On the Tamagawa Number Conjecture for Motives Attached to Modular Forms

Thesis by

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To my grandparents

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Abstract

We carry out certain automorphic and l-adic computations, the former extending results of Beilinson and Scholl and the latter using ideas of Kato and Kings, related to explicit motivic cohomology classes on modular varieties. Under mild local and global conditions on a modular form, these give exactly the coordinates of the Deligne and l-adic realizations of said motivic cohomology class in the eigenspace attached to the modular form (Theorem 4.1.1). Assuming the Kato's Main Conjecture and a Leopoldt-type conjecture, we deduce (a weak version of) the Tamagawa Number Conjecture for the motive attached to a modular form, twisted by a negative integer.

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Chapter 1

Introduction

1.1 Historical remarks

Among all mathematical objects that have been defined and studied over the past centuries, some are generally considered to be "arithmetic" (with stress on the third syllable, as opposed to the subject from grade school). There is no precise definition of criteria for differentiating arithmetic questions from nonarithmetic. However, generally speaking, many arithmetic objects have the property that they may be studied "one prime at a time," or, conversely, are deemed arithmetic because their study gives insight on the distribution or other properties of prime numbers. As an archetypical example of the former, given a polynomial $Q \in \mathbb{Z}[X_1, \ldots, X_n]$ in some number of variables with integer coefficients, one may try to count the number of solutions to Q modulo p for each prime p.

To many different types of arithmetic objects X, one may attach what is known as an L-function, $L(X, s)$, which is an analytic function of a complex variable s. The rough recipe for defining L-functions is, for each prime p, to study the behavior of X "at p," and use some invariants to define a local L-factor $L_p(X, s)$. This should be a rational function in p^{-s}. Formally define

$$L(X, s) = \prod_p L_p(X, s).$$

The definitions are such that the product tends to converge absolutely when $\mathrm{Re}(s)$ is large, so $L(X, s)$ is an actual function. Surprisingly, functions so constructed often have meromorphic, or even holomorphic, continuation to all of $\mathbb{C}$.

For example, consider the simplest object of study, a single point. To think of a point arithmetically, at each prime p, we obtain a set of one element. With the benefit of historical hindsight, the correct local L-factor to assign to a single point is

$$L_p(*, s) = \frac{1}{1 - p^{-s}}.$$

And so

$$L(*, s) = \prod_p \frac{1}{1 - p^{-s}} = \sum_{n \geq 1} \frac{1}{n^2} = \zeta(s),$$

the Riemann zeta function. So general L-functions are some vast generalizations of the Riemann zeta function. As values of $\zeta(s)$ are neither well-understood on the critical line $s = \frac{1}{2} + i\tau$, nor at $s = 2n + 1$ a positive odd integer, questions about values of L-functions are already guaranteed to be nontrivial.

To any number field K, an arithmetic object to be sure, there is defined a Dedekind zeta function

$$\zeta_K(s) = \sum_{\mathfrak{p} \subset K} \frac{1}{1 - \mathbb{N}\mathfrak{p}^{-s}}.$$

A nineteenth century theorem, the analytic class number formula (ACNF), expresses $\zeta_K(0)$ in terms of the values

$$\log |u|, u \in \mathcal{O}_K^*,$$

along with other more easily understood terms. As the 20th century progressed, many general L-functions were defined, but as of the early 1970s, it was not clear what an analogues of the ACNF should be. In the first place, what should be the analogue of the units? And what should play the role of $\log|\cdot|$, let alone ask how to interpret the other terms in the the expression for $\zeta_K(0)$.

In the early 1970s, Quillen succeeded in defining the higher algebraic K-groups of a regular scheme. The group of units of a number field is $K_1(\operatorname{Spec} \mathcal{O}_K)$, so there was hope that the K_i could provide appropriate generalizations of units in whatever formula one hoped to find. Furthermore, (certain summands of) algebraic K-groups define a universal cohomology theory, which carries realization maps into any other reasonable cohomology theory of a scheme. For a number field K, $\operatorname{Spec}(\mathcal{O}_K \otimes \mathbb{R})$ is a collection of points, and so the singular ("Betti") cohomology

$$H^0_B(\operatorname{Spec}(\mathcal{O}_K \otimes \mathbb{R}), \mathbb{R})$$

is a real vector space indexed by places of K. The realization map is exactly $\log|\cdot|$.

In the late 1970s, Bloch studied realizations from the K-theory of an elliptic curve, and conjectured relations to L-values [Bl00]. Inspired by Bloch, Beilinson developed some additional cohomological machinery, and was able to formulate a very general conjecture generalizing the ACNF (up to algebraic factors) to many schemes X over number fields. Furthermore, Beilinson [Bei86] gave compelling evidence for his conjecture by verifying parts in the case where X is associated to a modular form.

While there was now a concrete conjecture relating special values of L-functions to cohomology groups of X, the algebraic factors were still unexplained. Finally, in the early 1990s, work by Bloch-Kato, Fontaine-Perrin-Riou, and others, formulated a precise conjecture predicting a precise formula (perhaps up to powers of 2, which remain more difficult to pin down). This strengthening of Beilinson's work is known as the Tamagawa Number Conjecture (TNC). There has been various progress on such conjectures for number fields, but, aside from a theorem of Kings on CM elliptic curves [Ki01], there has been little progress in the intervening years in more geometric situations.

The subject of this thesis is the TNC in the case Beilinson studied, when X is associated to a modular form. Because the conjectures involve K-groups, which are not known to be finitely generated in any reasonable generality, the full strength of the statements necessarily can not be addressed. However, at heart the TNC is a formula describing how certain explicit elements are positioned with respect to one another in analytic and l-adic contexts, and such statements can be formulated unconditionally. We complement Beilinson's calculations to remove all indeterminant algebraic factors from his formulae, and are also able to relate explicit motivic cohomology classes to Kato's Euler system (which is of some interest in its own right). In the end, under some reasonable hypothesis, we find that the TNC is implied by Kato's Main Conjecture, in accordance with a philosophy of Huber and Kings.

1.2 What is a motive?

The word "motive" appears throughout this thesis. In most cases, a "motive" is an object that is expected to exist as a Chow motive if one assumes enough conjectures. On the other hand, the cohomlogy of a motive will always be given as a well-defined cohomology group, the L-function attached to a motive will always be a well-defined L-function, and so on. For the most part, the role played by the "motive" itself is merely to index various objects that occur, and to underline what geometry may be attached to each. Thus the reader will hopefully not take umbrage when idempotents are split or duals of motives taken; all theorems and propositions are made at the level of cohomology, where each construction makes sense. Much of this paper could also have been formulated in terms of Voevodsky's category of motives. We have avoided this approach because the proofs of theorems ultimately reduce to some very concrete and explicit computation, and we did not want to introduce a more abstract (and less familiar) framework when the goal is to do exactly the opposite. Of course, the reader is invited to dream that all motives exist in whatever context she feels justified.

1.3 The Tamagawa Number Conjecture

We will spend the rest of the introduction stating the Tamagawa Number Conjecture in the case of interest to us. In this range the conjecture can be formulated somewhat more simply than the most general one; we use a formulation by Kato and which is used by Kings [Ki01]. However, we adopt some notational differences. For simplicity, we are interested only in the middle dimensional cohomology of X. We also interchange the role played by a motive and its dual, so that the L-function that arises is the L-function attached to X, and we allow more leeway in choosing an integral basis for the Betti cohomology. Comparing to other formulations in print, these changes should be evident.

Let X be a smooth proper variety of dimension d over a number field K, with ring of integers $\mathcal{O}_K$. Choose a finite set of primes S containing all places of bad reduction, and an odd prime number l. For each integer r, one may consider the (hypothetical) Chow motive $M = h^d(X)(-r)$, with formal dual $M^*(1) = h^d(X)(d+1+r)$. Whether or not M can be physically constructed via correspondences, there is a well-defined motivic cohomology group

$$H^1(M^*(1)) := H^{d+1}_{\mathcal{M}}(X, d+1+r) := K_{d+1+2r}(X)^{(d+1+r)}_{\mathbb{Q}},$$

the r^{th} eigenspace for Adams operators acting on the rational vector space $K_{d+1+2r}(X)_{\mathbb{Q}}$. The formulation we use is given only at negative integers; that is, assume $r > 0$. (So $-r$ is the negative integer. In fact, one may take $r = 0$ if $\dim X > 0$, but we will not consider this range for technical simplicity)

In this range, the Deligne cohomology group may be computed as

$$H^1_{\mathcal{D}}(M^*(1)) := H^{d+1}_{\mathcal{D}}(X, \mathbb{R}(d+1+r)) = H^d_B(X \times_{\mathbb{Q}} \mathbb{C}, (2\pi i)^{d+r}\mathbb{R})^+,$$

while the relevant l-adic cohomology group is

$$H^1_{et}(M^*(1)) := H^1(\mathcal{O}_K[\frac{1}{Sl}], V_l^*(1)),$$

where

$$V_l = H^d(X \times_K \overline{K}, \mathbb{Q}_l(-r))$$

$$V_l^*(1) = H^d_{et}(X \times_K \overline{K}, \mathbb{Q}_l(d+r+1))$$

and the superscript "+" denotes the eigenspace fixed by complex conjugation. There exist regulator maps

$$r_{\mathcal{D}} : H^1(M^*(1)) \otimes \mathbb{R} \to H^1_{\mathcal{D}}(M^*(1))$$

$$r_l : H^1(M^*(1)) \otimes \mathbb{Q}_l \to H^1_{et}(M^*(1)).$$

For each prime $\mathfrak{p} \nmid l$ of K, the local L-factor is defined as usual as the determinant of Frobenius acting on the inertial invariants of V_l

$$L_{\mathfrak{p}}(M, s) = \det_{\mathbb{Q}_l}(1 - \mathrm{Fr}_{\mathfrak{p}} \mathbb{N}\mathfrak{p}^{-s} \mid V_l^{I_{\mathfrak{p}}}).$$

For $p \mid l$, take

$$L_{\mathfrak{p}}(M, s) = \det_{\mathbb{Q}_l}(1 - \phi_{\mathfrak{p}}^{-1} \mathbb{N}\mathfrak{p}^{-s} \mid D_{\mathrm{cris}}(V_l)).$$

These rational functions are expected to be independent of the choice of l, hence dependence on l is suppressed from the notation. Anyway, define

$$L_S(M, s) = \prod_{\mathfrak{p} \notin S} L_{\mathfrak{p}}(M, s).$$

Let δ be a $\mathbb{Q}$-basis for $H^d_B(X(\mathbb{C}), (2\pi i)^{d+r}\mathbb{Q})^+$, and let $T_l^*(1)$ be a $\mathbb{Z}_l$-lattice in $V_l^*(1)$ so that, under the standard comparison isomorphism, δ gives a $\mathbb{Z}_l$-basis for $T_l^*(1)^+$. Then the Tamagawa Number Conjecture at l states

Conjecture 1.3.1. *[Tamagawa Number Conjecture at l] Let l be an odd prime, X, S, $-r < 0$ as above. Assume that $L_S(M, s)$ has analytic continuation to all of $\mathbb{C}$, and that, for all $\mathfrak{p} \in S$,*

$$L_{\mathfrak{p}}(M, 0) \neq 0$$

Then

(a) $r_{\mathcal{D}}$ and r_l are isomorphisms, and

$$\# H^2(\mathcal{O}_K[\frac{1}{Sl}], T_l^*(1)) < \infty$$

(b) $\dim H^1(M^*(1)) = \mathrm{ord}_{s=0} L_S(M, s)$

(c) There exists a $\mathbb{Q}$-basis ξ of $H^1(M^*(1))$ so that, comparing two bases of a real vector space,

$$r_{\mathcal{D}}(\xi) = L^*_{Sl}(M, 0)\delta$$

(d) With the $[\cdot : \cdot]$ denoting the generalized index of one lattice with respect to another in a $\mathbb{Q}_l$-vector space containing both,

$$[H^1(\mathcal{O}_K[\frac{1}{Sl}], T_l^*(1)) : r_l(\xi)\mathbb{Z}_l] = \# H^2(\mathcal{O}_K[\frac{1}{Sl}], T_l^*(1))$$

This conjecture is easily seen to be independent of all choices. Notice that we have only given a statement at l; the full TNC would be the compositum of such statements for all primes l, which would predict the equality of integers up to a sign. The formulation here is preferable, first, because the question at 2 is significantly more difficult, and, second, because in any case in practice one verifies prime by prime. Note that one may use the same or similar ξ, and δ for all l, which is in fact what we will do.

Because there is no hope at the present of proving that the motivic cohomology groups are of any reasonable size, the following weak Tamagawa Number Conjecture is commonly made

Conjecture 1.3.2 (weak TNC at l)**.** *There exists an explicit subspace $H^1_{\text{constr}}(M^*(1)) \subset H^1(M^*(1))$ so that all statements in Conjecture 1.3.1 hold with $H^1(M^*(1))$ replaced by $H^1_{\text{constr}}(M^*(1))$.*

One expects of course that the explicit subspace of constructible elements appearing in this conjecture is in fact the whole of the motivic cohomology. This conjecture is established by Kings in [Ki01] when K is a quadratic imaginary field, $X = E$ is an elliptic curve with CM by the full ring of integers in K, and $l \neq 2, 3$, under the assumption that the H^2 group appearing is already known to be finite. The argument uses as a key ingredient Rubin's work on the main conjecture for imaginary quadratic fields.

1.4 A somewhat equivariant Tamagawa Number Conjecture

We actually wish to prove a slight and obvious variant of the above conjecture, that is better adapted to what one means by the motive attached to a modular form. Let X, S, $r > 0$ be as above, and again write formally $M = h^d(X)(-r)$. Let A be a commutative subring of $CH^d(X \times X)_{\mathbb{Q}}$, the ring of endomorphisms of X as a Chow motive. Suppose A is stable under transposition, that $\lambda : A \to F$ is a character of A valued in a number field F, and let $\bar{\lambda} : A \to F$ be the conjugate character given by precomposition of λ by transposition. Fix embeddings of F into $\mathbb{C}$ and $\overline{\mathbb{Q}_l}$ for each l.

Philosophically, λ and $\bar{\lambda}$ ought to define idempotents on the Chow motive X, and so one expects to have direct summands $M \otimes_A \lambda$ of M, $\mathcal{M}^*(1) \otimes_A \bar{\lambda}$ of $M^*(1)$. On a more concrete level, A acts on all the cohomology groups of M defined in the previous section, and one may consider the $\lambda-$ or $\bar{\lambda}-$eigenquotients of these groups, taken with respect to the chosen embeddings according to the coefficients used. Similarly one may define a partial L-function $L_S(M \otimes_A \lambda, s)$. As the regulator maps are also A-equivariant and descend to these quotients, we may formulate the analogue of Conj. 1.3.1 as

Conjecture 1.4.1. *Notation as above. Let $\delta \otimes_A \bar{\lambda}$ be an F-basis for $H^d(X(\mathbb{C}), (2\pi i)^{d+r}\mathbb{Q})^+ \otimes_A \bar{\lambda}$, and let $T_l^*(1) \otimes_A \bar{\lambda}$ be a lattice in $V_l^*(1) \otimes_A \bar{\lambda}$ whose $+$-part is generated by $\delta \otimes_A \bar{\lambda}$. Assume $L(M \otimes_A \lambda, s)$*

has analytic continuation to all of $\mathbb{C}$ *and that, for all* $\mathfrak{p} \in S$,

$$L_{\mathfrak{p}}(M \otimes_A \lambda, 0)$$

Then

(a) $r_{\mathcal{D}} \otimes_A \bar{\lambda}$ *and* $r_l \otimes_A \bar{\lambda}$ *are isomorphisms, and*

$$\#H^2(\mathcal{O}_K[\frac{1}{Sl}], T_l^*(1) \otimes_A \bar{\lambda}) < \infty$$

(b) The dimension of $H^1(M^*(1) \otimes_A \bar{\lambda})$ *is equal to the order of vanishing at* $s = 0$ *of each component of* $L_S(M \otimes_A \lambda, s)$

(c) There exists an F*-basis* $\xi \otimes_A \bar{\lambda}$ *of* $H^1(M^*(1) \otimes_A \bar{\lambda})$ *so that, comparing two bases of an real or complex vector space,*

$$r_{\mathcal{D}}(\xi) \otimes_A \bar{\lambda} = L_{Sl}^*(M \otimes_A \lambda, 0)(\delta \otimes_A \bar{\lambda})$$

(d) With the $[\cdot : \cdot]$ *denoting the generalized index of one lattice with respect to another in a* $\mathbb{Q}_l$*-vector space containing both,*

$$[H^1(\mathcal{O}_K[\frac{1}{Sl}], T_l^*(1) \otimes_A \bar{\lambda}) : r_l(\xi \otimes_A \bar{\lambda})\mathcal{O}_{F_l}] = \#H^2(\mathcal{O}_K[\frac{1}{Sl}], T_l^*(1) \otimes_A \bar{\lambda})$$

Finally, we have a conjecture in the form needed, namely

Conjecture 1.4.2. *Same as conjecture 1.4.1, but with* $H^1(M^*(1) \otimes_A \bar{\lambda})$ *replaced by an explicit subspace* H^1_{constr}

As the title of the section suggests, this conjecture is equivariant in the sense that it involves a motive with an action of its endomorphism algebra. What is usally termed the Equivariant Tamagawa Number Conjecture, as formulated by Burns and Flach, is a statement about the various cohomology groups appearing, and how they are related as modules over such an algebra (see [Fl04]). For a commutative algebra, the formulation here is roughly asking for the λ-component of this full equivariant conjecture. However, the full conjecture is much stronger than the sum of its eigencomponents, because the full conjecture also encodes information about congruences between different components.

In chapter 2, we will review what is meant by the motive attached to modular form, in the context just given. In chapter 3, we construct the explicit classes ξ and δ that are to be used in verifying the conjecture. Chapter 4 states the main theorem, giving the results of explicit calculations that together verify the conjecture un the case at hand. Chapters 5 to 8 carry out the analytic computations relevant to part (c) of the conjecture, and chapters 9 to 11 address part (d).

Chapter 2

Review of the Motive $M(f)$

2.1 Geometry of $M(f)$

Let f be a new cuspidal eigenform of weight $k+2 \geq 2$ and level $N \geq 5$, $f = \sum a_n q^n$, and take j to be a positive integer. It will be convenient to set $t = k+2+j$. We write $\check{f} = \sum \overline{a_n} q^n$ for the complex conjugate form and $F = \mathbb{Q}(\{a_n\})$ the field of coefficients. Let $\mathbb{T}$ be the $\mathbb{Q}$-algebra generated by Hecke operators T_p, $p \nmid N$, acting on $S_{k+2}(\Gamma_1(N))$, and $\mathbb{T}'$ the $\mathbb{Q}$-algebra of good Hecke operators acting on $M_{k+2}(\Gamma_1(N))$; $\mathbb{T}' \twoheadrightarrow \mathbb{T}$. Write $\lambda : \mathbb{T} \to F$ for the character attached to f, so $\bar{\lambda}$ is the character attached to $\check{f}$. We consider TNC for the motive $M = M(f)(-j)$, which has (at least formally) $M^*(1) = M(\check{f})(k+j+2) = M(\check{f})(t)$.

We begin by reviewing some geometry. Let $Y = Y(N)$ be the modular curve of full level N structure (We have chosen $N \geq 5$ to avoid any possible difficulties with the existence of such fine moduli schemes). It is geometrically connected over $\mathbb{Q}(\zeta_N)$, but we consider it as a variety $/_{\mathbb{Q}}$. As usual, $X = X(N)_{/\mathbb{Q}}$ is the completion of Y, and **Cusps** will denote the reduced complement of Y in X. We will let **Isom** denote the μ_2-torsor whose geometric points consist of a geometric point of **Cusps** along with an orientation of the generalized elliptic curve sitting over that cusp.

Let $\mathfrak{X}_{\Gamma(N)} \to Y$ be the universal elliptic curve, and $\lambda : \mathfrak{X}^k_{\Gamma(N)} \to Y$ its k-fold self-product over Y. The $(k+1)$-dimensional Kuga-Sato variety $KS^k_{\Gamma(N)}$ is a smooth compactification of $\mathfrak{X}^k_{\Gamma(N)}$, semistable over X. $KS^k_{\Gamma(N)}$ was first constructed by Deligne [De71], or it may be viewed as the toroidal compactification of the mixed Shimura variety $\mathfrak{X}^k_{\Gamma(N)}$. Similarly, we have $\lambda : \mathfrak{X}^k_{\Gamma_1(N)} \to Y_1(N)$ and the toroidal compactification $KS^k_{\Gamma_1(N)}$, semistable over $X_1(N)$. The standard idempotent ϵ [Sch90] acts fiberwise, hence compatibly on $\mathfrak{X}^k_{\Gamma(N)}$, $KS^k_{\Gamma(N)}$, $\mathfrak{X}^k_{\Gamma_1(N)}$, and $KS^k_{\Gamma_1(N)}$. The reduced complement of $\mathfrak{X}^k_{\Gamma(N)}$ in $KS^k_{\Gamma(N)}$ will be denoted $KS^{k,\infty}_{\Gamma(N)}$, and likewise for $KS^{k,\infty}_{\Gamma_1(N)}$; these are disjoint unions of toric varieties.

2.2 Cohomology of $M(f)$

For any regular scheme X, write as usual $H^i_{\mathcal{M}}(X,\bullet) = K_{2\bullet-i}(X)^{\bullet}_{\mathbb{Q}}$. One has Gysin sequences

$$\cdots \to H^{k+2}_{\mathcal{M}}(KS^k_{\Gamma_1(N)},t) \xrightarrow{i^*} H^{k+2}_{\mathcal{M}}(\mathfrak{X}^k_{\Gamma_1(N)},t) \xrightarrow{\alpha} H^{k+1}_{\mathcal{M}}(KS^{k,\infty}_{\Gamma_1(N)},t-1) \to \cdots$$

and

$$\cdots \to H^{k+2}_{\mathcal{M}}(KS^k_{\Gamma_1(N)},t)(\epsilon) \xrightarrow{i^*} H^{k+2}_{\mathcal{M}}(\mathfrak{X}^k_{\Gamma_1(N)},t)(\epsilon)$$
$$\xrightarrow{\alpha} H^{k+1}_{\mathcal{M}}(KS^{k,\infty}_{\Gamma_1(N)},t-1)(\epsilon) \to \cdots$$

Formally, define

$$H^1(M_{eq,cusp}(t)) = H^{k+2}_{\mathcal{M}}(KS^k_{\Gamma_1(N)},t)(\epsilon)$$

$$H^1(M_{eq}(t)) = H^{k+2}_{\mathcal{M}}(\mathfrak{X}^k_{\Gamma_1(N)},t)(\epsilon)$$

The usual Hecke correspondences on $\mathfrak{X}^k_{\Gamma_1(N)}$ extend to $KS^k_{\Gamma_1(N)}$, and these restrict to correspondences on $KS^{k,\infty}_{\Gamma_1(N)}$. Let $\widetilde{\mathbb{T}}$ denote the endomorphism $\mathbb{Q}$-algebra of the Chow motive $KS^k_{\Gamma_1(N)}(\epsilon)$ generated by the good T_p. (the notation M_{eq}, resp. $M_{eq,cusp}$, is to suggest a motive that is the sum of all, resp. all cuspidal, $M(f)$, considered as an equivariant object under the Hecke algebra). Then λ, $\bar{\lambda}$ define characters of $\widetilde{\mathbb{T}}$, and the Gysin sequence carries a $\widetilde{\mathbb{T}}$-action.

Lemma 2.2.1. *The natural map*

$$H^1(M_{eq}(t)) \otimes_{\widetilde{\mathbb{T}}} \bar{\lambda} \to H^1(M_{eq,cusp}(t)) \otimes_{\widetilde{\mathbb{T}}} \bar{\lambda}$$

is an isomorphism.

Proof. We need to show that

$$H^k_{\mathcal{M}}(KS^{k,\infty}_{\Gamma_1(N)},t-1)(\epsilon) \otimes_{\widetilde{\mathbb{T}}} \bar{\lambda} = H^{k+1}_{\mathcal{M}}(KS^{k,\infty}_{\Gamma_1(N)},t-1)(\epsilon) \otimes_{\widetilde{\mathbb{T}}} \bar{\lambda} = 0.$$

Suppose Z is a component of $KS^{k,\infty}_{\Gamma_1(N)}$, that sits over a cusp of $Y_1(N)$ defined over $\mathbb{Q}(\zeta_M)$. As in the proof of [Sch90], Theorem 1.3.3,

$$H^{k+1}_{\mathcal{M}}(Z,k+1+\bullet)(\epsilon) \cong H^{\bullet}_{\mathcal{M}}(\mathbb{Q}(\zeta_M),j+1).$$

Then

$$H^k_{\mathcal{M}}(KS^{k,\infty}_{\Gamma_1(N)},t-1)(\epsilon) \cong H^0_{\mathcal{M}}(\mathbf{Cusps}_{\Gamma_1(N)},j+1) = 0$$

and

$$H^{k+1}_{\mathcal{M}}(KS^{k,\infty}_{\Gamma_1(N)}, t-1)(\epsilon) \cong H^1_{\mathcal{M}}(\mathbf{Cusps}_{\Gamma_1(N)}, j+1).$$

By Borel's theorem, the latter space is a $\mathbb{Q}$-lattice in $H^0_B(\mathbf{Cusps}_{\Gamma_1(N)}(\mathbb{C}), \mathbb{R}(j))^+$. The action of $\widetilde{\mathbb{T}}$ is the action of $\mathbb{T}'$ on weight $k+2$ Eisenstein series, where $\bar{\lambda}$ does not appear. $\square$

We set

$$H^1_{\mathcal{M}}(M^*(1)) := H^1(M_{eq,cusp}(t)) \otimes_{\widetilde{\mathbb{T}}} \bar{\lambda} \cong H^1(M_{eq}(t)) \otimes_{\widetilde{\mathbb{T}}} \bar{\lambda}.$$

The isomorphism will be made more explicit later. The above arguments also hold in l-adic and Deligne cohomology; set

$$\begin{aligned} H^1_{et}(M^*(1)) &:= H^1(\mathbb{Q}, H^{k+1}(KS^k_{\Gamma_1(N),\overline{\mathbb{Q}}}, \mathbb{Q}_l)(t)(\epsilon) \otimes_{\widetilde{\mathbb{T}}} \bar{\lambda}) \\ &\cong H^1(\mathbb{Q}, H^{k+1}(\mathfrak{X}^k_{\Gamma_1(N),\overline{\mathbb{Q}}}, \mathbb{Q}_l)(t)(\epsilon) \otimes_{\widetilde{\mathbb{T}}} \bar{\lambda}) \end{aligned}$$

and

$$\begin{aligned} H^1_{\mathcal{D}}(M^*(1)) &:= H^{k+2}_{\mathcal{D}}(KS^k_{\Gamma_1(N)}/\mathbb{R}, \mathbb{R}(t))(\epsilon) \otimes_{\widetilde{\mathbb{T}}} \bar{\lambda} \\ &\cong H^{k+2}_{\mathcal{D}}(\mathfrak{X}^k_{\Gamma_1(N)}/\mathbb{R}, \mathbb{R}(t))(\epsilon) \otimes_{\widetilde{\mathbb{T}}} \bar{\lambda}. \end{aligned}$$

Let $\rho_{\mathcal{D}}$ and ρ_l be the Deligne and l-adic realization maps taking the algebraic K-theory of KS^k or $\mathfrak{X}^k$ to the respective cohomology theories. We prefer to think of $\rho_{\mathcal{D}}$ and ρ_l as being defined by Chern classes, as due to Soulé, Gillet, and Beilinson, although the reader may prefer to think of motivic cohomology and regulators in terms of Voevodsky's triangulated categories. Either way, compatibility with algebraic correspondences induces

$$\rho_{l,M^*(1)} : H^1_{\mathcal{M}}(M^*(1)) \to H^1_l(M^*(1))$$

$$\rho_{\mathcal{D},M^*(1)} : H^1_{\mathcal{M}}(M^*(1)) \to H^1_{\mathcal{D}}(M^*(1)).$$

To summarize what we have just done: the general TNC is formulated for motives $h^i(X)$, X smooth projective, and generalizes in an obvious manner to direct summands thereof. The motive $M(f)$ is expected to be a direct summand of $h^{k+1}(KS^k_{\Gamma_1(N)})$, cut out using ϵ and λ. Unfortunately, while $h^{k+1}(KS^k)$ is at least known to exist by [GHM02], attaching an idempotent to λ in the category of Chow motives is still an open problem in general. However, it is clear that such a construction would yield exactly the proposed cohomology groups, and we can thus avoid these technical difficulties by working directly on realizations. For weight 2, the situation is better, but we will not make use of this. As it will be convenient to work over the open modular curve, we have reformulated our groups in terms of $\mathfrak{X}^k$. (Another approach to working over the open modular curve is to attempt to split the inclusion i^* via Eisenstein symbols, which will be touched on in §5.3)

To finish this chapter, we give more concrete descriptions of the cohomology groups. Let $c \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ be the complex conjugation acting on geometric points; for a real variety X, $H^i(X(\mathbb{C}))^{\pm}$ denotes the $\pm$-eigenspace for the action of c. According to [Sch88], p. 9 and [Ka04], 11.2.2, setting $\mathcal{H} = R^1\lambda(\mathbb{C})_*\mathbb{Z}$,

$$\begin{aligned} H^1_{\mathcal{D}}(M^*(1)) &= H^{k+2}_{\mathcal{D}}(\mathfrak{X}^k_{\Gamma_1(N)}/\mathbb{R}, \mathbb{R}(t))(\epsilon) \otimes_{\widetilde{\mathbb{T}}} \bar{\lambda} \\ &= H^{k+1}_B(\mathfrak{X}^k_{\Gamma_1(N)}(\mathbb{C}), \mathbb{R}(t-1))(\epsilon)^+ \otimes_{\widetilde{\mathbb{T}}} \bar{\lambda} \\ &= H^1_B(Y_1(N)(\mathbb{C}), (\mathrm{Sym}^k\mathcal{H}_{\mathbb{R}})(t-1))^+ \otimes_{\widetilde{\mathbb{T}}} \bar{\lambda} \end{aligned}$$

and

$$\begin{aligned} H^1_l(M^*(1)) &= H^1(\mathbb{Q}, H^{k+1}(\mathfrak{X}^k_{\overline{\mathbb{Q}}}, \mathbb{Q}_l)(t))(\epsilon)^{\Gamma_1(N)} \otimes_{\widetilde{\mathbb{T}}} \bar{\lambda} \\ &= H^1(\mathbb{Q}, H^1(Y_1(N)_{\overline{\mathbb{Q}}}, \mathrm{Sym}^k\mathcal{H}_{\mathbb{Q}_l})(t) \otimes_{\widetilde{\mathbb{T}}} \bar{\lambda}) \\ &= \bigoplus_{v|l} H^1(\mathbb{Q}, H^1(Y_1(N)_{\overline{\mathbb{Q}}}, \mathrm{Sym}^k\mathcal{H}_{\mathbb{Q}_l})(t) \otimes_{\widetilde{\mathbb{T}},v} \bar{\lambda}). \end{aligned}$$

Here v ranges over the places of F dividing l, and for such a v, we abuse notation and write F_l for the v-adic completion of F. The tensor product in the last line is over $\mathbb{Q}_l$, with λ taking values in F_l, the Galois representation appearing there is well known to be the two-dimensional F_l-representation V_l attached to $\check{f}$.

Chapter 3

Construction of Explicit Elements

Choose a partition $k = k_1 + k_2$, $k_1 \geq 0$, which will remain fixed throughout. Consider the following global hypothesis:

(G): $k_1 + j$ is even, and $L(f, k_1 + 1) \neq 0$.

According to Rankin (see [JS81], §5.4) and the functional equation, $L(f, s)$ does not vanish on $(0, \frac{k+1}{2})$ or $(\frac{k+3}{2}, k+2)$. Thus if $k \neq 0, 2$, there always exists a partition satisfying **(G)**. If $k = 2$, **(G)** holds for j even, and when j is odd, it requires that $L(f, 2) \neq 0$. For $k = 0$, **(G)** can only hold if j is even and $L(f, 1) \neq 0$.

3.1 A Betti cohomology class and l-adic lattices

Consider the $k+1$-chain

$$\Delta = \Delta_{k_1} : (0, i\infty) \times [0,1]^k \to SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^{2k} \backslash \mathcal{H} \times \mathbb{C}^k$$

$$(\tau, t_1, \ldots, t_k) \mapsto (\tau, \tau \cdot t_1, \ldots, \tau \cdot t_{k_1}, t_{k_1+1}, \ldots, t_{k_1+k_2}).$$

Then Δ represents a class in

$$\begin{aligned} H_{k+1}(KS^k(\mathbb{C}), KS^{k,\infty}(\mathbb{C}), \mathbb{Q})^{(-1)^{k_1}} &\cong H^{BM}_{k+1}(\mathfrak{X}^k(\mathbb{C}), \mathbb{Q})^{(-1)^{k_1}} \\ &\cong H^{k+1}(\mathfrak{X}^k(\mathbb{C}), \mathbb{Q}(k+1))^{(-1)^{k_1}} \end{aligned}$$

(for more details on analytic parameterizations of modular varieties, see §5.2). Assuming **(G)**, we have

$$(2\pi i)^j \epsilon \circ \Delta \in H^{k+1}(\mathfrak{X}^k(\mathbb{C}), \mathbb{R}(t-1))(\epsilon)^+.$$

Write $\bar{\delta}_{k_1}$ for its image in $H^1_{\mathcal{D}}(M^*(1))$.

Set $\pm = (-1)^{k_2}$. By standard comparison theorems, $\epsilon \circ \Delta$ also gives rise to a class $(\epsilon \circ \Delta)_l \in H^1_{et}(Y_{\overline{\mathbb{Q}}}, \mathrm{Sym}^k \mathcal{H}_{\mathbb{Q}_l})$, with image $\bar{\delta}_l = \bar{\delta}_{l,k_1} \in V_l^{\pm}$. We will assume throughout the paper that $\bar{\delta}_l \neq 0$, which we will see later is implied by **(G)**. Let $\tilde{T}_l$ be any Galois-stable lattice in V_l. When $\bar{\delta}_l \neq 0$ there is a unique way to replace $\tilde{T}_l$ by a scalar multiple T_l so that $\bar{\delta}_l$ generates $T_l^{\pm}$. For each odd l, such a choice will remain fixed throughout.

Remark: When **(G)** can not be made to hold, one can more generally choose other modular symbols δ that have been twisted by finite order Dirichlet characters χ. In the analogue of condition **(G)**, one requires the parity of j be given in terms of the parity of χ, and that the twisted L-value $L(f, k_1 + 1, \chi) \neq 0$; certainly many such χ exist for any f, even uniformly across all forms of a given weight and level. However, each choice of δ must be accompanied by a compatible choice of ξ in the following chapter, resulting in another layer of complexity on all computations that follow (the kernel functions K of Lemma 7.2.1 will depend on χ). Since condition **(G)** is automatic in weight $\neq 2, 4$, we have not attempted these computations, but one expects the same techniques to work.

3.2 Beilinson's motivic cohomology class

As Y is the modular curve with full level N structure, there is a canonical identification of $\mathfrak{X}[N]$, the N-torsion of the universal elliptic over Y, with $(\mathbb{Z}/N)^2$. Let e_1 and e_2 be the canonical basis of $\mathfrak{X}[N]$, corresponding to $(1,0)$ and $(0,1)$ in $(\mathbb{Z}/N)^2$. It is convenient to interpret e_1 and e_2 as defining elements of $\mathbb{Q}[(\mathbb{Z}/N)^2]$. For $l \geq 0$, set

$$\mathbb{Q}^{(l)}[(\mathbb{Z}/N)^2] = \{\phi : (\mathbb{Z}/N)^2 \to \mathbb{Q} : \phi(-c,-d) = (-1)^l \phi(c,d)\}$$

and

$$\mathbb{Q}[\mathbf{Isom}]^{(l)} = H^0_{\mathcal{M}}(\mathbf{Isom}, 0)^{(l)} = \{f : \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \backslash GL_2(\mathbb{Z}/N) \to \mathbb{Q} : f(-g) = (-1)^l f(g)\}.$$

Then one has the $GL_2(\mathbb{Z}/N)$-equivariant horospherical map ([HK99b] Def. 7.5, or elsewhere; note that the normalization given here differs from the usual one by a factor of $N(l!)$)

$$\varrho^l : \mathbb{Q}^{(l)}[(\mathbb{Z}/N)^2] \to \mathbb{Q}[\mathbf{Isom}]^{(l)}$$

$$\varrho^l(\psi)(g) = \frac{N^l + 1}{(l+2)} \sum_{(t_1,t_2) \in (\mathbb{Z}/N)^2} \psi(g^{-1}t) B_{l+2}\left(\frac{t_2}{N}\right).$$

The same name will be given to the composite

$$\varrho : \mathbb{Q}[(\mathbb{Z}/N)^2] \to \mathbb{Q}^{(l)}[(\mathbb{Z}/N)^2] \to \mathbb{Q}[\mathbf{Isom}]^{(l)},$$

where the first map is projection to the $(-1)^l$-eigenspace. Let $\phi_i = \varrho^{j+k_1}(e_i)$. Recall that there is a GL_2-equivariant residue map $\mathrm{Res}^l : H^{l+1}_{\mathcal{M}}(\mathfrak{X}^l, l+1) \to \mathbb{Q}[\mathbf{Isom}]^{(l)}$, and that it has a canonical right inverse, the Eisenstein symbol map $\mathcal{E}is^l$ [Bei86], [HK99b], §2. Then $\mathcal{E}is^{j+k_i}(\phi_i) \in H^{k_1+j+1}_{\mathcal{M}}(\mathfrak{X}^{j+k_i}, j+k_i+1)$. Let $\pi_1 : \mathfrak{X}^{k_1+j+k_2} \to \mathfrak{X}^{k_1+j}$ be the projection to the first k_1+j coordinates, let $\pi_2 : \mathfrak{X}^{k_1+j+k_2} \to \mathfrak{X}^{j+k_2}$ be the projection to the last $j+k_2$ coordinates, and let $\pi : \mathfrak{X}^{k_1+j+k_2} \to \mathfrak{X}^k$ be defined by omitting the "middle" j coordinates. We consider $j : Y(N) \to Y_1(N)$ to be given on moduli problems by forgetting the first canonical torsion section.

Define

$$\xi_{k_1} = \epsilon \circ \pi_*(\pi_1^* \mathcal{E}is^{k_1+j}(\phi_1) \cup \pi_2^* \mathcal{E}is^{j+k_2}(\phi_2)) \in H^{2+k}_{\mathcal{M}}(\mathfrak{X}^k_{\Gamma(N)}, t)(\epsilon)$$

and take

$$\bar{\xi}_{k_i} = j_*(\xi_{k_i}) \otimes \bar{\lambda} \in H^{k+2}_{\mathcal{M}}(\mathfrak{X}^k_{\Gamma_1(N)}, t)(\epsilon).$$

The subscript k_i will often be suppressed; further, we will write $\xi^N_{k_i} = \xi^N$ if we need to make explicit the level N appearing in the definition. These motivic cohomology classes are a slight generalization of those appearing in [DS91], §5.7 (where $k_1 = k$, $k_2 = 0$).

Chapter 4

The Main Theorem

The abstract modular form f defines an $F \otimes \mathbb{C}$-valued holomorphic $(k+1)$-form ω_0' on $\mathfrak{X}^k_{\Gamma_1(N)}(\mathbb{C})$, and let ω_0 be its pullback to $\mathfrak{X}^k_{\Gamma(N)}(\mathbb{C})$. Since f has rapid decay at the cusps, integration against these forms defines functionals (see §5.3 for more details)

$$\langle \cdot, \omega_0 \rangle : H^{k+2}_{\mathcal{D}}(\mathfrak{X}^k_{\Gamma(N)}/\mathbb{R}, \mathbb{R}(t))(\epsilon) \to F \otimes \mathbb{C}$$

$$\langle \cdot, \omega_0' \rangle : H^{k+2}_{\mathcal{D}}(\mathfrak{X}^k_{\Gamma_1(N)}/\mathbb{R}, \mathbb{R}(t))(\epsilon) \to F \otimes \mathbb{C}.$$

Because ω_0' has eigensystem λ under the action of $\mathbb{T}$ and is holomorphic, the pairing $\langle \cdot, \omega_0' \rangle$ descends to an F-linear map

$$\langle \cdot, f \rangle_{M^*(1)} : H^1_{\mathcal{D}}(M^*(1)) \to \mathbb{C} \otimes F$$

in such a manner that $\langle j_* x \otimes \bar{\lambda}, f \rangle_{M^*(1)} = \langle x, \omega_0 \rangle$. As $H^1_{\mathcal{D}}(M^*(1))$ is a rank 1 $\mathbb{R} \otimes F$-module, its elements are completely determined by pairing against f.

4.1 Statement of theorem

Note: While every effort has been made to ensure that all stated equalities are exactly true, it is perhaps safer to regard them as holding up to powers of 2 and -1. As we do not consider the TNC at the prime 2, such discrepancies anyway do not affect the final result.

Theorem 4.1.1. *Let f be a new cuspidal eigenform of weight $k \geq 2$, level $N \geq 5$ and coefficients in F. Assume that $j > 0$ and that* **(G)** *holds, and let $L^{(M)}(f, s)$ denote the partial L function with Euler factors at primes $p \mid M$ removed. Then*

(i) If the local 2-adic representation attached to f is not supercuspidal, then the leading coefficient

$L^{(N),*}(f,-j) \in \mathbb{C} \otimes F$ is equal to

$$(-1)^{k^2+\frac{j^2+j}{2}} \frac{2\langle \rho_{\mathcal{D},M^*(1)}(\bar{\xi}), f\rangle_{M^*(1)}}{\langle \bar{\delta}, f\rangle_{M^*(1)}}$$

(ii) Take l odd and $v \mid l$. Then $\rho_{l,M^*(1)}(\bar{\xi}) \in H^1(\mathbb{Z}[\frac{1}{Nl}], V_l(t)) \subset H^1(\mathbb{Q}, V_l(t))$. Assume Kato's Main conjecture for $\check{f}$ and the Leopoldt-type conjecture that $H^2(\mathbb{Z}[\frac{1}{Nl}], T_l(t))$ is finite. Then $H^1(\mathbb{Z}[\frac{1}{Nl}], T_l(t))$ is a rank 1 $\mathcal{O}_l$-module. If $l \nmid N$, the index of $L_l(f,-j)^{-1}\rho_{l,M^*(1)}(\bar{\xi})$ with respect to this lattice is $\#H^2(\mathbb{Z}[\frac{1}{Nl}], T_l(t))$. If $l \mid N$, the index of $\rho_{l,M^*(1)}(\bar{\xi})$ in $H^1(\mathbb{Z}[\frac{1}{Nl}], T_l(t))$ is equal to $\#H^2(\mathbb{Z}[\frac{1}{Nl}], T_l(t))$

Considering Conj. 1.4.2 in this case, part (b) is obvious, and part (a) follows from the hypothesis already made, as in Kings. The calculations of the theorem exactly address parts (c) and (d), so

Corollary 4.1.1. *Let f be a new cuspidal eigenform on $\Gamma_1(N)$, of weight $k+2 \geq 2$ and level $N \geq 5$, whose local representation at 2 is not supercuspidal. Let $j > 0$ and l odd. Assume that $H^2(\mathbb{Z}[\frac{1}{Nl}], T_l(t))$ is finite and that* **(G)** *holds. Then the weak TNC for $M(f)(-j)$ at l is implied by Kato's l-adic main conjecture for $\check{f}$.*

We make a few remarks on this result. As mentioned in the introduction, the main result of [Ki01] is a proof of TNC for the motive attached to a CM elliptic curve, which is the motive $M(\psi)$ attached to its Hecke character ψ of the CM field. Let f_ψ be the associated weight 2 CM form. The constructions of $M(f_\psi)$ and $M(\psi)$ are not directly comparable, and the statement of Cor. 4.1.1 for f_ψ is not a simple reworking of Kings's result for ψ. However, the same L-series arises in both statements, and the realizations of these motives are closely related, as can be seen in the treatment of CM forms in [Ka04], §15. In this sense, the above result may be viewed as a generalization of Kings's theorem to all modular forms, as well as to Hecke characters of any (geometric) weight. In his context the necessary Main Conjecture is known by Rubin; when f has CM, the same result is relevant here. For progress on the main conjecture for more general general modular forms see [SU02]. The key observation to proving (ii) is in fact the same technical lemma used in Kings's paper. Comments on condition **(G)** were made in §3.1. A treatment of TNC for finite order Hecke characters of quadratic imaginary fields is given in [Jo05].

The restriction on the level is imposed solely to keep the geometry nice, and does not play a role in the local calculations. The hypothesis on the 2-adic representation is somewhat more serious from a technical point of view. For a dihedral supercuspidal representation, essentially the same proof works, with just a few parameters changed. But for a nondihedral supercuspidal representation, it does not appear that the newform in the Whittaker model is known explicitly enough to carry out the necessary computations! Philosophically, though, this hypothesis is superfluous.

When $k = 0$, up to rational multiple, the result in (i) is exactly Beilinson's theorem in [Bei86]. However, obtaining an exact formula requires making a good choice of the Eisenstein symbols in ξ, and up to now it was not known in general which explicit choices gave an equality. The new content here is really that the choices made in §3.1 and §3.2 give a manageable computation; the idea that bad zeta integrals could be tractable with a good choice of data was motivated by [Wa02] and [Pr04]. Notice that one may verify the formula embedding by embedding, so choose $F \hookrightarrow \mathbb{C}$. With respect to this embedding, f defines a classical automorphic form.

Chapter 5

Definitions and Preliminary Results

The next four chapters comprise the proof of Theorem 4.1.1(i). Broadly speaking, the proof is nothing more that Rankin's trick, along with the evaluation of some local zeta integrals. However, due to the need to keep track of all constants, and due to the lack of other comprehensive discussions in the literature, we will go into full detail (in particular, we provide the missing details from [DS91]). This chapter provides all definitions, and the main results that appear in the computation. Chapter 6 is solid computation, reducing (i) to a statement about certain p-adic integrals. Chapter 7 contains statements and proofs of a number of small computational results remaining from the previous. Finally, chapter 8 covers the evaluation of the local zeta integrals, with is the only truly new input here.

The reduction of (i) to evaluating local zeta integrals proceeds in roughly four steps: the expression $\langle \omega_0, \xi \rangle$ must be interpreted as an integral of concrete differential forms on a complex manifold, the simplified to an integral of classical modular forms. We translate from classical into adelic language, obtaining an expression involving adelic automorphic forms. Right-invariance of certain terms under diagonal matrices will be exploited to somewhat simplify the expression, and then the Rankin-Selberg method will be applied, obtaining the desired result

5.1 Conventions and notation

First, let us fix notation on subgroups of GL_2. $B = \left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right)$ shall denote the upper triangular Borel, and $U = \left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right)$ its unipotent radical. Write Z for the center of GL_2, $T^1 = \left(\begin{smallmatrix} * & \\ & 1 \end{smallmatrix}\right)$ and $T^2 = \left(\begin{smallmatrix} 1 & \\ & * \end{smallmatrix}\right)$. Given $N > 0$, we write

$$K(N) \subset K_1(N) \subset K_0(N) \subset GL_2(\hat{\mathbb{Z}})$$

for the groups of matrices congruent to $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}*&*\\0&1\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix}*&*\\0&*\end{smallmatrix}\right)$ modulo N, respectively. Then the usual arithmetic subgroups $\Gamma(N)$, $\Gamma_1(N)$, and $\Gamma_0(N)$ are the intersections of the K with $SL_2(\mathbb{Z})$. For $? \in \{\varnothing, 0, 1\}$, write $K_?(N)_p$ for the image of $K_?(N)$ in $GL_2(\mathbb{Z}_p)$; we will often suppress the index p when clear from context. Finally we let K_p, resp. $\prod_{p\leq\infty} K_p$, denote the standard maximal compact subgroups of $GL_2(\mathbb{Q}_p)$, resp. $GL_2(\mathbb{A})$. So $K_p = K(1)_p = GL_2(\mathbb{Z}_p)$ for p finite, and $K_\infty = O_2(\mathbb{R})$.

5.2 Analytic parameterizations of modular curves

First, let us fix conventions regarding the analytic parameterization of the complex points of modular curves. We write $Y(N)$ for the modular curve parameterizing elliptic curves with full level N structure. $Y(N)$ is geometrically connected over $\mathbb{Q}(\zeta_N)$, but will normally be considered as a scheme over $\mathbb{Q}$. The group $GL_2(\mathbb{Z}/N)$ has a natural left action on $Y(N)$, given on the moduli problem by

$$g \cdot (E, \left(\begin{smallmatrix}e_1\\e_2\end{smallmatrix}\right)) = (E, g \cdot \left(\begin{smallmatrix}e_1\\e_2\end{smallmatrix}\right)).$$

Define $Y_1(N) = Y(1, N) = \{\left(\begin{smallmatrix}*&*\\&1\end{smallmatrix}\right)\}\backslash Y(N)$. Then $Y_1(N)$ is geometrically connected over $\mathbb{Q}$, and has complex points

$$\begin{aligned} Y_1(N)(\mathbb{C}) &\cong \Gamma_1(N)\backslash\mathcal{H} \\ &\cong SL_2(\mathbb{Z})\backslash\mathcal{H} \times (SL_2(\mathbb{Z})/\Gamma_1(N)), [(g^{-1}\cdot\tau)] \mapsto [\tau, g] \\ &\cong SL_2(\mathbb{Z})\backslash\mathcal{H} \times GL_2(\mathbb{Z}/N)/\left(\begin{smallmatrix}*&*\\&1\end{smallmatrix}\right), \end{aligned}$$

the last map being induced from the canonical projection $SL_2(\mathbb{Z}) \to GL_2(\mathbb{Z}/N)$.

To uniformize one component of $Y(N)(\mathbb{C})$, take

$$\iota : \mathcal{H} \to Y(N)(\mathbb{C}),\ \tau \mapsto (\mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}, \frac{\tau}{N}, \frac{1}{N}).$$

It is easy to check that if we identify

$$Y(N)(\mathbb{C}) \xrightarrow{\sim} SL_2(\mathbb{Z})\backslash\mathcal{H} \times GL_2(\mathbb{Z}/N)$$

$$g^{-1} \cdot (\iota(\tau)) \leftarrow (\tau, g),$$

then the algebraic map $Y(N) \to Y_1(N) \cong \left(\begin{smallmatrix}*&*\\&1\end{smallmatrix}\right)\backslash Y(N)$ and the projection

$GL_2(\mathbb{Z}/N) \to GL_2(\mathbb{Z}/N)/\left(\begin{smallmatrix} * & * \\ & 1 \end{smallmatrix}\right)$ induce a commutative diagram

$$\begin{array}{ccc} Y(N)(\mathbb{C}) & \xrightarrow{\sim} & SL_2(\mathbb{Z})\backslash\mathcal{H} \times GL_2(\mathbb{Z}/N) \\ \big\downarrow & & \big\downarrow \\ Y_1(N)(\mathbb{C}) & \xrightarrow{\sim} & SL_2(\mathbb{Z})\backslash\mathcal{H} \times (GL_2(\mathbb{Z}/N)/\left(\begin{smallmatrix} * & * \\ & 1 \end{smallmatrix}\right)) \end{array}$$

Let $\widetilde{Y(N)(\mathbb{C})} = \pm U(\mathbb{Z})\backslash\mathcal{H} \times GL_2(\mathbb{Z}/N)$, and $p : \widetilde{Y(N)(\mathbb{C})} \to Y(N)(\mathbb{C})$ be the obvious map. Coordinates on $\mathfrak{X}^n_{\Gamma(N)}(\mathbb{C})$ are given by $(\tau, g, z_1, \ldots, z_n)$, and pulling back along p induces a morphism $p : \widetilde{\mathfrak{X}^n_{\Gamma(N)}}(\mathbb{C}) \to \mathfrak{X}^n_{\Gamma(N)}(\mathbb{C})$.

5.3 Deligne cohomology and realizations of Eisenstein symbols

Definitions and first properties of Deligne cohomology can be found in [DS91], §2. Let X be a variety over $\mathbb{R}$, $a \geq 1$. Then an element ϕ_a of the Deligne cohomology group $H^a_{\mathcal{D}}(X, \mathbb{R}(a))$ is represented by smooth $\mathbb{R}(a-1)$-valued $(a-1)$-forms $[\phi_a]$ on $X(\mathbb{C})$, such that $2d[\phi_a] = \omega_a + (-1)^{a-1}\overline{\omega_a}$ for some holomorphic a-form ω_a. The cup product on these groups is given by

Lemma 5.3.1. *([DS91], p. 180) Let X be a variety over $\mathbb{R}$, and let*

$$\phi_a \in H^a_{\mathcal{D}}(X, \mathbb{R}(a)), \phi_b \in H^b(X, \mathbb{R}(b)).$$

Let ω_a and ω_b be holomorphic forms as above. Then

$$[\phi_a \cup \phi_b] + (-1)^{a-1} d([\phi_a] \wedge [\phi_b]) = \overline{\omega_a} \wedge [\phi_b] + [\phi_a] \wedge \omega_b.$$

More specifically, the construction of ξ involves certain motivic cohomology classes $\mathcal{E}is^{j+k_i}(\phi_i)$, where

$$\phi_i : \left(\begin{smallmatrix} 1 & * \\ & * \end{smallmatrix}\right)\backslash GL_2(\mathbb{Z}/N) \to \mathbb{Q}$$

is defined by

$$\phi_i(g) = \rho^{k_1+j}(e_i)(g^{-1} \cdot i\infty).$$

Explicitly, if $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in GL_2(\mathbb{Z}/N)$,

$$\begin{aligned} \phi_1(g) &= \frac{N^{k_1+j+1}}{(k_1+j+2)} B_{k_1+j+2}\left(\frac{D \det g^{-1}}{N}\right) \\ \phi_2(g) &= \frac{N^{j+k_2+1}}{(j+k_2+2)} B_{j+k_2+2}\left(\frac{-C \det g^{-1}}{N}\right). \end{aligned}$$

Here B_k is the k^{th} Bernoulli polynomial, taking arguments in $[0,1)$ (mod 1).

As in [Bei86], §2.2 or [HK99b], §7, we have the following analytic formulae for $\rho_{\mathcal{D}}\ \mathcal{E}is^{j+k_i}(\phi_i)$: given $\tau \in \mathcal{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{R})$, write $j(\gamma,\tau) = c\tau + d$ for the usual factor of automorphy. Letting S_n denote the symmetric group on n letters, and given nonnegative integers r, s, t, define

$$\eta_s^{r,t} = \sum_{\sigma \in S_t/S_r S_{t-r}} dz_{\sigma(0)} \wedge \cdots \wedge \hat{dz_s} \wedge \cdots \wedge \overline{dz_{\sigma(t)}}$$

where there are r holomorphic differentials, $t-r$ antiholomorphic differentials, and S_t acts on $\{0,\ldots,t\} - \{s\}$. Then the class $\rho_{\mathcal{D}}\mathcal{E}is^{k_i+j}(\phi_i)$ is represented by a smooth k_i+j-form on $\mathfrak{X}^{k_i+j}(\mathbb{C})$ of the form

$$(\tau, g, z_1, \ldots, z_{k_i+j}) \mapsto \frac{(2\pi i)^{k_i+j+1} y}{(k_i+j+1)} \sum_{l=0}^{k_i+j} (l!(k_i+j-l)!)\eta_0^{k_i+j-l,k_i+j}.$$

$$\left(\sum_{\gamma \in \pm U(\mathbb{Z})\backslash SL_2(\mathbb{Z})} \phi_i(\gamma g) \frac{(c\tau+d)^l (c\overline{\tau}+d)^{k_i+j-l}}{|c\tau+d|^{2(k_i+j+1)}} \right) + (d\tau, \overline{d\tau} \text{ terms}).$$

Let $\text{Eis}^{k_i+j}(\phi_i)$ be the weight k_i+j+2 holomorphic Eisenstein series defined by

$$(\tau, g, z_1, \ldots, z_{k_i+j}) \mapsto (2\pi i)^{k_i+j}(k_1+j)! \sum_{\gamma \in \pm U(\mathbb{Z})\backslash SL_2(\mathbb{Z})} \frac{\phi_i(\gamma g)}{j(\gamma,\tau)^{k_i+j+2}} \frac{dq}{q} \wedge dz_1 \wedge \cdots \wedge dz_{k_i+j}.$$

Then Eis^{k_i+j} is the real or imaginary part of $d[\rho_D \mathcal{E}is^{k_i+j}]$.

We also explain the pairings appearing in Theorem 4.1.1. Recall that for any real variety X, the Deligne cohomology groups fit into exact sequences

$$\cdots \to H_B^i(X(\mathbb{C}), \mathbb{R}(j-1))^+ \to H_{\mathcal{D}}^{i+1}(X_{\mathbb{R}}, \mathbb{R}(j)) \to \text{Fil}^j H_{DR}^{i+1}(X_{\mathbb{R}}) \to \cdots$$

As $F^{k+j+2}H^{k+1}(\mathfrak{X}^k(\mathbb{C}),\mathbb{C}) = 0$ and $F^{k+j+2}H^{k+2}(\mathfrak{X}^k(\mathbb{C}),\mathbb{C}) = 0$, the standard exact sequences show that (functorially for π_*, even)

$$H_{\mathcal{D}}^{k+2}(\mathfrak{X}^k/\mathbb{R}, \mathbb{R}(t)) \cong H_B^{k+1}(\mathfrak{X}^k(\mathbb{C}), \mathbb{R}(t-1))^+.$$

There are compatible duality pairings

$$\begin{array}{ccc} H^{k+1}(\mathfrak{X}^k_{\Gamma(N)}(\mathbb{C}),\mathbb{C}) \times H_c^{k+1}(\mathfrak{X}^k_{\Gamma(N)}(\mathbb{C}),\mathbb{C}) & \xrightarrow{\cup} & H_c^{2k+2}(\mathfrak{X}^k_{\Gamma(N)}(\mathbb{C}),\mathbb{C}) \\ \Big\downarrow {\scriptstyle \beta \times \mathrm{Id}} & & \Big\downarrow {\scriptstyle \mathrm{Tr} = \frac{1}{(2\pi i)^{k+1}} \int_{\mathfrak{X}^k_{\Gamma(N)}(\mathbb{C})}} \\ H_{k+1}^{\mathrm{BM}}(\mathfrak{X}^k_{\Gamma(N)}(\mathbb{C}),\mathbb{C}) \times H_c^{k+1}(\mathfrak{X}^k_{\Gamma(N)}(\mathbb{C}),\mathbb{C}) & \xrightarrow{\int} & \mathbb{C} \end{array}$$

Let f be a cusp form of weight $k+2$. The form

$$\omega_0 = f(\tau, g)\frac{dq}{q} \wedge dz_1 \wedge \cdots \wedge dz_k$$

is holomorphic on $KS^k_{\Gamma(N)}(\mathbb{C})$ of top holomorphic degree, and thus represents classes

$$\omega_0 \in H^{k+1}(KS^k_{\Gamma(N)}(\mathbb{C}), \mathbb{C}), \qquad \omega_{0,c} \in H^{k+1}_c(\mathfrak{X}^k_{\Gamma(N)}(\mathbb{C}), \mathbb{C}).$$

On the other hand, the inclusion

$$i^* : H^{k+1}(KS^k_{\Gamma(N)}(\mathbb{C}), \mathbb{C}) \to H^{k+1}(\mathfrak{X}^k_{\Gamma(N)}(\mathbb{C}), \mathbb{C})$$

has a splitting pr with kernel $F^{k+1} \cap \overline{F}^{k+1}$, where $F^{k+1} = H^0(KS^k_{\Gamma(N)}(\mathbb{C}), \Omega^{k+1}\langle KS^{k,\infty}\rangle)(\epsilon)$. As ω_0 is holomorphic of top degree, $\langle \eta, \omega_{0,c}\rangle = 0$ for all $\eta \in \ker(\mathrm{pr})$, and so

$$\langle \Delta, \omega_{0,c}\rangle = \langle i^*\mathrm{pr}\, \Delta, \omega_{0,c}\rangle = \langle \mathrm{pr}\, \Delta, \omega_0\rangle.$$

I.e., applying pr eliminates any concerns about convergence of the integrals. For the pairing $\langle \rho_{\mathcal{D}}(\xi), \omega_{0,c}\rangle$, we regard both $\rho_{\mathcal{D}}(\xi)$ and $\omega_{0,c}$ as smooth $(k+1)$-forms, and the cup product is a wedge product. As $\omega_{0,c}$ has exponential decay and $\rho_{\mathcal{D}}(\xi)$ only moderate growth, again, there is no difficulty in convergence of the integral.

5.4 Some representation theory of GL_1 and GL_2

For readers who are not so familiar with the theory of automorphic forms, we will briefly review some representation theory of GL_1 and GL_2. A reference is [Bu98].

First, given any finite order Dirichlet character χ, we may associate a finite order Hecke character $\chi : \mathbb{Q}^*\backslash\mathbb{A}^* \to \mathbb{C}^*$, and vice versa; we will not distinguish notationally. For any Hecke character $\chi : \mathbb{Q}^*\backslash\mathbb{A}^* \to \mathbb{C}^*$, write $\chi = \prod_p \chi_p$. To each χ_p one can attach a local L-factor. If χ_p is ramified, $L_p(\chi_p, s) = 1$, and if χ_p is unramified, $L_p(\chi_p, s) = (1 - \chi_p(p)p^{-s})^{-1}$. If $\chi_\infty(-1) = (-1)^m$, $m = 0, 1$, then $L_\infty(\chi_\infty, s) = \Gamma_{\mathbb{R}}(s+m) = \pi^{-\frac{s+m}{2}}\Gamma(\frac{s+m}{2})$. One has the global L-function $L(\chi, s) = \prod_{p<\infty} L_p(\chi_p, s)$ and the completed global L-function $\Lambda(\chi, s) = \prod_{p\leq\infty} L_p(\chi_p, s)$. These have analytic continuation to all of $\mathbb{C}$ (with a pole as $s+1$ if $\chi = |\cdot|^s$), and one has the functional equation

$$\Lambda(\chi, s) = \epsilon(s, \chi)\Lambda(\chi^{-1}, 1-s).$$

Writing

$$G(s) = \frac{\Gamma(s)}{(2\pi)^s},\ \ G^*(n) = \frac{\Gamma^*(n)}{(2\pi)^n},$$

we may rewrite the functional equation as

$$L(\chi, s) = \frac{1}{2G(n)\cos(\pi(\frac{s-m}{2}))}\epsilon(s,\chi)L(\chi^{-1}, 1-s).$$

Let $\psi = \prod_p \psi_p : \mathbb{A}/\mathbb{Q} \to \mathbb{C}$ be the standard unramified additive character

$$\psi_\infty(x_\infty) = e^{2\pi i x_\infty},\ \psi_p(x_p) = e^{-2\pi i x_p}.$$

Then $\epsilon(s,\chi)$ is a product of local ϵ-factors

$$\epsilon(s,\chi) = \prod_{p\leq\infty} \epsilon_p(s,\chi_p,\psi_p).$$

If p is finite and χ_p is unramified, $\epsilon_p(s,\chi_p,\psi_p) = 1$, while if χ_p is ramified of conductor $n > 0$,

$$\epsilon_p(s,\chi_p,\psi_p) = \int\limits_{p^{-n}\mathbb{Z}_p^*} |x|^{-s}\chi_p^{-1}(x)\psi_p(x)dx.$$

Meanwhile, for $\chi_\infty(-1) = (-1)^l$, (see [JL70], above Theorem 15)

$$\epsilon_\infty(s,\chi_\infty,\psi_\infty) = (i)^{l^2} = \begin{cases} 1 & l \text{ even} \\ i & l \text{ odd} \end{cases}$$

Now let f be a classical cuspidal modular form. Via the standard method, which will be reviewed more thoroughly in the next section, f may be viewed as a smooth vector in $C^\infty(GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}))$. Let ϖ be the automorphic representation spanned by the right translates of f. If f is a new eigenform, it is well known that ϖ is an irreducible automorphic representation, and that $\varpi = \otimes'\varpi_p$ is a restricted tensor product of irreducible pre-unitary local representations.

To each local representation ϖ_p, one can attach a local L-factor $L_p(\varpi_p, s)$ (the exact definition need not concern us now). Given an irreducible automorphic representation ϖ of $GL_2(\mathbb{Q})$ (the only field we consider here), one again has a global L-function $L(\varpi, s) = \prod_{p<\infty} L_p(\varpi_p, s)$ and a completed global L-function $\Lambda(\varpi, s) = \prod_{p\leq\infty} L_p(\varpi_p, s)$. These converge for $s >> 0$ and have analytic continuation, with a functional equation

$$\Lambda(\varpi, s) = \epsilon(s,\varpi)\Lambda(\hat{\varpi}, 1-s) = \left(\prod_p \epsilon(s,\varpi_p,\psi_p)\right)\Lambda(\hat{\varpi}, 1-s).$$

Here $\hat{\varpi}$ is the dual representation to ϖ; if ϖ is generated by f, $\hat{\varpi}$ is generated by $\check{f}$. If f is a new eigen cusp form of weight l, $L(f, s + \frac{l-1}{2}) = L(\varpi, s)$, as we normalize automorphic L-functions to have center of symmetry $\frac{1}{2}$, while motivic L-functions are normalized according to the weights of the Chow motives from which they arise. Again, $L_\infty(f, s) = \Gamma_{\mathbb{C}}(s) = \frac{2\Gamma(s)}{(2\pi)^s}$, so that

$$L(f, s) = \frac{G(l-s)}{G(s)}\epsilon(s - \frac{l-1}{2})L(\check{f}, l-s).$$

Also,

$$\epsilon_\infty(s, \varpi_\infty, \psi_\infty) = i^l.$$

We will not give precise definitions of the finite epsilon factors at the moment.

We shall also need to fix conventions for the Haar measures on $GL_2(\mathbb{R})$ and $GL_2(\mathbb{A})$. As in [Bo97], §2.9, consider $K_\infty \subset SL_2(\mathbb{R})$, and let dk be the Haar measure on K_∞ with total volume 1. Then $SL_2(\mathbb{R})/K_\infty \cong \mathcal{H}$, and a Haar measure on $SL_2(\mathbb{R})$ is $y^{-2}(dx \wedge dy)dk$. As $GL_2(\mathbb{R})$ is a topological direct product $Z^0(\mathbb{R})SL_2(\mathbb{R})$, it has Haar measure $dg_{\mathbb{R},Haar} = y^{-2}dz(dx \wedge dy)dk$. We equip each $GL_2(\mathbb{Q}_p)$ with the Haar measure $dg_{p,Haar}$, for which $GL_2(\mathbb{Z}_p)$ has volume 1, and then $dg = dg_{\mathbb{R},Haar} \otimes \bigotimes dg_{p,Haar}$ is our Haar measure on $GL_2(\mathbb{A})$.

For parameters z, $\bar{z}$, x, y on $\mathbb{C}$, $dz \wedge \overline{dz} = -2idx \wedge dy$, and with $q = e^{2\pi iz}$,

$$\frac{dq}{q} \wedge \overline{\frac{dq}{q}} = 4\pi^2 dz \wedge \overline{dz} = -8\pi^2 idx \wedge dy.$$

5.5 From classical to adelic automorphic forms

We can mostly follow [Bu98], although the notation is slightly different because we work on the full $Y(N)$. Let h be a modular form of weight l for $Y(N)$. Then $h : \mathcal{H} \times GL_2(\mathbb{Z}/N) \to \mathbb{C}$ is a function satisfying, for any $\gamma \in SL_2(\mathbb{Z})$,

$$h(\gamma \cdot \tau, \gamma g) = j(\gamma, \tau)^l h(\tau, g).$$

Then $y^{\frac{l}{2}} h(\tau, g)$ is a Maass form, in the sense of [Bu98], §3.2. Define

$$F_h : SL_2(\mathbb{Z})\backslash GL_2^+(\mathbb{R}) \times GL_2(\mathbb{Z}/N) \to \mathbb{C}$$

by (here $g_\infty = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$)

$$F_h(g_\infty, g) = h(g_\infty \cdot i, g)\mathrm{Im}(g_\infty \cdot i)^{\frac{l}{2}} \left(\frac{-ci + d}{|ci + d|}\right)^l := h(g_\infty \cdot i, g)\mathrm{Im}(g_\infty \cdot i)^{\frac{l}{2}} \vartheta(g_\infty)^l.$$

It is easy to check from this formula that F is indeed invariant under left (diagonal) multiplication by $SL_2(\mathbb{Z})$. Finally, given any $g \in GL_2(\mathbb{A})$, one may write $g = \gamma g_\infty \kappa$ with $\gamma \in GL_2(\mathbb{Q})$, $g_\infty \in GL_2^+(\mathbb{R})$, and $\kappa \in GL_2(\hat{\mathbb{Z}})$. Let $\bar{\kappa}$ be the reduction of κ in $GL_2(\mathbb{Z}/N)$. Then the adelic modular form attached to h is $\varphi(g) = \varphi_h(g) : GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}) \to \mathbb{C}$,

$$\varphi(g) = F_h(g_\infty, \bar{\kappa}).$$

If one chooses a different decomposition $g = \gamma' g'_\infty \kappa'$, then $\gamma'\gamma^{-1} \in SL_2(\mathbb{Z})$, and so φ is well-defined. As usual, the association $f \rightarrowtail \varphi_h$ is equivariant for the respective Hecke algebra actions. Note that, to be slightly more explicit, suppose we were to write $g \in GL_2(\mathbb{A})$ as $(g_\mathbb{R}, g_f)$. Then set $c = \left(\begin{smallmatrix} 1 & \\ & \operatorname{sgn}(\det g_\mathbb{R}) \end{smallmatrix}\right)$. There is a unique $b = \left(\begin{smallmatrix} b_1 & * \\ & b_2 \end{smallmatrix}\right) \in U(\mathbb{Z})\backslash B(\mathbb{Q})$ with $b_i > 0$ such that

$$g_\infty = cb^{-1}g_\mathbb{R} \in GL_2^+(\mathbb{R}), \text{ and } \kappa = cb^{-1}g_f \in GL_2(\hat{\mathbb{Z}}).$$

Also, notice that

$$\begin{aligned} &\phi_2(\kappa)\mathrm{Im}(g_\infty \cdot i)^{\frac{j+k_2+2}{2}} \vartheta(g_\infty)^{j-k_2} = \\ &\phi_2(b^{-1}g_f) \left(\frac{b_2}{b_1}\right)^{\frac{j+k_2+2}{2}} \mathrm{Im}(cg_\mathbb{R} \cdot i)^{\frac{j+k_2+2}{2}} \vartheta(cg_\mathbb{R})^{j-k_2} \\ &:= \widetilde{\phi_2}(g_f)\phi_{2,\mathbb{R}}(g_\mathbb{R}). \end{aligned}$$

5.6 Whittaker functions

Each irreducible admissible local representation ϖ_p has a conductor n_p; this is the minimum non-negative integer such that ϖ_p possesses a vector fixed by

$$K_1(p^{n_p}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{Z}_p) : c, d-1 \in p^{n_p}\mathbb{Z}_p\}.$$

According to Casselman [Ca73], $\dim \varpi_p^{K_1(p^{n_p})} = 1$, and for $n \geq n_p$, $\dim \varpi_p^{K_1(p^n)} = n - n_p + 1$. A global irreducible automorphic representation ϖ has $n_p = 0$ for almost all p, and the conductor of ϖ is $\prod_p p^{n_p}$. If ϖ is generated by f, the level of f and the conductor of ϖ coincide [Car86].

We also review some facts about Whittaker functions (at least for finite primes; the situation at the infinite prime is somewhat more technical, as in [JL70]). Recall that ψ_p is the standard unramified character of $\mathbb{Q}_p$. The space of local Whittaker functions is

$$\mathcal{W}_{\psi_p} = \{f : GL_2(\mathbb{Q}_p) \to \mathbb{C} : f(\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)g) = \psi_p(x)f(g)\}.$$

Since all representations of $GL_2(\mathbb{Q}_p)$ are generic, each ϖ_p occurs in $\mathcal{W}_{\psi_p}$ as a representation of $GL_2(\mathbb{Q}_p)$. By multiplicity 1, each ϖ_p occurs exactly once. The unique element

$$W_{\varpi_p} \in \varpi_p^{K_1(p^{n_p})} \subset \mathcal{W}_{\psi_p},\ W_{\varpi_p}(\left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right)) = 1$$

is called the newvector in the space of local Whittaker functions. When ϖ_p is spherical, that is, for $n_p = 0$,

$$\int\limits_{\mathbb{Q}_p^*} W_{\varpi_p}(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right))|a|^{s-\frac{1}{2}} d^*a = L_p(\varpi_p, s).$$

For a global irreducible automorphic representation ϖ, there is a global Whittaker model $\mathcal{W}_\psi = \otimes' \mathcal{W}_{\psi_p}$, and ϖ occurs exactly once in $\mathcal{W}_\psi$. There is a newvector $W_\varpi = \otimes' W_{\varpi_p} \in \varpi \subset \mathcal{W}_\psi$. When ϖ is generated by f, we will write W_f and $W_{f,p}$ in place of W_ϖ and W_{ϖ_p}; W_f is (in an appropriate sense) the Fourier expansion of f. Outside of the case of supercuspidal representations at $p = 2$, formulae for newforms in models of the ϖ_p are well-known; a concise compilation of results can be found in [Sc02]. This can be taken as a standard reference for later chapters, where such formulae will be used heavily.

Chapter 6

The Rankin-Selberg Method

Let ω_0 be the $k+1$-form on $\mathfrak{X}^k$ representing a cusp form (for now, we need not assume any more). In local coordinates,

$$\omega_0 = f(\tau, g)\frac{dq}{q} \wedge dz_1 \wedge \cdots \wedge dz_k$$

and f has rapid decay at the cusps; moreover, $\epsilon \circ \omega_0 = \omega_0$. Then

$$\begin{aligned}
&\langle \omega_0, \epsilon \circ \pi_*(\pi_1^* \rho_{\mathcal{D}} \mathcal{E}is^{k_1+j}(\phi_1) \cup \pi_2^* \rho_{\mathcal{D}} \mathcal{E}is^{j+k_2}(\phi_2))\rangle \\
&= \langle \omega_0, \pi_*(\pi_1^* \rho_{\mathcal{D}} \mathcal{E}is^{k_1+j}(\phi_1) \cup \pi_2^* \rho_{\mathcal{D}} \mathcal{E}is^{j+k_2}(\phi_2))\rangle \\
&= \langle \pi^*\omega_0, \pi_1^* \rho_{\mathcal{D}} \mathcal{E}is^{k_1+j}(\phi_1) \cup \pi_2^* \rho_{\mathcal{D}} \mathcal{E}is^{j+k_2}(\phi_2)\rangle
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(2\pi i)^{j+k+1}} \int_{\mathfrak{X}^{j+k}_{\Gamma(N)}(\mathbb{C})} \pi^*\omega_0 \wedge [\pi_1^* \rho_{\mathcal{D}}\ \mathcal{E}is^{k_1+j}(\phi_1) \cup \pi_2^* \rho_{\mathcal{D}}\ \mathcal{E}is^{j+k_2}(\phi_2)] \\
&= \frac{1}{(2\pi i)^{j+k+1}} \int_{\mathfrak{X}^{j+k}_{\Gamma(N)}(\mathbb{C})} \pi^*\omega_0 \wedge (\overline{\pi_1^* \mathrm{Eis}^{k_1+j}(\phi_1)} \wedge [\pi_2^* \rho_{\mathcal{D}}\ \mathcal{E}is^{j+k_2}(\phi_2)] \\
&\qquad\qquad + [\pi_1^* \rho_{\mathcal{D}}\ \mathcal{E}is^{k_1+j}(\phi_1)] \wedge \pi_2^* \mathrm{Eis}^{j+k_2}(\phi_2)) \\
&= \frac{1}{(2\pi i)^{j+k+1}} \int_{\mathfrak{X}^{j+k}_{\Gamma(N)}(\mathbb{C})} \pi^*\omega_0 \wedge \overline{\pi_1^* \mathrm{Eis}^{k_1+j}(\phi_1)} \wedge [\pi_2^* \rho_{\mathcal{D}}\ \mathcal{E}is^{j+k_2}(\phi_2)] \\
&= \frac{(2\pi i)^{-k_1} k_2! j!}{N(j+k_2+1)!} \int_{\mathfrak{X}^{j+k}_{\Gamma(N)}(\mathbb{C})} \pi^*\omega_0 \wedge \overline{\pi_1^* \mathrm{Eis}^{k_1+j}(\phi_1)} \wedge p_*(y\phi_2(g) \\
&\qquad\qquad \times dz_{k_1+1} \wedge \cdots \wedge dz_{k_1+j} \wedge \overline{dz_{k_1+j+1}} \wedge \cdots \wedge \overline{dz_{k+j}}) \\
&= \frac{(2\pi i)^{-k_1} k_2! j!}{N(j+k_2+1)!} \int_{\widetilde{\mathfrak{X}^{j+k}_{\Gamma(N)}}(\mathbb{C})} p^*\pi^*\omega_0 \wedge \overline{p^*\pi_1^* \mathrm{Eis}^{k_1+j}(\phi_1)} \wedge (y\phi_2(g) \\
&\qquad\qquad \times dz_{k_1+1} \wedge \cdots \wedge dz_{k_1+j} \wedge \overline{dz_{k_1+j+1}} \wedge \cdots \wedge \overline{dz_{k+j}}
\end{aligned}$$

(rearranging the differentials)

$$= (-1)^{\frac{k^2+k+j^2+j}{2}} \frac{2^j \pi^j i^{-j} k_2! j!}{N^2 (j+k_2+1)!} \int\limits_{\widetilde{\mathfrak{X}^{j+k}_{\Gamma(N)}(\mathbb{C})}} f(\tau, g) \overline{\left(\sum_{\gamma} \frac{\phi_1(\gamma g)}{j(\gamma,\tau)^{k_1+j+2}} \right)} y \phi_2(g)$$

$$\times \frac{dq}{q} \wedge \overline{\frac{dq}{q}} \wedge dz_1 \wedge \overline{dz_1} \wedge \cdots \wedge dz_{k_1+j+k_2} \wedge \overline{dz_{k_1+j+k_2}}$$

$$= \frac{2^{j+k} i^{k^2+2k+j^2+j+1} G(k_2+1)(j)!}{(2\pi i) N^2 G(j+k_2+2)} \int\limits_{\widetilde{Y(N)(\mathbb{C})}} f(\tau, g) \overline{\left(\sum_{\gamma} \frac{\phi_1(\gamma g)}{(c\tau+d)^{k_1+j+2}} \right)} y^{j+k+1} \phi_2(g) \frac{dq}{q} \wedge \overline{\frac{dq}{q}}.$$

Set

$$C_1 = \frac{2^{j+k+2} \pi i^{k^2+2k+j^2+j-1} G(k_2+1)(j)!}{G(j+k_2+2)}.$$

We may rewrite the integral as

$$\frac{C_1}{N^2} \int\limits_{\pm U(\mathbb{Z}) \backslash \mathcal{H} \times GL_2(\mathbb{Z}/N)} f(\tau, g) \overline{\left(\sum_{\gamma} \frac{\phi_1(\gamma g)}{(c\tau+d)^{k_1+j+2}} \right)} y^{j+k+1} \phi_2(g) dx \wedge dy$$

$$= \frac{C_1}{N^2} \int\limits_{\pm U(\mathbb{Z}) \backslash SL_2(\mathbb{R}) \times GL_2(\hat{\mathbb{Z}}) / K_\infty^+ K(N)} f(g_\infty \cdot i, \kappa) \overline{E_{\phi_1}(g_\infty \cdot i, \kappa)} \phi_2(\kappa) y^{j+k+3} y^{-2} dx \wedge dy$$

$$= \frac{C_1}{N^2} \int\limits_{\pm U(\mathbb{Z}) Z(\mathbb{R}) \backslash GL_2(\mathbb{R}) \times GL_2(\hat{\mathbb{Z}}) / K_\infty K(N)} F_f(g_\infty, \kappa) \overline{F_E(g_\infty, \kappa)} \phi_2(\kappa) \mathrm{Im}(g_\infty \cdot i)^{\frac{j+k_2+2}{2}} \vartheta(g_\infty)^{j-k_2} dg_\infty$$

$$= \frac{C_1}{N^2} \int\limits_{B(\mathbb{Q}) Z(\mathbb{R}) \backslash GL_2(\mathbb{A}) / K_\infty K(N)} \varphi_f(g) \overline{\varphi_E(g)} \widetilde{\phi_2}(g_f) \phi_{2,\mathbb{R}}(g_\mathbb{R}) dg_\mathbb{R}$$

$$= C_1 \frac{[GL_2(\hat{\mathbb{Z}}) : K(N)]}{N^2} \int\limits_{B(\mathbb{Q}) Z(\mathbb{R}) \backslash GL_2(\mathbb{A})} \varphi_f(g) \overline{\varphi_E(g)} \widetilde{\phi_2}(g_f) \phi_{2,\mathbb{R}}(g_\mathbb{R}) dg.$$

Up to this point, we have interpreted the original pairing in Betti cohomology in terms of an integral of classical automorphic forms, and have translated into adelic language. Notice that $\varphi_f(g)$ and $\widetilde{\phi_2}(g_f)$ are right-invariant by $T^1(\hat{\mathbb{Z}})$, while $\varphi_E(g)$ is right-invariant under $T^2(\hat{\mathbb{Z}})$. Let us assume that φ_f comes from $\Gamma_1(N)$ and generates a representation with central character ω. The above is equal to

$$
\begin{aligned}
& C_1 \frac{[GL_2(\hat{\mathbb{Z}}):K(N)]}{N^2} \int\limits_{B(\mathbb{Q})Z(\mathbb{R})\backslash GL_2(\mathbb{A})} \varphi_f(g) \left(\overline{\int\limits_{T^1(\hat{\mathbb{Z}})} \varphi_E(gt)dt} \right) \widetilde{\phi_2}(g_f)\phi_{2,\mathbb{R}}(g_{\mathbb{R}})dg \\
&= C_1 \frac{[GL_2(\hat{\mathbb{Z}}):K(N)]}{N^2} \int\limits_{B(\mathbb{Q})Z(\mathbb{R})\backslash GL_2(\mathbb{A})} \varphi_f(g) \left(\overline{\int\limits_{Z(\hat{\mathbb{Z}})} \varphi_E(tg)dt} \right) \widetilde{\phi_2}(g_f)\phi_{2,\mathbb{R}}(g_{\mathbb{R}})dg \\
&= C_1 \frac{[GL_2(\hat{\mathbb{Z}}):K(N)]}{N^2} \int\limits_{B(\mathbb{Q})Z(\mathbb{R})\backslash GL_2(\mathbb{A})} \varphi_f(g) \left(\overline{\int\limits_{Z(\hat{\mathbb{Z}})} \varphi_E(tg)dt} \right) \\
&\qquad \times \left(\int\limits_{Z(\hat{\mathbb{Z}})} \widetilde{\phi}_2(tg_f)\omega(t)dt \right) \phi_{2,\mathbb{R}}(g_{\mathbb{R}})dg \\
&:= C_1 \frac{[GL_2(\hat{\mathbb{Z}}):K(N)]}{N^2} \int\limits_{B(\mathbb{Q})Z(\mathbb{R})\backslash GL_2(\mathbb{A})} \varphi_f(g)\overline{\varphi_{E,1}(g)}\widetilde{\phi}_{2,\omega^{-1}}(g_f)\phi_{2,\mathbb{R}}(g_{\mathbb{R}})dg \\
&= C_1 \frac{[GL_2(\hat{\mathbb{Z}}):K(N)]}{N^2} \int\limits_{B(\mathbb{Q})Z(\mathbb{A})\backslash GL_2(\mathbb{A})} \varphi_f(g)\overline{\varphi_{E,1}(g)}\widetilde{\phi}_{2,\omega^{-1}}(g_f)\phi_{2,\mathbb{R}}(g_{\mathbb{R}})dg.
\end{aligned}
$$

This is exactly the sort of expression to which the automorphic formulation of Rankin's trick applies. As in §3.8 of [Bu98], the above is equal to

$$
C_1 \frac{[GL_2(\hat{\mathbb{Z}}):K(N)]}{N^2} \int\limits_{\mathbb{A}^* \times K} W_f(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)\kappa)\overline{W_{E,1}(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)\kappa)}(\widetilde{\phi}_{2,\omega^{-1}}\phi_{2,\mathbb{R}})(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)\kappa)|a|^{-1}d\kappa d^*a.
$$

6.1 Pure tensor decomposition

Finally, we assume that f is a normalized new eigenform on $\Gamma_1(N)$, with nebentypus ω; recall the parity of ω is equal to the parity of the weight $k+2$. Then the Whittaker function W_f is a product of local Whittaker functions $\prod_{p\leq\infty} W_{f,p}$. In the following section, we will show

Lemma 6.1.1. *(i) There exist local functions $\widetilde{\phi}_{2,\omega^{-1},p}$ such that*

$$
\widetilde{\phi}_{2,\omega^{-1}}(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)\kappa) = \frac{-L(\omega,-j-k_2-1)\omega(-1)}{N} \prod_{p<\infty} \widetilde{\phi}_{2,\omega^{-1},p}(\left(\begin{smallmatrix} a_p & \\ & 1 \end{smallmatrix}\right)\kappa_p).
$$

We also write $\widetilde{\phi}_{2,\omega^{-1},\infty} = \phi_{2,\mathbb{R}}$.

(ii) There exist local Whittaker functions $W_{E,1,p}$ such that

$$
W_{E,1}(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)\kappa) = -\frac{\zeta(-k_1-j-1)}{N} \prod_{p\leq\infty} W_{E,1,p}(\left(\begin{smallmatrix} a_p & \\ & 1 \end{smallmatrix}\right)\kappa_p).
$$

Precise formulae for these local factors will be given in the proof.

Thus

$$\begin{aligned}&\langle \omega_0, \epsilon\circ\pi_*(\pi_1^*\rho_{\mathcal{D}}\mathcal{E}is^{k_1+j}(\phi_1)\cup\pi_2^*\rho_{\mathcal{D}}\mathcal{E}is^{j+k_2}(\phi_2))\rangle\\
=&\, C_1\frac{[GL_2(\hat{\mathbb{Z}}):K(N)]}{N^4}\omega(-1)L(\omega,-j-k_2-1)\zeta(-k_1-j-1)\\
&\times\int_{\mathbb{A}^*\times K}\prod_{p\le\infty} W_{f,p}((\begin{smallmatrix}a_p&\\&1\end{smallmatrix})\kappa_p)\overline{W_{E,1,p}((\begin{smallmatrix}a_p&\\&1\end{smallmatrix})\kappa_p)}\widetilde{\phi}_{2,\omega^{-1},p}((\begin{smallmatrix}a_p&\\&1\end{smallmatrix})\kappa_p)|a_p|^{-1}d\kappa.\end{aligned}$$

We cannot quite exchange the order of product and integration, because the product of the local integrals may not converge. However, the usual trick of analytic continuation applies. Namely, write

$$F(s)=\int_{\mathbb{A}^*\times K}\prod_{p\le\infty} W_{f,p}((\begin{smallmatrix}a_p&\\&1\end{smallmatrix})\kappa_p)\overline{W_{E,1,p}((\begin{smallmatrix}a_p&\\&1\end{smallmatrix})\kappa_p)}\widetilde{\phi}_{2,\omega^{-1},p}((\begin{smallmatrix}a_p&\\&1\end{smallmatrix})\kappa_p)|a_p|^{s-1}d\kappa d^*a.$$

For $s>>0$, we are justified in writing

$$F(s)=\prod_{p\le\infty}\int_{\mathbb{Q}_p^*\times K_p} W_{f,p}((\begin{smallmatrix}a_p&\\&1\end{smallmatrix})\kappa_p)\overline{W_{E,1,p}((\begin{smallmatrix}a_p&\\&1\end{smallmatrix})\kappa_p)}\widetilde{\phi}_{2,\omega^{-1},p}((\begin{smallmatrix}a_p&\\&1\end{smallmatrix})\kappa_p)|a_p|^{s-1}d\kappa_p d^*a_p$$

$$:=\prod_{p\le\infty}\Psi_p(f,s).$$

In chapter 8 we shall prove the following proposition

Proposition 6.1.1. *(i) If p is finite, $p\nmid N$,*

$$\Psi_p(f,s)=\frac{L_p(\check{f},1+k_2+s)L_p(\check{f},j+k+2+s)}{L_p(\omega^{-1},j+k_2+2+2s)\zeta_p(k_1+j+2)}.$$

(ii) Suppose $p\mid N$, and that if $p=2$ then the local representation ϖ_p is not supercuspidal. Then $\Psi_p(f,s)$ is a rational function of p^{-s}, and

$$\left(\frac{p-1}{p}\right)^2\frac{p+1}{p}\Psi_p(f,0)=\frac{L_p(\check{f},1+k_2)L_p(\check{f},j+k+2)}{L_p(\omega^{-1},j+k_2+2)\zeta_p(k_1+j+2)}$$

$$\times\frac{\epsilon_p(-j-\frac{k}{2}-\frac{1}{2},\varpi_p,\psi_p)\epsilon_p(\frac{k_1-k_2+1}{2},\varpi_p,\psi_p)}{\epsilon_p(-j-k_2-1,\omega_p,\overline{\psi_p})}\frac{1}{L_p(f,-j)}.$$

(iii) The factor at infinity is

$$\Psi_\infty(f,0)=\frac{i^{k_1+j+2}G(j+k+2)}{2^{j+k+2}G(k_1+j+2)}.$$

Granting this, it is clear that $F(s)$ has analytic continuation to all of $\mathbb{C}$, and that

$$\frac{[GL_2(\hat{\mathbb{Z}}):K(N)]}{N^4}F(0)=\frac{G(j+k+2)}{2^{j+k+2}G(k_1+j+2)}\frac{L(\check{f},1+k_2)L(\check{f},j+k+2)}{L(\omega^{-1},j+k_2+2)\zeta(k_1+j+2)}$$

$$\times\frac{i^{k^2+k_1+j+2}}{i^{2k}}\frac{\epsilon(-j-\frac{k}{2}-\frac{1}{2},\varpi)\epsilon(\frac{k_1-k_2+1}{2},\varpi)}{\omega(-1)\epsilon(-j-k_2-1,\omega)}\prod_{p|N}L_p(f,-j)^{-1}.$$

Recalling that

$$C_1=\frac{2^{j+k+2}\pi i^{k^2+2k+j^2+j-1}G(k_2+1)(j)!}{G(j+k_2+2)},$$

the value to be computed is

$$\begin{aligned}
&\frac{\pi j!G(k_2+1)G(j+k+2)}{N^2G(j+k_2+2)G(k_1+j+2)}\omega(-1)L(\omega,-j-k_2-1)\zeta(-k_1-j-1)i^{2k^2+j^2+2j+k_1+1}\\
\times&\frac{L(\check{f},1+k_2)L(\check{f},j+k+2)}{L(\omega^{-1},j+k_2+2)\zeta(k_1+j+2)}\frac{\epsilon(-j-\frac{k}{2}-\frac{1}{2},\varpi)\epsilon(\frac{k_1-k_2+1}{2},\varpi)}{\epsilon(-j-k_2-1,\omega)}\prod_{p|N}L_p(f,-j)^{-1}\\
=&(-1)^{k^2+\frac{j^2+j}{2}}(i)^{k_1+j+1}\pi j!G(k_1+1)G^*(-j)L(f,k_1+1)L(f,-j)\prod_{p|N}L_p(f,-j)^{-1}\\
=&(-1)^{k^2+\frac{j^2+j}{2}}i^{k_1+1}\pi(2\pi i)^jG(k_1+1)L(f,k_1+1)L^{(N),*}(f,-j).
\end{aligned}$$

Finally,

$$\begin{aligned}
\langle\bar{\delta}_{k_1},f\rangle=&\langle(2\pi i)^j\Delta,\omega_0\rangle=(2\pi i)^j\int_{\Delta}\omega_0=(2\pi i)^j\int_0^{\infty}f(iy)(iy)^{k_1}(2\pi i)dy\\
=&(2\pi i)^{j+1}i^{k_1}G(k_1+1)L(f,k_1+1),
\end{aligned}$$

and thus the ratio $\frac{\langle\rho_{\mathcal{D},M^*(1)}(\bar{\xi}),f\rangle_{M^*(1)}}{\langle\bar{\delta},f\rangle_{M^*(1)}}$ is given by

$$\frac{(-1)^{k^2+\frac{j^2+j}{2}}L^{(N),*}(f,-j)}{2},$$

, which is precisely the statement of Theorem 4.1.1(i).

Chapter 7

Lemmas on Eisenstein Series

The purpose of this chapter is to prove Lemma 6.1.1, and to give more explicit formulae for the $\Psi_p(f, s)$. We factor $N = \prod p^{n_p}$, and n_ω will denote the conductor of the Dirichlet character ω. We will sometimes abuse notation and write $\omega = \omega \circ \det : GL_2(\mathbb{Z}/N) \to \mathbb{C}$.

7.1 Bernoulli numbers

We wish to give an explicit formula for $\phi_{2,\omega^{-1}}$. Let

$$\kappa = ((\begin{smallmatrix} a_p & b_p \\ c_p & d_p \end{smallmatrix})_p) \in GL_2(\hat{\mathbb{Z}})$$

with reduction $\bar{\kappa} \in GL_2(\mathbb{Z}/N)$. With the level N fixed, let $\gamma = \prod_{p|N} \max(|c|_p, |N|_p)$, and write $c' = \gamma c$, $N' = \gamma N$. Recall also that

$$\mathbb{A}_f/\hat{\mathbb{Z}} \cong \mathbb{Q}/\mathbb{Z} \to [0, 1).$$

In this we, we view the Bernoulli polynomials as functions on $\mathbb{A}_f$. By definition,

$$\begin{aligned} \phi_{2,\omega^{-1}}(\kappa) &= \int_{\hat{\mathbb{Z}}^*} \phi_2(\bar{\kappa}(\begin{smallmatrix} 1 & \\ & \bar{x} \end{smallmatrix}))\omega(x)d^*x \\ &= \frac{N^{j+k_2+1}}{j+k_2+2} \int_{\hat{\mathbb{Z}}^*} B_{j+k_2+2}(\frac{-cx^{-1} \det \kappa^{-1}}{N})\omega(x)d^*x \\ &= \frac{N^{j+k_2+1}}{j+k_2+2} \int_{\hat{\mathbb{Z}}^*} B_{j+k_2+2}(\frac{-c'x^{-1} \det \kappa^{-1}}{N'})\omega(x)d^*x \end{aligned}$$

$$= \frac{N^{j+k_2+1}}{j+k_2+2}\mathbb{I}_{\mathrm{cond}(\omega)|N'}\left(\prod_{p|N'}\frac{p}{p-1}\right)\int\limits_{\prod_{p|N'}\mathbb{Z}_p^*} B_{j+k_2+2}(\frac{-c'x\det\kappa^{-1}}{N'})\omega^{-1}(x)dx$$

$$= \frac{N^{j+k_2+1}}{j+k_2+2}\mathbb{I}_{\mathrm{cond}(\omega)|N'}\left(\prod_{p|N'}\frac{p}{p-1}\omega_p(-c'_p\det\kappa_p^{-1})\right)\int\limits_{\prod_{p|N'}\mathbb{Z}_p^*} B_{j+k_2+2}(\frac{x}{N'})\omega^{-1}(x)dx.$$

For any N, l, the standard distribution relations for Bernoulli numbers may be written as

$$\frac{N^l}{l}\int\limits_{\prod_{p|N}\mathbb{Z}_p} B_l(\frac{x}{N})\omega^{-1}(x)dx = -L(\omega,-l+1).$$

Using inclusion-exclusion, we obtain that $\phi_{2,\omega^{-1}}(\kappa) =$

$$-\frac{1}{N}(\frac{N}{N'})^{j+k_2+2}L(\omega,-j-k_2-1)\mathbb{I}_{\mathrm{cond}(\omega)|N'}\left(\prod_{p|N'}\frac{p}{p-1}\omega_p(-c_p\det\kappa_p^{-1})L_p(\omega,-j-k_2-1)\right).$$

In other words, if for any Hecke character χ we define a function $h_{\chi,p}:\mathbb{Z}_p\to\mathbb{C}$ by

$$h_{\chi,p}(x) = \begin{cases}\chi_p(p^n)|p^n|^{-j-k_2-2}\ \ \mathbb{I}_{\mathrm{cond}_p(\chi_p)=0} & \text{if } |x|_p\le|N|_p=p^{-n}\\ \frac{p}{p-1}\chi_p(x)|x|^{-j-k_2-2}L_p(\chi,-j-k_2-1)^{-1}\ \ \mathbb{I}_{\mathrm{cond}_p(\chi_p)+\mathrm{ord}_p x\le n} & \text{if } |x|_p>|N|_p\end{cases}$$

Then

Lemma 7.1.1. *With notations as above,*

$$\phi_{2,\omega^{-1}}(\bar\kappa) = \frac{-L(\omega,-j-k_2-1)\omega(-1)}{N}\prod_p h_{\omega,p}(c_p)\omega_p^{-1}(\kappa_p).$$

For an arbitrary element $g_f = ((\begin{smallmatrix}a_p & b_p\\ c_p & d_p\end{smallmatrix})_p)\in GL_2(\mathbb{A}_f)$, we chose $b = (\begin{smallmatrix}b_1 & *\\ & b_2\end{smallmatrix})\in B(\mathbb{Q})$, with $b_i>0$, and $\kappa = b^{-1}g_f\in GL_2(\hat{\mathbb{Z}})$. Set $e_p = \max(|c_p|,|d_p|)$, and $\bar c_p = c_pe_p\in\mathbb{Z}_p$. Then $|b_2|_p = e_p$, $|\det b|_p = |\det g_p|_p$. We defined

$$\widetilde{\phi}_{2,\omega^{-1}}(g_f) = \phi_{2,\omega^{-1}}(\kappa)|\frac{b_1}{b_2}|_f^{\frac{j+k_2+2}{2}}$$

$$= \frac{-L(\omega,-j-k_2-1)\omega(-1)}{N}\Big|\frac{\det b}{(b_2)^2}\Big|_f^{\frac{j+k_2+2}{2}} \prod_p h_{\omega,p}(c_p \prod_l e_l)\omega_p^{-1}(b^{-1}g_p)$$
$$= \frac{-L(\omega,-j-k_2-1)\omega(-1)}{N} \prod_p (\frac{|\det g_p|}{e_p^2})^{\frac{j+k_2+2}{2}} h_{\omega,p}(\bar{c}_p \prod_{l\neq p} e_l)\omega_p^{-1}(g_p)$$
$$= \frac{-L(\omega,-j-k_2-1)\omega(-1)}{N} \prod_p (\frac{|\det g_p|}{e_p^2})^{\frac{j+k_2+2}{2}} h_{\omega,p}(\bar{c}_p)\omega_p^{-1}(g_p)\omega_p(\prod_{l\neq p} e_l)$$
$$= \frac{-L(\omega,-j-k_2-1)\omega(-1)}{N} \prod_p (\frac{|\det g_p|}{e_p^2})^{\frac{j+k_2+2}{2}} h_{\omega,p}(\bar{c}_p)\omega_p^{-1}(g_p)\omega_p^{-1}(e_p).$$

Observe that, given $a \in \mathbb{A}_f^*$, $\kappa \in GL_2(\hat{\mathbb{Z}})$,

$$\widetilde{\phi}_{2,\omega^{-1}}((\begin{smallmatrix} a & \\ & 1\end{smallmatrix})\kappa) = \frac{-L(\omega,-j-k_2-1)\omega(-1)}{N} \prod_p |a_p|_p^{\frac{j+k_2+2}{2}} h_{\omega,p}(\bar{c_p})\omega_p^{-1}(a_p)$$
$$:= \frac{-L(\omega,-j-k_2-1)\omega(-1)}{N} \prod_p \widetilde{\phi}_{2,\omega^{-1},p}((\begin{smallmatrix} a_p & \\ & 1\end{smallmatrix})\kappa_p).$$

This is the expression in (i) of Lemma 6.1.1.

So now we need to calculate out $W_{E,1}$. As for ϕ_2, extend ϕ_1 to a function $\widetilde{\phi_1} : GL_2(\mathbb{A}_f) \to \mathbb{C}$ by

$$\widetilde{\phi}_1(g_f) = \phi_1(b^{-1}g_f)(\frac{b_2}{b_1})^{\frac{k_1+j+2}{2}}.$$

Arguing as above, and with the adelic notation there, the same computation yields

$$\widetilde{\phi}_{1,1}(g_f) = \frac{-\zeta(-k_1-j-1)}{N} \prod_p (\frac{|\det g_p|}{e_p^2})^{\frac{k_1+j+2}{2}} h_{1,p}(\bar{d}_p)$$
$$:= \frac{-\zeta(-k_1-j-1)}{N} \prod_p \widetilde{\phi}_{1,1,p}(g_p).$$

For good measure, we shall review how to calculate the Whittaker function attached to an Eisenstein series. Recalling that $\pm U(\mathbb{Z})\backslash SL_2(\mathbb{Z}) \cong B(\mathbb{Q})\backslash GL_2(\mathbb{Q})$, one can check that

$$\sum_{\gamma\in\pm U(\mathbb{Z})\backslash SL_2(\mathbb{Z})} \frac{\phi_1(\gamma\kappa)}{j(\gamma,\tau)^{k_1+j+2}} = \sum_{B(\mathbb{Q})\backslash GL_2(\mathbb{Q})} \frac{\widetilde{\phi}_1(\gamma\kappa)(\det\gamma)^{\frac{k_1+j+2}{2}}}{j(\gamma,\tau)^{k_1+j+2}}$$

Given $g_\infty \in GL_2(\mathbb{R})$, write again $c = \left(\begin{smallmatrix} 1 & \\ & \mathrm{sgn}(\det g_\infty) \end{smallmatrix}\right)$, $g'_\infty = cg\infty$ and $b' = cbc$. Then

$$\begin{aligned}
&\phi_E(g_f, g_\infty) = F_E(b'^{-1}g'_\infty, cb^{-1}g_f)\\
&= E_{\phi_1}(b'^{-1}g'_\infty, cb^{-1}g_f)\mathrm{Im}(b'^{-1}g_\infty \cdot i)^{\frac{k_1+j+2}{2}}\vartheta(b'^{-1}g'_\infty)^{\frac{k_1+j+2}{2}}\\
&= \sum_{B(\mathbb{Q})\backslash GL_2(\mathbb{Q})} \frac{\widetilde{\phi}_1(\gamma cb^{-1}g_f)(\det\gamma)^{\frac{k_1+j+2}{2}}}{j(\gamma, b'^{-1}g'_\infty)^{k_1+j+2}}(\frac{b_2}{b_1})^{\frac{k_1+j+2}{2}}\mathrm{Im}(g_\infty \cdot i)^{\frac{k_1+j+2}{2}}\vartheta(g'_\infty)^{\frac{k_1+j+2}{2}}\\
&= \pm \sum_{B(\mathbb{Q})\backslash GL_2(\mathbb{Q})} \frac{\widetilde{\phi}_1(\gamma g_f)(\det\gamma)^{\frac{k_1+j+2}{2}}}{j(\gamma c, g'_\infty)^{k_1+j+2}}\mathrm{Im}(g_\infty \cdot i)^{\frac{k_1+j+2}{2}}\vartheta(g'_\infty)^{k_1+j+2}.
\end{aligned}$$

A complete set of coset representatives for $B(\mathbb{Q})\backslash GL_2(\mathbb{Q})$ are given by $\left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right)$ and $\{\left(\begin{smallmatrix} & -1 \\ 1 & \lambda \end{smallmatrix}\right), \lambda \in \mathbb{Q}\}$, so

$$\begin{aligned}
&W_E(g_\infty, g_f) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \phi_E(m_\infty g_\infty, m_f g_f)\psi(m)^{-1}dm\\
&= \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \sum_{B(\mathbb{Q})\backslash GL_2(\mathbb{Q})} \frac{\widetilde{\phi}_1(\gamma m_f g_f)(\det\gamma)^{\frac{k_1+j+2}{2}}}{j(\gamma c, cm_\infty g_\infty)^{k_1+j+2}}\\
&\qquad\qquad \times \mathrm{Im}(cm_\infty g_\infty \cdot i)^{\frac{k_1+j+2}{2}}\vartheta(cm_\infty g_\infty)^{k_1+j+2}\psi(m)^{-1}dm\\
&= \left[\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \sum_{B(\mathbb{Q})\backslash GL_2(\mathbb{Q})} \frac{\widetilde{\phi}_1(\gamma m_f g_f)(\det\gamma)^{\frac{k_1+j+2}{2}}}{j(\gamma m_\infty c, g'_\infty)^{k_1+j+2}}\psi(m)^{-1}dm\right]\\
&\qquad\qquad \times \mathrm{Im}(g'_\infty \cdot i)^{\frac{k_1+j+2}{2}}\vartheta(g'_\infty)^{k_1+j+2}.
\end{aligned}$$

The $\left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right)$ coset does not contribute to the integral, so we get

$$\begin{aligned}
&\left[\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \sum_{U(\mathbb{Q})} \frac{\widetilde{\phi}_1(w_0\gamma m_f g_f)}{j(w_0\gamma m_\infty c, g'_\infty)^{k_1+j+2}}\psi(m)^{-1}dm\right]\mathrm{Im}(g'_\infty \cdot i)^{\frac{k_1+j+2}{2}}\vartheta(g'_\infty)^{k_1+j+2}\\
&= \left[\int_{U(\mathbb{A})} \frac{\widetilde{\phi}_1(w_0 m_f g_f)}{j(w_0 m_\infty c, g'_\infty)^{k_1+j+2}}\psi(m)^{-1}dm\right]\mathrm{Im}(g'_\infty \cdot i)^{\frac{k_1+j+2}{2}}\vartheta(g'_\infty)^{k_1+j+2}\\
&= \left[\mathrm{Im}(g'_\infty \cdot i)^{\frac{k_1+j+2}{2}}\vartheta(g'_\infty)^{k_1+j+2}\int_{U(\mathbb{R})} \frac{\psi_{\mathbb{R}}(m_{\mathbb{R}})^{-1}dm_{\mathbb{R}}}{j(w_0 m_\infty c, g'_\infty)^{k_1+j+2}}\right]\\
&\qquad\qquad \times \left[\int_{U(\mathbb{A}_f)} \widetilde{\phi}_1(w_0 m_f g_f)\psi_f(m_f)^{-1}dm_f\right].
\end{aligned}$$

Writing $W_{E,\infty}$ for the term at the infinite prime, one has immediately

$$\begin{aligned}
W_{E,1}(g_\infty, g_f) &= W_{E,\infty}(g_\infty)\int_{\mathbb{A}_f} \widetilde{\phi}_{1,1}(\left(\begin{smallmatrix} & -1 \\ 1 & x \end{smallmatrix}\right) g_f)\psi_f(x)^{-1}dx \\
&= \frac{-\zeta(-k_1-j-1)}{N} W_{E,\infty}(g_\infty)\prod_p \int_{\mathbb{Q}_p} \widetilde{\phi}_{1,1,p}(\left(\begin{smallmatrix} & -1 \\ 1 & x \end{smallmatrix}\right) g_p)\psi_f(x)^{-1}dx \\
&:= \frac{-\zeta(-k_1-j-1)}{N} \prod_{p\le\infty} W_{E,1,p}(g_p).
\end{aligned}$$

Thus Proposition 6.1.1 is proven.

7.2 An explicit kernel

The rest of the chapter is devoted to making more explicit the integrals

$$\Psi_p(f,s) = \int_{\mathbb{Q}_p^*\times K_p} W_{f,p}(\left(\begin{smallmatrix} a_p & \\ & 1 \end{smallmatrix}\right)\kappa_p)\overline{W_{E,1,p}(\left(\begin{smallmatrix} a_p & \\ & 1 \end{smallmatrix}\right)\kappa_p)}\widetilde{\phi}_{2,\omega^{-1},p}(\left(\begin{smallmatrix} a_p & \\ & 1 \end{smallmatrix}\right)\kappa_p)|a_p|^{s-1}d\kappa,$$

at least for $p \mid N$. With p fixed, $n = n_p$ denotes the exponent of p in N, and $|\cdot| = |\cdot|_p$. In this situation, the product $W_{f,p}\widetilde{\phi}_{2,\omega^{-1},p}$ is invariant under right translation by $K_0(p^n)$, so

$$\begin{aligned}
\Psi_p(f,s) = \int_{\mathbb{Q}_p^*\times GL_2(\mathbb{Z}_p)/K_0(p^n)} & W_{f,p}(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)\kappa)\widetilde{\phi}_{2,\omega^{-1},p}(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)\kappa)|a|_p^{s-1} \\
& \times \int_{K_0(p^n)} \overline{W_{E,1,p}(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)\kappa\gamma)}d\gamma d\kappa d^*a.
\end{aligned}$$

Let us write

$$K(a,\kappa,s) = \widetilde{\phi}_{2,\omega^{-1},p}(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)\kappa)|a|_p^{s-1}\int_{K_0(p^n)} \overline{W_{E,1,p}(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)\kappa\gamma)}$$

for the kernel appearing, so that

$$\Psi_p(f,s) = \int_{\mathbb{Q}_p^*\times GL_2(\mathbb{Z}_p)/K_0(p^n)} W_{f,p}(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)\kappa)K(a,\kappa,s)d\kappa d^*a. \tag{7.1}$$

Now a complete set of coset representatives for $GL_2(\mathbb{Z}_p)/K_0(p^n)$ is given by

$$\{\left(\begin{smallmatrix} x & -1 \\ 1 & \end{smallmatrix}\right), 0 < x \le p^n\}, \{\left(\begin{smallmatrix} 1 & \\ x & 1 \end{smallmatrix}\right), 0 < x \le p^n, p \mid x\}.$$

Lemma 7.2.1. *For $0 < x \leq p^n$,*

$$K(a, \left(\begin{smallmatrix} x & -1 \\ 1 & \end{smallmatrix}\right), s) = \psi(-ax)|a|^{s+\frac{k_2-k_1}{2}}(\frac{p}{p-1})^2 \frac{\zeta_p(k_1+j+1)}{L_p(\omega, -j-k_2-1)\zeta_p(k_1+j+2)}$$
$$\times \mathbb{I}_{\mathbb{Z}_p}(pa)\omega^{-1}(a)\left[1 - p^{k_1+j} - \frac{p-1}{p}|pa|^{k_1+j+1}\right].$$

For $0 < x < p^n$, $p \mid x$,

$$K(a, \left(\begin{smallmatrix} 1 & \\ x & 1 \end{smallmatrix}\right), s) = |a|^{s+j+\frac{k}{2}+1}(\frac{p}{p-1})^2 \frac{1}{L_p(\omega, -j-k_2-1)\zeta_p(k_1+j+2)} \mathbb{I}_{\mathbb{Z}_p}(a)$$
$$\times \mathbb{I}_{|x||n_\omega| \geq p^{-n}}\omega(xa^{-1})|x|^{-j-k_2-2}$$

and

$$K(a, \left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right), s) = |a|^{s+j+\frac{k}{2}+1} \frac{p}{p-1} \frac{1}{\zeta_p(k_1+j+2)} \mathbb{I}_{\mathbb{Z}_p}(a)\mathbb{I}_{|n_\omega|=1}\omega(p^n a^{-1})|p^n|^{-j-k_2-2}.$$

Combining this lemma with the known formulae for normalized newforms W_f, the equation (7.1) is completely explicit, and will be evaluated in chapter 8.

Proof.

$$K(a, \kappa, s) = \widetilde{\phi}_{2,\omega^{-1},p}(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)\kappa)|a|_p^{s-1} \int_{K_0(p^n)} \overline{W_{E,1,p}(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)\kappa\gamma)}d\gamma.$$

From above, for $0 \leq x < p^n$, $0 < y < p^n$

$$\begin{aligned}
\widetilde{\phi}_{2,\omega^{-1},p}(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} x & -1 \\ 1 & \end{smallmatrix}\right)) &= |a|^{\frac{j+k_2+2}{2}} \frac{p}{p-1} L_p(\omega, -j-k_2-1)^{-1}\omega^{-1}(a) \\
\widetilde{\phi}_{2,\omega^{-1},p}(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \\ y & 1 \end{smallmatrix}\right)) &= |a|^{\frac{j+k_2+2}{2}} \mathbb{I}_{|yn_\omega| \geq p^{-n}}\omega(ya^{-1})|y|^{-j-k_2-2} \frac{p}{p-1} L_p(\omega, -j-k_2-1)^{-1} \\
\widetilde{\phi}_{2,\omega^{-1},p}\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right) &= |a|^{\frac{j+k_2+2}{2}} \mathbb{I}_{|n_\omega|=1}\omega(p^n a^{-1})|p^n|^{-j-k_2-2}.
\end{aligned}$$

So we just need to calculate

$$\int_{K_0(p^n)} W_{E,1,p}(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} x & -1 \\ 1 & \end{smallmatrix}\right)\gamma)d\gamma$$

and

$$\int_{K_0(p^n)} W_{E,1,p}(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \\ x & 1 \end{smallmatrix}\right)\gamma)d\gamma.$$

First, a quick calculation yields

Lemma 7.2.2. *For* $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $p \mid N$,

$$\frac{1}{\mathrm{vol}(K_0(p^n))} \int\limits_{K_0(p^n)} \phi_{1,1,p}(g\gamma)d\gamma = \begin{cases} 1 & c \in \mathbb{Z}_p^* \\ \frac{p}{p-1}\zeta_p(-k_1-j-1)^{-1} & c \notin \mathbb{Z}_p^* \end{cases}$$

Now, for the former,

$$\begin{aligned}
&\psi(ax)^{-1}|a|^{\frac{k_1+j}{2}} \int\limits_{K_0(p^n)} W_{E,1,p}(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} x & -1 \\ 1 & \end{smallmatrix}\right)\gamma)d\gamma \\
&= \psi(ax)^{-1} \int\limits_{K_0(p^n)} \int\limits_{\mathbb{Q}_p} \widetilde{\phi}_{1,1,p}(\left(\begin{smallmatrix} & -1 \\ 1 & y \end{smallmatrix}\right)\left(\begin{smallmatrix} x & -1 \\ 1 & \end{smallmatrix}\right)\gamma)\overline{\psi(ay)}dyd\gamma \\
&= \int\limits_{K_0(p^n)} \int\limits_{\mathbb{Q}_p} \widetilde{\phi}_{1,1,p}(\left(\begin{smallmatrix} 1 & \\ y & 1 \end{smallmatrix}\right)\gamma)\psi(ay)dyd\gamma \\
&= \int\limits_{K_0(p^n)} \Bigg[\int\limits_{p\mathbb{Z}_p} \widetilde{\phi}_{1,1,p}(\left(\begin{smallmatrix} 1 & \\ y & 1 \end{smallmatrix}\right)\gamma)\psi(ay)dy \\
&\qquad\qquad + \int\limits_{\mathbb{Q}_p - p\mathbb{Z}_p} \widetilde{\phi}_{1,1,p}(\left(\begin{smallmatrix} * & * \\ 1 & * \end{smallmatrix}\right)\gamma)|y|^{-(k_1+j+2)}\psi(ay)dy\Bigg]d\gamma \\
&= \int\limits_{p\mathbb{Z}_p} \Bigg[\int\limits_{K_0(p^n)} \phi_{1,1,p}(\left(\begin{smallmatrix} 1 & \\ y & 1 \end{smallmatrix}\right)\gamma)d\gamma\Bigg]\psi(ay)dy \\
&\qquad\qquad + \int\limits_{\mathbb{Q}_p - p\mathbb{Z}_p} \Bigg[\int\limits_{K_0(p^n)} \phi_{1,1,p}(\left(\begin{smallmatrix} * & * \\ 1 & * \end{smallmatrix}\right)\gamma)d\gamma\Bigg] |y|^{-(k_1+j+2)}\psi(ay)dy \\
&= \int\limits_{p\mathbb{Z}_p} \frac{p}{p-1}\zeta_p(-k_1-j-1)^{-1}\psi_p(ay)dy + \int\limits_{\mathbb{Q}_p - p\mathbb{Z}_p} |y|^{-(k_1+j+2)}\psi(ay)dy \\
&= \frac{\zeta_p(-k_1-j-1)^{-1}}{p-1}\mathbb{I}_{\mathbb{Z}_p}(pa) + \sum_{m=-\infty}^{0} |p|^{-m(k_1+j+1)} \int\limits_{\mathbb{Z}_p^*} \psi(ap^m y)dy \\
&= \frac{\zeta_p(-k_1-j-1)^{-1}}{p-1}\mathbb{I}_{\mathbb{Z}_p}(pa) + \sum_{m=-\infty}^{0} |p|^{-m(k_1+j+1)} \left[\mathbb{I}_{\mathbb{Z}_p}(p^m a) - \frac{1}{p}\mathbb{I}_{\mathbb{Z}_p}(p^{m+1}a)\right] \\
&= \frac{\zeta_p(-k_1-j-1)^{-1}}{p-1}\mathbb{I}_{\mathbb{Z}_p}(pa) + \sum_{m=-\mathrm{ord}a}^{0} |p|^{-m(k_1+j+1)} - \frac{1}{p} \sum_{m=-\mathrm{ord}a-1}^{0} |p|^{-m(k_1+j+1)} \\
&= \mathbb{I}_{\mathbb{Z}_p}(pa) \left[\frac{\zeta_p(-k_1-j-1)^{-1}}{p-1} + \frac{1 - |p|^{(\mathrm{ord}a+1)(k_1+j+1)} - \frac{1}{p} + \frac{1}{p}|p|^{(\mathrm{ord}a+2)(k_1+j+2)}}{1-p^{-k_1-j-1}}\right] \\
&= \mathbb{I}_{\mathbb{Z}_p}(pa) \left[\frac{p(1-p^{k_1+j})(1-p^{-k_1-j-2})}{(p-1)(1-p^{-(k_1+j+1)})} - \frac{|pa|^{k_1+j+1}(1-p^{-k_1-j-2})}{1-p^{-k_1-j-1}}\right],
\end{aligned}$$

as claimed. Similarly, for $x \in p\mathbb{Z}_p$,

$$\begin{aligned}
&|a|^{\frac{k_1+j}{2}} \int\limits_{K_0(p^n)} W_{E,1,p}(\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \\ x & 1\end{smallmatrix}\right)\gamma)d\gamma \\
&= \int\limits_{K_0(p^n)} \int\limits_{\mathbb{Q}_p} \widetilde{\phi}_{1,1,p}(\left(\begin{smallmatrix} & -1 \\ 1 & y\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \\ x & 1\end{smallmatrix}\right)\gamma)\overline{\psi(ay)}dyd\gamma \\
&= \int\limits_{K_0(p^n)} \int\limits_{\mathbb{Q}_p} \widetilde{\phi}_{1,1,p}(\left(\begin{smallmatrix} -x & -1 \\ 1+xy & y\end{smallmatrix}\right)\gamma)\overline{\psi(ay)}dyd\gamma \\
&= \int\limits_{\mathbb{Z}_p} \left[\int\limits_{K_0(p^n)} \widetilde{\phi}_{1,1,p}(\left(\begin{smallmatrix} * & * \\ 1 & *\end{smallmatrix}\right)\gamma)d\gamma\right]\overline{\psi(ay)}dy \\
&\qquad\qquad + \int\limits_{\mathbb{Q}_p-\mathbb{Z}_p} \left[\int\limits_{K_0(p^n)} \widetilde{\phi}_{1,1,p}(\left(\begin{smallmatrix} * & * \\ y^{-1}+x & *\end{smallmatrix}\right)\gamma)d\gamma\right]|y|^{-k_1-j-2}\overline{\psi(ay)}dy \\
&= \int\limits_{\mathbb{Z}_p} \overline{\psi(ay)}dy + \int\limits_{\mathbb{Q}_p-\mathbb{Z}_p} \frac{p}{p-1}\zeta_p(-k_1-j-1)^{-1}|y|^{-k_1-j-2}\overline{\psi(ay)}dy \\
&= \mathbb{I}_{\mathbb{Z}_p}(a) + \frac{p}{p-1}\zeta_p(-k_1-j-1)^{-1}\left[\sum_{m=-\infty}^{-1} |p|^{-m(k_1+j+1)} \int\limits_{\mathbb{Z}_p^*} \overline{\psi_p(ap^m y)}dy\right] \\
&= \mathbb{I}_{\mathbb{Z}_p}(a)|a|^{k_1+j+1}(1-p^{-k_1-j-2})\frac{p}{p-1}.
\end{aligned}$$

$\square$

Chapter 8

The Local Computations

In this chapter we prove Proposition 6.1.1. The computations 6.1.1(i) and 6.1.1(iii) (which do not depend on the choice of residue data ϕ_i) are well-known:

8.1 Classical calculations

8.1.1 The unramified computation

Suppose now that the local representation ϖ_p is unramified. Then the integral in question is merely

$$\int_{\mathbb{Q}_p^*} W_f\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right)\overline{W_{E,1}\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right)}|a|^{s+\frac{j+k_2}{2}}\omega^{-1}(a)d^*a.$$

This may be evaluated as in, for example, Lemma 15.9.4 of [Ja72], and one obtains

$$\begin{aligned}
&\frac{L_p(\varpi_p\otimes\omega^{-1}, s+\frac{k_2-k_1}{2})L_p(\varpi_p\otimes\omega^{-1}, s+j+\frac{k}{2})}{L_p(\omega^{-1}, j+k_2+2+2s)}\\
&=\frac{L_p(\hat{\varpi}_p, s+\frac{k_2-k_1+1}{2})L_p(\hat{\varpi}_p, s+j+k_2+\frac{3}{2})}{L_p(\omega^{-1}, j+k_2+2+2s)}\\
&=\frac{L_p(\check{f}, 1+k_2+s)L_p(\check{f}, j+k+2+s)}{L_p(\omega^{-1}, j+k_2+2+2s)}.
\end{aligned}$$

8.1.2 Computation at infinity

The integral to be computed is $\Psi_\infty(f,s) =$

$$\int_{\mathbb{R}^*\times SO(2)} W_{f,\infty}\left(\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right)\theta\right)\overline{W_{E,1,\infty}\left(\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right)\theta\right)}\widetilde{\phi}_{2,\omega^{-1},\infty}\left(\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right)\theta\right)|a|^{s-1}d\theta d^*a.$$

A formula for $W_{f,\infty}$ can be found in [Bu98], §2.8, although one must modify for our choice of ψ_∞. One finds

$$W_{f,\infty}\left(\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right)\theta\right) = \mathbb{I}_{\mathbb{R}_+^*}(a)a^{\frac{k+2}{2}}e^{-2\pi a}\theta^{k+2}.$$

Along with the formula for $\phi_{2,\mathbb{R}}$ and $W_{E,\infty}$,

$$\Psi_\infty(f,s) = \int\limits_{\mathbb{R}_+^*\times SO(2)} a^{\frac{k+2}{2}}e^{-2\pi a}\overline{\left[a^{\frac{k_1+j+2}{2}}\int\limits_{\mathbb{R}}\frac{\psi_\infty(x)^{-1}dx}{(ai+x)^{k_1+j+2}}\right]}a^{\frac{j+k_2+2}{2}}a^{s-1}d\theta d^*a$$

$$= \int\limits_0^\infty a^{k+j+s+2}e^{-4\pi a}\overline{\int\limits_{\mathbb{R}+ai}\frac{\psi_\infty(x)^{-1}dx}{x^{k_1+j+2}}}d^*a$$

$$= \int\limits_0^\infty a^{k+j+s+2}e^{-4\pi a}\overline{\frac{-i^{k_1+j+2}}{G(k_1+j+2)}}d^*a$$

$$= \frac{i^{k_1+j+2}}{G(k_1+j+2)(4\pi)^{k+j+s+2}}\int\limits_0^\infty a^{k+j+s+2}e^{-a}d^*a$$

$$= \frac{i^{k_1+j+2}}{G(k_1+j+2)(4\pi)^{k+j+s+2}}\Gamma(k+j+s+2).$$

This function is clearly analytic in s, and taking $s=0$ gives the asserted value.

8.2 The bad integrals

We are left to do the bad local zeta integrals of 6.1.1(ii). Fix a prime $p \mid N$, and set $n = \mathrm{ord}_p N$. It will also be convenient to change notation and write n_ω for $\mathrm{ord}_p(\mathrm{cond}\,\omega)$. Then

$$\Psi_p(f,s) = \int\limits_{\mathbb{Q}_p^*\times GL_2(\mathbb{Z}_p)/K_0(p^n)} W_f\left(\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right)\kappa\right)K(a,\kappa,s)d\kappa d^*a,$$

where K is the kernel given in Lemma 7.2.1. It is well-known that such integrals give rational functions of p^{-s}. Note also that there is an analogous definition of K for larger p-adic fields, and in fact the computations that follow make no special use of the fact that the ground field is $\mathbb{Q}$.

A complete set of left cosets for $K_0(p^n)$ in $GL_2(\mathbb{Z}_p)$ is given by $\{\left(\begin{smallmatrix} x & -1\\ 1 & \end{smallmatrix}\right), 1\le x\le p^n\}$ and $\{\left(\begin{smallmatrix} 1 & \\ px & 1\end{smallmatrix}\right), 1\le x\le p^{n-1}\}$. The latter set stratifies naturally as $\cup_{l=1}^n S_l$, $S_l = \{\left(\begin{smallmatrix} 1 & \\ x & 1\end{smallmatrix}\right), 1\le x\le p^n, \mathrm{ord}_p(x) = l\}$, and it is further natural to write $S_0 = \{\left(\begin{smallmatrix} x & -1\\ 1 & \end{smallmatrix}\right), 1\le x\le p^n\}$. Thus the integral in question may be written

$$\sum_{l=0}^n\sum_{m=-\infty}^\infty\int\limits_{S_l}\int\limits_{p^m\mathbb{Z}_p^*} W_f\left(\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right)\kappa\right)K((a,\kappa,s)d^*a d\kappa.$$

The volume normalizations are such that $\int\limits_{\mathbb{Z}_p} dx = \int\limits_{\mathbb{Z}_p^*} d^*x = 1$, and, given any function H on $GL_2(\mathbb{Z}_p)/\Gamma_0(p^n)$,

$$\int\limits_{S_0} H(\kappa)d\kappa = \frac{p}{p+1}\int\limits_{\mathbb{Z}_p} H\left(\begin{smallmatrix} x & -1 \\ 1 & \end{smallmatrix}\right)dx$$
$$\int\limits_{S_n} H(\kappa)d\kappa = \frac{p}{p+1}\frac{p}{p-1}\int\limits_{p^n\mathbb{Z}_p^*} H\left(\begin{smallmatrix} 1 & \\ x & 1 \end{smallmatrix}\right)dx = \frac{p}{(p+1)p^n}H\left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right),$$

and, for $0 < l < n$,

$$\int\limits_{S_l} H(\kappa)d\kappa = \frac{p}{p+1}\int\limits_{p^l\mathbb{Z}_p^*} H\left(\begin{smallmatrix} 1 & \\ x & 1 \end{smallmatrix}\right)dx.$$

8.2.1 Supercuspidal case

Suppose ϖ_p is supercuspidal, and restrict to the situation where p is odd. This means (e.g., [Sc02]) that one is given a quadratic extension E of $F = \mathbb{Q}_p$ and a character $\xi : E^* \to \mathbb{C}^*$ that does not factor through $N_{E/F}$. The contragredient representation $\hat{\varpi}_p$ is defined by using ξ^{-1}. We fix the additive character $\psi_F = \psi_p$ from above, $\psi_E = \psi_F \circ \mathrm{Tr}_{E/F}$, and let $\chi_{E/F} : F^* \to \mathbb{C}^*$ be the quadratic character attached to the extension E/F.

The Weil representation Ω_E of $SL_2(F)$ on $C_c^\infty(E)$ is given by

$$(\Omega_E\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)f)(v) = \psi_F(xN_{E/F}(v))f(v)$$
$$(\Omega_E\left(\begin{smallmatrix} a & \\ & a^{-1} \end{smallmatrix}\right)f)(v) = |a|_F\chi_{E/F}(a)f(av)$$
$$(\Omega_E\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)f)(v) = \gamma\int\limits_E f(u)\psi_E(u\overline{v})du$$

where, according to [JL70], Theorem 4.6 and earlier,

$$\gamma = \frac{\epsilon_p(s, \hat{\varpi}_p, \psi_p)}{\epsilon_p(s, \xi^{-1}, \psi_E)} \qquad \gamma^2 = \chi_{E/F}(-1).$$

Set $U_\xi \subset C_c^\infty(E)$ to be the set of functions f such that

$$f(yv) = \xi(y)^{-1}f(v), \ \ y \in \ker(N_{E/F}), v \in E.$$

The action of $SL_2(F)$ on U_ξ extends to a representation Ω_{ξ,χ_F} of $GL_2^+(F)$ (the set of matrices with determinant in $N_{E/F}E^*$) by

$$(\Omega_{\xi,\psi_F}\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)f)(v) = |a|_F^{\frac{1}{2}}\xi(b)f(bv), \ \ N_{E/F}b = a.$$

The representation Ω_ξ of $GL_2(F)$ obtained by induction is the dihedral supercuspidal ϖ_p, and the vector space of functions $f : GL_2(F) \to U_\xi$ on which it acts will be denoted V_ξ. The central character of Ω_ξ is known to be $\chi_{E/F}\xi|_{F^*}$. Let $\phi_0 \in U_\xi$ be given by

$$\phi_0(v) = \xi^{-1}(v)\mathbb{I}_{\mathcal{O}_E^*}(v).$$

Then the normalized newvector in the model V_ξ is given as

Proposition 8.2.1. *([Sc02], Theorem 2.3.5) If E/F is unramified, the normalized newvector in the model V_ξ of Ω_ξ is defined by*

$$f(g) = \begin{cases} \Omega_{\xi,\psi_F}(g)\phi_0 & \text{if } g \in GL_2^+(F) \\ 0 & \text{if } g \notin GL_2^+(F). \end{cases}$$

If E/F is ramified, choose a unit $x \in \mathcal{O}_F^* - N_{E/F}\mathcal{O}_E^*$. The normalized newvector in the model V_ξ of Ω_ξ is

$$f(g) = \begin{cases} \Omega_{\xi,\psi_F}(g)\phi_0 & \text{if } g \in GL_2^+(F) \\ \Omega_{\xi,\psi_F}(g\left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right))\phi_0 & \text{if } g \notin GL_2^+(F). \end{cases}$$

A Whittaker functional $V_\xi \to \mathbb{C}$ is given by

$$\Lambda_{\psi_F}(h) = h\left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right)(1_E),$$

and the normalized Whittaker function for $\varpi_p = \Omega_\xi$ is thus

$$W_f(g) = f(g)(1_E).$$

Recall that the goal is to calculate

$$\Psi_p(f, s) = \sum_{l=0}^{n} \sum_{m=-\infty}^{\infty} \int_{S_l} \int_{p^m\mathbb{Z}_p^*} W_f(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)\kappa)K(a, \kappa, s)d^*a d\kappa.$$

For $\kappa \in S_0$, $K(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)\kappa)$ has support on $a \in p^{-1}\mathbb{Z}_p$, while $W_f(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)\kappa)$ has support on $a \in p^{-n}\mathbb{Z}_p^*$. Since $n \geq 2$, these supports are disjoint, and the large stratum S_0 does not contribute to the integral.

Hence

$$
\begin{aligned}
&(\frac{p-1}{p})^2\frac{p+1}{p}\Psi_p(f,s)\\
&=(\frac{p-1}{p})^2\sum_{l=1}^{n-1}\sum_{m=-\infty}^{\infty}\int_{p^l\mathbb{Z}_p^*}\int_{p^m\mathbb{Z}_p^*}W_f((\begin{smallmatrix}a&\\&1\end{smallmatrix})(\begin{smallmatrix}1&\\x&1\end{smallmatrix}))K(a,(\begin{smallmatrix}1&\\x&1\end{smallmatrix}),s)d^*adx\\
&\qquad+(\frac{p-1}{p})^2|p|^n\sum_{m=-\infty}^{\infty}\int_{p^m\mathbb{Z}_p^*}W_f(\begin{smallmatrix}a&\\&1\end{smallmatrix})K(a,(\begin{smallmatrix}1&\\&1\end{smallmatrix}),s)d^*adx\\
&=\sum_{l=1}^{n-n_\omega}\sum_{m\geq 0}C_{l,m}\int_{p^l\mathbb{Z}_p^*}\int_{p^m\mathbb{Z}_p^*}W_f((\begin{smallmatrix}a&\\&1\end{smallmatrix})(\begin{smallmatrix}1&\\x&1\end{smallmatrix}))\omega(xa^{-1})d^*adx
\end{aligned}
$$

where for $l<n$,

$$C_{l,m}=\frac{|p|^{-l(j+k_2+2)}|p|^{m(s+j+\frac{k}{2}+2)}}{\zeta_p(k_1+j+2)L_p(\omega,-j-k_2-1)}$$

while

$$C_{n,m}=\frac{|p|^{-n(j+k_2+2)}|p|^{m(s+j+\frac{k}{2}+2)}}{\zeta_p(k_1+j+2)}.$$

By elementary algebra, Proposition 6.1.1(ii) in this case follows immediately from

Lemma 8.2.1. *If the central character ω is ramified, then for $1\leq l\leq n-n_\omega$, $0\leq m$,*

$$\Psi_{p,l,m}(f,s)=\int_{p^l\mathbb{Z}_p^*}\int_{p^m\mathbb{Z}_p^*}W_f((\begin{smallmatrix}a&\\&1\end{smallmatrix})(\begin{smallmatrix}1&\\x&1\end{smallmatrix}))\omega(xa^{-1})d^*adx$$

vanishes unless $l=n-n_\omega$, $m=0$, in which case it is equal to

$$\frac{\epsilon(1,\varpi_p,\psi_F)^2}{\epsilon(1,\omega,\overline{\psi_F})}.$$

If ω is unramified,

$$\Psi_{p,l,m}(f,s)=\begin{cases}\epsilon_p(1,\varpi_p,\psi_F)^2\frac{p-1}{p} & \text{if } l=n, m=0\\ -\epsilon_p(1,\varpi_p,\psi_F)^2\omega(p)^{-1} & \text{if } l=n-1, m=0\\ 0 & \text{else.}\end{cases}$$

Proof. For uniformity of notation, if E/F is ramified, we will write $p^{\frac{1}{2}}\mathcal{O}_E^*=\mathfrak{p}^k\mathcal{O}_E-\mathfrak{p}^{k+1}\mathcal{O}_E$. As another bit of notation, given $a\in NE^*\subset F^*$, we will write $b=b(a)$ for any choice of $b\in E^*$ with $Nb=a$. We may choose b to be a locally continuous function of a.

First, suppose E/F is ramified. Choose a $t \in F\backslash NE^*$ of valuation 1. Then

$$\int\limits_{p^l\mathbb{Z}_p^*}\int\limits_{p^m\mathbb{Z}_p^*} W_f((\begin{smallmatrix} a & \\ & 1\end{smallmatrix})(\begin{smallmatrix} 1 & \\ x & 1\end{smallmatrix}))\omega(xa^{-1})d^*adx$$

$$= \int\limits_{p^l\mathbb{Z}_p^*}\int\limits_{p^m\mathbb{Z}_p^*} W_f((\begin{smallmatrix} -1 & \\ & -1\end{smallmatrix})(\begin{smallmatrix} a & \\ & 1\end{smallmatrix})(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix})(\begin{smallmatrix} 1 & -x \\ & 1\end{smallmatrix})(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}))\omega(xa^{-1})d^*adx$$

$$= \int\limits_{p^l\mathbb{Z}_p^*}\int\limits_{p^m\mathbb{Z}_p^*\cap NE^*} \Omega_{\xi,\psi_f}((\begin{smallmatrix} -1 & \\ & -1\end{smallmatrix})(\begin{smallmatrix} a & \\ & 1\end{smallmatrix})(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix})(\begin{smallmatrix} 1 & -x \\ & 1\end{smallmatrix})(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}))\phi_0(1)\omega(xa^{-1})d^*adx$$

$$+ \int\limits_{p^l\mathbb{Z}_p^*}\int\limits_{p^m\mathbb{Z}_p^*\cap(F^*-NE^*)} \Omega_{\xi,\psi_F}((\begin{smallmatrix} -1 & \\ & -1\end{smallmatrix})(\begin{smallmatrix} at^{-1} & \\ & 1\end{smallmatrix})(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix})(\begin{smallmatrix} 1 & -xt^{-1} \\ & 1\end{smallmatrix})(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}))\phi_0(1)\omega(xa^{-1})d^*adx$$

$$= 2\int\limits_{p^l\mathbb{Z}_p^*}\int\limits_{p^m\mathbb{Z}_p^*\cap NE^*} \Omega_{\xi,\psi_F}((\begin{smallmatrix} -1 & \\ & -1\end{smallmatrix})(\begin{smallmatrix} a & \\ & 1\end{smallmatrix})(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix})(\begin{smallmatrix} 1 & -x \\ & 1\end{smallmatrix})(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}))\phi_0(1)\omega(xa^{-1})d^*adx$$

$$= 2\int\limits_{p^l\mathbb{Z}_p}\int\limits_{p^m\mathbb{Z}_p\cap NE*}\int\limits_{E}\int\limits_{\mathcal{O}_E} \xi(bv^{-1})\psi_F(-xN(u))\psi_E(u\bar{v})\psi_E(-u\bar{b})\omega(xa^{-1})\frac{dv}{|v|}dud^*adx.$$

Replacing $x \mapsto xa$, $v \mapsto vb$, $u \mapsto u\bar{b}^{-1}$, this is equal to

$$2\int\limits_{p^{(l-m)}\mathbb{Z}_p^*}\int\limits_{p^m\mathbb{Z}_p^*\cap NE*}\int\limits_{E}\int\limits_{p^{-\frac{m}{2}}\mathcal{O}_E^*} \xi(v^{-1})\psi_F(-xN(u))\psi_E(u\bar{v})\psi_E(-u)\omega(x)\frac{dv}{|v|}dud^*adx$$

$$= \int\limits_{p^{(l-m)}\mathbb{Z}_p^*}\int\limits_{E}\int\limits_{p^{-\frac{m}{2}}\mathcal{O}_E^*} \xi(v^{-1})\psi_F(-xN(u))\psi_E(u\bar{v})\psi_E(-u)\omega(x)\frac{dv}{|v|}dudx$$

$$= |p^{l-m}|\sum_{t=-\infty}^{\infty}\left[\int\limits_{\mathfrak{p}^t\mathcal{O}_E^*}\int\limits_{p^{-\frac{m}{2}}\mathcal{O}_E^*} \xi(v^{-1})\omega(-N(u)^{-1})\psi_E(u\bar{v})\psi_E(-u)\frac{dv}{|v|}du\right]\left[\int\limits_{p^{l-m-t}\mathbb{Z}_p^*}\psi_F(x)\omega(x)\frac{dx}{|x|}\right]$$

$$= |p^{l-m}|\gamma^2\sum_{t=-\infty}^{\infty}\left[\int\limits_{\mathfrak{p}^t\mathcal{O}_E^*}\int\limits_{\mathfrak{p}^{t-m}\mathcal{O}_E^*} \xi^{-1}(uv)\psi_E(v)\psi_E(u)\frac{dv}{|v|}du\right]\left[\int\limits_{p^{l-m-t}\mathbb{Z}_p^*}\psi_F(x)\omega(x)\frac{dx}{|x|}\right]$$

The first integral vanishes unless $t = -n, m = 0$, so

$$\Psi_{p,l,m}(f,s) = |p|^{l-n}\gamma^2\epsilon(1,\xi,\psi_E)^2\int\limits_{p^{l-n}\mathbb{Z}_p^*}\omega(x)\psi_F(x)\frac{dx}{|x|}$$

$$= \epsilon(1,\varpi_p,\psi_E)^2\int\limits_{p^{l-n}\mathbb{Z}_p^*}\omega(x)\psi_F(x)dx.$$

On the other hand, if E/F is unramified, the same basic calculation yields

$$\int\limits_{p^l\mathbb{Z}_p^*}\int\limits_{p^m\mathbb{Z}_p^*} W_f((\begin{smallmatrix} a & \\ & 1\end{smallmatrix})(\begin{smallmatrix} 1 & \\ x & 1\end{smallmatrix}))\omega(xa^{-1})d^*adx =$$

$$\mathbb{I}_{m \text{ even}} \int\limits_{p^{(l-m)}\mathbb{Z}_p^*} \int\limits_{E} \int\limits_{p^{-\frac{m}{2}}\mathcal{O}_E^*} \xi(v^{-1})\psi_F(-xN(u))\psi_E(u\bar{v})\psi_E(-u)\omega(x)\frac{dv}{|v|}dudx,$$

and one may proceed in the same manner. □

8.2.2 Principal series or special with two ramified characters

Suppose $\varpi_p \cong \pi(\chi_1, \chi_2)$ is principal series, where both χ_1 and χ_2 are ramified, or that $\varpi_p \cong \sigma(\chi_1, \chi_2)$ is special with ramified characters. The Whittaker functions, L-, and ϵ-factors are given by identical formulae ([Sc02], pp. 126, 130), so we can assume ϖ_p is principal series. Write $n_i > 0$ for the conductor of χ_i, so that $n = n_1 + n_2$. We may take $n_1 \geq n_2$. The central character is $\omega = \chi_1\chi_2$, so $n_\omega \leq n_1$, with equality unless $n_1 = n_2$.

We need to compute

$$\Psi_p(f,s) = \sum_{l=0}^{n} \sum_{m=-\infty}^{\infty} \int\limits_{S_l} \int\limits_{p^m\mathbb{Z}_p^*} W_f((\begin{smallmatrix} a & \\ & 1 \end{smallmatrix})\kappa)K((\begin{smallmatrix} a & \\ & 1 \end{smallmatrix})\kappa)d^*a d\kappa.$$

For $\kappa \in S_0$, $K((\begin{smallmatrix} a & \\ & 1 \end{smallmatrix})\kappa)$ has support on $a \in p^{-1}\mathbb{Z}_p$, while $W_f((\begin{smallmatrix} a & \\ & 1 \end{smallmatrix})\kappa)$ has support on $a \in p^{-n}\mathbb{Z}_p^*$. Since $n \geq 2$, these supports are disjoint, and the large stratum S_0 does not contribute to the integral. Thus, as in the supercuspidal case, and with the same constants,

$$(\frac{p-1}{p})^2\frac{p+1}{p}\Psi_p(f,s) = \sum_{l=1}^{n-n_\omega} \sum_{m\geq 0} C_{l,m} \int\limits_{p^l\mathbb{Z}_p^*} \int\limits_{p^m\mathbb{Z}_p^*} W_f((\begin{smallmatrix} a & \\ & 1 \end{smallmatrix})(\begin{smallmatrix} 1 & \\ x & 1 \end{smallmatrix}))\omega(xa^{-1})d^*a dx,$$

and Proposition 6.1.1(ii) will again follow from

Lemma 8.2.2. *If the central character ω is ramified, then for $1 \leq l \leq n - n_\omega$, $0 \leq m$, $\Psi_{p,l,m}(f,s)$ vanishes unless $l = n - n_\omega$, $m = 0$, in which case it is equal to*

$$\frac{\epsilon(1, \varpi_p, \psi_p)^2}{\epsilon(1, \omega, \overline{\psi_p})}$$

If ω is unramified,

$$\Psi_{p,l,m}(f,s) = \begin{cases} \epsilon_p(1, \varpi_p, \psi_F)^2 \frac{p-1}{p} & \text{if } l = n, m = 0 \\ -\epsilon_p(1, \varpi_p, \psi_F)^2\omega(p)^{-1} & \text{if } l = n-1, m = 0 \\ 0 & \text{else.} \end{cases}$$

In the standard model of $\pi(\chi_1,\chi_2)$, one $K_1(p^n)$-invariant vector is given by

$$g_0\left(\begin{smallmatrix}1&\\x&1\end{smallmatrix}\right)=\chi_1(x)^{-1}\mathbb{I}_{p^{n_2}\mathbb{Z}_p^*}(x),$$

and the intertwining map from the standard model of $\pi(\chi_1,\chi_2)$ to the space of Whittaker functions is

$$g\mapsto W_g : h\to\int_{\mathbb{Q}_p} g(\left(\begin{smallmatrix}&-1\\1&y\end{smallmatrix}\right)h)\overline{\psi_p(y)}dy.$$

For our g_0,

$$W_{g_0}\left(\begin{smallmatrix}1&\\&1\end{smallmatrix}\right)=\epsilon_p(1,\chi_2^{-1},\overline{\psi_p}).$$

Thus

$$\begin{aligned}W_f(\left(\begin{smallmatrix}a&\\&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&\\x&1\end{smallmatrix}\right))&=\frac{1}{\epsilon_p(1,\chi_2^{-1},\overline{\psi_p})}\int_{\mathbb{Q}_p} g(\left(\begin{smallmatrix}&-1\\1&y\end{smallmatrix}\right)\left(\begin{smallmatrix}a&\\&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&\\x&1\end{smallmatrix}\right))\overline{\psi_p(y)}dy\\&=\frac{|a|}{\epsilon_p(1,\chi_2^{-1},\overline{\psi_p})}\int_{\mathbb{Q}_p} g(\left(\begin{smallmatrix}y^{-1}&1\\&ay\end{smallmatrix}\right)\left(\begin{smallmatrix}1&\\y^{-1}+x&1\end{smallmatrix}\right))\overline{\psi_p(ay)}dy\\&=\frac{\chi_2(a)|a|^{\frac12}}{\epsilon_p(1,\chi_2^{-1},\overline{\psi_p})}\int_{\mathbb{Q}_p}\chi_2(y)\chi_1(1+yx)^{-1}\mathbb{I}_{p^{n_2}\mathbb{Z}_p^*}(y^{-1}+x)\overline{\psi_p(ay)}\frac{dy}{|y|}.\end{aligned}$$

We may compute as follows:

Proof. For $1\le l\le n-n_\omega$,

$$\int_{p^l\mathbb{Z}_p^*}\int_{p^m\mathbb{Z}_p^*} W_f(\left(\begin{smallmatrix}a&\\&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&\\x&1\end{smallmatrix}\right))\omega(xa^{-1})d^*adx$$

$$=\int_{p^l\mathbb{Z}_p^*}\int_{p^m\mathbb{Z}_p^*}\int_{\mathbb{Q}_p}\frac{\chi_2(a)|a|^{\frac12}}{\epsilon_p(1,\chi_2^{-1},\overline{\psi_p})}\chi_2(y)\chi_1(1+yx)^{-1}$$

$$\mathbb{I}_{p^{n_2}\mathbb{Z}_p^*}(y^{-1}+x)\overline{\psi_p(ay)}\omega(xa^{-1})\frac{dy}{|y|}d^*adx$$

$$=\frac{|p|^{\frac{m}{2}}}{\epsilon_p(1,\chi_2^{-1},\overline{\psi_p})}\int_{p^l\mathbb{Z}_p^*}\int_{p^m\mathbb{Z}_p^*}\int_{\mathbb{Q}_p}\chi_2(ayx^{-1})\chi_1(1+y)^{-1}\overline{\psi_p(ayx^{-1})}$$

$$\mathbb{I}_{p^{n_2}\mathbb{Z}_p^*}(x(y^{-1}+1))\omega(xa^{-1})\frac{dy}{|y|}d^*adx$$

$$= \frac{|p|^{\frac{m}{2}}|p^l|}{\epsilon_p(1,\chi_2^{-1},\overline{\psi_p})} \int\limits_{p^m\mathbb{Z}_p^*} \int\limits_{\mathbb{Q}_p} \chi_2(ay)\chi_1(1+y)^{-1}\omega(a)^{-1}$$

$$\mathbb{I}_{p^{n_2-l}\mathbb{Z}_p^*}(y^{-1}+1) \int\limits_{p^l\mathbb{Z}_p^*} \chi_2^{-1}(x)\omega(x)\overline{\psi_p(ayx^{-1})}\frac{dx}{|x|}\frac{dy}{|y|}d^*a$$

$$= \frac{|p^{l+\frac{m}{2}}|}{\epsilon_p(1,\chi_2^{-1},\overline{\psi_p})} \int\limits_{p^m\mathbb{Z}_p^*} \int\limits_{\mathbb{Q}_p} \chi_2(ay)\chi_1(1+y)^{-1}\omega(a)^{-1}$$

$$\mathbb{I}_{p^{n_2-l}\mathbb{Z}_p^*}(y^{-1}+1) \int\limits_{p^{-l+\mathrm{ord}y+m}\mathbb{Z}_p^*} \chi_1(x^{-1}ay)\overline{\psi_p(x)}\frac{dx}{|x|}\frac{dy}{|y|}d^*a$$

$$= \frac{|p^{l+\frac{m}{2}}|\epsilon(1,\chi_1,\overline{\psi_p})}{\epsilon_p(1,\chi_2^{-1},\overline{\psi_p})} \int\limits_{p^m\mathbb{Z}_p^*} \int\limits_{p^{l-n_1-m}\mathbb{Z}_p^*} \omega(y)\chi_1(1+y)^{-1}\mathbb{I}_{p^{n_2-l}\mathbb{Z}_p^*}(y^{-1}+1)\frac{dy}{|y|}d^*a$$

$$= \frac{|p^{l+\frac{m}{2}}|\epsilon(1,\chi_1,\overline{\psi_p})}{\epsilon_p(1,\chi_2^{-1},\overline{\psi_p})} \int\limits_{p^{m+n_1-l}\mathbb{Z}_p^*} \omega^{-1}(y)\chi_1(1+y^{-1})^{-1}\mathbb{I}_{p^{n_2-l}\mathbb{Z}_p^*}(y+1)\frac{dy}{|y|}$$

$$= \frac{|p^{l+\frac{m}{2}}|\epsilon(1,\chi_1,\overline{\psi_p})}{\epsilon_p(1,\chi_2^{-1},\overline{\psi_p})} \int\limits_{p^{m+n_1-l}\mathbb{Z}_p^*} \chi_2^{-1}(y)\chi_1(1+y)^{-1}\mathbb{I}_{p^{n_2-l}\mathbb{Z}_p^*}(y+1)\frac{dy}{|y|}.$$

If $m+n_1 > n_2$, the integrand vanishes unless $l = n_2$, in which case we get (writing $t = m+n_1-n_2 > 0$)

$$\frac{|p^{n_2+\frac{m}{2}}|\epsilon(1,\chi_1,\overline{\psi_p})}{\epsilon_p(1,\chi_2^{-1},\overline{\psi_p})} \int\limits_{p^t} \chi_2^{-1}(y)\chi_1^{-1}(1+y)\frac{dy}{|y|}$$

$$= \frac{|p^{n_2+\frac{m}{2}}|\epsilon(1,\chi_1,\overline{\psi_p})}{\epsilon_p(1,\chi_2^{-1},\overline{\psi_p})} \int\limits_{p^{-t}} \omega(y)\chi_1^{-1}(1+y)\frac{dy}{|y|}.$$

And

$$\int\limits_{p^{-t}\mathbb{Z}_p^*} \omega(y)\chi_1^{-1}(1+y)\frac{dy}{|y|} \int\limits_{p^{-n_\omega}\mathbb{Z}_p^*} \omega^{-1}(\delta)\psi_p(\delta)\frac{d\delta}{|\delta|}$$

$$= \int\limits_{p^{-t}\mathbb{Z}_p^*} \int\limits_{p^{-n_\omega+t}\mathbb{Z}_p^*} \omega^{-1}(\delta)\chi_1^{-1}(1+y)\psi_p(\delta y)\frac{d\delta}{|\delta|}\frac{dy}{|y|}$$

$$= \int\limits_{p^{-t}\mathbb{Z}_p^*} \int\limits_{p^{-n_\omega+t}\mathbb{Z}_p^*} \omega^{-1}(\delta)\chi_1^{-1}(y)\psi_p(\delta y-\delta)\frac{d\delta}{|\delta|}\frac{dy}{|y|}$$

$$= \int\limits_{p^{-n^\chi}\mathbb{Z}_p^*} \chi_1^{-1}(y)\psi_p(y)\frac{dy}{|y|} \int\limits_{p^{-n_\omega+t}\mathbb{Z}_p^*} \chi_2^{-1}(\delta)\psi_p(-\delta)\frac{d\delta}{|\delta|}.$$

This last expression vanishes unless $n_\omega = n_1$ and $t = n^\chi - n_2$, so in this case $\Psi_{p,l,m} = 0$ unless

$l = n_2$, $m = 0$, where we get

$$\frac{\epsilon_p(1, \chi_1, \psi_p)\epsilon_p(1, \chi_2, \psi_p)}{\epsilon_p(1, \omega, \psi_p)} = \frac{\epsilon_p(1, \varpi_p, \psi_p)}{\epsilon_p(1, \omega, \psi_p)}.$$

The other possibility is that $m = 0$ and $n_1 = n_2$, when the integral

$$\int_{p^{n_1-l}\mathbb{Z}_p^*} \chi_2^{-1}(y)\chi_1^{-1}(1+y)\mathbb{I}_{p^{n_1-l}\mathbb{Z}_p^*}(y+1)\frac{dy}{|y|}$$

clearly vanishes unless $l \geq n_1$. If $l > n_1$, we must evaluate

$$\int_{p^{n_1-l}\mathbb{Z}_p^*} \chi_2^{-1}(y)\chi_1^{-1}(1+y)\frac{dy}{|y|}.$$

Multiplying by $\epsilon_p(1, \chi_2, \psi_p)$ and rearranging integrals as above, we obtain

$$\Psi_{p,l,m} = \mathbb{I}_{m=0}\epsilon_p(1, \varpi_p, \psi_p)^2 \int_{p^{l-n}\mathbb{Z}_p^*} \omega(y)\psi_p(y)dy,$$

which gives the desired answer. Finally, we must consider $m = 0$, $l = n_1 = n_2$, where the integral is

$$\int_{\mathbb{Z}_p^*} \chi_2^{-1}(y)\chi_1^{-1}(1+y)\mathbb{I}_{\mathbb{Z}_p^*}(1+y)\frac{dy}{|y|}.$$

Comparing conductors,

$$\int_{p\mathbb{Z}_p-1} \chi_2^{-1}(y)\chi_1^{-1}(1+y)dy = 0,$$

so that

$$\int_{\mathbb{Z}_p^*} \chi_2^{-1}(y)\chi_1^{-1}(1+y)\mathbb{I}_{\mathbb{Z}_p^*}(1+y)\frac{dy}{|y|} = \int_{\mathbb{Z}_p^*} \chi_2^{-1}(y)\chi_1^{-1}(1+y)\frac{dy}{|y|},$$

and the calculation proceeds as above.

□

8.2.3 Principal series with one ramified character

Suppose $\varpi_p \cong \pi(\chi_1, \chi_2)$, with χ_1 ramified (of conductor n) and χ_2 unramified. As usual, our goal is to compute

$$\Psi_p(f, s) = \sum_{l=0}^{n} \sum_{m=-\infty}^{\infty} \int_{S_l} \int_{p^m\mathbb{Z}_p^*} W_f(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)\kappa)K(a, \kappa, s)d^*a d\kappa.$$

A newvector in the standard model for ϖ_p is determined by

$$g\left(\begin{smallmatrix} 1 & \\ x & 1 \end{smallmatrix}\right) = \chi_1^{-1}(x)\chi_2(x)|x|^{-1}\mathbb{I}_{\mathrm{ord}x\leq 0},$$

and the Whittaker function attached to g is normalized. Since $n_\omega = n$, $K(a,\kappa,s)$ is supported on $\kappa \in S_0$; recalling the formula for K,

$$\begin{aligned}
&(\frac{p-1}{p})^2\frac{p+1}{p}\Psi_p(f,s)\\
&= |p|^n \frac{\zeta_p(k_1+j+1)}{\zeta_p(k_1+j+2)} \sum_{m=-1}^{\infty}\sum_{x=0}^{p^n-1} \int_{p^m\mathbb{Z}_p^*} W_f(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} x & 1 \\ 1 & \end{smallmatrix}\right))\psi(-ax)\\
&\qquad \times |a|^{s+\frac{k_2-k_1}{2}}\omega^{-1}(a)\left[1-p^{k_1+j}-\frac{p-1}{p}|pa|^{k_1+j+1}\right]d^*a\\
&= \frac{\zeta_p(k_1+j+1)}{\zeta_p(k_1+j+2)} \sum_{m=-1}^{\infty} |p|^{m(s+\frac{k_2-k_1}{2})}\left[1-p^{k_1+j}-\frac{p-1}{p}|p|^{(m+1)(k_1+j+1)}\right]\\
&\qquad \times \int_{p^m\mathbb{Z}_p^*} W_f(\left(\begin{smallmatrix} & a \\ -1 & \end{smallmatrix}\right))\omega^{-1}(a)d^*a.
\end{aligned}$$

Easily,

$$\begin{aligned}
&\int_{p^m\mathbb{Z}_p^*} W_f(\left(\begin{smallmatrix} & a \\ -1 & \end{smallmatrix}\right))\omega^{-1}(a)d^*a = W_f\left(\begin{smallmatrix} & 1 \\ -p^{-m} & \end{smallmatrix}\right)\\
&= |p|^{\frac{m}{2}} \int_{\mathbb{Q}_p} \chi_1(p^{-m})g\left(\begin{smallmatrix} 1 & \\ y & 1 \end{smallmatrix}\right)\psi(p^{-m}y)dy\\
&= |p|^{\frac{m}{2}} \int_{\mathbb{Q}_p-\mathbb{Z}_p} \chi_1^{-1}(p^m y)\chi_2(y)\psi(p^{-m}y)\frac{dy}{|y|}\\
&= |p|^{\frac{m}{2}}\chi_2(p^{-m}) \int_{\mathbb{Q}_p-p^m\mathbb{Z}_p} \chi_1^{-1}(y)\chi_2(y)\psi(y)\frac{dy}{|y|}\\
&= |p|^{\frac{m}{2}}\chi_2(p^{-m}) \int_{p^{-n_1}} \chi_1^{-1}(y)\chi_2(y)\psi(y)\frac{dy}{|y|}\\
&= |p|^{\frac{m}{2}}\chi_2(p^{-m})\frac{\epsilon_p(1,\chi_1,\psi_p)^2}{\epsilon_p(1,\omega,\psi_p)},
\end{aligned}$$

and so

$$
\begin{aligned}
&(\frac{p-1}{p})^2\frac{p+1}{p}\Psi_p(f,s)\\
=&\frac{\zeta_p(k_1+j+1)}{\zeta_p(k_1+j+2)}\frac{\epsilon_p(1,\chi_1,\psi_p)^2}{\epsilon_p(1,\omega,\psi_p)}\\
&\qquad\qquad\times\sum_{m=-1}^{\infty}\chi_2(p^{-m})|p|^{m(s+\frac{k_2-k_1}{2}+\frac{1}{2})}\left[1-p^{k_1+j}-\frac{p-1}{p}|p|^{(m+1)(k_1+j+1)}\right]\\
=&\frac{\zeta_p(k_1+j+1)}{\zeta_p(k_1+j+2)}\frac{\epsilon_p(1,\chi_1,\psi_p)^2}{\epsilon_p(1,\omega,\psi_p)}|p|^{-(s+\frac{k_2-k_1}{2}+\frac{1}{2})}\\
&\qquad\qquad\times\left[\frac{1-p^{k_1+j}}{1-\chi_2^{-1}(p)p^{-(s+\frac{k_2-k_1}{2}+\frac{1}{2})}}-\frac{(p-1)/p}{1-\chi_2^{-1}(p)p^{-(s+j+\frac{k}{2}+\frac{3}{2})}}\right]\\
=&\frac{\zeta_p(k_1+j+1)}{\zeta_p(k_1+j+2)}\frac{\epsilon_p(1,\chi_1,\psi_p)^2}{\epsilon_p(1,\omega,\psi_p)}|p|^{-(s+\frac{k_2-k_1}{2}+\frac{1}{2})}L_p(\chi_2^{-1},s+\frac{k_2-k_1}{2}+\frac{1}{2})\\
&\qquad\qquad\times L_p(\chi_2^{-1},s+j+\frac{k}{2}+\frac{3}{2})\frac{(1-p^{k_1+j+1})}{p}(1-\chi_2^{-1}(p)p^{-(s+j+\frac{k}{2}+\frac{1}{2})})\\
=&\frac{L_p(\chi_2^{-1},s+\frac{k_2-k_1}{2}+\frac{1}{2})L_p(\chi_2^{-1},s+j+\frac{k}{2}+\frac{3}{2})}{\zeta_p(k_1+j+2)}\frac{\epsilon_p(1,\chi_1,\psi_p)^2}{\epsilon_p(1,\omega,\psi_p)}\frac{1}{L_p(\chi_2,-s-j-\frac{k}{2}-\frac{1}{2})}\\
=&\frac{L_p(\check{f},s+k_2+1)L_p(\check{f},s+j+k+2)}{\zeta_p(k_1+j+2)}\frac{\epsilon_p(1,\varpi_p,\psi_p)^2}{\epsilon_p(1,\omega,\psi_p)}\frac{1}{L_p(f,-s-j)}.
\end{aligned}
$$

8.2.4 Special with conductor p

The only case remaining is where ϖ_p is the special representation $\sigma(\chi|\cdot|^{\frac{1}{2}},\chi|\cdot|^{-\frac{1}{2}})$ for an unramified character χ. This representation has conductor 1, and central character $\omega=\chi^2$. We have

$$\epsilon(s,\varpi_p,\psi_p)=-\chi^{-1}(p)p^{\frac{1}{2}-s}.$$

The normalized Whittaker function for ϖ_p is given by

$$
\begin{aligned}
W_f(\begin{smallmatrix}a&\\&1\end{smallmatrix})&=\chi(a)|a|\mathbb{I}_{\mathbb{Z}_p}(a)\\
W_f(\begin{smallmatrix}&a\\-1&\end{smallmatrix})&=-\chi(a)|pa|\mathbb{I}_{\mathbb{Z}_p}(pa).
\end{aligned}
$$

Combining this with the formulas for K given above,

$$
\begin{aligned}
&(\frac{p-1}{p})^2\frac{p+1}{p}\Psi_p(f,s)\\
&=(\frac{p-1}{p})^2\sum_{l=0}^{1}\sum_{m=-\infty}^{\infty}\int_{S_l}\int_{p^m\mathbb{Z}_p^*}W_f((\begin{smallmatrix}a&\\&1\end{smallmatrix})\kappa)K(a,\kappa,s)d^*a d\kappa\\
&=(\frac{p-1}{p})^2\Big[p^{-1}\sum_{m=0}^{\infty}\int_{p^m\mathbb{Z}_p^*}\chi(a)|a|K(a,(\begin{smallmatrix}1&\\&1\end{smallmatrix}),s)d^*a\\
&\qquad+\sum_{m=-1}^{\infty}\int_{p^m\mathbb{Z}_p^*}-\chi(a)|pa|K(a,(\begin{smallmatrix}&1\\-1&\end{smallmatrix}),s)d^*a\Big]\\
&:=I_1(s)+I_0(s)
\end{aligned}
$$

We compute

$$
\begin{aligned}
I_1(s)&=|p|\sum_{m=0}^{\infty}\int_{p^m\mathbb{Z}_p^*}\chi(a)|a|^{s+j+\frac{k}{2}+2}\frac{p-1}{p}\frac{1}{\zeta_p(k_1+j+2)}\omega(pa^{-1})|p|^{-j-k_2-2}\\
&=\omega(p)\frac{p-1}{p}\frac{1}{\zeta_p(k_1+j+2)}|p|^{-j-k_2-1}\sum_{m=0}^{\infty}\chi^{-1}(p^m)|p^m|^{s+j+\frac{k}{2}+2}\\
&=\omega(p)\frac{p-1}{p}\frac{1}{\zeta_p(k_1+j+2)}|p|^{-j-k_2-1}L_p(\chi^{-1},s+j+\frac{k}{2}+2)
\end{aligned}
$$

and

$$
\begin{aligned}
I_0(s)&=-\sum_{m=-1}^{\infty}\int_{p^m\mathbb{Z}_p^*}\chi(a)|pa||a|^{s+\frac{k_2-k_1}{2}}\frac{\zeta_p(k_1+j+1)}{L_p(\omega,-j-k_2-1)\zeta_p(k_1+j+2)}\\
&\qquad\times\omega^{-1}(a)\left[1-p^{k_1+j}-\frac{p-1}{p}|pa|^{k_1+j+1}\right]d^*a\\
&=-|p|^{-(s+\frac{k_2-k_1}{2})}\frac{\zeta_p(k_1+j+1)}{L_p(\omega,-j-k_2-1)\zeta_p(k_1+j+2)}\chi(p)\\
&\qquad\times\sum_{m=0}^{\infty}\chi^{-1}(p^m)|p^m|^{s+\frac{k_2-k_1}{2}+1}\left[1-p^{k_1+j}-\frac{p-1}{p}|a|^{k_1+j+1}\right]d^*a\\
&=-|p|^{-(s+\frac{k_2-k_1}{2})}\frac{\zeta_p(k_1+j+1)}{L_p(\omega,-j-k_2-1)\zeta_p(k_1+j+2)}\chi(p)\\
&\qquad\times\left[\frac{1-p^{k_1+j}}{1-\chi^{-1}(p)p^{-(s+\frac{k_2-k_1}{2}+1)}}-\frac{1-p^{-1}}{1-\chi^{-1}(p)p^{-(2+j\frac{k}{2}+2)}}\right]\\
&=-|p|^{-(s+\frac{k_2-k_1}{2})}\frac{\zeta_p(k_1+j+1)}{L_p(\omega,-j-k_2-1)\zeta_p(k_1+j+2)}\chi(p)\frac{(1-p^{k_1+j+1})}{p}\\
&\qquad\times L_p(\chi^{-1},s+\frac{k_2-k_1}{2}+1)L_p(\chi^{-1},s+j+\frac{k}{2}+2)(1-\chi^{-1}(p)p^{-(s+j+\frac{k}{2}+1)})\\
&=-|p|\frac{L_p(\chi^{-1},s+\frac{k_2-k_1}{2}+1)L_p(\chi^{-1},s+j+\frac{k}{2}+2)}{L_p(\omega,-j-k_2-1)\zeta_p(k_1+j+2)}(1-\chi(p)p^{s+j+\frac{k}{2}+1}).
\end{aligned}
$$

So

$$
\begin{aligned}
&\Psi_p(f,0) = I_1(0) + I_0(0) \\
&= \omega(p)|p|^{-j-k_2-1} \frac{L_p(\chi^{-1}, \frac{k_2-k_1}{2}+1)L_p(\chi^{-1}, j+\frac{k}{2}+2)}{\zeta_p(k_1+j+2)} \\
&\qquad\qquad \times \left[(\frac{p-1}{p})(1-\chi^{-1}(p)p^{-(\frac{k_2-k_1}{2}+1)}) + \frac{(1-\chi(p)p^{j+\frac{k}{2}+1})}{p}(1-\omega_p^{-1}(p)p^{-j-k_2-1}) \right] \\
&= (\frac{p}{p-1})^2 \omega(p)|p|^{-j-k_2-1} \frac{L_p(\chi^{-1}, \frac{k_2-k_1}{2}+1)L_p(\chi^{-1}, j+\frac{k}{2}+2)}{\zeta_p(k_1+j+2)} \\
&\qquad\qquad\qquad \times \left[1-\chi(p)p^{j+\frac{k}{2}} - \omega_p^{-1}(p)p^{-j-k_2-2} + \chi^{-1}(p)p^{-(\frac{k_2-k_1}{2}+2)} \right] \\
&= \omega(p)|p|^{-j-k_2-1} \frac{L_p(\chi^{-1}, \frac{k_2-k_1}{2}+1)L_p(\chi^{-1}, j+\frac{k}{2}+2)}{\zeta_p(k_1+j+2)L_p(\chi, -j-\frac{k}{2})L_p(\omega_p^{-1}, j+k_2+2)} \\
&= \frac{\epsilon_p(-j-\frac{k}{2}-\frac{1}{2}, \varpi_p, \psi_p)\epsilon_p(\frac{k_1-k_2+1}{2}, \varpi_p, \psi_p)}{\zeta_p(k_1+j+2)L_p(\omega_p^{-1}, j+k_2+2)} \frac{L_p(\chi^{-1}, \frac{k_2-k_1}{2}+1)L_p(\chi^{-1}, j+\frac{k}{2}+2)}{L_p(\chi, -j-\frac{k}{2})} \\
&= \frac{\epsilon_p(-j-\frac{k}{2}-\frac{1}{2}, \varpi_p, \psi_p)\epsilon_p(\frac{k_1-k_2+1}{2}, \varpi_p, \psi_p)}{\zeta_p(k_1+j+2)L_p(\omega_p^{-1}, j+k_2+2)} \frac{L_p(\check{f}, j+k+2)L_p(\check{f}, k_2+1)}{L_p(f, -j)}.
\end{aligned}
$$

This completes the case of a special representation with minimal conductor, and thus the formula for the bad local zeta integrals has been checked in all cases.

Chapter 9

Overview of Proof of Theorem 4.1.1(ii)

We begin with some notation. Recall that f is a new cuspidal eigenform of weight $k+2 \geq 2$, level $N \geq 5$, and we write ω for the nebentypus of f. Choose $l \nmid 2N$, and $v \mid l$ a place of F. F_l denotes the v-adic completion of F, and $\mathcal{O}_l$ is its ring of integers. V_l is the 2-dimensional F_l-Galois representation attached to f, and T_l in an $\mathcal{O}_l$-lattice chosen compatibly with the modular symbol δ. Recall that $\delta = \delta^{\pm}$ was a Betti cohomology class on $Y(N)$; in the notation of [Ka04], $\delta = \delta(k+2, k_1+1)$, and $\bar{\delta} = \delta_{k+2,k_1+1,\mathrm{Id}}$. In chapter 10, we will define

$$z_1^{(l)}(\check{f}, k_1+1, -j, \mathrm{Id}, Nl) \in H^1(\mathbb{Q}, V_l(t))$$

In the notation of that paper, σ_c acts on $H^1(\mathbb{Q}, V_l(t))$ by c^{-t}.

9.1 Key propositions

The statement (ii) of Theorem 4.1.1 is the actually the conjunction of the following

Proposition 9.1.1. *If $l \nmid N$, the l-adic realization $\rho_{l,M^*(1)}(\bar{\xi})$ is equal to*

$$L_l(f, -j) z_1^{(l)}(\check{f}, k_1+1, -j, \mathrm{Id}, Nl).$$

If $l \mid N$, the l-adic realization $\rho_{l,M^*(1)}(\bar{\xi})$ is equal to

$$z_1^{(l)}(f, k_1+1, -j, \mathrm{Id}, Nl).$$

Thus, according to [Ka04] §8, $\rho_{l,M^*(1)}(\bar{\xi}) \in H^1(\mathbb{Z}[\frac{1}{Nl}], V_l(t))$.

Proposition 9.1.2. *Assume the Main Conjecture for $\check{f}$ and the Leopoldt-type conjecture of the finiteness of $H^2(\mathbb{Z}[\frac{1}{l}], T_l(t))$. Then $H^1(\mathbb{Z}[\frac{1}{l}], T_l(t))$ is a rank 1 $\mathcal{O}_l$-module, and the index of‘ $\prod_{p|N}(1-$*

$a_p p^j)^{-1} z_1^{(l)}(\check{f}, k_1+1, -j, \mathrm{Id}, Nl) \cdot \mathcal{O}_l$ in $H^1(\mathbb{Z}[\frac{1}{l}], T_l(t))$ is equal to $\# H^2(\mathbb{Z}[\frac{1}{l}], T_l(t))$.

The former proposition is proved in chapter 10, and the latter in chapter 11. To deduce Theorem 4.1.1(ii) from these propositions, recall that under the hypothesis, the canonical map $H^1(\mathbb{Z}[\frac{1}{l}], T_l(t)) \to H^1(\mathbb{Z}[\frac{1}{Nl}], T_l(t))$ is in fact an isomorphism, and the difference between the orders of $H^2(\mathbb{Z}[\frac{1}{l}], T_l(t))$ and $H^2(\mathbb{Z}[\frac{1}{Nl}], T_l(t))$ is $\#(\mathcal{O}_l/(\prod_{l|N}(1-a_p p^j))\mathcal{O}_l)$.

Chapter 10

On Constructions of Kato

In preparation to introducing K. Kato's main conjecture for modular forms, we will review certain constructions from [Ka04]. We then present a new proof of what is essentially a lemma of G. Kings, introduced to study the TNC for CM elliptic curves. From this lemma, it will follow that certain of Kato's elements are precisely the l-adic realizations of the motivic cohomology classes constructed in §3.2; as a consequence, Proposition 9.1.1 holds.

10.1 Kato elements

As mentioned above, the notation in this paper generally follows [HK99b], which is rather different from the notation of [Ka04]. Furthermore, in [Ka04] there are many different cohomology classes in many different cohomology groups; as a rule they are all labeled z, and distinguished by the attached parameters. To maintain some degree of readability with the original, and for lack of any truly better schema, this last convention will be continued. That being said, we begin with modular units.

Theorem 10.1.1. *([Ka04], Proposition 1.3) Let $\pi : E \to S$ be an elliptic curve with identity section E. Let $c > 1$ be an integer prime to 6 and invertible on S, and $[c]$ the endomorphism of E over S given by multiplication by c. Then there is a unique section ${}_c\theta \in \mathcal{O}^*(E \setminus \ker[c])$, compatible with base change in S, so that*

1. $\mathrm{Div}({}_c\theta) = c^2(e) - \ker[c]$.
2. *For any integer b prime to c, $[b]_*({}_c\theta) = {}_c\theta$.*

Kato applies this result to the context where $S = Y = Y(M, N)$, $E = \mathfrak{X}$ is its universal elliptic curve, and c is prime to $6N$. Each choice of a pair $(\alpha, \beta) \in \frac{1}{M}\mathbb{Z}/\mathbb{Z} \times \frac{1}{N}\mathbb{Z}/\mathbb{Z}$ determines a section of π; let ${}_cg_{\alpha,\beta}$ be the pullback of ${}_c\theta$ along this section. For $c \equiv 1 (\mathrm{mod}\, 6MN)$, there is a well-defined element

$$g_{\alpha,\beta} = {}_cg_{\alpha,\beta} \otimes (c^2 - 1)^{-1} \in K_1(Y)_{\mathbb{Q}}.$$

The motivic classes commonly called "Beilinson's elements" will be denoted

$$ {}_{c,d}z_{M,N} = \{ \, {}_c g_{\frac{1}{M},0}, \; {}_d g_{0,\frac{1}{N}} \} \in K_2(Y(M,N)). $$

Lemma 10.1.1. *Assume that* $N|N'$, $M|M'$, $\mathrm{prime}(M') = \mathrm{prime}(M)$, $\mathrm{prime}(N') = \mathrm{prime}(N)$.

1. *The trace map* $K_1(Y(M,N')) \to K_1(Y(M,N))$ *takes* ${}_d g_{0,\frac{1}{N'}}$ *to* ${}_d g_{0,\frac{1}{N}}$.
2. *The trace map* $K_2(Y(M',N')) \to K_2(Y(M,N))$ *takes* ${}_{c,d}z_{M',N'}$ *to* ${}_{c,d}z_{M,N}$.

Proof. The second statement is [Ka04], Proposition 2.3, and the first can be found in the proof given there. $\square$

Let $\mathcal{H}_l = \mathcal{H} \otimes_{\mathbb{Z}} \mathbb{Z}_l = R^1\pi_*\mathbb{Z}_l$ be the relative first l-adic étale cohomology of $\mathfrak{X}$ over Y, and $\mathcal{H}_{\mathbb{Q}_l} = \mathcal{H} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. Recall that, just as the Galois representation attached to a form of weight 2, level N, can be located in $H^1(Y_{\overline{\mathbb{Q}}}, \mathbb{Q}_l)$, similarly the representation attached to a form of weight $k+2$ and level N can be located in $H^1(Y_{\overline{\mathbb{Q}}}, \mathrm{Sym}^k \mathcal{H}_{\mathbb{Q}_l})$. Furthermore, if T denotes the relative l-adic Tate module of $\mathfrak{X}$ over Y, then $T \cong \mathcal{H}_l(1)$. Let ξ_1 and ξ_2 denote the canonical sections of T/l^n over $Y(Nl^n, Nl^n)$, and note that ξ_2 still defines a section of T/l^n over $Y(N, Nl^n)$.

By norm-compatibility, for all M,

$$ \{ {}_c g_{0,\frac{1}{Ml^n}} \}_n \in \varprojlim K_1(Y(M, Ml^n)). $$

For $k \geq 0$, consider the composite $Ch_k = Ch_{M,k}$ (a "Chern Class map")

$$
\begin{aligned}
\varprojlim K_1(Y(M,Ml^n)) \xrightarrow{ch_{1,1}} \quad & \varprojlim H^1(Y(M,Ml^n), (\mathbb{Z}/l^n)(1)) \\
\xrightarrow{\cup \xi_2^k} \quad & \varprojlim H^1(Y(M,Ml^n), (\mathrm{Sym}^k T/l^n)(1)) \\
\xrightarrow{\mathrm{Tr}} \quad & \varprojlim H^1(Y(M), (\mathrm{Sym}^k T/l^n)(1)) \\
\cong \quad & H^1(Y(M), (\mathrm{Sym}^k T)(1)).
\end{aligned}
$$

The maps are as labelled; the first is the l-adic Chern class map, which in this context is just the Kummer map. For c prime to Ml, define

$$ {}_c Z_{M,k} = Ch_k(\{ {}_c g_{0,\frac{1}{Ml^n}} \}_n). $$

Following Kato, we do an analogous construction beginning with Beilinson's elements. By the lemma above, $\{ {}_{c,d}z_{Ml^n, Ml^n} \}_n \in \varprojlim K^2(Y(Ml^n))$. We take the same partition $k = k_1 + k_2$, and $j \geq 0$. (To compare to the conventions of [Ka04], §8, his k is our $k+2$, his r' is our $k_1 + 1$, and his r is our $-j$).

Recall that ζ_{l^n}, the primitive l^n-th root of unity, is defined on $Y(Ml^n)$. Consider the composite $Ch_{k+2,k_1+1,-j}$

$$\begin{array}{rcl}
\varprojlim K^2(Y(Ml^n)) & \xrightarrow{ch_{2,2}} & \varprojlim H^2(Y(Ml^n),(\mathbb{Z}/l^n)(2)) \\
& \xrightarrow{\cup} & \varprojlim H^2(Y(Ml^n),(\mathrm{Sym}^k T/l^n)(2+j)) \\
& \cong & \varprojlim H^2(Y(Ml^n),(\mathrm{Sym}^k \mathcal{H}/l^n)(t)) \\
& \xrightarrow{\mathrm{Tr}} & \varprojlim H^2(Y(M),(\mathrm{Sym}^k \mathcal{H}/l^n)(t)) \\
& \cong & H^2(Y(M),(\mathrm{Sym}^k \mathcal{H}_l)(t)).
\end{array}$$

Here the first labeled map is again a Chern class map, after Gillet, and the second arrow is cup product

$$x \mapsto x \cup \xi_1^{\otimes k_1} \otimes \xi_2^{\otimes k_2} \otimes \zeta_{l^n}^{\otimes j}.$$

For c, d prime to $6lM$, set

$${}_{c,d}z^{(l)}_{M,M}(k+2,k_1+1,-j) = Ch_{k+2,k_1+1,-j}\left(\{{}_{c,d}z_{Ml^n,Ml^n}\}_n\right).$$

This element can be found in [Ka04], §8. We will use the same name to denote its image under

$$H^2(Y(M),\mathrm{Sym}^k\mathcal{H}_l(t)) \to H^1(\mathbb{Q},H^1(Y(M)_{\overline{\mathbb{Q}}},\mathrm{Sym}^k\mathcal{H}_l)(t)).$$

Choose a $c, d > 1$, congruent to 1 modulo Nl. For f a new cuspidal eigenform of level N and weight $k+2 \geq 2$, the classes $z_1^{(l)}(\check{f}, k_1+1, -j, Id, Nl)$ in Proposition 9.1.1 are defined (independently of c, d as):

- If $l \mid N$, $z_1^{(l)}(\check{f}, k_1+1, -j, Id, Nl)$ is the image of

$$(c^2 - c^{-k_1-j})^{-1}(d^2 - d^{-j-k_2})^{-1}\,{}_{c,d}z^{(l)}_{N,N}(k+2,k_1+1,-j)$$

  under the quotient map $H^1(Y(N)_{\overline{\mathbb{Q}}},\mathrm{Sym}^k\mathcal{H}_{\mathbb{Q}_l}) \to V_l$.

- If $l \nmid N$, $z_1^{(l)}(\check{f}, k_1+1, -j, Id, Nl)$ is the image of

$$(c^2 - c^{-k_1-j})^{-1}(d^2 - d^{-j-k_2})^{-1}\,{}_{c,d}z^{(l)}_{Nl,Nl}(k+2,k_1+1,-j)$$

  under $\mathrm{Tr}^{Y(Nl)}_{Y(N)}$ and the same quotient map.

10.2 Relation between Galois cohomology classes

As before, $\lambda : \mathfrak{X} \to Y$, and let $\lambda^n : \mathfrak{X}^n \to Y$ be the n-fold fiber product of $\mathfrak{X}$ with itself over Y. The group $\mathcal{G} = (\mu_2) \wr \Sigma_n$ acts on $\mathfrak{X}^n$, where the i^{th} copy of μ_2 acts as multiplication by -1 on the i^{th} copy of $\mathfrak{X}$, and the symmetric group Σ_n acts by permuting the n copies of $\mathfrak{X}$. Let $\epsilon : \mathcal{G} \to \mu_2$ be the character given by multiplication on $(\mu_2)^k$, and by the sign character on Σ_n; the same symbol ϵ will also be used to denote the corresponding projector in $\mathbb{Q}\mathcal{G}$. This is the usual motivic projector on $\mathfrak{X}^n$, as in [Sch90] or [Ka04].

As in [De71], the Leray spectral sequence associated to λ^n_* degenerates at E_2, and one may compute $R\lambda^n_*$ by the Kunneth formula. Analyzing the situation yields, for any $j \geq 0$,

$$H^1(Y(M), \mathrm{Sym}^n T_{\mathbb{Q}_l}(j+1)) \cong H^{n+1}(\mathfrak{X}^n, \mathbb{Q}_l(n+j+1))(\epsilon) \subset H^{n+1}(\mathfrak{X}^n, \mathbb{Q}_l(n+j+1)).$$

As before, let $\pi_1 : \mathfrak{X}^{k_1+j+k_2} \to \mathfrak{X}^{k_1+j}$ be given by the first $k_1 + j$ coordinates; similarly, let $\pi_2 : \mathfrak{X}^{k_1+j+k_2} \to \mathfrak{X}^{j+k_2}$ be given by the last $j + k_2$ coordinates. Finally, let $\pi : \mathfrak{X}^{k_1+j+k_2} \to \mathfrak{X}^k$ be given by omitting the middle j coordinates. Let σ denote the involution of $Y(M)$ that exchanges the two canonical sections of $\mathfrak{X}$.

Proposition 10.2.1. *Assume $l \mid M$. For $j \geq 0$,*

$${}_{c,d}z^{(l)}_{M,M}(k+2, k_1+1, -j) \in H^2(Y(M), \mathrm{Sym}^k \mathcal{H}_{\mathbb{Q}_l}(t))$$

is given by

$$\epsilon \circ \pi_* \left(\pi_1^*(\sigma^* {}_cZ_{M,k_1+j}) \cup \pi_2^*({}_dZ_{M,j+k_2})\right).$$

Proof. After multiplying both sides by $k!$, they become elements of
$H^2(Y(M), (\mathrm{Sym}^k \mathcal{H}_l)(k+2+j))$, and it is enough to prove the equality for these mod l^n for all n. Recall that $R^1\lambda_*\mathbb{Z}_l \cong T(-1)$.

Consider first π_1^*, which maps

$$H^1(Y(M), \mathrm{Sym}^{k_1+j}\, T(1)) \subset H^1(Y(M), T^{\otimes(k_1+j)}(1))$$

into

$$H^1(Y(M), R^{k_1+j}\lambda_*^{k_1+j+k_2}\mathbb{Z}_l(k_1+j+1)) =$$

$$H^1(Y(M), \bigoplus_{i_1+\cdots i_{k_1+j+k_2}=k_1+j} R^{i_1}\lambda_*\mathbb{Z}_l \otimes \cdots \otimes R^{i_{k_1+j+k_2}}\lambda_*\mathbb{Z}_l(k_1+j+1)).$$

It is clear that the image of π_1^* actually lies in

$$H^1(Y(M), (R^1\lambda_*\mathbb{Z}_l)^{\otimes k_1+j} \otimes (R^0\lambda_*\mathbb{Z}_l)^{\otimes k_2}(k_1+j+1)).$$

Similarly, the image of π_2^* lies in

$$H^1(Y(M), (R^0\lambda_*\mathbb{Z}_l)^{\otimes k_1} \otimes (R^1\lambda_*\mathbb{Z}_l)^{\otimes j+k_2}(j+k+2+1)).$$

Thus $\pi_1^*(\sigma^*{}_cZ_{M,k_1+j}) \cup \pi_2^*({}_dZ_{M,j+k_2})$ is an element of

$$H^2(Y(M), (R^1\lambda_*\mathbb{Z}_l)^{\otimes k_1} \otimes (R^2\lambda_*\mathbb{Z}_l)^{\otimes j} \otimes (R^1\lambda_*\mathbb{Z}_l)^{\otimes k_2}(k+2j+2))$$

$$\cong H^2(Y(M), T^{\otimes k_1} \otimes \mathbb{Z}_l(1)^{\otimes j} \otimes T^{\otimes k_2}(2)).$$

Mod l^n, ${}_cZ_{M,k_1+j}$ is the trace from $Y(Ml^n, M)$ to $Y(M)$ of

$$ch_{1,1}({}_cg_{\frac{1}{Ml^n},0}) \cup (\xi_1^{\otimes(k_1+j)}) \in H^1(Y(Ml^n, M), (\mathrm{Sym}^{k_1+j}\, T/l^n)(1)).$$

Similarly, ${}_dZ_{M,j+k_2}$ (mod l^n) is the trace from $Y(M, Ml^n)$ to $Y(M)$ of

$$ch_{1,1}({}_dg_{0,\frac{1}{Ml^n}}) \cup (\xi_2^{\otimes(j+k_2)}) \in H^1(Y(M, Ml^n), (\mathrm{Sym}^{j+k_2}\, T/l^n)(1)).$$

As $l \mid M$,

$$\begin{array}{ccc} Y(Ml^n) & \longrightarrow & Y(M, Ml^n) \\ \big\downarrow & & \big\downarrow \\ Y(Ml^n, M) & \longrightarrow & Y(M) \end{array}$$

is Cartesian, and hence $\pi_1^*(\sigma^*{}_cZ_{M,k_1+j}) \cup \pi_2^*({}_dZ_{M,j+k_2})$ (mod l^n) is the trace from $Y(Ml^n)$ to $Y(M)$ of

$$ch_{2,2}(\{{}_cg_{\frac{1}{Ml^n},0}, {}_dg_{0,\frac{1}{Ml^n}}\}) \cup (\xi_1^{\otimes k_1} \otimes (\xi_1 \cup \xi_2)^{\otimes j} \otimes \xi_2^{\otimes k_2}) \in$$
$$H^2(Y(Ml^n), (T/l^n)^{\otimes k_1} \otimes (\mathbb{Z}/l^n(1))^{\otimes j} \otimes (T/l^n)^{\otimes k_2}(2)).$$

Applying π_* gives the obvious map to

$$H^2(Y(Ml^n), (T/l^n)^{\otimes k}(j+2)).$$

Finally, on this space, $k!\epsilon$ is the symmetrizing operator (with no denominators). In summary,

$$k!\epsilon \circ \pi_* \left(\pi_1^*(\sigma^*{}_cZ_{M,k_1+j}) \cup \pi_2^*({}_dZ_{M,j+k_2})\right)$$

is given (mod l^n) by the trace from $Y(Ml^n)$ to $Y(M)$ of

$$ch_{2,2}(\{{}_cg_{\frac{1}{Ml^n},0}, {}_dg_{0,\frac{1}{Ml^n}}\}) \cup \left(\sum_{\tau\in\Sigma_k} \tau^*(\xi_1^{\otimes k_1} \otimes \xi_2^{\otimes k_2})\right) \zeta_{l^n}^{\otimes j}.$$

This is exactly the formula defining ${}_{c,d}z_{M,M}^{(l)}(k+2, k_1+1, -j)$. □

10.3 A lemma of Kings

According to Proposition 10.2.1, in order to realize the ${}_{c,d}z_{M,M}^{(l)}(k+2, -j, k_1+1)$ as coming from motivic cohomology classes, it is enough to do so for the ${}_cZ_{M,\kappa}$. In fact, one has the following result

Lemma 10.3.1. *(Kings) Let e_2 denote the second standard section of $\mathfrak{X}[M]$ over $Y(M)$, and suppose that $l \mid M$, $\kappa \geq 0$, and $c \geq 1$ is prime to $6lM$. Then ${}_cZ_{M,\kappa}$ is the l-adic realization of the Eisenstein symbol*

$$\left[c^2 \mathcal{E}is(\varrho^\kappa(e_2)) - c^{-\kappa}\mathcal{E}is(\varrho^\kappa(ce_2))\right].$$

Compare to Lemma 4.2.9 of [Ki01], using Theorem 2.2.4 of [HK99a] (their conventions on the degree of the polylogarithm differ by 1). Using linearity and compatibility with base change and the GL_2-action, one can recover most of the statement given there. The result here is slightly weaker in that it does not address CM endomorphisms, but is slightly stronger in specifying the sign of both sides (necessary for potential application to equivariant questions). We will shortly provide an alternative proof of the lemma, whose directness hopefully compensates for the weakening. Notice that the ${}_cZ_{M,\kappa}$ have good trace-compatibility properties as the level M varies; this is one motivation for having rescaled the horospherical map in §3.2.

Before proving the lemma, we see how it implies Proposition 9.1.1. First,

Corollary 10.3.1. *When $l \mid M$ and $j \geq 0$, Kato's zeta element ${}_{c,d}z_{M,M}^{(l)}(k+2, k_1+1, -j)$ is the l-adic realization of the class in $H_{\mathcal{M}}^{k+2}(\mathfrak{X}^k(M), t)(\epsilon)$*

$$(c^2 - c^{-k_1-j}(\begin{smallmatrix} c & \\ & 1\end{smallmatrix})^*)(d^2 - d^{-k_2-j}(\begin{smallmatrix} 1 & \\ & d\end{smallmatrix})^*)\epsilon \circ \pi_* \left(\pi_1^*\mathcal{E}is(\varrho^{k_1+j}(e_1)) \cup \pi_2^*\mathcal{E}is(\varrho^{j+k_2}(e_2))\right).$$

Proof. Combining Lemma 10.3.1 and Proposition 10.2.1, ${}_{c,d}z_{M,M}^{(l)}(k+2, -j, k_1+1)$ is the l-adic realization of

$$\epsilon\circ\pi_*(\pi_1^*\sigma^*\left[(c^2 - c^{-k_1-j}(\begin{smallmatrix} 1 & \\ & c\end{smallmatrix})^*)\mathcal{E}is(\varrho^{k_1+j}(e_2))\right] \cup \pi_2^*(d^2 - d^{-k_2-j}(\begin{smallmatrix} 1 & \\ & d\end{smallmatrix})^*)\mathcal{E}is(\varrho^{j+k_2}(e_2))) =$$

$$\epsilon \circ \pi_* \left(\pi_1^*(c^2 - c^{-k_1-j}\left(\begin{smallmatrix} c & \\ & 1 \end{smallmatrix}\right)^*)\mathcal{E}is(\varrho^{k_1+j}(e_1)) \cup \pi_2^*(d^2 - d^{-k_2-j}\left(\begin{smallmatrix} 1 & \\ & d \end{smallmatrix}\right)^*)\mathcal{E}is(\varrho^{j+k_2}(e_2))\right).$$

Since $\left(\begin{smallmatrix} c & \\ & 1 \end{smallmatrix}\right)^*$ fixes $\mathcal{E}is(\varrho^{j+k_2}(e_2))$ and $\left(\begin{smallmatrix} 1 & \\ & d \end{smallmatrix}\right)^*$ fixes $\mathcal{E}is(\varrho^{k_1+j}(e_1))$, and since the $GL_2(\mathbb{Z}/M)$ action commutes with the π_i and π, the corollary follows. $\square$

We will also need

Lemma 10.3.2. *Suppose $l \nmid M$. Let* Tr *be the trace from $Y(Ml)$ to $Y(M)$, and T_l' the dual Hecke operator (as in [Ka04], §4.9). Then*

$$\rho_l(\mathrm{Tr}\,(\xi^{Ml})) = \rho_l(\left(1 - T_l'\left(\begin{smallmatrix} 1/l & \\ & 1 \end{smallmatrix}\right)^* l^j + \left(\begin{smallmatrix} 1/l & \\ & 1/l \end{smallmatrix}\right)^* l^{1+2j}\right)\xi^M).$$

Proof. As both sides are sums of cup products of Eisenstein symbols, it is enough to prove the equality with $\rho_{\mathcal{D}}$ in place of ρ_l. Write

$$\xi_\bullet = \pi_1^*\mathcal{E}is^{k_1+j}(\phi_1) \cup \pi_2^*\mathcal{E}is^{j+k_2}(\phi_2),$$

so that $\xi^M = \epsilon \circ \pi_*\xi_\bullet^M$. As π_* and ϵ commute with the $GL_2(\mathbb{Z}/N)$ and Hecke actions, we want to show

$$\mathrm{Tr}\,(\rho_{\mathcal{D}}\xi_\bullet^{Ml}) = \left(1 - T_l'\left(\begin{smallmatrix} 1/l & \\ & 1 \end{smallmatrix}\right)^* l^j + \left(\begin{smallmatrix} 1/l & \\ & 1/l \end{smallmatrix}\right)^* l^{1+2j}\right)\rho_{\mathcal{D}}\xi_\bullet^M$$

Proposition 4.4 of [Ka04] proves an analogous result for Eisenstein series, using certain norm-compatibility relations. According to [Sch98], these same relations are already satisfied in motivic cohomology, and *a fortiori* in the l-adic realization, so the same argument holds (again, mind that we have rescaled by a factor of the level). $\square$

Proof of Proposition 9.1.1. Recall the definitions of $z_1^{(l)}(f, k_1 + 1, -j, Id, Nl)$ from above. If $l \mid N$, the proposition follows immediately from Cor. 10.3.1 (with $M = N$). If $l \nmid N$, Cor. 10.3.1 gives that at least

$$\rho_l(\xi^{Nl}) = z_{Nl,Nl}^{(l)}(k + 2, k_1 + 1, -j).$$

By definition, the trace from $Y(Nl)$ to $Y(N)$ takes $z_{Nl,Nl}^{(l)}(k+2, k_1+1, -j)$ to $z_{N,N}^{(l)}(k+2, k_1+1, -j)$. Since l-adic realization commutes with trace maps, by Lemma 10.3.2,

$$\rho_l\left(\left(1 - T_l'\left(\begin{smallmatrix} 1/l & \\ & 1 \end{smallmatrix}\right)^* l^j + \left(\begin{smallmatrix} 1/l & \\ & 1/l \end{smallmatrix}\right)^* l^{1+2j}\right)\xi^N\right) = z_{N,N}^{(l)}(k + 2, k_1 + 1, -j).$$

According to [Ka04], 4.9.3, the $T_l'\left(\begin{smallmatrix}1/l & \\ & 1\end{smallmatrix}\right)^*$ commute with the $GL_2(\mathbb{Z}/N)$-action, so

$$\rho_l\left(\left(1 - T_l'\left(\begin{smallmatrix}1/l & \\ & 1\end{smallmatrix}\right)^* l^j + \left(\begin{smallmatrix}1/l & \\ & 1/l\end{smallmatrix}\right)^* l^{1+2j}\right) \operatorname{Tr}{}^{Y(N)}_{Y_1(N)} \xi^N\right) = \operatorname{Tr}{}^{Y(N)}_{Y_1(N)} z^{(l)}_{N,N}(k+2, k_1+1, -j).$$

Now apply $\otimes_{\widetilde{\mathbb{T}}}\overline{\lambda}$ to both sides. As $\left(\begin{smallmatrix}1/l & \\ & 1\end{smallmatrix}\right)$ acts trivially on $Y_1(N)$, we obtain

$$(1 - a_l l^j + \omega(l) l^{1+2j})\rho_l(\bar{\xi}) = z_1^{(l)}(f, k_1+1, -j, \mathrm{Id}\,, Nl).$$

The coefficient on the left is $L_l(f, -j)^{-1}$, and the proposition is shown. $\square$

10.4 Proof of Lemma 10.3.1

This proof was inspired by [HK99a]. If $\kappa = 0$, then the statement is essentially tautologous, so assume $\kappa \geq 1$. The Gysin sequence that defines the residue map in motivic cohomology defines a similar residue map in l-adic cohomology. The residue maps commute with realization, so one has a commutative diagram

$$\begin{array}{ccc} H^{\kappa+1}_{\mathcal{M}}(\mathfrak{X}^\kappa, \kappa+1) & \longrightarrow & \mathbb{Q}[\mathbf{Isom}]^{(\kappa)} \\ \downarrow & & \downarrow \\ H^{\kappa+1}_{et}(\mathfrak{X}^\kappa, \mathbb{Q}_l(\kappa+1)) & \longrightarrow & \mathbb{Q}_l[\mathbf{Isom}]^{(\kappa)} \end{array}$$

which restricts to

$$\begin{array}{ccc} H^{\kappa+1}_{\mathcal{M}}(\mathfrak{X}^\kappa, \kappa+1)(\epsilon) & \longrightarrow & \mathbb{Q}[\mathbf{Isom}]^{(\kappa)} \\ \downarrow & & \downarrow \\ H^{\kappa+1}_{et}(\mathfrak{X}^\kappa, \mathbb{Q}_l(\kappa+1))(\epsilon) & \xrightarrow{\cong} & \mathbb{Q}_l[\mathbf{Isom}]^{(\kappa)} \end{array}$$

For the isomorphism in the last row, see the proof of [HK99a], Theorem C.2.2. Since

$$H^{\kappa+1}_{et}(\mathfrak{X}^\kappa, \mathbb{Q}_l(\kappa+1))(\epsilon) \cong H^1(Y(M), (\mathrm{Sym}^\kappa T_{\mathbb{Q}_l})(1)),$$

and since $H^1(Y(M), (\mathrm{Sym}^\kappa T)(1)) \cong M_\kappa(Y(M)) \otimes \mathbb{Z}_l$ is torsion-free, at least ${}_cZ_{M,\kappa}$ is an element of the right space to be the l-adic realization of the given Eisenstein symbol. By the isomorphism above, by the fact that Res^κ is a left inverse to $\mathcal{E}is^\kappa$, and by the formulae for ϱ^κ (§3.2, §5.3), is is enough to show

Proposition 10.4.1. *Let $i\infty$ be the usual cusp at infinity with its usual orientation. Let $h = \left(\begin{smallmatrix} * & * \\ X & Y \end{smallmatrix}\right) \in GL_2(\mathbb{Z})$. Then, assuming $l \mid M$,*

$$\mathrm{Res}^\kappa({}_cZ^\kappa_{M,M})(h \cdot i\infty) = \frac{M^{\kappa+1}}{\kappa+2}(c^2 B_{\kappa+2}(\frac{X}{M}) - c^{-\kappa} B_{\kappa+2}(\frac{cX}{M})).$$

First, one needs a more explicit description of the l-adic residue map. Let Y_η be the completion of $Y(M)$ at the standard cusp ∞. Let $Y_{\eta,n}$ be the base change of $Y(Ml^n)$ to Y_η

Lemma 10.4.1. *Over Y_η, there exists a monodromy filtration*

$$0 \to \mathbb{Q}_l(1) \to T_{\mathbb{Q}_l} \to \mathbb{Q}_l \to 0.$$

The induced maps $\mathrm{Sym}^\kappa T_{\mathbb{Q}_l} \to \mathrm{Sym}^{\kappa-1} T_{\mathbb{Q}_l}$ induce isomorphisms on residue. Furthermore, the standard section ξ_1 of T/l^n over $Y_{\eta,n}$ gives a splitting of

$$0 \to \mathbb{Z}/l^n(1) \to T/l^n \to \mathbb{Z}/l^n \to 0.$$

There is a similar construction at each cusp

Proof. For the first two assertions, see [HK99a], 2.1.2, 2.1.3. The third follows from considering the situation over $\mathbb{C}$. □

Let j be the inclusion of Y into $\overline{Y}$, and let i be the inclusion of its reduced complement **Cusps**. By C.2.3 and 2.1.3 of [HK99a], the residue map on $H^1(Y, \mathrm{Sym}^\kappa T_{\mathbb{Q}_l}(1))$ is the composite

$$H^1(Y, \mathrm{Sym}^\kappa T_{\mathbb{Q}_l}(1)) \to H^0(\mathbf{Cusps}, i^*R^1j_*\mathrm{Sym}^\kappa T_{\mathbb{Q}_l}) \xrightarrow{\tau^\kappa} H^0(\mathbf{Isom}, \mathbb{Q}_l)$$

where τ^κ is, at each cusp, the κ-fold composite of the map τ defined above. We thus have a commutative diagram

$$\begin{tikzcd}
H^1(Y, \mathrm{Sym}^\kappa T_{\mathbb{Q}_l}(1)) \arrow[r] & H^0(\mathbf{Cusps}, i^*R^1j_*\mathrm{Sym}^\kappa T_{\mathbb{Q}_l}) \arrow[r, "\tau^\kappa"] & H^0(\mathbf{Isom}, \mathbb{Q}_l) \\
H^1(Y, \mathrm{Sym}^\kappa T(1)) \arrow[r] \arrow[u] \arrow[d] & H^0(\mathbf{Cusps}, i^*R^1j_*\mathrm{Sym}^\kappa T) \arrow[r, "\tau^\kappa"] \arrow[u] \arrow[d] & H^0(\mathbf{Isom}, \mathbb{Z}_l) \arrow[u] \arrow[d] \\
H^1(Y, \mathrm{Sym}^\kappa T/l^n(1)) \arrow[r] & H^0(\mathbf{Cusps}, i^*R^1j_*\mathrm{Sym}^\kappa T/l^n) \arrow[r, "\tau^\kappa"] & H^0(\mathbf{Isom}, \mathbb{Z}/l^n)
\end{tikzcd}$$

It is enough to compute the residue along the bottom row for each n. Let $\mathcal{G}$ be the Galois group of $Y(Ml^n)$ over $Y(M)$. The diagram is compatible with base change to $Y(Ml^n)$, so when computing residues mod l^n, we may consider our elements as $\mathcal{G}$-invariant elements of $H^1(Y(Ml^n), \mathrm{Sym}^\kappa T(1))$. These are the preliminaries necessary for the computation.

Let

$$0 \to \mathbb{Z}/l^n(1) \to \mathbb{G}_m \to \mathbb{G}_m \to 0$$

be the Kummer sequence over $Y = Y(M), Y(M, Ml^n)$, or $Y(Ml^n)$, and let

$$\delta : H^0(Y, \mathbb{G}_m) \to H^1(Y, \mathbb{Z}/l^n(1))$$

be the first coboundary map. Recall that ξ_1 and ξ_2 define global sections of T/l^n over $Y(Ml^n)$. Examining the definitions, one sees that

$${}_cZ_{M,\kappa}(\operatorname{mod} l^n) \in H^1(Y(M), \operatorname{Sym}^\kappa T/l^n(1))$$

is equal to

$$\sum_{A,B=0}^{l^n-1} \delta\left({}_cg_{\frac{AM}{Ml^n},\frac{BM+1}{Ml^n}}\right) \otimes [AM\xi_1 + (BM+1)\xi_2]^{\otimes\kappa}.$$

By $GL_2(\mathbb{Z})$-equivariance, the residue of this at $h \cdot i\infty$ is equal to the residue at $i\infty$ of

$$\sum_{A,B=0}^{l^n-1} \delta\left(h^*{}_cg_{\frac{AM}{Ml^n},\frac{BM+1}{Ml^n}}\right) \otimes [h^*(AM\xi_1 + (BM+1)\xi_2)]^{\otimes\kappa}.$$

Rearranging the summation, this equals

$$\sum_{A,B=0}^{l^n-1} \delta\left({}_cg_{\frac{AM+X}{Ml^n},\frac{BM+Y}{Ml^n}}\right) \otimes [(AM+X)\xi_1 + (BM+Y)\xi_2]^{\otimes\kappa}.$$

Now base change to Y_η and apply the composite $\operatorname{Sym}^\kappa T/l^n \xrightarrow{\tau^\kappa} \mathbb{Z}/l^n$. Since τ take $\xi_1 \mapsto 1$ and $\xi_2 \mapsto 0$, the residue of ${}_cZ_{M,\kappa}$ at h is equal to the residue at ∞ of

$$\sum_{A,B=0}^{l^n-1} \delta\left({}_cg_{\frac{AM+X}{Ml^n},\frac{BM+Y}{Ml^n}}\right) \cdot (AM+X)^\kappa \in H^1(Y_{\eta,n}, \mathbb{Z}/l^n(1))^{\mathcal{G}}.$$

Lemma 10.4.2. *The residue of $\delta({}_cg_{\alpha_1,\alpha_2})$ at ∞ is $\frac{M}{2}(c^2B_2(\alpha_1) - B_2(c\alpha_1))$.*

Proof. As discussed in [HK99a], below Lemma C.3.1, given a global section f of $\mathbb{G}_m$ over Y_η, the residue of $\delta(f)$ is exactly the obstruction to taking a l^n-th root of f at that cusp, i.e., the order of vanishing of f at the cusp. The power series expressions for $f = {}_cg_{\alpha_1,\alpha_2}$ are given, for example, by Kato [Ka04]. □

Thus

$$\operatorname{Res}^\kappa({}_cZ_{M,\kappa})(h \cdot i\infty)(\operatorname{mod} l^n)$$

$$= \frac{M}{2} \sum_{A,B=0}^{l^n-1} \left(c^2B_2(\frac{AM+X}{Ml^n}) - B_2(\frac{c(AM+X)}{Ml^n})\right) \cdot (AM+X)^\kappa.$$

To prove the proposition, hence the lemma, it is enough to show

Lemma 10.4.3. *As $n \to \infty$,*

$$\frac{l^n}{2}\sum_{A=0}^{l^n-1}\left(c^2B_2(\frac{AM+X}{Ml^n}) - B_2(\frac{c(AM+X)}{Ml^n})\right)\cdot(AM+X)^\kappa$$

converges l-adically to $\frac{M^\kappa}{\kappa+2}(c^2B_{\kappa+2}(\frac{X}{M}) - c^{-\kappa}B_{\kappa+2}(\frac{cX}{M}))$.

Proof. This result is very similiar to [Wa97], Theorem 2.2, and can be proved directly as therein. For a quicker argument, we may use results of [L76], §XIII.2,3. In the notation of loc. cit., the limit of the left side is

$$\begin{aligned}\frac{c^2}{2M}\int \mathbb{I}_X(Y)Y^\kappa dE_{2,c^{-1}} &= \frac{c^2}{M(\kappa+2)}\int \mathbb{I}_X(Y)dE_{\kappa+2,c^{-1}} \\ &= \frac{c^2}{M(\kappa+2)}\left[E_{\kappa+2}^{(M)}(\frac{X}{M}) - c^{-\kappa-2}E_{\kappa+2}^{(M)}(\frac{cX}{M})\right] \\ &= \frac{M^\kappa}{\kappa+2}(c^2B_{\kappa+2}(\frac{X}{M}) - c^{-\kappa}B_{\kappa+2}(\frac{cX}{M})).\end{aligned}$$

□

Chapter 11

On Kato's Main Conjecture

11.1 Some Iwasawa theory

Let $G_n = \mathrm{Gal}(\mathbb{Q}(\zeta_{l^n})/\mathbb{Q})$, and $G_\infty = \varprojlim G_n = \mathrm{Gal}(\mathbb{Q}(\zeta_{l^\infty})/\mathbb{Q})$. The cyclotomic character

$$\chi_{cyclo} : G_\infty \xrightarrow{\cong} \mathbb{Z}_l^*$$

is defined by the action on the Tate module of l-torsion roots of unity, and for $c \in \mathbb{Z}_l^*$, let $\sigma_c = \chi_{cyclo}^{-1}(c)$. The Iwasawa algebra Λ is defined, as usual, to be

$$\Lambda = \varprojlim \mathbb{Z}_l[G_n].$$

Since $l \neq 2$, one has

$$\Lambda \cong \mathbb{Z}_l[(\mathbb{Z}/l)^*][[T]].$$

For the Galois module T_l chosen in §3.1, or more generally for any l-adic Galois representation, define

$$H^m_{Iw}(T_l) = \varprojlim H^m(\mathbb{Z}[\zeta_{l^n}, \frac{1}{p}], T_l).$$

For any $i \in \mathbb{Z}$, there is a canonical isomorphism $H^m_{Iw}(T_l) \cong H^m_{Iw}(T_l(i))$, and hence a canonical map

$$s_i : H^m_{Iw}(T_l) \to H^m(\mathbb{Z}[\frac{1}{p}], T_l(i)).$$

In the previous chapter, there were constructed classes

$$(k!)_{c,d}z^{(l)}_{Nl^m,Nl^m}(k+2, k_1+1, -j) \in H^2(Y(Nl^m), \mathrm{Sym}^k \mathcal{H}_l(t)).$$

By their definition, these are compatible under the natural trace maps. The Hochschild-Serre spectral sequence yields natural arrows

$$\beta : H^2(Y(Nl^m), \mathrm{Sym}^k \mathcal{H}_l(t)) \to H^1(\mathbb{Q}, H^1(Y(Nl^m)_{\overline{\mathbb{Q}}}, \mathrm{Sym}^k \mathcal{H}_l)(t)),$$

and one may consider the $(k!)_{c,d} z^{(l)}_{Nl^m,Nl^m}(k+2, k_1+1, -j)$ as norm-compatible elements of the latter. The norm-compatibility forces the classes to be unramified outside of l (see [Ka04], Lemma 8.5), and so

$$\{(k!)_{c,d} z^{(l)}_{Nl^m,Nl^m}(k+2, k_1+1, -j)\}_m \in \varprojlim H^1(\mathbb{Z}[\frac{1}{l}], H^1(Y(Nl^m)_{\overline{\mathbb{Q}}}, \mathrm{Sym}^k \mathcal{H}_l)(t)).$$

The trace maps from $Y(Nl^m)$ to $Y(N) \otimes \mathbb{Q}(\zeta_{l^m})$ induce

$$\begin{aligned}\varprojlim H^1(\mathbb{Z}[\frac{1}{l}], H^1(Y(Nl^m)_{\overline{\mathbb{Q}}}, \mathrm{Sym}^k \mathcal{H}_l)(t)) &\to \varprojlim H^1(\mathbb{Z}[\zeta_{l^n}, \frac{1}{l}], H^1(Y(N)_{\overline{\mathbb{Q}}}, \mathrm{Sym}^k \mathcal{H}_l)(t)) \\ &\to H^1_{Iw}(T_l) \otimes \mathbb{Q}.\end{aligned}$$

We will write ${}_{c,d}\tilde{z}^{(l)}_\delta(t)$ for the image of $\{{}_{c,d} z^{(l)}_{Nl^m,Nl^m}(k+2, k_1+1, -j)\}_m$ in $H^1_{Iw}(T_l) \otimes \mathbb{Q}$, and set

$$z^{(l)}_\delta(t) = (c^2 - c^{k_2+2}\sigma_c)^{-1}(d^2 - d^{k_1+2}\sigma_d)^{-1} \prod_{p|N}(1 - a_p p^{-k-2}\sigma_p^{-1})^{-1}{}_{c,d}\tilde{z}^{(l)}_\delta(t).$$

Here $\delta = \delta_{k_i}$ is the same δ appearing in §3.1, and the definition of $z^{(l)}_\delta$ agrees with Kato's from §13.9 after a Tate twist of $k+2$. A priori, $z^{(l)}_\delta$ is defined as an element of $H^1_{Iw}(T_l) \otimes \mathrm{Quot}(\Lambda)$, but [Ka04], §13.12 shows that in fact $z^{(l)}_\delta \in H^1_{Iw}(T_l) \otimes \mathbb{Q}$. Note also that for $j \geq 0$, by definition

$$\beta({}_{c,d} z^{(l)}_{N,N}(k+2, k_1+1, -j)) = s_t({}_{c,d}\tilde{z}^{(l)}_\delta(t)) \in H^1(\mathbb{Z}[\frac{1}{l}], V_l(t)).$$

11.2 Statement of the Main Conjecture

By choice of the lattice T_l, $\overline{\delta}$ is a generator of $T_l^{\pm}$. Let us also choose a generator γ of $T_l^{\mp}$. Just as we defined $z^{(l)}_\delta$, one may also define $z^{(l)}_\gamma \in H^1_{Iw}(T_l) \otimes \mathbb{Q}$ as in [Ka04], §13.9, except with the noted difference in convention on twisting. The exact definition of $z^{(l)}_\gamma$ is not relevant, but for correctness we need elements coming from both the plus and minus part of T_l in order to generate a submodule that can be related to specialization at both even and odd twists.

As in [Ka04], Th. 12.5.(4), let $Z(f,T)$ be the Λ-submodule of $H^1_{Iw}(T_l) \otimes \mathbb{Q}$ generated by $z^{(l)}_\delta(t)$ and $z^{(l)}_\gamma(t)$. As σ_{-1} acts on $z^{(l)}_\delta(t)$ as 1 and on $z^{(l)}_\gamma(t)$ as -1, $Z(f,T)$ is generated over Λ by the single element $z^{(l)}_{\delta+\gamma}(t) = z^{(l)}_\delta(t) + z^{(l)}_\gamma(t)$. Kato's l-adic main conjecture for f $(l \neq 2)$ now reads

Conjecture 11.2.1 (Kato's Main Conjecture, 12.10 of [Ka04]). *Let $\mathfrak{p}$ be a prime ideal of Λ of height 1. Then $Z(f,T)_{\mathfrak{p}} \subset H^1_{Iw}(T_l)_{\mathfrak{p}}$, and*

$$\mathrm{length}_{\Lambda_{\mathfrak{p}}}(H^2_{Iw}(T_l)_{\mathfrak{p}}) = \mathrm{length}_{\Lambda_{\mathfrak{p}}}(H^1_{Iw}(T_l)_{\mathfrak{p}}/Z(f,T)_{\mathfrak{p}}).$$

We remark that, for all f, the module $Z(f,T)$ just defined really is the module that appears in the statement of the Main Conjecture. When f has CM, one prefers to work with elements coming from elliptic units, but the connection between the two constructions is given in §15.16. By Rubin [Ru91], the main conjecture for imaginary quadratic fields is known to hold in many cases; see [Ka04] §15 for a discussion of how to relate the two results in the CM case.

11.3 Descent to twist (t)

We finish by briefly giving a descent argument, following Kato to reduce from the Main Conjecture given above to a finite level statement.

Lemma 11.3.1. *([Ka04], Lemma 14.15) Let A be a Noetherian commutative ring, let C be the category of f.g. A-modules M such that the support of M in* Spec *A is of codimension ≥ 2, and let $G(C)$ be the Grothendieck group of the abelian category C. Let M be a finitely generated A-module whose support is of codimension ≥ 1, let $a \in A$, and assume that $M_{\mathfrak{p}} = 0$ for any prime ideal of height 1 that contains a. Then M/aM and ${}_aM = \ker(a : M \to M)$ belong to C, and we have*

$$[M/aM] - [{}_aM] = \sum_{\mathfrak{q}} \mathrm{length}_{A_{\mathfrak{q}}}(M_{\mathfrak{q}}) \cdot [A/(\mathfrak{q} + aA)]$$

in $G(C)$, where $\mathfrak{q}$ ranges over all prime ideals of A of height 1 that do not contain a, and where $[\cdot]$ denotes the class in $G(C)$.

In the case $A = \Lambda$, C is the category of finite A-modules, and equality in $G(C)$ implies an equality of the orders of the groups involved. Continuing Kato's argument, let $\mathfrak{p}$ be the augmentation ideal of Λ. The $\mathfrak{p}$ is principal; let a be a generator. One has an exact sequence

$$0 \to H^1_{Iw}(T_l(t))/aH^1_{Iw}(T_l(t)) \to H^1(\mathbb{Z}[\frac{1}{l}], T_l(t)) \to {}_aH^2_{Iw}(T_l(t))$$

and the isomorphism

$$H^2_{Iw}(T_l(t))/aH^2_{Iw}(T_l(t)) \cong H^2(\mathbb{Z}[\frac{1}{l}], T_l(t)).$$

Assume for the moment that the hypotheses of Lemma 11.3.1 are met for either choice of $M =$

$H^2_{Iw}(T_l(t))$ or $M' = H^1_{Iw}(T_l(t))/Z(f,T)$. The operator σ_c acts on $V_l(t)$ by c^{-t}; it follows that

$$z^{(l)}_{\gamma}(t) \in aH^1_{Iw}(T_l(t)),$$

and the image of $z^{(l)}_{\delta+\gamma}(t)$ in $H^1_{Iw}(T_l(t))/aH^1_{Iw}(T_l(t))$ is equal to

$$z = \prod_{p|N}(1-a_pp^j)^{-1}z_1^{(l)}(f, k_1+1, -j, \mathrm{Id}\,, Nl).$$

According to the Main Conjecture, one has an equality in $G(C)$

$$[M/aM] - [{}_aM] = [M'/aM'] - [{}_aM'].$$

Under our hypotheses, z can not be $\mathbb{Z}_l$-torsion in $H^1(\mathbb{Z}[\frac{1}{l}], T_l(t))$, so there do not exist $w \in H^1_{Iw}(T_l(t))$, $b \in \Lambda - (a)$ such that $aw = bz^{(l)}_{\delta+\gamma}(t)$. As $H^1_{Iw}(T_l(t))$ is torsion-free, $[{}_aM'] = 0$. Forgetting about all but the orders of the groups, one has concretely

$$\#(H^2_{Iw}(T_l(t))/aH^2_{Iw}(T_l(t))) \cdot \#({}_aH^2_{Iw}(T_l(t)))^{-1} = [H^1_{Iw}(T_l(t)) : aH^1_{Iw}(T_l(t)) : z],$$

and so

$$\begin{aligned}\#(H^2(\mathbb{Z}[\frac{1}{l}], T_l(t))) &= \#(H^2_{Iw}(T_l(t))/aH^2_{Iw}(T_l(t))) \\ &= \#({}_aH^2_{Iw}(T_l(t)))[H^1_{Iw}(T_l(t))/aH^1_{Iw}(T_l(t)) : z] \\ &= [H^1(\mathbb{Z}[\frac{1}{l}], T_l(t)) : z].\end{aligned}$$

This is the desired finite level result. What does it mean that M and M' satisfy the hypothesis of Lemma 11.3.1? According to the Main Conjecture, M and M' have the same support. The dimension of the support of M is unchanged if M is replaced by some Tate twist, so twisting into Kato's range, M has support in codimension ≥ 1. It remains to require that $M_{\mathfrak{p}} = 0$ for any prime $\mathfrak{p}$ of height 1 that contains a. The only such prime is the augmentation ideal, for which asking that $M_{\mathfrak{p}} = 0$ amounts to requiring that $H^2(\mathbb{Z}[\frac{1}{l}], T_l(t))$ be finite. In summary,

Proposition 11.3.1. *Assume the Main Conjecture for f, and assume the Leopoldt-type hypothesis that*

$$H^2(\mathbb{Z}[\frac{1}{l}], T_l(t)) \otimes \mathbb{Q} = 0.$$

Then the order of $H^2(\mathbb{Z}[\frac{1}{l}], T_l(t))$ is equal to the index of z in $H^1(\mathbb{Z}[\frac{1}{l}], T_l(t))$. This is exactly the statement of Proposition 9.1.2.

This completes the proof of Theorem 4.1.1. Let us conclude with a few remarks on this Leopoldt-type hypothesis. There is an Euler system at the Iwasawa level, (whose elements were used above, and due to Kato) which proves that the H^2_{Iw} is a torsion Λ-module. One perhaps expects that the most natural way to prove the finiteness of $H^2(\mathbb{Z}[\frac{1}{l}], T_l(t))$ is to show that that elements comprising this Euler system are nontrivial when specialized to twist t. When t is in the critical range, Kato can relate these specializations to certain L-values, and thus prove nonvanishing. Here the interest has been to the right of the critical strip, where the specializations are related to values of a l-adic L-function ([Ka04], §16). And, unfortunately, the question of nonvanishing of l-adic L-functions is still very hard.

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