ON MAGNETOHYDRODYNAMIC FLOW OVER SOLIDS

Thesis by

M. C. Gourdine

In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1960

ACKNOWLEDGEMENTS

The author is grateful for the advice and encouragement offered by Professor J. Cole, thesis adviser, and Dr. R. Meghreblian. He thanks Professor H. Liepmann for originally stimulating his interest in this class of problems, and appreciates the help of Miss R. Winkel and Mrs. M. Pennington in the preparation of the manuscript.

The author acknowledges and appreciates the financial assistance provided by the following sources:

1. The Daniel and Florence Guggenheim Fellowship in Jet Propulsion, 1955-1957.
2. The Ramo-Wooldridge Fellowship, 1957-1958.
3. The Jet Propulsion Laboratory, California Institute of Technology, 1959-1960.

ABSTRACT

The steady flow of a viscous, incompressible and electrically conducting fluid over a solid, in the presence of an applied magnetic field parallel to the main flow, is considered. The equations of magnetohydrodynamics (MHD) are linearized by assuming that the solid only slightly perturbs the velocity and magnetic field. Fundamental solutions of the linearized equations are derived, and they are used to construct MHD flows over solids. The MHD drag formulas for the finite flat plate and the sphere are derived. The special cases of zero viscosity and infinite conductivity are studied, and general formulas for MHD forces on a solid are presented. The problem is generalized to include an electrical generator in the body.

Steady flow over a flat, circular, broadside-on disk in the presence of a parallel magnetic field is solved as a boundary value problem. The flow solution and drag formulas are valid for all values of the three parameters, Reynolds number, Magnetic Reynolds number, and Alfven number. The drag is calculated for large and small magnetic interaction; in the latter case the drag is proportional to the Alfven number. A special diffusion model applicable for large Hartmann number flows is also presented.

TABLE OF CONTENTS

LIST OF SYMBOLS

$\vec{U}$	velocity
$\vec{u}$	perturbation velocity (induced field)
$U_\infty \vec{i}$	velocity at infinity
$\vec{H}$	magnetic intensity
$\vec{h}$	perturbation magnetic intensity (induced field)
$H_\infty \vec{i}$	magnetic intensity at infinity
σ	conductivity
μ	permeability
ν	kinematic viscosity
ρ	density
$\vec{J}$	current density
$\vec{\Omega}$	vorticity
p	pressure
$\bar{p}$	<u>net</u> pressure
(L)	longitudinal field
(T)	transversal field
α	Alfvén number
$\vec{i}, \vec{j}, \vec{k}$	unit normal vectors parallel to (x,y,z)
Re	Reynolds number
Rm	Magnetic Reynolds number
$\lambda_{1,2}$	MHD Reynolds numbers
$K_{1,2}$	Magnetic mode strengths
Pr_m	Magnetic Prandtl number
$\vec{E}$	Electric intensity

SUMMARY

A general class of problems in magnetohydrodynamics (MHD) is considered: the steady flow of a viscous, incompressible and electrically conducting fluid over a solid, in the presence of an applied magnetic field parallel to the main flow. The equations of MHD are linearized by assuming that the solid only slightly perturbs the velocity and magnetic fields. There are only three independent dimensionless parameters in the equations: Reynolds number R_e, Magnetic Reynolds number R_m, and Alfvén number α.

A. General Solution

The general solution of the linearized equations is found by splitting the fields into their longitudinal (irrotational) and transverse (rotational) parts; the transverse fields split further into two modes. Both of these transverse modes satisfy equations of the Oseen type whose solutions are well known. Therefore, the general solutions are expressable in terms of Oseen's hydrodynamic solutions. The splitting of the transverse velocity mode into two parts is due to magnetic interaction; the extent of the interaction is determined by the MHD Reynolds numbers λ_1 and λ_2 which are combinations of Reynolds number, Magnetic Reynolds number and Alfvén number -

$$\lambda_1 = \frac{1}{\alpha}\left[(R_e + R_m) + \sqrt{(R_e + R_m)^2 - 4 R_e R_m (1-\alpha^2)}\right]$$

$$\lambda_2 = \frac{1}{\alpha}\left[(R_e + R_m) - \sqrt{(R_e + R_m)^2 - 4 R_e R_m (1-\alpha^2)}\right]$$

In each transverse mode there is a magnetic field proportional to the velocity field through the parameters K_1 and K_2, where

$$K_1 = \frac{1}{2 R_e \alpha^2}\left[(R_e - R_m) - \sqrt{(R_e + R_m)^2 - 4 R_e R_m (1-\alpha^2)}\right]$$

$$K_2 = \frac{1}{2 R_e \alpha^2}\left[(R_e - R_m) + \sqrt{(R_e + R_m)^2 - 4 R_e R_m (1-\alpha^2)}\right]$$

Thus, the general solution for the perturbation velocity and magnetic fields have the form:

$$\vec{u} = \sum_{n=0}^{2} U_n \vec{u}_n$$

$$\vec{h} = \sum_{n=0}^{2} K_n U_n \vec{u}_n$$

where $K_0 = 1$, and the mode strengths U_0, U_1 and U_2 must be determined by boundary conditions. The functions $\vec{u}_n$ $(n = 0, 1, 2)$ satisfy Oseen's equations

$$\nabla^2 \vec{u}_n - \lambda_n \frac{\partial \vec{u}_n}{\partial x} = 0, \qquad \nabla \cdot \vec{u}_n = 0$$

where $\lambda_0 = 0$. The two transverse modes represent parabolic wakes: the first mode wake is always downstream; the second mode wake is downstream when $\alpha < 1$, and upstream when $\alpha > 1$. The net pressure in the fluid--hydrostatic plus magnetic pressure--is proportional to the x-component of the longitudinal velocity field, that is,

$$\bar{p} = -(1-\alpha^2)\, u_{0_x}$$

B. Fundamental Solutions

The mode constants are evaluated for two types of singularities in the fluid: the velocity singularity, and the magnetic singularity. The velocity singularity generates velocity disturbances at a point in the fluid, but far from the singularity there is also a magnetic disturbance caused by MHD coupling. The mode strengths for the velocity singularity are:

$$U_1^{(u)} = \frac{R_e}{2\pi} \, \frac{K_2}{K_2 - K_1}$$

$$U_2^{(u)} = \frac{R_e}{2\pi} \, \frac{-K_1}{K_2 - K_1}$$

The magnetic singularity generates magnetic disturbances at a point in the fluid, but far from the singularity there is also a velocity disturbance caused by MHD coupling. The mode strengths for the magnetic singularity are:

$$U_1^{(h)} = \frac{R_e}{2\pi} \frac{-P_{r_m}}{K_2 - K_1}$$

$$U_2^{(h)} = \frac{R_e}{2\pi} \frac{P_{r_m}}{K_2 - K_1}$$

where $P_{r_m} = \sigma\mu\nu$ is the magnetic Prandtl number. The current density in the vicinity of a velocity singularity is zero, while the vorticity in the vicinity of a magnetic singularity is zero. Therefore, the boundary condition that current density must vanish on a body which does not carry on electrical generator is automatically satisfied by superimposing velocity singularities.

C. Infinite Conductivity and Zero Viscosity

Hasimoto (5) has studied the MHD equations in the limit of infinite conductivity. His solution indicates that a single wake exists which is upstream if $\alpha > 1$, and downstream if $\alpha < 1$. Passing formally to the limit $\sigma \rightarrow \infty$ in the fundamental solution shows that the first mode wake vanishes; and the second mode wake is upstream for $\alpha > 1$, and downstream for $\alpha < 1$ (figure 2). In this limiting case, a magnetic singularity has a large effect on the flow because the magnetic Prandtl number is large.

Lary (4) has studied the MHD equations in the limit of zero viscosity. He finds a single wake having the above mentioned property. Passing formally to the limit shows that the first mode wake degenerates to a singularity along the x-axis, (figure 1). It is the irrotational flow associated with this degenerate wake and the second mode wake that is responsible for the force on a solid: lift is due to potential vortices, and drag is due to potential sources, just as it is in hydrodynamics. In this limiting case a magnetic singularity has no effect on the flow because .

D. Flow Over Solids

As examples of the use of the fundamental solutions of MHD, flow over a finite flat plate and flow over a sphere are constructed. The procedure is analogous to that used in hydrodynamics, and requires the additional assumption that the MHD Reynolds numbers are small; that is, the magnetic interaction must be weak and the Reynolds number low.

The drag coefficient formula for the finite flat plate is:

$$C_D = C_{D_0}\left(1 + \frac{P_{r_m}}{(1-P_{r_m})^2}\,\alpha^2 + \cdots\right), \qquad P_{r_m} \neq 1$$

where C_{D_0} is the hydrodynamic drag coefficient. Therefore the correction to drag due to weak magnetic interaction is second order in the Alfven number, and usually negligible because $P_{r_m} \ll 1$ for laboratory flows.

The drag coefficient for the sphere is calculated up to terms of order λ, or terms of order Ha; the result is $C_D = C_{D_0}\left(1 + \frac{3}{8}K + \cdots\right)$ where

$$K = Re \qquad (\alpha < 1)$$

$$= \frac{2Ha^2 + Re^2 - Re\,Rm}{\sqrt{(Re - R_m)^2 + 4Ha^2}} \qquad (\alpha > 1)$$

$$= Ha \qquad (\alpha \to \infty;\ R_e, R_m \to 0)$$

These results are in agreement with those of Chester (7) and Ludford (9).

E. MHD Forces on Solids

Some general formulas for the forces on a solid, including those due to Maxwell stresses, are derived. The lift per unit length of a flat plate is

$$C_L = -\rho V_\infty \oint_c \vec{u}_0 \cdot d\vec{l} + \mu H_\infty \oint_c \vec{h}_0 \cdot d\vec{l}$$

where c is a large contour around the body. The drag per unit length of a flat plate is

$$C_D = -\rho V_\infty \iint_S \vec{u}_0 \cdot d\vec{S} + \mu H_\infty \iint_S \vec{h}_0 \cdot d\vec{S}$$

where S is a large surface enclosing the body. Since the longitudinal fields are parallel, the lift and drag coefficients are

$$c_L = -(1-\alpha^2)\oint_C \vec{u}_0 \cdot d\vec{l} = -(1-\alpha^2)\Gamma_0$$

$$c_D = -(1-\alpha^2)\oiint_S \vec{u}_0 \cdot d\vec{s} = -(1-\alpha^2) S_0$$

where Γ_0 is the circulation of the potential vortices, and S_0 is the strength of the potential sources. The forces on the body do not vanish as $\alpha \to 1$, because as $\alpha \to 1$ the strengths of the potential singularities become infinite like $(1-\alpha^2)^{-1}$.

Another formula for the drag of a solid in terms of vorticity and current density is presented:

$$D = \frac{1}{U_\infty}\left[\iiint_V \rho\nu\Omega^2\,dV + \iiint_V \frac{J^2}{\sigma}\,dV\right]$$

where V is the entire volume (infinite) of the fluid. This formula states that the rate at which mechanical energy is put into the fluid equals the rate of viscous and ohmic dissipation.

F. Some Comments on the Linearized Equations of MHD

Oseen's hydrodynamic solutions may be generalized to include the effects of a parallel magnetic field in a formal way. The Oseen linearization procedure yields a symmetric set of coupled equations for the perturbation velocity and magnetic fields. The symmetry in the equations goes deeper, as the formulas for lift and drag show. In this class of problems there seems to be an advantage in working with the full set of MHD equations, rather than part of the equations, because the symmetry in the equations helps to simplify the task of finding the solution.

G. MHD Flow over a Disk

In some cases it is more convenient to solve problems of this class as boundary value problems; this is true for the broadside-on disk. The flow solution is found in terms of an axi-symmetric stream function, and drag formulas are derived in terms of the stream function. The development of upstream and downstream wakes as the Alfven number goes from zero to infinity is clearly shown in the solution (figures 6 and 7).

The drag of the disk for small α is $D = 0.85D$, where D_s is Stokes' drag for a sphere of the same radius. There is no first order drag correction when α is less than unity. When α is large, the drag formula is $D = 2\pi\sqrt{Pr_m}\,\alpha$, where Pr_m is the magnetic Prandtl number.

A useful approximation for large Hartmann number flows is developed: Vorticity diffuses from the body like heat; x is analogous to time, and vorticity is analogous to temperature. This approach also yields the same expression for the drag of the disk for $Ha \to \infty$.

INTRODUCTION

The class of problems in which an incompressible, viscous and electrically conducting fluid flows steadily over a solid, in the presence of a parallel applied magnetic field, has been considered by several authors. Chester (7) considers low Reynolds number flow over a sphere with small Hartmann number, while Stewartson (8) considers infinite Reynolds number flow ($\nu \to 0$) with infinite Hartmann number ($Ha \to \infty$). Hasimoto (5) finds solutions of the MHD equations in the limit of infinite conductivity, while Lary (4) finds solutions in the limit of zero viscosity. The purpose of this thesis is to present a technique for constructing solutions for this class of problems which have no restriction on the value of the parameters.

The basic assumption underlying this technique is that the body only slightly perturbs the velocity and magnetic field; this allows the MHD equations to be linearized. No further assumptions are necessary. Criticisms of this assumption are well known, and are discussed. It is also assumed that the fluid is infinite in extent.

The class of problems considered here is generalized to include a solid which carries an electrical generator. In this case, current may flow in the body and induce fields in the fluid.

II. BASIC EQUATIONS AND BOUNDARY CONDITIONS

A. Basic Equations

The basic equations of MHD are well known for an incompressible, viscous fluid with a simple scalar conductivity (3). They are the equations of Navier-Stokes, Maxwell, and Ohm, and are summarized below:

$$(\vec{U}\cdot\nabla)\vec{U} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\vec{U} + \frac{1}{\rho}(\vec{J}\times\mu\vec{H}) \tag{1}$$

$$\nabla\cdot\vec{U} = 0 \tag{2}$$

$$\nabla\times\vec{E} = 0 \tag{3}$$

$$\nabla\times\vec{H} = \vec{J} \tag{4}$$

$$\nabla\cdot\vec{E} = 0 \tag{5}$$

$$\nabla\cdot\vec{H} = 0 \tag{6}$$

$$\vec{J} = \sigma(\vec{E} + \vec{U}\times\mu\vec{H}) \tag{7}$$

The flow is assumed to be steady, and without excess charge. Equations 3 and 5 show that the electric field is derivable from a potential which satisfies LaPlace's equation; that is,

$$\vec{E} = -\nabla\bar{\phi}\,, \qquad \nabla^2\bar{\phi} = 0 \tag{8}$$

Using Equation 4 in Equations 1 and 7 eliminates the current density. Taking the curl of Equation 7, and using certain well known vector identities yields the following equations for the velocity and magnetic fields:

$$(\vec{U}\cdot\nabla)\vec{U} - \nu\nabla^2\vec{U} - \frac{\mu}{\rho}(\vec{H}\cdot\nabla)\vec{H} = -\frac{1}{\rho}\nabla\bar{p} \tag{9a}$$

$$(\vec{U}\cdot\nabla)\vec{H} - \frac{1}{\sigma\mu}\nabla^2\vec{H} - (\vec{H}\cdot\nabla)\vec{U} = 0 \tag{9b}$$

$$\nabla\cdot\vec{U} = \nabla\cdot\vec{H} = 0 \tag{9c}$$

where $\bar{p}$ is the net pressure--the sum of magnetic and hydrostatic pressure -

$$\bar{p} = p + \mu\frac{H^2}{2} \tag{10}$$

It is convenient to introduce the following dimensionless variables: $\vec{U}' = \vec{U}/U_\infty$, $\vec{H}' = \vec{H}/H_\infty$, $\vec{E}' = \vec{E}/\mu H_\infty U_\infty$, $\bar{p}' = \bar{p}/\rho U_\infty^2$ where l is an arbitrary length. The resulting dimensionless equations are:*

*The primes are suppressed for convenience, dimensionless quantities are to be understood except when stated otherwise.

$$(\vec{U}\cdot\nabla)\vec{U} - \frac{1}{R_e}\nabla^2\vec{U} - \alpha^2(\vec{H}\cdot\nabla)\vec{H} = -\nabla\bar{p} \tag{11a}$$

$$(\vec{U}\cdot\nabla)\vec{H} - \frac{1}{R_m}\nabla^2\vec{H} - (\vec{H}\cdot\nabla)\vec{U} = 0 \tag{11b}$$

$$\nabla\cdot\vec{U} = \nabla\cdot\vec{H} = 0 \tag{11c}$$

$$\vec{E} = -\nabla\bar{\phi}, \qquad \nabla^2\bar{\phi} = 0 \tag{12}$$

There are three independent dimensionless parameters: (1) the Reynolds number $R_e = \frac{U_\infty l}{\nu}$, (2) the magnetic Reynolds number $R_m = \sigma\mu U_\infty l$, (3) the Alfvén number $\alpha = \left(\frac{\mu H_\infty^2}{\rho U_\infty^2}\right)^{1/2}$. Sometimes it is useful to replace one or more of these parameters by other related parameters such as the Hartmann number ($H_a = [R_e R_m \alpha^2]^{1/2}$) or the magnetic Prandtl number ($Pr_m = R_m/R_e = \sigma\mu\nu$).

B. Linearization

Equations 11 can be linearized by assuming that the body slightly perturbs the velocity and magnetic fields by the amounts $\vec{u}$ and $\vec{h}$, respectively. Then substituting $\vec{U} = \vec{i} + \vec{u}$ and $\vec{H} = \vec{i} + \vec{h}$ into Equations 11, and neglecting second order terms like $u_x \frac{\partial u_x}{\partial x}$, $u_y \frac{\partial u_y}{\partial y}$, etc., the following linearized equations of MHD are obtained:

$$\frac{\partial \vec{u}}{\partial x} - \frac{1}{R_e} \nabla^2 \vec{u} - \alpha^2 \frac{\partial \vec{h}}{\partial x} = - \nabla p \tag{13a}$$

$$\frac{\partial \vec{h}}{\partial x} - \frac{1}{R_m} \nabla^2 \vec{h} - \frac{\partial \vec{u}}{\partial x} = 0 \tag{13b}$$

$$\nabla \cdot \vec{u} = \nabla \cdot \vec{h} = 0 \tag{13c}$$

This is the linearization procedure used by Oseen in ordinary hydrodynamic problems of this type (ref. 2). The main criticism of this procedure is that the convection terms $(\vec{U} \cdot \nabla)\vec{U}$ and $(\vec{U} \cdot \nabla)\vec{H}$ actually vanish at the body because $\vec{U}$ vanishes; however, the terms $\frac{\partial \vec{u}}{\partial x}$ and $\frac{\partial \vec{h}}{\partial x}$ retained in this approximation do not vanish at the body. Therefore the flow field in the vicinity of the body is not expected to be correct; however the fields computed far from the body should closely approximate the physical situation. There can be no criticism of this linearization procedure for the operator $(\vec{H} \cdot \nabla)$ except when the conductivity is very large. In this extreme case the flow lines and the magnetic lines are almost frozen, hence the magnetic field also vanishes at the body. The perturbation in both the fields is therefore 100%, and the linearization assumptions are violated; nevertheless, the solutions are still qualitatively correct, and for small R_e and R_m they are also quantitatively correct.*

---

*For a more detailed discussion of this linearization procedure see reference 2.

C. Boundary Conditions

At infinity, $U = U_\infty \vec{i}$, $H = H_\infty \vec{i}$ and $p = p_\infty$. The electric field $\vec{E}$ and potential $\vec{\phi}$ is zero at infinity. Assuming there is no excess charge in the fluid or on the body,* then the electric potential satisfies Laplace's equation and must be zero everywhere. Hence, the electric field is identically zero.

At the body, the velocity vanishes due to the "no-slip" condition. However, the magnetic field penetrates the body and is continuous at the body since there are no surface currents. In general, the solution for the magnetic field inside the body must be matched at the body surface with the solution for the magnetic field in the fluid. This complicated procedure may be avoided by solving for the current density instead. The current density $\vec{J}$ satisfies the simple boundary condition $\vec{J} = 0$ at the body, and it is shown in the Appendix that this boundary condition defines the solution uniquely.

It is also interesting to consider a body which contains an electrical generator that induces currents in itself. In this case the electric field is not identically zero, but it must be continuous at the interface. The current density at the interface is $\vec{J} = \sigma \vec{E}$, and it is discontinuous if the conductivity of the body differs from that of the fluid. The power output of the generator is eventually dissipated as heat in the fluid.

---

*It can be shown from Maxwell's equations that the excess charge in a medium of conductivity σ and permittivity ϵ dies out exponentially with the time constant ϵ/σ --a very small duration for good conductors.

III. SOLUTION OF THE LINEARIZED EQUATIONS

The general solution of Equations 13 can be written in terms of three modes: one longitudinal, and two transverse. The magnetic and velocity fields are parallel in each mode.

A. Field Splitting

The perturbation fields may be split into two parts: the longitudinal part (L) is irrotational, and the transverse part (T) is rotational. That is,

$$\vec{u} = \vec{u}^L + \vec{u}^T \tag{14a}$$

$$\vec{h} = \vec{h}^L + \vec{H}^T \tag{14b}$$

where,

$$\vec{u}^L = -\nabla\phi, \qquad \nabla^2\phi = 0 \tag{15a}$$

$$\vec{h}^L = -\nabla\varphi, \qquad \nabla^2\varphi = 0 \tag{15b}$$

Substitution of these equations into Equations 13 shows that the following is a particular solution for the longitudinal fields;

$$\vec{u}^L = \vec{h}^L = \vec{u_0} \tag{16a}$$

$$\bar{p} = -u_x^L + \alpha^2 h_x^L = -(1-\alpha^2)u_{0_x} \tag{16b}$$

The transverse fields satisfy the following homogeneous equations:

$$\frac{\partial \vec{u}^T}{\partial x} - \frac{1}{R_e} \nabla^2 \vec{u}^T - \alpha^2 \frac{\partial \vec{h}^T}{\partial x} = 0 \tag{17a}$$

$$\frac{\partial \vec{h}^T}{\partial x} - \frac{1}{R_m} \nabla^2 \vec{h}^T - \frac{\partial \vec{u}^T}{\partial x} = 0 \tag{17b}$$

$$\nabla \cdot \vec{u}^T = \nabla \cdot \vec{h}^T = 0 \tag{17c}$$

By substituting $\vec{h}^T = K \vec{u}^T$ into Equations 17a and 17b it can be shown that the transverse fields split into two independent modes as follows:*

$$R_e (1 - K\alpha^2) \frac{\partial \vec{u}^T}{\partial x} - \nabla^2 \vec{u}^T = 0 \tag{18a}$$

$$R_m \left(1 - \frac{1}{K}\right) \frac{\partial \vec{u}^T}{\partial x} - \nabla^2 \vec{u}^T = 0 \tag{18b}$$

This substitution is valid only if,

$$R_e (1 - K\alpha^2) = R_m \left(1 - \frac{1}{K}\right) \tag{19}$$

or, when K has the following two values:

$$K_1 = \frac{1}{2 R_e \alpha^2} \left[(R_e - R_m) - \sqrt{(R_e + R_m)^2 - 4 R_e R_m (1 - \alpha^2)}\right] \tag{20a}$$

$$K_2 = \frac{1}{2 R_e \alpha^2} \left[(R_e - R_m) + \sqrt{(R_e + R_m)^2 - 4 R_e R_m (1 - \alpha^2)}\right] \tag{20b}$$

---

*See Appendix.

The corresponding values of the coefficients in Equations 18a and 18b are:

$$\lambda_1 = \frac{1}{2}\left[(R_e + R_m) + \sqrt{(R_e + R_m)^2 - 4R_e R_m (1-\alpha^2)}\right] \tag{21a}$$

$$\lambda_2 = \frac{1}{2}\left[(R_e + R_m) - \sqrt{(R_e + R_m)^2 - 4R_e R_m (1-\alpha^2)}\right] \tag{21b}$$

Hence, the transverse fields may be written as the sum of two modes,

$$\vec{u}^T = U_1 \vec{u}_1 + U_2 \vec{u}_2 \tag{22a}$$

$$\vec{h}^T = K_1 U_1 \vec{u}_1 + K_2 U_2 \vec{u}_2 \tag{22b}$$

and the total perturbation fields can be written as sums of three modes,

$$\vec{u} = \sum_{n=0}^{2} U_n \vec{u}_n \tag{23a}$$

$$\vec{h} = \sum_{n=0}^{2} K_n U_n \vec{u}_n \tag{23b}$$

The constant U_n is the strength of the nth velocity mode, and $K_n U_n$ is the strength of the nth magnetic mode, where $K_0 = 1$. The nth velocity mode satisfies the following equations:

$$\nabla^2 \vec{u}_n = \lambda_n \frac{\partial \vec{u}_n}{\partial x} \tag{24a}$$

$$\nabla \cdot \vec{u}_n = 0 \tag{24b}$$

where $\lambda_0 = 0$.

B. MHD Reynolds Numbers

The λ 's may be thought of as generalized Reynolds numbers, or as MHD Reynolds numbers, since they are combinations of R_e and R_m . The parameter $\alpha^2 = \mu H_\infty^2 / \rho U_\infty^2$ is the ratio of magnetic to dynamic pressure. When this quantity is larger than unity, $\alpha^2 > 1$, it is convenient to introduce the Hartmann number $Ha^2 = \frac{\mu^2 H_\infty^2 l^2}{\rho \nu}$.*

C. Generalization of Oseen's Solutions to MHD

Solutions of 24 were obtained by Oseen in connection with perturbation flow over bodies in ordinary hydrodynamics. His solution for a singular flat plate with unit drag ($\vec{f} = -\vec{i}$) is,**

$$\vec{u} = U\left[\frac{1}{\lambda} \nabla(\chi - \phi) - \chi \vec{i} \right] \tag{25}$$

where, $U = R_e/2\pi$ and $\lambda = R_e$. The function, $\phi(r) = \ln r$, is the potential of a two-dimensional source; it is the longitudinal part of the solution, and is chosen such that the flow is divergence free near the origin. The Oseen function in two-dimensions is, $\chi = e^{\lambda x/2} K_0(\lambda r/2)$; where, K_0 is the modified Bessel function of the second kind. At large distances from the origin, χ has the asymptotic behavior, $\chi \approx e^{-\lambda(r-x)/2} \Big/ \left(\frac{\lambda r}{2}\right)^{1/2}$. Near the origin $\chi \approx \ln r$.

---

*See, for example, Reference 3, page 14.

**See Reference 2.

Therefore, χ behaves like a potential source near the origin; but for large positive x, it is essentially zero outside of a parabolic region bounded by $y = c\left(\frac{x}{\lambda}\right)^{1/2}$. The strength of χ inside the parabola vanishes slowly, like $\left(\frac{1}{x}\right)^{1/2}$. However, for negative x, χ vanishes exponentially. Thus, Oseen's solution contains a parabolic wake extending downstream from the body.

According to Equations 24 there are two wakes in this problem. If $\lambda_2 > 0$, there is a second wake downstream; and if $\lambda_2 < 0$, there is a second wake upstream. Equation 21 shows that $\lambda_2 < 0$, if $\alpha > 1$; and $\lambda_2 > 0$, if $\alpha < 1$. The mode strengths U_1 and U_2 are determined by the boundary conditions.

IV. FUNDAMENTAL SOLUTIONS OF MHD

Let $u_{ij}(x, y, z; \xi, \eta, \zeta)$ be the ith component of velocity at $P(x, y, z)$ due to the jth component of a unit singularity at $Q(\xi, \eta, \zeta)$, then the velocity at $P(x, y, z)$ due to a distribution of singularities of strength $S_j(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta$ is given by the fundamental integral, (tensor notation)

$$u_i(x, y, z) = \iiint_{-\infty}^{\infty} u_{ij}(x, y, z; \xi, \eta, \zeta)\, S_j(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta \tag{26}$$

where u_{ij} is the fundamental solution.

In MHD there are two types of singularities and two corresponding fundamental solutions: (1) the velocity singularity, or singular body, (2) the magnetic singularity, or singular current element. The fundamental solutions corresponding to these singularities are denoted by $\overset{(u)}{u}_{ij}$ and $\overset{(h)}{u}_{ij}$, respectively. The tensor $\overset{(u)}{u}_{ij}$ is the solution of Equations 17 with the singular driving function $\delta(r)\vec{f}$ on the right hand side of Equation 17a; while $\overset{(h)}{u}_{ij}$ is the solution of Equations 17 with $\delta(r)\vec{g}$ on the right hand side of Equation 17b. Where $\vec{f}$ and $\vec{g}$ are unit vectors, and $\delta(r)$ is the Dirac delta function.

A. Velocity Singularities

The physical meaning of the velocity singularity $\delta(r)\vec{f}$ is obvious; it is an infinite body-force at the origin and equals the rate of change of momentum of the fluid per unit volume i.e. $\vec{f} = \frac{1}{R_e}\left(\frac{\partial \vec{u}}{\partial t}\right)_{r=0}$. Integration of this body-force over a small finite volume surrounding the origin yields, $\vec{F} = \iiint_V \delta(r)\vec{f}\,dV = \vec{f}$. Therefore, this singular body-force generates a flow which, far from the origin, approximates the flow that would be generated by a finite body exerting the same net force $\vec{F}$ on the fluid. Due to the coupling of the fields there is also an induced magnetic field associated with the velocity singularity.

As an example of how Oseen's solutions can be used to express fundamental solutions of MHD, consider the singular MHD flate plate with unit drag ($\vec{f}=-\vec{i}$). Equations 23 and 25 show that the solution is

$$\vec{u} = U_1^{(u)}\left[\frac{1}{\lambda_1}\nabla(\chi_1-\phi)-\chi_1\vec{i}\right]+U_2^{(u)}\left[\frac{1}{\lambda_2}\nabla(\chi_2-\phi)-\chi_2\vec{i}\right] \quad (27a)$$

$$\vec{h} = K_1U_1^{(u)}\left[\frac{1}{\lambda_1}\nabla(\chi_1-\phi)-\chi_1\vec{i}\right]+K_2U_2^{(u)}\left[\frac{1}{\lambda_2}\nabla(\chi_2-\phi)-\chi_2\vec{i}\right] \quad (27b)$$

The mode strengths $U_1^{(u)}$ and $U_2^{(u)}$ are found as follows: substitute these solutions into Equations 17, integrate the equations over V', then let $V' \to 0$. The only non-zero contributions come from the Laplacian terms in the x-components of the equations; the result is,

$$\frac{2\pi}{R_e} U_1^{(u)} + \frac{2\pi}{R_e} U_2^{(u)} = 1 \tag{28a}$$

$$\frac{2\pi}{R_m} K_1 U_1^{(u)} + \frac{2\pi}{R_m} K_2 U_2^{(u)} = 0 \tag{28b}$$

Now solve for $U_1^{(u)}$ and $U_2^{(u)}$ to get,

$$U_1^{(u)} = \frac{R_e}{2\pi} \frac{K_2}{K_2 - K_1} \tag{29a}$$

$$U_2^{(u)} = \frac{R_e}{2\pi} \frac{-K_1}{K_2 - K_1} \tag{29b}$$

In a similar manner, Oseen's solution for a singular flat plate with unit lift ($\vec{f} = -\vec{j}$) can be used to write the solution for a singular MHD flat plate with unit lift.

$$\vec{u} = \frac{U_1^{(u)}}{\lambda_1} \nabla \times \left[(\chi_1 - \phi)\vec{k}\right] + \frac{U_2^{(u)}}{\lambda_2} \nabla \times \left[(\chi_2 - \phi)\vec{k}\right] \tag{30a}$$

$$\vec{h} = \frac{K_1 U_1^{(u)}}{\lambda_1} \nabla \times \left[(\chi_1 - \phi)\vec{k}\right] + \frac{K_2 U_2^{(u)}}{\lambda_2} \nabla \times \left[(\chi_2 - \phi)\vec{k}\right] \tag{30b}$$

Again, substituting these equations into Equations 17 and performing the same limiting process, Equations 29 are obtained. This time the non-zero contributions come from the laplacian terms in the y-components of the equations.

Generalization of this technique to three-dimensions is straightforward. The only change in Equations 27 is that 2π is replaced by 4π. For example, Oseen's solution for the singular needle with unit drag immediately gives the following solution for the singular MHD needle with unit drag:

$$\vec{u} = U_1^{(u)} \left\{ \frac{1}{\lambda_1} \nabla \left[\frac{e^{-\lambda_1 (r-x)/2} - 1}{r} \right] - \left[\frac{e^{-\lambda_1 (r-x)/2}}{r} \right] \vec{i} \right\}$$

$$+ U_2^{(u)} \left\{ \frac{1}{\lambda_2} \nabla \left[\frac{e^{-\lambda_2 (r-x)/2} - 1}{r} \right] - \left[\frac{e^{-\lambda_2 (r-x)/2}}{r} \right] \vec{i} \right\} \qquad (31a)$$

$$\vec{h} = K_1 U_1^{(u)} \left\{ \frac{1}{\lambda_1} \nabla \left[\frac{e^{-\lambda_1 (r-x)/2} - 1}{r} \right] - \left[\frac{e^{-\lambda_1 (r-x)/2}}{r} \right] \vec{i} \right\}$$

$$+ K_2 U_2^{(u)} \left\{ \frac{1}{\lambda_2} \nabla \left[\frac{e^{-\lambda_2 (r-x)/2} - 1}{r} \right] - \left[\frac{e^{-\lambda_2 (r-x)/2}}{r} \right] \vec{i} \right\} \qquad (31b)$$

where

$$U_1^{(u)} = \frac{Re}{4\pi} \; \frac{K_2}{K_2 - K_1} \qquad (32a)$$

$$U_2^{(u)} = \frac{Re}{4\pi} \; \frac{-K_1}{K_2 - K_1} \qquad (32b)$$

There is one additional rule which must be remembered when employing this technique; that is, as α becomes greater than one and λ_2 becomes negative, the sign of x and the sign of λ_2 in the exponent must be changed. This procedure assures that the second mode represents an upstream wake and vanishes at infinity.

The fundamental solutions due to velocity singularities have one other important property: at the singularity the current density vanishes, therefore the current density is zero on finite bodies made up of these singularities. This statement can be proven in each case by considering $\vec{J} = \text{curl}\,\vec{h}$, letting $\not{x} \to \phi$ and then using Equation 27b.

B. Magnetic Singularities

The physical meaning of the magnetic singularity $\delta(r)\vec{g}$ is also obvious; it is the rate at which magnetic lines of force are being produced at the origin, i.e.

$\vec{g} = \frac{1}{R_m}\left(\frac{\partial \vec{h}}{\partial t}\right)_{r=0}$. By analogy with the velocity singularity, an integration of $\delta(r)\vec{g}$ over a small finite volume V' yields the vector $\vec{G} = \iiint_{V'} \delta(r)\vec{g}\,dV = \vec{g}$. This means that the magnetic singularity is a singular current element that generates a magnetic field which, far from the origin, approximates the field that would be generated by a distribution of current over the finite volume V' having the same vector $\vec{G}$. But what is $\vec{G}$? Consider Equation 11b which states that

$$\vec{G} = \iiint_{V'} \delta(r)\vec{g}\, dV = \iiint_{V'} [(\vec{U}\cdot\nabla)\vec{H} - \frac{1}{R_m}\nabla^2\vec{H} - (\vec{H}\cdot\nabla)\vec{H}]\, dV$$

$$= \iiint_{V'} [\frac{1}{R_m}\nabla\times\vec{J} - \nabla\times(\vec{U}\times\vec{H})]\, dV$$

$$= \iiint_{V'} \nabla\times[\vec{J} - R_m(\vec{U}\times\vec{H})]\, dV$$

$$= \frac{1}{R_m}\iiint_{V'} \text{curl}\,\vec{E}\, dV$$

But according to Maxwell's equation, curl $\vec{E} = -\frac{\partial\vec{H}}{\partial t}$; therefore,

$$\vec{G} = -\frac{1}{R_m}\iiint_{V'} \frac{\partial\vec{H}}{\partial t}\delta(r)\, dV = \frac{1}{R_m}\left(\frac{\partial\vec{H}}{\partial t}\right)_{r=0} = \vec{g}$$

This is analogous to the force $\vec{F}$ equaling the rate of change of momentum of the fluid, or

$$\vec{F} = \frac{1}{R_e}\iiint_{V'} \frac{\partial\vec{U}}{\partial t}\delta(r)\, dV = \frac{1}{R_e}\left(\frac{\partial\vec{U}}{\partial t}\right)_{r=0} = \vec{f}$$

The mode strengths $U_1^{(h)}$ and $U_2^{(h)}$ are determined in the same way as before; and it can be shown that they satisfy the following equations:

$$\frac{2\pi}{R_e}U_1^{(h)} + \frac{2\pi}{R_e}U_2^{(h)} = 0 \tag{33a}$$

$$\frac{2\pi}{R_e}U_1^{(h)} + \frac{2\pi}{R_e}U_2^{(h)} = 1 \tag{33b}$$

The solution is

$$U_1^{(h)} = \frac{Re}{2\pi} \quad \frac{-Pr_m}{K_2 - K_1} \tag{34a}$$

$$U_2^{(h)} = \frac{Re}{2\pi} \quad \frac{Pr_m}{K_2 - K_1} \tag{34b}$$

where $Pr_m = \sigma\mu\nu = Rm/Re$ is the magnetic Prandtl number. Hence, if $Pr_m \ll 1$, current elements in the body have very little effect on the flow.

The magnetic singularity has the important property that vorticity vanishes at the singularity. Again, this can be shown by considering $\vec{\Omega} = \text{curl}\,\vec{u}$, letting $\chi \to \phi$, and using Equation 35a.

It has already been mentioned that the laplacian terms balance the singular body-force; this means that

$\frac{1}{Re} \nabla^2 \vec{u} = -\frac{1}{Re}\,\text{curl}\,\vec{\Omega}$ is singular at the origin. In the case of the magnetic singularity, again the laplacian terms balance the singular driving force; therefore the terms

$\frac{1}{Rm} \nabla^2 \vec{h} = -\frac{1}{Rm}\,\text{curl}\,\vec{J} = -\frac{1}{Rm}\,\text{curl}\,\vec{E}$ are singular. It may be concluded that the velocity singularity generates vorticity and transverse velocity fields directly, and through coupling of the fields it also generates current density and transverse magnetic fields; however, at the singularity the current density is zero. The magnetic singularity generates current density and transverse magnetic fields directly, and through coupling of the fields it also generates vorticity and transverse velocity fields; however at the singularity the vorticity is zero.

V. ON THE ROLE OF CONDUCTIVITY AND VISCOSITY IN MHD

Sears and Lary (4) have investigated these equations in the limit of zero viscosity. They report the existence of an upstream wake-but no downstream wake--when $\alpha > 1$. Hasimoto (5) also finds the same phenomena in his study of the equations for the special case of infinite conductivity. These two extreme cases are also considered here as limits of the fundamental solutions. It is shown that when $\alpha > 1$, and either resistivity or viscosity vanishes, the downstream wake vanishes. Actually, in the case of vanishing viscosity, the downstream wake becomes very intense but is confined to the +x-axis.

A. Fundamental Solutions in the Limit of Infinte Conductivity

Consider a velocity singularity in the limit of infinite conductivity. As conductivity tends to infinity;

$\lambda_1 \to R_m \to \infty$, $\lambda_2 \to -R_e(1-\alpha^2)$, $K_1 \to -\frac{R_m}{R_e}\frac{1}{\alpha^2} \to -\infty$, $K_2 \to 1$,
$U_1^{(u)} \to 0$, $U_2^{(u)} \to \frac{R_e}{2\pi}$, $U_1^{(u)}/\lambda_1 \to 0$, and $U_2^{(u)}/\lambda_2 \to -\frac{1}{2\pi(1-\alpha^2)}$.

Therefore, the perturbation fields for the singular flat plate with drag ($\vec{f} = -\vec{i}$) are:

$$\vec{u} = \vec{h} = -\frac{1}{2\pi(1-\alpha^2)}\nabla(\chi_2 - \phi) - \frac{R_e}{2\pi}\chi_2\vec{i} \tag{36}$$

$$\Omega\vec{k} = J\vec{k} = -\frac{R_e}{2\pi}\nabla\times(\chi_2\vec{i}) \tag{37}$$

The perturbation fields for the lifting singular flat plate $(\vec{f} = -\vec{j})$ are:

$$\vec{u} = \vec{h} = -\frac{1}{2\pi(1-\alpha^2)} \nabla \times \left[(\chi_2 - \phi)\vec{k} \right] \tag{38}$$

$$\Omega\vec{k} = J\vec{k} = \frac{1}{2\pi(1-\alpha^2)} \nabla^2 \left[(\chi_2 - \phi)\vec{k} \right] \tag{39}$$

Figure 1 shows the location and relative sizes of the wakes.

The net pressure in the lift and drag case, respectively, is

$$\bar{p} = -\frac{1}{2\pi} \frac{\partial \phi}{\partial x} \tag{40}$$

$$\bar{p} = -\frac{1}{2\pi} \frac{\partial \phi}{\partial y} \tag{41}$$

Since the effect of a magnetic singularity is proportional to R_m it has a large effect on the mode strengths. As $R_m \to \infty$, $U_1^{(h)} \to \frac{Re\alpha^2}{2\pi}$ and $U_2^{(h)} \to -\frac{Re\alpha^2}{2\pi}$; therefore, for example, with $\alpha = 1$ a magnetic singularity can entirely cancel out the second mode of a corresponding velocity singularity.

B. Comparison with the Work of Hasimoto

The present results are in agreement with those of Hasimoto (5). Actually, Hasimoto's work is more general because it deals with a compressible fluid. Notice that for

$\sigma \to \infty$ the entire first mode vanishes, leaving only the second mode in which the fields are "frozen" together (parallel). It is interesting to note that in this special case the magnetic field vanishes at a finite body because the velocity vanishes; and the vorticity vanishes at the body because the current density vanishes.

C. Fundamental Solutions in the Limit of Zero Viscosity

Consider a velocity singularity in the limit of zero viscosity. As viscosity tends to zero; $\lambda_1 \to R_e \to \infty$, $\lambda_2 \to -R_m(1-\alpha^2)$, $K_1 \to -\sigma\mu\nu \to 0$, $K_2 \to 1/\alpha^2$, $U_1^{(u)} \to \frac{R_e}{2\pi} \to \infty$, $U_2^{(u)} \to \frac{R_m \alpha^2}{2\pi}$, $\frac{U_1^{(u)}}{\lambda_1} \to \frac{1}{2\pi}$, and $\frac{U_2^{(u)}}{\lambda_2} \to -\frac{\alpha^2}{2\pi(1-\alpha^2)}$.

The perturbation fields, for the singular flat plate with drag $(\vec{f} = -\vec{i})$, are: (except on + x axis)

$$\vec{u} = -\frac{1}{2\pi}\nabla\phi - \frac{\alpha^2}{2\pi(1-\alpha^2)}\nabla(\chi_2 - \phi) - \frac{R_m \alpha^2}{2\pi}\chi_2 \vec{i} \tag{42}$$

$$\vec{h} = -\frac{1}{2\pi(1-\alpha^2)}\nabla(\chi_2 - \phi) - \frac{R_m}{2\pi}\chi_2 \vec{i} \tag{43}$$

The vorticity and current density, is given by,

$$\Omega \vec{k} = \nabla \times \vec{u} = -\frac{R_m \alpha^2}{2\pi}\nabla \times (\chi_2 \vec{i}) \tag{44}$$

$$J\vec{k} = \nabla \times \vec{h} = -\frac{R_m}{2\pi}\nabla \times (\chi_2 \vec{i}) \tag{45}$$

$$J = \frac{1}{\alpha^2}\Omega \tag{46}$$

Note that the irrotational fields are parallel as they should be. In general, the current density and vorticity are proportional in each mode. However, in this limiting case, the first mode degenerates to a singularity along the + x-axis (see Figure 2).

It is also important to note that for $\nu \to 0$ a magnetic singularity has no effect on the mode strengths because $P_{r_m} = 0$.

In the same limit, the perturbation fields for the lifting singular flat plate ($\vec{f} = -\vec{j}$) are:

$$\vec{u} = -\frac{1}{2\pi} \nabla \times (\phi \vec{k}) - \frac{\alpha^2}{2\pi(1-\alpha^2)} \nabla \times [(\chi_2 - \phi)\vec{k}] \tag{47}$$

$$\vec{h} = \qquad - \frac{1}{2\pi(1-\alpha^2)} \nabla \times [(\chi_2 - \phi)\vec{k}] \tag{48}$$

$$\Omega \vec{k} = \frac{\alpha^2}{2\pi(1-\alpha^2)} \nabla^2 (\chi_2 \vec{k}) \tag{49}$$

$$J\vec{k} = \frac{1}{2\pi(1-\alpha^2)} \nabla^2 (\chi_2 \vec{k}) \tag{50}$$

$$J = \frac{1}{\alpha^2} \Omega \tag{51}$$

Note that none of these equations are singular at $\alpha = 1$; because as $\alpha \to 1$, $\lambda_2 \to 0$, $\chi_2 \to \phi$, and $\nabla^2 \phi = 0$.

The <u>net</u> pressure in the lift and drag case, respectively, is,

$$\bar{p} = -\frac{1}{2\pi}\frac{\partial\phi}{\partial x} \tag{52}$$

$$\bar{p} = -\frac{1}{2\pi}\frac{\partial\phi}{\partial y} \tag{53}$$

It is interesting to note that these are the same expressions obtained by Oseen for fluid pressure in ordinary hydrodynamics.

D. Comparison with the Work of Lary

This analysis shows that Lary, by neglecting viscosity in his initial formulation of the problem, obtains only the second transverse mode as a solution. Since the "no slip" condition on a finite body does not apply for an inviscid fluid; the body cannot have viscous drag. However, a finite body can have lift, and the lift is due to the potential vortices which must be added to satisfy boundary conditions.

Lary neglects viscosity for laboratory type problems on the grounds that viscous dissipation is only important in a "boundary layer" that is much thinner than the so-called, "magnetic boundary layer." He uses the following concept of the "magnetic Prandtl number":

$$P_{r_m} = \frac{(\text{viscous diffusion length})^2}{(\text{magnetic diffusion length})^2} = \sigma\mu\nu \tag{54}$$

where, $\sigma\mu\nu$ depends only on fluid properties, and is usually a very small number. According to this analysis, the ratio of the square of the wake sizes, is,

$$\frac{(\text{wake size of mode 1})^2}{(\text{wake size of mode 2})^2} = \sqrt{1-\alpha^2/\sigma\mu\nu} \qquad (55)$$

Thus, the first mode wake is usually small compared to the second mode wake, <u>except when α^2 is very large</u>. It is important to note that in each wake current density and vorticity exist together; in fact, due to coupling they produce each other.

The present results agree with Lary except in one respect. Lary gives for the fluid pressure,

$$p = -u_x^{(L)} \qquad (56)$$

This analysis shows that the <u>net</u> pressure is, $\bar{p} = -u_x^{(L)} + \alpha^2 h_x^{(L)}$, where the <u>net</u> pressure is defined by, $\bar{p} \equiv p + \alpha^2 h_x$. Therefore, the fluid pressure is,

$$p = -u_x^{(L)} - h_x^{(T)} \qquad (57)$$

Since the force on a finite body equals the integral of the <u>net</u> stresses (including Maxwell stresses) over the surface; it is convenient to work with <u>net</u> pressure rather than fluid pressure. It is important to remember that the Maxwell stresses on the body contribute nothing to the force unless there are currents within the body.

VI. SOME SIMPLE FLOW CONSTRUCTIONS WITH FUNDAMENTAL SOLUTIONS

Finding the distribution of magnetic and velocity singularities that satisfies the boundary conditions on a solid requires solving an integral equation whose kernel is quite complicated. In hydrodynamics the kernel can be simplified if the Reynolds number is small. In MHD the kernel can also be simplified if both MHD Reynolds numbers are small. As examples of MHD flow constructions with the fundamental solutions the following problems are solved: (1) drag of a finite flat plate, (2) drag of a sphere.

A. MHD Drag of a Finite Flat Plate

Let $f(\xi)\,d\xi$ be the strength of the velocity singularity in the range $\xi < x < \xi + d\xi$, and let $Mf(\xi)\,d\xi$ be the strength of the magnetic singularity in the same range. This is a special case in which the distribution of magnetic singularities is assumed to be proportional to the distribution of velocity singularities in order to demonstrate in a simple way the effects of currents in the body. The fundamental solution for this case can be written immediately in terms of Oseen's solution, that is

$$u_{ij} = U_1 \Gamma_{ij}(\lambda_1, x, y) + U_2 \Gamma_{ij}(\lambda_2, x, y) \tag{58}$$

where Oseen's solution for the flat plate with drag is,

$$\Gamma_{ij}(\lambda x, y) = \frac{1}{\lambda}\nabla\left[e^{\lambda x/2} K_0(\lambda r/2) - \ln r\right] - e^{\lambda x/2} K_0(\lambda r/2)\,\vec{i} \qquad (59)$$

and the mode strengths are given by

$$U_1 = U_1^{(w)} + M U_1^{(h)} \qquad (60a)$$

$$U_2 = U_2^{(w)} + M U_2^{(h)} \qquad (60b)$$

The distribution function $f(\xi)$ is such that the x-component of velocity on the plate vanishes; that is,

$$u_x(x,0) = -1 = \int_{-1}^{1}\left\{U_1\Gamma_{11}[\lambda_1(x-\xi),0] + U_2\Gamma_{11}[\lambda_2(x-\xi),0]\right\} f(\xi)\,d\xi \qquad (61)$$

On the plate, the argument $\lambda(x-\xi) < \lambda < 1$, therefore

$$\Gamma_{11}[\lambda(x-\xi),0] \simeq 1 - \ln\frac{\gamma_0\lambda|x-\xi|}{4} \qquad (62)$$

where $\gamma_0 = e^{\gamma}$ and $\gamma = .577$ (Euler's constant). Substitution of this into Equation 61 yields

$$-1 = \int_{-1}^{1}\left\{U_1[c_1 - \ln|x-\xi|] + U_2[c_2 - \ln|x-\xi|]\right\} f(\xi)\,d\xi \qquad (63)$$

where

$$c_1 = 1 - \ln\frac{\gamma_0\lambda_1}{4} \qquad (64a)$$

$$C_2 = 1 - \ln \frac{\gamma_0 \lambda_2}{4} \tag{64b}$$

The solution of Equation 81 is*

$$f(\xi) = \frac{1}{\pi} \frac{1}{U_1(C_1 + \ln 2) + U_2(C_2 + \ln 2)} \frac{1}{\sqrt{1-\xi^2}} \tag{65}$$

The drag coefficient per unit length is therefore,

$$C_D = \frac{D}{\frac{1}{2}\rho U_\infty^2 (2a)} = \int_{-1}^{1} f(\xi)\, d\xi \tag{66a}$$

$$= \frac{1}{U_1(C_1 + \ln 2) + U_2(C_2 + \ln 2)} \tag{66b}$$

$$= \frac{2\pi}{Re} \frac{1}{C + \ln 2} \tag{66c}$$

where

$$C = \frac{C_1 U_1 + C_2 U_2}{U_1 + U_2} \tag{66d}$$

In the limit of zero magnetic interaction C_D approaches the classical hydrodynamic value,**

$$C_{D_0} = \frac{2\pi}{Re} \frac{1}{1 - \ln \frac{\gamma_0 Re}{8}} \tag{67}$$

---

*See, for example, Reference 6.

**See, for example, Reference 2.

where $R_e = \frac{U_\infty a}{\nu}$.

If the magnetic interaction is small, and $Ha^2 \ll (R_e - R_m)^2$; then

$$\lambda_1 \approx R_e + \frac{Ha^2}{|R_e - R_m|} , \qquad R_e \neq R_m \tag{68a}$$

$$\lambda_2 \approx R_m - \frac{Ha^2}{|R_e - R_m|} \tag{68b}$$

Substituting this into Equation 66b and expanding the logarithms gives,

$$C_D = \frac{2\pi}{R_e} \quad \frac{U_1 + U_2}{U_1\left(1 - \ln\frac{\gamma_0 R_e}{8} - \frac{Ha^2}{R_e|R_e - R_m|}\right) + U_2\left(1 - \ln\frac{\gamma_0 R_m}{8} + \frac{Ha^2}{R_m|R_e - R_m|}\right)} \tag{69}$$

If, furthermore, $M = 0$ and $U_2/U_1 < 1$,

$$C_D \approx \frac{2\pi}{R_e} \quad \frac{1}{1 - \ln\frac{\gamma_0 R_e}{8}} \quad \left(1 + \frac{U_2^{(u)}}{U_1^{(u)}} + \dots\right) \tag{70a}$$

$$\approx C_{D_0}\left(1 - \frac{K_1}{K_2} + \dots\right) \tag{70b}$$

$$\approx C_{D_0}\left(1 + \frac{P_{r_m}}{(1 - P_{r_m})^2}\alpha^2 + \dots\right), \qquad P_{r_m} \neq 1 \tag{70c}$$

Consider the effect of the magnetic singularities on the mode strengths U_1 and U_2 .

$$U_1 = \frac{2\pi}{Re} \frac{K_2 - M Pr_m}{K_2 - K_1} \tag{71a}$$

$$U_2 = \frac{2\pi}{Re} \frac{-K_1 + M Pr_m}{K_2 - K_1} \tag{71b}$$

The effect of a positive M is to decrease U_1 and increase U_2, and vice versa for a negative M. In fact if $M Pr_m = K_2$ the first mode is cancelled out entirely, and if $M Pr_m = K_1$, the second mode is cancelled out. In the former case, with only the second mode remaining,

$$c_D \to \frac{2\pi}{Re} \frac{1}{c_2 + \ln 2} = \frac{2\pi}{Re} \frac{1}{1 - \ln \frac{r_0 R_m}{8}} \tag{72}$$

and in the latter case, with only the first mode remaining,

$$c_D \to \frac{2\pi}{Re} \frac{1}{c_1 + \ln 2} = \frac{2\pi}{Re} \frac{1}{1 - \ln \frac{r_0 Re}{8}} \tag{73}$$

Notice, that the effect of currents in the body is proportional to the magnetic Prandt number Pr_m.

B. MHD Drag of a Sphere

In ordinary hydrodynamics, low Reynolds number flow over a sphere is constructed approximately by superimposing a uniform flow, the flow of a singular needle, and the flow of a potential dipole;* therefore, it is reasonable

---

*See, for example, Reference 2.

to expect that low MHD Reynolds number flow over a sphere can be approximated by replacing the singular needle by a singular MHD needle as follows:

$$\vec{U} = \vec{i} + U_1 \left\{ -\frac{1}{\lambda_1} \nabla \left[\frac{e^{-\lambda_1 (r-x)/2} - 1}{r} \right] + \frac{e^{-\lambda_1 (r-x)/2}}{r} \vec{i} \right\}$$
$$+ U_2 \left\{ -\frac{1}{\lambda_2} \nabla \left[\frac{e^{-\lambda_2 (r-x)/2} - 1}{r} \right] + \frac{e^{-\lambda_2 (r-x)/2}}{r} \vec{i} \right\}$$
$$+ U_3 \nabla \left[\frac{\partial}{\partial x} \left(\frac{1}{r} \right) \right] \tag{74}$$

where the ratio of mode strengths for the singular MHD needle is

$$\frac{U_2}{U_1} = -\frac{K_1}{K_2} \tag{75}$$

and $\lambda_2 > 0$ because $\alpha < 1$. The mode strengths U_1 and U_2 are determined by the boundary condition on the sphere which demands that $\vec{U} = 0$.

Following the procedure used in ordinary hydrodynamics, an expansion of the exponential terms in Equation 74 yields,

$$\vec{U} \approx \vec{i} + U_1 \left\{ \left[\frac{x}{2r^2} - \frac{\lambda_1}{8} + \frac{\lambda_1}{8} \frac{x^2}{r^2} \right] \nabla r + \left[\frac{1}{2r} - \frac{\lambda_1}{4} + \frac{\lambda_1}{4} \frac{x}{r} \right] \vec{i} \right\}$$
$$- \frac{K_1}{K_2} U_1 \left\{ \left[\frac{x}{2r^2} - \frac{\lambda_2}{8} + \frac{\lambda_2}{8} \frac{x^2}{r^2} \right] \nabla r + \left[\frac{1}{2r} - \frac{\lambda_2}{4} + \frac{\lambda_2}{4} \frac{x}{r} \right] \vec{i} \right\}$$
$$+ U_3 \left\{ \frac{3x}{r^4} \nabla r - \frac{1}{r^3} \vec{i} \right\} \tag{76}$$

The y-component of the velocity on the sphere (r = 1) must vanish, therefore

$$0 \approx U_1 \left\{ \left[\frac{x}{2} - \frac{\lambda_1}{8} + \frac{\lambda_1}{8} x^2 \right] y \right\}$$

$$- \frac{K_1}{K_2} U_1 \left\{ \left[\frac{x}{2} - \frac{\lambda_2}{8} + \frac{\lambda_2}{8} x^2 \right] y \right\}$$

$$+ U_3 \left\{ 3xy \right\} \qquad (77)$$

Matching odd terms in x yields,

$$U_3 \approx -\frac{1}{6}\left(1 - \frac{K_1}{K_2}\right) U_1 \qquad (78)$$

The z-component is zero by symmetry. The x-component of velocity must also vanish at the sphere, therefore

$$-1 = U_1 \left\{ \left[\frac{x}{2} - \frac{\lambda_1}{8} + \frac{\lambda_1}{8} x^2 \right] x + \left[\frac{1}{2} - \frac{\lambda_1}{4} + \frac{\lambda_1}{4} x \right] \right\}$$

$$- \frac{K_1}{K_2} U_1 \left\{ \left[\frac{x}{2} - \frac{\lambda_2}{8} + \frac{\lambda_2}{8} x^2 \right] x + \left[\frac{1}{2} - \frac{\lambda_2}{4} + \frac{\lambda_2}{4} x \right] \right\}$$

$$- \frac{1}{6}\left(1 - \frac{K_1}{K_2}\right) U_1 \left\{ 3x^2 - 1 \right\} \qquad (79)$$

Matching the even terms in x, and solving for U_1 gives*

*This is the same procedure followed in hydrodynamics.

$$U_1 \approx -\frac{3}{2}\,\frac{K_1}{K_2-K_1}\,\frac{1}{1-\frac{3}{8}\left(\frac{K_2\lambda_2-K_1\lambda_1}{K_2-K_1}\right)} \tag{80a}$$

$$U_3 \approx \frac{1}{4}\,\frac{1}{1-\frac{3}{8}\left(\frac{K_2\lambda_2-K_1\lambda_1}{K_2-K_1}\right)} \tag{80b}$$

where, from Equations 42 and 43,

$$\frac{K_2\lambda_2-K_1\lambda_1}{K_2-K_1} = R_e \tag{80c}$$

Therefore,

$$\vec{U} = \vec{i} - \frac{3}{2}\left(\frac{4\pi}{R_e}\right)\left(1+\frac{3}{8}R_e\right)\vec{u}^{(1)} + \frac{1}{4}\left(1+\frac{3}{8}R_e\right)\vec{u}^{(2)} \tag{81}$$

where $\vec{u}^{(1)}$ is the flow due to a singular MHD needle of unit strength,

$$\begin{aligned}\vec{u}^{(1)} = {} & \frac{R_e}{4\pi}\,\frac{K_2}{K_2-K_1}\left\{-\frac{1}{\lambda_1}\nabla\left[\frac{e^{-\lambda_1(r-x)/2}-1}{r}\right]+\left[\frac{e^{-\lambda_1(r-x)/2}}{r}\right]\vec{i}\right\} \\ & -\frac{R_e}{4\pi}\,\frac{K_1}{K_2-K_1}\left\{-\frac{1}{\lambda_2}\nabla\left[\frac{e^{-\lambda_2(r-x)/2}-1}{r}\right]+\left[\frac{e^{-\lambda_2(r-x)/2}}{r}\right]\vec{i}\right\}\end{aligned} \tag{82}$$

and $\vec{u}^{(2)}$ is the longitudinal flow due to a potential dipole of unit strength,

$$\vec{u}^{(2)} = \nabla\left[\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\right] \tag{83}$$

The drag of the sphere is due to a singular MHD needle of strength $\frac{3}{2}\left(\frac{4\pi}{R_e}\right)\left(1+\frac{3}{8}R_e\right)$, therefore the drag is

$$D = \left(\rho U_\infty^2 a^2\right)\frac{3}{2}\left(\frac{4\pi}{R_e}\right)\left(1+\frac{3}{8}R_e\right)$$

$$= 6\pi\rho\nu U_\infty a\left(1+\frac{3}{8}R_e\right) \tag{84}$$

This is precisely the same drag obtained without magnetic interaction. It appears that, up to terms of order λ, the magnetic interaction has no effect on drag.

It is easy to obtain the solution for $\alpha > 1$ simply by changing the sign of x and λ_2 in equation 92 and repeating the above procedure to obtain

$$U_1 = -\frac{3}{2}\frac{K_1}{K_2-K_1}\frac{1}{1-\frac{3}{8}\left(\frac{-K_2\lambda_2-K_1\lambda_1}{K_2-K_1}\right)} \tag{85a}$$

$$U_3 = \frac{1}{4}\frac{1}{1-\frac{3}{8}\left(\frac{-K_2\lambda_2-K_1\lambda_1}{K_2-K_1}\right)} \tag{85b}$$

where

$$-\frac{K_2\lambda_2-K_1\lambda_1}{K_2-K_1} = \frac{2Ha^2+R_e^2-R_eR_m}{\sqrt{(R_e-R_m)^2+4Ha^2}} \equiv \overline{K} \tag{85c}$$

Now the second mode wake is upstream and the drag is

$$D = 6\pi\rho\nu U_\infty a\left(1+\frac{3}{8}\overline{K}\right) \tag{86}$$

provided that Ha, Re, Rm are all less than unity. In the limit $Re, Rm \rightarrow 0$ with $\alpha \rightarrow \infty$, such that $Ha^2 = Re\,Rm\,\alpha^2$ is finite but less than unity, Chester's (7) result is obtained:

$$D = 6\pi\rho\nu U_\infty a \left(1 + \frac{3}{8} Ha\right) \tag{87}$$

VII. MAGNETOHYDRODYNAMIC FORCES ON A SOLID

A body in a MHD flow is subjected to Maxwell stresses as well as ordinary fluid stresses. The forces on the body depend on the applied magnetic field as well as the applied velocity field. The following derivations yield approximate expressions for MHD forces on a body in terms of the fields at large distances from the body.

A. Momentum Balance

Consider the situation shown in figure 3. For steady flow, the momentum law for the fluid in volume V states: (in cartesian tensor rotation)

$$0 = \iiint_V \frac{\partial}{\partial t} \rho U_i \, dV = \iint_\Sigma \sigma_{ij} n_j \, dS - \iint_S \sigma_{ij} n_j \, dS - \iint_\Sigma \rho U_i U_j n_j \, dS + \iiint_V f_i^m \, dV \tag{88}$$

The volume integral of the Lorentz body-force f_i^m can be written as a surface integral of Maxwell stresses:

$$\iiint_V f_i^m \, dV = \iint_\Sigma \sigma_{ij}^m n_j \, dS - \iint_S \sigma_{ij}^m n_j \, dS \tag{89}$$

The net force on the body is, therefore,

$$F_i = \iint_S (\sigma_{ij} + \sigma_{ij}^m) n_j \, dS = \iint_\Sigma (\sigma_{ij} + \sigma_{ij}^m - \rho U_i U_j) n_j \, dS \tag{90}$$

Where, the fluid stress tensor is given by

$$\sigma_{ij} = \tau_{ij} - \delta_{ij}\, p, \qquad (i, j = x, y, z) \tag{91}$$

and the Maxwell stress tensor by

$$\begin{aligned} \sigma_{ij}^{m} &= \mu H_i H_j - \delta_{ij}\, p^{m} \\ p^{m} &= \frac{\mu H^2}{2} \end{aligned} \tag{92}$$

It is important to note that the contribution of Maxwell stresses to the net force on the body is zero unless there are currents in the body; i.e.

$$\iint_S \sigma_{ij}^{m} n_j\, ds = \iiint_{V'} f_i^{m}\, dV = \iiint_{V'} (\bar{J} \times \bar{B})_i\, dV = 0 \tag{93}$$

because the Lorentz force in the body (V') is identically zero.

B. Two Dimensional Drag Formula

The drag of a two-dimensional body is defined by

$$\begin{aligned} D = F_x &= \iint_{\Sigma} (\sigma_{xj} + \sigma_{xj}^{m} - \rho V_x V_j)\, n_j\, dS \\ &= \iint_{\Sigma} \tau_{xj}\, n_j\, ds + \iint_{\Sigma} -(p + p^{m})\, \delta_{ij}\, n_j\, ds \\ &\quad + \iint_{\Sigma} \mu (H_\infty + h_x)\, H_j\, n_j\, ds - \iint_{\Sigma} \rho (V_\infty + u_x)\, V_j\, n_j\, ds \end{aligned} \tag{94}$$

For large Σ we can neglect the contribution of shear stresses τ_{ij}. Also, since the fields have zero divergence,

$$\mu H_\infty \iint_\Sigma H_j n_j \, ds = \rho V_\infty \iint_\Sigma V_j n_j \, ds = 0 \tag{95}$$

Then, neglecting squares of the perturbation quantities, we get

$$D = \iint_\Sigma -(p+p^m) n_x \, ds + \iint_\Sigma \mu H_\infty h_x n_x \, ds - \iint_\Sigma \rho V_\infty u_x n_x \, ds \tag{96}$$

Finally, noting that $p^m = \mu H_\infty h_x^L$ and $p = -\rho V_\infty u_x^L$; we have,

$$D = \iint_\Sigma \left[-\rho V_\infty u_x^T + \mu H_\infty h_x^T \right] n_x \, ds \tag{97}$$

Since the transverse fields are confined to parabolic wakes, the only contribution to these surface integrals are found in planes normal to the x-axis; say, at $x = \pm x_1$, where x_1 is large; thus,

$$D = \iint_{-\infty\,(x=x_1)}^{\infty} \left[-\rho V_\infty u_x^T + \mu H_\infty h_x^T \right] dy\,dz - \iint_{-\infty\,(x=-x_1)}^{\infty} \left[-\rho V_\infty u_x^T + \mu H_\infty h_x^T \right] dy\,dz \tag{98}$$

C. Two Dimensional Lift Formula

The lift of a two-dimensional body is defined by

$$L = F_y = \iint_\Sigma (\sigma_{yj} + \sigma_{yj}^m - \rho V_y V_j) n_j \, ds \tag{99}$$

Expanding $\sigma_{yj} n_j$, $\sigma^m_{yj} n_j$, and $V_y V_j n_j$, it can be shown that,

$$L = \iint_A [\tau_{xy} + \mu H_x H_y - \rho V_x V_y] dy\, dz + \iint_{C_2} [-(\tau_{yy} - p) - (\mu H_y H_y - p^m) - \rho V_y V_y] dx\, dz$$

$$+ \iint_B [\tau_{xy} + \mu H_x H_y - \rho V_x V_y] dy\, dz + \iint_{C_1} [-(\tau_{yy} - p) - (\mu H_y H_y - p^m) - \rho V_y V_y] dx\, dz \qquad (100)$$

Where the surfaces A, B, C_1 and C_2 are shown in Figure 4. If we make the same assumptions that we made for the drag formula, the result is

$$L = \iint_A [\mu H_\infty h_y - \rho V_\infty u_y] dy\, dz + \iint_{C_2} (p + p^m) dx\, dz$$

$$+ \iint_B [\mu H_\infty h_y - \rho V_\infty u_y] dy\, dz + \iint_{C_2} (p + p^m) dx\, dz \qquad (101)$$

Substituting for p and p^m, yields

$$L = \int_{-\infty}^{\infty} dz \Big\{ \int_A (\mu H_\infty h_y - \rho V_\infty u_y) dy + \int_{C_2} (\mu H_\infty h_x - \rho V_\infty u_x) dx$$

$$+ \int_B (\mu H_\infty h_y - \rho V_\infty u_y) dy + \int_{C_1} (\mu H_\infty h_x - \rho V_\infty u_x) dx \qquad (102)$$

Note that we have added $\mu H_\infty h_x^T$ and $-\rho V_\infty u_x^T$ to the integrals along C_1 and C_2. This is justified since h_x^T and u_x^T are both zero on C_1 and C_2 if the contour is large.

Hence,

$$L = -\rho V_\infty \int_{-\infty}^{\infty} \Gamma\, dz + \mu H_\infty \int_{-\infty}^{\infty} \Gamma^m\, dz \qquad (103)$$

where Γ is the velocity circulation and Γ^m is the magnetic circulation defined by

$$\Gamma = \oint_C u_i \, dl_i = \iint_S \Omega \, dx \, dy \tag{104}$$

$$\Gamma^m = \oint_C h_i \, dl_i = \iint_S J \, dx \, dy \tag{105}$$

and C is any large contour encircling the body; while S is the surface area enclosed by C.

D. Generalization to Three-Dimensions

For further details on the assumptions used in the derivation of these formulas see reference 2 chapter 7. This reference also shows that generalization to three dimensions is straightforward. In general the force formula may be written

$$F_i = \iint_\Sigma \left[-\delta_{ij} (p + p^m) + \mu H_i H_j - \rho V_i V_j \right] n_j \, dS \tag{106}$$

The drag formula for three-dimensions is the same as the one for two-dimensions (98). The lift formula for three-dimensions is somewhat more complicated. The net force normal to the drag is given by,

$$\vec{F} - D\vec{i} = -\rho V_\infty (\vec{i} \times \vec{\Omega}) + \mu H_\infty (\vec{i} \times \vec{J}) \tag{107}$$

E. Drag Formula in Terms of Vorticity and Current Density

The rate at which energy is dissipated in the fluid by viscous and ohmic heating must equal the rate at which energy is put into the fluid;

$$\iiint_V \rho\nu\Omega^2 dV + \iiint_V \frac{J^2}{\sigma} dV = DU_\infty + \iint_{S'} (\vec{E}\times\vec{H})\cdot d\vec{S} \tag{108}$$

where DU_∞ is the rate at which the solid delivers mechanical energy to the fluid, and $\iint_{S'}(\vec{E}\times\vec{H})\cdot d\vec{S}$ is the rate at which the body delivers electro-magnetic energy to the fluid. (S' is the surface area of the body, V is the volume of the fluid). Therefore, the drag is given by

$$D = \frac{1}{U_\infty}\left[\iiint_V \left(\rho\nu\Omega^2 + \frac{J^2}{\sigma}\right) dV - \iint_{S'} (\vec{E}\times\vec{H})\cdot d\vec{S}\right] \tag{109}$$

F. Contributions of the Fundamental Modes to Lift and Drag

The following formula for the lift force per unit length is obtained by introducing dimensionless notation:

$$L' \equiv L/\rho U_\infty^2 a^2 = -\oint_c \vec{u}\cdot d\vec{l} + \alpha^2 \oint_c \vec{h}\cdot d\vec{l} \tag{110}$$

where c is a large contour around the body. Since the transverse fields are confined to parabolas, their contribution to the circulation integrals can be neglected; so Equation 110, becomes,

$$L = -\oint_c \vec{u}^{\,i} \cdot d\vec{l} + \alpha^2 \oint_c \vec{h}^{\,i} \cdot d\vec{l} \tag{111}$$

But, $\oint_c \vec{u}^{\,i} \cdot d\vec{l}$ equals the strength of the potential vortex filament at the origin, $2\pi U_1/\lambda_1 + 2\pi U_2/\lambda_2$ (see Equation 30a). Therefore,

$$L = -2\pi (1-\alpha^2 K_1) U_1/\lambda_1 - 2\pi (1-\alpha^2 K_2) U_2/\lambda_2 \tag{112}$$

The following formula for drag per unit length is also derived by introducing dimensionless notation:

$$D' = D/\rho U_\infty^2 a^2 = (-1+\alpha^2 K_2) \int_{-\infty}^{\infty} u_x^T \, dy \Big|_{(x \to -\infty)} - (-1+\alpha^2 K_1) \int_{-\infty}^{\infty} u_x^T \, dy \Big|_{(x \to \infty)} \tag{113}$$

Figure 5 serves as a guide in evaluating these integrals. The transverse flow in the downstream wake is due to an Oseen sink at the origin which is just balanced by a potential source of strength $2\pi U_1/\lambda_1$ (see Equation 28a). The transverse flow in the upstream wake is due to an Oseen source at the origin which is just balanced by a potential sink of strength $2\pi U_2/\lambda_2$. Therefore, the drag formula can be written,

$$D = -2\pi (1-\alpha^2 K_1) U_1/\lambda_1 - 2\pi (1-\alpha^2 K_2) U_2/\lambda_2 \tag{114}$$

Substituting for U_1/λ_1 and U_2/λ_2 in 114 gives, after some algebra, L = D = -1, as it should. The first term in Equations 112 and 114 is the direct contribution from the first mode to the lift and drag of the singular flat plate. The second term is the contribution from the second mode. In the first mode, the relative contribution to lift and drag from Maxwell stresses is $\alpha^2 K_1$; in the second mode it is $\alpha^2 K_2$. In general, lift is due to potential vortices, and drag is due to potential sources, just as in ordinary hydrodynamics.

IX. CONCLUSIONS ABOUT THE FUNDAMENTAL SOLUTIONS OF MHD

The existence of an upstream wake when $\alpha > 1$ has been predicted before and is predicted again by this analysis (experimental evidence is still lacking). A downstream wake also exists when the viscosity and resistivity of the fluid are finite. The zero viscosity approximation of Lary is valid when $Pr_m = \sigma\mu\nu \ll 1$; in this case, the first mode wake (downstream) vanishes. The infinite conductivity approximation of Hasimoto is valid when $Pr_m = \sigma\mu\nu \gg 1$; in this case, the first mode wake (downstream) also vanishes. Hasimoto's case is of great academic interest, but of little interest experimentally; in the laboratory $\sigma\mu\nu \approx 10^{-6}$.

The approach of Lary should be useful for constructing low magnetic Prandtl number flows over bodies. Since the "no-slip" condition must be abandoned, only <u>one</u> boundary condition must be satisfied: the normal component of velocity must vanish on the body. This is sufficient to determine the potential flow that goes with the second mode wake. When the fluid has finite viscosity, there are <u>two</u> boundary conditions to be satisfied on a finite body: (1) the velocity must vanish on the body, (2) the current density must vanish on the body--a consequence of (1). Note that condition (2) is automatically satisfied by the singular flat plate solutions for lift and drag if there is no generator in the body ($\vec{g} = 0$).

The expressions for current density in the lift and drag cases are:

$$J\vec{k} = -K_1 \frac{U_1}{\lambda_1} \nabla^2(\chi_1 \vec{k}) - K_2 \frac{U_2}{\lambda_2} \nabla^2(\chi_2 \vec{k}) \tag{115}$$

$$J\vec{k} = -K_1 U_1 \nabla \times (\chi_1 \vec{i}) - K_2 U_2 \nabla \times (\chi_2 \vec{i}) \tag{116}$$

These vanish as r $\rightarrow$ 0 because $\chi_1 \rightarrow \chi_2 \rightarrow \phi$; and since $-K_1U_1 - K_2U_2 = 0$ (equation 49b), $J(0) = 0$. Thus, in constructing flows over a finite body, a distribution of singular bodies must be found such that the velocity vanishes on the body; the current density on the body will automatically be zero unless there is an electric generator in the body. In the Lary approximation, the current density on the body need not be zero even if there is no generator because the velocity does not vanish.

X. MHD FLOW OVER A DISK (BROADSIDE-ON)

Some MHD flow solutions in this class can be found more conveniently by treating the problem as a boundary value problem instead of using the fundamental solution approach. MHD flow over a disk is a good example.

A. History and Description of the Problem

Three authors have considered the problem of steady flow over a sphere in the presence of a magnetic field parallel to the main motion: (1) Chester (7) finds the perturbation on Stokes' flow over a sphere due to weak magnetic interaction; this is a special limiting case in which $Re \to \infty$, $R_m \to 0$, $\alpha^2 \to \infty$, and $Ha < 1$. (2) Ludford (9) finds the perturbation on Oseen's flow over a sphere due to weak magnetic interaction; this solution is valid when $Re < 1$, $R_m < 1$, $Ha < 1$, but α can be either less than or greater than unity. (3) Stewartson (8) gives a solution for the special case in which the applied magnetic field is infinite and the viscosity is zero; that is, $Re \to \infty$, $Ha \to \infty$, and $\alpha \to \infty$.

If the sphere is at rest at the center of a cylindrical coordinate system (x, ϖ, θ), then Chester's solution is symmetric about the plane $x = 0$, and his drag formula is $D = D_s\left(1 + \frac{3}{8} Ha\right)$; where $D_s = 6\pi\rho\nu U_\infty a$ --Stokes classical drag formula for the sphere. Ludford's solution is not symmetric about the plane $x = 0$; it contains the classical downstream parabolic wake of Oseen. However, in addition, there is a second parabolic wake due to magnetic interaction which is downstream when $\alpha < 1$ and upstream when $\alpha > 1$. Ludford's drag formula is $D = D_s(1 + \frac{3}{8}\overline{K})$, where

$$\overline{K} = R_e \qquad (\alpha < 1)$$

$$= \frac{2H_a^2 + R_e^2 - R_e R_m}{\sqrt{(R_e - R_m)^2 + 4H_a^2}} \qquad (\alpha > 1)$$

$$= H_a \qquad (\alpha \to \infty,\ R_e \to 0,\ R_m \to 0)$$

Stewartson's solution shows that an infinite magnetic field prevents radial motion of the fluid; therefore, the fluid in the cylindrical column upstream and downstream of the sphere remains at rest. Only the fluid outside this cylinder can move in the axial direction. The velocity at the edge of the cylinder is singular, but this is probably due to neglecting viscosity. No drag formula can be derived for the same reason. Thus, in the limit $\alpha \to \infty$ the two wakes become cylinders which are symmetric about the plane $x = 0$.

It would be useful to have an analytic solution valid over the entire range of the three parameters R_e, R_m and α. Ludford points out that such a solution exists for the sphere, but it is not obtainable without overcoming great mathematical difficulties. However, these mathematical difficulties are removed if a broadside-on, flat circular disk is considered instead of the sphere. Certainly in Stewartson's problem ($\alpha \to \infty$) the shape of the body is immaterial; and it is well-known that the shape of the body has a relatively small effect on Stokes' or Oseen's solutions. For example, Stokes' drag of a broadside-on disk is 0.85 that of a sphere having the same diameter. The qualitative features of the flow are certainly the same. Thus, MHD flow over a disk is worthy of consideration.

B. Solution

A solution of Equations 44 for those modes $n = 0,1,2$ may be obtained conveniently by expressing $\vec{u}_n$ and $\vec{h}_n$ in terms of the curl of vector potentials $\vec{P}_n$ and $\vec{Q}_n$; that is, let

$$\vec{u}_n = \text{curl}\, \vec{P}_n \tag{117a}$$

$$\vec{h}_n = \text{curl}\, \vec{Q}_n \tag{117b}$$

By considering the vorticity ($\vec{\Omega}_n = \text{curl}\, \vec{u}_n = \text{curl}^2 \vec{P}_n$) and the current density ($\vec{J}_n = \text{curl}\, \vec{h}_n = \text{curl}^2 \vec{Q}_n$), it is easy to show that $\vec{P}_n$ and $\vec{Q}_n$ like $\vec{\Omega}_n$ and $\vec{J}_n$ have only one component in the azimuthal direction ($\vec{k}$). This is a consequence of the axial symmetry of the problem. Substitution of Equation 117a into Equation 22 yields

$$\Delta^2 P_n - \lambda_n \frac{\partial P_n}{\partial x} = 0 \tag{118}$$

where the operators Δ^2 and ∇^2 are defined as follows:

$$\Delta^2 = \left(\frac{\partial}{\partial \varpi}\right)\left(\frac{1}{\varpi}\frac{\partial}{\partial \varpi}\varpi\right) + \frac{\partial^2}{\partial x^2} \tag{119a}$$

$$P^2 = \left(\frac{1}{\varpi}\frac{\partial}{\partial \varpi}\varpi\right)\left(\frac{\partial}{\partial \varpi}\right) + \frac{\partial^2}{\partial x^2} \tag{119b}$$

Thus, the total vector potentials are:

$$P = \sum_{n=0}^{2} P_n \tag{120a}$$

$$Q = \sum_{n=0}^{2}{}' Q_n \tag{120b}$$

The velocity vector potential P is related to Stokes' stream function ψ, because

$$u_x = \frac{1}{\varpi}\frac{\partial}{\partial \varpi}\varpi P \qquad u_\varpi = -\frac{\partial P}{\partial x} \tag{121a}$$

$$u_x = -\frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi} \qquad u_\varpi = \frac{1}{\varpi}\frac{\partial \psi}{\partial x} \tag{121b}$$

Therefore, $\psi = -\varpi P$.

The solution of Equation 118 may be found by taking the Hankel transform of order one to obtain

$$\frac{\partial^2 \tilde{P}_n}{\partial x^2} - \lambda_n \frac{\partial \tilde{P}_n}{\partial x} - m^2 \tilde{P}_n = 0 \tag{122}$$

where the transform is defined by

$$\tilde{P}_n(x, m) = \int_0^\infty P_n(x, \varpi)\, J_1(m\varpi)\, \varpi\, d\varpi \tag{123}$$

The solution of Equation 122 is

$$\tilde{P}_n(x, m) = A_n(m)\, e^{-k_n(m)x}, \qquad x > 0 \tag{124a}$$

$$= \bar{A}_n(m)\, e^{\bar{k}_n(m)x}, \qquad x < 0 \tag{124b}$$

This assures that $\tilde{P}_n$ and P_n vanish as $x \to \pm\infty$ --a sufficient condition which assures that $\vec{u}$ and $\vec{h}$ vanish as $x \to \pm\infty$. Since A_n and $\bar{A}_n$ are not necessarily equal, $\tilde{P}_n(0, m)$ remains undefined. The exponential damping coefficients k_n and $\bar{k}_n$ are given by

$$k_n(m) = \sqrt{(\lambda_n/2)^2 + m^2} - \lambda_n/2 \tag{125a}$$

$$\bar{k}_n(m) = \sqrt{(\lambda_n/2)^2 + m^2} + \lambda_n/2 \tag{125b}$$

Taking the inverse Hankel transform of first order of Equation 124 yields the solution

$$P(x, \varpi) = \int_0^\infty \sum_{n=0}^{2} A_n(m)\, e^{-k_n(m)x} J_1(m\varpi)\, m\, dm, \qquad x > 0 \tag{126a}$$

$$= \int_0^\infty \sum_{n=0}^{2} \bar{A}_n(m)\, e^{\bar{k}_n(m)x} J_1(m\varpi)\, m\, dm, \qquad x < 0 \tag{126b}$$

where the inverse transform is defined by

$$P_n(x, \varpi) = \int_0^\infty \tilde{P}_n(x, m)\, J_1(m\varpi)\, m\, dm \tag{127}$$

C. Boundary Conditions

There are three conditions to be satisfied at the disk: (1) the total axial component of velocity must vanish, (2) the total radial component of velocity must vanish, (3) the current density must vanish. These conditions are satisfied on the downstream side of the disk ($x=0+$) if,

$$u_x(0+,\varpi) = \left(\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\varpi P\right)_{x=0+} = -1 = \int_0^\infty \sum_{n=0}^{2} A_n J_0(m\varpi)\, m^2\, dm \tag{128a}$$

$$u_\varpi(0+,\varpi) = -\left(\frac{\partial P}{\partial x}\right)_{x=0+} = 0 = \int_0^\infty \sum_{n=0}^{2} k_n A_n J_1(m\varpi)\, m\, dm \tag{128b}$$

$$J(0+,\varpi) = -\left(\Delta^2 Q\right)_{x=0+} = 0 = \int_0^\infty \sum_{n=0}^{2} \lambda_n K_n k_n A_n J_1(m\varpi)\, m\, dm \tag{128c}$$

These conditions are also satisfied on the upstream side of the disk ($x=0-$) if,

$$u_x(0-,\varpi) = -1 = \int_0^\infty \sum_{n=0}^{2} \bar{A}_n J_0(m\varpi)\, m^2\, dm \tag{129a}$$

$$u_\varpi(0-,\varpi) = 0 = \int_0^\infty \sum_{n=0}^{2} -\bar{k}_n \bar{A}_n J_1(m\varpi)\, m\, dm \tag{129b}$$

$$J(0-,\varpi) = 0 = \int_0^\infty \sum_{n=0}^{2} -\lambda_n K_n \bar{k}_n \bar{A}_n J_1(m\varpi)\, m\, dm \tag{129c}$$

Equations 128 and 129 are all satisfied if*

*See Appendix for derivation.

$$\sum_{n=0}^{2} A_n = \sum_{n=0}^{2} \bar{A}_n = f(m) = -\frac{2}{\pi}\frac{\sin m}{m^3} \tag{130a}$$

$$\sum_{n=0}^{2} k_n A_n = \sum_{n=0}^{2} \bar{k}_n \bar{A}_n = 0 \tag{130b}$$

$$\sum_{n=0}^{2} \lambda_n K_n k_n A_n = \sum_{n=0}^{2} \lambda_n K_n \bar{k}_n \bar{A}_n = 0 \tag{130c}$$

Or, solving these equations simultaneously,

$$A_0 = f(m)\,\frac{k_1 k_2 (\lambda_2 K_2 - \lambda_1 K_1)}{d(m)} \tag{131a}$$

$$A_1 = -f(m)\,\frac{\lambda_2 K_2 k_2}{d(m)} \tag{131b}$$

$$A_2 = f(m)\,\frac{\lambda_1 K_1 k_1}{d(m)} \tag{131c}$$

and

$$\bar{A}_0 = f(m)\,\frac{\bar{k}_1 \bar{k}_2 (\lambda_2 K_2 - \lambda_1 K_1)}{\bar{d}(m)} \tag{132a}$$

$$\bar{A}_1 = -f(m)\,\frac{\lambda_2 K_2 \bar{k}_2}{\bar{d}(m)} \tag{132b}$$

$$\bar{A}_2 = f(m)\,\frac{\lambda_1 K_1 \bar{k}_1}{\bar{d}(m)} \tag{132c}$$

where

$$d(m) = m(\lambda_1 K_1 k_1 - \lambda_2 K_2 k_2) - k_1 k_2 (\lambda_1 K_1 - \lambda_2 K_2) \quad (133a)$$

$$\bar{d}(m) = m(\lambda_1 K_1 \bar{k}_1 - \lambda_2 K_2 \bar{k}_2) - \bar{k}_1 \bar{k}_2 (\lambda_1 K_1 - \lambda_2 K_2) \quad (133b)$$

These results imply that the radial velocity and current density vanish in the plane of the disk (x = 0). This is shown to be true in the Appendix where $f(m)$ is derived from matching conditions off the disk, and in the plane x = 0.

D. Physical Interpretation of the Solution

It can be shown that the fundamental solution of the equation

$$\Delta^2 P_n - \lambda_n \frac{\partial P_n}{\partial x} = 0 \quad ; \qquad n = 0, 1, 2 \quad (134)$$

is the velocity potential of a ring of simple sources of the Oseen type ($\lambda_n \neq 0$) at $\varpi = \varpi'$. The fundamental solution is*

$$P_n^{(f)}(x, \varpi, \varpi') = \int_0^\infty e^{-k_n(m)x} J_0(m\varpi') J_1(m\varpi)\, dm, \qquad x > 0 \quad (135)$$

The vector potential due to a source distribution $S_n(\varpi')$ over

---

*See, for example, Reference 10, page 138.

the downstream side of the disk ($x=0+$) is

$$P_n(x,\varpi) = \int_0^1 S_n(\varpi')\left[e^{-k_n(m)x} J_0(m\varpi')J_1(m\varpi)dm\right]2\pi\varpi'd\varpi' \qquad (136)$$
$$(x>0)$$

Upon interchanging the order of integration,

$$P_n(x,\varpi) = \int_0^\infty \left[\int_0^1 S_n(\varpi')J_0(m\varpi')2\pi\varpi'd\varpi'\right]e^{-k_n(m)x} J_1(m\varpi)dm \qquad (137)$$
$$(x>0)$$

Therefore, the mode strength $A_n(m)$ and the distribution function $S_n(\varpi')$ are related by the integral equation

$$mA_n(m) = \int_0^1 S_n(\varpi')J_0(m\varpi')2\pi\varpi'd\varpi' \qquad (138)$$

There is a different distribution of these singularities on the other side of the disk ($x=0-$), given by

$$m\bar{A}_n(m) = \int_0^1 \bar{S}_n(\varpi')J_0(m\varpi')2\pi\varpi'd\varpi' \qquad (139)$$

A distribution of simple sources on the disk cannot produce an axial velocity in the plane of the disk for $\varpi>1$.* They do produce a radial velocity in the plane of the disk for $\varpi>1$; however, the three types of singularities are distributed in such a way that the net radial velocity vanishes

---

*The small axial velocity in the plane x = 0 for $\varpi>1$ is due to higher order singularities. These are unimportant in our discussion since they do not affect the flow far from the disk, or the drag of the disk.

in the plane of the disk for $\bar{\omega} > 1$.

The perturbation flow downstream is due to Oseen sinks of type one (λ_1), Oseen sources of type two (λ_2), and potential sources ($\lambda_0 = 0$) distributed over the downstream side of the disk. Since the damping coefficients of the three modes are unequal ($k_1 < m < k_2$), the first mode persists farther downstream; it is responsible for the downstream wake (region of high vorticity) and its flow is opposed to the free stream flow. (See figure 7). This is true when $\alpha > 1$; but when $\alpha < 1$, $k_1 < k_2 < m$, and there is another wake downstream due to magnetic interaction. (See figure 6).

The perturbation flow upstream is due to Oseen sinks of type one (λ_1), Oseen sources of type two (λ_2), and potential sources ($\lambda_0 = 0$) distributed over the upstream side of the disk. For $\alpha > 1$ the second mode persists farther upstream because $\bar{k}_2 < m < \bar{k}_1$; it is responsible for the upstream wake, and its flow is opposed to the free stream flow. However for $\alpha < 1$, $m < \bar{k}_2 < \bar{k}_1$, and neither of the transverse modes persists far upstream; therefore there is no upstream wake for $\alpha < 1$. (See figures 6 & 7).

E. Drag Formulas

It is shown in section VII that the MHD drag of a solid can be written in terms of the transverse fields at infinity, i.e. (in dimensionless notation)

$$\frac{c}{D} = \lim_{\substack{x \to -\infty \\ \varpi \to \infty}} \int_0^{\varpi} (-u_x^T + \alpha^2 h_x^T)\, 2\pi \varpi \, d\varpi$$

$$- \lim_{\substack{x \to \infty \\ \varpi \to \infty}} \int_0^{\varpi} (-u_x^T + \alpha^2 h_x^T)\, 2\pi \varpi \, d\varpi$$

$$\frac{c}{D} = \lim_{\substack{x \to -\infty \\ \varpi \to \infty}} \left\{ -(1-\alpha^2 K_1) \int_0^{\varpi} u_{x_1} 2\pi \varpi \, d\varpi - (1-\alpha^2 K_2) \int_0^{\varpi} u_{x_2} 2\pi \varpi \, d\varpi \right\}$$

$$- \lim_{\substack{x \to \infty \\ \varpi \to \infty}} \left\{ -(1-\alpha^2 K_1) \int_0^{\varpi} u_{x_1} 2\pi \varpi \, d\varpi - (1-\alpha^2 K_2) \int_0^{\varpi} u_{x_2} 2\pi \varpi \, d\varpi \right\} \tag{140}$$

Since the flow at infinity comes from singularities on the disk, (figures 6 & 7)

$$\frac{c}{D} = \lim_{\substack{x \to 0- \\ \varpi \to 1}} \left\{ -(1-\alpha^2 K_1) \int_0^{\varpi} u_{x_1} 2\pi \varpi \, d\varpi - (1-\alpha^2 K_2) \int_0^{\varpi} u_{x_2} 2\pi \varpi \, d\varpi \right\}$$

$$- \lim_{\substack{x \to 0+ \\ \varpi \to 1}} \left\{ -(1-\alpha^2 K_1) \int_0^{\varpi} u_{x_1} 2\pi \varpi \, d\varpi - (1-\alpha^2 K_2) \int_0^{\varpi} u_{x_2} 2\pi \varpi \, d\varpi \right\} \tag{141}$$

But,

$$\int_0^{\varpi} u_{x_n} 2\pi \varpi \, d\varpi = \int_0^{\varpi} \frac{1}{\varpi} \frac{\partial}{\partial \varpi} \varpi P_n \, 2\pi \varpi \, d\varpi = 2\pi \varpi P_n(x, \varpi) \tag{142}$$

$$n = 0, 1, 2$$

Therefore,

$$C_D = 2\pi\left[-(1-\alpha^2K_1)P_1(0-,1) - (1-\alpha^2K_2)P_2(0-,1)\right]$$

$$-2\pi\left[-(1-\alpha^2K_1)P_1(0+,1) - (1-\alpha^2K_2)P_2(0+,1)\right] \tag{143}$$

And finally after substituting for P_1 and P_2,

$$C_D = 2\pi(1-\alpha^2K_1)\int_0^\infty [A_1(m) - \bar{A}_1(m)]J_1(m)\,m\,dm$$

$$+2\pi(1-\alpha^2K_2)\int_0^\infty [A_2(m) - \bar{A}_2(m)]J_1(m)\,m\,dm \tag{144}$$

Another formula for drag can be written by noting that the net force on the disk equals the integral of the net pressure over the two sides of the disk;* that is,

$$C_D = \int_0^1 \bar{p}(0+,\varpi)\,2\pi\varpi\,d\varpi - \int_0^1 \bar{p}(0-,\varpi)\,2\pi\varpi\,d\varpi \tag{145}$$

$$= \int_0^1 (1-\alpha^2)u_{x_0}(0+,\varpi)\,2\pi\varpi\,d\varpi - \int_0^1 (1-\alpha^2)u_{x_0}(0-,\varpi)\,2\pi\varpi\,d\varpi$$

$$C_D = 2\pi(1-\alpha^2)\left[P_0(0+,\varpi) - P_0(0-,\varpi)\right] \tag{146}$$

$$C_D = 2\pi(1-\alpha^2)\int_0^\infty \left[A_0(m) - \bar{A}_0(m)\right]J_1(m)\,m\,dm \tag{147}$$

---

*Shear stresses contribute nothing to the drag.

Equations 144 and 147 can be shown to be equivalent by direct use of Equations 131 and 132.

In principle, at least, the drag of the disk can be calculated from Equation 144 or 147 by integration; however, in general the integration cannot be performed analytically because of the complex form of the A_n's and $\bar{A}_n$'s. Nevertheless, in certain special cases, the A_n's and $\bar{A}_n$'s may be approximated by simple algebraic expressions and analytic integration becomes possible.

The following cases are considered in the Appendix and elsewhere, and the results are summarized below:

$\underline{\alpha = 0}$ a) $R_e << 1$, $C_D = 0.85\, C_{D_0} = 0.85\left(\frac{6\pi}{R_e}\right)$

b) $R_e >> 1$, $C_D = \pi$

$\underline{\alpha = 1}$ a) $R_e << 1$ $C_D \approx 0.85\, C_{D_0}$

b) $R_e >> 1$ $C_D \approx \pi$

$\underline{\alpha > 1}$ a) $R_e << 1$ $(H_a < 1)$ $C_D \approx 0.85\, C_{D_0}\left(1 + \frac{3}{8} H_a\right)$

b) $R_e >> 1$ $(H_a >> 1)$ $C_D \approx 2\pi \frac{H_a}{R_e} = 2\pi \sqrt{P_m}\, \alpha$

It is also interesting to note that the flow field far from the disk and the drag of the disk, are relatively insensitive to the function $f(m)$, or the axial velocity in the plane x = 0. For example, the same values of drag are obtained if the function $f(m) = -\frac{2}{\pi}\frac{\sin m}{m^3}$ is replaced by $f(m) = -\frac{J_1(m)}{m^2}$. This corresponds to approximating $u_x(0,\varpi)$ by a step function, and $\Omega(0,\varpi)$ by a delta function.

F. The Diffusion Approximation for $H_a \to \infty$

In general the vorticity and current density are given by

$$\Omega = \Omega_1 + \Omega_2 \tag{148}$$

$$J = K_1\Omega_1 + K_2\Omega_2 \tag{149}$$

where,

$$\Delta^2\Omega_1 - \lambda_1 \frac{\partial\Omega_1}{\partial x} = 0 \tag{150a}$$

$$\Delta^2\Omega_2 - \lambda_2 \frac{\partial\Omega_2}{\partial x} = 0 \tag{150b}$$

However, when $H_a \to \infty$, $\lambda_{1,2} \to \pm H_a$, and $K_{1,2} \to \mp H_a / R_e\,\alpha^2$. Equations 157 may then be approximated by

$$\left(\frac{\partial}{\partial\varpi}\right)\left(\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\varpi\right)\Omega_n - H_a\frac{\partial\Omega_n}{\partial x} = 0, \qquad n = 1, 2 \tag{151}$$

These are analogous to heat diffusion equations where Ω_n is temperature and x is time. The term $\frac{\partial^2}{\partial x^2}$ has been neglected on the grounds that x-derivatives must be small if the term $H_a\frac{\partial\Omega_n}{\partial x}$ is to be of order unity. Furthermore, since

$u_x(0,\varpi) = -1$ for $\varpi \leq 1$, and approximately zero for $\varpi > 1$, then

$$\Omega(0,\varpi) = \left(\frac{\partial u_\varpi}{\partial x}\right)_{x=0} - \left(\frac{\partial u_x}{\partial \varpi}\right)_{x=0} \approx -\delta(\varpi - 1) \qquad (152)$$

That is, the vorticity distribution in the plane x = 0 is a dirac delta function infinite at the edge of the disk, and Ω diffuses like heat away from the cylinder $\varpi = 1$ as $|x|$ increases. The current density is zero in the plane x = 0 because $K_1 = -K_2$.

The solution of Equations 151 are easily shown to be

$$\Omega_1(x,\varpi) = \int_0^\infty e^{-\frac{m^2}{Ha}x} J_1(m) J_1(m\varpi)\, m\, dm, \qquad x \geq 0 \qquad (153a)$$

$$= \int_0^\infty e^{\frac{m^2}{Ha}x} J_1(m) J_1(m\varpi)\, m\, dm, \qquad x \leq 0 \qquad (153b)$$

Or,*

$$\Omega(x,\varpi) = \frac{Ha}{2|x|} e^{-\frac{(\varpi^2+1)Ha}{4x}} I_1\left(\frac{\varpi Ha}{2|x|}\right) \qquad (154)$$

where I_1 is the modified Bessel function.

For $x < \frac{Ha}{2}$, and $\varpi \approx 1$, the asymptotic expansion for I_1 can be used, and

---

*See, for example, Reference 11, page 51.

$$\Omega(x, \varpi) \approx \frac{1}{2}\sqrt{\frac{Ha}{\pi\,\varpi\,|x|}}\; e^{-\frac{Ha}{4|x|}(\varpi-1)^2} \tag{155a}$$

$$J(x, \varpi) \approx -\frac{R_m}{2Ha}\sqrt{\frac{Ha}{\pi\,\varpi\,|x|}}\; e^{-\frac{Ha}{4|x|}(\varpi-1)^2} \qquad (x>0) \tag{155b}$$

$$\approx \frac{R_m}{2Ha}\sqrt{\frac{Ha}{\pi\,\varpi\,|x|}}\; e^{-\frac{Ha}{4|x|}(\varpi-1)^2} \qquad (x<0)$$

The drag of the disk in this special case can be calculated using Equation 109; in dimensionless notation,

$$c_D = \frac{1}{R_e}\iiint_V \Omega^2 dV + \frac{\alpha^2}{R_m}\iiint_V J^2 dV \tag{156}$$

It is difficult to integrate these expressions for Ω^2 and J^2, but it is easy to show how the integral depends on the parameters, and the relative effects of obmic and viscous dissipation. Since the flow is symmetric in x,

$$c_D = 2\left(\frac{1}{R_e} + \frac{\alpha^2}{R_m}K_1^2\right)\int_0^\infty\!\!\int_0^\infty \Omega_1^2\left(\frac{x}{H_a}, \varpi\right) 2\pi\varpi\, d\varpi\, dx \tag{157}$$

Now let $\xi = \frac{x}{Ha}$, and notice that

$$\left(\frac{1}{Re} + \frac{\alpha^2}{R_m} K_1^2\right) = \left(\frac{1}{Re} + \frac{\alpha^2}{R_m} \frac{Ha^2}{Re^2 \alpha^4}\right) = \frac{2}{Re} \tag{158}$$

ohmic and viscous dissipation rates are equal, and

Therefore

$$C_D = 4 \frac{Ha}{Re} \int_0^\infty \int_0^\infty \Omega_1^2(\xi, \varpi)\, 2\pi \varpi \, d\varpi \, d\xi \tag{159}$$

Since the integral is independent of the parameters, $C_D \propto Ha/Re$, which agrees with the previous result.

X. CONCLUDING REMARKS

A. Comparison with other Work

The main results of the MHD disk problem are:

1. Magnetic interaction has little effect on drag until α exceeds unity.
2. When α is large, there is an upstream and downstream wake in which ohmic and viscous dissipation occur at the same rate.
3. The drag coefficient of the disk is $(Ha > Re)$

   $$C_D = 2\pi\, Ha/Re \quad = 2\pi \sqrt{P_{r_m}}\ \alpha$$

4. The diffusion model shows that vorticity and current density are concentrated on the surface of the cylinder generated by the disk. It spreads out parabolically at large distances from the disk.

Stewartson (8) obtained some of these qualitative results, but did not give the drag because he considered inviscid flow. There is nothing in the literature, at present, on high Hartmann number drag.

The results mentioned in this thesis are in general agreement, with the work of other researchers in this field (references 4,5,7,8,9,14 and 15). Although, in most cases different techniques are used to obtain the results.

B. Experiments

Chester (7) points out that a one millimeter sphere in Hg subjected to a magnetic field of 100 gauss results in a Hartmann number of 0.1. Attempts by the author and others to check Chester's drag formula in the laboratory have not been successful because of the following restrictions:

$$Ha^2 \approx \frac{\text{Magnetic Forces}}{\text{Viscous Forces}} < 1 \tag{1}$$

$$Re \approx \frac{\text{Inertia Forces}}{\text{Viscous Forces}} < 1 \tag{2}$$

$$N \approx \frac{\text{Magnetic Forces}}{\text{Inertia Forces}} \approx \frac{Ha^2}{Re} > 1 \tag{3}$$

Since Chester's drag formula $D = D_s \left(1 + \frac{3}{8} Ha\right)$ is only valid for $Ha \leq 0.1$, flows with $Re \leq 0.01$ must be produced in the laboratory. This is almost impossible. If restriction (1) is removed, and high Hartmann number flows are considered, then reasonable Reynolds number flows can be studied without violating restriction (3).

The author proposes the use of sodium rather than mercury as the fluid because higher Reynolds number flows can be studied with reasonable magnetic fields, without violating restriction (3). For example, if the flow is $Re = 10^3$, then $Ha^2 \geq 10^3$. If a 1cm. sphere is used a magnetic field of 10,000 gauss is required with Hg, while a magnetic field of 1,000 gauss is required with Na. There are other advantages and disadvantages to be weighed of course; but Na looks attractive because a column of it can be suspended between two oils, and the pellet can be seen entering and leaving. The wakes may also be visible at the interface.

It has been shown experimentally (17) that the parameter N is important in controlling turbulence in a pipe flow of a conducting fluid with a parallel magnetic field. Undoubtedly the same effects exist in the trailing turbulent wake of a high Reynolds number flow. The interaction between the turbulent wake and the magnetic field should result in a change in drag and time of flight through a column of Na.

APPENDIX

I. Boundary Conditions for the Disk

Consider first the problem without magnetic interaction. Since separate representations of the solution are given for the $+x$ -half plane and the -x-half plane, the boundary conditions must be satisfied on both sides of the disk, and the flow fields must be matched off the disk in the plane $x = 0$. Those conditions are summarized below:

$\underline{\varpi \leq 1}$

$$-1 = u_x(0+, \varpi) = u_x(0-, \varpi) \tag{160a}$$

$$0 = u_\varpi(0+, \varpi) = u_\varpi(0-, \varpi) \tag{160b}$$

$\underline{\varpi > 1}$

$$u_x(0+, \varpi) = u_x(0-, \varpi) \tag{160c}$$

$$u_\varpi(0+, \varpi) = u_\varpi(0-, \varpi) \tag{160d}$$

$$\frac{\partial}{\partial x} u_x(0+, \varpi) = \frac{\partial}{\partial x} u_x(0-, \varpi) \tag{160e}$$

$$\frac{\partial}{\partial x} u_\varpi(0+, \varpi) = \frac{\partial}{\partial x} u_\varpi(0-, \varpi) \tag{160f}$$

The corresponding integral equations are:

$\underline{\varpi \leq 1}$

$$-1 = \int_0^\infty \sum_n A_n J_0(m\varpi) m^2 dm = \int_0^\infty \sum_n \bar{A}_n J_0(m\varpi) m^2 dm \tag{161a}$$

$$0 = \int_0^\infty \sum_n k_n A_n J_1(m\varpi) m\, dm = \int_0^\infty \sum_n -\bar{k}_n \bar{A}_n J_1(m\varpi) m^2 dm \tag{161b}$$

$\underline{\varpi > 1}$

$$\int_0^\infty \sum_n A_n J_0(m\varpi) m^2 dm = \int_0^\infty \sum_n \bar{A}_n J_0(m\varpi) m^2 dm \tag{161c}$$

$$\int_0^\infty \sum_n k_n A_n J_1(m\varpi) m\, dm = \int_0^\infty \sum_n -\bar{k}_n \bar{A}_n J_1(m\varpi) m\, dm \tag{161d}$$

$$\int_0^\infty \sum_n -k_n A_n J_0(m\varpi) m^2 dm = \int_0^\infty \bar{k}_n \bar{A}_n J_0(m\varpi) m^2 dm \tag{161e}$$

$$\int_0^\infty \sum_n k_n^2 A_n J_1(m\varpi) m\, dm = \int_0^\infty \sum_n \bar{k}_n^2 \bar{A}_n J_1(m\varpi) m\, dm \tag{161f}$$

Equations 161a, b, c, and d show that

$$\sum_n A_n = \sum_n \bar{A}_n \equiv f(m) \tag{162a}$$

$$\sum_n k_n A_n = \sum_n -\bar{k}_n \bar{A}_n \equiv g(m) \tag{162b}$$

Furthermore, since Equation 161e is obtained from 161d by the operation $\left(\frac{1}{\varpi} \frac{\partial}{\partial \varpi} \varpi\right)$, they are satisfied simultaneously.

It can be shown by the following arguments that $g(m) = 0$; which means that the radial velocity in the plane of the disk is zero, $u_\varpi(0, \varpi) = 0$: The divergence equation states that

$$\left(\frac{\partial u_x}{\partial x}\right)_{x=0} + \left(\frac{1}{\varpi} \frac{\partial}{\partial \varpi} \varpi u_\varpi\right)_{x=0} = 0$$

Assuming $\left(\frac{\partial u_x}{\partial x}\right)_{x=0}$ is an analytic function of ϖ -- and there is no physical reason to suspect otherwise -- then $u_\varpi(0,\varpi)$ must also be an analytic function of ϖ. For $\varpi \leq 1$, $u_\varpi(0,\varpi) = 0$, and it may be analytically continued to $\varpi > 1$. Therefore, $u_\varpi(0,\varpi) = 0$ for all ϖ, and $g(m) = 0$. Q.E.D.

Using Equations 162; A_0; A_1, $\bar{A}_0$ and $\bar{A}_1$, may be expressed in terms of $f(m)$ as follows:

$$A_0 = \frac{k_1}{k_1 - m} f(m) \qquad A_1 = \frac{m}{k_1 - m} f(m) \tag{163a}$$

$$\bar{A}_0 = \frac{\bar{k}_1}{\bar{k}_1 - m} f(m) \qquad \bar{A}_1 = \frac{m}{\bar{k}_1 - m} f(m) \tag{163b}$$

Substitution of 163 into 161f the remaining condition to be satisfied yields the following set of dual integral equations for $f(m)$:

$$-1 = \int_0^\infty f(m) J_0(m\varpi)\, m^2\, dm, \qquad \varpi \leq 1 \tag{164a}$$

$$0 = \int_0^\infty m f(m) J_0(m\varpi)\, m^2\, dm, \qquad \varpi > 1 \tag{164b}$$

The solution of 164 is well known,*

---

*See, for example, Reference 13, p. 179.

$$f(m) = -\frac{2}{\pi}\,\frac{\sin m}{m^3} \tag{165}$$

The axial velocity in the plane of the disk is*

$$u_x(0,\varpi) = -\frac{2}{\pi}\int_0^\infty \frac{\sin m}{m^3}\,J_0(m\varpi)\,dm$$

$$\begin{aligned} u_x(0,\varpi) &= -1 && \varpi \le 1 \\ &= -\frac{2}{\pi}\arcsin\frac{1}{\varpi} && \varpi > 1 \end{aligned} \tag{166}$$

The drag of the disk may be calculated by integrating the net pressure over both sides of the disk; that is,

$$\begin{aligned} c_D &= \int_0^1 \bar{p}(0+,\varpi)\,2\pi\varpi\,d\varpi - \int_0^1 \bar{p}(0-,\varpi)\,2\pi\varpi\,d\varpi \\ &= \int_0^1 u_{x_0}(0+,\varpi)\,2\pi\varpi\,d\varpi - \int_0^1 u_{x_0}(0-,\varpi)\,2\pi\varpi\,d\varpi \end{aligned} \tag{167}$$

Thus, the drag is given in terms of the irrotational stream functions,

$$c_D = 2\pi\left[P_0(0+,1) - P_0(0-,1)\right] \tag{168}$$

$$= 2\pi\int_0^\infty \left[A_0(m) - \bar{A}_0(m)\right] J_1(m)\,m\,dm \tag{169}$$

Using Equations 163 and 158, it can be shown that

$$A_0 - \bar{A}_0 = \frac{\lambda_1}{2}\,\frac{f(m)}{m - \sqrt{(\lambda_1/2)^2 + m^2}}$$

---

*See, for example, Reference 6, p. 36.

In the limit of small Reynolds number ($\lambda_1 \to 0$),

$$A_0 - \overline{A}_1 \approx - \frac{4}{\lambda_1} m f(m)$$

And,*

$$S_D \approx 2\pi \left(\frac{4}{\lambda_1}\right)\left(\frac{2}{\pi}\right) \int_0^\infty \frac{\sin m}{m} J_1(m)\, dm$$

$$\approx \left(\frac{6\pi}{\lambda_1}\right)(0.85)$$

Thus, the low Reynolds number drag of a disk is just 0.85 times the drag of a sphere having the same radius.** .

The generalization of the boundary conditions to include magnetic interaction is straightforward. Since two additional unknown functions (A_2 and $\overline{A}_2$) have entered the problem, two new boundary conditions must be employed. They are:

$$0 = J(0_+, \varpi) = J(0_-, \varpi) \qquad \varpi \leq 1 \qquad (170a)$$

$$J(0_+, \varpi) = J(0_-, \varpi) \qquad \overline{\varpi} \leq 1 \qquad (170b)$$

These conditions state that the current density vanishes on the disk, and is continuous across the plane $x = 0$.

---

*See, for example, Reference 6, p. 36.

**This checks precisely with Reference 10, p. 605.

Therefore, the integral equations corresponding to conditions 167 are:

$$0 = \int_0^{\infty} \sum_n \lambda_n K_n k_n A_n J_1(m\varpi) m\, dm = \int_0^{\infty} \sum_n \lambda_n K_n \bar{k}_n \bar{A}_n J_1(m\varpi) m\, dm, \quad \varpi < 1 \tag{171a}$$

$$\int_0^{\infty} \sum_n \lambda_n K_n k_n A_n J_1(m\varpi) m\, dm = \int_0^{\infty} \sum_n \lambda_n K_n \bar{k}_n \bar{A}_n J_1(m\varpi) m\, dm, \quad \varpi > 1 \tag{171b}$$

This means that

$$\sum_n \lambda_n K_n k_n A_n = \sum_n \lambda_n K_n \bar{k}_n \bar{A}_n = \ell(m) \tag{172}$$

It can be shown that $\ell(m)$ is identically zero as follows: The linearized form of Ohm's law, Equation 7, is

$$J = R_m \left(u_\varpi - h_\varpi \right) \tag{173}$$

Since it has been shown that $u_\varpi(0, \varpi) = 0$, and it can be shown by precisely the same arguments that $h_\varpi(0, \varpi) = 0$; then $J(0, \varpi) = 0$ and $\ell(m) = 0$. Q.E.D.

Corresponding to Equations 123, the following six simultaneous equations determine A_0, A_1, A_2, $\bar{A}_0$, $\bar{A}_1$ and $\bar{A}_2$ in terms of $f(m)$:

$$\sum_n A_n = \sum_n \bar{A}_n = f(m) \tag{174a}$$

$$\sum_n k_n A_n = \sum_n \bar{k}_n \bar{A}_n = 0 \tag{174b}$$

$$\sum_n \lambda_n K_n k_n A_n = \sum_n \lambda_n K_n \bar{k}_n \bar{A}_n = 0 \tag{174c}$$

The solution of these equations is:

$$A_0(m) = f(m) \frac{k_2 k_1 (\lambda_2 K_1 - \lambda_1 K_1)}{d(m)} \tag{175a}$$

$$A_1(m) = f(m) \frac{-m \lambda_2 K_2 k_2}{d(m)} \tag{175b}$$

$$A_2(m) = f(m) \frac{m \lambda_1 K_1 k_1}{d(m)} \tag{175c}$$

where

$$d(m) = m(\lambda_1 K_1 k_1 - \lambda_2 K_2 k_2) - k_1 k_2 (\lambda_1 K_1 - \lambda_2 K_2) \tag{175d}$$

It follows from the form of 174 that

$$\bar{A}_n(m, \bar{k}_n) = A_n(m, \bar{k}_n) \tag{176}$$

The dual integral equations for $f(m)$ are again obtained by substituting Equations 175 into Equations 121a and 121f. This is a very involved algebraic procedure, but it can, in principle, be carried out and can be determined just as before. However, a simple physical argument suggests that must be the same as it is in the Oseen problem without magnetic interaction: The axial component of the velocity in the plane $x = 0$ determines $f(m)$; and vice versa (see Equation 126). Since the current density and Lorentz force vanishes in the plane $x = 0$, it is reasonable to assume that the flow field in the plane $x = 0$ is the same with or without magnetic interaction elsewhere in the fluid. This assumption can be checked by obtaining the integral equation for $f(m)$ in the limit of large magnetic interaction ($\alpha \to \infty$), where the algebra is less complicated.

In the limit of $\alpha \to \infty$,

$$\lambda_{1,2} \to \pm Ha + \frac{1}{2}(R_e + R_m) \to \infty$$

$$K_{1,2} \to \mp R_m + \frac{R_e - R_m}{2 R_e \alpha^2}$$

$$k_1 \to \frac{m^2}{\lambda_1}$$

$$k_2 \to |\lambda_2|$$

$$\bar{k}_1 \to \lambda_1$$

$$\bar{k}_2 \to \frac{m^2}{|\lambda_2|}$$

$$\sum_n k_n^2 A_n \rightarrow m f(m) \frac{m^2}{\lambda_1} |\lambda_2| \frac{\lambda_2 R_m m - \lambda_1 R_m |\lambda_2|}{-\lambda_2 R_m |\lambda_2| m + \lambda_1 R_m \frac{m^2}{\lambda_1} |\lambda_2|}$$

$$\rightarrow - m^3 f(m)$$

$$\sum_n \bar{k}_n^2 \bar{A}_n \rightarrow m f(m) \lambda_1 \frac{m^2}{|\lambda_2|} \frac{\lambda_2 R_m \lambda_1 - \lambda_1 (-R_m) m}{\lambda_2 R_m \frac{m^2}{|\lambda_2|} \lambda_1 - \lambda_1 (-R_m) \lambda_1 (-m)}$$

$$\rightarrow m^3 f(m)$$

$$\sum_n \left[k_n^2 A_n - \bar{k}_n^2 \bar{A}_n \right] = -2 m^3 f(m)$$

Therefore, the dual integral equation for $f(m)$ in the limit of $\alpha \rightarrow \infty$ are:

$$-1 = \int_0^\infty f(m) J_0(m\varpi)\, m^2\, dm , \qquad \varpi \leq 1$$

$$0 = \int_0^\infty m^3 f(m) J_0(m\varpi)\, m^2\, dm , \qquad \varpi > 1$$

The second integral may be transformed into

$$0 = \int_0^\infty m f(m)\, J_1(m\varpi) m^2\, dm, \qquad \varpi > 1$$

because the original integral is obtained from this by the operation $\left(\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\varpi\right)$. This integral may now be transformed into

$$0 = \int_0^\infty m f(m)\, J_0(m\varpi)\, m^2\, dm, \qquad \varpi > 1$$

by integrating with respect to ϖ. Hence, again

$$f(m) = -\frac{2}{\pi}\frac{\sin m}{m^3} \qquad \text{Q.E.D.}$$

II. <u>Drag Formulas</u>

The basic formulas for drag are equations 144 and 147. Equation 144 is more convenient for handling flows with low magnetic Prandtl number which are the most common type in laboratory experiments ($P_m = \sigma\mu\nu \lesssim 10^{-6}$ for Hg). The important approximation for these flows is,

$$\lambda_1 K_1 = \lambda_2 K_2$$

because the equations for the mode strengths A_1, A_2, $\overline{A}_1$, and $\overline{A}_2$ simplify to

$$A_1 \approx \frac{2}{\pi}\frac{\sin m}{m^3}\frac{k_2}{k_2 - k_1}$$

$$A_2 \approx -\frac{2}{\pi}\frac{\sin m}{m^3}\frac{k_1}{k_2 - k_1}$$

$$\bar{A}_1 \approx \frac{2}{\pi} \frac{\sin m}{m^3} \frac{\bar{k}_2}{\bar{k}_2 - \bar{k}_1}$$

$$\bar{A}_2 \approx -\frac{2}{\pi} \frac{\sin m}{m^3} \frac{\bar{k}_1}{\bar{k}_2 - \bar{k}_1}$$

One case of interest that can be handled analytically is high Hartmann number flow, or $\alpha^2 \to \infty$ with R_e and R_m finite. In this case,

$$\lambda_{1,2} \approx \pm Ha, \qquad Ha >> 1$$

$$K_{1,2} \approx \mp Ha / Re\, \alpha^2$$

$$k_{1,2} \approx \sqrt{(Ha/2)^2 + m^2} \mp Ha/2$$

$$\bar{k}_{1,2} \approx \sqrt{(Ha/2)^2 + m^2} \pm Ha/2$$

Therefore,

$$(A_1 - \bar{A}_1) = -(A_2 - \bar{A}_2) = \frac{2}{\pi} \frac{\sin m}{m^3} \frac{2}{Ha} \sqrt{(Ha/2)^2 + m^2}$$

and since $\alpha^2 K_1 = -\alpha^2 K_2 = -Ha/Re$

$$c_D \approx \frac{8\pi}{Re} \frac{2}{\pi} \int_0^\infty \sqrt{(Ha/2)^2 + m^2}\ \frac{\sin m}{m^2} J_1(m)\, dm$$

For $Ha \to \infty$,

$$c_D \approx 4\pi \frac{Ha}{Re} \frac{2}{\pi} \int_0^\infty \frac{\sin m}{m^2} J_1(m)\, dm$$

$$\delta \approx 2\pi \frac{Ha}{Re} \quad *$$

$$\approx 2\pi \sqrt{Pr_m}\, \alpha \tag{177}$$

Thus, for large Hartmann number flows the drag is proportional to Ha/Re, or to the Alfvén number α .

Another case of interest which can be handled analytically is $\alpha = 1$, where

$$\lambda_1 \approx Re + Rm \qquad \lambda_2 \approx 0$$

$$\kappa_1 \approx -Pr_m \qquad \kappa_2 \approx 1$$

$$k_1 \approx \sqrt{(\lambda_1/2)^2 + m^2} - \lambda_1/2 \qquad k_2 \approx m$$

$$\bar{k}_1 \approx \sqrt{(\lambda_1/2)^2 + m^2} + \lambda_1/2 \qquad \bar{k}_2 \approx m$$

---

*See, for example, Reference 6, p. 36.

$$A_1 \approx \frac{2}{\pi}\frac{\sin m}{m^3}\frac{m}{m-\left(\sqrt{(\lambda_1/2)^2+m^2}-\lambda_1/2\right)}$$

$$A_2 \approx -\frac{2}{\pi}\frac{\sin m}{m^3}\frac{\sqrt{(\lambda_1/2)^2+m^2}-\lambda_1/2}{m-\left(\sqrt{(\lambda_1/2)^2+m^2}-\lambda_1/2\right)}$$

$$\bar{A}_1 \approx \frac{2}{\pi}\frac{\sin m}{m^3}\frac{m}{m-\left(\sqrt{(\lambda_1/2)^2+m^2}+\lambda_1/2\right)}$$

$$\bar{A}_2 \approx -\frac{2}{\pi}\frac{\sin m}{m^3}\frac{\sqrt{(\lambda_1/2)^2+m^2}+\lambda_1/2}{m-\left(\sqrt{(\lambda_1/2)^2+m^2}+\lambda_1/2\right)}$$

The integration is simple in the two extreme cases, $\lambda_1 \to 0$ and $\lambda_1 \to \infty$.

For $\lambda_1 \to 0$:

$$A_1 - \bar{A}_1 = -(A_2 - \bar{A}_2) = \frac{8}{\pi\lambda_1}\frac{\sin m}{m^2}$$

$$C_D = 2\pi(-K_1 + K_2)\int_0^\infty \frac{8}{\pi\lambda_1}\sin m\, J_1(m)\frac{dm}{m}$$

$$= \left(\frac{6\pi}{Re}\right)\left(\frac{8}{3\pi}\right)\frac{(1+Pr_m)}{(1+Pr_m)} = 0.85\, C_{D_0}$$

Thus, when $\alpha = 1$ in low R_e, low Pr_m flows, the drag is unaffected by the magnetic interaction.

For $\lambda_1 \to \infty$:

$$A_1 - \bar{A}_1 \approx \frac{2}{\pi} \frac{\sin m}{m^3} \qquad A_2 - \bar{A}_2 \approx \frac{4}{\pi} \frac{\sin m}{m^3}$$

$$C_D \approx 2\pi \left(1 - 3\sqrt{P_{r_m}}\right) \int_0^\infty \frac{2}{\pi} \frac{\sin m}{m^2} J_1(m)\, dm$$

$$C_D \approx \pi \left(1 - 3\sqrt{P_{r_m}}\right)$$

Thus, for large R_e and $\alpha = 1$, the change to the high R_e drag coefficient is proportional to $\sqrt{P_{r_m}}$, a very small number in most cases. It may be concluded that magnetic interaction in low P_{r_m} flows has negligible effect on the drag when $\alpha \lesssim 1$.

III. Uniqueness Proof

Equations:

$$\frac{\partial \vec{u}}{\partial x} - \frac{1}{R_e} \nabla^2 \vec{u} - \alpha^2 \frac{\partial \vec{h}}{\partial x} = - \nabla \bar{p}$$

$$\frac{\partial \vec{h}}{\partial x} - \frac{1}{R_m} \nabla^2 \vec{h} - \frac{\partial \vec{u}}{\partial x} = 0$$

$$\nabla \cdot \vec{u} = \nabla \cdot \vec{h} = 0$$

Boundary Conditions:

$$\vec{r} \to \infty ; \quad \vec{u} \to 0, \quad \vec{h} \to 0, \quad \bar{p} \to 0$$

$$\text{on body;} \quad \vec{u} \to -\vec{i} \quad , \quad \vec{J} = \text{curl}\, \vec{h} = 0 \quad ,$$

Suppose there are two solutions of the above equations and boundary conditions, $\vec{h}_1$, $\vec{u}_1$, $\bar{p}_1$ and $\vec{h}_2$, $\vec{u}_2$, $\bar{p}_2$.

then the following functions must also satisfy the equations:

$$\vec{h}_3 = \vec{h}_2 - \vec{h}_1$$
$$\vec{u}_3 = \vec{u}_2 - \vec{u}_1$$
$$\bar{p}_3 = \bar{p}_2 - \bar{p}_1$$

However, these functions satisfy the following boundary conditions:

$$\vec{r} \to \infty; \ \vec{u}_3 \to 0, \quad \vec{h}_3 \to 0, \quad \bar{p}_3 \to 0$$

$$\text{on body;} \ \vec{u}_3 = 0, \quad \vec{J}_3 = \text{curl}\, \vec{h}_3 = 0, \quad \bar{p}_3 = 0$$

Furthermore, since $\operatorname{div} \vec{h}_3 = 0$ everywhere, on the body $\vec{h}_3 = 0$. Hence, $\vec{u}_3$, $\vec{h}_3$, $\bar{p}_3$ all vanish at the boundaries of the fluid. Now it can be shown that $\vec{u}_3$, $\vec{h}_3$, $\bar{p}_3$ must vanish everywhere in the fluid, so that $\vec{u}_2 = \vec{u}_1$, $\vec{h}_2 = \vec{h}_1$, $\bar{p}_2 = \bar{p}_1$, and therefore the solution is unique. Proof:

The divergence of the momentum equation shows that $\nabla^2 \bar{p}_3 = 0$. Since $\bar{p}_3$ vanishes on the boundaries it must vanish everywhere within the boundaries, so $\bar{p}_3 = 0$.

The functions $\vec{u}_3$ and $\vec{h}_3$ satisfy the following homogeneous equations:

$$\frac{\partial \vec{u}_3}{\partial x} - \frac{1}{R_e} \nabla^2 \vec{u}_3 - \alpha^2 \frac{\partial \vec{h}_3}{\partial x} = 0$$

$$\frac{\partial \vec{h}_3}{\partial x} - \frac{1}{R_m} \nabla^2 \vec{h}_3 - \frac{\partial \vec{u}_3}{\partial x} = 0$$

$$\nabla \cdot \vec{u}_3 = \nabla \cdot \vec{h}_3 = 0$$

Multiply the first by $\vec{u}_3$ and the second by $\vec{h}_3$, add, and integrate over the volume to get

$$\iiint_{-\infty}^{\infty} \frac{\partial}{\partial x}\left(\frac{u_3^2}{2}+\frac{h_3^2}{2}\right)dx\,dy\,dz - \alpha^2\iiint_{-\infty}^{\infty}\frac{\partial}{\partial x}(\vec{u}_3\cdot\vec{h}_3)\,dx\,dy\,dz$$

$$=\iiint_{-\infty}^{\infty}\vec{u}_3\cdot(\nabla^2\vec{u}_3)\,dx\,dy\,dz + \alpha^2\iiint_{-\infty}^{\infty}\vec{h}_3\cdot(\nabla^2\vec{h}_3)\,dx\,dy\,dz$$

Now, integrate by parts and use the fact that $\vec{u}_3$ and $\vec{h}_3$ vanish at the boundaries of the fluid to get

$$0=\iiint_{-\infty}^{\infty}(\nabla\vec{u}_3)^2\,dx\,dy\,dz + \alpha^2\iiint_{-\infty}^{\infty}(\nabla\vec{h}_3)^2\,dx\,dy\,dz$$

This means that $\nabla\vec{u}_3=0$ and $\nabla\vec{h}_3=0$, or that $\vec{u}_3=\vec{h}_3=0$ everywhere. Q.E.D.

IV. Splitting Theorem

The equations to be considered are:

$$\frac{\partial\vec{u}}{\partial x}-\frac{1}{R_e}\nabla^2\vec{u}-\alpha^2\frac{\partial\vec{h}}{\partial x}=-\nabla\vec{p}+\vec{F}$$

$$\frac{\partial\vec{h}}{\partial x}-\frac{1}{R_m}\nabla^2\vec{h}-\frac{\partial\vec{u}}{\partial x}=\vec{G}$$

$$\nabla\cdot\vec{u}=\nabla\cdot\vec{h}=0$$

where $\vec{F}$ and $\vec{G}$ represent the effects of a finite body and current elements in the body on the fluid, respectively.

Any vector field may be split into a longitudinal (L) and transverse (T) part, that is,

$$\vec{u}=\vec{u}^L+\vec{u}^T,\quad \vec{h}=\vec{h}^L+\vec{h}^T,\quad \vec{F}=\vec{F}^L+\vec{F}^T,\quad \vec{G}=\vec{G}^L+\vec{G}^T$$

where,

$\text{curl}\, \vec{u}^{L} = 0$ $\qquad$ $\text{div}\, \vec{u}^{T} = 0$

$\text{curl}\, \vec{h}^{L} = 0$ $\qquad$ $\text{div}\, \vec{h}^{T} = 0$

$\text{curl}\, \vec{F}^{L} = 0$ $\qquad$ $\text{div}\, \vec{F}^{T} = 0$

$\text{curl}\, \vec{G}^{L} = 0$ $\qquad$ $\text{div}\, \vec{G}^{T} = 0$

The longitudinal fields may be written as gradients of a scaler potential, and the transverse fields as curls of a vector potential.

$$\vec{u}^{L} = -\text{grad}\, \phi_u \qquad \vec{u}^{T} = -\text{curl}\, A_u$$

$$\vec{h}^{L} = -\text{grad}\, \phi_h \qquad \vec{h}^{T} = -\text{curl}\, A_h$$

$$\vec{F}^{L} = -\text{grad}\, \phi_F \qquad \vec{F}^{T} = -\text{curl}\, A_F$$

$$\vec{G}^{L} = -\text{grad}\, \phi_G \qquad \vec{G}^{T} = -\text{curl}\, A_G$$

If it is assumed that $\nabla \cdot \vec{F} = \nabla \cdot \vec{G} = 0$ in choosing the fields $\vec{F}$ and $\vec{G}$ that represent the effects of the body; then it follows that

and that the longitudinal functions are irrotational and solenoidal (harmonic). Therefore,

$$\nabla^2 \phi_u = \nabla^2 \phi_h = \nabla^2 \phi_F = \nabla^2 \phi_G = 0$$

Substituting for $\vec{h}$, $\vec{u}$, $\bar{p}$, $\vec{F}$ and $\vec{G}$ in the original equations and collecting terms yields,

$$\nabla\left[\frac{\partial\phi_u}{\partial x} - \alpha^2\frac{\partial\phi_h}{\partial x} - \bar{p} - \phi_F\right] + \nabla\times\left\{\frac{\partial A_u}{\partial x} - \frac{1}{R_e}\nabla^2 A_u - \alpha^2\frac{\partial A_h}{\partial x} - A_F\right\} = 0$$

$$\nabla\left[\frac{\partial\phi_h}{\partial x} - \alpha^2\frac{\partial\phi_u}{\partial x} - \phi_G\right] + \nabla\times\left\{\frac{\partial A_h}{\partial x} - \frac{1}{R_m}\nabla^2 A_h - \frac{\partial A_u}{\partial x} - A_G\right\} = 0$$

It can be shown that the terms in brackets must vanish by taking the divergence of the equations. This results in $\nabla^2[\quad] = 0$; but at infinity the term in brackets vanishes, so it must be zero everywhere inside. By the same argument, $\phi_F = \phi_G = 0$ because $\nabla^2\phi_F = \nabla^2\phi_G = 0$ and $\phi_F = \phi_G = 0$ at infinity. Hence,

$$\phi_h = \phi_u = 0$$

$$\bar{p} = (1-\alpha^2)\frac{\partial\phi}{\partial x}$$

The curl terms in the equation are the equations for the transverse fields, that is,

$$\frac{\partial\vec{u}^T}{\partial x} - \frac{1}{R_e}\nabla^2\vec{u}^T - \alpha^2\frac{\partial\vec{h}^T}{\partial x} = \vec{F}^T$$

$$\frac{\partial\vec{h}^T}{\partial x} - \frac{1}{R_m}\nabla^2\vec{h}^T - \frac{\partial\vec{u}^T}{\partial x} = \vec{G}^T$$

$$\nabla\cdot\vec{u}^T = \nabla\cdot\vec{h}^T = 0$$

It is clear from the above analysis that this splitting is unique. Even making $\vec{F}$ and $\vec{G}$ divergence free was not arbitrary as can be seen by taking the divergence of the original equations.

$$0 = -\nabla^2 \bar{p} + \nabla \cdot \vec{F}$$
$$0 = \nabla \cdot \vec{G}$$

Since $\bar{p}$ is harmonic and continuous in the fluid, it follows that $\nabla \cdot \vec{F} = 0$.

Splitting into Modes

The homogeneous equations for the transverse fields are

$$\begin{pmatrix} \frac{\partial}{\partial x} - \frac{1}{R_e} \nabla^2 & -\alpha^2 \frac{\partial}{\partial x} \\ -\frac{\partial}{\partial x} & \frac{\partial}{\partial x} - \frac{1}{R_m} \nabla^2 \end{pmatrix} \begin{pmatrix} \vec{u}^T \\ \vec{h}^T \end{pmatrix} = 0$$

The equations for $\vec{u}^T$ and $\vec{h}^T$ are therefore identical; that is,

$$\left[\left(\frac{\partial}{\partial x} - \frac{1}{R_e} \nabla^2 \right) \left(\frac{\partial}{\partial x} - \frac{1}{R_m} \nabla^2 \right) - \alpha^2 \frac{\partial^2}{\partial x^2} \right] \vec{u}^T, \vec{h}^T = 0$$

The fourth order operator factors into two second order operators

$$\left[\nabla^2 - \lambda_1 \frac{\partial}{\partial x} \right] \left[\nabla^2 - \lambda_2 \frac{\partial}{\partial x} \right] \vec{u}^T, \vec{h}^T = 0$$

This suggests solutions of the type

$$\vec{u}^T = \vec{u}_1 + \vec{u}_2$$
$$\vec{h}^T = \vec{h}_1 + \vec{h}_2 = K_1 \vec{u}_1 + K_2 \vec{u}_2$$

where

$$\left[\nabla^2 - \lambda_1 \frac{\partial}{\partial x} \right] \vec{u}_1 = 0 \qquad \nabla \cdot \vec{u}_1 = 0$$
$$\left[\nabla^2 - \lambda_2 \frac{\partial}{\partial x} \right] \vec{u}_2 = 0 \qquad \nabla \cdot \vec{u}_2 = 0$$

This justifies the substitution used on page 8.

REFERENCES

1. Oseen, C. W., Neure Methoden und Ergebnisse in der Hydrodynamik, Liepzig (1927).

2. Lagerstrom, P. A., High Speed Aerodynamics and Jet Propulsion, Vol. IV: "Laminar Flows and Transition to Turbulence," Section B: "Laminar Flow" (Chapter 7).

3. Cowling, T. G., Magnetohydrodynamics, Interscience Publishers, Inc., New York, N. Y. (1957).

4. Lary, E. C., "A Theory of Thin Airfoils and Slender Bodies in Fluids of Arbitrary Electrical Conductivity." Ph.D. Thesis, Cornell University, Ithaca, N.Y., Feb. 1960.

5. Hasimoto, H., "Viscous Flow of a Perfectly Conducting Fluid with a Frozen Magnetic Field," J. Phys. Fluids 2, 337-338 (1959).

6. Magnus & Oberhettinger, Functions of Mathematical Physics, Chelsea Publishing Co., N.Y. (1954).

7. Chester, W., "The Effect of a Magnetic Field on Stokes Flow in a Conducting Fluid," Journal of Fluid Mechanics, Vol. 3, Part 3, December, 1957.

8. Stewartson, "Motion of a Sphere Through a Conducting Fluid in the Presence of a Strong Magnetic Field," Cambridge Philos. Soc. 52, 1956.

9. Ludford, G. S. S., "The Effect of an Aligned Magnetic Field on Oseen Flow of a Conducting Fluid," Interim Technical Report #43, Dept. of Mathematics, University of Maryland, College Park, Maryland, Nov. 1959.

10. Lamb, Hydrodynamics, Sixth Edit., Dover Publications, N.Y., 1932.

11. Table of Integral Transforms, Vol. 2, McGraw-Hill, N.Y., 1954.

12. Adams and Hippisley, Smithsonian Mathematical Formulae and Tables of Mathematical Formulae, Smithsonian Institution, Washington, D.C., 1947.

13. Sneddon, I., Elements of Partial Differential Equations, McGraw-Hill Book Co., New York, 1957.

14. Gotoh, K., "Magnetohydrodynamic Flow Past a Sphere," Phys. Soc. Japan J., Jan. 1960, pp. 189-196.

15. Yosinobu, H., "A Linearized Theory of Magnetohydrodynamic Flow Past a Fixed Body in a Parallel Magnetic Field," Phys. Soc. Japan J., Jan. 1960, pp. 175-188.

16. Imai, I., "Some Remarks on Flows of Conducting Fluids Past Bodies," presented at the International Symposium on Magneto-Fluid-Dynamics, Jan. 17-23, 1960, Washington, D.C.

17. Globe, S., "The Suppression of Turbulence in Pipe Flow of Mercury by an Axial Magnetic Field," Research and Advanced Development Division, Avco Manufacturing Corporation, Wilmington, Massachusetts.

FIGURES

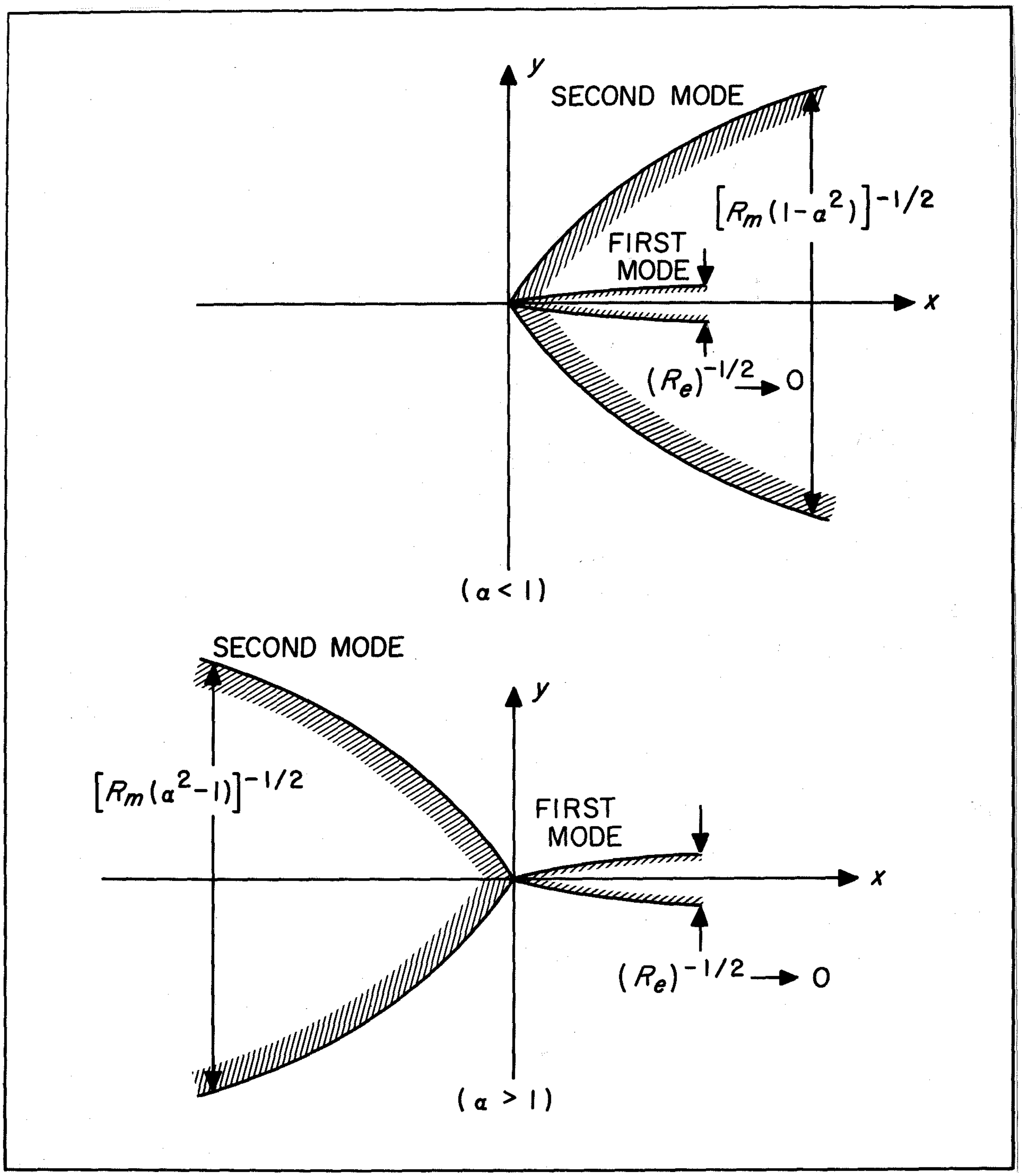

Figure 1

Qualitative behavior of the wakes in the limit of zero viscosity

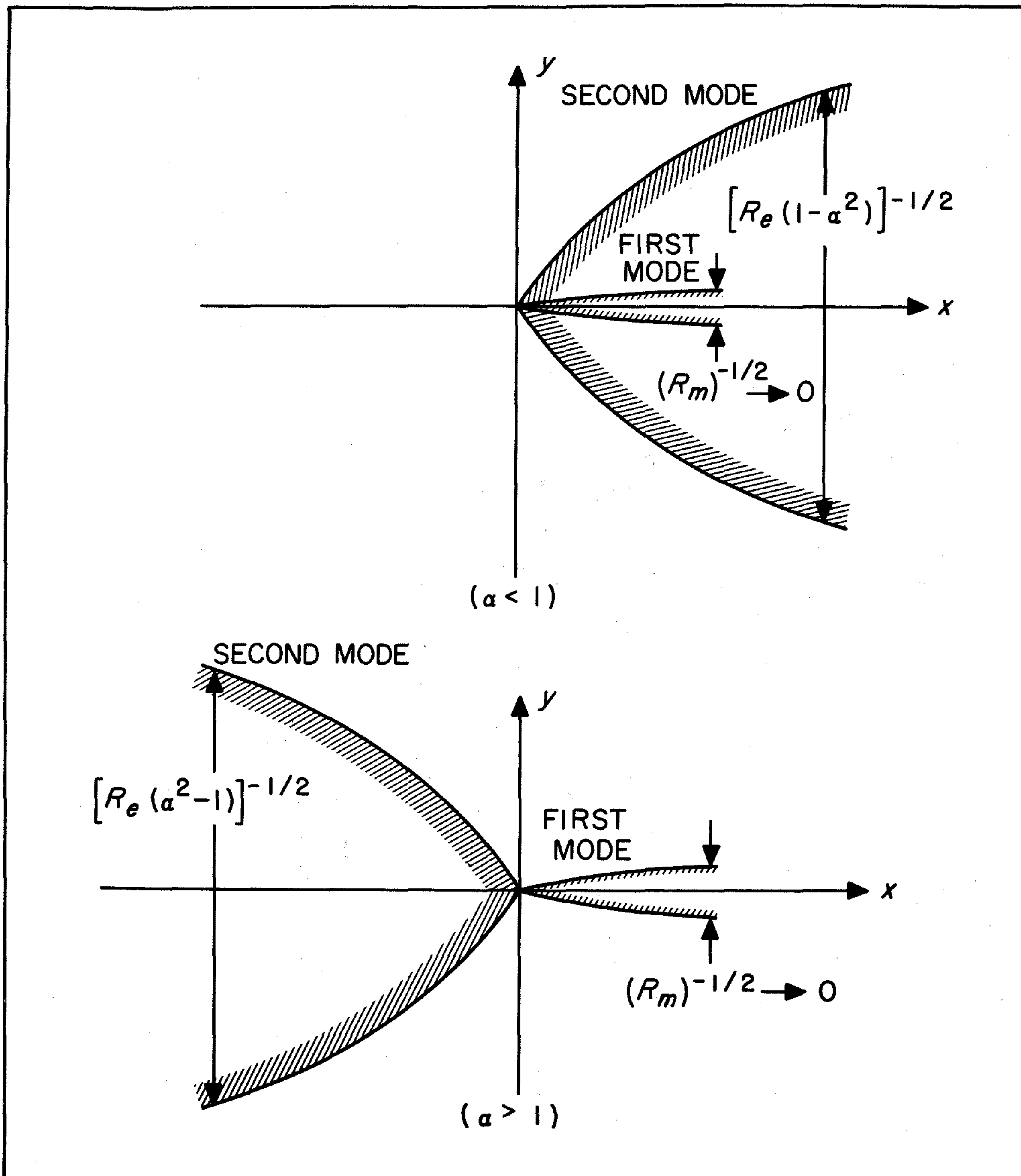

Figure 2

Qualitative behavior of the wakes in the limit of infinite conductivity

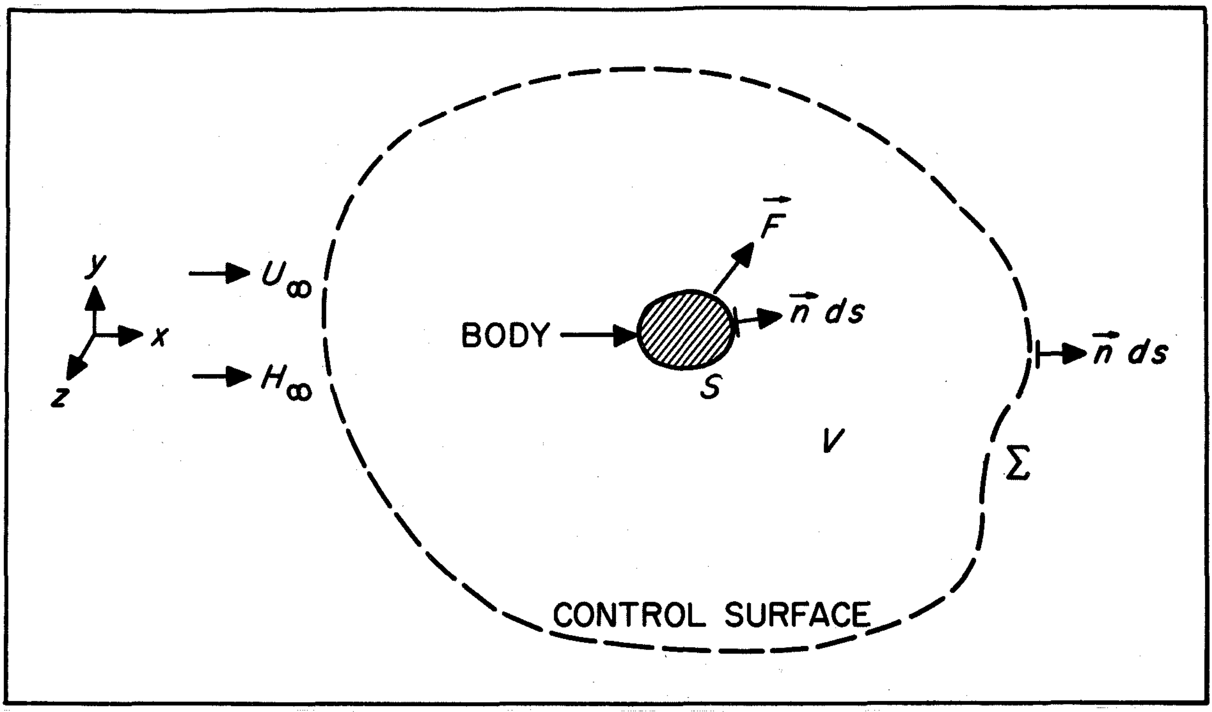

Figure 3

Momentum balance diagram

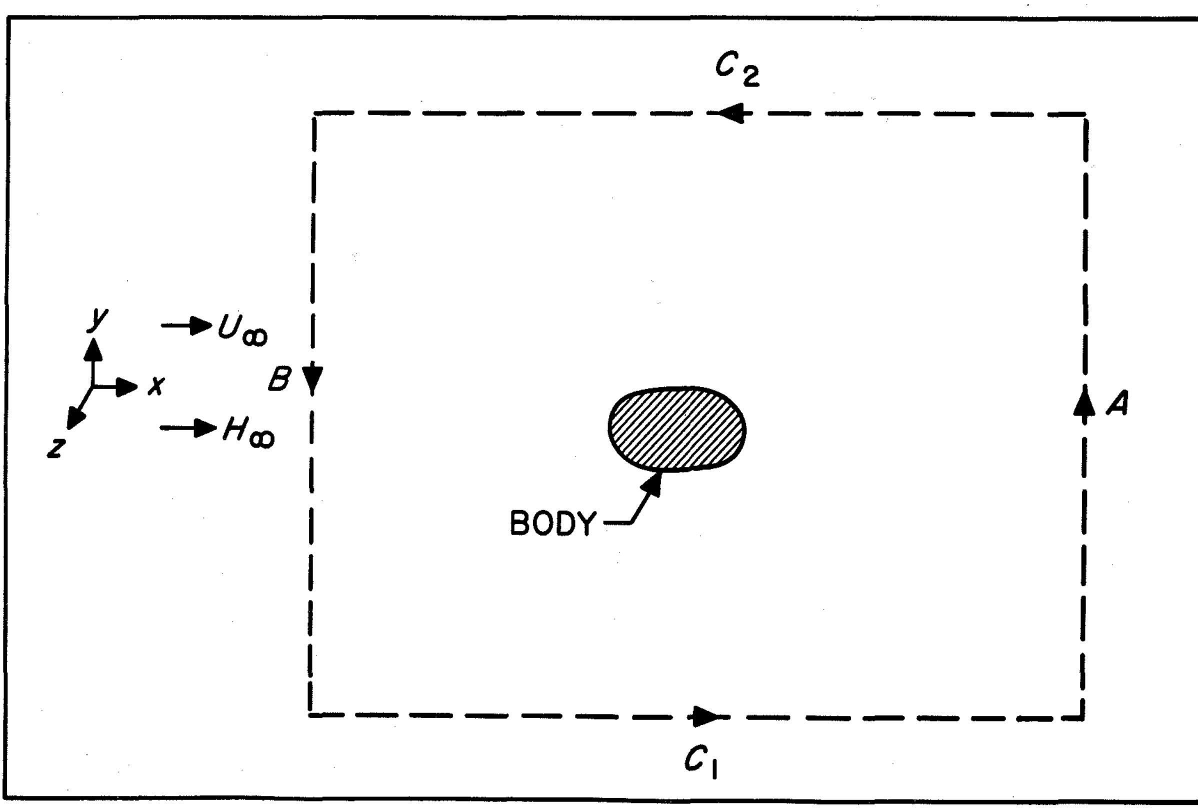

Figure 4

Circulation diagram

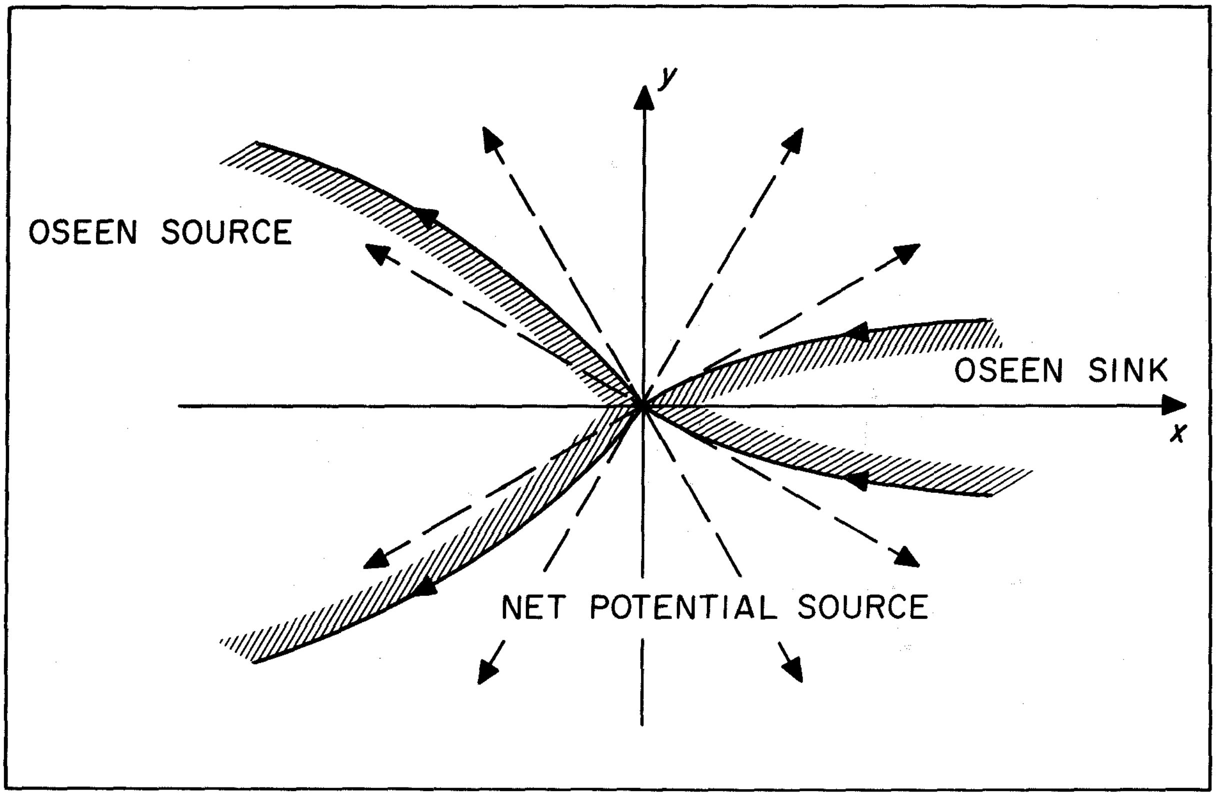

Figure 5

Qualitative behavior of the perturbation velocity field

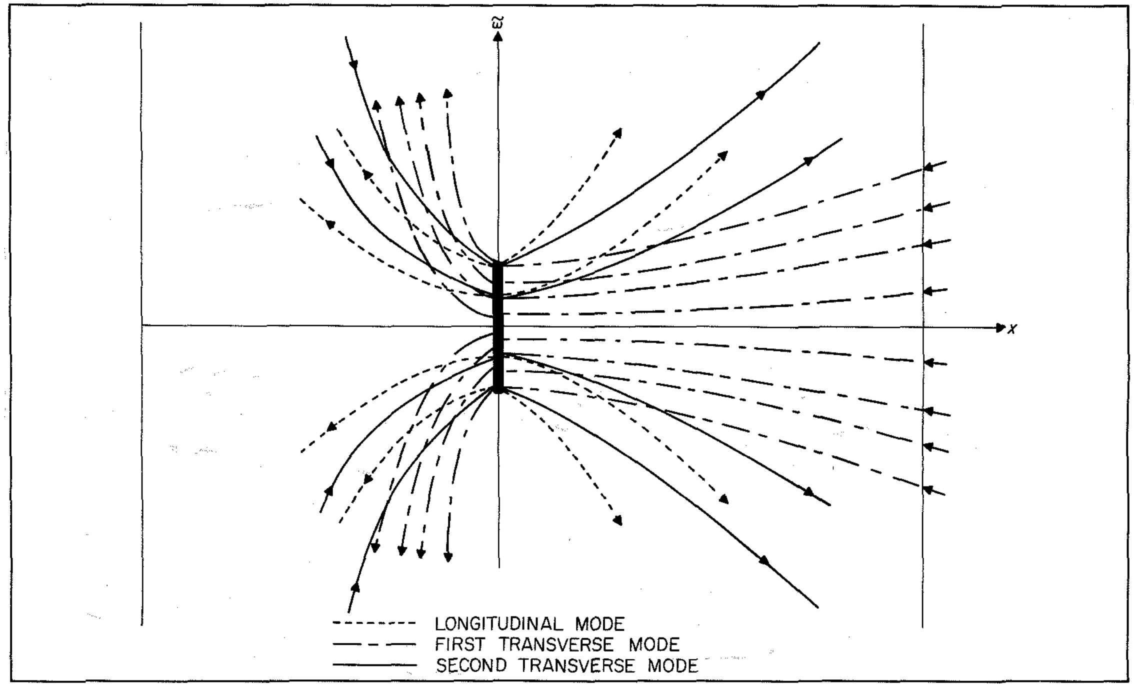

Figure 6

MHD Flow over a Disk $(\alpha < 1)$

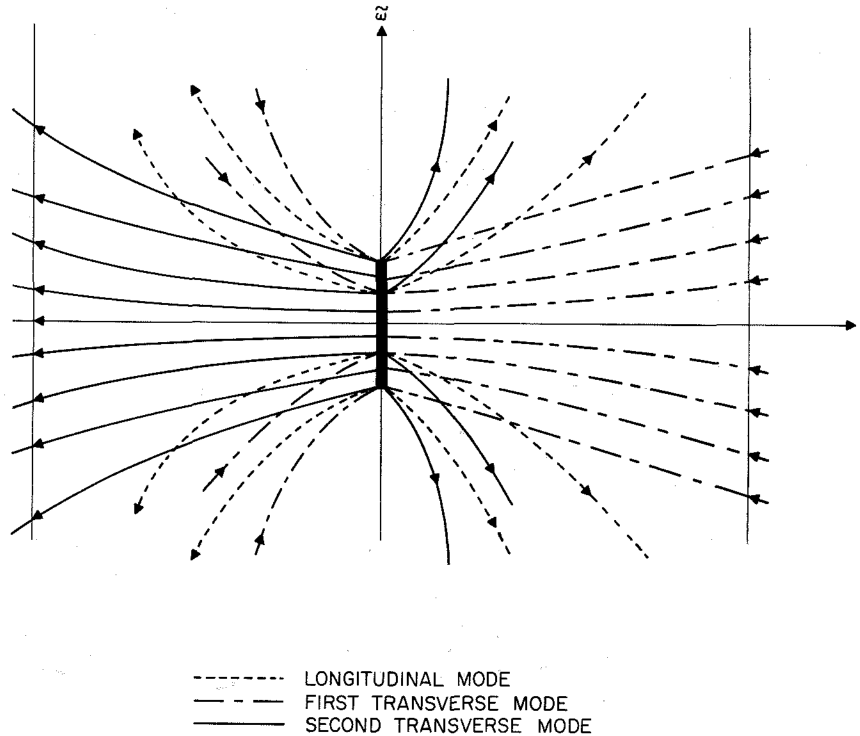

Figure 7

MHD Flow over a Disk ($\alpha > 1$)

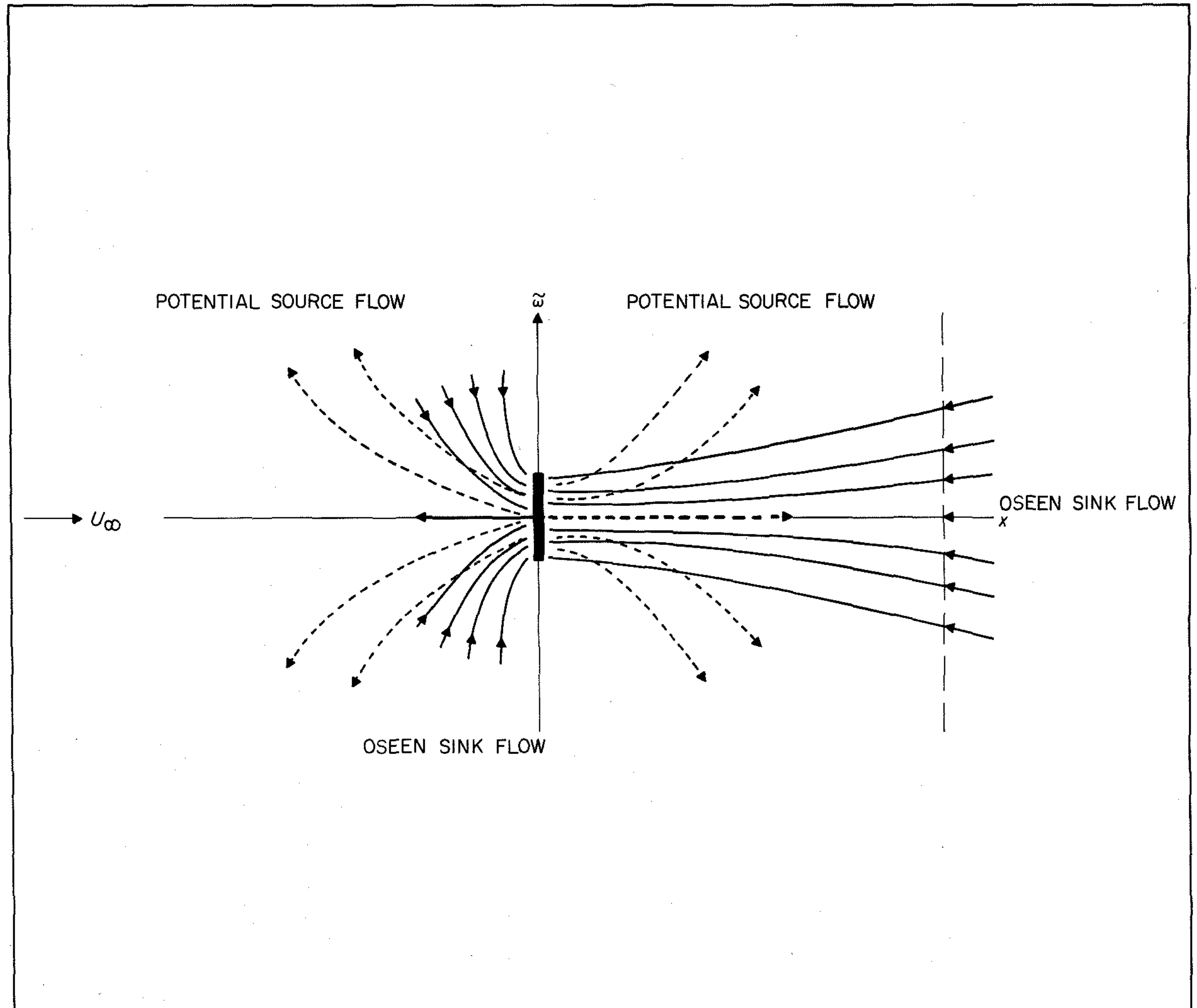

Figure 8

Oseen Flow over a Disk