m^{th} space harmonic with the beam absence is denoted by β_m and the propagation constant of the m^{th} space harmonic with the beam present is denoted simply by β, S can be approximated by

$$S = \left(\frac{\partial S}{\partial \beta}\right)_{\beta_m} (\beta - \beta_m) + \frac{1}{2}\left(\frac{\partial^2 S}{\partial \beta^2}\right)_{\beta_m} (\beta - \beta_m)^2 \qquad \text{IV.42}$$

in a small neighborhood around β_m. Except when $\beta_m D \approx (2m \pm 1)\pi$, i.e., when operation is near the upper cutoff frequency of the circuit and the first derivative is small, the first derivative term by itself is a satisfactory approximation to S.

S is a periodic function of βD with period 2π. This is verified by noting that replacing $\beta_o D$ by $\beta_o D + 2\pi$ in IV.41 is equivalent to replacing n by $n+1$ in the first summation and does not affect the second summation. Since the first summation is over all values of n, this sum is also unchanged. Furthermore, S is an even function of βD. This may be demonstrated either by examining the terms in the sum in detail or by appealing to the symmetry properties of the circuit. Similarly $\frac{\partial S}{\partial \beta}$ is an odd periodic function of $\beta_o D$ with period 2π. Thus $\left(\frac{\partial S}{\partial \beta}\right)_{\beta_m}$ does not depend on m since the different values of $\beta_m D$ differ by 2π.

As a result, if $-\pi < \beta_o D < 0$ so that the net energy flow of the wave is in the negative z direction, $\left(\frac{\partial S}{\partial \beta}\right)_{\beta_m}$ will be equal in magnitude but opposite in sign from the value of $\left(\frac{\partial S}{\partial \beta}\right)_{\beta_m}$ for the corresponding wave with energy flow in the positive z direction $(0 < \beta_o D < \pi)$ at the same frequency. Figure 11 shows $\left(\frac{\partial S}{\partial \beta}\right)_{\beta_m}$ for the circuit whose characteristics are shown in Figure 10.

Thus when the electron beam has a velocity approximately equal to the phase velocity of the m^{th} space harmonic of the slow wave circuit, the circuit presents a susceptance to the beam given approximately by

$$B_m - B_{mo} = \frac{1}{M_m^2} \left(\frac{\partial S}{\partial \beta}\right)_{\beta_m} (\beta - \beta_m) \qquad \text{IV.43}$$

where the sign of $\left(\frac{\partial S}{\partial \beta}\right)_{\beta_m}$ is negative for forward space harmonic interaction and positive for backward space harmonic operation. This is the principal result of this section. It gives a simple representation of the susceptance presented to the electron beam by the slow wave

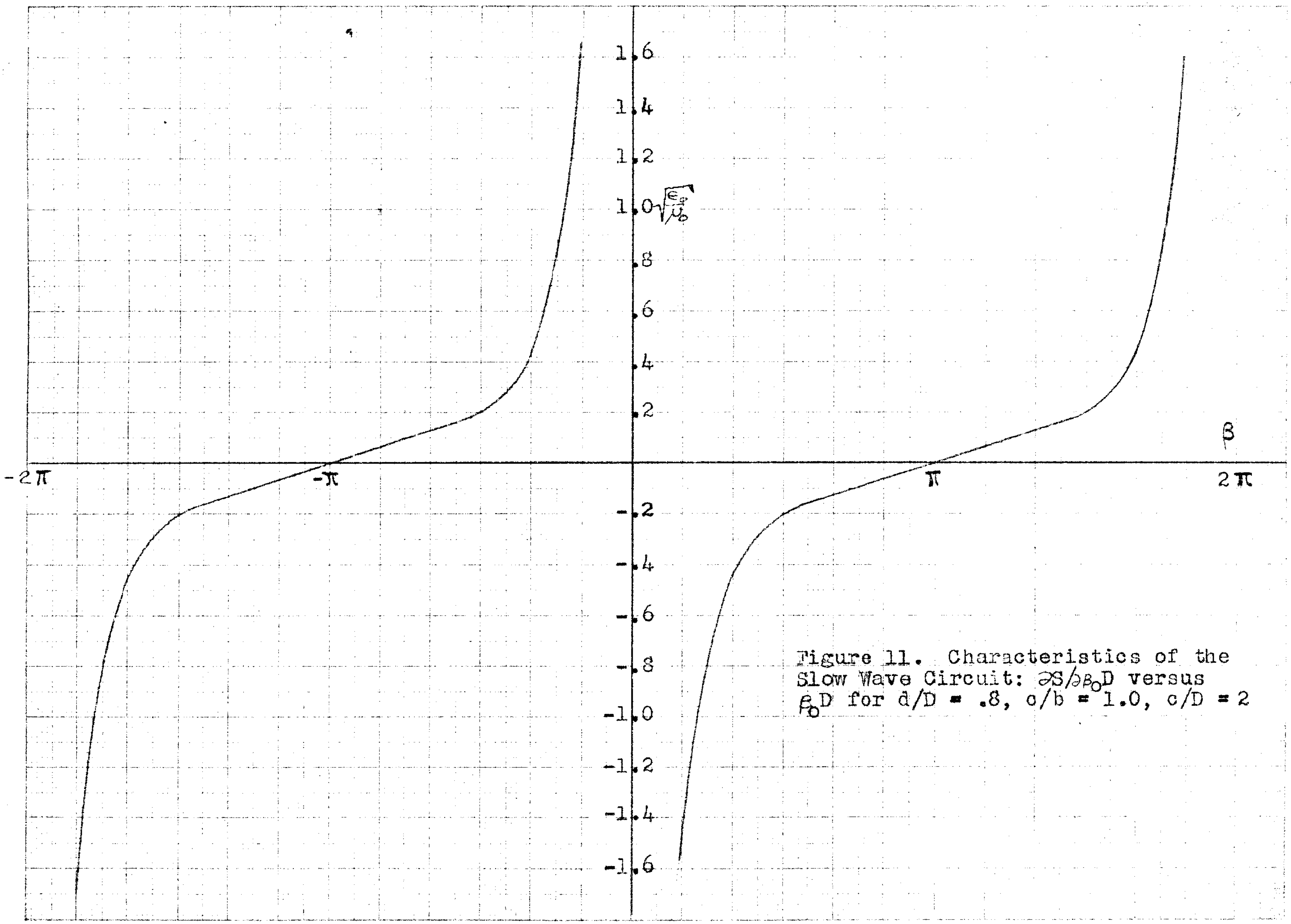

Figure 11. Characteristics of the Slow Wave Circuit: $\partial S/\partial \beta_o D$ versus $\beta_o D$ for d/D = .8, c/b = 1.0, c/D = 2

circuit. Fletcher (5) has shown that it is possible to represent the susceptance of a sheath helix, at the surface of the helix, in this manner. He has also demonstrated for the sheath helix, the equivalence between this type of representation and the Pierce equivalent circuit representation. It seems likely that most slow wave circuits can be represented approximately in this manner. Pierce (8) for example, assumes that this is possible. The treatment of this section demonstrates by means of a field analysis that this is possible for the loaded strip transmission line, a space harmonic circuit, and determines the pertinent constants of the representation. It is also possible to determine these constants experimentally (25) (26).

In Section V it will be assumed that the slow wave circuit may be represented by a susceptance

$$B_m = B_{mo} + C_m \frac{\beta - \beta_m}{\beta_m} \qquad \text{IV.44}$$

placed at the plane of the circuit. The values of C_m and β_m obtained in this section will not be used specifically in the computations, but rather a wider variety of values such as might be obtained with other types of slow wave circuits will be assumed.

<u>Comparison with the Pierce Circuit Equation.</u> Pierce derives a similar circuit equation using the normal mode theory (8) but the constants in his circuit equation must be evaluated from a field analysis (4),(5) or by experiment (25),(26). The difference in sign in the circuit equation between forward space harmonic operation and backward space harmonic operation can be deduced from the equivalent circuit approach, but the above analysis constitutes a proof of the validity of this type of circuit representation. The relationship between the

constant C_m in IV.49 and Pierce's traveling wave tube impedance parameter K will now be determined. Pierce's circuit equation for a thin beam is (see Figure 12)

$$\frac{V}{i} = \frac{\beta\,\beta_m\,K}{\beta^2 - \beta_m^2} + \frac{\beta}{\omega C_1} \qquad \text{IV.45}$$

where V is the voltage at the electron beam, i is the convection current of the beam, and C_1 is the capacitance between beam and circuit in a unit length of circuit.

IV.45 gives the circuit impedance at the surface of the circuit if the second term is neglected.

$$\frac{V}{i} = \frac{\beta\,\beta_m}{\beta^2 - \beta_m^2}\,K \quad .$$

Furthermore, if $\beta \approx \beta_m$ this may be written

$$\frac{V}{i} = \frac{\beta_m}{2(\beta - \beta_m)}\,K \quad \text{or} \quad \frac{i}{V} = \frac{2}{K}\,\frac{\beta - \beta_m}{\beta_m} \quad . \qquad \text{IV.46}$$

IV.46 is similar in form to IV.44. If the beam is a thin sheet beam of width w, the convection current in the z direction is related to the discontinuity in magnetic field by

$$i_z = \left[(H_{1x})_{below} - (H_{1x})_{above} \right] \cdot w \quad .$$

The longitudinal field is related to the voltage V by

$$V = \frac{E_{1z}}{j\beta} \approx \frac{E_{1z}}{j\beta_m}$$

so that IV.46 may be written

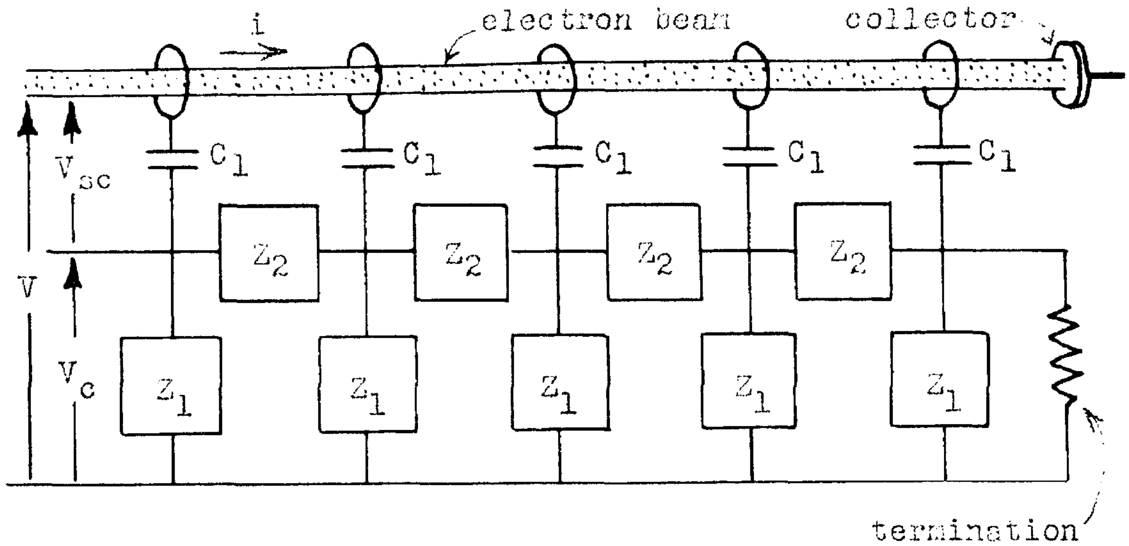

Figure 12. The Pierce Traveling Wave Tube Equivalent Circuit

$$\frac{i}{V} = \frac{2}{K}\frac{\beta - \beta_m}{\beta_m} = j\,\beta_m\,w\,\frac{(H_{1x})_{below} - (H_{1x})_{above}}{E_{1z}}$$

$$= j\,\beta_m\,w\left[Y_m - Y_{mo}\right] \qquad \text{IV.47}$$

$$= \beta_m\,w\left[B_{mo} - B_m\right] \qquad (Y_m = j\,\beta_m)\ .$$

Solving for $B_m - B_{mo}$,

$$B_m - B_{mo} = -\frac{2}{K}\,\frac{1}{\beta_m w}\,\frac{\beta - \beta_m}{\beta_m}\ . \qquad \text{IV.48}$$

Comparison of IV.48 with IV.44 shows that the Pierce traveling wave tube impedance parameter is inversely proportional to C_m ,

$$K = -\frac{2}{\beta_m w}\,\frac{1}{C_m}\ . \qquad \text{IV.49}$$

K is the value of the impedance parameter at a plane which just grazes the circuit. Thus IV.44 can also be written

$$B_m - B_{mo} = -\frac{2}{K}\,\frac{1}{\beta_m w}\,\frac{\beta - \beta_m}{\beta_m} \qquad \text{IV.50}$$

and, in terms of the normalized susceptance,

$$Q_m - Q_{mo} = \frac{2}{k\,w}\,\frac{\sqrt{\frac{\mu_o}{\epsilon_o}}}{K}\,\frac{(\beta - \beta_m)}{\beta_m}\ . \qquad \text{IV.51}$$

IV.51 can also be expressed in terms of the variable ν of the preceding section (III).

$$Q_m - Q_{mo} = \frac{1}{\chi^2}(\nu - b) \qquad \text{IV.52}$$

where

$$\chi^2 = \frac{kw}{2}\,\frac{K}{\sqrt{\frac{\mu_o}{\epsilon_o}}}\,\frac{1}{r^2{}_m} \qquad \text{IV.53}$$

and b is defined by

$$\beta_m = \beta_+ (1 + r^2 m b) \quad . \qquad \text{IV.54}$$

b is a measure of the difference in velocity of upper edge electrons and the phase velocity of the m^{th} space harmonic in the absence of the electron beam. When $b > 0$ the upper edge electrons travel faster than the space harmonic field. $\varkappa^2$ is a dimensionless parameter which is a measure of the strength of circuit field.

In Section V the circuit equation, IV.52, will be combined with the electronic equation, III.68, to find the waves of the electron beam in the presence of the circuit.

Operation Near the upper Cutoff Frequency, $\beta_m \approx \frac{2m+1}{D}\pi$. In this region two space harmonics are important; one is a forward space harmonic and one is a backward space harmonic. It is necessary to retain both terms of IV.42 in approximating S . The consequences of the extra term and the conditions under which it must be retained will now be examined.

When the electron beam is absent, the phase constants of the circuit space harmonics may be found by setting S, as given by IV.42, equal to zero. Clearly, one solution is $\beta = \beta_m$, but this equation has two solutions. To the extent that third and higher order terms can be neglected, the other solution must represent the nearby space harmonic which has a propagation constant

$$\begin{aligned} \beta &= \frac{2m + 1}{D}\pi + \left(\frac{2m + 1}{D}\pi - \beta_m\right) \\ &= \frac{2m \pm 1}{D} 2\pi - \beta_m \quad . \end{aligned} \qquad \text{IV.55}$$

It should be noted that these two space harmonics are not space harmonics of the same wave. One is a space harmonic of a wave whose net energy flow is in the positive z direction and the other is a space harmonic of a wave whose net energy flow is in the negative z direction. As such, these space harmonics, together with the waves with which they are associated, can exist on the circuit independently of each other. With the aid of IV.55, IV.42 can be rewritten

$$S = \left(\frac{\partial S}{\partial \beta}\right)_{\beta_m} (\beta - \beta_m)\left(1 + \frac{\beta - \beta_m}{2\beta_m - \frac{2m \pm 1}{D}\, 2\pi}\right) \quad . \qquad \text{IV.56}$$

The second factor has been adjusted so as to vanish at the value given by IV.55 and still have the form given by IV.42. From IV.47 it can be seen that the one term approximation will be valid when the electron beam is present if all the propagation constants, β , which are determined using the one term approximation satisfy the condition

$$|\beta - \beta_m| << 2\left(\beta_m - \frac{2m \pm 1}{D}\pi\right) \quad . \qquad \text{IV.57}$$

For electron beams which have approximately the same velocity as the m^{th} space harmonic $|\beta - \beta_m|$ is generally between $.01\ \beta_m$ and $.1\ \beta_m$. Finally, it should be noted that as $\beta_m \rightarrow \frac{2m \pm 1}{D}\pi$ the second term of the second factor in IV.56 becomes large compared with the first term, but that the factor $\left(\frac{\partial S}{\partial \beta}\right)_{\beta_m}$ approaches zero in such a manner that $\left(\frac{\partial^2 S}{\partial \beta^2}\right)_{\beta_m}$ is approximately constant.

The two term approximation is not used in subsequent analysis but is necessary for a correct analysis of operation near the high frequency cutoff.

Simultaneous Interaction of the Electron Beam with More Than One Space Harmonic of the Same Wave. This effect has been termed the rising sun effect by French workers (6), because of its similarity to an effect which occurs in rising sun magnetrons. It is observed to cause slight anomalies in the starting conditions. Although this effect is not studied in Section V, the formal modifications which would be required in the theory are presented here. It is possible to have simultaneous interaction with three space harmonics of the same wave if

$$\frac{\omega_c}{u} \approx \frac{2\pi}{D}\ell \qquad \ell = \text{integer} , \qquad \text{IV.58}$$

for under these circumstances when electrons have a velocity approximately equal to the phase velocity of the m^{th} space harmonic, the slow cyclotron wave is approximately in synchronism with the $m+\ell$ space harmonic and the fast cyclotron wave is in synchronism with the $m-\ell$ space harmonic. It is then necessary to treat three terms separately in IV.29 ,

$$\left[Y_{(m-\ell)} - Y_{(m-\ell)o}\right] M^2_{(m-\ell)} + \left[Y_m - Y_{mo}\right] M^2_m + \left[Y_{(m+\ell)} - Y_{(m+\ell)o}\right] M^2_{(m+\ell)} + jS = 0 . \qquad \text{IV.59}$$

The various admittances $Y_{m-\ell}$, Y_m, $Y_{m+\ell}$, etc. are functions of the propagation constants

$$\beta - \frac{2\pi\ell}{D} , \quad \beta , \quad \beta + \frac{2\pi\ell}{D}$$

respectively, where β is the propagation constant which is approximately equal to β_m , and must be determined from the theory given in the previous section. The approximation IV.42 for S may still be used.

V START OSCILLATION CONDITIONS FOR THE BACKWARD WAVE OSCILLATOR

Characteristic Waves of the System. The electronic equation, III.69 must be combined with the circuit equation IV.52 to determine the propagation constants of the waves of the beam in the presence of the circuit. The resulting equation is

$$\frac{1}{\gamma^2}(\nu - b) = \frac{-\nu\left[\frac{2T}{1+T}\frac{1-T'}{1+T'}\right] + \frac{1-T'}{1+T'}\left[s\beta_- t - \frac{T}{1+T}\right]}{\nu^2 + \nu\left[\frac{T}{1+T}\frac{1-T'}{1+T'} - s\beta_- t\right] + \frac{T'}{1+T'}\left[s\beta_- t - \frac{T}{1+T}\right]} . \qquad \text{V.1}$$

Since $T = \tanh \beta t$ and $T' = \tanh \beta d$ are slowly varying functions of β, they may be considered as constants in this equation. They are to be evaluated at $\beta = \frac{\omega}{\sqrt{u_+ u_-}} \approx \beta_+ \approx \beta_-$. IV.1 has three solutions, since it is of third degree in ν . When the coupling between the circuit and beam is weak, it is expected that these solutions will represent a circuit wave and an upper and lower surface wave of the beam. The solutions of this equation will be discussed in more detail in connection with the numerical examples.

The analysis which leads to V.1 neglects all waves except those which have a phase velocity nearly equal to the electron velocity. Thus many of the characteristic waves will not be obtained from V.1; the cyclotron waves and the waves with density modulation in the interior of the beam, found in the non-slipping beam, are omitted. Higher order waves of the circuit are also omitted.

Boundary Conditions at $z = 0, L$. The boundary conditions which must be satisfied at $z = 0$ and $z = L$ can be described loosely as

(a) The beam enters unmodulated at a plane $z = 0$. The y and z velocity and displacement of each electron from its equilibrium position is zero. The electric field at $z = 0$ depends on the exact details of the coupling of circuit to an external waveguide or transmission line.

(b) The circuit is terminated with a perfect absorber of electromagnetic energy at $z = L$. In practice the circuit termination is confined to the immediate vicinity of the slow wave circuit and spread over a short distance in the z direction.

It is clear that even if these conditions could be stated more precisely, such as by specifying the exact shapes of all conductors and absorbers, it is only possible to meet three boundary conditions with the three waves that have been studied. For this reason this analysis is far short of a complete field analysis. Only certain aspects of the problem have been studied from the field point of view. It is, for example, possible to give a complete description of the fields associated with the three waves which have been studied.

Since the analysis has been reduced to a treatment of what are thought to be the three most important waves of the system, it is necessary to select the three most important boundary conditions. The electron beam propagates two waves in the absence of the circuit. One is associated with the upper boundary and one is associated with the lower boundary. Thus two conditions should probably be applied to the electron beam at $z = 0$ in such a manner as to determine the strength of these two surface waves. It is more important to have the a.c. current in the electron beam be zero at $z = 0$ than to have the velocity be zero there since current modulation produces a direct effect in the circuit while velocity modulation produces an effect only after it has been converted to a current. In Section III it was shown that when $\omega_p^2 << \omega_c^2$ most of the a.c. beam current is in the form of surface current. Hence it seems reasonable to require that the surface current at the two beam edges be zero at $z = 0$.

It is interesting to note in this regard that J. W. Sedin (27) has shown that in the theory of Pierce and Muller it is sufficient to specify the current in the beam at $z = 0$. In this case there are only two waves near the electron velocity rather than three, so that only one condition is required at $z = 0$. Sedin also finds that the cyclotron waves, although they are excited only to a small extent and produce very little electric field, produce a velocity modulation of the beam which is comparable with that produced by the waves near the electron velocity. These waves are necessary to meet the boundary conditions on the velocity at $z = 0$. However, little error in the start oscillation conditions results by neglecting these waves and not satisfying the initial velocity conditions.

The third wave which is included in V.1 is the wave of the slow wave circuit whose group velocity and power flow are in the negative z direction. Thus the third boundary condition should probably be applied at $z = L$ in such a manner as to determine the strength of the circuit wave. This boundary condition is most easily formulated by analogy with the Pierce theory of traveling wave tubes. The Pierce equivalent circuit is shown in Figure 12. Since only the circuit wave with energy flow to the left is required, the absence of energy flow to the left at $z = L$ can be insured by taking V_c , the circuit voltage, to be zero at this point. To be sure, the wave to the left is excited in the region between $z = 0$ and $z = L$ by the electron beam, and must be employed in that region.

By analogy it may be argued that the boundary condition which represents a termination at $z = L$ in the field analysis is: $E_z(d) = 0$, i.e., the electric field at the surface of the circuit must be zero. It is permissible to use the electric field in this case since it differs from the potential only by the factor $+j\beta$, which is nearly the same for all waves. Notice also that zero longitudinal field at the circuit does not imply zero longitudinal field at the beam, since these points are generally separated by a small distance. Similarly, in the Pierce equivalent circuit, $V_c = 0$ does not imply $V = 0$ since there can be a voltage drop V_{sc} across the capacitance C_1, which represents the space between the beam and the circuit.

The Starting Conditions. The amplitudes of the three waves will be specified by giving the strength of the longitudinal electric field of each wave at the upper edge of the beam. Denoting these amplitudes by $E^{(1)}$, $E^{(2)}$, and $E^{(3)}$, the total longitudinal field at the upper edge may be written,

$$E_{1z}(o)=E^{(1)}\, e^{-j\beta_1 z} + E^{(2)}\, e^{-j\beta_2 z} + E^{(3)}\, e^{-j\beta_3 z} \quad . \qquad \text{V.3}$$

The surface current density on the upper edge of the beam for a single wave of propagation constant β is

$$i_+ = u_+\sigma_+ = -\frac{\omega_p^2 u_+ \epsilon_o E_{1z}(o)}{j(\omega - \beta u_+)\omega_c} \quad .$$

This may be rewritten in terms of the variable ν as

$$i_+ = -j \frac{u_+ \epsilon_o E_{1z}(o)}{\nu} \quad . \qquad \text{V.4}$$

Hence the total surface current of the upper edge of the beam when all three waves are present is

$$i_+ = -j u_+ \epsilon_o \sum_{i=1}^{3} \frac{E^{(i)} e^{-j\beta_i z}}{\nu_i} \quad . \qquad \text{V.5}$$

Since this must vanish at $z = 0$, one of the three boundary conditions is expressed by

$$\sum_{i=1}^{3} \frac{E^{(i)}}{\nu_i} = 0 \quad . \qquad \text{V.6}$$

To express the other two boundary conditions, E_{1z} at the lower edge of the beam and at the circuit must be expressed in terms of $E^{(i)}$. To do this note that, for a single wave, E_{1z} and E_{1y} in the interior of the beam are given by

$$E_{1z} = E^{(1)} \left[\cosh \beta_1 y + A_1 \sinh \beta_1 y\right] \qquad \text{V.7}$$

$$E_{1y} = jE^{(1)} \left[\sinh \beta_1 y + A_1 \cosh \beta_1 y\right] . \qquad \text{V.8}$$

A_1 may be determined as follows. At $y = -t$, the lower edge of the beam,

$$E_{1z} = E^{(1)} \left[\cosh \beta_1 t - A_1 \sinh \beta_1 t\right] \qquad \text{V.9}$$

$$E_{1y} = jE^{(1)} \left[-\sinh \beta_1 t + A_1 \cosh \beta_1 t\right]. \qquad \text{V.10}$$

Equating $\frac{E_{1y}}{E_{1z}}$ obtained from the quotient of these two expressions to

$$j \left[1 - \frac{\omega_p^2}{\omega_c(\omega - \beta_1 u_-)}\right] \approx j \left[1 - \frac{1}{s\, \beta_- t - \quad 1}\right] \qquad \text{V.11}$$

the admittance just above the lower edge of the beam, the following relationship is obtained

$$\frac{-T + A_1}{1 - TA_1} = 1 - \frac{1}{s\,\beta\_t - \nu_1} = \frac{s\,\beta\_t - \nu_1 - 1}{s\beta\_t - \nu_1} \, .$$

Solving for A_1 ,

$$A_1 = \frac{\nu_1 - s\beta\_t + \frac{1}{1+T}}{\nu_1 - s\beta\_t + \frac{T}{1+T}} \, . \qquad \text{V.12}$$

The total longitudinal field of lower edge of the beam $E_{1z}(-t)$ is found by substituting V.12 into V.9 and summing over all three waves

$$E_{1z} = \cosh \beta t \left[1 - T\right] \sum_{i=1}^{3} E^{(i)} \, e^{-j\beta_i z} \, \frac{\nu_i - s\beta\_t}{\nu_i - s\beta\_t + \frac{T}{1+T}} \, . \qquad \text{V.13}$$

The surface current density at the lower edge of the beam when a single wave is present, is given by

$$i\_ = u\_\sigma\_ = -j \, \frac{u\_ \, \omega_p^2 \, \epsilon_o \, E_z}{(\omega - \beta u\_) \, \omega_c} \, ,$$

which may be rewritten

$$i\_ = j \, \frac{u\_ \, \epsilon_o \, E_{1z}}{\nu - s\beta\_t} \, . \qquad \text{V.14}$$

The total surface current density at the lower edge of the beam is given by summing the contributions from the three waves

$$i\_ = j \, u\_ \epsilon_o \cosh \beta t \left[1 - T\right] \sum_{i=1}^{3} \frac{E^{(i)} \, e^{j\beta_i z}}{\nu_i - s\beta\_t + \frac{T}{1+T}} \, . \qquad \text{V.15}$$

Since this also must vanish at $z = 0$, the second of the three boundary conditions is expressed by,

$$\sum_{i=1}^{3} \frac{E^{(i)}}{\nu_i - s\beta_- t + \frac{T}{1+T}} = 0 \quad . \qquad \text{V.16}$$

The fields in the region above the beam may be written

$$E_{1z} = E^{(1)} \left[\cosh \beta_1 y + A_2 \sinh \beta_1 y\right] \qquad \text{V.17}$$

$$E_{1y} = jE^{(1)} \left[\sinh \beta_1 y + A_2 \cosh \beta_1 y\right] \qquad \text{V.18}$$

where A_2 may be determined as follows. From V.7, V.8, V.17, and V.18

$$\left.\frac{E_{1y}}{E_{1z}}\right|_{y=0_+} = jA_2 \quad \text{and} \quad \left.\frac{E_{1y}}{E_{1z}}\right|_{y=0_-} = jA_1 \quad .$$

These ratios differ by the normalized admittance of the equivalent surface charge density at $y = 0$

$$jA_2 = jA_1 + j\frac{\omega_p^2}{(\omega - \beta_1 u_+)\omega_o} \quad .$$

Using the vallue of A_1 given by V.12

$$A_2 = A_1 - \frac{1}{\nu_1} = \frac{\nu_1 - s\beta_- t + \frac{1}{1+T}}{\nu_1 - s\beta_- t + \frac{T}{1+T}} - \frac{1}{\nu_1} \quad . \qquad \text{V.19}$$

Denoting A_2, which is a function of ν, by $A_2(\nu)$, the total longitudinal field at the circuit may be written as

$$E_{1z} = \cosh \beta d \sum_{i=1}^{3} \left[1 + A_2(\nu_i)T'\right] E^{(i)} e^{-j\beta_i z} \qquad \text{V.20}$$

where $T' \equiv \tanh \beta d$.

This must vanish at $z = L$. Thus the third boundary condition is expressed by

$$\sum_{i=1}^{3} \left[1 + A_2(\nu_i)T'\right] E^{(i)} e^{-j\beta_i L} = 0 \qquad \text{V.21}$$

To obtain the start oscillation condition V.6, V.16 and V.21 must be satisfied simultaneously with non-zero $E^{(1)}$, $E^{(2)}$, or $E^{(3)}$. This will only be possible for certain values of b and L.

From V.6 and V.16 it is easily established that $E^{(1)}$, $E^{(2)}$, and $E^{(3)}$ must be in the ratio

$$\frac{1}{\nu_2(\nu_3 - s\beta_- t + \frac{T}{1+T})} - \frac{1}{\nu_3(\nu_2 - s\beta_- t + \frac{T}{1+T})} ,$$

$$\frac{1}{\nu_3(\nu_1 - s\beta_- t + \frac{T}{1+T})} - \frac{1}{\nu_1(\nu_3 - s\beta_- t + \frac{T}{1+T})} ,$$

$$\frac{1}{\nu_1(\nu_2 - s\beta_- t + \frac{T}{1+T})} - \frac{1}{\nu_2(\nu_1 - s\beta_- t + \frac{T}{1+T})} ,$$

so that V.21 may be written

$$\sum_{i=1}^{3} \left[1 + A_2(\nu_i)T'\right]\left[\frac{1}{\nu_j(\nu_k - s\beta_- t + \frac{T}{1+T})} - \frac{1}{\nu_k(\nu_j - s\beta_- t + \frac{T}{1+T})}\right] e^{-j\beta_i L} = 0 \qquad \text{V.22}$$

where $i\ j\ k$ are cyclical permutations of 1, 2, and 3. Since

$$\beta_i = \beta_+ (1 + mr^2 \nu_i)$$

a common, non-zero factor $e^{-j\beta_+ L}$ can be removed from V.22 leaving

$$\sum_{i=1}^{3}\left[1+A_2(\nu_i)T'\right]\left[\frac{1}{\nu_j(\nu_k - s\beta_- t + \frac{T}{1+T})} - \frac{1}{\nu_k(\nu_j - s\beta_- t + \frac{T}{1+T})}\right]e^{-j\nu_i\theta} = 0 \qquad \text{V.23}$$

where $\theta \equiv \beta_+ L\, mr^2$. The roots, ν_i, of V.1 are functions of the geometrical parameters of the circuit and the beam, the beam current, and the parameter b , which denotes the difference in velocity between the upper edge electrons and the circuit space harmonic in the absence of the electrons. The procedure which will be adopted here is to assume that the beam current and all the geometrical parameters, except the length of the tube, are known. The length of the tube at which oscillation begins to occur is found by adjusting θ , the length parameter, and b , the velocity difference parameter until V.23 is satisfied.

Numerical Solution of the Start Oscillation Conditions. The following more or less typical conditions have been assumed for the numerical work which follows

$$\begin{array}{llr} \beta t = .50 & T = .4621 & \\ \beta d = .50 & T' = .4621 & \qquad\qquad \text{V.24} \\ \beta a = \infty & C = 1.0000 \; . & \end{array}$$

The circuit impedance, represented in dimensionless form by $\mathcal{X}^2$, is varied since it is expected that when it is small space charge effects will be important and when it is large space charge effects will be unimportant. Although the circuit analyzed in Section IV is capable of providing only a limited range of values of $\mathcal{X}^2$, a rather wide range of $\mathcal{X}^2$ is likely to be encountered when other circuits are also considered. Calculations have been made for the non-slipping case $(s = 0)$ as well as the slipping beam case $(s = 1)$.

The first stage of the computation is to solve V.1 for ν_1, ν_2, and ν_3, for a number of values of b. Figures 13, 14, and 15 are plots of the solutions of V.1 versus b for $\gamma^2 = -.1$, $-.01$, $-.001$ and the remaining parameters given by V.24. As $|b|$ increases all curves are asymptotic to straight lines. Two of the asymptotes are the solutions which would be obtained by replacing the circuit by a conducting plane and the third asymptote is $\nu = b$, the circuit solution in the absence of the electron beam. Thus in these regions the solutions represent waves similar to those studied in Sections III and IV. In the intermediate regions where the curves deviate considerably from the asymptotes, the waves of the beam and the wave of the circuit are coupled together and are of an intermediate nature.

It is interesting to note that in the slipping stream case the waves of the beam are described by complex conjugate values of ν, hence one wave increases and one wave decreases with z. This effect was described in Section III and it is found to affect the starting conditions significantly. In the non-slipping case (s = 0) the waves of the beam are constant amplitude waves, one faster than the electrons and one slower than the electrons. In Figures 13b, 14b, and 15b, it can be seen that when the phase velocity of the unperturbed circuit wave is approximately equal to the phase velocity of the faster of the two beam waves (ν_3), an increasing and decreasing pair of waves results. A similar situation occurs in the ordinary backward wave oscillator (28), (29).

To find the zeros of V.23 it is convenient to plot the function

$$\frac{F(b,\theta)}{F(b,0)}$$

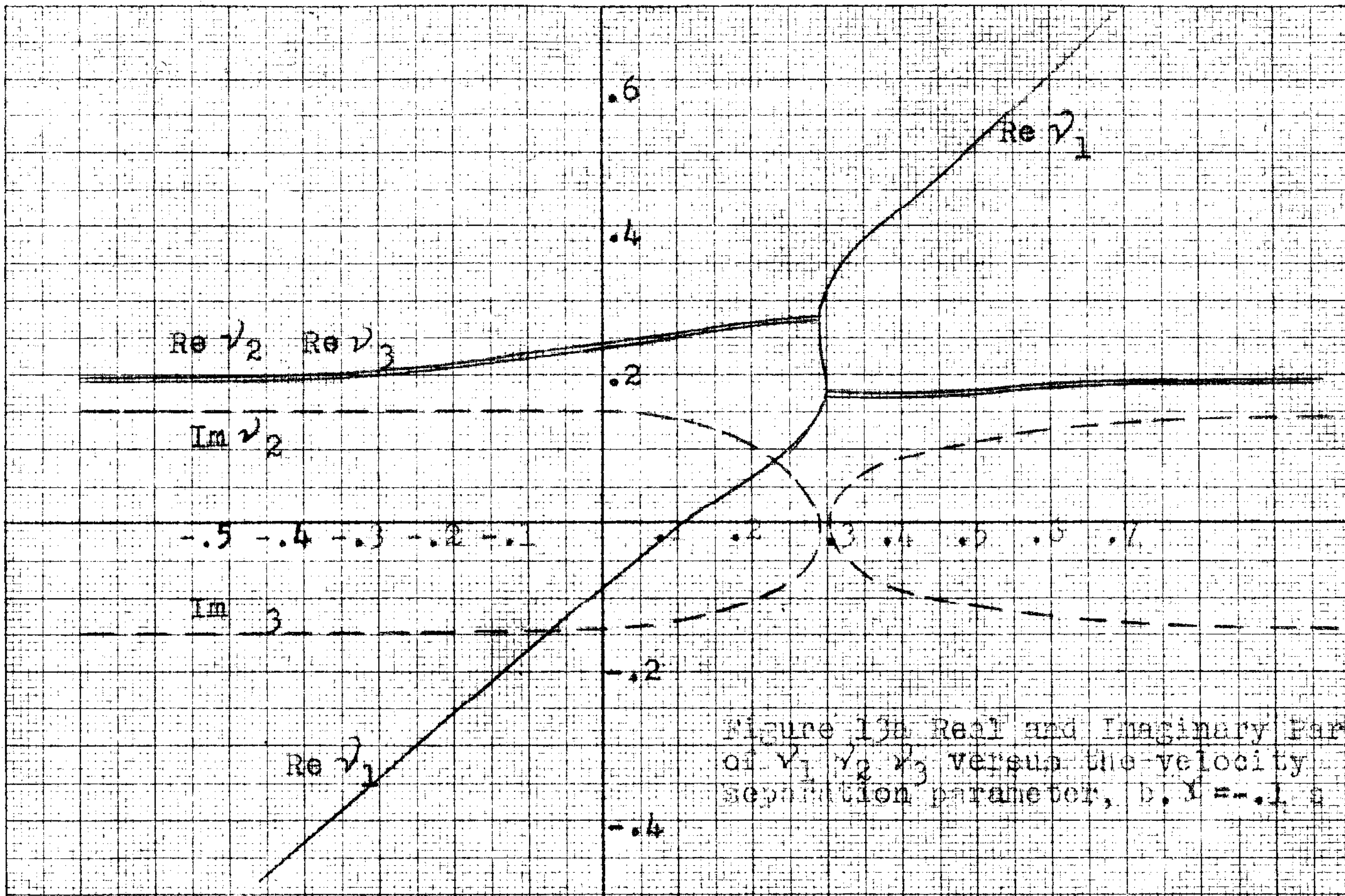

Figure 13a Real and Imaginary Parts of ν_1 ν_2 ν_3 versus the velocity separation parameter, b. γ = -.1 s = 1

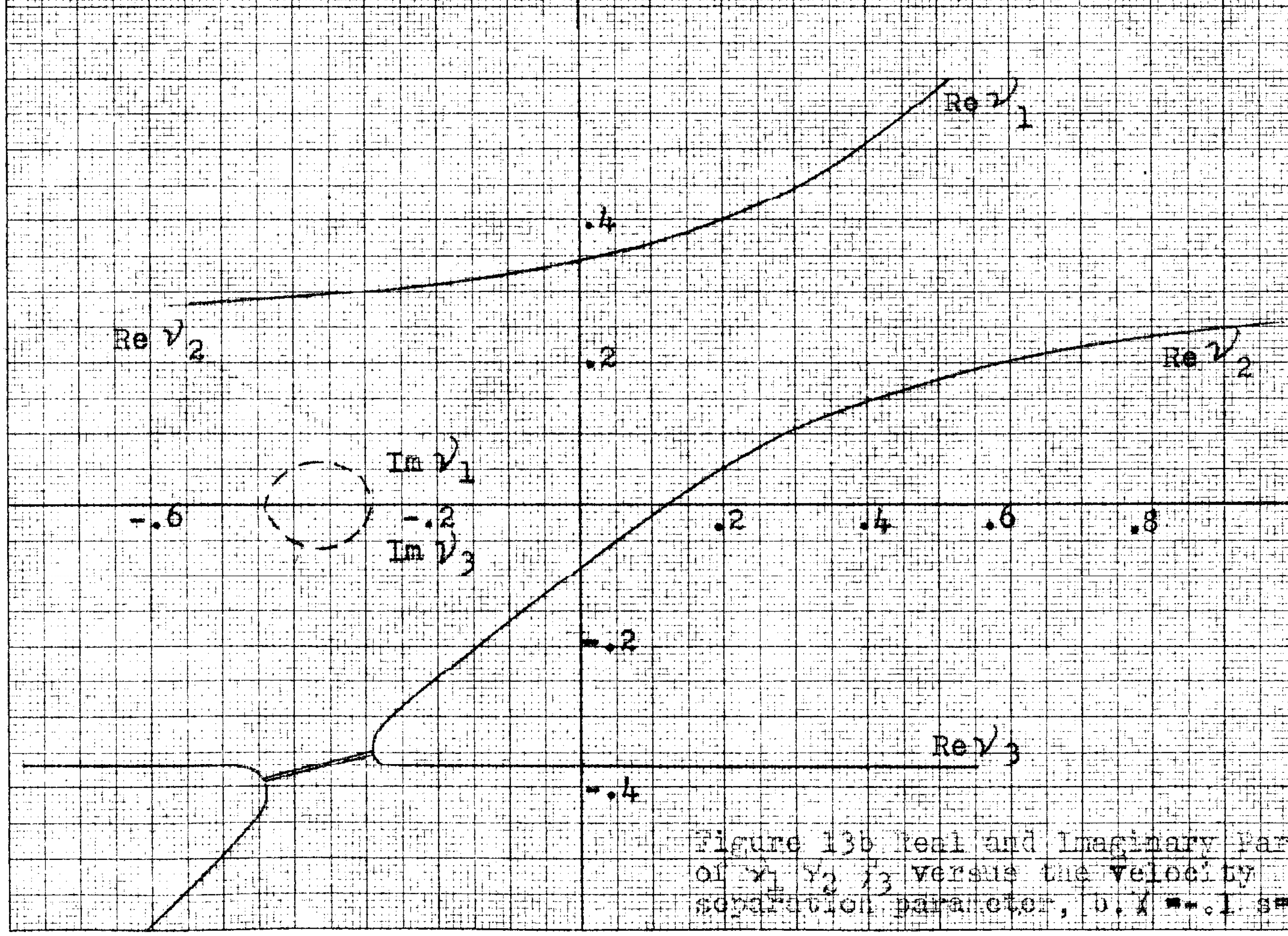

Figure 13b Real and Imaginary Parts of ν_1 ν_2 ν_3 versus the velocity separation parameter, b. γ = -.1 s=0

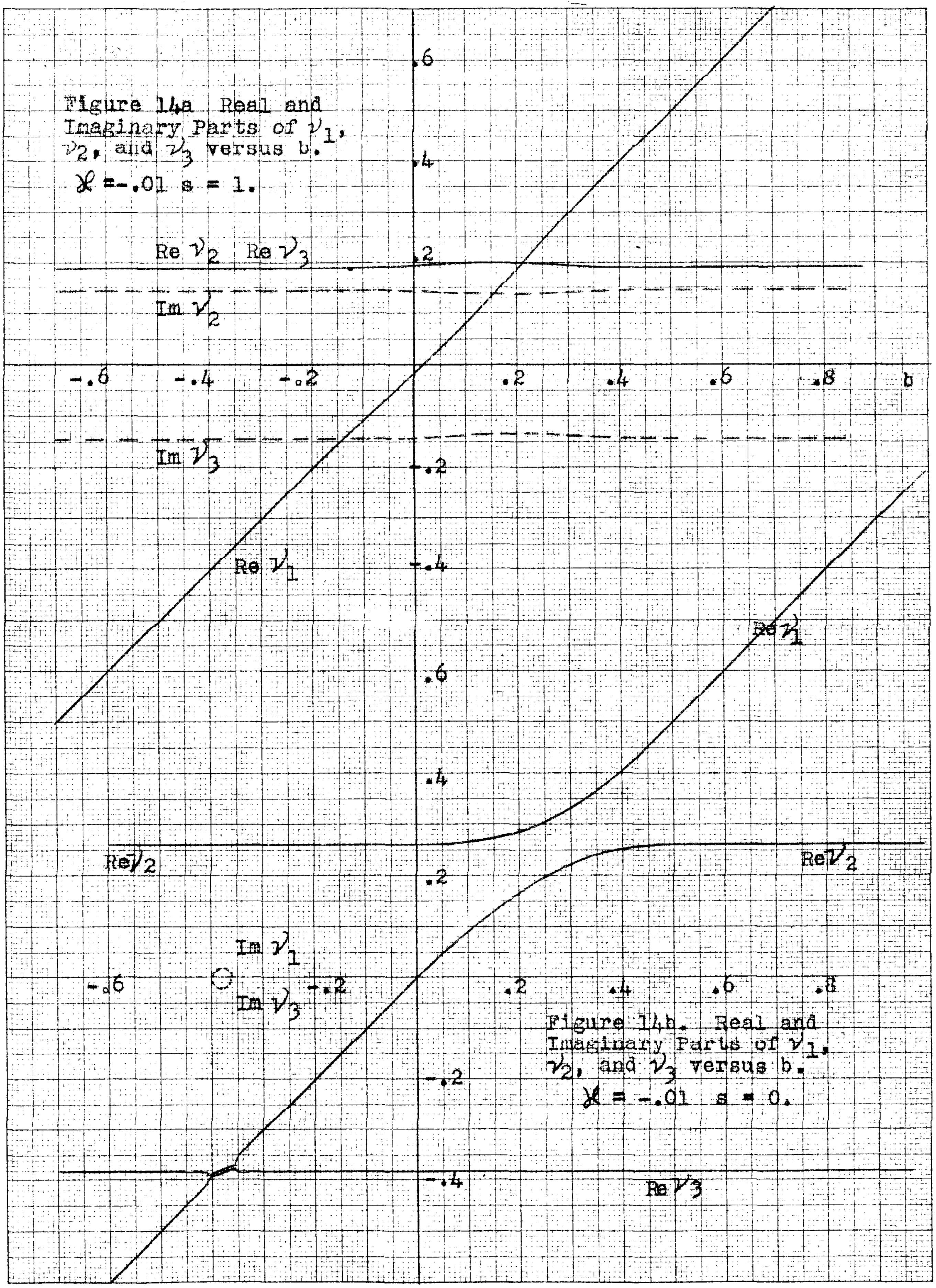

Figure 14a Real and Imaginary Parts of ν_1, ν_2, and ν_3 versus b. $\varkappa = -.01$ s = 1.

Figure 14b. Real and Imaginary Parts of ν_1, ν_2, and ν_3 versus b. $\varkappa = -.01$ s = 0.

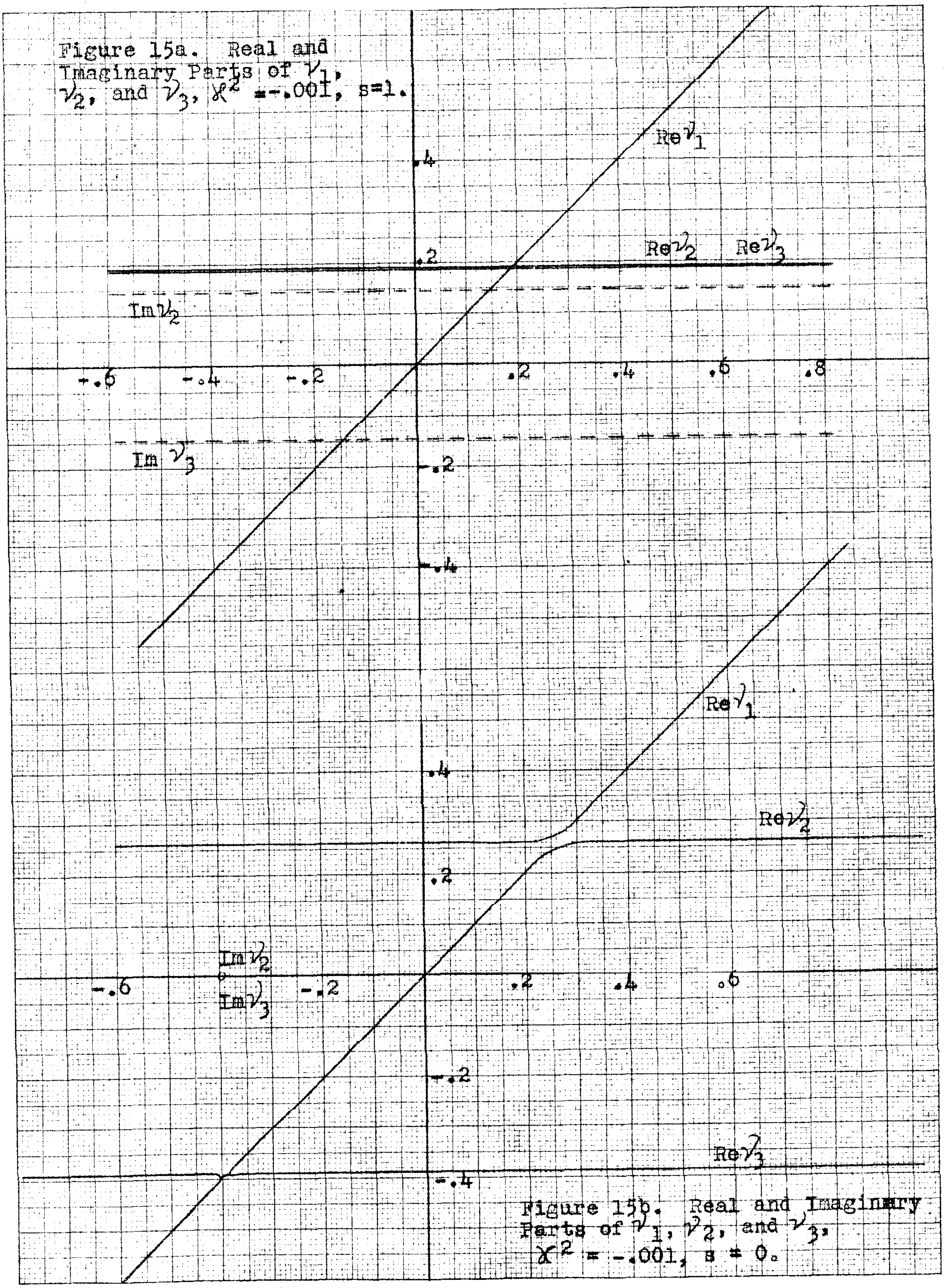

Figure 15a. Real and Imaginary Parts of ν_1, ν_2, and ν_3, $\chi^2 = -.001$, s=1.

Figure 15b. Real and Imaginary Parts of ν_1, ν_2, and ν_3, $\chi^2 = -.001$, s = 0.

where

$$F(b,\theta) = \sum_{i=1}^{3} \left[1 + C_2(\nu_i)T'\right]\left[\frac{1}{\nu_j(\nu_k - s\beta_- t + \frac{T}{1+T})} - \frac{1}{\nu_k(\nu_j - s\beta_- t + \frac{T}{1+T})}\right] e^{-j\nu_i\theta} \qquad V.16$$

in the complex plane with b and θ as parameters. Figure 16 shows a typical plot from which it is possible to determine the values of b and θ that make $F(b,\theta)$ equal to zero. Although there is more than one pair of b and θ which make $F(b,\theta)$ equal to zero (not shown in Figure 16), only the solution with the smallest value of θ has been obtained. This corresponds to the minimum length of tube for oscillation. If the tube is considerably longer than this minimum length, other modes of oscillation are possible, but these will not be discussed.

Solutions have been obtained for $\chi^2 = -.10$, $-.01$, and $-.001$ (χ^2 is negative for backward space harmonic operation) with $s = 0$ and $s = 1$. The results are summarized in Table I.

<u>Comparison with the Pierce-Muller Theory</u>. Muller (9) has adapted Pierce's (8) theory of interaction between a forward wave circuit and a thin electron beam focused with crossed electric and magnetic fields to backward wave interaction and finds for the M type backward wave oscillator starting conditions

$$-(\beta_e L)^2 \frac{\omega}{\omega_c} \alpha \phi^2 \frac{KI_o}{2V_o} = \left(\frac{\pi}{2}\right)^2 \qquad V.17$$

where I_o is the beam current, V_o is beam voltage (not the circuit voltage), K is impedance parameter at the plane of the circuit, ϕ is the ratio of the field (E_{1z}) at the beam to the field at the circuit and $\alpha = \frac{E_{1y}}{jE_{1z}}$ at the beam. Space charge effects were neglected in

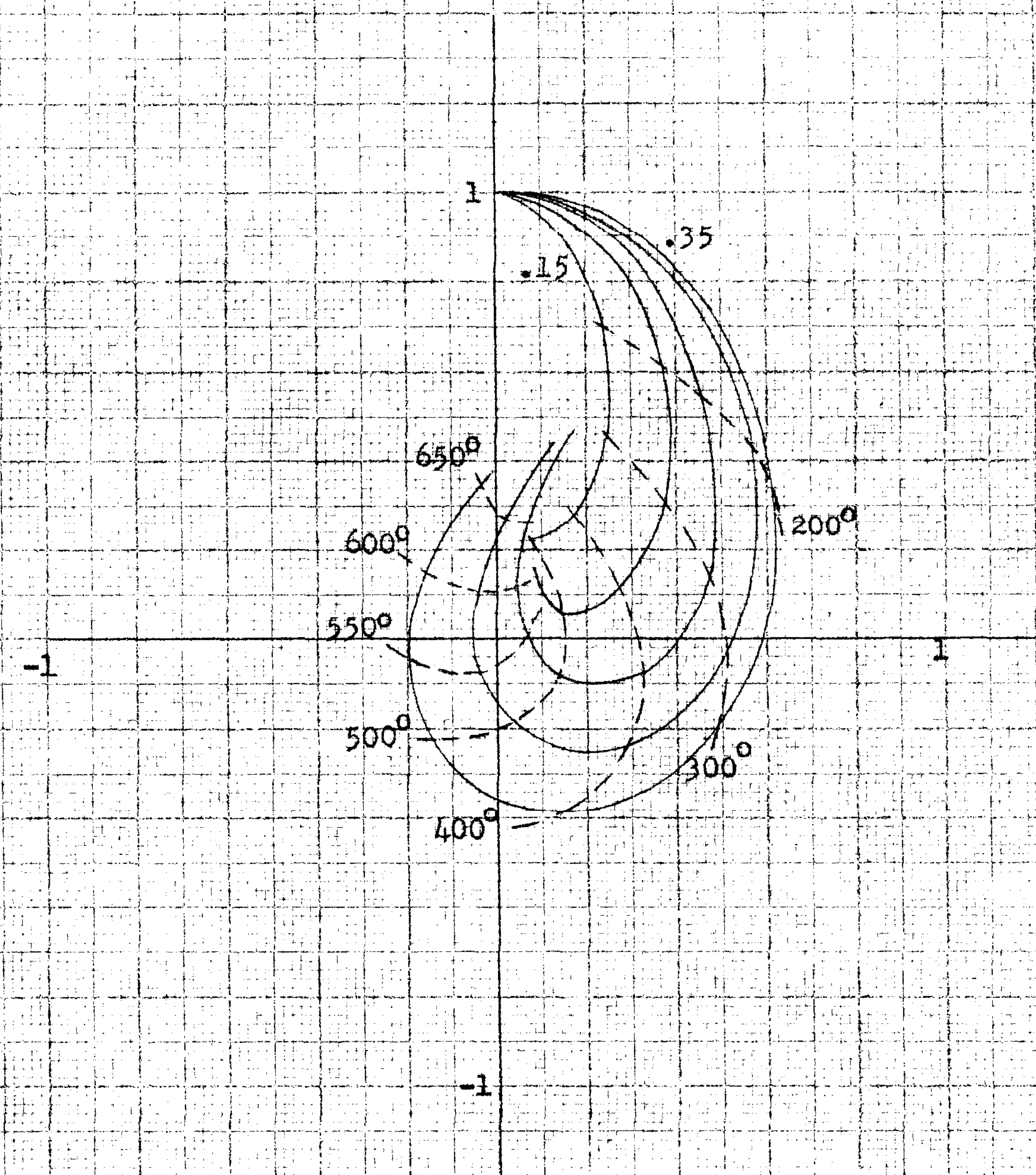

Figure 16. A Plot of the Function $\frac{F(b,\theta)}{F(b,0)}$ for $\chi^2 = -.1$, s=0.
The solid lines are lines of constant b (b = .15, .20, .25, .30, .35) and the dashed lines are lines of constant θ.
The function is zero when b ≅ .28, θ ≅ 565°.

TABLE I

M-Type Backward Wave Oscillator Start Oscillation Conditions

Summary of Numerical Computations

case	b	θ	$E^{(1)}$	$E^{(2)}$	$E^{(3)}$
$\varkappa^2 = -.1$, $s = 1$	.11	550°	j.736	$.209\angle 55^\circ$	$.209\angle -55^\circ$
$\varkappa^2 = -.01$, $s = 1$	.16	1250°	j1.016	$.042\angle -12^\circ$	$.042\angle 12^\circ$
$\varkappa^2 = -.001$, $s = 1$	.165	1975°	j1.009	$.0064\angle -35^\circ$	$.0064\angle 35^\circ$
$\varkappa^2 = -.1$, $s = 0$	.28	565°	.47	.45	.08
$\varkappa^2 = -.01$, $s = 0$	.27	1720°	.50	.50	.001
$\varkappa^2 = -.001$, $s = 0$	.265	5430°	.50	.50	0

deriving this expression. V.17 can be written in terms of the parameters of this paper as

$$-\mathcal{K}^2 (\beta_e L\, mr^2)^2\, 2\beta t\ \alpha \phi^2 = (\tfrac{\pi}{2})^2 \qquad \text{V.18}$$

or

$$\theta_o = \beta_e L\, mr^2 = \frac{\pi/2}{\sqrt{-\mathcal{K}^2\,(2\beta t)\,\alpha\,\phi^2}} \qquad \text{V.19}$$

The subscript 0 is appended to θ to distinguish the value of θ calculated in this manner from that obtained from the field analysis. When the beam is not thin it is appropriate to average ϕ^2 over the beam cross section,

$$\bar{\phi}^2 = \frac{1}{t}\int_{-t}^{0} \phi^2\, dy$$

$$= \frac{1}{t}\int_{-t}^{0} e^{+2\beta(y-d)}\, dy = e^{-2\beta d}\,\frac{1-e^{-2\beta t}}{2\beta t} .$$

Taking $\alpha = 1$, and substituting for $\bar{\phi}^2$ V.17 becomes

$$-\mathcal{K}^2(\beta_e L\, mr^2)^2\ e^{-2\beta d}(1-e^{-2\beta t}) = (\tfrac{\pi}{2})^2$$

or

$$\theta_o = \frac{\pi/2}{\sqrt{-\mathcal{K}'^2}} ,$$

where

$$\mathcal{K}'^2 = e^{-2\beta d}(1-e^{-2\beta t})\mathcal{K}^2 = \frac{2T}{1+T}\,\frac{1-T'}{1+T'}\,\mathcal{K}^2 . \qquad \text{V.20}$$

For the numerical examples $\beta t = .5$ $\beta d = .5$ this becomes

$$\theta_o = \frac{\pi/2}{\sqrt{-.2325\mathcal{K}^2}} \qquad \text{V.21}$$

The results of the field analysis given in Table I have been compared to the thin beam result of Pierce and Muller given by V.21. This comparison is shown in Figure 17 where θ/θ_0 is plotted versus $\mathcal{K}^2$. θ/θ_0 is the length of the tube at start oscillation as predicted by the field analysis of this paper divided by length predicted by the Pierce-Muller thin beam theory. The comparison is made when the beam slips (s = 1) and when it does not slip (s = 0).

Also shown is the result of an approximate analysis of space charge effects by Epsztein. His result is discussed at the end of this section.

From the definition of $\mathcal{K}^2$ (IV.53) it is seen that the $\mathcal{K}^2$ is increased by increasing the circuit impedance and decreasing the plasma frequency of the beam (and hence by decreasing the current). In the region of large $\mathcal{K}^2$ the circuit fields are much stronger than the fields produced by the space charge so that space charge effects are negligible. In this region the field analysis and the Pierce-Muller analysis should agree, and they do.

Figure 17 shows that when the slipping of the beam is ignored (s = 0) the results of the field analysis agree relatively well with the Pierce-Muller theory for all values of $\mathcal{K}^2$ whereas if the slipping is taken into account and $\mathcal{K}^2$ is small, the field analysis predicts a considerably shorter length of tube. Although no specific experimental data is available to check this curve, the magnitude of the effect is large enough to explain the French observations. Roughly speaking, the factor by which the current is reduced is the square of the factor by which the length is reduced.

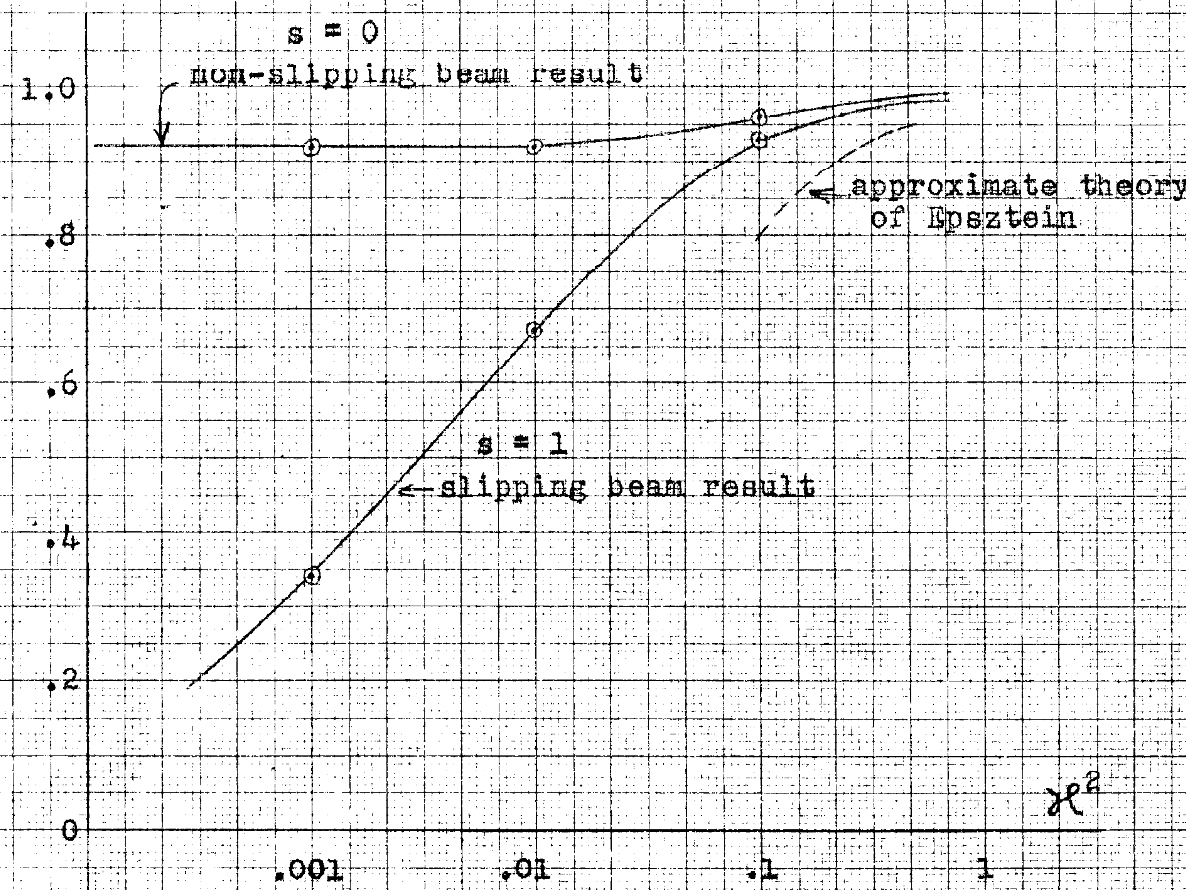

Figure 17. A comparision of the Results of the Field Analysis with the Results of the Thin Beam Theory of Pierce and Muller. The ordinate is Tube Length for Start Oscillation, as predicted by the Field Analysis, divided by the Length as predicted by the Thin Beam Theory. The Dashed Curve shows the Results of Epsztein's Theory.

This result seems plausible on the following grounds. The backward wave oscillator may be considered as a continuous amplifier with modulation at the electron beam by circuit fields along the entire length of the tube. The slow wave circuit feeds back energy to the beginning of the tube to sustain this modulation. Due to the slipping beam effect small perturbations in the beam conditions at $z = 0$ modulate the beam more and more as it drifts through the tube, even in the absence of the slow wave circuit. Thus when the slipping beam effect is present the circuit can be coupled more loosely to the beam than is otherwise possible, and still have oscillation occur.

Discussion of the French Theory of Space Charge Effects. Recently Epsztein (6),(10), has modified the thin beam theory to include the effects of space charge. In this theory the electrons in the thin beam are assumed to move in the field of the circuit plus the field generated by the electronic charge. The beam is assumed to have a finite but small thickness and the field at the center of the beam is computed by assuming that the density of the beam is constant and using an equivalent surface charge density to represent deformations of the beam boundaries. The space charge field computed in this manner is then assumed to act on all electrons. This causes a modification of the propagation constants of the two waves in thin beam theory. This in turn causes a decrease in the length (or current) of the tube required for start oscillation. The reduction computed in this manner is said to be in agreement with the experimental results.

Although Epsztein's theory applies only to thin beams, it may be compared with the results of this paper by replacing the thick beam of this paper by a thin beam with the same current at the average position of the thick beam. His result can be written in terms of the parameters of this paper as follows:

$$\frac{L}{L_o} = \frac{\theta}{\theta_o} = 1 + \frac{3}{32} \frac{m^2(\beta t)^2}{\gamma'^2} . \qquad \text{V.22}$$

Only the first two terms in a series expansion are given and the formula applies only when the second term is small compared to unity. This result is plotted in Figure 17.

Epsztein's treatment is an attempt to include a.c. space charge effects. It does not take into account the d.c. space charge effect or slipping beam effect, since when the beam is thin all electrons have substantially the same velocity. The $s = 0$ analysis of this paper is undoubtedly a better description of the same situation and it does not predict an appreciable reduction in starting length or current. Only when the slipping beam, or diocotron, effect is taken into account are the predicted starting currents much less than given by the simple thin beam theory. Thus there is considerable doubt as to the correctness of Epsztein's explanation of space charge effects.

The analysis presented in this paper, although requiring lengthier calculations, offers the possibility of studying the effect of other parameters, such as beam thickness, distance from the circuit, etc. on the characteristics of the M-type backward wave oscillator. Although it has not been discussed here, the magnetron amplifier can be analyzed in a similar manner.

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LIST OF SYMBOLS

α attenuation constant in z direction

β propagation constant in z direction

$$\beta_e = \frac{\omega}{u}, \quad \beta_c = \frac{\omega_c}{u}, \quad \beta_p = \frac{\omega_p}{u}, \quad \beta_+ = \frac{\omega}{u_+}, \quad \beta_- = \frac{\omega}{u_-}$$

β_o propagation constant of the fundamental space harmonic of the periodic slow-wave circuit

β_n propagation constant of the n^{th} space harmonic of the periodic slow wave circuit

$\gamma_n = \sqrt{\beta_n^2 - k^2}$, transverse separation constant for the n^{th} space harmonic of the periodic slow wave circuit

$\delta = \beta/\beta_e$, normalized propagation constant

$\Delta = \frac{\partial u}{\partial y}$, transverse velocity gradient in electron beam

$\epsilon = \frac{u_+ - u_-}{u_-}$, fractional velocity difference in electron beam

ϵ_o permittivity of free space

$\mathcal{E}(z)$ z component of electric field at a plane which just grazes the slow wave circuit

η charge to mass ratio of the electron $(\eta > 0)$

$\theta = \beta_+ Lmr^2$, length of backward wave oscillator at start oscillation in dimensionless units

θ_o length of backward wave oscillator at start oscillation as predicted by the thin beam theory, in dimensionless units

$\mathcal{K}^2$ dimensionless constant of coupling between circuit waves and space charge waves of the electron beam

μ_o permeability of free space

$\nu = \frac{\beta - \beta_+}{r^2 m \beta_+}$, dimensionless variable representing the propagation constant β

ξ $= \frac{\beta u - \omega}{\Delta}$, dimensionless variable which is a linear function of the y coordinate in the electron beam

ρ_o steady or d.c. component of the charge density of the electron beam

ρ_1 perturbation or a.c. component of the charge density of the electron beam

σ_+ equivalent surface charge density which represents the deformation of the upper boundary of the electron beam

σ_- equivalent surface charge density which represents the deformation of the lower boundary of the electron beam

ω radian frequency of sinusoidal oscillations

ω_c = B_{ox} , radian cyclotron frequency of electrons in a magnetic field B_{ox}

ω_p = $\sqrt{-\frac{\rho_o}{\epsilon_o}\eta}$, radian plasma frequency of the electron beam

Ω = $\sqrt{(\omega - \beta u)^2 - \omega_c(\omega_c - \Delta)}$, a quantity with the dimensions of frequency (sec^{-1}) which appears in the solution of the electronic equations.

a distance from lower edge of beam to conducting plane

b $= \frac{\beta_m - \beta_+}{mr^2 \beta_+}$ difference in the m^{th} space harmonic phase velocity and electron velocity in dimensionless units

B_E E-mode, or transverse magnetic, surface susceptance

B_m E-mode surface susceptance for the m^{th} space harmonic at the plane of the slow wave circuit, of the space below the circuit with the electron beam present

B_{mo} E-mode surface susceptance for the m^{th} space harmonic at the plane of the slow wave circuit, of the space below the circuit with the electron beam absent

$\underline{B}_o$ steady part of the magnetic field vector

B_{ox} x component of steady magnetic field

$C = \coth \beta a$

C_1 "space charge" capacitance in the Pierce equivalent circuit

D period of the slow wave circuit

$\underline{E}_o$ steady part of the electric field vector

$\underline{E}_1$ time-varying part of the electric field vector

E_{oy} y component of the steady electric field

E_{1y} y component of the time-varying electric field

E_{1z} z component of the time-varying electric field

i_+ equivalent surface current on upper beam edge

i_- equivalent surface current on lower beam edge

$k = \omega\sqrt{\mu_o \epsilon_o}$, free space wave number

K Pierce traveling wave tube circuit impedance

L length of backward wave oscillator at start oscillation

$m = \dfrac{\omega_c}{\omega}$ ratio of cyclotron frequency to oscillation frequency

P normalized surface conductance

Q normalized surface susceptance

$\underline{r}$ vector position of an electron in the Lagrangian description

$\underline{r}_o$ unperturbed position of an electron in the Lagrangian description

$\underline{r}_1$ perturbation position of an electron in the Lagrangian description

$r = \frac{\omega_p}{\omega_c}$ ratio of plasma frequency to cyclotron frequency

s slip parameter

S sum which arises in determining propagation characteristics of the slow wave circuit

t thickness of electron beam

$T = \tanh \beta t$

$T' = \tanh \beta d$

$\underline{u}$ steady part of the velocity field

$u = u(y)$, z component of the steady velocity field

u_+ z component of velocity of the upper-beam edge

u_- z component of velocity of the lower-beam edge

$\underline{v}_1$ time-varying component of the velocity field

v_{1y} y component of the time-varying velocity field

v_{1z} z component of the time-varying velocity field

w width, or x dimension, of the tube

y_1 perturbation in y position of electron in Lagrangian description

z_1 perturbation in z position of electron in Lagrangian description

Y_E E-mode, or transverse magnetic, surface admittance

Y_m E-mode surface admittance for the mth space harmonic, at the plane of the slow wave circuit, of the space below the circuit with the electron beam present

Y_{mo} E-mode surface admittance for the mth space harmonic, at the plane of the slow wave circuit, of the space below the circuit with the electron beam absent

Y'_n Admittance of slow wave circuit slots to nth harmonic