REPRESENTATIONS FOR DICATEGORIES

Thesis by

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In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1954

ACKNOWLEDGMENTS

It is my great pleasure to extend humble and sincere thanks to the following persons for their gracious and invaluable assistance. Space permits only a listing of names, not the many reasons for which thanks are due. These persons are:

Professor Emeritus E. T. Bell;

Professor H. F. Bohnenblust;

Professor Saunders MacLane, University of Chicago;

Dorothy Hughart, my wife.

ABSTRACT

This thesis is concerned with functions and group homomorphisms. The tool system employed is the dicategory, an algebra of mappings with operation that of composition and in which decomposition into composites of mappings onto, isomorphisms into, identities into, and so on, is possible. The dicategory axioms are abstractions of certain properties common to functions, group and ring homomorphisms, continuous functions between topological spaces, and so on. The problems solved are those of faithful representations of abstract dicategories by particular dicategories.

Chapter I reviews the notion of category and defines the dicategory. By addition of one further axiom, a representation by classes and functions is obtained. The connections between this representation and two well-known ones, one for groups and one for partially ordered sets, are noted.

Chapter II presents axioms for a system which is shown to be representable as a dicategory of abelian semigroup homomorphisms.

Chapter III exhibits axioms for an abelian dicategory and shows that each such dicategory is isomorphic to a dicategory of abelian group homomorphisms. The availability of a second representation and its connection with that of Chapter II are noted.

Chapter IV studies homomorphisms of arbitrary groups. After developing a theorem on associative operations in groups, axioms are presented which allow representation for certain dicategories by particular ones consisting of group homomorphisms. The representation is not faithful, but a remedy which will achieve faithfulness is indicated.

CHAPTER I

A REPRESENTATION BY CLASSES

1. Preliminary Remarks. In this chapter we begin a sequence of four representation theorems. Their purpose is to show that the axioms upon which they are built completely characterize certain properties of functions and of homomorphisms between groups, abelian in the case of the second and third representations, arbitrary in the fourth. The basic axioms are those for a dicategory and are modelled after those of a bicategory (MacLane [8])[a]. Underlying both these systems is the category, originally announced in MacLane [7] and Eilenberg and MacLane [4], but revised in [8]. These axioms are, for completeness, reproduced in Section 2. The genesis of our problem is found in these latter two papers, in which representations are given for a category, one by sets and functions, the other by abelian semigroups and their homomorphisms (after addition of further axioms). Our first two representation theorems extend those of the category by allowing not only for composition of homomorphisms but also for special kinds of homomorphisms such as isomorphisms into and homomorphisms onto and for the decomposition of mappings into composites of these particular ones. Our last two representation theorems concern groups, abelian and general.

In Section 2 we select from [8] the properties of categories

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[a] Numbers in brackets refer to the list of references at the end of the paper.

most useful for this paper; the repetition is made only for the sake of completeness.

In Section 3 we define a dicategory and derive some elementary results concerning partial order, zero objects, and zero maps. Most of these results are stated, though not generally proved, in [8].

In Sections 4 and 5 we deal with dicategories satisfying one additional axiom. The first of these sections includes the preliminary results which find their application in the representation given in the latter section.

Finally, in the Appendix, we note a connection between our representation and well-known representations for groups and for partially ordered sets and indicate an application to lattices.

2. Categories. The concepts of category and dicategory arise from the formal properties of the class of all transformations $\alpha : X \dashrightarrow Y$ of any set X into another set Y, of homomorphisms of one group into another, of continuous mappings of one topological space into another, and so on. By a transformation on sets we mean an ordered triple $\alpha = (\alpha_f, X, Y)$ composed of a function α_f and two sets X and Y such that the function maps all of X into Y. Similarly a transformation on groups (spaces) is a triple α in which X and Y are groups (spaces) and α_f is a homomorphism (continuous mapping) on X into Y. Two transformations $\alpha = (\alpha_f, X, Y)$ and $\alpha' = (\alpha'_f, X', Y')$ are equal if and only if corresponding components are equal. The product $\alpha\alpha'$ is defined just in case the domain X of α is the range Y' of α'; when defined, $\alpha\alpha' = (\alpha_f \alpha'_f, X', Y)$.

Some of the formal properties of transformations which refer to equality and products are listed below as axioms for a category.

DEFINITION 1.1. A category $\mathcal{C}$ is a class of elements α, β, γ, ... , called "maps" in which certain pairs have a product $\alpha\beta \in \mathcal{C}$ defined, subject to axioms C-0 to C-4.

C-0. (Equality axiom). If $\alpha = \alpha'$, $\beta = \beta'$, and the product $\alpha\beta$ is defined, then the product $\alpha'\beta'$ is defined and $\alpha\beta = \alpha'\beta'$.

C-1. If the products $\alpha\beta$ and $(\alpha\beta)\gamma$ are defined, so is the product $\beta\gamma$.

C-1'. If the products $\beta\gamma$ and $\alpha(\beta\gamma)$ are defined, so is the product $\alpha\beta$.

C-2. (Associative law). If the products $\alpha\beta$ and $\beta\gamma$ are defined, then the products $(\alpha\beta)\gamma$ and $\alpha(\beta\gamma)$ are defined and are equal.

A map I of $\mathcal{C}$ is called an identity of $\mathcal{C}$ if (i) II is defined, (ii) $I\alpha = \alpha$ whenever $I\alpha$ is defined, and (iii) $\beta I = \beta$ whenever βI is defined.

C-3. (Existence of domain I and range I'). For each $\alpha \in \mathcal{C}$ there exist identities I and I' such that both αI and $I'\alpha$ are defined.

C-4. For each pair of identities, I and I', the class of all maps $\alpha \in \mathcal{C}$ such that both αI and $I'\alpha$ are defined is a set[a].

We note that for each map $\alpha \in \mathcal{C}$, the identities I and I' such that $I'\alpha = \alpha I = \alpha$ are unique.

We define the objects of the category to be any class of elements A, B, C, . . . in one-to-one correspondence A <---> I_A with the class of identity maps of $\mathcal{C}$. Also if $I_B\alpha$ and αI_A are defined,

[a]The distinction between "set" and "class" made here and throughout this paper is that of the von Neumann-Bernays-Gödel axiomatic set theory [2; 5]. For a discussion of foundations appropriate to categories we refer the reader to [4, §6] and [8, §8].

we take the domain of α to be $D(\alpha) = A$, the range of α to be $R(\alpha) = B$, and we write $\alpha : A \dashrightarrow B$. Then for any maps $\alpha, \beta \in \mathcal{C}$, $\alpha\beta$ is defined if and only if $D(\alpha) = R(\beta)$; in this case, $D(\alpha\beta) = D(\beta)$ and $R(\alpha\beta) = R(\alpha)$.

A map $\theta \in \mathcal{C}$ is an equivalence in $\mathcal{C}$ if there exist maps ϕ and ψ such that $\phi\theta$ and $\theta\psi$ are defined and are identities; in such cases $\phi = \psi$ and each is the unique inverse θ^{-1} of θ. If θ is an equivalence in $\mathcal{C}$, then so is θ^{-1}, and

$$R(\theta^{-1}) = D(\theta), \quad D(\theta^{-1}) = R(\theta), \quad (\theta^{-1})^{-1} = \theta.$$

Also if θ_1 and θ_2 are equivalences in $\mathcal{C}$ and $\theta_1\theta_2$ is defined, then $\theta_1\theta_2$ is an equivalence in $\mathcal{C}$ and $(\theta_1\theta_2)^{-1} = \theta_2^{-1}\theta_1^{-1}$. If there is an equivalence map $\theta: A \dashrightarrow B$, we call the objects A and B equivalent.

3. Dicategories. In the applications each function (homomorphism, continuous mapping, etc.) α can be expressed uniquely as a composite mapping $\kappa\rho$ in which ρ is a mapping onto and κ an identity mapping into. Turning to transformations, we say that $\alpha = (\alpha_f, X, Y)$ is an injection transformation if α_f is the identity function on X and if X is a subset (subgroup, subspace) of Y. We call α a supermap transformation if α_f is a mapping of X onto Y. Thus, for example, each subgroup H of a group G determines a unique injection transformation of H into G and each homomorphic image $K \cong G/N$ determines at least one supermap transformation of G onto K. We now formalize these special transformations and adopt some of their properties as axioms for a dicategory.

DEFINITION 1.2. *A dicategory $\mathcal{C}$ is a category with two given subclasses of maps, the class of "injections" (κ) and the class of* "supermaps" (ρ) satisfying axioms D-1 to D-7.

D-1. *Every identity is an injection and every equivalence is a supermap.*

D-2. *If* α_1 *and* α_2 *are injections (supermaps) and* $\alpha_1\alpha_2$ *is defined, then* $\alpha_1\alpha_2$ *is an injection (supermap).*

D-3. (Canonical decomposition). *Each map* $\alpha \varepsilon \mathcal{C}$ *has a unique representation* $\alpha = \kappa\rho$ *as a product of an injection* κ *and a supermap* ρ.

D-4. *If* κ_1 *and* κ_2 *are injections having the same domain and the same range, then* $\kappa_1 = \kappa_2$.

D-5. *Let* ρ_1, ρ_2, ρ_3 *be supermaps. If* $\rho_1\rho_3 = \rho_2\rho_3$, *then* $\rho_1 = \rho_2$; *if* $\rho_1\rho_3$ *is an identity, then* ρ_1 *is an equivalence.*

D-6. *If* κ *is an injection and* ρ *a supermap for which* $R(\kappa) = R(\rho)$, *then an injection* $\bar{\kappa}$ *and a supermap* $\bar{\rho}$ *exist so that* $\rho\bar{\kappa} = \kappa\bar{\rho}$.

D-7. *For each object* A, *the class of all injections with range* A *is a set.*

For groups, axiom D-6 asserts that if K is a subgroup of G/N, then K = H/N for a suitable subgroup H of G.

The axioms D-1 to D-7 are modifications of those for a bicategory [8]. Our axioms hold for groups and their homomorphisms without the special precautions for equality required in the interpretations of bicategories and demanded for the explanation of duality phenomena for groups. Our concern is not with duality; the dicategory axioms are therefore somewhat more natural. We may, for example, interpret our axioms in the category of all topological spaces by taking the maps $\alpha : A \dashrightarrow B$ to be the continuous transformations of space A into space B, the injection $\kappa : C \dashrightarrow D$ the identity mapping of C into D for C a subspace of D under the usual relative topology, and the supermaps $\rho : P \dashrightarrow Q$ the continuous mappings of P onto Q. In such an interpretation an analysis of equality is not required.

Any map $\lambda \varepsilon \mathcal{C}$ having the canonical decomposition $\lambda = \kappa\theta$, θ an equivalence, is called a submap. In the interpretations, submaps are the isomorphisms into; for them we have a result analogous to axiom D-2.

LEMMA 1.1. If λ_1 and λ_2 are submaps and $\lambda_1\lambda_2$ is defined, then $\lambda_1\lambda_2$ is a submap.

Proof. It is sufficient to establish the special case in which $\lambda_1 = \theta$ is an equivalence, $\lambda_2 = \kappa$ is an injection, and $\theta\kappa$ is defined. By D-3 and category axioms we have the canonical decompositions

$$\theta\kappa = \kappa_1\rho_1 ,$$
$$\theta^{-1}\kappa_1 = \kappa_2\rho_2 ,$$
$$\theta\kappa_2 = \kappa_3\rho_3 .$$

Let $D(\kappa) = A$ and $D(\kappa_1) = B$. Then

$$\theta^{-1}\theta\kappa = \theta^{-1}\kappa_1\rho_1 = \kappa_2\rho_2\rho_1 = \kappa I_A ,$$
$$\theta\theta^{-1}\kappa_1 = \theta\kappa_2\rho_2 = \kappa_3\rho_3\rho_2 = \kappa_1 I_B .$$

Since the canonical decompositions $\kappa_2(\rho_2\rho_1) = \kappa I_A$ and $\kappa_3(\rho_3\rho_2) = \kappa_1 I_B$ are each unique, $\rho_2\rho_1 = I_A$ and $\rho_3\rho_2 = I_B$. Hence ρ_2, having both a left and a right inverse, is an equivalence. But $\rho_1 = \rho_2^{-1}$ is also an equivalence and $\theta\kappa = \kappa_1\rho_1$, establishing the lemma.

The injections serve to introduce naturally a partial order[a] for the objects of a dicategory. We write $A \subset B$ provided there is an injection $\kappa : A \dashrightarrow B$ and, in this case, call A a <u>subobject</u> of B. We prove first a lemma sufficient to insure anti-symmetry of the inclusion relation "$\subset$."

LEMMA 1.2. *If κ_1 and κ_2 are injections and I an identity, then $\kappa_1 \kappa_2 = I$ implies $\kappa_1 = \kappa_2 = I$.*

Proof. $\kappa_1 \kappa_2 = I$ and $R(\kappa_1) = R(I) = D(I) = D(\kappa_2)$. Thus $\kappa_2 \kappa_1$ is defined and $R(\kappa_2 \kappa_1) = D(\kappa_2 \kappa_1)$. By D-4, $\kappa_2 \kappa_1$ is an identity. Hence κ_1 and κ_2 are equivalences and, by D-3, identities. From the definition of an identity, $\kappa_1 = \kappa_2 = I$.

THEOREM 1.1. *The class of objects in a dicategory is partially ordered by the inclusion relation "$\subset$."*

Proof. $A \subset A$ since I_A is an injection. If $A \subset B$ and $B \subset A$, then there exist injections $\kappa : A \dashrightarrow B$ and $\kappa' : B \dashrightarrow A$; hence $\kappa\kappa' = I_B$, $\kappa = \kappa' = I_B = I_A$, $A = B$. If $A \subset B$ and $B \subset C$, the injections $\kappa_1 : A \dashrightarrow B$ and $\kappa_2 : B \dashrightarrow C$ have a product $\kappa_2 \kappa_1 : A \dashrightarrow C$ which is an injection, whence $A \subset C$.

For an injection $\kappa : A \dashrightarrow B$ it is convenient, and justifiable by axiom D-4, to write $\kappa = [B \supset A]$. With this notation, $[A \supset B][B \supset C] = [A \supset C]$ and $[A \supset A] = I_A$.

We shall at various times assume the presence of objects which behave as do one-element groups. Formally we say with [8] that a category $\mathcal{C}$ has a zero if it satisfies the following axiom.

Z. *There is an object Z such that for all objects A of $\mathcal{C}$ there*

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[a] The results stated here for partial order are given without proof in [8, §11].

exists a unique map $\zeta : A \longrightarrow Z$ and a unique map $\eta : Z \longrightarrow A$. We call Z a zero object. For a category with zero one shows easily that an object is a zero object if and only if it is equivalent to a zero object, that for each pair of objects A and B there exists a unique zero map[a] $0_{BA} : A \longrightarrow B$, that $0_{CB}0_{BA} = 0_{CA}$, and that if α is a zero map, then any product having α as a factor is also a zero map. For a dicategory with zero one also proves that for any zero object Z and any object A, 0_{AZ} is a submap and 0_{ZA} a supermap, that each object A has a unique zero subobject Z_A, that S(A), the set of all subobjects of A, is a partially ordered set having "unit" A and "zero" Z_A and that $0_{BA} = [B \supset Z_A]0_{Z_A A}$ is a canonical decomposition for all A and B.

4. Special Dicategories. If ρ_1 and ρ_2 are supermaps having a common domain and σ_1 and σ_2 are supermaps such that $\sigma_1 \rho_1 = \sigma_2 \rho_2$, then the ordered pair (σ_1, σ_2) is called a lower bound for the ordered pair (ρ_1, ρ_2). If (σ_1, σ_2) is a lower bound for (ρ_1, ρ_2) and if, whenever (σ_1', σ_2') is a lower bound for (ρ_1, ρ_2), there exists a supermap τ such that $\sigma_i' = \tau\sigma_i$, $i = 1, 2$, then (σ_1, σ_2) is called a maximal lower bound for the pair (ρ_1, ρ_2) of supermaps having common domain. With this terminology we may state the following axiom SD.

SD. *If ρ_1 and ρ_2 are supermaps and $D(\rho_1) = D(\rho_2)$, there exists a maximal lower bound for (ρ_1, ρ_2).*

COROLLARY. *If both (σ_1, σ_2) and (σ_1', σ_2') are maximal lower bounds for (ρ_1, ρ_2), the supermap τ such that $\sigma_i' = \tau\sigma_i$,*

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[a] α is a zero map if there exist maps β and γ and a zero object Z such that $\alpha = \beta\gamma$ and $R(\gamma) = D(\beta) = Z$.

$i = 1, 2$, <u>is unique and is an equivalence</u>.

Proof. If $\sigma'_i = \tau\sigma_i = \bar{\tau}\sigma_i$, $i = 1, 2$, then $\tau = \bar{\tau}$ by axiom D-5. Since (σ_1, σ_2) is a maximal lower bound, there exists a supermap τ' such that $\sigma_i = \tau'\sigma'_i$, $i = 1, 2$. Thus $\tau\tau'\sigma'_1 = \sigma'_1$, $\tau'\tau\sigma_1 = \sigma_1$, and, by axiom D-5, both $\tau\tau'$ and $\tau'\tau$ are identities.

We shall call a dicategory satisfying also axiom SD a <u>special dicategory</u>. For the remainder of this chapter the only dicategories we consider are special dicategories. Note that axiom Z is not assumed for the present.

We now derive some results for special dicategories which prove useful for the representation in Section 5 below. It is convenient to consider first an equivalence relation defined for any category.

Let (α_1, α_2) and (β_1, β_2) be ordered pairs of maps of a category $\mathcal{C}$ such that $R(\alpha_1) = R(\alpha_2)$ and $R(\beta_1) = R(\beta_2)$. We write $(\alpha_1, \alpha_2) \sim (\beta_1, \beta_2)$ just in case there exists an equivalence $\theta \varepsilon \mathcal{C}$ such that $\alpha_1 = \theta\beta_1$ and $\alpha_2 = \theta\beta_2$. Since an identity, the inverse of an equivalence, and the product of equivalences are each equivalences, it follows readily that $\sim$ is an equivalence relation in the class of all ordered pairs (α_1, α_2) of maps of a category $\mathcal{C}$ for which $R(\alpha_1) = R(\alpha_2)$. We denote by $[\alpha_1, \alpha_2]$ the equivalence class of all pairs (β_1, β_2) for which $(\alpha_1, \alpha_2) \sim (\beta_1, \beta_2)$.

We note that for a special dicategory $\mathcal{C}$ any lower bound (σ_1, σ_2) for a pair (ρ_1, ρ_2) of supermaps having common domain belongs to the class of all pairs (α_1, α_2) of $\mathcal{C}$ having $R(\alpha_1) = R(\alpha_2)$. Hence we may legitimately speak of the equivalence class $[\sigma_1, \sigma_2]$ determined by such a lower bound.

LEMMA 1.3. (SD)[a] _If_ (ρ_1, ρ_2) _is a pair of supermaps with common domain and_ (σ_1, σ_2) _is a maximal lower bound for_ (ρ_1, ρ_2), _then_ $(\bar{\sigma}_1, \bar{\sigma}_2)$ _is a maximal lower bound for_ (ρ_1, ρ_2) _if and only if_ $[\bar{\sigma}_1, \bar{\sigma}_2] = [\sigma_1, \sigma_2]$; _that is, just in case_ $(\bar{\sigma}_1, \bar{\sigma}_2) \sim (\sigma_1, \sigma_2)$.

Proof. Throughout this argument let i range over the set 1, 2. Suppose $(\bar{\sigma}_1, \bar{\sigma}_2) \sim (\sigma_1, \sigma_2)$. Then $\bar{\sigma}_i = \Theta\sigma_i$ for an equivalence Θ. But (σ_1, σ_2) is a maximal lower bound for (ρ_1, ρ_2) so that $\sigma_1\rho_1 = \sigma_2\rho_2$. Hence $\Theta\sigma_1\rho_1 = \Theta\sigma_2\rho_2 = \bar{\sigma}_1\rho_1 = \bar{\sigma}_2\rho_2$ and $(\bar{\sigma}_1, \bar{\sigma}_2)$ is a lower bound for (ρ_1, ρ_2). If (σ_1', σ_2') is also a lower bound for (ρ_1, ρ_2), then there exists a (unique) supermap τ such that $\sigma_i' = \tau\sigma_i$ by the assumption for (σ_1, σ_2). Because $\tau\sigma_1$ and $\Theta\sigma_1$ are defined, so is $\bar{\tau} = \tau\Theta^{-1}$. Also, by D-2, $\bar{\tau}$ is a supermap. But $\bar{\tau}\bar{\sigma}_i = (\tau\Theta^{-1})(\Theta\sigma_i) = \tau\sigma_i = \sigma_i'$. These latter equations and the fact that $(\bar{\sigma}_1, \bar{\sigma}_2)$ is a lower bound for (ρ_1, ρ_2) show that $(\bar{\sigma}_1, \bar{\sigma}_2)$ is a maximal lower bound for (ρ_1, ρ_2). Conversely, suppose $(\bar{\sigma}_1, \bar{\sigma}_2)$ and (σ_1, σ_2) are each maximal lower bounds for (ρ_1, ρ_2). By the Corollary to axiom SD, there is an equivalence Θ such that $\bar{\sigma}_i = \Theta\sigma_i$. Thus $(\bar{\sigma}_1, \bar{\sigma}_2) \sim (\sigma_1, \sigma_2)$.

By agreeing to call $[\sigma_1, \sigma_2]$ a _greatest lower bound_ for (ρ_1, ρ_2) just in case (σ_1, σ_2) is a maximal lower bound for (ρ_1, ρ_2), we have immediately from axiom SD and Lemma 1.3 the following result.

LEMMA 1.4. (SD) _For each pair_ (ρ_1, ρ_2) _of supermaps with common domain there exists a unique greatest lower bound_ $[\sigma_1, \sigma_2]$.

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[a]The initials SD indicate that the result is valid under the axioms for a special dicategory. Other initials elsewhere are to be similarly interpreted.

We may thus speak of $[\sigma_1, \sigma_2]$ as the greatest lower bound for (ρ_1, ρ_2) and shall write $[\sigma_1, \sigma_2] = \rho_1 \cap \rho_2$. Note that the existence of $\rho_1 \cap \rho_2$ implies the existence of $\rho_2 \cap \rho_1$, but that these lower bounds are not generally equal. The following three lemmas state some useful results concerning greatest lower bounds.

LEMMA 1.5. (SD) Let ρ_1 and ρ_2 be supermaps having a common domain and θ_1 and θ_2 equivalences such that $\theta_1\rho_1$ and $\theta_2\rho_2$ are defined. Then $[\sigma_1, \sigma_2] = \rho_1 \cap \rho_2$ if and only if $[\sigma_1\theta_1^{-1}, \sigma_2\theta_2^{-1}] = (\theta_1\rho_1) \cap (\theta_2\rho_2)$.

Proof. Assume first $[\sigma_1, \sigma_2] = \rho_1 \cap \rho_2$. Then $\sigma_1\rho_1 = \sigma_2\rho_2$ and $(\sigma_1\theta_1^{-1})(\theta_1\rho_1) = (\sigma_2\theta_2^{-1})(\theta_2\rho_2)$; whence $(\sigma_1\theta_1^{-1}, \sigma_2\theta_2^{-1})$ is a lower bound for $(\theta_1\rho_1, \theta_2\rho_2)$. If (σ_1', σ_2') is also a lower bound for $(\theta_1\rho_1, \theta_2\rho_2)$, then $\sigma_1'\theta_1\rho_1 = \sigma_2'\theta_2\rho_2$ and $(\sigma_1'\theta_1, \sigma_2'\theta_2)$ is a lower bound for (ρ_1, ρ_2). Hence there is a supermap τ such that $\sigma_i'\theta_i = \tau\sigma_i$, that is, $\sigma_i' = \tau(\sigma_i\theta_i^{-1})$, $i = 1, 2$. By definition $(\sigma_1\theta_1^{-1}, \sigma_2\theta_2^{-1})$ is a maximal lower bound for $(\theta_1\rho_1, \theta_2\rho_2)$ and $[\sigma_1\theta_1^{-1}, \sigma_2\theta_2^{-1}] = (\theta_1\rho_1) \cap (\theta_2\rho_2)$. The converse follows directly from the preceding argument.

LEMMA 1.6. (SD) If ρ_1 and ρ_2 are supermaps and $\rho_1\rho_2$ is defined, $[I_{R(\rho_1)}, \rho_1] = \rho_1\rho_2 \cap \rho_2$.

Proof. Letting $I = I_{R(\rho_1)}$ we have $I(\rho_1\rho_2) = (\rho_1)\rho_2$, so that (I, ρ_1) is a lower bound for $(\rho_1\rho_2, \rho_2)$. For any lower bound (σ_1', σ_2') of $(\rho_1\rho_2, \rho_2)$, $\sigma_1'\rho_1\rho_2 = \sigma_2'\rho_2$ and, by D-5, $\sigma_1'\rho_1 = \sigma_2'$; thus with $\tau = \sigma_1'$, we have $\tau I = \sigma_1'$ and $\tau\rho_1 = \sigma_2'$. Hence (I, ρ_1) is a maximal lower bound for $(\rho_1\rho_2, \rho_2)$ and $[I, \rho_1] = \rho_1\rho_2 \cap \rho_2$.

COROLLARY 1. $[I_{R(\rho)}, \rho] = \rho \cap I_{D(\rho)}$ for all supermaps ρ.

COROLLARY 2. <u>For any supermap</u> ρ <u>and identity</u> I <u>with</u> $I\rho$ <u>defined</u>, $[I, I] = \rho \cap \rho$.

Proofs. For the first corollary, take $\rho_2 = I_{D(\rho_1)}$; for the second, let $\rho_1 = I_{R(\rho_2)}$.

LEMMA 1.7. (SD) <u>Let</u> ρ_1, ρ_2, ρ_3 <u>be supermaps and suppose</u> $\rho_3 \rho_2$ <u>is defined and</u> $D(\rho_1) = D(\rho_2)$. <u>Then</u> $[\sigma_1, \sigma_2] = \rho_1 \cap \rho_2$ <u>and</u> $[\bar{\sigma}_1, \bar{\sigma}_2] = \sigma_2 \cap \rho_3$ <u>imply</u> $[\bar{\sigma}_1 \sigma_1, \bar{\sigma}_2] = \rho_1 \cap \rho_3 \rho_2$.

Proof. Since $D(\rho_1) = D(\rho_2) = D(\rho_3 \rho_2)$, we conclude from Lemma 1.4 that $\rho_1 \cap \rho_3 \rho_2$ exists. Also $(\bar{\sigma}_1 \sigma_1, \bar{\sigma}_2)$ is a lower bound for $(\rho_1, \rho_3 \rho_2)$; for $\sigma_1 \rho_1 = \sigma_2 \rho_2$ and $\bar{\sigma}_1 \sigma_2 = \bar{\sigma}_2 \rho_3$ by hypothesis, so that $(\bar{\sigma}_1 \sigma_1) \rho_1 = (\bar{\sigma}_1 \sigma_2) \rho_2 = \bar{\sigma}_2 (\rho_3 \rho_2)$. If (σ_1', σ_2') is any lower bound for $(\rho_1, \rho_3 \rho_2)$, then $\sigma_1' \rho_1 = (\sigma_2' \rho_3) \rho_2$ and $(\sigma_1', \sigma_2' \rho_3)$ is a lower bound for (ρ_1, ρ_2). Since $[\sigma_1, \sigma_2] = \rho_1 \cap \rho_2$ is the greatest lower bound for (ρ_1, ρ_2), there exists a supermap τ with the properties

$$(1.1) \qquad \sigma_1' = \tau \sigma_1, \quad \sigma_2' \rho_3 = \tau \sigma_2 .$$

By the second of equations (1.1), (τ, σ_2') is a lower bound for (σ_2, ρ_3); this result and the hypothesis concerning $\sigma_2 \cap \rho_3$ imply the existence of a supermap $\bar{\tau}$ for which

$$(1.2) \qquad \tau = \bar{\tau} \bar{\sigma}_1, \quad \sigma_2' = \bar{\tau} \bar{\sigma}_2 .$$

From (1.1) and (1.2) it follows that $\bar{\tau}(\bar{\sigma}_1 \sigma_1) = \sigma_1'$ and $\bar{\tau} \bar{\sigma}_2 = \sigma_2'$. Hence $[\bar{\sigma}_1 \sigma_1, \bar{\sigma}_2] = \rho_1 \cap \rho_3 \rho_2$.

5. <u>A Representation for Special Dicategories</u>. If $\mathcal{C}$ is a dicategory, we understand a <u>representation for</u> $\mathcal{C}$ <u>by classes</u> to be a

function Σ which assigns to each object A of $\mathcal{C}$ a class Σ_A and to each map $\alpha : A \dashrightarrow B$ a transformation $\Sigma_\alpha = (\alpha_f, \Sigma_A, \Sigma_B)$ in such a way that

(1) $\alpha\beta$ defined implies $\Sigma_\alpha \Sigma_\beta$ is defined and $\Sigma_{\alpha\beta} = \Sigma_\alpha \Sigma_\beta$;

(2) if κ is the injection $[B \supset A]$, then κ_f is the identity mapping on Σ_A and $\Sigma_A \subseteq \Sigma_B$ [a];

(3) if $\rho : A \dashrightarrow C$ is a supermap, then ρ_f maps Σ_A onto Σ_C.

A representation Σ is called <u>faithful</u> if it has the additional property

(4) $\Sigma_\alpha = \Sigma_\beta$ implies $\alpha = \beta$, for all maps α, $\beta \in \mathcal{C}$.

For any representation Σ of $\mathcal{C}$ by classes, $\alpha = I_A$ implies α_f is the identity function on Σ_A and $\Theta: A \dashrightarrow B$ an equivalence implies Θ_f is a one-to-one mapping of Σ_A onto Σ_B. For a faithful representation, $\Sigma_A = \Sigma_B$ implies A = B, for all objects A and B.

To obtain a representation for a special dicategory we construct for each object A a corresponding class S_A and for each map $\alpha : A \dashrightarrow B$ a function α_f mapping S_A into S_B. Following these constructions we prove the requisite properties for a representation.

We take S_A to be the class of all equivalence classes $[\rho_1, \rho_2]$ for which (i) ρ_1 and ρ_2 are supermaps having a common range and (ii) the domain of ρ_2 is a subobject of A. Clearly S_A is nonempty since $[I_A, I_A] \in S_A$.

In order to avoid numerous references to axiom D-3, we denote

---

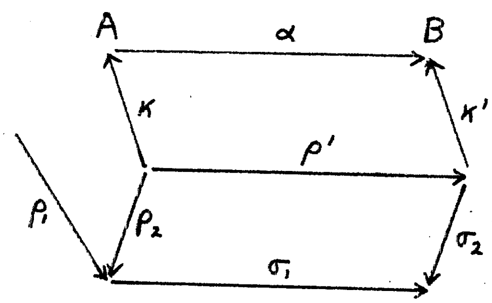
[a] By the definition of a transformation the condition imposed on κ_f guarantees $\Sigma_A \subseteq \Sigma_B$. We retain the two conditions on injections for clarity. Note that if representations were defined so as to send objects into classes and maps into <u>functions</u>, the added condition would be needed.

by inj(α) and sup(α), respectively, the injection and supermap factors in the canonical decomposition of α. Thus $\alpha = \mathrm{inj}(\alpha)\mathrm{sup}(\alpha)$ is canonical for all maps α. Note that $R(\mathrm{sup}(\alpha)) \subset R(\alpha)$ and $D(\mathrm{sup}(\alpha)) = D(\alpha)$.

Suppose now $\alpha : A \dashrightarrow B$ and $x = [\rho_1, \rho_2] \in S_A$. Then $\rho_2 \cap \mathrm{sup}(\alpha[A \supset D(\rho_2)])$ exists since $D(\mathrm{sup}(\gamma)) = D(\gamma) = D(\beta\gamma)$ whenever $\beta\gamma$ is defined. With this notation we let

$$(1.3)\qquad \begin{aligned} [\sigma_1, \sigma_2] &= \rho_2 \cap \mathrm{sup}(\alpha[A \supset D(\rho_2)]) , \\ \alpha_f(x) &= [\sigma_1\rho_1, \sigma_2] . \end{aligned}$$

The accompanying diagram indicates the maps which enter in the description of α_f. Constructions such as this and subsequent proofs may be visualized readily by drawing the appropriate diagrams.



$$\kappa = [A \supset D(\rho_2)] ,$$

$$\alpha\kappa = \kappa'\rho' .$$

To verify that α_f is a function we may proceed as follows. $[\sigma_1, \sigma_2] = \rho_2 \cap \mathrm{sup}(\alpha[A \supset D(\rho_2)])$ exists uniquely for each $\alpha : A \dashrightarrow B$ and for each choice of "representative" (ρ_1, ρ_2) in $x = [\rho_1, \rho_2] \in S_A$. Therefore, for each choice of a representative (ρ_1, ρ_2) of x and a representative (σ_1, σ_2) of $[\sigma_1, \sigma_2]$ there is a unique $y = [\sigma_1 \rho_1, \sigma_2]$ corresponding (under α_f) to x. Suppose possibly different representatives (ρ_1', ρ_2') and (σ_1', σ_2') are chosen; let $x' = [\rho_1', \rho_2']$, so that the correspondent of x' is $y' = [\sigma_1' \rho_1', \sigma_2']$. Clearly $x' = x$; we show $y' = y$. Since $x' = x$, there is an equivalence θ such that $\rho_1' = \theta \rho_1$ and $\rho_2' = \theta \rho_2$. But

$$\begin{aligned} [\sigma_1', \sigma_2'] &= \rho_2' \cap \sup(\alpha [A \supset D(\rho_2')]) \\ &= \theta \rho_2 \cap \sup(\alpha [A \supset D(\rho_2)]) \\ &= [\sigma_1 \theta^{-1}, \sigma_2], \end{aligned}$$

the last equation by Lemma 1.5. Thus for some equivalence ϕ, $\sigma_1' = \phi \sigma_1 \theta^{-1}$ and $\sigma_2' = \phi \sigma_2$. Hence $y' = [(\phi \sigma_1 \theta^{-1})\theta \rho_1, \phi \sigma_2] = [\phi \sigma_1 \rho_1, \phi \sigma_2] = [\sigma_1 \rho_1, \sigma_2]$. Thus $y' = y$, and α_f is a function defined on S_A. We see also that α_f maps S_A into S_B. For with x and y as above, x may be considered as arbitrary in S_A and $y = \alpha_f(x) = [\sigma_1 \rho_1, \sigma_2]$. Evidently $\sigma_1 \rho_1$ and σ_2 are supermaps. Since, by (1.3), (σ_1, σ_2) is a lower bound, $\sigma_1 \rho_2 = \sigma_2 \sup(\alpha [A \supset D(\rho_2)])$ and $R(\sigma_1 \rho_2) = R(\sigma_2)$. Finally, since $R(\sup(\beta)) \subset R(\beta)$ for every map β, $D(\sigma_2) = R(\sup(\alpha [A \supset D(\rho_2)])) \subset R(\alpha [A \supset D(\rho_2)]) = R(\alpha) = B$. Thus $y = [\sigma_1 \rho_1, \sigma_2] \varepsilon S_B$, and α_f is a function mapping S_A into S_B.

For each map $\alpha : A \dashrightarrow B$ let S_α be the transformation $S_\alpha = (\alpha_f, S_A, S_B)$. Here, as in the following theorem, S_A and α_f denote the class and function previously specified.

THEOREM 1.2. <u>If $\mathcal{C}$ is a special dicategory, then the mapping</u>

$$\Sigma : \begin{cases} A \rightarrow S_A \\ \alpha \rightarrow S_\alpha \end{cases}$$

<u>is a faithful representation for $\mathcal{C}$ by classes.</u>

Proof. We establish, in order, the conditions (1) - (4) listed at the beginning of this section.

First, suppose $\alpha\beta$ is defined. Then for appropriate objects A, B, and C, $\beta : A \dashrightarrow B$ and $\alpha : B \dashrightarrow C$. Let $x = [\rho_1, \rho_2]$ be an arbitrary element of S_A. Then by (1.3),

$$[\sigma_1, \sigma_2] = \rho_2 \cap \sup(\beta[A \supset D(\rho_2)]) ,$$

$$\beta_f(x) = [\sigma_1\rho_1, \sigma_2] ,$$

(1.4)
$$[\sigma_1', \sigma_2'] = \sigma_2 \cap \sup(\alpha[B \supset D(\sigma_2)]) ,$$

$$\alpha_f(\beta_f(x)) = [\sigma_1'\sigma_1\rho_1, \sigma_2'] ,$$

$$[\sigma_1^*, \sigma_2^*] = \rho_2 \cap \sup(\alpha\beta[A \supset D(\rho_2)]) ,$$

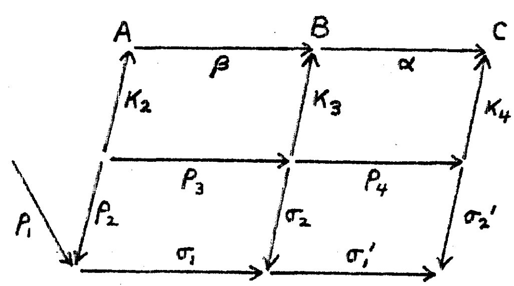
$$(\alpha\beta)_f(x) = [\sigma_1^*\rho_1, \sigma_2^*] .$$

Consider now the following canonical decompositions,

(1.5)
$$\beta[A \supset D(\rho_2)] = [B \supset R(\rho_3)]\rho_3 ,$$

$$\alpha[B \supset R(\rho_3)] = [C \supset R(\rho_4)]\rho_4 .$$

The following diagram illustrates some of the maps involved in this part of the proof.



$$\kappa_2 = [A \supset D(\rho_2)] ,$$

$$\kappa_3 = [B \supset R(\rho_3)] ,$$

$$\kappa_4 = [C \supset R(\rho_4)] .$$

From (1.5) there follows $\alpha\beta[A \supset D(\rho_2)] = \alpha[B \supset R(\rho_3)]\rho_3 =$ $[C \supset R(\rho_4)]\rho_4\rho_3$. By this last result, (1.4), and (1.5),

$$
\begin{aligned}
[\sigma_1, \sigma_2] &= \rho_2 \cap \rho_3 , \\
[\sigma_1', \sigma_2'] &= \sigma_2 \cap \rho_4 , \qquad (1.6) \\
[\sigma_1^*, \sigma_2^*] &= \rho_2 \cap \rho_4\rho_3 .
\end{aligned}
$$

By (1.6) and Lemma 1.7, $[\sigma_1'\sigma_1, \sigma_2'] = \rho_2 \cap \rho_4\rho_3 = [\sigma_1^*, \sigma_2^*]$. Thus there is an equivalence Θ such that $\sigma_1'\sigma_1 = \Theta\sigma_1^*$ and $\sigma_2' = \Theta\sigma_2^*$. By (1.4),

$$
\begin{aligned}
\alpha_f(\beta_f(x)) &= [\Theta\sigma_1^*\rho_1, \Theta\sigma_2^*] \\
&= [\sigma_1^*\rho_1, \sigma_2^*] \\
&= (\alpha\beta)_f(x) .
\end{aligned}
$$

Since x was arbitrary in S_A, $(\alpha\beta)_f = \alpha_f\beta_f$ and $S_{\alpha\beta} = S_\alpha S_\beta$.

Secondly, let again $x = [\rho_1, \rho_2] \in S_A$ and suppose $\kappa : A \dashrightarrow B$ is an injection. Since $D(\rho_2) \subset A \subset B$, $x \in S_B$. Thus $S_A \subseteq S_B$. By (1.3), $\kappa_f(x) = [\bar\sigma_1\rho_1, \bar\sigma_2]$, where $[\bar\sigma_1, \bar\sigma_2] = \rho_2 \cap \sup(\kappa[A \supset D(\rho_2)]) = \rho_2 \cap I_{D(\rho_2)}$. By Corollary 1, Lemma 1.6, $[\bar\sigma_1, \bar\sigma_2] = [I_{R(\rho_2)}, \rho_2]$. Thus there exists an equivalence ϕ for which $\bar\sigma_1 = \phi I_{R(\rho_2)} = \phi$, $\bar\sigma_2 = \phi\rho_2$. Hence $\kappa_f(x) = [\phi\rho_1, \phi\rho_2] = [\rho_1, \rho_2] = x$; κ_f is the identity function on S_A.

Thirdly, suppose $\rho : A \dashrightarrow C$ is a supermap. Since ρ_f maps S_A into S_C, it suffices to show ρ_f is onto. Let $y = [\tau_1, \tau_2] \in S_C$. By axiom SD and the fact that $R(\rho) = C$, there is an injection $\bar\kappa$ and a supermap $\hat\rho$ such that

$$(1.7) \qquad \rho\bar{\kappa} = [C \supset D(\tau_2)]\bar{\rho} .$$

Consider now $x = [\tau_1, \tau_2\bar{\rho}]$. Since $y \varepsilon S_C$, $R(\tau_1) = R(\tau_2\bar{\rho})$; also by (1.7), $D(\tau_2\bar{\rho}) \subset D(\rho) = A$. Hence $x \varepsilon S_A$. Now $\rho_f(x) = [\tilde{\sigma}_1 \tau_1, \tilde{\sigma}_2]$, where $[\tilde{\sigma}_1, \tilde{\sigma}_2] = \tau_2\bar{\rho} \cap \sup(\rho[A \supset D(\tau_2\bar{\rho})])$. But $\rho[A \supset D(\tau_2\bar{\rho})] = \rho\bar{\kappa} = [C \supset D(\tau_2)]\bar{\rho}$; whence $[\tilde{\sigma}_1, \tilde{\sigma}_2] = \tau_2\bar{\rho} \cap \bar{\rho}$. By Lemma 1.6, $[I_{R(\tau_2)}, \tau_2] = \tau_2\bar{\rho} \cap \bar{\rho}$. Thus, for some equivalence ψ, $\tilde{\sigma}_1 = \psi$ and $\tilde{\sigma}_2 = \psi\tau_2$. Therefore $\rho_f(x) = [\psi\tau_1, \psi\tau_2] = [\tau_1, \tau_2] = y$.

Next we prove the representation faithful. Suppose $S_A = S_B$. Then $A = B$. For $[I_A, I_A] \varepsilon S_A$, $[I_A, I_A] \varepsilon S_B$, and $D(I_A) = A \subset B$ by the definition of S_B. Similarly $[I_B, I_B] \varepsilon S_A$, $B \subset A$, and, by Theorem 1.1, $A = B$. Finally suppose $S_\alpha = S_\beta$, where $\alpha : A \longrightarrow B$, $\beta : A' \longrightarrow B'$. Since $S_\alpha = S_\beta$, we have $(\alpha_f, S_A, S_B) = (\beta_f, S_{A'}, S_{B'})$, $\alpha_f = \beta_f$, $S_A = S_{A'}$, $S_B = S_{B'}$, $A = A'$, and $B = B'$. Let $\rho = \sup(\alpha)$, $\rho' = \sup(\beta)$, and $I = I_{R(\rho)}$. Then $x = [I, \rho] \varepsilon S_A$. Now by (1.3), and recalling that $\alpha_f = \beta_f$,

$$(1.8) \qquad \alpha_f(x) = [\bar{\bar{\sigma}}_1 I, \bar{\bar{\sigma}}_2] = [\bar{\bar{\sigma}}_1, \bar{\bar{\sigma}}_2] = \rho \cap \sup(\alpha[A \supset D(\rho)]) ,$$

and

$$(1.9) \qquad \beta_f(x) = [\bar{\bar{\sigma}}_1, \bar{\bar{\sigma}}_2] = \rho \cap \sup(\beta[A \supset D(\rho)]) .$$

Since $\sup(\alpha[A \supset D(\rho)]) = \sup(\rho I_A) = \rho$ and $\sup(\beta[A \supset D(\rho)]) = \sup(\rho') = \rho'$, we have from (1.8) and (1.9),

$$(1.10) \qquad [\bar{\bar{\sigma}}_1, \bar{\bar{\sigma}}_2] = \rho \cap \rho = \rho \cap \rho' .$$

But, by Corollary 2, Lemma 1.6, $[I, I] = \rho \cap \rho$. By (1.10), $[I, I] = \rho \cap \rho'$ and $I\rho = I\rho'$. Hence $\rho = \rho'$. Thus $inj(\alpha)$ and $inj(\beta)$ are coterminal, that is, have the same range and the same domain, and therefore are equal. Consequently, $inj(\alpha)sup(\alpha) = inj(\beta)sup(\beta)$, that is, $\alpha = \beta$. The proof of Theorem 1.2. is complete.

6. Appendix. The representation given by Theorem 1.2 constitutes essentially a simultaneous generalization of the Cayley representation for groups and the representation for partially ordered sets given by Birkhoff [3]. For a group G is a special dicategory in which the maps are elements of G, all maps are equivalences, and only the identity element of G is an identity map. Also, a set P partially ordered by " $\leq$ " determines a special dicategory $\mathcal{C}_P$ if we take as objects the elements of P, as maps those ordered pairs (x, y) for which $x \leq y$, and if we consider each map (x, y) an injection with domain x and range y. In the category $\mathcal{C}_P$ the product of two maps (when defined) is therefore given by $(y, z)(x, y) = (x, z)$, $x \leq y \leq z$. Conversely, for every special dicategory $\mathcal{C}$ all of whose maps are injections and having the class of identities a set there is a partially ordered set P such that $\mathcal{C}$ is isomorphic (in the sense of preservation of products and injections) to $\mathcal{C}_P$ as constructed above.

In the case of a group G with identity element I, our representation yields $S_I = \{[I, \theta] \mid \theta \in G\}$ and, for $\phi \in G$ and $[I, \theta] \in S_I$, $S_\phi[I, \theta] = [I, \theta\phi^{-1}]$. Since $[I, \theta] = [I, \theta']$ implies $\theta = \theta'$, $S_I \cong G$ and S_ϕ is essentially the permutation $\theta \dashrightarrow \theta\phi^{-1}$.

In the case of a partially ordered set P, we have $I_x = (x, x)$ and find that the only equivalence classes occurring in the representing sets are of the form $[I_y, I_y]$ and that $[I_y, I_y]$ consists of the single pair (I_y, I_y). Thus the representation reduces to

$$S_x = \{(I_w, I_w) \mid w \leq x\} ,$$

$$S_{(x,y)}(I_z, I_z) = (I_z, I_z) \qquad \text{for each } z \leqq x,$$

and is therefore essentially that which sends $x \varepsilon P$ into $(x^*)^+$, the principal ideal of all $w \leqq x$. Both representations send greatest lower bounds (in the lattice-theoretic sense), when they exist, into intersections. If P is a complemented lattice L, then the representing sets form a complemented lattice S(L); if L has unique complements, so does S(L).

Lattices, but not partially ordered sets generally, may also be interpreted as special dicategories in which all maps are supermaps, $\rho : x \dashrightarrow y$ if and only if $y \leqq x$. Our theorem provides in this case a representation by sets and by functions mapping S_x onto S_y whenever $y \leqq x$. This representation is somewhat more complicated than that by principal ideals in the "injection" interpretation.

CHAPTER II

A REPRESENTATION BY ABELIAN SEMIGROUPS

1. Preliminary Remarks. In this and subsequent chapters we presuppose a familiarity on the part of the reader with the Theorem 1.1 and the sections 14, 17, 18, and 19 of [8]. We shall define an abelian semigroup dicategory (ASD) and show that any ASD is isomorphic to a dicategory of abelian semigroups, thereby extending [8, Th. 20.1] from a category to a dicategory and, incidentally, justifying our terminology. Before stating our definition, we find it convenient to repeat certain definitions and observations from [8]. These are included in the following paragraph.

Let $\mathcal{C}$ be a dicategory and $\alpha : A \dashrightarrow B$ a map (element) of $\mathcal{C}$. Then α has the canonical decomposition $\alpha = [B \supset C]\rho$ for a uniquely determined object C. We call C the image of α and write $\mathrm{Im}(\alpha) = C \subset R(\alpha)$. Note that $\mathrm{Im}(\alpha) = R(\sup(\alpha))$. If $\alpha : A \dashrightarrow B$ and $T \subset A$, then T has an "image" $\alpha_s(T) = \mathrm{Im}(\alpha[A \supset T])$. α_s is order preserving in the sense that $T_1 \subset T_2 \subset A$ implies $\alpha_s(T_1) \subset \alpha_s(T_2) \subset B$. For each object A we denote by $\mathcal{S}(A)$ the set of all subobjects $T \subset A$. Evidently, by our Theorem 1.1, $\mathcal{S}(A)$ is partially ordered by the inclusion relation. Finally, an object J is said to be an integral object if it has the following three properties:

(i) For two distinct maps $\alpha_1, \alpha_2 : G \dashrightarrow H$ of $\mathcal{C}$ with the same domain G and the same range H there exists a map β: J ---> G such that $\alpha_1\beta \neq \alpha_2\beta$.

(ii) If J' is another object of $\mathcal{C}$ with the property (i), there exists in $\mathcal{C}$ a map σ : J' ---> J such that $\sigma\sigma' = I_J$ for some $\sigma' \in \mathcal{C}$.

(iii) If $\alpha_1\alpha_2 = I_J$ for two maps α_1, α_2: J ---> J, then α_1 and α_2 are equivalences.

These properties are seen to characterize, up to isomorphism, both the object J in the category $\mathcal{C}$ and the additive group of integers in the dicategory of all abelian groups. Certain other properties of the group of integers are noted in Definitions 2.1 and 2.2 below.

2. Abelian Semigroup Dicategories.

The following definition is based on [8, Th. 1.1].

DEFINITION 2.1. *An object F of a dicategory $\mathcal{C}$ is a free object if, whenever ρ: B ---> A is a supermap and α: F ---> A is a map of $\mathcal{C}$, there exists a map β: F ---> B such that $\rho\beta = \alpha$.*

We now state as axioms for an abelian semigroup dicategory certain properties which can be verified in the dicategory of all abelian groups.

DEFINITION 2.2. *An abelian semigroup dicategory $\mathcal{C}$ is a dicategory satisfying axiom Z (existence of zero) and the following additional axioms.*

ASD-1. *There exists an object in $\mathcal{C}$ which is both an integral object and a free object.*

ASD-2. *There exists in $\mathcal{C}$ a free-and-direct product diagram for any two objects of $\mathcal{C}$.*

ASD-3. *For each object A and every pair of subobjects B, C, of*

A <u>there exists a least upper bound</u> B $\cup_A$ C <u>in the partially ordered set</u> $\mathcal{S}(A)$.

ASD-4. <u>If</u> α : A ---> B, <u>then, for any two subobjects</u> C, D, <u>of</u> A,

$$(2.1) \qquad \alpha_s(C \cup_A D) \subset \alpha_s(C) \cup_B \alpha_s(D) .$$

As in [8], it may be noted that equality holds in (2.1) since α_s is an order preserving operation. Also our axioms ASD-1, -2, -3, -4 correspond, respectively, to MacLane's axioms AC-1, AC-2, LC-1, and LC-2. In ASD-2 we have added the word "diagram" to his AC-2 simply for the purpose of clarification; in the others we have considerably weaker statements.

Let $\mathcal{C}$ be an abelian semigroup dicategory and consider two maps with common domain, α_1: A ---> B_1 and α_2: A ---> B_2, having ranges included in a common object; that is, $B_1 \subset T$ and $B_2 \subset T$ for some T. Then $B = B_1 \cup_T B_2$ exists uniquely in $\mathcal{S}(T)$ and the maps $[B \supset B_1]\alpha_1$ and $[B \supset B_2]\alpha_2$ are coterminal. Using the addition of [8][a], $[B \supset B_1]\alpha_1 + [B \supset B_2]\alpha_2$ exists uniquely. Hence the operation

$$(2.2) \qquad \alpha_1 \oplus_T \alpha_2 = [B \supset B_1]\alpha_1 + [B \supset B_2]\alpha_2$$

is well-defined. This operation generalizes slightly the notion of addition of maps to what we shall call T-<u>addition</u> of maps. We conclude this section with a result stating properties of T-addition which will emerge as tools for the representation argument to follow.

---

[a]From this paper it is apparent that the operation + , which is not primitive in the notion of a dicategory, may be defined in any dicategory which assumes axiom ASD-2.

Throughout we use freely facts from [8] concerning addition (+).

LEMMA 2.1. (ASD) <u>If</u> $\alpha_1: A \dashrightarrow B_1$, $\alpha_2: A \dashrightarrow B_2$, $\alpha_3: A \dashrightarrow B_3$, <u>and</u> T <u>is an object containing each of</u> B_1, B_2, B_3 <u>as subobject, then</u>

(A) $\alpha_1 \oplus_T \alpha_2 = \alpha_2 \oplus_T \alpha_1$,

(B) $(\alpha_1 \oplus_T \alpha_2) \oplus_T \alpha_3 = \alpha_1 \oplus_T (\alpha_2 \oplus_T \alpha_3)$,

(C) $\alpha_1 \oplus_T 0_{Z_T A} = \alpha_1$.

Proof. (A) follows directly from (2.2) and the commutative property of addition (+). (B) follows from (2.2) and the associative property of addition by a rather lengthy but completely straightforward argument. To prove (C), we observe first that $Z_{B_1} = Z_T$ since each is the "zero" of the partially ordered set $\mathcal{S}(T)$ whose "unit" is T. Hence $Z_T \cup_T B_1 = Z_{B_1} \cup_T B_1 = B_1$ and $\alpha_1 \oplus_T 0_{Z_T A} = [B_1 \supset B_1]\alpha_1 + [B_1 \supset Z_T]0_{Z_T A} = \alpha_1 + 0_{B_1 A} = \alpha_1$.

3. <u>A Representation for Abelian Semigroup Dicategories</u>. For an ASD $\mathcal{C}$ we understand a <u>representation for $\mathcal{C}$ by abelian semigroups</u> to be first a representation Σ by classes such that, for each object A, Σ_A is an abelian semigroup in the sense that Σ_A is closed relative to some commutative and associative operation and contains a zero for this operation and, for each map α, Σ_α is a transformation whose functional component is a semigroup homomorphism; secondly we require that for each injection $\kappa: A \dashrightarrow B$, Σ_A is a subsemigroup of Σ_B.

Our representation is obtained by constructing for each object A a set G_A, taking A-addition as the operation, and determining for each map $\alpha : A \dashrightarrow B$ a homomorphism of G_A (considered as a set with an operation) into G_B.

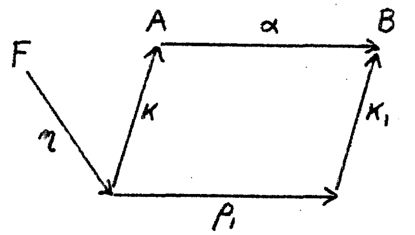
Let F be any one integral and free object. Since all such objects are categorically equivalent, we may interpret F as the group of integers. The set G_A we construct is, in the case of groups, the set of homomorphisms of the integers into (a subgroup of) the group A. Each such homomorphism η has as image a cyclic subgroup of A so that η may be identified with the pair (a, C), where $\eta(1) = a \in A$ and $R(\eta) = C$. We take, then, G_A to be the following set of maps of the dicategory $\mathcal{C}$:

$$(2.3) \qquad G_A = \left\{ \eta \mid \eta : F \dashrightarrow C \text{ for some } C \subset A \right\}.$$

Note that G_A is nonempty since $0_{AF} \in G_A$. Now for each $\alpha : A \dashrightarrow B$ and each $\eta \in G_A$, the map $\sup(\alpha\,[A \supset R(\eta)\,])\eta$ is uniquely determined. We may take, then, G_α to be the <u>function</u> such that

$$(2.4) \qquad G_\alpha(\eta) = \sup(\alpha\,[D(\alpha) \supset R(\eta)\,])\,\eta \quad \text{for all } \alpha \in \mathcal{C},\ \eta \in G_{D(\alpha)}.$$

The behavior of G_α may be visualized as in the accompanying diagram.



$$\kappa = [D(\alpha) \supset R(\eta)\,],$$

$$\alpha\kappa = \kappa_1 \rho_1,$$

$$G_\alpha(\eta) = \rho_1 \eta.$$

LEMMA 2.2. (ASD) <u>The set G_A is closed under the commutative and associative operation $\oplus_A$ and contains $0_{Z_A F}$ as the unique zero for</u>

this operation.

Proof. Consider elements $\eta_1: F \dashrightarrow C_1 \subset A$ and $\eta_2: F \dashrightarrow C_2 \subset A$ of G_A. Then $C = C_1 \cup_A C_2 \subset A$. By (2.2), $\eta_1 \oplus_A \eta_2: F \dashrightarrow C$; hence G_A is closed. By Lemma 2.1, the operation $\oplus_A$ is commutative and associative and $0_{Z_A F}$, which is clearly a member of G_A, serves as zero. By a standard argument, the zero element is unique.

LEMMA 2.3. (ASD) *If* $\alpha: A \dashrightarrow B$, *the function* G_α *is a homomorphism on the abelian semigroup* $\overline{G}_A = [G_A, \oplus_A]$ *to the abelian semigroup* $\overline{G}_B$.

Proof. Let $\eta_1: F \dashrightarrow C_1$, $\eta_2: F \dashrightarrow C_2$ be arbitrary elements of G_A. By (2.4),

$$G_\alpha(\eta_1): F \dashrightarrow R\left\{\sup(\alpha[A \supset R(\eta_1)]\right\} \subset R(\alpha[A \supset R(\eta_1)] = R(\alpha).$$

Hence G_α is a mapping into G_B. Taking $C = C_1 \cup_A C_2$, we have from (2.2),

$$\eta_1 \oplus_A \eta_2 = [C \supset C_1]\eta_1 + [C \supset C_2]\eta_2: F \dashrightarrow C\,. \tag{2.5}$$

Let $\rho = \sup(\alpha)$. By D-3, we may assume the canonical decompositions

$$\begin{aligned} \rho[A \supset C] &= \overline{\kappa}\overline{\rho}\,, \\ \rho[A \supset C_1] &= \kappa_1\rho_1\,, \\ \rho[A \supset C_2] &= \kappa_2\rho_2\,. \end{aligned} \tag{2.6}$$

Also let $\overline{D} = R(\overline{\rho})$, $D_1 = R(\rho_1)$, $D_2 = R(\rho_2)$, $E = R(\rho)$. We show $\overline{D} = D_1 \cup_B D_2$. By axiom ASD-4 and (2.6),

$$\begin{aligned}(2.7)\qquad \rho_s(C_1 \cup_A C_2) &= \rho_s(C_1) \cup_E \rho_s(C_2) \\ &= \mathrm{Im}(\rho[A \supset C_1]) \cup_E \mathrm{Im}(\rho[A \supset C_2]) \\ &= R(\rho_1) \cup_E R(\rho_2) \\ &= D_1 \cup_B D_2 \,.\end{aligned}$$

But $\rho_s(C_1 \cup_A C_2) = \rho_s(C) = \mathrm{Im}(\rho[A \supset C]) = R(\bar{\rho}) = \bar{D}$. Hence

$$(2.8)\qquad \bar{D} = D_1 \cup_B D_2 \,.$$

Now by (2.4), (2.5), and the distributive property for addition,

$$\begin{aligned}(2.9)\qquad G_\alpha(\eta_1 \oplus_A \eta_2) &= \sup(\alpha[A \supset C])(\eta_1 \oplus_A \eta_2) \\ &= \sup(\rho[A \supset C])(\eta_1 \oplus_A \eta_2) \\ &= \bar{\rho}([C \supset C_1]\eta_1 + [C \supset C_2]\eta_2) \\ &= \bar{\rho}[C \supset C_1]\eta_1 + \bar{\rho}[C \supset C_2]\eta_2 \,.\end{aligned}$$

But $G_\alpha(\eta_1) = \sup(\alpha[A \supset R(\eta_1)])\eta_1 = \sup(\rho[A \supset C_1])\eta_1 = \rho_1\eta_1$; similarly, $G_\alpha(\eta_2) = \rho_2\eta_2$. By (2.2) and (2.8),

$$(2.10)\qquad G_\alpha(\eta_1) \oplus_B G_\alpha(\eta_2) = [\bar{D} \supset D_1]\rho_1\eta_1 + [\bar{D} \supset D_2]\rho_2\eta_2 \,.$$

We shall show $\bar{\rho}[C \supset C_i] = [\bar{D} \supset D_i]\rho_i$, $i = 1, 2$; the lemma then follows from (2.9) and (2.10).

By (2.6),

$$\bar{\kappa}\bar{\rho}[C \supset C_i] = \rho[A \supset C][C \supset C_i]$$

(2.11) $$= \rho[A \supset C_i]$$

$$= \kappa_i \rho_i , \qquad i = 1, 2 .$$

Using the canonical decompositions

(2.12) $$\bar{\rho}[C \supset C_i] = \kappa_i^* \rho_i^* , \qquad i = 1, 2 ,$$

we have

(2.13) $$\bar{\kappa}\bar{\rho}[C \supset C_i] = \bar{\kappa}\kappa_i^* \rho_i^* , \qquad i = 1, 2 .$$

By (2.11) and (2.13), $\kappa_1 = \bar{\kappa}\kappa_1^*$, $\kappa_2 = \bar{\kappa}\kappa_2^*$, $\rho_1 = \rho_1^*$, $\rho_2 = \rho_2^*$. A consideration of ranges and domains shows $\kappa_1^* = [\bar{D} \supset D_1]$, $\kappa_2^* = [\bar{D} \supset D_2]$. By (2.12), $\bar{\rho}[C \supset C_1] = \kappa_1^* \rho_1^* = [\bar{D} \supset D_1]\rho_1$ and $\bar{\rho}[C \supset C_2] = \kappa_2^* \rho_2^* = [\bar{D} \supset D_2]\rho_2$, as we wished to show.

We have already used the notation $\bar{G}_A$ to denote the abelian semigroup $[G_A, \oplus_A]$. For each map $\alpha : A \dashrightarrow B$ let $\bar{G}_\alpha$ be the transformation $(G_\alpha, \bar{G}_A, \bar{G}_B)$.

THEOREM 2.1. <u>If $\mathcal{C}$ is an abelian semigroup dicategory, the mapping</u>

$$\Sigma : \begin{cases} A \longrightarrow \bar{G}_A \\ \alpha \longrightarrow \bar{G}_\alpha \end{cases}$$

<u>is a faithful representation for $\mathcal{C}$ by abelian semigroups.</u>

Proof. In the previous two lemmas we have proved that $\bar{G}_A$ is an abelian semigroup and that G_α is a homomorphism. For the mapping Σ we demonstrate in order the preservation of products, of injections, and of supermaps, and the property of faithfulness.

First suppose $\alpha\beta$ is defined; then $\beta : A \dashrightarrow B$ and $\alpha : B \dashrightarrow C$ for appropriate objects A, B, and C. Letting $\eta \in G_A$, we have $\eta : F \dashrightarrow D$ for some $D \subset A$. For appropriate supermaps ρ_1 and ρ_2 there exist canonical decompositions,

$$\beta[A \supset D] = [B \supset R(\rho_1)]\rho_1 ,$$

$$\alpha[B \supset R(\rho_1)] = [C \supset R(\rho_2)]\rho_2 ,$$

and, consequently, the further decomposition,

$$\alpha\beta[A \supset D] = \alpha[B \supset R(\rho_1)]\rho_1 = [C \supset R(\rho_2)]\rho_2\rho_1 .$$

Thus

$$G_\beta(\eta) = \sup(\beta[A \supset R(\eta)])\eta = \rho_1\eta ,$$

$$G_\alpha G_\beta(\eta) = \sup(\alpha[B \supset R(\rho_1\eta)])\rho_1\eta = \rho_2\rho_1\eta ,$$

$$G_{\alpha\beta}(\eta) = \sup(\alpha\beta[A \supset R(\eta)])\eta = \rho_2\rho_1\eta .$$

Since η was arbitrary, $G_{\alpha\beta} = G_\alpha G_\beta$ and $\overline{G}_{\alpha\beta} = \overline{G}_\alpha\overline{G}_\beta$.

Secondly, suppose $\kappa : P \dashrightarrow Q$ is an injection and $\gamma \in G_P$, $\gamma : F \dashrightarrow M \subset P$. Then $G_\kappa(\gamma) = \sup(\kappa[P \supset M])\gamma = I_M\gamma = \gamma$. Also, for any two elements of G_P, $\gamma_1 : F \dashrightarrow M_1$, $\gamma_2 : F \dashrightarrow M_2$, we have $M_1 \cup_P M_2 = M_1 \cup_Q M_2$ (since $P \subset Q$) and, by the definition (2.2), $\gamma_1 \oplus_P \gamma_2 = \gamma_1 \oplus_Q \gamma_2$. Thus P-addition agrees with Q-addition on G_P. By Lemma 2.2, G_P and G_Q have the same zero element since $Z_P = Z_Q$. In summary, $\overline{G}_P$ is a subsemigroup of $\overline{G}_Q$ and G_κ is the identity homomorphism on $\overline{G}_P$.

Thirdly, suppose $\rho : K \dashrightarrow L$ is a supermap and $\xi \in G_L$, $\xi : F \dashrightarrow N \subset L$. By axiom D-6, there is an injection κ_1 and a supermap ρ_1 such that

(2.14) $$\rho\kappa_1 = [L \supset N]\rho_1 .$$

Letting $D(\kappa_1) = J$, $\kappa_1 = [K \supset J]$. Since F is a <u>free</u> object and ρ_1 a supermap, $\rho_1 : J \dashrightarrow N$, there is a map $\eta : F \dashrightarrow J$ such that $\rho_1\eta = \xi$. We show $G_\rho(\eta) = \xi$. Note first that $\eta \in G_K$; for $R(\eta) = J \subset K$. Then by (2.4) and (2.14), $G_\rho(\eta) = \sup(\rho[K \supset R(\eta)])\eta = \sup(\rho\kappa_1)\eta = \rho_1\eta = \xi$. Hence G_ρ is onto G_L.

Finally, suppose $\alpha_1 : A_1 \dashrightarrow B_1$, $\alpha_2 : A_2 \dashrightarrow B_2$. We wish to show $\bar{G}_{\alpha_1} = \bar{G}_{\alpha_2}$ implies $\alpha_1 = \alpha_2$. Hence we may assume $G_{A_1} = G_{A_2}$ and $G_{B_1} = G_{B_2}$. But $G_X = G_Y$ implies $X = Y$ for all objects X and Y. For if $G_X = G_Y$, then $O_{XF} \in G_Y$, $O_{YF} \in G_X$, $X \subset Y \subset X$, and $X = Y$. It suffices, then, to consider α_1 and α_2 coterminal. This we do and write $\alpha_1, \alpha_2 : A \dashrightarrow B$. We now show $\alpha_1 \neq \alpha_2$ implies $G_{\alpha_1} \neq G_{\alpha_2}$. Suppose $\alpha_1 \neq \alpha_2$. By property (i) of the <u>integral</u> object F, there exists a map $\beta : F \dashrightarrow A$ such that $\alpha_1\beta \neq \alpha_2\beta$. Clearly $\beta \in G_A$; by (2.4), $G_{\alpha_1}(\beta) = \sup(\alpha_1)\beta$ and $G_{\alpha_2}(\beta) = \sup(\alpha_2)\beta$. If $R(\sup(\alpha_1)) \neq R(\sup(\alpha_2))$, then $G_{\alpha_1}(\beta) \neq G_{\alpha_2}(\beta)$; in this case the proof would be complete. Assume, then, $R(\sup(\alpha_1)) = R(\sup(\alpha_2))$, that is, $\mathrm{Im}(\alpha_1) = \mathrm{Im}(\alpha_2)$. We have $\mathrm{inj}(\alpha_1) = \mathrm{inj}(\alpha_2)$ and consequently, since $\alpha_1\beta \neq \alpha_2\beta$, $\sup(\alpha_1)\beta \neq \sup(\alpha_2)\beta$. Again $G_{\alpha_1}(\beta) \neq G_{\alpha_2}(\beta)$. Therefore $G_{\alpha_1} \neq G_{\alpha_2}$ and $\bar{G}_{\alpha_1} \neq \bar{G}_{\alpha_2}$.

CHAPTER III

A REPRESENTATION BY ABELIAN GROUPS

1. Preliminary Remarks. In this chapter we define an abelian dicategory (AD) and show that every abelian dicategory is isomorphic to a dicategory of abelian groups. Only one of the AD axioms is new; it reflects a simple property of groups, namely, that groups which are distinct have either distinct sets of elements or distinct group operations. Of the remaining axioms all but one have been used previously and that one has been borrowed from [8]. Each axiom is valid in the dicategory of all abelian groups.

The constructions and arguments for the representation resemble those of Chapter II. In the appendix to this chapter we indicate how a representation even more like that of the preceding chapter can be obtained; for these, an exact relationship can be formulated. Roughly speaking, the representation of Theorem 2.1 yields semigroups rather than groups because the sets are too "big" and because the operation of A-addition is dependent upon joins. One may obtain groups by "cutting down" the sets and foregoing use of l.u.b.

2. Abelian Dicategories.

DEFINITION 3.1. *An abelian dicategory* $\mathcal{C}$ *is a dicategory satisfying axioms* Z, ASD-1, ASD-2, *and two further axioms:*

AD-1[a]. <u>For each object</u> A <u>there exists a map</u> $V_A: A \dashrightarrow A$ <u>such that</u> $V_A + I_A = O_{AA}$;

AD-2. <u>If</u> J <u>is an integral and free object and if the objects</u> A_1 <u>and</u> A_2 <u>are distinct, then there exist maps</u> $\alpha_1: J \times J \dashrightarrow A_1$ <u>and</u> $\alpha_2: J \times J \dashrightarrow A_2$ <u>such that</u> $\sup(\alpha_1) \neq \sup(\alpha_2)$.

The only consequences of axiom AD-1 which we shall need are given in the first sentence of the paragraph following the statement of axiom ABC-1 in [8, p. 513]. For simplicity of reference we quote this sentence:

(3.1) <u>It follows readily that</u> V_A <u>is unique, and that</u> $V_A V_A = I_A$.

The object $J \times J$ involved in axiom AD-2 is the object in a free-and-direct product diagram constructed on J. For groups, J is the additive group of integers and $J \times J$ the usual cartesian product.

Let $\mathcal{C}$ be an abelian dicategory and $\sigma_1: F \dashrightarrow A_1$, $\sigma_2: F \dashrightarrow A_2$ two supermaps of $\mathcal{C}$ with common domain F and with ranges $A_1, A_2 \subset T$ for some object T. We define for such supermaps the operation F:T-<u>addition</u> by

$$(3.2) \qquad \sigma_1 \;{}^{F}{+}_{T}\; \sigma_2 = \sup \{ [T \supset A_1] \sigma_1 + [T \supset A_2] \sigma_2 \} .$$

We use the symbol "F" in the names of the operation both to remind ourselves that $D(\sigma_1) = D(\sigma_2) = F$ and to make easy the transition to the application of F:T-addition in which F is a <u>free</u> object. For the moment, F is an arbitrary object. We note $\sigma_1 \;{}^{F}{+}_{T}\; \sigma_2: F \dashrightarrow A_3 \subset T$

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[a]This is the axiom ABC-1 of [8].

for some A_3. For if $\beta = [T \supset A_1]\sigma_1 + [T \supset A_2]\sigma_2$, then $\beta : F \dashrightarrow T$ and $\sup(\beta) : F \dashrightarrow R(\sup(\beta)) \subset T$. Also, it is convenient in the following lemma to note that for any maps $\alpha_1, \alpha_2 : F \dashrightarrow T$, there is the canonical decomposition

$$(3.3) \qquad \alpha_1 + \alpha_2 = \kappa(\sup(\alpha_1) \ {}^F{+}_T \ \sup(\alpha_2)) .$$

LEMMA 3.1. <u>For any two objects</u> F <u>and</u> T <u>and for any three supermaps</u> $\sigma_i : F \dashrightarrow A_i \subset T$, $i = 1, 2, 3$,

(A) $\sigma_1 \ {}^F{+}_T \ \sigma_2 = \sigma_2 \ {}^F{+}_T \ \sigma_1$,

(B) $\sigma_1 \ {}^F{+}_T \ (\sigma_2 \ {}^F{+}_T \ \sigma_3) = (\sigma_1 \ {}^F{+}_T \ \sigma_2) \ {}^F{+}_T \ \sigma_3$,

(C) $\sigma_1 \ {}^F{+}_T \ 0_{Z_T F} = \sigma_1$,

(D) $\sigma_1 \ {}^F{+}_T \ \sigma_1 V_F = 0_{Z_T F}$.

Proof. Let $\kappa_i = [T \supset A_i]$, $i = 1, 2, 3$. (A) follows directly from (3.2) and the fact that $\kappa_1\sigma_1 + \kappa_2\sigma_2 = \kappa_2\sigma_2 + \kappa_1\sigma_1$. To prove (B), we consider the canonical decompositions

$$(3.4) \qquad \begin{aligned} \kappa_1\sigma_1 + \kappa_2\sigma_2 &= \kappa_4(\sigma_1 \ {}^F{+}_T \ \sigma_2) , \\ \kappa_2\sigma_2 + \kappa_3\sigma_3 &= \kappa_5(\sigma_2 \ {}^F{+}_T \ \sigma_3) . \end{aligned}$$

Then by (3.3) (or (3.2)), (3.4), and the associative property of addition (+), there exist injections κ_6 and κ_7, each having range T, such that

$$\kappa_4(\sigma_1 \;{}^{F}{+}_{T}\; \sigma_2) + \kappa_3\sigma_3 = \kappa_6\left\{(\sigma_1 \;{}^{F}{+}_{T}\; \sigma_2) \;{}^{F}{+}_{T}\; \sigma_3\right\}$$

$$(3.5) \qquad = \kappa_1\sigma_1 + \kappa_2\sigma_2 + \kappa_3\sigma_3$$

$$= \kappa_1\sigma_1 + \kappa_5(\sigma_2 \;{}^{F}{+}_{T}\; \sigma_3)$$

$$= \kappa_7\left\{\sigma_1 \;{}^{F}{+}_{T}\; (\sigma_2 \;{}^{F}{+}_{T}\; \sigma_3)\right\}.$$

The associativity of F:T-addition follows by uniqueness of canonical decompositions from the first and last of the equations (3.5).

To prove (C), we observe first that $\sigma_1 \;{}^{F}{+}_{T}\; 0_{Z_TF}$ makes sense since 0_{Z_TF} is a supermap. Next we use [8, eq. (19.2)] to obtain

$$\kappa_1\sigma_1 + 0_{TZ_T}0_{Z_TF} = \kappa_1\sigma_1 + 0_{TF} = \kappa_1\sigma_1 .$$

By definition (3.2), $\sigma_1 \;{}^{F}{+}_{T}\; 0_{Z_TF} = \sigma_1$.

Finally, to prove (D) we rely on (3.1) to see that V_F is an equivalence and hence a supermap. Thus $\sigma_1 \;{}^{F}{+}_{T}\; \sigma_1 V_F$ is meaningful. By the distributive property of addition (+) and axiom AD-1,

$$\sigma_1 + \sigma_1V_F = \sigma_1(I_F + V_F) = \sigma_1 0_{FF} = 0_{A_1F} = 0_{A_1Z_T}0_{Z_TF} .$$

By (3.3), $\sigma_1 \;{}^{F}{+}_{T}\; (\sigma_1V_F) = 0_{Z_TF}$.

It is convenient to note here the following result which will be used in establishing our representation.

LEMMA 3.2. _If_ J _is a free object, then_ $J \times J$ _is a free object._

Proof. We make use of results provable from axiom ASD-2 and stated in [8: eq. (18.5'), Th. 18.5, eq. (18.8)].

Suppose there is a map $\alpha : J \times J \dashrightarrow A$ and a supermap $\rho : B \dashrightarrow A$. We wish to find a map $\beta : J \times J \dashrightarrow B$ such that $\rho\beta = \alpha$.

But $\alpha = \Delta_A(\gamma \times \delta)$ for some $\gamma, \delta : J \dashrightarrow A$. Since J is free, there exist maps $\beta_1, \beta_2 : J \dashrightarrow B$ such that $\rho\beta_1 = \gamma$, $\rho\beta_2 = \delta$. Hence

$$\begin{aligned} \alpha &= \Delta_A(\gamma \times \delta) \\ &= \Delta_A(\rho\beta_1 \times \rho\beta_2) \\ &= \Delta_A(\rho \times \rho)(\beta_1 \times \beta_2) \\ &= \rho\, \Delta_B(\beta_1 \times \beta_2) . \end{aligned}$$

Thus, for $\beta = \Delta_B(\beta_1 \times \beta_2)$, $\beta : J \times J \dashrightarrow B$ and $\rho\beta = \alpha$. Therefore $J \times J$ is free.

3. A Representation for Abelian Dicategories. If $\mathscr{C}$ is an abelian dicategory, a representation for $\mathscr{C}$ by abelian groups is a representation Σ by abelian semigroups in which each Σ_A is an abelian group.

Our representation is obtained, in part, by constructing within $\mathscr{C}$ a set G_A for each object A which, for the particular dicategory $\mathscr{C}_{G+}$ of all abelian groups, is just the set of homomorphisms of the cartesian product of the integers with themselves onto subgroups of the group A. The operation of F:A-addition already described and the homomorphism G_α we shall construct for the representation are similarly abstracted from properties of this free abelian group built from the integers and the dicategory $\mathscr{C}_{G+}$.

Let, then, J be any one integral and free object of $\mathscr{C}$ and select some fixed free-and-direct product object $F = J \times J$. Consider the set

$$\mathcal{G}_A = \{\sigma \mid \sigma = \sup(\sigma),\ D(\sigma) = F,\ R(\sigma) \subset A\}.$$

We have immediately, for each object A, the group properties for $\mathcal{G}_A$.

LEMMA 3.3. $\mathcal{G}_A$ <u>is an abelian group relative to the operation F:A-addition having the map</u> $0_{Z_A F}$ <u>as zero element</u>.

We have also a one-to-one correspondence A <---> $\mathcal{G}_A$ between the objects of $\mathcal{C}$ and the groups constructed within $\mathcal{C}$.

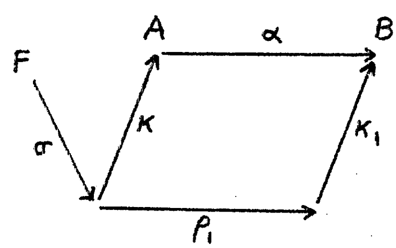
LEMMA 3.4. <u>If</u> $\mathcal{G}_{A_1} = \mathcal{G}_{A_2}$, <u>then</u> $A_1 = A_2$.

Proof. Suppose $A_1 \neq A_2$. By axiom AD-2, there exist maps $\alpha_1 : F \longrightarrow A_1$ and $\alpha_2 : F \longrightarrow A_2$ such that $\sup(\alpha_1) \neq \sup(\alpha_2)$. Since $\sup(\alpha_1) \in \mathcal{G}_{A_1}$ and $\sup(\alpha_2) \in \mathcal{G}_{A_2}$, $\mathcal{G}_{A_1} \neq \mathcal{G}_{A_2}$.

For each map $\alpha : A \longrightarrow B$ and each supermap $\sigma \in \mathcal{G}_A$ we take $\mathcal{G}_\alpha$ to be the function such that

$$\mathcal{G}_\alpha(\sigma) = \sup(\alpha[D(\alpha) \supset R(\sigma)]\sigma). \tag{3.6}$$

The following diagram illustrates $\mathcal{G}_\alpha$ for the dicategory $\mathcal{C}$.



$$\kappa = [D(\alpha) \supset R(\sigma)],$$

$$\alpha\kappa = \kappa_1\rho_1,$$

$$\mathcal{G}_\alpha(\sigma) = \rho_1\sigma.$$

For groups, the action of $\mathcal{G}_\alpha$ may be seen as follows. Taking J to be the integers and σ a mapping of $J \times J$ onto a subgroup of A, $\sigma(1, 0) = a_1$ and $\sigma(0, 1) = a_2$ for some $a_1, a_2 \in A$. If $\alpha(a_1) = b_1$ and $\alpha(a_2) = b_2$, then $\mathcal{G}_\alpha(\sigma)$ sends $(m, n) \in J \times J$ into $\alpha(\sigma(m, n)) = mb_1 + nb_2$. Thus $\mathcal{G}_\alpha(\sigma) = \tau$, where τ is the homomorphism of $J \times J$ onto the subgroup of B generated by $\tau(1, 0) = b_1$ and $\tau(0, 1) = b_2$.

LEMMA 3.5. <u>For each map</u> $\alpha : A \dashrightarrow B$, G_α <u>is a homomorphism of</u> G_A <u>into</u> G_B.

Proof. Consider any $\sigma \varepsilon G_A$. Then $\alpha[A \supset R(\sigma)]\sigma : F \dashrightarrow B$. Hence, by (3.6), $G_\alpha(\sigma)$ is a supermap with domain F and range a subobject of B. Thus $G_\alpha(\sigma) \varepsilon G_B$, and G_α (which is clearly defined on G_A) has values in G_B. Consider next any two maps $\sigma_1, \sigma_2 \varepsilon G_A$. Then there are supermaps ρ_1, ρ_2 which yield the canonical decompositions

$$\alpha[A \supset R(\sigma_1)] = [B \supset R(\rho_1)]\rho_1 ,$$

(3.7)

$$\alpha[A \supset R(\sigma_2)] = [B \supset R(\rho_2)]\rho_2 .$$

By (3.3) there is the additional decomposition

(3.8) $$[A \supset R(\sigma_1)]\sigma_1 + [A \supset R(\sigma_2)]\sigma_2 = \kappa_4(\sigma_1 \ {}^F{+}_A \ \sigma_2) .$$

By (3.7), (3.8), and the distributive property of +,

$$[B \supset R(\rho_1)]G_\alpha(\sigma_1) + [B \supset R(\rho_2)]G_\alpha(\sigma_2) = [B \supset R(\rho_1)]\rho_1\sigma_1 + [B \supset R(\rho_2)]\rho_2\sigma_2$$

$$= \alpha[A \supset R(\sigma_1)]\sigma_1 + \alpha[A \supset R(\sigma_2)]\sigma_2$$

(3.9) $$= \alpha([A \supset R(\sigma_1)]\sigma_1 + [A \supset R(\sigma_2)]\sigma_2)$$

$$= \alpha\kappa_4(\sigma_1 \ {}^F{+}_A \ \sigma_2)$$

$$= \alpha[A \supset R(\sigma_1 \ {}^F{+}_A \ \sigma_2)](\sigma_1 \ {}^F{+}_A \ \sigma_2) .$$

Taking the supermap factors of the first and last members of equation (3.9), we have, by (3.2) and (3.6), the desired result,

$$G_\alpha(\sigma_1) \;{}^{F}+_B\; G_\alpha(\sigma_2) = G_\alpha(\sigma_1 \;{}^{F}+_A\; \sigma_2) .$$

If $\alpha : A \dashrightarrow B$, we denote by $\mathcal{G}_\alpha$ the transformation (G_α, G_A, G_B).

THEOREM 3.1. If $\mathcal{C}$ is an abelian dicategory, the mapping

$$\Sigma : \begin{cases} A \longrightarrow G_A \\ \alpha \longrightarrow \mathcal{G}_\alpha \end{cases}$$

is a faithful representation for $\mathcal{C}$ by abelian groups.

Proof. Suppose $\alpha\beta$ is defined, with $\beta : A \dashrightarrow B$, $\alpha : B \dashrightarrow C$, and suppose $\sigma \in G_A$. For appropriate supermaps ρ_1 and ρ_2 there are the canonical decompositions

$$(3.10) \qquad \begin{aligned} \beta[A \supset R(\sigma)]\sigma &= [B \supset R(\rho_1)]\rho_1 , \\ \alpha[B \supset R(\rho_1)]\rho_1 &= [C \supset R(\rho_2)]\rho_2 , \end{aligned}$$

from which there follows

$$(3.11) \quad \alpha\beta[A \supset R(\sigma)]\sigma = \alpha[B \supset R(\rho_1)]\rho_1 = [C \supset R(\rho_2)]\rho_2 .$$

From (3.10) and (3.6), $G_\beta(\sigma) = \rho_1$ and $G_\alpha G_\beta(\sigma) = G_\alpha(\rho_1) = \rho_2$. But, by (3.11) and (3.6), $G_{\alpha\beta}(\sigma) = \rho_2$. Hence $G_{\alpha\beta} = G_\alpha G_\beta$, and $\mathcal{G}_{\alpha\beta} = \mathcal{G}_\alpha \mathcal{G}_\beta$.

Next, consider any injection $\kappa = [E \supset D]$, and suppose $\sigma \in G_D$. Then $G_\kappa(\sigma) = \sup(\kappa[D \supset R(\sigma)]\sigma) = \sup(\sigma) = \sigma$. Thus, under the identity homomorphism G_κ, the group G_D is its own image. Therefore G_D is a subgroup of G_E.

Next, suppose $\rho : P \dashrightarrow Q$ is a supermap and $\tau \in G_Q$. By axiom D-6, there exists an injection $\bar{\kappa}$ and a supermap $\bar{\rho}$ such that

$$\rho\bar{\kappa} = [Q \supset R(\tau)]\bar{\rho} . \tag{3.12}$$

Let $D(\bar{\kappa}) = N$; then $\bar{\kappa} = [P \supset N]$ and $\bar{\rho} : N \dashrightarrow R(\tau)$. But τ is a map, $\bar{\rho}$ is a supermap, and F is (by Lemma 3.2) a free object such that $\tau : F \dashrightarrow R(\tau)$, $\bar{\rho} : N \dashrightarrow R(\tau)$. Hence, by definition, there exists a map $\beta : F \dashrightarrow N$ such that

$$\bar{\rho}\beta = \tau . \tag{3.13}$$

Consider now $\sigma = \sup(\beta)$, and note that $\text{inj}(\beta) = [N \supset R(\sigma)]$. We shall show $G_\rho(\sigma) = \tau$. Now $\sigma \in G_P$; for $\sigma : F \dashrightarrow R(\sup(\beta)) \subset R(\beta) = N \subset P$. By (3.12) and (3.13),

$$\begin{aligned} \rho[P \supset R(\sigma)]\sigma &= \rho[P \supset N][N \supset R(\sigma)]\sigma \\ &= \rho\bar{\kappa}[N \supset R(\sigma)]\sigma \\ &= [Q \supset R(\tau)]\bar{\rho}[N \supset R(\sigma)]\sigma \\ &= [Q \supset R(\tau)]\bar{\rho}\,\text{inj}(\beta)\sup(\beta) \\ &= [Q \supset R(\tau)]\bar{\rho}\beta \\ &= [Q \supset R(\tau)]\tau. \end{aligned} \tag{3.14}$$

By (3.6) and (3.14), $G_\rho(\sigma) = \tau$; therefore G_ρ is a homomorphism onto G_Q.

Finally, suppose $\alpha_1, \alpha_2 \in \mathcal{C}$, $\alpha_1 \neq \alpha_2$. We wish to prove $\mathcal{G}_{\alpha_1} \neq \mathcal{G}_{\alpha_2}$. If $\mathcal{G}_{\alpha_1} = \mathcal{G}_{\alpha_2}$, then the group components of

$\mathcal{G}_{\alpha_1}$ and $\mathcal{G}_{\alpha_2}$ are equal; hence, by Lemma 3.4, α_1 and α_2 are coterminal. We may assume, then, $\alpha_1, \alpha_2 : A \longrightarrow B$ and $\alpha_1 \neq \alpha_2$. By property (i) of the integral object J (stated in Chapter II), there exists a map $\beta : J \longrightarrow A$ such that $\alpha_1 \beta \neq \alpha_2 \beta$. Then, using the maps which enter in the definition [8, §18] of the free-and-direct product diagram for $F = J \times J$, $\alpha_1 \beta \Gamma_F^1 \neq \alpha_2 \beta \Gamma_F^1$. For let $\gamma_i = \alpha_i \beta \Gamma_F^1$, i = 1, 2, and suppose $\gamma_1 = \gamma_2$; then
$\gamma_1 \Lambda_F^1 = \gamma_2 \Lambda_F^1 = \alpha_1 \beta \Gamma_F^1 \Lambda_F^1 = \alpha_2 \beta \Gamma_F^1 \Lambda_F^1 = \alpha_1 \beta = \alpha_2 \beta$.
Now let $\sigma = \sup(\beta \Gamma_F^1)$. Since $\beta \Gamma_F^1 : F \longrightarrow A$, $\sigma \in G_A$. Also $G_{\alpha_1}(\sigma) = \sup(\alpha_1 [A \supset R(\sigma)] \sigma) = \sup(\alpha_1 \beta \Gamma_F^1) = \sup(\gamma_1)$; similarly $G_{\alpha_2}(\sigma) = \sup(\gamma_2)$. We may assume $R(\sup(\gamma_1)) = R(\sup(\gamma_2))$; for otherwise $G_{\alpha_1}(\sigma) \neq G_{\alpha_2}(\sigma)$, and the proof would be complete. But with this assumption, $D(inj(\gamma_1)) = D(inj(\gamma_2))$, $inj(\gamma_1)$ and $inj(\gamma_2)$ are coterminal, and $inj(\gamma_1) = inj(\gamma_2)$. Since $\sup(\gamma_1) = \sup(\gamma_2)$ would imply $\gamma_1 = \gamma_2$, we must have $\sup(\gamma_1) \neq \sup(\gamma_2)$; that is $G_{\alpha_1}(\sigma) \neq G_{\alpha_2}(\sigma)$. Therefore $G_{\alpha_1} \neq G_{\alpha_2}$ and $\mathcal{G}_{\alpha_1} \neq \mathcal{G}_{\alpha_2}$.

4. Appendix. The representation Σ of Theorem 3.1 can, of course, be applied to the particular abelian dicategory of all abelian groups. Since every infinite cyclic group is an "integral and free object" there are many ways in which the application can be made, but no generality is lost in interpreting J as the group of integers. Then it is easily shown that for each abelian group G the representing group Σ_G is isomorphic to the cartesian product $G \times G$.

The constructions and arguments used in the proof of Theorem 3.1 may be generalized in the following way.

Let $\mathcal{C}$ be a dicategory and P any object of $\mathcal{C}$. In the definition of the set G_A given in Section 3, replace "F" by "P" and call the resulting set G_A^P. Then

$$G_A^P = \{\sigma \mid \sigma = \sup(\sigma),\ D(\sigma) = P,\ R(\sigma) \subset A\}.$$

For each $\alpha \in \mathcal{C}$ let G_α^P be defined by the formula of (3.6); that is,

$$G_\alpha^P(\sigma) = \sup(\alpha\,[D(\alpha) \supset R(\sigma)]\,\sigma)$$

for each $\sigma \in G_{D(\alpha)}^P$. Then, by arguments like those of Section 3, with "P" replacing "F", G_α^P is, for each map $\alpha : A \dashrightarrow B$, a function defined on G_A^P having values in G_B^P such that

(A) $\alpha\beta$ defined implies $G_{\alpha\beta}^P = G_\alpha^P G_\beta^P$;

(B) for each injection $\kappa : A \dashrightarrow B$, G_κ^P is the identity function on G_A^P, a set which, consequently, is a subset of G_B^P .

If P is a <u>free</u> object, then

(C) for each supermap $\rho : C \dashrightarrow D$, the function G_ρ^P maps G_C^P onto G_D^P .

Also, if P has property (i) of an <u>integral</u> object (regardless of whether or not P is free), then

(D) if α_1, α_2 are coterminal, $\alpha_1 \neq \alpha_2$ implies $G_{\alpha_1}^P \neq G_{\alpha_2}^P$.

Now if $\mathcal{C}$ satisfies also axioms Z, ASD-1, and ASD-2, the operation of P:A-addition is definable as in Section 2 and converts G_A^P into an abelian semigroup. Furthermore G_α^P is a homomorphism which, if P is selected as a fixed free and integral object, has the properties (A) - (D) listed directly above.

If we require further that $\mathcal{C}$ satisfies axiom AD-1, and therefore all axioms for AD-dicategories except possibly AD-2, then there is a representation Σ' for $\mathcal{C}$ by abelian groups in which

$$\Sigma'_A = G^P_A, \qquad \Sigma'_\alpha = (G^P_\alpha, G^P_A, G^P_B)$$

whenever $\alpha : A \dashrightarrow B$. Applied to $\mathcal{C}$, Σ' preserves direct products in the sense that $\Sigma'_{A \times B} \cong \Sigma'_A \times \Sigma'_B$ for all objects A and B; applied to the dicategory $\mathcal{C}_{G+}$ of all abelian groups, Σ' is object-faithful ($\Sigma'_G = \Sigma'_H$ implies G = H for all groups G and H) and $\Sigma'_G \cong G$. Σ' is not generally faithful, but becomes so under the hypothesis of object-faithfulness. This deficiency may be removed by the assumption of the following axiom for   :

AD-2'. <u>If</u> J <u>is an integral and free object and if</u> A_1 <u>and</u> A_2 <u>are objects,</u> $A_1 \neq A_2$, <u>then there exist maps</u> $\alpha_1 : J \dashrightarrow A_1$ <u>and</u> $\alpha_2 : J \dashrightarrow A_2$ <u>with canonical decompositions</u> $\alpha_1 = \kappa_1 \rho_1$, $\alpha_2 = \kappa_2 \rho_2$ <u>such that whenever</u> $\tau_1 = \sup(\alpha_1 + [A_1 \supset R(\rho_2)]\rho_2)$ <u>and</u> $\tau_2 = \sup([A_2 \supset R(\rho_1)]\rho_1 + \alpha_2)$ <u>exist as maps of</u> $\mathcal{C}$, <u>then</u> $\tau_1 \neq \tau_2$.

This axiom, like AD-2, is valid for $\mathcal{C}_{G+}$ and has an interpretation stating that distinct groups differ either in their elements or in their operations.

Let $\mathcal{C}$ be called an AD'-<u>dicategory</u> if it satisfies axiom AD-2' and those axioms for an AD-dicategory other than AD-2. Then the representation Σ' is a faithful representation by abelian groups for any AD'-dicategory which, as noted above, preserves direct products. Counterexamples show that this property does not generally hold for faithful representations of AD'-dicategories.

Finally, we note briefly the decomposability of the representation given by Theorem 2.1. If $\mathcal{C}$ is an ASD-dicategory satisfying axiom AD-2' but not necessarily axiom AD-1, then the representation Σ' can be "cut down" to provide for $\mathcal{C}$ a faithful representation $\Sigma^{(2)}$ by abelian semigroups. Also there is a somewhat trivial (non-faithful) representation $\Sigma^{(1)}$ for $\mathcal{C}$ by abelian semigroups in which $\Sigma_A^{(1)}$ has as set all subobjects $C \subset A$ such that (with J a fixed integral and free object) there is a map $\eta : J \longrightarrow C$ and in which the functional component of $\Sigma_\alpha^{(1)}$ is the function α_s cited in Section 1 of Chapter II. If we denote the representation for $\mathcal{C}$ of Theorem 2.1 by $\Sigma^{(0)}$, then it may be shown that $\Sigma^{(0)}$ is a subdirect sum of $\Sigma^{(1)}$ and $\Sigma^{(2)}$ in the sense that $\Sigma_A^{(0)}$ is isomorphic to a subdirect sum of $\Sigma_A^{(1)}$ and $\Sigma_A^{(2)}$ and that $\Sigma_\alpha^{(0)}$ has functional component which is the cartesian product of the functional components of $\Sigma_\alpha^{(1)}$ and $\Sigma_\alpha^{(2)}$ on its domain. Furthermore $\Sigma_A^{(0)}$ is isomorphic to the particular subdirect sum of $\Sigma_A^{(1)}$ and $\Sigma_A^{(2)}$ consisting of all ordered pairs (C, σ) for which σ is a supermap such that $\sigma : J \longrightarrow R(\sigma) \subset C \subset A$.

CHAPTER IV

A REPRESENTATION BY GROUPS

1. Preliminary Remarks. For each of the particular types of dicategories discussed in the previous chapters, the verification of axioms within the dicategory of all abelian groups proceeds by standard and often routine arguments. In this chapter we present a set of axioms valid in the dicategory of all groups. With one exception the verification of axioms seems to involve no new result or method. The exception is axiom DFP-3 below, an assertion whose interpretation for groups implies that if the formula

$$xoy = x^{a_1} y^{b_1} x^{a_2} y^{b_2} \ldots x^{a_n} y^{b_n} ,$$

with integral exponents and finite n, yields an associative operation $z = xoy$ in every group G, then for each g, h ε G, goh is either g, h, gh, hg, or the identity element of G. We believe this implication for groups is a new result; its proof is given in Section 3 following the preliminaries on free groups needed in the proof and presented in Section 2. The remainder of the chapter concerns so-called dicategories with free products and concludes with a representation for such dicategories by groups.

2. The Group F_3. We consider certain features of a free group with three generators. For the general case see Kurosch [6],

Reidemeister [9], Schreier [10], van der Waerden [11], and, particularly, Artin [1]. We first review certain well-known facts and then introduce a notation appropriate to subsequent arguments.

Let F_3 be the free group with the three free generators $x^{\pm 1}$, $y^{\pm 1}$, $z^{\pm 1}$. The elements $w \in F_3$ are sequences, or words, ordinarily taken to be formal products of a finite, possibly zero, number of factors, each of which is a generator. If the word w is empty, we denote it by 1. A word is said to be in reduced form if it contains no pairs of adjacent factors of the type $t^{-1}t$ or tt^{-1} ($t = x, y$, or z). Moreover each word w has a unique reduced form $\bar{w}$ [a]: if w is 1, so is $\bar{w}$; if w is not reduced, then $\bar{w}$ is obtained by removing pairs of adjacent factors of the types cited and iterating this process until no such pairs remain. Consider now two elements $w_1, w_2 \in F_3$. We recall that $\bar{w}_1 = \bar{w}_2$ asserts the typographical identity of the reduced forms: each is 1 or each is nonempty and is the same sequence of symbols. We shall say w_1 is congruent to w_2 ($w_1 \equiv w_2$) in case $\bar{w}_1 = \bar{w}_2$. Also the product w_1w_2 is the word obtained by juxtaposition of its two factors and the inverse w_1^{-1} is the sequence for w_1 put in reverse order with each exponent changed in sign.

It is convenient to have a notation which indicates, for any nonempty word w, the initial and final generators in its reduced form. We shall, for example, use the symbol $[x,w,y]$ both to stand for $\bar{w}$ and to indicate further that w, when put in reduced form, begins with $x^{\pm 1}$ and ends with $y^{\pm 1}$. Other typical symbols used in other cases

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[a] See, for example, Reidemeister [9, pp. 29-34].

for $\bar{w}$ are the following:

$[\hat{x},w,y]$ if $\bar{w}$ ends with $y^{\pm1}$ but does not begin with $x^{\pm1}$;

$[x,w,\hat{y}]$ if $\bar{w}$ begins with $x^{\pm1}$ but does not end with $y^{\pm1}$;

$[\hat{x},w,\hat{y}]$ if $\bar{w}$ neither begins with $x^{\pm1}$ nor ends with $y^{\pm1}$.

The notion of congruence and the bracket symbols apply, of course, to any free group.

3. Associative Operations in Groups. For any natural number n and any integers a_i, b_j $(1 \leqq i, j \leqq n)$ there is in all groups G an operation o defined by

$$goh = \prod_{i=1}^{n} g^{a_i} h^{b_i} = g^{a_1} h^{b_1} g^{a_2} h^{b_2} \ldots g^{a_n} h^{b_n}, \quad g, h \in G.$$

Clearly we need consider only those exponents for which $a_i = 0$ implies $i = 1$ and $b_j = 0$ implies $j = n$. Henceforth in speaking of the operation o, we understand these restrictions to hold on the exponents.

THEOREM 4.1 *The operation* o *is an associative operation in every group if and only if* $n = 1$ *and* $a_1^2 - a_1 = b_1^2 - b_1 = 0$, *or* $n = 2$, $a_1 = b_2 = 0$, *and* $a_2 = b_1 = 1$.

Proof. The solutions a_1 and b_1 of the equations given for $n = 1$ yield for goh either gh, g, h, or the identity. For the second alternative, goh = hg.

Conversely, suppose o is associative in all groups, hence in F_3. We consider two cases: $n = 1$ and $n > 1$. For $n = 1$ a straightforward computation shows, with subscripts of a_1 and b_1 omitted for notational simplicity,

$$(4.1) \qquad xo(xox^{-1}) = x^{a+b(a-b)} = (xox)ox^{-1} = x^{a(a+b)-b},$$

$$(4.2) \qquad xo(xox) = x^{a+b(a+b)} = (xox)ox = x^{a(a+b)+a} .$$

Equating the exponents in each of (4.1) and (4.2) yields $a^2 - a = b^2 - b = 0$, proving the first alternative of the theorem.

Before investigating $n > 1$ we state several lemmas which are useful for the remainder of the proof and which concern free groups and the operation o. To avoid repetition we state as General Hypotheses the assumptions common to the four results.

GENERAL HYPOTHESES. F _is a free group with at least two free generators_ $X^{\pm 1}$ _and_ $Y^{\pm 1}$. XoY _is an element of_ F _such that_ $XoY \equiv X^{a_1} Y^{b_1} \ldots X^{a_n} Y^{b_n}$, _where_ $n \geqq 2$ _and the exponents are integers restricted as above. The exponent_ k _is a positive integer._

LEMMA 4.1. _If_ $a_1 = 0$ _and_ $b_n = 0$, _then for some_ $w_1, w_2 \in F$,

$$(XoY)^k \equiv Y^{b_1} X^{a_2}, \qquad \text{if } n = 2 \text{ and } k = 1 ,$$

$$(XoY)^k \equiv Y^{b_1} X^{a_2} [Y, w_1, X] \equiv [Y, w_2, X] Y^{b_{n-1}} X^{a_n}, \text{ otherwise.}$$

LEMMA 4.2. _If_ $a_1 = 0$ _and_ $b_n \neq 0$, _then for some_ $w_1, w_2 \in F$,

$$(XoY)^k \equiv Y^{b_1} X^{ka_2} Y^{b_2} , \qquad \text{if } n = 2 \text{ and } b_1 + b_2 = 0 ,$$

$$(XoY)^k \equiv Y^{b_1} X^{a_2} [Y, w_1, Y] \equiv [Y, w_2, Y] X^{a_n} Y^{b_n} , \quad \text{otherwise.}$$

LEMMA 4.3. _If_ $a_1 \neq 0$ _and_ $b_n = 0$, _then for some_ $w_1, w_2 \in F$,

$$(XoY)^k \equiv X^{a_1} Y^{kb_1} X^{a_2} , \qquad \text{if } n = 2 \text{ and } a_1 + a_2 = 0 ,$$

$$(XoY)^k \equiv X^{a_1} Y^{b_1} [X, w_1, X] \equiv [X, w_2, X] Y^{b_{n-1}} X^{a_n} , \text{ otherwise.}$$

LEMMA 4.4. <u>If</u> $a_1 \neq 0$ <u>and</u> $b_n \neq 0$, <u>then</u> <u>for</u> <u>some</u> $w_1, w_2 \in F$,

$$(XoY)^k \equiv X^{a_1} Y^{b_1} [X,w_1,Y] \equiv [X,w_2,Y]X^{a_n} Y^{b_n} .$$

Proofs of Lemmas. We exhibit a proof only for Lemma 4.2. The remaining proofs proceed by either parallel or considerably easier arguments. We verify directly the case $n = 2$ and then use induction on n for $n \geqq 3$.

If $n = 2$ and $b_1 + b_2 = 0$, then $XoY \equiv Y^{b_1} X^{a_2} Y^{-b_1}$. Thus XoY is conjugate to X^{a_2} and a trivial induction on k shows the first case of the Lemma.

If $n = 2$ and $b_1 + b_2 \neq 0$, then $XoY \equiv Y^{b_1} X^{a_2} (Y^{b_2})$ and $XoY \equiv (Y^{b_1})X^{a_2} Y^{b_2}$. Thus the "otherwise" case of the Lemma holds for $k = 1$. Also $(XoY)^2 \equiv Y^{b_1} X^{a_2} Y^{b_1+b_2} X^{a_2} Y^{b_2}$. Hence the following statement holds for $k = 2$ and may serve as inductive hypothesis for $2 \leqq k \leqq r$:

$$(4.3) \qquad (XoY)^k \equiv Y^{b_1} X^{a_2} [Y,w,Y]X^{a_2} Y^{b_2} \qquad \text{for some } w \in F .$$

But $(XoY)^{r+1} \equiv (XoY)^r(XoY) \equiv Y^{b_1} X^{a_2} [Y,w,Y]X^{a_2} Y^{b_1+b_2} X^{a_2} Y^{b_2}$; whence, by taking $[Y,w,Y]X^{a_2} Y^{b_1+b_2}$ as the word w',

$$(XoY)^{r+1} \equiv Y^{b_1} X^{a_2} [Y,w',Y]X^{a_2} Y^{b_2} .$$

Then (4.3) is valid for $k \geqq 2$. Since $Y^{b_1} X^{a_2} [Y,w,Y] \equiv [Y,w_1,Y]$ and $[Y,w,Y]X^{a_2} Y^{b_2} \equiv [Y,w_2,Y]$ for some $w_1, w_2 \in F$, the "otherwise" case holds for $k \geqq 2$ and, from before, for $k = 1$. Thus Lemma 4.2 is proved for $n = 2$.

For $n > 2$ it is convenient to replace Lemma 4.2 by the following statement.

LEMMA 4.2'. *If* $n > 2$, $W \varepsilon F$, *and* $W \equiv Y^{b_1} X^{a_2} Y^{b_2} \ldots X^{a_n} Y^{b_n}$ *for nonzero integers* $b_1, a_2, b_2, \ldots, a_n, b_n$, *then for each positive integer* k *there exists* $w \varepsilon F$ *such that*

$$W^k \equiv Y^{b_1} X^{a_2} [Y,w,Y] X^{a_n} Y^{b_n} . \tag{4.4}$$

We prove Lemma 4.2' by induction on n. If $n = 3$ and $b_1 + b_3 \neq 0$, then a straightforward induction on k proves (4.4). If $n = 3$, $b_1 + b_3 = 0$, and $a_2 + a_3 \neq 0$, then for $k > 0$,

$$W \equiv Y^{b_1} (X^{a_2} Y^{b_2} X^{a_3}) Y^{b_3} ,$$

$$W^k \equiv Y^{b_1} (X^{a_2} [Y,w,Y] X^{a_3})^k Y^{b_3} , \qquad \text{for some } w \varepsilon F .$$

By an easy induction on k, $(X^{a_2} [Y,w,Y] X^{a_3})^k \equiv X^{a_2} [Y,w_1,Y] X^{a_3}$ for some $w_1 \varepsilon F$. Again (4.4) holds for all $k > 0$. If $n = 3$ and $b_1 + b_3 = a_2 + a_3 = 0$, then

$$W \equiv (Y^{b_1} X^{a_2}) Y^{b_2} (X^{-a_2} Y^{-b_1}) ,$$

$$W^k \equiv (Y^{b_1} X^{a_2}) Y^{kb_2} (X^{-a_2} Y^{-b_1}) ,$$

and (4.4) is valid. Thus Lemma 4.2' holds for $n = 3$.

Assume (4.4) for $3 \leqq n \leqq s$ and consider the case $n = s + 1$. Since $n = s + 1 > 3$, any word W' such that

$$W' \equiv Y^{b_2} X^{a_3} \ldots X^{a_s} Y^{b_s}$$

has the further properties $W' \not\equiv 1$, $W' \equiv [Y,w,Y]$ for some $w \varepsilon F$, and

(4.5) $$W \equiv Y^{b_1} X^{a_2} W' X^{a_n} Y^{b_n} .$$

Also W' (in place of XoY) satisfies Lemma 4.2 if s = 3, and W' satisfies both the hypotheses of Lemma 4.2' and the inductive hypothesis if s > 3. Hence, for s = 3, there exist w_1, w_2 ε F such that

$$(W')^k \equiv Y^{b_2} X^{ka_2} Y^{b_3} , \qquad \text{if } b_2 + b_3 = 0 ,$$

$$(W')^k \equiv Y^{b_2} X^{a_2} [Y, w_1, Y] \equiv [Y, w_2, Y] X^{a_3} Y^{b_3} , \text{ otherwise} ;$$

and, for s > 3, there exists w ε F such that

$$(W')^k \equiv Y^{b_2} X^{a_3} [Y, w, Y] X^{a_s} Y^{b_s} .$$

Thus, in any case, there exists, for each k > 0, an element w ε F such that

(4.6) $$(W')^k \equiv [Y, w, Y] .$$

Return now to a consideration of W. If $b_1 + b_n \neq 0$ or if $a_2 + a_n \neq 0$, then by (4.5), the fact that $W' \equiv [Y, w, Y]$, and straightforward induction on k not involving powers of W' higher than the first, we obtain (4.4). Unless $b_1 + b_n = a_2 + a_n$, the induction on n is complete.

Assume, finally, $b_1 + b_n = a_2 + a_n = 0$. By (4.5) and (4.6),

$$W^k \equiv Y^{b_1} X^{a_2} (W')^k X^{-a_2} Y^{-b_1}$$

$$\equiv Y^{b_1} X^{a_2} [Y, w, Y] X^{-a_2} Y^{-b_1} \qquad \text{for some } w \in F .$$

Since (4.4) is valid for n = s + 1, Lemma 4.2', and hence Lemma 4.2, is proved.

By taking inverses results similar to those of Lemmas 4.1 - 4.4 are seen to hold for $k < 0$. It is also useful to observe that the first congruence in each of Lemmas 4.2 and 4.3 is valid as it stands for all integers $k \neq 0$.

To continue with the proof of Theorem 4.1, we suppose $n \geqq 2$. Let P and Q be elements of F_3 such that $P \equiv xo(yoz)$ and $Q \equiv (xoy)oz$. Then

$$(4.7) \qquad P \equiv x^{a_1} (yoz)^{b_1} \left(\prod_{i=2}^{n} (x^{a_i} (yoz)^{b_i}) \right) \equiv \left(\prod_{i=1}^{n-1} (x^{a_i} (yoz)^{b_i}) \right) x^{a_n} (yoz)^{b_n} ,$$

$$(4.8) \qquad Q \equiv (xoy)^{a_1} z^{b_1} \left(\prod_{i=2}^{n} ((xoy)^{a_i} z^{b_i}) \right) \equiv \left(\prod_{i=1}^{n-1} ((xoy)^{a_i} z^{b_i}) \right) (xoy)^{a_n} z^{b_n} .$$

By the assumption of associativity, $P \equiv Q$; since the exponents are restricted, none is zero except possibly a_1 and b_n. We consider three main cases: case 1, $a_1 < 0$; case 2, $a_1 > 0$; case 3, $a_1 = 0$. Subcases are indicated by decimal numbers. In each case but the last we derive the contradiction $P \not\equiv Q$.

<u>Case</u> 1. $a_1 < 0$. If $b_n \neq 0$, then by Lemma 4.4, (4.7), and (4.8),

$$(xoy)^{a_1} \equiv y^{-b_n} [x,w_1,x] , \qquad \text{for some } w_1 \in F_3 ,$$

$$P \equiv x^{a_1} [\hat{x},w_2,\hat{x}] , \qquad \text{for some } w_2 \in F_3 ,$$

$$Q \equiv y^{-b_n} [x, w_3, z] , \qquad \text{for some } w_3 \in F_3 .$$

Thus $b_n = 0$; for otherwise $P \not\equiv Q$. Now for $b_n = 0$, by Lemma 4.3,

$$(xoy)^{a_1} \equiv x^{-a_n} [y, w_1, x] , \qquad \text{for some } w_1 \in F_3 ,$$

$$P \equiv x^{a_1} [\hat{x}, w_2, x] , \qquad \text{for some } w_2 \in F_3 ,$$

$$Q \equiv x^{-a_n} [y, w_3, \hat{z}] , \qquad \text{for some } w_3 \in F_3 .$$

Hence $b_n = 0$ and $a_1 = -a_n$. We consider now two subcases, $b_{n-1} < 0$ and $b_{n-1} > 0$.

<u>Case</u> 1.1. $b_{n-1} < 0$ (and $a_1 < 0$, $b_n = 0$, $-a_1 = a_n > 0$). By Lemma 4.3, letting $b = a_n b_{n-1}$ if $n = 2$ and $b = b_{n-1}$ if $n > 2$,

$$(yoz)^{b_{n-1}} \equiv [y, w_1, z] y^{-a_1} , \qquad \text{for some } w_1 \in F_3 ,$$

$$(4.9) \qquad (xoy)^{a_n} \equiv [x, w_2, x] y^{b} x^{a_n} , \qquad \text{for some } w_2 \in F_3 .$$

But

$$(4.10) \qquad P \equiv \left(\prod_{i=1}^{n-2} (x^{a_i} (yoz)^{b_i}) \right) x^{a_{n-1}} (yoz)^{b_{n-1}} x^{a_n} , [a]$$

$$P \equiv [x, w_3, x][y, w_1, z] y^{-a_1} x^{a_n} \equiv [x, w_5, z] y^{-a_1} x^{a_n} ,$$

$$\text{for some } w_1,\ w_3,\ w_5 \in F \ .$$

By (4.8) and (4.9),

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[a]It is understood that for all words $w \in F_3$ and all integers α_i, k, m, $(k > 0,\ m \geqq 0)$, $\prod_{i=k}^{m} w^{\alpha_i} \equiv 1$ if $m < k$.

(4.11) $$Q \equiv [\hat{z}, w_4, z][x, w_2, x] y^{b} x^{a_n} \equiv [\hat{z}, w_6, x] y^{b} x^{a_n},$$

for some w_2, w_4, $w_6 \varepsilon F_3$.

Since $P \not\equiv Q$, we must have $b_{n-1} > 0$.

Case 1.2. $b_{n-1} > 0$ (and $a_1 < 0$, $b_n = 0$, $-a_1 = a_n > 0$). Congruences (4.9) and (4.11) remain unchanged. By Lemma 4.3 and (4.10),

$$(yoz)^{b_{n-1}} \equiv [y, w_1, z] y^{a_n}, \qquad \text{for some } w_1 ,$$

(4.12)
$$P \equiv [x, w_7, x][y, w_1, z] y^{a_n} x^{a_n} \equiv [x, w_8, z] y^{a_n} x^{a_n}, \text{ for some } w_1, w_7, w_8 .$$

By (4.11) and (4.12), $P \not\equiv Q$. Case 1 is impossible.

Case 2. $a_1 > 0$. Suppose $b_n \neq 0$. Then there exist elements $w_1, \ldots, w_6 \varepsilon F_3$ such that

$$(yoz)^{b_1} \equiv \begin{cases} y^{a_1} z^{b_1} [y, w_1, z], & \text{if } b_1 > 0, \\ z^{-b_n} y^{-a_n} [z, w_2, y], & \text{if } b_1 < 0, \end{cases}$$

$$(xoy)^{a_1} \equiv x^{a_1} y^{b_1} [x, w_3, y],$$

$$P \equiv \begin{cases} x^{a_1} y^{a_1} z^{b_1} [y, w_4, \hat{x}], & \text{if } b_1 > 0, \\ x^{a_1} z^{-b_n} y^{-a_n} [z, w_5, \hat{x}], & \text{if } b_1 < 0, \end{cases}$$

$$Q \equiv x^{a_1} y^{b_1} [x, w_6, z] .$$

Thus $P \not\equiv Q$ and $b_n = 0$.

Suppose, then, $a_1 > 0$ and $b_n = 0$. Then there are elements $w_1, \ldots, w_5 \varepsilon F_3$ such that

$$(yoz)^{b_1} \equiv \begin{cases} y^{a_1} [z,w_1,y] , & \text{if } b_1 > 0, \\ y^{-a_n} [z,w_2,y] , & \text{if } b_1 < 0, \end{cases}$$

$$(xoy)^{a_1} \equiv \begin{cases} x^{a_1} y^{a_1 b_1} x^{a_2} , & \text{if } n = 2 \text{ and } a_1 + a_2 = 0 , \\ x^{a_1} y^{b_1} [x,w_3,x] , & \text{otherwise} , \end{cases}$$

$$P \equiv x^{a_1} y^{a} [z,w_4,x] , \qquad \text{for some integer } a \neq 0 ,$$

$$Q \equiv x^{a_1} y^{b} [x,w_5,\hat{z}] , \qquad \text{for some integer } b \neq 0 .$$

Thus $P \not\equiv Q$. Case 2 is impossible.

Case 3. $a_1 = 0$. Suppose $b_n \neq 0$ and $b_1 > 0$. Then there are elements w_1, w_2, $w_3 \in F_3$ such that

$$(4.13) \qquad (yoz)^{b_1} \equiv \begin{cases} z^{b_1} y^{b_1 a_2} z^{b_2} , & \text{if } n = 2 \text{ and } b_1 + b_2 = 0 , \\ z^{b_1} y^{a_2} [z,w_1,z] , & \text{otherwise} , \end{cases}$$

$$(4.14) \qquad (xoy)^{a_2} \equiv \begin{cases} y^{b_1} [x,w_2,y] , & \text{if } a_2 > 0 , \\ y^{-b_n} [x,w_3,y] , & \text{if } a_2 < 0 . \end{cases}$$

But since $a_1 = 0$, we may write

$$(4.15) \qquad Q \equiv z^{b_1} (xoy)^{a_2} z^{b_2} \left(\prod_{i=3}^{n} ((xoy)^{a_i} z^{b_i}) \right) .$$

By (4.7) and (4.13), there exists $w \in F_3$ such that

$$P \equiv z^{b_1} y^{a} [z,w,\hat{x}] , \qquad \text{for some integer } a \neq 0 .$$

By (4.14) and (4.15), we have for some w,

$$(4.16) \qquad Q \equiv z^{b_1} y^{b} [x,w,z] , \quad \text{for some integer } b \neq 0 .$$

Again $P \not\equiv Q$. Thus $b_n \neq 0$ implies $b_1 < 0$.

Suppose, then, $a_1 = 0$, $b_n \neq 0$, and $b_1 < 0$. The congruences (4.14), (4.15), and (4.16) remain valid, but for some w,

$$(yoz)^{b_1} \equiv \begin{cases} z^{b_1} y^{b_1 a_2} z^{b_2} , & \text{if } n = 2 \text{ and } b_1 + b_2 = 0 , \\ z^{-b_n} y^{-a_n} [z,w,z] , & \text{otherwise} . \end{cases}$$

Thus, for some $w \varepsilon F_3$ and nonzero integers a and c,

$$(4.17) \qquad P \equiv z^{c} y^{a} [z,w,\hat{x}] .$$

By (4.16) and (4.17), $P \not\equiv Q$. Hence $b_n = 0$.

Suppose next $a_n < 0$. Then by Lemma 4.1 there exist $w_1, \ldots, w_4 \varepsilon F_3$ such that

$$(xoy)^{a_n} \equiv [x,w_1,y] ,$$

$$P \equiv [\hat{x},w_2,x] ,$$

$$Q \equiv [z,w_3,z][x,w_1,y] \equiv [z,w_4,y] .$$

Since in this case $P \not\equiv Q$, we have $a_n > 0$ and, from before, $b_n = 0$.

Suppose now $b_1 < 0$. Then there exist w_1, w_2, w_3 such that

$$(yoz)^{b_1} \equiv [y,w_1,z] ,$$

$$P \equiv [y,w_2,x] ,$$

$$Q \equiv [z,w_3,\hat{z}] .$$

Thus, $P \not\equiv Q$ and $b_1 > 0$. We have, then, $a_1 = b_n = 0$, $a_n > 0$, $b_1 > 0$.

Suppose $a_2 < 0$. There must exist w_1, w_2, w_3, w_4 such that

$$(xoy)^{a_2} \equiv [x, w_1, y] ,$$

$$(yoz)^{b_1} \equiv z^{b_1} [y, w_2, y] ,$$

$$P \equiv z^{b_1} [y, w_3, x] ,$$

$$Q \equiv z^{b_1} [x, w_4, \hat{z}] .$$

Again $P \not\equiv Q$. Summarizing, we have $a_1 = b_n = 0$ and b_1, a_2, a_n positive. We consider now two subcases, $n > 2$ and $n = 2$.

Case 3.1. $n > 2$. By Lemma 4.1, (4.7), and (4.8), there exist w_1, w_2, w_3, $w_4 \in F_3$ such that

$$(yoz)^{b_1} \equiv z^{b_1} y^{a_2} [z, w_1, y] ,$$

$$(xoy)^{a_2} \equiv y^{b_1} x^{a_2} [y, w_2, x] ,$$

$$P \equiv z^{b_1} y^{a_2} [z, w_3, x] ,$$

$$Q \equiv z^{b_1} y^{b_1} (x^{a_2} [y, w_4, \hat{z}]) .$$

Thus $n > 2$ implies $P \not\equiv Q$, a contradiction.

Case 3.2. $n = 2$. Since $a_2 > 0$ and $b_1 > 0$, there exist w_1, w_2, $w_3 \in F_3$ such that

$$(yoz)^{b_1} \equiv \begin{cases} z^{b_1} y^{a_2} , & \text{if } b_1 = 1 , \\ z^{b_1} y^{a_2} [z, w_1, y] , & \text{if } b_1 > 1 , \end{cases}$$

$$(xoy)^{a_2} \equiv \begin{cases} y^{b_1} x^{a_2} , & \text{if } a_2 = 1 , \\ y^{b_1} x^{a_2} [y, w_2, x] , & \text{if } a_2 > 1 , \end{cases}$$

$$P \equiv \begin{cases} z^{b_1} y^{a_2} x^{a_2} , & \text{if } b_1 = 1 , \\ z^{b_1} y^{a_2} [z, w_3, x] , & \text{if } b_1 > 1 , \end{cases}$$

$$Q \equiv \begin{cases} z^{b_1} y^{b_1} x^{a_2} , & \text{if } a_2 = 1 , \\ z^{b_1} y^{b_1} x^{a_2} [y, w_2, x] , & \text{if } a_2 > 1 . \end{cases}$$

Hence $P \not\equiv Q$ unless $a_2 = b_1 = 1$. Since all cases but the last are impossible, the proof is complete.

4. Free Products. A categorical description by mapping diagrams which characterize the free product of two groups has been given in [8, §3]. Our purposes require somewhat more elaborate diagrams similar to those given in [8, §18] for the free-and-direct products of abelian groups. We now state the definition of a free product diagram in terms of an axiom and indicate some simple results as a matter of convenience in writing the further axioms required for the representation by groups. The definition, axiom, results, and proofs parallel those for free-and-direct products. We give, therefore, only one proof; it is typical of those not presented.

Let $\mathcal{C}$ be a dicategory with zero satisfying the following axiom.

DFP-1. For each two objects A and B there exists a free product (f.p.) diagram on A with B; that is, an object A*B and four maps

$$\Psi^1_{A*B} : A \dashrightarrow A*B , \qquad \Psi^2_{A*B} : B \dashrightarrow A*B ,$$

$$\Phi^1_{A*B} : A*B \dashrightarrow A , \qquad \Phi^2_{A*B} : A*B \dashrightarrow B ,$$

such that (omitting for simplicity the subscripts $A*B$)

(i) $\Phi^1 \Psi^1 = I_A$, $\Phi^1 \Psi^2 = 0_{AB}$, $\Phi^2 \Psi^1 = 0_{BA}$, $\Phi^2 \Psi^2 = I_B$;

(ii) *for each object* C *and each pair of maps* $\beta_1 : A \dashrightarrow C$, $\beta_2 : B \dashrightarrow C$ *there exists a unique map* $\gamma : A*B \dashrightarrow C$ *such that* $\gamma \Psi^1 = \beta_1$ *and* $\gamma \Psi^2 = \beta_2$.

For groups A and B the homomorphism Ψ^1_{A*B}, for example, sends the element $a \varepsilon A$ into the word "a" [8]; the mapping Φ^1_{A*B} sends the word "$a_1 b_1 a_2 b_2 . . . a_n b_n$," where $a_i \varepsilon A$, $b_i \varepsilon B$, into the element $a_1 a_2 . . . a_n \varepsilon A$, and γ sends this word into the element

$$\beta_1(a_1)\, \beta_2(b_1)\, \beta_1(a_2)\, \beta_2(b_2) . . . \beta_1(a_n)\, \beta_2(b_n).$$

Two free product diagrams on A with B having objects $A*B$ and $A \bullet B$ are called *equivalent* if there exists an equivalence map $\Theta : A*B \dashrightarrow A \bullet B$ such that $\Theta \Psi^i_{A*B} = \Psi^i_{A \bullet B}$ and $\Phi^i_{A \bullet B} \Theta = \Phi^i_{A*B}$, $i = 1, 2$. It is easily shown that any two free product diagrams on A with B are equivalent and, conversely, that for any object C equivalent to $A*B$ there is an f.p. diagram with object C equivalent to that with object $A*B$. Even more easily one may show that the objects of $\mathcal{C}$ have a zero under the operation of free product formation in the following sense.

LEMMA 4.5. (DFP) *For any object* A *and any zero object* Z, *the diagram*

$$A \rightleftarrows A \leftrightarrows Z ,$$

in which the maps on the left are identities and those on the right are zero maps, is a free product diagram on A with Z having object $A*Z = A$.

LEMMA 4.6. (DFP) *If* $\alpha_1 : A \longrightarrow A'$ *and* $\alpha_2 : B \longrightarrow B'$, *there is, for each free product diagram on A with B and on A' with B', a unique map* $\alpha_1 * \alpha_2 : A*B \longrightarrow A'*B'$ *such that*

$$(4.18) \qquad (\alpha_1 * \alpha_2)\Psi^i_{A*B} = \Psi^i_{A'*B'}\alpha_i , \qquad i = 1, 2 .$$

This map has the further property

$$(4.19) \qquad \Phi^i_{A'*B'}(\alpha_1 * \alpha_2) = \alpha_i \Phi^i_{A*B} , \qquad i = 1, 2 .$$

Proof. Since for $C = A'*B'$, $\Psi^1_{A'*B'}\alpha_1 : A \longrightarrow C$ and $\Psi^2_{A'*B'}\alpha_2 : B \longrightarrow C$, property (ii) of axiom DFP-1 yields a unique map $\gamma = \alpha_1 * \alpha_2 : A*B \longrightarrow A'*B'$ such that $\gamma \Psi^i_{A*B} = \Psi^i_{A'*B'}\alpha_i$, $i = 1, 2$. Also

$$\Phi^1_{A'*B'}(\alpha_1 * \alpha_2)\Psi^j_{A*B} = \Phi^1_{A'*B'}\Psi^j_{A'*B'}\alpha_j = \alpha_1 \quad \text{if } j = 1 ,$$

$$= 0_{A'B} \quad \text{if } j = 2 .$$

But

$$\alpha_1 \Phi^1_{A*B}\Psi^j_{A*B} = \alpha_1 \quad \text{if } j = 1 ,$$

$$= 0_{A'B} \quad \text{if } j = 2 .$$

Thus $\left(\Phi^1_{A'*B'}(\alpha_1 * \alpha_2)\right)\Psi^j_{A*B} = (\alpha_1 \Phi^1_{A*B})\Psi^j_{A*B}$, $j = 1, 2$; hence, by the uniqueness property (ii), (4.19) is valid for $i = 1$. A similar argument for $i = 2$ completes the argument.

For groups, $\alpha_1 * \alpha_2$ is the homomorphism which sends a word

"$a_1b_1a_2b_2 \ldots a_nb_n$" of A*B into the word "$\alpha_1(a_1)\alpha_2(b_1) \ldots \alpha_1(a_n)\alpha_2(b_n)$" of A'*B'.

LEMMA 4.7. (DFP) For each object A and each free product diagram on A with A, there exists a unique map $\overline{\Delta}_A$: A*A ---> A such that

$$\overline{\Delta}_A \Psi^i_{A*A} = I_A, \qquad i = 1, 2.$$

5. Dicategories with Free Products.

DEFINITION 4.1. A dicategory with free products (DFP) $\mathcal{C}$ is a dicategory satisfying axioms Z (existence of zero), ASD-1 (on the existence of integral and free objects), and DFP-1 subject to the additional requirements listed below.

DFP-2. If J is an integral (and free) object, there exists a unique equivalence Θ_J: J ---> J such that $\Theta_J \neq I_J$.

As for groups, Θ_J has period two and generates the "automorphism" group of J.

DFP-3. If J is an integral (and free) object and A is any object, there exist free product diagrams on J with J and on A with A for which there are exactly two maps $\overline{\overline{\nabla}}_J^{\,1}$, $\overline{\overline{\nabla}}_J^{\,2}$: J ---> J*J having the properties:

(i) $\Phi^i_{J*J}\overline{\overline{\nabla}}_J^{\,1} = \Phi^i_{J*J}\overline{\overline{\nabla}}_J^{\,2} = I_J$, $\qquad i = 1, 2;$

(ii) $\overline{\Delta}_J(I_J * \Theta_J)\overline{\overline{\nabla}}_J^{\,k} = 0_{JJ}$, $\qquad k = 1, 2;$

(iii) for all maps $\alpha_1, \alpha_2, \alpha_3$: J ---> A,

$$\overline{\Delta}_A\left\{(\overline{\Delta}_A(\alpha_1 * \alpha_2)\overline{\overline{\nabla}}_J^{\,k}) * \alpha_3\right\}\overline{\overline{\nabla}}_J^{\,k} = \overline{\Delta}_A\left\{\alpha_1 * (\overline{\Delta}_A(\alpha_2 * \alpha_3)\overline{\overline{\nabla}}_J^{\,k}\right\}\overline{\overline{\nabla}}_J^{\,k},$$
$$k = 1, 2.$$

For groups we intend J to be the (additive) group of integers and J*J the usual free product, hence a free (non-abelian) group on two generators. Now the reduced words of J*J may be taken to be just those formal products

$$w = \prod_{i=1}^{n} a_i b_i = a_1 b_1 a_2 b_2 \cdot \cdot \cdot a_n b_n , \qquad n \geqq 0 ,$$

in which a_i and b_i are integers, such that w is the empty word if $n = 0$, and no a_i, b_i is zero except possibly a_1 and b_n if $n > 0$. With this notation we may describe, for groups, the maps $\overline{\nabla}_J^1$ and $\overline{\nabla}_J^2$. If $\overline{\nabla}_J$: J ---> J*J, then $\overline{\nabla}_J(1) = \prod_{i=1}^{n} a_i b_i$ for some integers a_i, b_i and some non-negative integer n. Part (i) of axiom DFP-3 is satisfied with $\overline{\nabla}_J$ only if $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i = 1$; part (iii) is satisfied only if, in the notation of Section 3 and with $g_i = \alpha_i(1)$ for i = 1, 2, 3, $(g_1 o g_2) o g_3 = g_1 o (g_2 o g_3)$. By Theorem 4.1, both parts (i) and (iii) are satisfied only in case $\overline{\nabla}_J(1)$ is either the word "11" or the word "0110"; these words are $\overline{\nabla}_J^1(1)$ and $\overline{\nabla}_J^2(1)$.

Before discussing a representation for DFP-dicategories, we note several useful results.

LEMMA 4.8. (DFP) <u>If</u> $\alpha_1 \beta_1$ <u>and</u> $\alpha_2 \beta_2$ <u>are defined, then</u> $(\alpha_1 * \alpha_2)(\beta_1 * \beta_2) = (\alpha_1 \beta_1) * (\alpha_2 \beta_2)$.

Proof. We may suppose A_i, B_i, C_i are objects such that α_i: B_i ---> C_i, β_i: A_i ---> B_i, i = 1, 2. For notational simplicity we write Ψ_A^i for $\Psi_{A_1 * A_2}^i$, and so on. Then, by Lemma 4.6,

$$(\beta_1 * \beta_2) \Psi_A^i = \Psi_B^i \beta_i ,$$

$$(\alpha_1 * \alpha_2) \Psi_B^i = \Psi_C^i \alpha_i ,$$

$$(\alpha_1 \beta_1 * \alpha_2 \beta_2) \Psi_A^i = \Psi_C^i \alpha_i \beta_i , \qquad i = 1, 2 .$$

Thus

$(\alpha_1 * \alpha_2)(\beta_1 * \beta_2)\Psi^i_A = (\alpha_1 * \alpha_2)\Psi^i_B \beta_i = \Psi^i_C \alpha_i \beta_i = (\alpha_1 \beta_1 * \alpha_2 \beta_2)\Psi^i_A$, $i = 1, 2$. By the uniqueness assertion (ii) for the f.p. diagram on A_1 with A_2, the conclusion holds.

LEMMA 4.9. (DFP) <u>If</u> $\alpha : A \dashrightarrow B$, <u>then</u> $\overline{\Delta}_B(\alpha * \alpha) = \alpha \overline{\Delta}_A$.

Proof. Let i have range 1, 2 throughout. By Lemmas 4.6 and 4.7,

$$\overline{\Delta}_B(\alpha * \alpha)\Psi^i_{A*A} = \overline{\Delta}_B \Psi^i_{B*B}\alpha = I_B\alpha = \alpha ,$$

$$\alpha \overline{\Delta}_A \Psi^i_{A*A} = \alpha I_A = \alpha .$$

Thus $\overline{\Delta}_B(\alpha * \alpha)\Psi^i_{A*A} = \alpha \overline{\Delta}_A \Psi^i_{A*A}$. The desired result follows by uniqueness (ii) for the f.p. diagram on A with A.

LEMMA 4.10. (DFP) <u>If</u> $\alpha : J \dashrightarrow A$, <u>then</u>

$$\overline{\Delta}_A(\alpha * \alpha\Theta_J)\overline{\nabla}^k_J = 0_{AJ} , \qquad k = 1, 2 .$$

Proof. Let k have range 1, 2. Then, by Lemmas 4.8 and 4.9 and axiom DFP-3, (ii),

$$\begin{aligned} \overline{\Delta}_A(\alpha * \alpha\Theta_J)\overline{\nabla}^k_J &= \overline{\Delta}_A(\alpha * \alpha)(I_J * \Theta_J)\overline{\nabla}^k_J \\ &= \alpha \overline{\Delta}_J(I_J * \Theta_J)\overline{\nabla}^k_J \\ &= \alpha 0_{JJ} \\ &= 0_{AJ} . \end{aligned}$$

6. <u>A Representation for Dicategories with Free Products.</u>

We take as the definition of a <u>representation of a</u> DFP-<u>dicategory</u> $\mathcal{C}$ <u>by</u>

groups the statement which results from the definition in Chapter III by deleting "abelian" and replacing "AD-dicategory" by "DFP-dicategory." Also we take unaltered the definition of the function $\mathcal{G}_\alpha$ and a simple modification of the definition of $\mathcal{G}_A$ from the preceding chapter. Explicitly we select any one integral and free object $J \in \mathcal{C}$ and take, for each object A and each map α,

$$(4.20) \qquad \mathcal{G}_A = \{\sigma \mid \sigma = \sup(\sigma), \ D(\sigma) = J, \ R(\sigma) \subset A\},$$

$$(4.21) \qquad \mathcal{G}_\alpha(\sigma) = \sup(\alpha\,[D(\alpha) \supset R(\sigma)]\,\sigma), \text{ for each } \sigma \in \mathcal{G}_{D(\alpha)} .$$

In several cases, therefore, the previous proofs may be carried over. The new arguments concern the operation which converts $\mathcal{G}_A$ into a group and which arises from the properties of free products of arbitrary groups rather than from the direct product of abelian groups.

To obtain a group operation for $\mathcal{G}_A$ we select for each object B any one f.p. diagram appropriate to axiom DFP-3 and further fix $\overline{\nabla}_J$ as one of the two maps $\overline{\nabla}_J^1$, $\overline{\nabla}_J^2$. We define first an operation applicable to any two coterminal maps $\alpha_1, \alpha_2 \colon J \dashrightarrow A$ having domain J by

$$(4.22) \qquad \alpha_1 \times_A \alpha_2 = \overline{\Delta}_A(\alpha_1 * \alpha_2)\,\overline{\nabla}_J .$$

Then, for any $\sigma_1, \sigma_2 \in \mathcal{G}_A$, we define

$$(4.23) \qquad \sigma_1 \circledcirc_A \sigma_2 = \sup([A \supset R(\sigma_1)]\,\sigma_1 \times_A [A \supset R(\sigma_2)]\,\sigma_2) .$$

Clearly (4.22) and (4.23) define binary operations. Note that for any maps $\alpha_1, \alpha_2 \colon J \dashrightarrow A$ there is, for some injection κ, the canonical decomposition

(4.24) $$\alpha_1 \times_A \alpha_2 = \kappa\{\sup(\alpha_1) \circledcirc_A \sup(\alpha_2)\} .$$

LEMMA 4.11. (DFP) G_A <u>is a group relative to the operation</u> $\circledcirc_A$ <u>and has</u> $\sup(0_{AJ})$ <u>as unit element.</u>

Proof. Closure is immediate from definitions (4.20) and (4.23). Suppose $\sigma_i \in G_A$ and let $\alpha_i = [A \supset R(\sigma_i)]\sigma_i$, $i = 1, 2, 3$. Then, by definition (4.22) and axiom DFP-3, (iii), $\alpha_1 \times_A(\alpha_2 \times_A \alpha_3) = (\alpha_1 \times_A \alpha_2) \times_A \alpha_3$. Hence, by (4.24), there exist injections κ_4, κ_5, κ_6, and κ_7 such that

$$\alpha_1 \times_A \alpha_2 = \kappa_4(\sigma_1 \circledcirc_A \sigma_2) ,$$

$$\alpha_2 \times_A \alpha_3 = \kappa_5(\sigma_2 \circledcirc_A \sigma_3) ,$$

$$\begin{aligned}\{\kappa_4(\sigma_1 \circledcirc_A \sigma_2)\} \times_A \alpha_3 &= \kappa_6\{(\sigma_1 \circledcirc_A \sigma_2) \circledcirc_A \sigma_3\} \\ &= (\alpha_1 \times_A \alpha_2) \times_A \alpha_3 \\ &= \alpha_1 \times_A (\alpha_2 \times_A \alpha_3) \\ &= \alpha_1 \times_A \kappa_5(\sigma_2 \circledcirc_A \sigma_3) \\ &= \kappa_7\{\sigma_1 \circledcirc_A (\sigma_2 \circledcirc_A \sigma_3)\} .\end{aligned}$$

The associative property for $\circledcirc_A$ follows from the two canonical decompositions above whose injection factors are κ_6 and κ_7.

Next, let Z be any zero object. By Lemma 4.8,

$$\begin{aligned}\alpha_1 \times_A 0_{AJ} &= \overline{\Delta}_A(\alpha_1 * 0_{AJ})\overline{\nabla}_J \\ &= \overline{\Delta}_A(\alpha_1 * 0_{AZ})(I_J * 0_{ZJ})\overline{\nabla}_J .\end{aligned}$$

But, by Lemma 4.5, there is an f.p. diagram in which $J*Z = J$ and $\Psi^1_{J*Z} = \Phi^1_{J*Z} = I_J$. With these results and Lemmas 4.6 and 4.7, we have

$$\alpha_1 \times_A 0_{AJ} = \left\{\overline{\Delta}_A(\alpha_1 * 0_{AZ})\,\Psi^1_{J*Z}\right\}\left\{\Phi^1_{J*Z}(I_J * 0_{ZJ})\,\overline{\nabla}_J\right\}$$

$$= (\overline{\Delta}_A\,\Psi^1_{A*A}\,\alpha_1)(I_J\,\Phi^1_{J*J}\,\overline{\nabla}_J)$$

$$= (I_A\,\alpha_1)(I_J\,\Phi^1_{J*J}\,\overline{\nabla}_J)\ .$$

By axiom DFP-3, (i), $\Phi^1_{J*J}\overline{\nabla}_J = I_J$; hence $\alpha_1 \times_A 0_{AJ} = I_A\alpha_1 I_J = \alpha_1$. Thus, by definition (4.23), $\sigma_1 \odot_A \sup(0_{AJ}) = \sup(\alpha_1 \times_A 0_{AJ}) = \sup(\alpha_1) = \sigma_1$. Therefore $\sup(0_{AJ})$ is a right unit element in G_A.

Consider now $\sigma_1\Theta_J$, where Θ_J is the unique equivalence described in axiom DFP-2. By appropriate definitions and Lemma 4.10,

$$\sigma_1 \odot_A \sigma_1\Theta_J = \sup(\alpha_1 \times_A \alpha_1\Theta_J)$$

$$= \sup(\overline{\Delta}_A(\alpha_1 * \alpha_1\Theta_J)\,\overline{\nabla}_J)$$

$$= \sup(0_{AJ}).$$

By a standard argument, $\sup(0_{AJ})$ is the unique two-sided unit of G_A and $\sigma_1\Theta_J$ is the unique inverse of σ_1. G_A is a group.

LEMMA 4.12. (DFP) <u>For each map</u> $\alpha : A \longrightarrow B$, G_α <u>is a homomorphism of the group</u> G_A <u>into the group</u> G_B.

Proof. It is clear that G_α is a function defined on G_A and that G_α has values in G_B. Suppose $\sigma_1, \sigma_2 \in G_A$ and let $\alpha_i = [A \supset R(\sigma_i)]\,\sigma_i$, $i = 1, 2$. We may assume the canonical decompositions

$$\alpha[A \supset R(\sigma_i)] = [B \supset R(\rho_i)]\rho_i , \qquad i = 1, 2 .$$

Thus $G_\alpha(\sigma_i) = \sup([B \supset R(\rho_i)]\rho_i \sigma_i) = \rho_i \sigma_i$, $i = 1, 2$. By appropriate definitions and Lemmas 4.8 and 4.9,

$$[B \supset R(\rho_1)] G_\alpha(\sigma_1) \times_B [B \supset R(\rho_2)] G_\alpha(\sigma_2)$$

$$= ([B \supset R(\rho_1)]\rho_1 \sigma_1) \times_B ([B \supset R(\rho_2)]\rho_2 \sigma_2)$$

$$= (\alpha[A \supset R(\sigma_1)]\sigma_1) \times_B (\alpha[A \supset R(\sigma_2)]\sigma_2)$$

$$(4.25) \qquad = \alpha\alpha_1 \times_B \alpha\alpha_2$$

$$= \overline{\Delta}_B(\alpha\alpha_1 * \alpha\alpha_2)\overline{\nabla}_J$$

$$= \overline{\Delta}_B(\alpha * \alpha)(\alpha_1 * \alpha_2)\overline{\nabla}_J$$

$$= \alpha\overline{\Delta}_A(\alpha_1 * \alpha_2)\overline{\nabla}_J$$

$$= \alpha(\alpha_1 \times_A \alpha_2) .$$

By (4.25) and definition,

$$(4.26) \qquad G_\alpha(\sigma_1) \circledcirc_B G_\alpha(\sigma_2) = \sup(\alpha(\alpha_1 \times_A \alpha_2)) .$$

But $\sigma_1 \circledcirc_A \sigma_2 = \sup(\alpha_1 \times_A \alpha_2)$. Hence, by definitions and (4.24), there is an injection κ such that

$$(4.27) \qquad G_\alpha(\sigma_1 \circledcirc_A \sigma_2) = \sup(\alpha\kappa(\sigma_1 \circledcirc_A \sigma_2))$$

$$= \sup(\alpha(\alpha_1 \times_A \alpha_2)) .$$

The Lemma follows from (4.26) and (4.27).

For each map $\alpha : A \dashrightarrow B$, let $\mathcal{G}_\alpha$ be the transformation (G_α, G_A, G_B).

THEOREM 4.2. <u>For any</u> DFP-<u>dicategory</u> $\mathcal{C}$, <u>the function</u>

$$\Sigma : \begin{cases} A \longrightarrow G_A \\ \alpha \longrightarrow \mathcal{G}_\alpha \end{cases}$$

<u>is a faithful representation for</u> $\mathcal{C}$ <u>by groups provided it is object-faithful; that is, provided</u> $G_A = G_B$ <u>implies</u> $A = B$ <u>for all objects</u> A <u>and</u> B.

Proof. Consider the statements:

(A) $\alpha\beta$ defined implies $G_{\alpha\beta} = G_\alpha G_\beta$;

(B) for each injection $\kappa : A \dashrightarrow B$, G_κ is the identity function on G_A;

(C) for each supermap $\rho : C \dashrightarrow D$, G_ρ maps G_C onto G_D;

(D) if α_1 and α_2 are coterminal, then $\alpha_1 \neq \alpha_2$ implies $G_{\alpha_1} \neq G_{\alpha_2}$.

As was noted in the Appendix to Chapter III, these statements are valid for the <u>functions</u> G_α and the <u>sets</u> G_A. For J is an integral and free object and, in the notation of that appendix, these functions and sets are G^J_α and G^J_A, respectively. Finally, it is easily shown that these statements extend to sufficient conditions for a representation; for, by Lemmas 4.11 and 4.12, each G_A is a group and each G_α is a homomorphism.

7. <u>Appendix</u>. The axioms for DFP-dicategories have been obtained by abstracting certain properties of the dicategory $\mathcal{C}_G$ of all groups. In the absence of any other criterion for the selection

of axioms one may attempt to remove the restrictive hypothesis of Theorem 4.2 by adding as axiom a statement whose interpretation for groups is valid and which implies object-faithfulness for the representing function Σ. Since the representation of Theorem 4.2 is in fact object-faithful when applied to $\mathcal{C}_G$, such a statement is available. We have chosen not to include an axiom of this sort because the phrasings which we have been able to achieve entail awkwardness: if stated in the language of the other axioms, clarity is lacking; put in simplest form, none of the terms primitive in a dicategory are used.

Finally, we note that the representation of Theorem 4.2 applies to the dicategory $\mathcal{C}_G$ of all groups to yield for each group A an isomorphism ϕ_A mapping A onto G_A for one choice of $\overline{\nabla}_J$ and an anti-isomorphism for the other choice of $\overline{\nabla}_J$. In the first case, there is for each homomorphism α of the group A into the group B, the commutativity relation $\phi_B \alpha = G_\alpha \phi_A$.

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