ROLLING MOMENT DUE TO YAW OF FLAT WINGS
OF TRAPEZOIDAL PLANFORM
AT SUPERSONIC SPEEDS.

Thesis by

Seymour Lampert

In Partial Fulfillment of the Requirements

For the Degree of

Aeronautical Engineer

California Institute of Technology

Pasadena, California

1948

ACKNOWLEDGEMENT

The author wishes to thank Paco Lagerstrom for his help in preparing this thesis. The methods of conical flow that are used were discussed in Dr. Lagerstrom's A.E. 268 class.

The author also wishes to thank his brother, Edward, for his aid.

Summary:

The purpose of this thesis is to determine the effect of yaw on the rolling moment characteristics of rectangular and trapezoidal planforms of zero thickness. The method used is that of applying the superposition of solutions of linearized conical flow to the problem of the yawed wing. The solutions particularly applicable are those where $W = \frac{\partial \phi}{\partial x_3}$ is symmetrical with respect to the plane of the wing. i.e. (lifting case). For the wings considered in this report it was necessary to determine uniquely $W = W(b, \sigma, t)$ in the following regions: 1) subsonic leading edge, 2) subsonic trailing edge, 3) infinitely swept wing, 4) supersonic leading edges. The rolling moment and lift coefficients were calculated and the effect of varying the angle of side edge and yaw angle of the wing in respect to these parameters can be computed. The rolling moment coefficient for a rectangular wing of different aspect ratios was computed and it was found that the variation of rolling moment with angle of yaw Λ increases almost linearly to a maximum and then falls off. There is also an increase in $\frac{dC_{M_R}}{d\Lambda}$ with increasing aspect ratio.

SYMBOLS

α = angle of attack

$A\!\!R$ = aspect ratio $\left(\sigma/S\right)^2$; $A\!\!R = \left(\frac{\sigma}{c}\right)$ for rectangular wing

c = chord

σ = span

C_L = lift coefficient $\left(\frac{L}{q S}\right)$

C_{M_1} = rolling moment coefficient $\left(\frac{M_1}{q S}\right)$

b = Tan $\beta = \left(\frac{x_1}{x_3}\right)$ Side edge of wing

t = Tan $\tau = \left(\frac{x_1}{x_3}\right)$ In plane of wing

Λ = $\mathrm{Tan}^{-1}\gamma$ Angle of yaw

W = $\frac{\partial \phi}{\partial x_3}$ perturbation velocity in x_3 direction

W_∞ = free stream velocity

S = area

Cp(b, t) = $-\frac{2W(b,t)}{W_\infty}$ pressure coefficient

ϵ = $R e^{i\vartheta}$ $\qquad \zeta = r e^{i\theta} = x_1 + i x_2$

R = $\frac{1-\sqrt{1-r^2}}{r}$; $T = \frac{1-\sqrt{1-t^2}}{t}$; $T = (R)_{x_2=0}$

η = $\frac{B-\epsilon}{1-B\epsilon} = |\eta| e^{i\psi}$

ν = $(\eta)^{1/2} = |\eta|^{1/2} e^{i\psi/2}$

r = $\sqrt{x_1^2 + x_2^2}$

M = Mach number (for convenience $M = \sqrt{2}$ throughout report

USEFUL FORMULAE

1) $\frac{B^2+1}{2B} = \frac{1}{b}$

2) $\frac{1-B^2}{2B} = \frac{\sqrt{1-b^2}}{b}$

3) $\frac{(1-b)(1+b)^2}{4B\sqrt{-B}} = \frac{1+b}{b}\sqrt{\frac{1-b}{-b}}$

4) $t^* = \frac{t-b}{1-bt}$

5) $i\left(\nu+\frac{1}{\nu}\right) = \sqrt{\frac{2(1-b)(1+t)}{b-t}}$

HOMOGRAPHIC TRANSFORMATIONS

$$x_1^* = \frac{x_1 - b x_3}{\sqrt{1-b^2}} \qquad t^* = \frac{x_1^*}{x_3^*} = \frac{t-b}{1-bt}$$

$$x_2^* = x_2$$

$$x_3^* = \frac{x_3 - b x_1}{\sqrt{1-b^2}}$$

$\cos\psi^* = \mathrm{Tan}\,\Lambda^* = \frac{\mathrm{Tan}\,\Lambda - b}{1 - b\,\mathrm{Tan}\,\Lambda}$ $\qquad ()^* \quad b = 0$

Side edge parallel to stream

$\cos\psi = \mathrm{Tan}\,\Lambda$

INTRODUCTION

The first planform considered is shown in (Fig. 1(a)) as wing A. The wing has parallel leading and trailing edges and side edges which may be set at some arbitrary angle β with the x_3 direction; the only restriction being that in region 1 $\mu > \beta > 0$, and in region 2 $-\mu < \beta < 0$. The wing shown in Fig. 1(b) is similar to wing A except now in addition to the three regions considered there is the region where the leading edges are both supersonic.

In Part I of this paper the values of W(b, γ) and the corresponding Cp(b,γ) are derived for the four principal regions. In Part II the rolling moment and lift coefficients are determined for arbitrary b and γ, and the C.P. of each region determined about some suitable value X_1 = constant.

In all cases considered $M = \sqrt{2}$ for ease in calculation, and A.R. was taken large enough so that the regions where the Mach waves from the tips intersect are off the wing. Conical flow methods considered here are restricted to the plane of the wing ((x_3, x_1) plane).

PART II. CALCULATION OF Cp(b, δ) (PRESSURE COEFFICIENT)

Region 1. Subsonic leading edge inside Mach cone $\mu > \beta > 0$

In this region Cp(b, δ) is calculated for an arbitrary angle of yaw, Λ, (Tan $\Lambda = \delta$) and side edge angle β (Tan β = b). See (Fig. 1.)

From methods of conical flow (Ref. 1, 2) it is known that the rays from the side edge inside the Mach cone on the wing are isobars, lines of constant pressure. These lines are characterized by Tan $\tau = t$ $t = \frac{x_1}{x_3}$. From this fact it follows that if we pass a plane through the Mach cone at any x_3 = constant, we can completely describe the pressure field in terms of the one parameter (t). The projection of the Mach cone and the trace of the wing are shown in (Fig. 2) for $x_3 = 1$, and $\zeta = x_1 + i x_2 = r e^{i\theta}$.

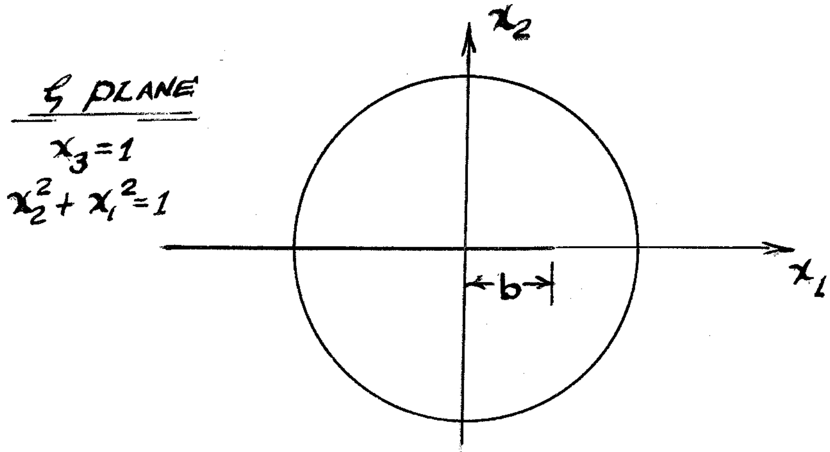


Fig. 2

The ζ plane now may be transformed by a suitable one to one mapping as described in (Ref. 1) where unit circle maps into the unit circle and where everything is regular inside the unit circle. Such a transformation is $\zeta \rightarrow \epsilon = Re^{i\vartheta}$ where $R = \frac{1-\sqrt{1-r^2}}{r}$. We may now construct solutions that are analytic for $R<1$ and have a great number of functions at our disposal. The boundary conditions inside the unit circle are similar to those in subsonic potential theory while for $R \geq 1$ solutions are gotten by method of characteristics. See(Ref. 2 .) The boundary conditions in the ϵ plane are given for $\mathbb{W}(\epsilon) = W + iW'$ where W' is the harmonic conjugate and it is introduced to show that the Cauchy Riemann Equations are satisfied. The proper boundary conditions are shown in (Fig. 3.)

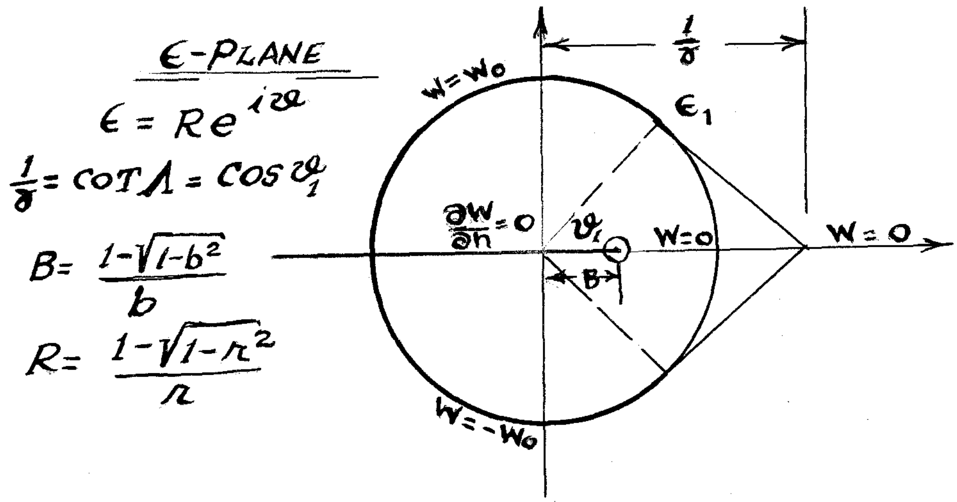


FIG. 3

From subsonic potential theory it is known that $W(B) = \infty$ or goes continuously to zero. In this case $W(B) = \infty$ and $W \sim \log R$. However by placing an infinity (pole) of the same strength at $\frac{1}{B}$ W is made completely regular for $R<1$ and the $(W)_{R=1} = 0$ for this singularity. However we must also take care of the jump condition at the side edge. First by a linear fraction transformation, B is mapped into the origin. We now have the condition of a side edge parallel to the flow, and solutions of this problem are known. With sufficient manipulation we get the value of W(b,t) to be (for complete solution see Appendix A) THE FOLLOWING

1.1

$$W(b,\gamma,t) = \frac{2W_0}{\pi}\left\{\frac{b}{b+1}\sqrt{\frac{(1+\gamma)(1-\gamma b)(1+t)}{(b-t)}} + \tan^{-1}\sqrt{\frac{(b-t)(1+\gamma)}{(1+t)(1-\gamma b)}}\right\}$$

$$b \leq t \leq 1$$

$$C_p(b,\gamma,t) = -\frac{2W}{W_\infty} = \frac{-4\alpha}{\pi\sqrt{1-\gamma^2}}\left\{\frac{b}{b+1}\sqrt{\frac{N(1+t)}{(b-t)}} + \tan^{-1}\sqrt{\frac{K(b-t)}{(1+t)}}\right\}$$

$$N = (1+\gamma)(1-\gamma b) \qquad K = \left(\frac{1+\gamma}{1-\gamma b}\right)$$

Region 2. Subsonic trailing edge. $0 > \beta > -\mu$

The value for $W(b,\gamma,t)$ is essentially the same as that for Region 1 except that now the Kutta condition must be imposed. That is that W(B) = 0 and must go to zero continuously. The solution in this case is

1.2

$$W(b,\gamma,t) = \frac{2W_0}{\pi}\tan^{-1}\sqrt{\frac{K(b-t)}{(1+t)}} \qquad -1 \leq t \leq b$$

$$C_p(b,\gamma,t) = \frac{-4\alpha}{\pi\sqrt{1-\gamma^2}}\tan^{-1}\sqrt{\frac{K(b-t)}{(1+t)}}$$

Region 3. 2-dimensional sweep back.

In this region we apply the 2-dimensional sweep back theory

1.3

$$W = W_0 = \frac{W_\infty\alpha}{\sqrt{1-\gamma^2}}$$

$$C_p = -\frac{2\alpha}{\sqrt{1-\gamma^2}}$$

Region 4. Supersonic leading edges.

1.4

$$W(t,\gamma_1,\gamma_2) = \frac{2}{\pi}\left[W_1 \tan^{-1}\sqrt{\frac{(1+t)(1+\gamma_1)}{(1-t)(1-\gamma_1)}} + W_2 \tan^{-1}\sqrt{\frac{(1-t)(1+\gamma_2)}{(1+t)(1-\gamma_2)}}\right]$$

$$C_p(t,\gamma_1,\gamma_2) = -\frac{2}{\pi}\left[\frac{1}{\sqrt{1-\gamma_1^2}} \tan^{-1}\sqrt{\frac{(1+t)(1+\gamma_1)}{(1-t)(1-\gamma_1)}} + \frac{1}{\sqrt{1-\gamma_2^2}} \tan^{-1}\sqrt{\frac{(1-t)(1+\gamma_2)}{(1+t)(1-\gamma_2)}}\right]$$

A three dimensional sketch of Cp in regions 1 and 2 are shown in (Figs. 4 and 5.).

PART III. CALCULATION OF C_L AND C_{M_R}

Region 1. $-1 \leq b \leq 0 \qquad -b \leq t \leq 1$

The region of integration is shown in Fig. 6.

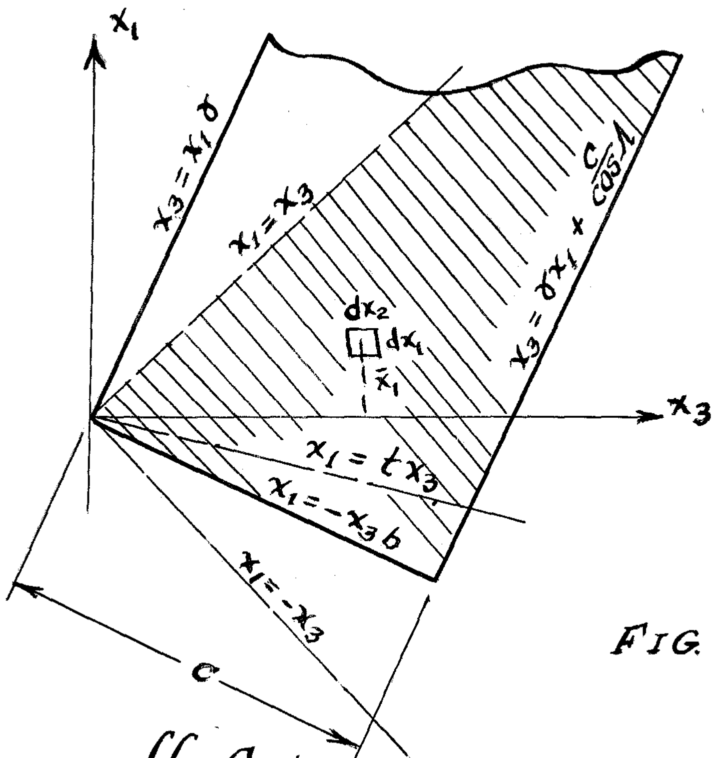


FIG. 6

2.1
$$C_L = \frac{\iint_\Delta -2C_p \, dx_1 dx_3}{\iint dx_1 \, dx_3}$$

Since Cp is constant along rays from the origin, THE numerator of Eq. 2:1 BECOMES

$$\iint_\Delta -2C_p \, dx_1 dx_3 = 2\iint_\Delta C_p \frac{x_1}{t^2} dx_1 dt = 2\int_{-b}^{1} C_p \frac{x_1^2}{2t^2} dt$$

Substituting $x_1 = \frac{c}{\cos \Lambda} \frac{t}{[1-t\gamma]}$

2.2
$$C_L = \frac{1}{S}\int_{-b}^{1} C_p(\gamma, b, t) \frac{dt}{[1-t\gamma]^2}$$

Substituting Cp of Eq. 1.1 we get that

2.3
$$C_L = -\frac{4\alpha c^2}{\pi\sqrt{1-\delta^2}\cos^2\Lambda S}\left[\frac{b}{b+1}\int_b^1\sqrt{\frac{N(1+t)}{(b-t)}} + \tan^{-1}\int_b^1\sqrt{\frac{k(b-t)}{(1+t)}}\right]$$
$$= -\frac{4\alpha c^2}{\pi\sqrt{1-\delta^2}\cos^2\Lambda S}\left[I[b,\delta] + II(b,\delta)\right]$$

The above integrals are evaluated in Appendix B. The solution of C_L becomes:

2.6
$$C_L = \frac{4\alpha}{\sqrt{1-\delta^2}}\left[\frac{b(1-\delta)}{b+1} + \frac{1}{2}\right] \qquad \begin{array}{l} -1 \leq b \leq 0 \\ 1 \geqslant \delta \geqslant 0 \end{array}$$

2.7
$$C_{M_R} = \frac{\iint_\Delta -2C_p x_1\, dx_1\, dx_3}{\iint_\Delta dS}$$

The integration of C_{M_R} is carried out over the same region as that for C_L. C_{M_R} is found around the x_3 axis as shown in Fig. 6. The evaluation of C_{M_R} is also given in Appendix B. The solution for $(C_{M_R})_{x_3}$ is as follows:

2.8
$$C_{M_R} = \frac{c\alpha}{6\cos\Lambda(b+1)(1-\delta b)\sqrt{1-\delta^2}}\left\{\frac{5(1+b)(1+b\delta)}{(1-\delta)} - b^2(1+6\delta) - 7b\right\}$$

The distance to C.P. becomes

$$\bar{x}_1 = \frac{c}{12\cos\Lambda(1-\delta b)(3b-2b\delta+1)}\left\{\frac{5(1+b)(1+b\delta)}{(1-\delta)} - b(b+6b\delta+7)\right\}$$

2.9 Region 2. The range of integration is shown in Fig. 7.

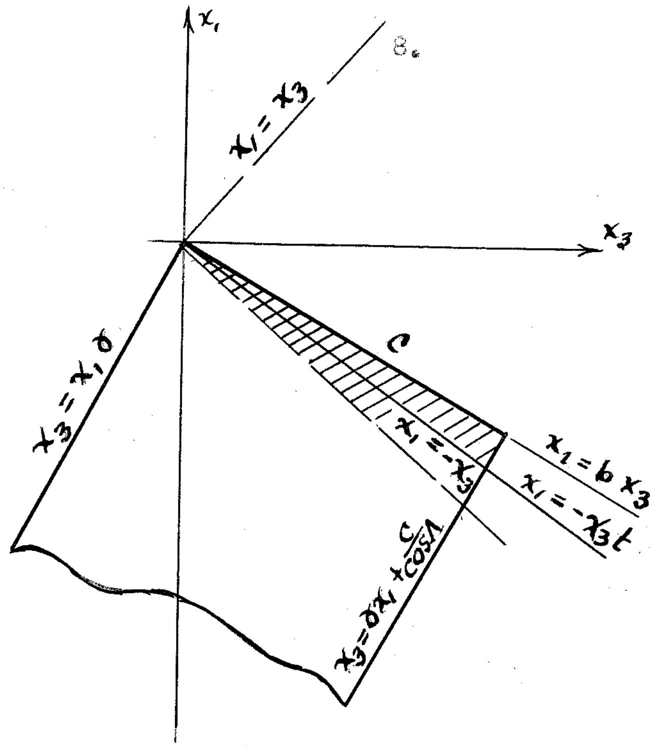


Fig. 7.

2.11 $$C_L = -\frac{4\alpha c^2}{\pi\sqrt{1-\gamma^2}\,S\cos^2\Lambda}\int_b^{-1}\tan^{-1}\sqrt{\frac{k(b-t)}{(1+t)}}\;\frac{dt}{[1-t\gamma]^2}$$

This integral is evaluated in Appendix B.

2.12 $$C_L = \frac{2\alpha}{\sqrt{1-\gamma^2}}$$

2.13 $$C_{M_R} = \frac{8c^3\alpha}{3\pi\cos^3\Lambda\sqrt{1-\gamma^2}}\int\tan^{-1}\sqrt{\frac{k(b-t)}{1+t}}\;\frac{t\,dt}{[1-t\gamma]^3}$$

The distance to the C.P. for this region becomes:

2.14 $$\bar{x}_l = -\frac{c}{12\cos\Lambda}\left[\frac{3b}{1-b\gamma} - \frac{5}{1+\gamma}\right]$$

WHERE

$$C_{M_R} = \frac{c\alpha}{6\cos\Lambda\sqrt{1-\gamma^2}}\left[\frac{3b}{1-b\gamma} - \frac{5}{1+\gamma}\right]$$

Region 3.

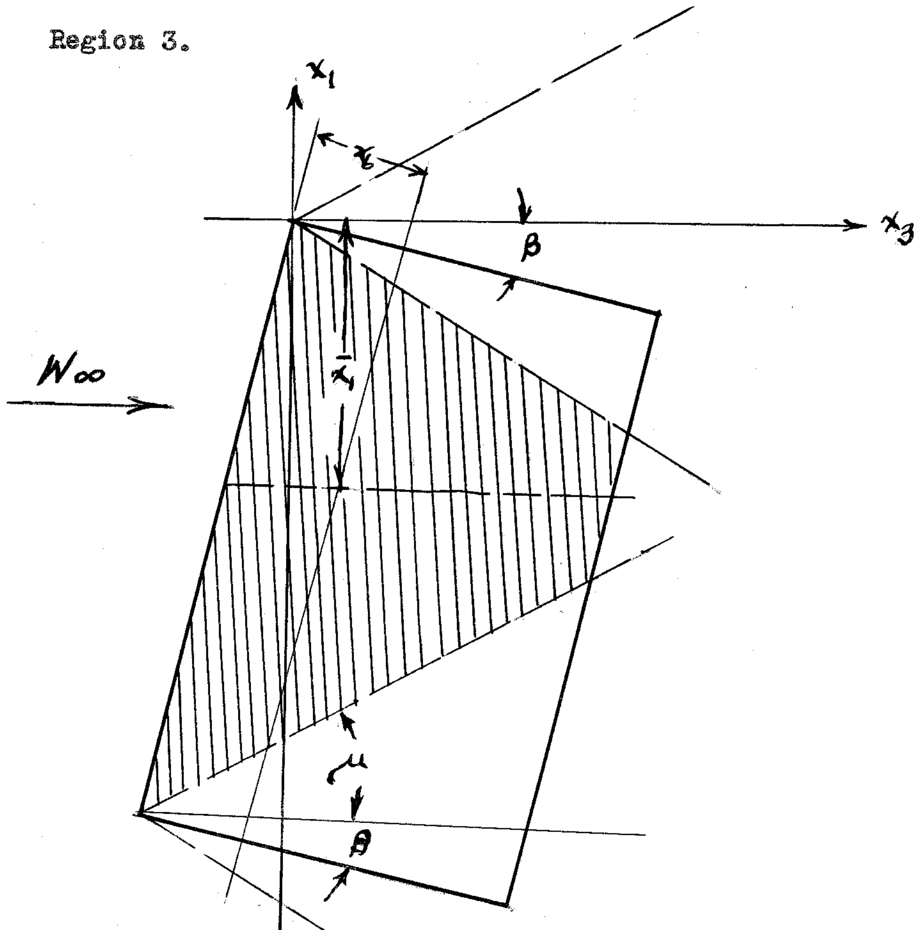


Fig. 8

2.15 $$C_L = \frac{4\alpha}{\sqrt{1-\gamma^2}}$$

2.16 $$\bar{x}_1 = \frac{1}{(1-\gamma^2)}\left[\frac{x_b\gamma}{\cos\Lambda} - \frac{\sigma}{2}\cos\Lambda + \frac{\sigma}{2}\gamma\sin\Lambda\right]$$

2.17 $$C_{M_R} = \frac{2\alpha c\cos\Lambda}{(1-\gamma^2)^{3/2}}\left[\frac{\gamma}{6\cos^2\Lambda}\left(4 - \frac{AR}{AR-\left(\frac{1+\gamma^2}{1-\gamma^2}\right)}\right) - AR(1-\gamma^2)\right]$$

PART IV. CALCULATIONS FOR ROLLING MOMENT COEFFICIENT FOR A RECTANGULAR WING ($\gamma = -b$)

The rolling moment will be computed about an axis passing through the geometric center of the wing and parallel to the direction of motion of the wing. See (Fig. 9). The distances to the C.p.'s of the respective regions are indicated as $\bar{x}_{11}$, $\bar{x}_{12}$, and $\bar{x}_{13}$ respectively.

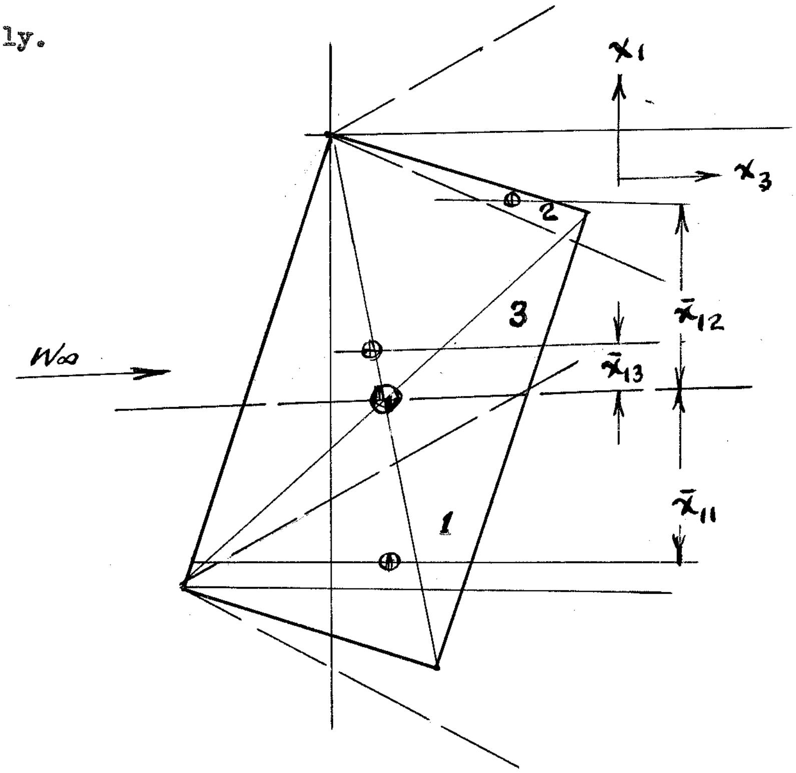


Fig. 9

THE CORRESPONDING MOMENT COEFFICIENTS ARE

3.1 $$C_{M_{R_{11}}} = \frac{2\alpha c}{\sqrt{1-\gamma^2}}\left[\frac{b(1-\gamma)}{b+1}+\frac{1}{2}\right]\cos\Lambda(\mathit{AR}-\gamma) - \frac{\alpha c\left[\frac{5(1+b)(1+b\gamma)}{1-\gamma} - b^2(1+6\gamma)-7b\right]}{6\cos\Lambda\sqrt{1-\gamma^2}\,(b+1)(1-\gamma b)}$$

$$C_{M_{R_{12}}} = \frac{\alpha c}{\sqrt{1-\gamma^2}}\cos\Lambda(\mathit{AR}+\gamma) + \frac{\alpha c}{6\cos\Lambda\sqrt{1-\gamma^2}}\left[\frac{3b}{1-b\gamma}-\frac{5}{1+\gamma}\right]$$

$$C_{M_{R_{13}}} = \frac{2\alpha c\gamma\cos\Lambda}{\sqrt{1-\gamma^2}}\left\{1+\frac{1}{3(1-\gamma^2)\cos^2\Lambda}\left(4-\frac{1}{1-\frac{1}{\mathit{AR}}\left(\frac{1+\gamma^2}{1-\gamma^2}\right)}\right)\right\}$$

The rolling moment coefficient for rectangular wing about the X_3 axis as shown in Fig. 9 becomes

$$C_{M_{R_T}} = C_{R_{12}} + C_{R_{13}} - C_{R_{11}}$$

3.2 $$C_{M_{R_T}} = \frac{2c\alpha\cos\Lambda}{\sqrt{1-\gamma^2}}\left\{2\gamma\left(2-2\gamma+AR\right)+\frac{1}{6\cos^2\Lambda}\left[\frac{4\gamma\left(4-\frac{1}{1-\frac{1}{AR}\left(\frac{1+\gamma^2}{1-\gamma^2}\right)}\right)}{(1-\gamma^2)}+\frac{(5+7\gamma-6\gamma^2-6\gamma^3)}{(1-\gamma)(1-\gamma^2)}-\left(\frac{3\gamma}{1+\gamma^2}+\frac{5}{1+\gamma}\right)\right]\right\}$$

The rolling moment computed from Eq. 3.1 is taken about an axis passing through the geometric center of the wing and parallel to the free stream velocity W_∞. C_{M_R} for AR = 2,3,4,8,6 against yaw angle Λ is shown in Fig. 10. Positive rolling moment is defined as a counter-clockwise rotation in the X_2-X_1 plane; X_2 directed upward. Equations 3.1 are derived for a general wing plan-form where the leading and trailing edges are parallel, and the rectangular wing is merely a special case of this family of trapezoidal wings. By varying the side edge, b = Tan β we can get the rolling moment for any trapezoidal wing of this category.

CONCLUDING REMARKS

1. C_{M_R} for various aspect ratios indicates that the rolling moment increases almost linearly with angle of yaw Λ.

2. Trapezoidal wings with parallel leading and trailing edges may be calculated in a manner similar to that of the rectangular wing using equations derived in this thesis.

3. By superimposing a pointed leading edge on the trapezoidal plane forms another family of planforms may be examined. The C_p for this region has been calculated in this paper.

4. The equations derived here remain valid only within the restrictions of the linearized theory.

REFERENCES

1. Lagerstrom, Paco A.: Linearized Supersonic Theory of Conical Wings. Jet Propulsion Laboratory, California Institute of Technology, Dec. 1947.

2. Hayes, W. D.: Ph.D. Thesis, California Institute of Technology, 1947.

3. Hayes, W. D., Browne, S. H., and Lew, R. J.: Linearized Theory of Conical Supersonic Flow with Applications to Triangular Wings, North American Aviation Report No. NA-46-818, 1946.

4. Evvard, John C.: Distribution of Wave Drag and Lift in the Vicinity of the Wing Tips at Supersonic Speeds. N.A.C.A. T.N. 1382, July, 1947.

5. Heaslet, Max A., Lomax, Harvard, and Jones, Arthur L.: Volterra Solution of the Wave Equation as Applied to Three Dimensional Supersonic Airfoil Problems; N.A.C.A. T.N. 1412, Sept., 1947.

6. Lagerstrom, P. A., and Graham, M. E.: Downwash and Sidewash Induced by Three-Dimensional Lifting Wings in Supersonic Flow, Douglas Aircraft Report SM-13007, April, 1947.

7. Jones, Robert T.,: Thin Oblique Airfoils at Supersonic Speed. N.A.C.A. T.N. 1107, 1946.

8. Stewart, H. J.: The Lift of a Delta Wing at Supersonic Speeds. Quarterly of Applied Mathematics, Vol. IV, No. 3 Oct., 1946.

9. Bonney, E. Arthur: Aerodynamic Characteristics of Rectangular Wings at Supersonic Speeds. Jour. Aero. Sci. Vol. 14, No. 2, Feb., 1947.

THE CASES CONSIDERED ARE SHOWN IN FIG. 1a AND 1b

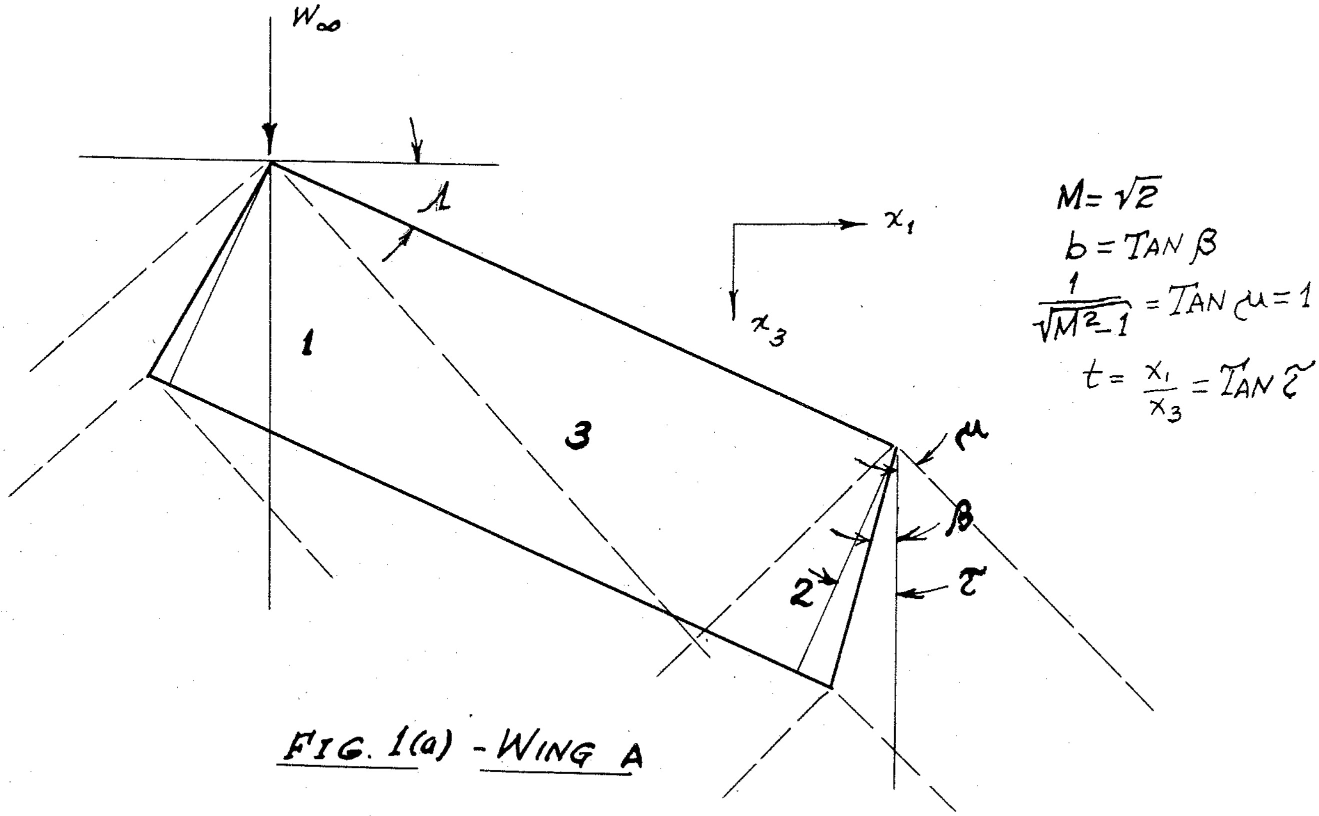


FIG. 1(a) - WING A

REGION 1 ~ SUBSONIC LEADING EDGE
REGION 2 ~ SUBSONIC TRAILING EDGE
REGION 3 ~ INFINITELY SWEPT WING

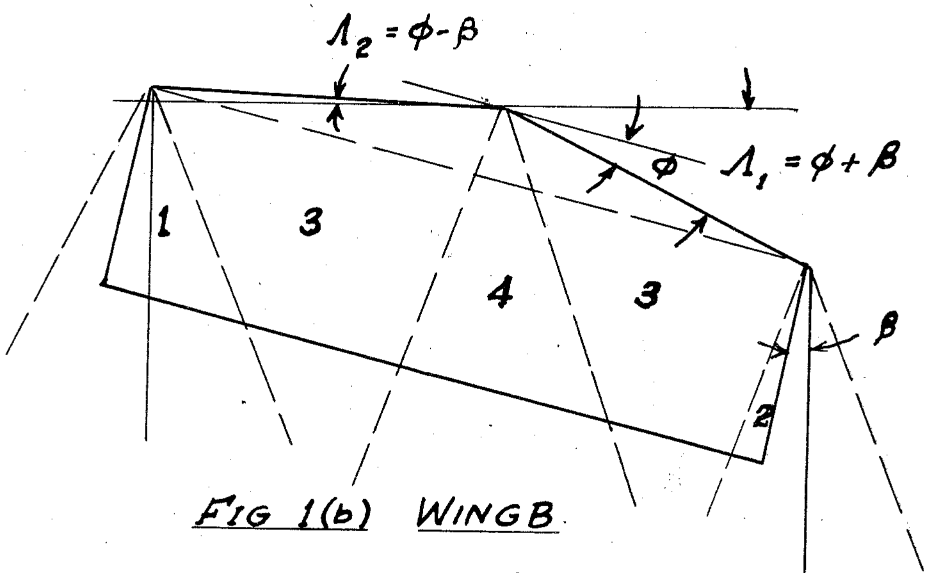


FIG 1(b) WING B

REGION 4 ~ SUPERSONIC LEADING EDGES

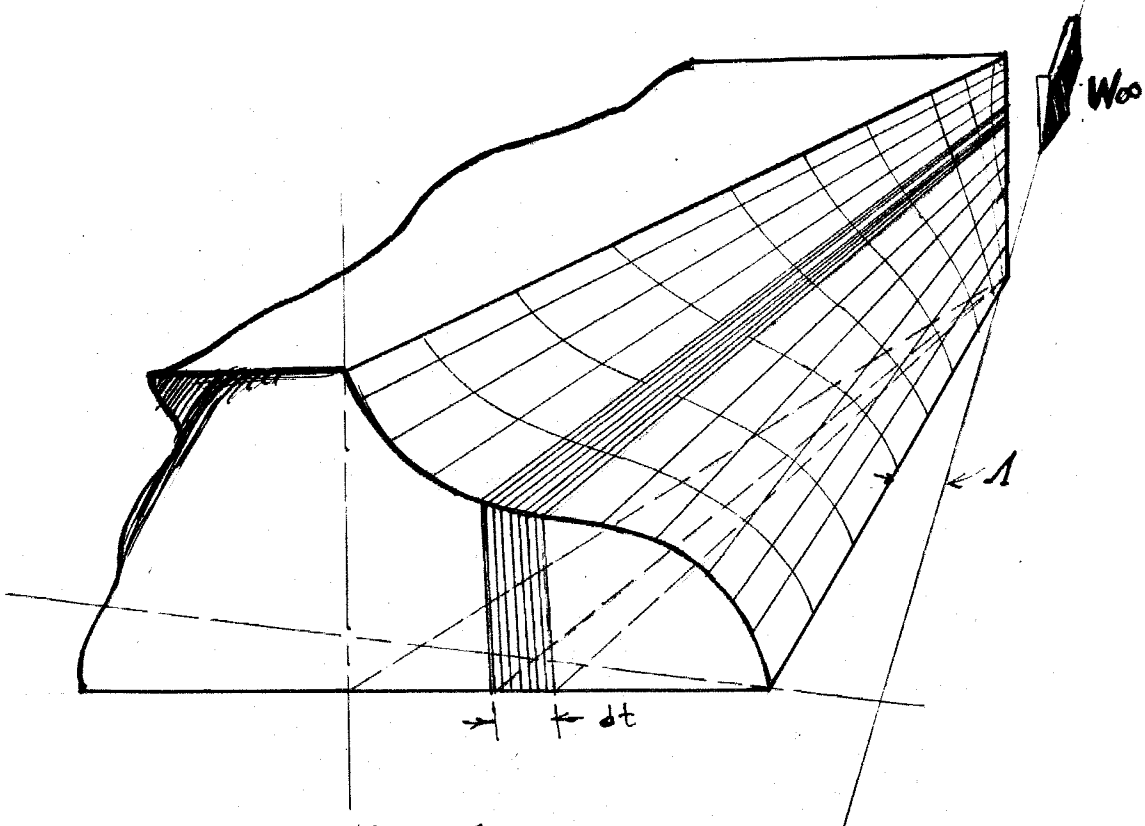


FIG. 4

REGION 2 SUBSONIC TRAILING EDGE

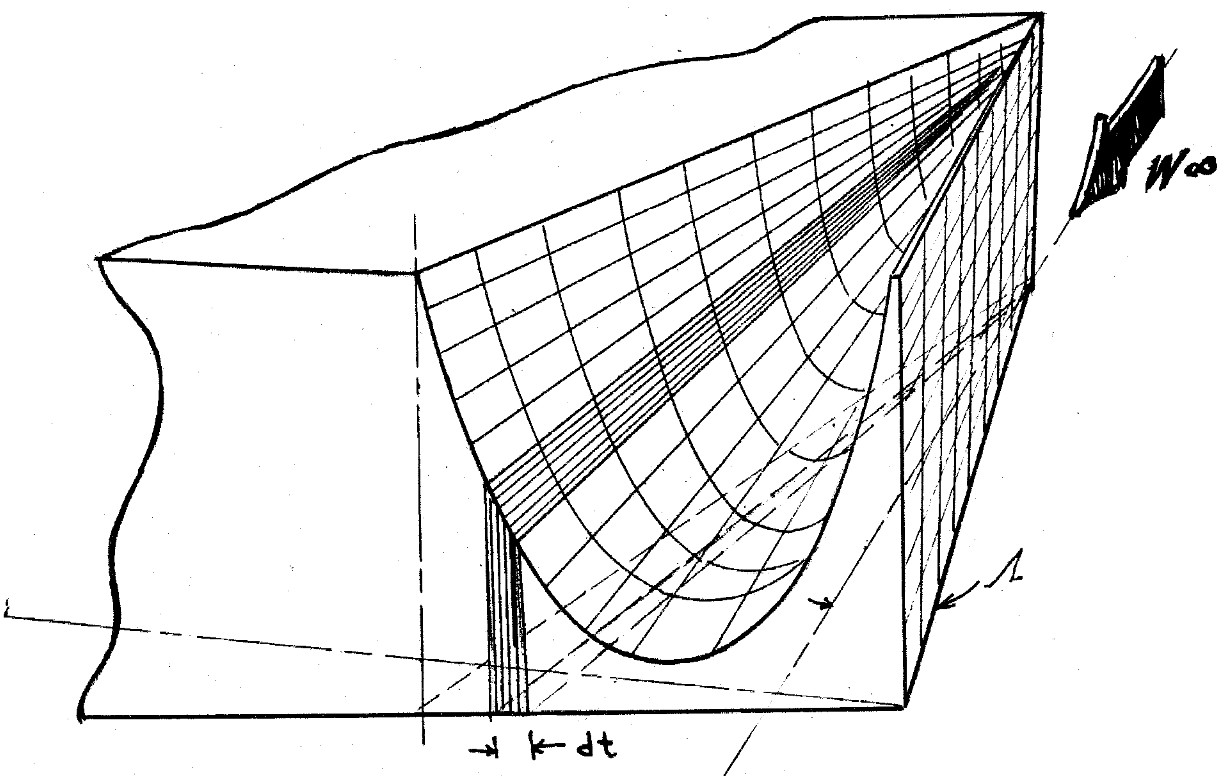


FIG 5

REGION 1 SUBSONIC LEADING EDGE

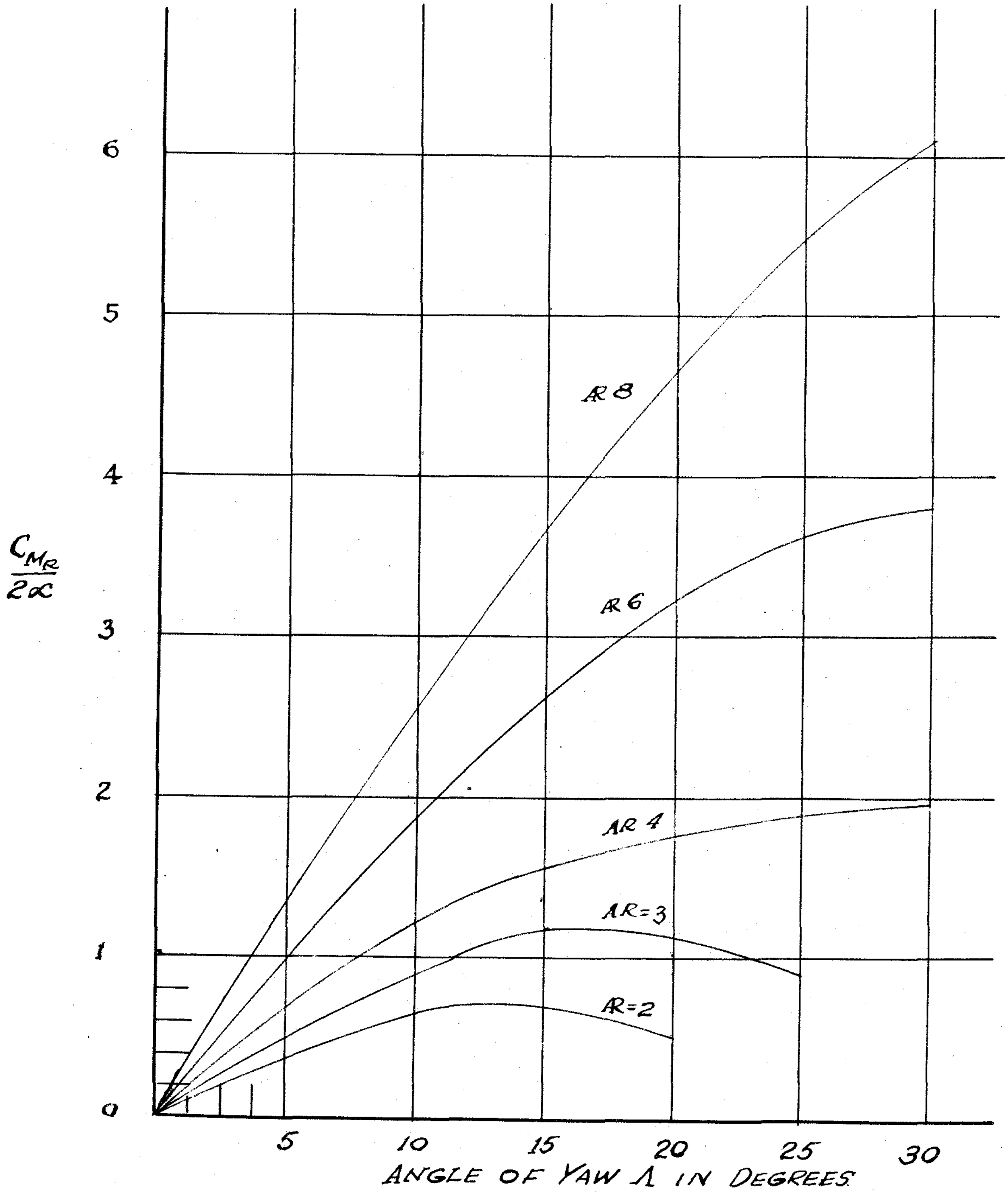
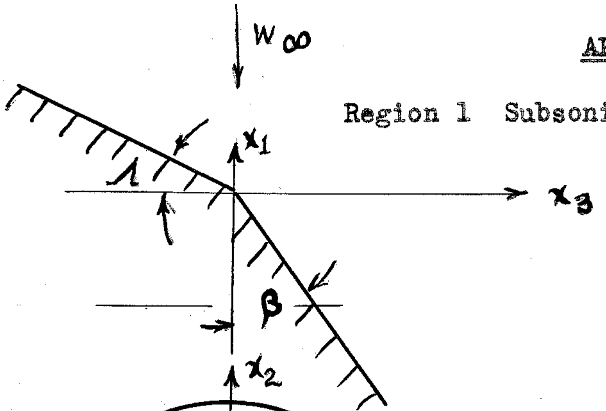


FIG. 10 VARIATION OF ROLLING MOMENT COEF. C_{M_R} WITH ANGLE OF YAW Λ FOR A RECTANGULAR WING AT $M=\sqrt{2}$

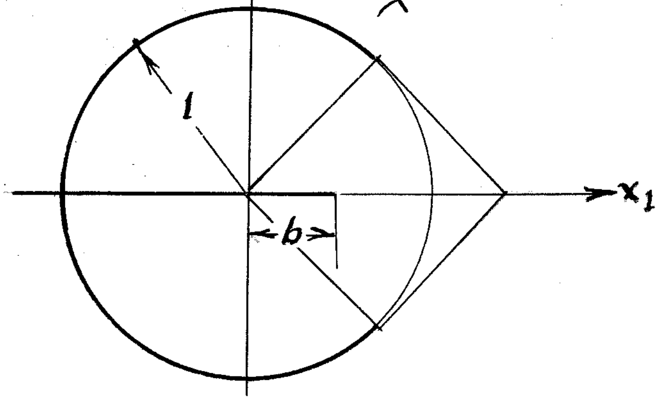
APPENDIX

APPENDIX A

Region 1 Subsonic Leading Edge



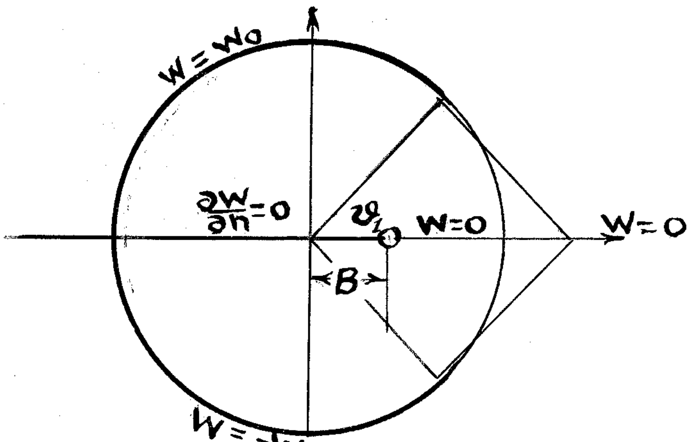
FOR A COMPLETE DESCRIPTION OF THESE MAPPINGS SEE (REF. 1)



Physical Plane – ς

$$\varsigma = r e^{i\theta} = x_1 + i x_2$$

mapped into

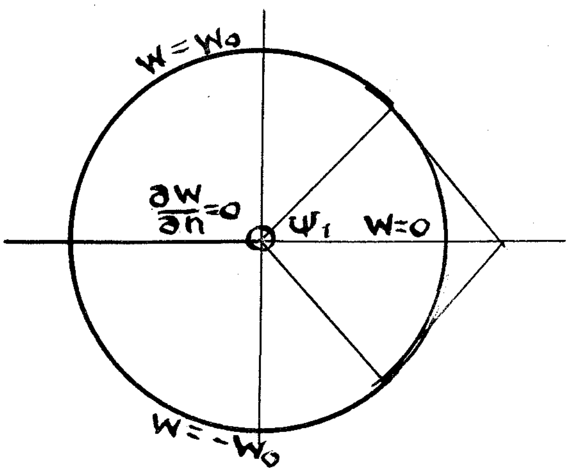


ϵ Plane

$$\epsilon = R e^{i\vartheta}$$

$$R = \frac{1-\sqrt{1-r^2}}{r}$$

$$\cos\vartheta_1 = \cot\Lambda$$

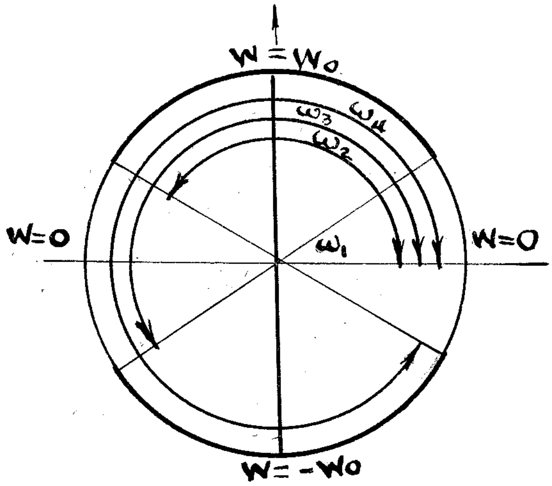


η Plane

$$\eta = \frac{B-\epsilon}{1-\epsilon B}$$

$$\cos\psi_1 = \cos\psi^* = \tan\Lambda^*$$

$$\tan\Lambda^* = \frac{\tan\Lambda - b}{1 - b\tan\Lambda}$$



ν - PLANE

$$\nu = (\eta)^{1/2} = |\eta|^{1/2} e^{i\psi/2} = |\nu| e^{i\omega}$$

Evaluating W(r,ω) on the wing by means of Poisson's integral

$$W(r,\omega) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r_1^2}{1-2r_1\cos(\phi-\omega)} f(r,\omega)\,d\omega$$

$$f(r,\omega) = W_0 \qquad \omega_1 \leq \omega \leq \omega_2$$

$$f(r,\omega) = -W_0 \qquad \omega_3 \leq \omega \leq \omega_4$$

$$\phi = \pi/2 \qquad \text{EVALUATE } W \text{ ON THE WING}$$

$$W(r,\omega) = \frac{W_0}{2\pi}\left(\frac{1-r_1^2}{1+r_1^2}\right)\left[\int_{\omega_1}^{\pi-\omega_1} \frac{d\omega}{1-\frac{2r_1}{[1+r_1^2]}\sin\omega} - \int_{\pi+\omega_1}^{2\pi-\omega_1} \frac{d\omega}{1-\frac{2r_1}{[1+r_1^2]}\sin\omega}\right]$$

$$W(r,\omega) = \frac{2W_0}{\pi}\left[\tan^{-1}\frac{[1+r_1][1-\tan\frac{\omega_1}{2}]}{[1-r_1][1+\tan\frac{\omega_1}{2}]} - \tan^{-1}\frac{[1-r_1][1-\tan\frac{\omega_1}{2}]}{[1+r_1][1+\tan\frac{\omega_1}{2}]}\right]$$

Adding up the angles we get

$$W(\nu) = \frac{2W_0}{\pi}\left[\tan^{-1}\left(\frac{2r_1\cos\omega_1}{[1-r_1^2]}\right)\right]$$

$$W(\eta) = \frac{2W_0}{\pi}\left[\tan^{-1}\left(\frac{2|\eta_1|^{1/2}\cos\frac{\vartheta}{2}}{1-|\eta_1|}\right)\right]$$

$$W(\epsilon) = \frac{2W_0}{\pi}\tan^{-1}\left[\frac{2\left|\frac{B-\epsilon_1}{1-B\epsilon_1}\right|^{1/2}\cos\frac{\vartheta_1}{2}}{1-\left|\frac{B-\epsilon_1}{1-B\epsilon_1}\right|}\right]$$

$$\cos\frac{\vartheta}{2}=\sqrt{\frac{1+\cos\vartheta}{2}}=\sqrt{\frac{(1-b)(1+\cos\psi)}{2(1-b\cos\psi)}}$$

AND SUBSTITUTING WE GET

$$W(\mathcal{E})=\frac{2W_0}{\pi}\tan^{-1}\frac{2\left|\frac{B-R}{1-BR}\right|^{1/2}\sqrt{\frac{(1-b)(1+\gamma)}{2(1-b\gamma)}}}{\left[1-\left|\frac{B-R}{1-BR}\right|\right]}$$

and in the physical plane

1.2 $$W(b,\gamma,t)=\frac{2W_0}{\pi}\tan^{-1}\sqrt{\frac{(b-t)(1+\gamma)}{(1+t)(1-b\gamma)}}$$

We must add to this solution the infinity at the side edge which is a leading edge, such that the function is zero on the Mach cone. Such a solution is:

$$W=C_1\left(\nu+\frac{1}{\nu}\right)=C_1\sqrt{\frac{2(1-b)(1+t)}{b-t}}$$

$$C_1=\frac{W_0 b}{\pi(b+1)}\sqrt{\frac{2(1+\gamma)(1-b\gamma)}{(1-b)}}$$

$$W=\frac{2W_0 b}{\pi(b+1)}\sqrt{\frac{(1+\gamma)(1-\gamma b)(1-b)(1+t)}{(1-b)(b-t)}}$$

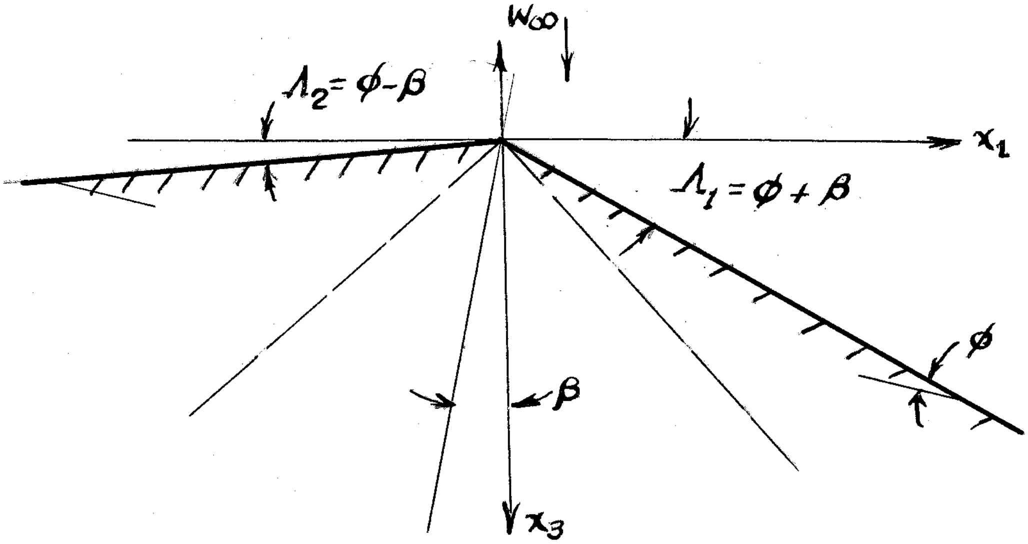
$$=\frac{2W_0 b}{\pi(b+1)}\sqrt{\frac{(1+\gamma)(1-\gamma b)(1+t)}{(b-t)}}$$

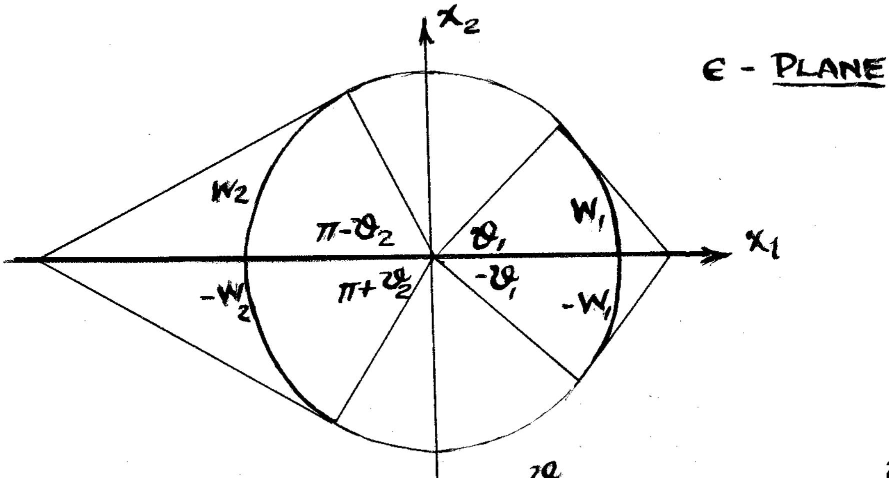
and THE complete solution is

1.1 $$W(b,t,\gamma)=\frac{2W_0}{\pi}\left[\frac{b}{b+1}\sqrt{\frac{(1+\gamma)(1-\gamma b)(1+t)}{(b-t)}}+\tan^{-1}\sqrt{\frac{(b-t)(1+\gamma)}{(1+t)(1-b\gamma)}}\right]$$

Care must be taken as to the range of b AND t in Eq. 1.2 which satisfies the Kutta condition; t is less than b and greater than -1, while in Eq. 1.1 $b<t<1$

Supersonic Leading Edges. <u>Region 4</u>





$$W(\epsilon) = \frac{1}{2\pi}\left(\frac{1-\lambda_1^2}{1+\lambda_1^2}\right)\left\{W_1\int_0^{\vartheta_1}\frac{da}{1-\frac{2\lambda}{1+\lambda^2}\cos a} + W_2\int_0^{\vartheta_2}\frac{da}{1+\frac{2\lambda}{1+\lambda^2}\cos a}\right\}$$

1.4 EVALUATING WE GET

$$W(t,\gamma_1,\gamma_2) = \frac{2}{\pi}\left[W_1 \tan^{-1}\sqrt{\frac{(1+t)(1+\gamma_1)}{(1-t)(1-\gamma_1)}} + W_2 \tan^{-1}\sqrt{\frac{(1-t)(1+\gamma_2)}{(1+t)(1-\gamma_2)}}\right]$$

APPENDIX B

Evaluation of Integrals in Part II

A.
$$A(b,\gamma) = \int \tan^{-1}\sqrt{\frac{K(b-t)}{(1+t)}}\,\frac{dt}{[1-t\gamma]^2}$$

Integrating by parts

$$A = \frac{1}{\gamma(1-t\gamma)}\tan^{-1}\sqrt{\frac{K(b-t)}{(1+t)}}\Bigg|_{t_1}^{t_2} - \frac{1}{2\gamma(1+\gamma)}\int_{\theta_1}^{\theta_2}[(1+K)+(K-1)\cos 2\theta]\,d\theta$$

$$\tan\theta = \sqrt{\frac{K(b-t)}{1+t}} \qquad \begin{cases} t_1 = b & \theta_1 = 0 \\ t_2 = -1 & \theta_2 = \frac{\pi}{2} \end{cases}$$

$$A = \frac{1}{\gamma}\frac{1}{(1+\gamma)}\frac{\pi}{2} - \frac{1}{2\gamma(1+\gamma)}\left[(1+K)\theta + \frac{K-1}{2}\sin 2\theta\right]\Bigg|_0^{\pi/2}$$

$$= \frac{1}{\gamma}\frac{1}{(1+\gamma)}\frac{\pi}{2}\left[1 - \frac{2+\gamma(1-b)}{2(1-b\gamma)}\right]$$

$$A = -\frac{1}{\gamma}\frac{1}{(1+\gamma)}\frac{\pi}{4}\frac{(b+1)}{(1-b\gamma)}$$

B.
$$B(b,\gamma) = \int \tan^{-1}\sqrt{\frac{K(b-t)}{(1+t)}}\,\frac{t\,dt}{[1-t\gamma]^3}$$

Integrating by parts

$$B = \frac{1}{\gamma^2}\frac{2t\gamma-1}{2(1-t\gamma)^2}\Bigg|_{t_1}^{t_2} - \frac{1}{2\gamma^2}\int_{\theta_1}^{\theta_2}\frac{2t\gamma-1}{(1-t\gamma)^2}\,d\theta$$

Using the same substitution of

$$\tan\theta = \sqrt{\frac{K(b-t)}{1+t}}$$

$$B = -\frac{1}{2\gamma^2}\frac{2\gamma+1}{(1+\gamma)^2}\frac{\pi}{2} - \frac{1}{2\gamma^2}\int_{\theta_1}^{\theta_2}\frac{[K+\tan^2\theta]^2}{(1+\gamma)^2\sec^4\theta}\left[1 - \frac{2(1+\gamma)\sec^2\theta}{K+\tan^2\theta}\right]d\theta$$

$$= -\frac{\pi(1+b)[8b\gamma + (3b-5)]}{32(1+\gamma)^2(1-b\gamma)^2}$$

C.

$$I(b,\gamma) = \frac{b}{b+1}\int_{t_1}^{t_2}\sqrt{\frac{N(1-t)}{b+t}}\,\frac{dt}{[1-t\gamma]^2}$$

Making the substitution

$$\zeta^2 = \frac{N(1-t)}{(b+t)} \qquad t = \frac{L-b\zeta^2}{L+\zeta^2}$$

$$\frac{dt}{d\zeta^2}\,d\zeta^2 = \frac{-N(1+b)}{(N+\zeta^2)^2}\,d\zeta^2$$

$$I = -\frac{b}{(1+\gamma b)^2}\int\frac{\zeta\,d\zeta^2}{[NK+\zeta^2]^2}$$

$$= -\frac{bN}{[1+\gamma b]^2}\left[\frac{1}{\sqrt{NK}}\,\mathrm{TAN}^{-1}\frac{\zeta}{\sqrt{NK}} - \frac{\zeta}{(KN+\zeta^2)}\right]\Bigg|_{\infty}^{0}$$

$$I = -\frac{bN}{[1+\gamma b]^2}\left[-\frac{1}{\sqrt{NK}}\,\frac{\pi}{2} - \lim_{S\to\infty}\frac{S}{(KN+S^2)}\right]$$

$$\lim_{S\to\infty}\frac{\frac{S}{S^2}}{\left(\frac{NK}{S^2}+1\right)} \to 0$$

$$I = \frac{bN}{[1+\gamma b]^2}\,\frac{1}{\sqrt{NK}}\,\frac{\pi}{2} = \frac{b(1-\gamma)(1+\gamma b)\,\pi}{(1+\gamma b)^2(1-\gamma)\,2}$$

$$= \frac{b\pi}{2}\,\frac{1}{(1+\gamma b)}$$

II $[b,\gamma]$ ∿ Same type as integral in A $(A(b,\gamma))$

D. $$I_1(b,\gamma) = \frac{b}{b+1}\int_{-b}^{1}\sqrt{\frac{N(1-t)}{(b+t)}}\,\frac{t\,dt}{(1-\gamma t)^3}$$

Making the same substitution of $\varsigma^2 = N\frac{(1-t)}{(1+t)}$

$I_1[b,\gamma)$ becomes

$$I_1 = \frac{-2Nb}{(1+\gamma b)^3}\int_{-b}^{1}\frac{\varsigma^2(N-b\varsigma^2)\,d\varsigma}{(NK+\varsigma^2)^3}$$

Break the integral down into partial fractions; we get

$$I_1 = -\frac{2Nb}{(1+\gamma b)^3}\left[\int_{\infty}^{0}\frac{-b\,d\varsigma}{[NK+\varsigma^2]} + \int_{\infty}^{0}\frac{(N+2bNK)d\varsigma}{[NK+\varsigma^2]^2} - \int_{\infty}^{0}\frac{N^2K+bN^2K^2\,d\varsigma}{[NK+\varsigma^2]^3}\right.$$

and

$$\int\frac{d\varsigma}{(NK-\varsigma^2)^3} = \frac{1}{4(1-\gamma)^2}\,\frac{\varsigma}{[(1-\gamma)^2+\varsigma^2]} + \frac{3}{4(1-\gamma)^2}\left[\frac{\varsigma}{2NK(NK+\varsigma^2)} + \frac{1}{2(NK)^{3/2}}\,\mathrm{TAN}^{-1}\frac{\varsigma}{\sqrt{NK}}\right]$$

$$I_1 = -\frac{2Nb}{(1+\gamma b)^3}\left\{-\frac{b}{(1-\gamma)}\,\mathrm{TAN}^{-1}\frac{\varsigma}{1-\gamma} + \frac{1}{2}\frac{(1+2bk)}{k}\frac{1}{1-\gamma}\,\mathrm{TAN}^{-1}\frac{\varsigma}{(1-\gamma)} + \frac{1(1+2bk)\varsigma}{2(NK+\varsigma^2)}\right.$$

$$- \frac{N^2K(1+bk)}{4(1-\gamma)^2}\left[\frac{\varsigma}{[(1-\gamma)^2+\varsigma^2]} + \frac{3\varsigma}{2NK(NK+\varsigma^2)}\right.$$

$$\left.\left. + \frac{3}{2(1-\gamma)^3}\,\mathrm{TAN}^{-1}\frac{\varsigma}{(1-\gamma)}\right]\right\}\Bigg|_{\infty}^{0}$$

Substituting the limits

$$I_1 = \frac{b\pi}{(1+\gamma b)^2}\left\{-b + \frac{1}{2}\,\frac{(1+2b-b\gamma)}{1-\gamma} - \frac{3(1+b)}{8(1-\gamma)}\right\}$$

$$I_1(b,\gamma) = \frac{b\pi\,(4b\gamma - 3b + 1)}{8(1-\gamma)(1+\gamma b)^2}$$