THE GROWTH OF WATER WAVES DUE TO THE ACTION OF THE WIND

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ABSTRACT

This study consists of an attempt to understand the mechanism whereby the wind causes the growth of waves in water. A simple geometrical model is suggested to describe, in part, the turbulent flow of air over a sinusoidal surface. The model permits the calculation of the magnitude of that component of air pressure which is responsible for wave growth. Interpreting the model as a rough picture of a separation phenomenon permits the calculation of the magnitude of that component of surface shear which also contributes to wave growth. Pressures computed from the model coincide reasonably well with some measured in wind tunnel tests using solid test-models.

As a preliminary to the application of the model to water surfaces, a general wave-growth equation is developed under the condition that winds extend over only a finite region of space. Use of this growth equation, together with a condition limiting the wave-height (obtained from the theory of a single-frequency wave system), leads to an encouraging comparison between the predictions of the model and observations on the growth of water waves over a wide range of experimental conditions.

Effects limiting wave growth in multiple-frequency systems are considered briefly, and a result is obtained which describes one such effect.

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I. INTRODUCTION.

How the wind makes waves in water has never been satisfactorily explained. Observations of wave characteristics date from the days of Leonardo da Vinci, who reported an apparent increase in wave-length of ocean waves with distance of wave travel. In the last few years quite accurate semi-empirical relations have been developed which relate wave-height and length to wind speed, fetch (distance from beginning of the storm to the point of observation), and duration [(1)]. With these relations the waves resulting from any given storm may be predicted; or, conversely, if the waves only are observed, the position and strength of the storm which produced them may be determined, even though the waves may have traveled through a long region of calm since leaving the storm area. But such relations give no clear understanding of the interaction between wind and water.

Why should such a problem still exist, with all the abilities of modern physics and the accomplishments of modern aerodynamics? The nature of the problem can be understood from the simplest picture of water waves and air motion. For the two dimensional case, with the y-axis pointing vertically upwards and the x-axis lying along the undisturbed water surface, gravity waves of small amplitude are adequately represented by (Cf. Ref. 2, p. 363)

$$\eta = a \cos k(x - Ct) \qquad (1)$$

where η is the height of the surface, measured parallel to the y-axis, and wave length and speed are related by

$$kC^2 = g = \text{the acceleration of gravity} \qquad (2)$$

Beneath the surface of the wave the motion of the water particles at any point is described by the x and y components of velocity, which are, respectively,

$$u = akCe^{ky} \cos k(x-Ct) \tag{3}$$

$$v = akCe^{ky} \sin k(x-Ct) \tag{4}$$

If a variable air pressure applied at the surface (thus a normal stress) is to do work on the wave system, the pressure must be 180° out of phase with the outward normal component of water particle velocity at the surface. For small amplitude waves, the normal component of velocity may be set equal to the y component, given by eq. (4). Thus the variable part of the air pressure must be 90° out of phase with the surface shape.

An analysis of the air motion is given in Section III, where it is shown that for pure potential flow the variable part of the air pressure is in phase with the surface shape. Thus if the air is moving in potential flow over the wave surface, variations in air pressure cannot cause wave growth.

Shear forces applied at the surface might do work on the wave system (Cf. Ref. 2, p. 629). This possibility is analyzed in Section III. Shear forces of the right phase exist for air in potential flow, and if the motion of the water immediately below the surface is laminar, work could be done on the wave system. However, the resulting shear force depends linearly upon wave amplitude, and if this is the only force present, wave growth would be exponential. Such growth rates are not observed.

Thus an idealized picture of air and water motion completely fails to describe the phenomenon of wave growth in a wind. It is possible to understand then why the problem is still unsolved. Only recently have non-ideal fluid motions (e.g. turbulent motion) been carefully investigated. Even now only a very little bit is known of this extremely complex field.

Using some early results on non-ideal fluid motion, Jeffreys, in 1925 [3], suggested a mechanism to explain wave growth. He based his idea on analogies with two results. First, the motion of air around a sphere placed in an air stream was observed to be non-potential behind the sphere. The pressure on the back of the sphere was found to be less than it would have been had potential motion been followed. Thus the sphere experienced a drag force. Second, the pressure on a flat plate held at an angle to an air stream was found to be proportional to that angle. From these results Jeffreys reasoned that the motion behind a wave crest is not potential motion, but such that the pressure on the front (weather) slope of the crest is greater than that on the rear (lee) slope. Further, the variation of pressure over the surface must go as the slope of the surface. Thus the variable part of the air pressure, according to Jeffreys' idea, is

$$\Delta p = s\rho_a (W-C)^2 \frac{\partial \eta}{\partial x} \qquad (5)$$

where ρ_a = density of the air

W = speed of the wind

s = arbitrary constant (the "sheltering coefficient")

Actually, there is no reason to believe that there is any similarity between air flow around a flat plate held in an air stream and air flow over a wavy surface. Thus Jeffreys was probably not justified in using this analogy as a reason for selecting the term $\frac{\partial \eta}{\partial x}$ in his pressure law. A more reasonable way to approach his result is as follows: Certainly the pressure must be proportional to the dynamic pressure of the wind relative to the wave shape, thus $\rho_a (W - C)^2$. It is reasonable to suppose that there is some break-down of potential flow behind the crests, and the resulting pressure distribution undoubtedly has a component 90° out of phase with the wave shape. Also, the magnitude of this component must depend on the size of the wave. The simplest mathematical expression which is 90° out of phase with a sine wave and is proportional to the size of the sine wave is its derivative. Thus the pressure is proportional to $\rho_a (W - C)^2 \frac{\partial \eta}{\partial x}$. Inclusion of an arbitrary constant is reasonable in the hope that many of the unknown factors may be thus disposed of, and the result is the simplest possible explanation of wave growth. Indeed, in 1925 it was probably the only reasonable explanation. Unfortunately, the predictions of this idea do not agree with observations. First, the pressure depends linearly upon wave amplitude. Thus the resulting wave growth is exponential in time. As mentioned earlier, this does not agree with observations. Second, Jeffreys evaluated the constant, s, by measuring the smallest wind which would maintain waves against viscosity, and measuring the wave length of the waves so maintained. (Cf. Ref. 2, pp. 623-625 for the relations involved.) The resulting value of s found by him and others is $s \simeq 0.27$. If this value is used for large waves, for instance at sea in a storm, the wave growth is at least ten times faster than observed.

Although today the picture is far from complete, enough additional information is available to make important improvements in Jeffreys' preliminary idea.

Recently another, and completely different, wave-growth mechanism has been suggested by Carl Eckart [4]. The motion of air in a storm is far from regular. Many local pressure fluctuations occur, and seem to move with the mean air velocity in their vicinity until they disappear. Eckart has analyzed the response of the sea surface to such pressure fluctuation and found that a wave pattern is produced whose spectrum appears similar to that observed. Increase in wave height is a result of the addition of waves arising from various pressure fluctuations, or "gusts". Thus the rate of wave growth depends upon the correlation among gusts. There is no direct amplification of wave height; that is, the existence of a wave does not result in any local pressure variation which causes that same wave to grow still further. Thus there is a distinct difference between this theory of Eckart and that of Jeffreys. Unfortunately, preliminary measurements of the strength of such pressure fluctuations indicate that wave growth from such a cause will proceed at about 1/10 of the observed rate.

The present study is principally concerned with the description and analysis of a model of air motion over a sinusoidal surface. Thus it might be considered an extension of Jeffreys' idea. However, before this matter is considered, a preliminary result is obtained. The author was unable to find in the literature the solution to the problem of the response of a water surface to a sinusoidal pressure applied over a limited region of space. Since both storms and oceans are finite in extent, it is felt that such a result should be obtained for the analysis of the effect of wind on water. The next section of this paper is devoted to that problem.

Since a growth mechanism can only give half the description of water waves, a portion of this paper is devoted to consideration of effects limiting wave growth. Only very preliminary results are obtained for this part of the problem. The understanding of growth-limiting mechanisms is a different and perhaps even more complex problem than the understanding of growth mechanisms.

II. RESPONSE TO A TRAVELING DISTURBANCE.

In explaining wave growth in a wind two types of forces will be considered. These are normal pressure and tangential shear applied at the surface. These forces will be periodic, with a period equal to that of the water waves. The magnitude of the forces will depend upon the amplitude and wave length of the water waves, and the velocity of the wind. Thus it is necessary to determine the growth rate of waves in the presence of a traveling periodic disturbance of either normal pressure or tangential shear when the extent of the disturbance is limited in space, i.e. limited to the region over which the wind is blowing. The basis of the method used to solve this problem is given by Ref. 2, p. 396.

A. Forces Normal to the Surface. Consider first the application of a normal pressure given by

$$p = p_0 \operatorname{Cos}(Kx - \omega t) \quad \text{for} \quad 0 \leq t \text{ and } -D \leq x \leq 0$$
$$\text{and} \quad p = 0 \quad \text{otherwise.} \tag{6}$$

The response to such a function will be computed by a Green's function method. Suppose an initial pressure impulse applied at the surface and having a value Cos kx per unit length causes oscillations wherein the water surface is given by the real part of

$$\eta = f(k) \cdot \operatorname{Cos} kx \cdot e^{i\sigma t} \tag{7}$$

where $\sigma^2 = kg$. Then the response to a δ-function impulse applied at the point $x = \alpha$ is

$$\eta = \frac{1}{\pi}\int_0^\infty f(k) e^{i\sigma t} \operatorname{Cos} k(x-\alpha)\, dk \tag{8}$$

since

$$\delta(x-\alpha) = \frac{1}{\pi}\int_0^\infty \operatorname{Cos} k(x-\alpha)\, dk \tag{9}$$

The particular surface pressure under consideration (eq. (6)) can be thought of as a set of infinitesimal δ-function impulses of width dx applied at intervals of time dt. Thus at the point $x = \alpha$ and the time $t = \tau$, the magnitude of the applied impulse is

$$dp = p_o \operatorname{Cos}(K\alpha - \omega\tau)\,d\alpha\,d\tau \qquad (10)$$

for α and τ in the stipulated range. The cumulative effect of such impulses at the point x and the time t is given by the integral

$$\eta = \frac{i p_o}{4\pi \rho_w g}\int_0^t\int_{-D}^{0}\int_0^\infty \sigma e^{i\sigma(t-\tau)}\left[e^{ik(x-\alpha)} + e^{-ik(x-\alpha)}\right] \cdot\left[e^{i(K\alpha-\omega\tau)} + e^{-i(K\alpha-\omega\tau)}\right] dk\,d\alpha\,d\tau \qquad (11)$$

The correct form for f(k) obtained from Ref. (2), pp. 11 and 415, has been introduced into this expression,

$$f(k) = \frac{i\sigma p_o}{\rho_w g} \qquad (12)$$

where ρ_w = density of the water. Integrating first over α and then over τ gives

$$\eta = \frac{i p_o}{4\pi\rho_w g}\int_0^\infty \sigma e^{i\sigma t}\Big\{\Big(e^{ikx}\frac{1-e^{-i(K-k)D}}{i(K-k)} + e^{-ikx}\frac{1-e^{-i(K+k)D}}{i(K+k)}\Big)\frac{e^{-i(\omega+\sigma)t}-1}{-i(\omega+\sigma)} + \Big(e^{ikx}\frac{1-e^{i(K+k)D}}{-i(K+k)} + e^{-ikx}\frac{1-e^{i(K-k)D}}{-i(K-k)}\Big)\frac{e^{i(\omega-\sigma)t}-1}{i(\omega-\sigma)}\Big\}dk \qquad (13)$$

Making the substitutions

$$\left.\begin{array}{lll} dk = (2/g)\,\sigma\,d\sigma & & \sigma/\omega = y \\ KD = N & x\omega^2/g = f & \omega t = b \end{array}\right\} \qquad (14)$$

and assuming, first, $Kg = \omega^2$ (15)

permits the remaining integral to be written as

$$\eta = \frac{k}{2\pi\rho_w g}\frac{\partial}{\partial f}\int_0^\infty \Big\{ e^{ify^2}\frac{(e^{-ib}-e^{iby})(1-e^{-i(1-y^2)N})}{(1+y)(1-y^2)}$$

$$-e^{-ify^2}\frac{(e^{-ib}-e^{iby})(1-e^{-i(1+y^2)N})}{(1+y)(1+y^2)} + e^{ify^2}\frac{(e^{ib}-e^{iby})(1-e^{i(1+y^2)N})}{(1-y)(1+y^2)}$$

$$-e^{-ify^2}\frac{(e^{ib}-e^{iby})(1-e^{i(1-y^2)N})}{(1-y)(1-y^2)}\Big\}\,dy \qquad (16)$$

The first, third and fourth terms of this integral contain apparent singularities, and thus these terms will make the most important contributions to the result. In comparison with them, the second term may be neglected. The first term of the integral may be expressed as the sum of four integrals. Call the first of these I_1 and make the substitution $z = y - 1$. Then

$$I_1 = \int_{-1}^{\infty} \frac{e^{i[b+f(z^2+2z+1)]}}{(z+2)^2 z}\,dz \qquad (17)$$

Important contributions to the integral, for large values of f (i.e. a wave train many wave-lengths long), occur in the region of $z = 0$. Therefore a negligible error will be introduced by neglecting the variation of z in the term $(z + 2)^2$ and writing the exponent as $i\,(f - b + 2fz)$. Finally, extension of the lower limit of integration to $-\infty$ may be justified by the same argument, and the integral is approximately

$$I_1 \simeq \frac{1}{4} e^{i(f-b)}\int_{-\infty}^{\infty} e^{2ifz}\frac{dz}{z} \qquad (18)$$

The singular behavior of this integral is due to the fact that it represents only a part of the complete expression of eq. (16). The complete expression has no singularities, as may be verified by writing out the exponential terms as sines and cosines and taking limits. Thus in order to evaluate the integral term by term using contour integration, the principal parts of singular integrals must be taken, and the singularities must be left on the same side of the contour in all the integrals. The result for I_1 is

$$I_1 \simeq \pm \frac{\pi i}{4} e^{i(f-b)} \quad \begin{cases} + \text{ for } & f > 0 \\ - \text{ for } & f < 0 \end{cases} \tag{19a}$$

In the same manner the other three integrals of the first term are approximated by

$$I_2 \simeq \pm \frac{\pi i}{4} e^{i(f-b)} \quad \begin{cases} + \text{ for } & f > -N \\ - \text{ for } & f < -N \end{cases} \tag{19b}$$

$$I_3 \simeq \pm \frac{\pi i}{4} e^{i(f+b)} \quad \begin{cases} + \text{ for } & f > -b/2 \\ - \text{ for } & f < -b/2 \end{cases} \tag{19c}$$

$$I_4 \simeq \pm \frac{\pi i}{4} e^{i(f+b)} \quad \begin{cases} + \text{ for } & f > -b/2 - N \\ - \text{ for } & f < -b/2 - N \end{cases} \tag{19d}$$

The complete term is then $I = I_1 - I_2 - I_3 + I_4$. The third term may be handled in exactly similar manner, giving

$$J_1 \simeq \pm \frac{\pi i}{2} e^{i(f+b)} \quad \begin{cases} + \text{ for } & f > 0 \\ - \text{ for } & f < 0 \end{cases} \tag{20a}$$

$$J_2 \simeq \pm \frac{\pi i}{2} e^{i(f+b+2N)} \quad \begin{cases} + \text{ for } & f > -N \\ - \text{ for } & f < -N \end{cases} \tag{20b}$$

$$J_3 \simeq \pm \frac{\pi i}{2} e^{i(f+b)} \quad \begin{cases} + \text{ for } & f > -b/2 \\ - \text{ for } & f < -b/2 \end{cases} \tag{20c}$$

$$J_4 \simeq \pm \frac{\pi i}{2} e^{i(f+b+2N)} \quad \begin{cases} + \text{ for } & f > -N - b/2 \\ - \text{ for } & f < -N - b/2 \end{cases} \tag{20d}$$

where the complete term is $J = J_1 - J_2 - J_3 + J_4$. The only difference found in the fourth term is the existence of a double pole at the point $z = 0$, whereas single poles were encountered in the first and third terms. However the principal values of the integrals may still be obtained, and the results are

$$L_1 \simeq \pm \pi f e^{i(b-f)} \quad \begin{cases} + \text{ for} & f < 0 \\ - \text{ for} & f > 0 \end{cases} \tag{21a}$$

$$L_2 \simeq \pm \pi (f - \tfrac{b}{2}) e^{i(b-f)} \quad \begin{cases} + \text{ for} & f < b/2 \\ - \text{ for} & f > b/2 \end{cases} \tag{21b}$$

$$L_3 \simeq \pm \pi (f + N) e^{i(b-f)} \quad \begin{cases} + \text{ for} & f < -N \\ - \text{ for} & f > -N \end{cases} \tag{21c}$$

$$L_4 \simeq \pm \pi (f - \tfrac{b}{2} + N) e^{i(b-f)} \quad \begin{cases} + \text{ for} & f < -N + b/2 \\ - \text{ for} & f > -N + b/2 \end{cases} \tag{21d}$$

where the complete term is $L = L_1 - L_2 - L_3 + L_4$. The complete expression for the surface is approximately the real part of

$$\eta = \frac{-p_0}{2\pi \rho_w g} \frac{\partial}{\partial f} (I + J + L) \tag{22}$$

This expression turns out to have seven different values in seven different regions, as follows (taking real parts)

Region 1. $f < -N - b/2$

$$\eta = 0 \tag{23a}$$

Region 2. $-N - b/2 < f < -N$

$$\eta = -\frac{p_0}{4\rho_w g} \left[\text{Cos}(f+b) + 2\,\text{Cos}(f+b+N) \right] \tag{23b}$$

Region 3. $-N < f < -N - b/2$

$$\eta = -\frac{p_0}{4\rho_w g} \left[\text{Cos}(f+b) + 3\,\text{Cos}(f-b) + 4(f+N)\,\text{Sin}(f-b) \right] \tag{23c}$$

Region 4. $-N - b/2 < f < -b/2$

$$\eta = -\frac{p_0}{4\rho_w g}\left[\cos(f-b) - \cos(f+b) - 2b\sin(f-b)\right] \tag{23d}$$

Region 5. $-b/2 < f < 0$

$$\eta = -\frac{p_0}{4\rho_w g}\left[\cos(f-b) + 2\cos(f+b) - 2b\sin(f-b)\right] \tag{23e}$$

Region 6. $0 < f < b/2$

$$\eta = -\frac{p_0}{4\rho_w g}\left[-4\cos(f-b) + (4f-2b)\sin(f-b)\right] \tag{23f}$$

Region 7. $b/2 < f$

$$\eta = 0 \tag{23g}$$

It is tacitly assumed that $N > b$. The situation when this is not true will be treated later. The description of the surface at the boundary points of the regions cannot be obtained by the method just described. However other methods are available for obtaining approximate solutions near these points. Such solutions indicate that the amplitudes of the oscillations given above as straight lines in x (or f) and shown in fig. 1 are joined by smooth curves at the boundaries of the regions. The rounding effect of these curves extends over a few wave lengths. Therefore, for N and b large the straight lines describe the complete solution adequately, and no deeper understanding of the problem can be gained by careful attention to the special solutions in the vicinity of the boundary points. Therefore the methods of obtaining such solutions, and a detailed description of the results will not be given here.

Consideration should now be given to the meaning of the solutions just obtained. The first two terms of the solutions in regions 2, 3, 4, and 5, and the first term in region 6 have amplitudes which do not vary with either f or b (i.e. space or time). Such terms represent the transient parts of the solution, as may be verified by introducing an exponential damping term in the original expression; that is, substitute $\exp (i\sigma - p)(t - \tau)$ for $\exp\ i\sigma(t - \tau)$ in eq.(11). The term $\exp(-pt)$ will come through as a factor in the transient terms listed above, but will cancel out of the remaining terms in the limit of small p. Therefore it is the remaining terms which are of chief interest. The amplitudes of the oscillations in these remaining terms are listed below together with the regions over which they are applicable. The expressions have been rewritten in terms of x (fetch) and t (duration). The amplitudes are also shown in fig. 1.

Region I. $x < -D$ (includes Regions 1 and 2)

Amplitude = 0

Region II. $-D < x < -\frac{\omega t}{2K}$ (includes Region 3)

Amplitude = $\frac{p_o K}{\rho_w g}(x + D)$

Region III. $-\frac{\omega t}{2K} < x < 0$ (includes Regions 4 and 5)

Amplitude = $\frac{p_o}{2\rho_w g}\omega t$

Region IV. $0 < x < \frac{\omega t}{2K}$ (includes Region 6)

Amplitude = $\frac{p_o}{\rho_w g}\left(\frac{\omega t}{2} - Kx\right)$

Region V. $\frac{\omega t}{2} < Kx$ (includes Region 7)

Amplitude = 0

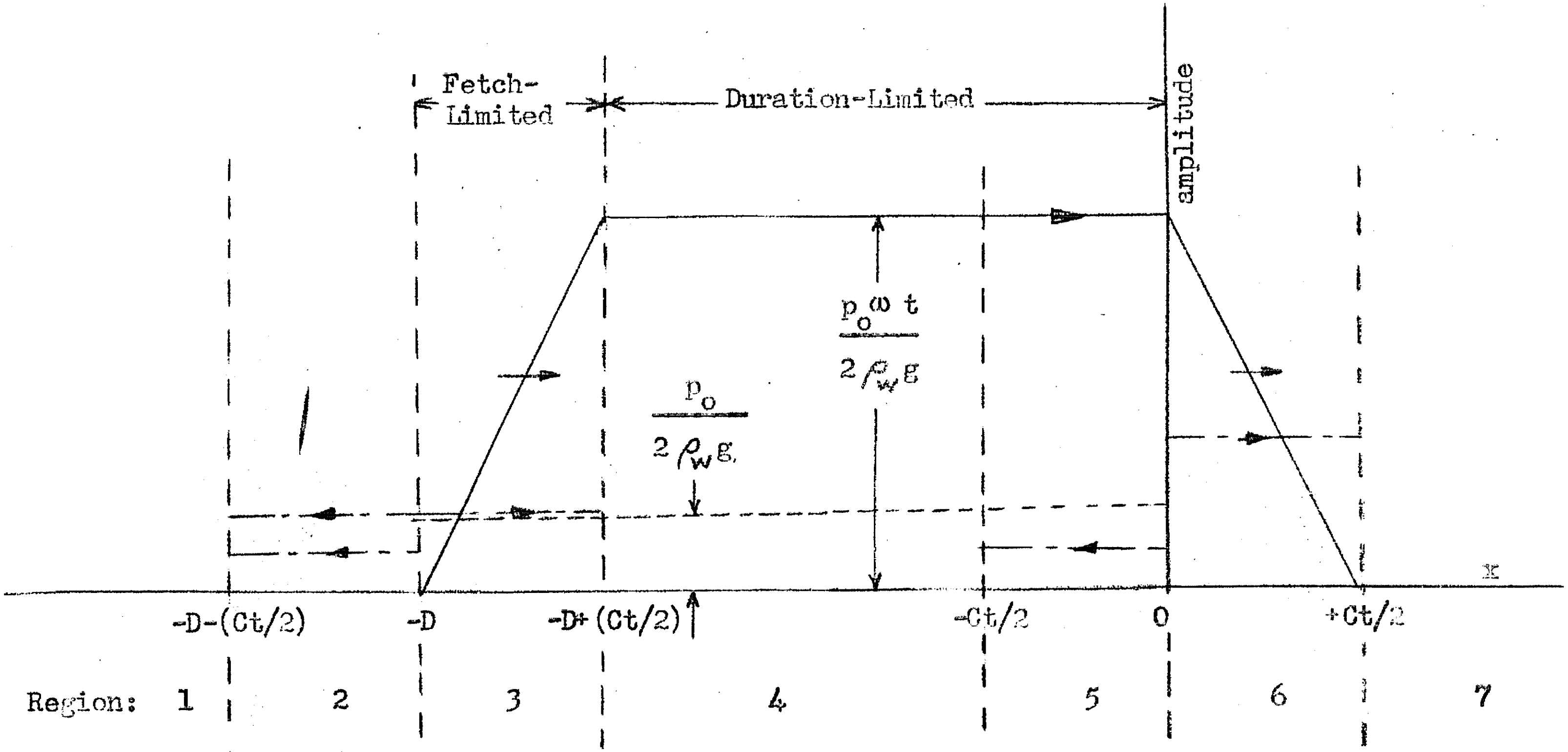
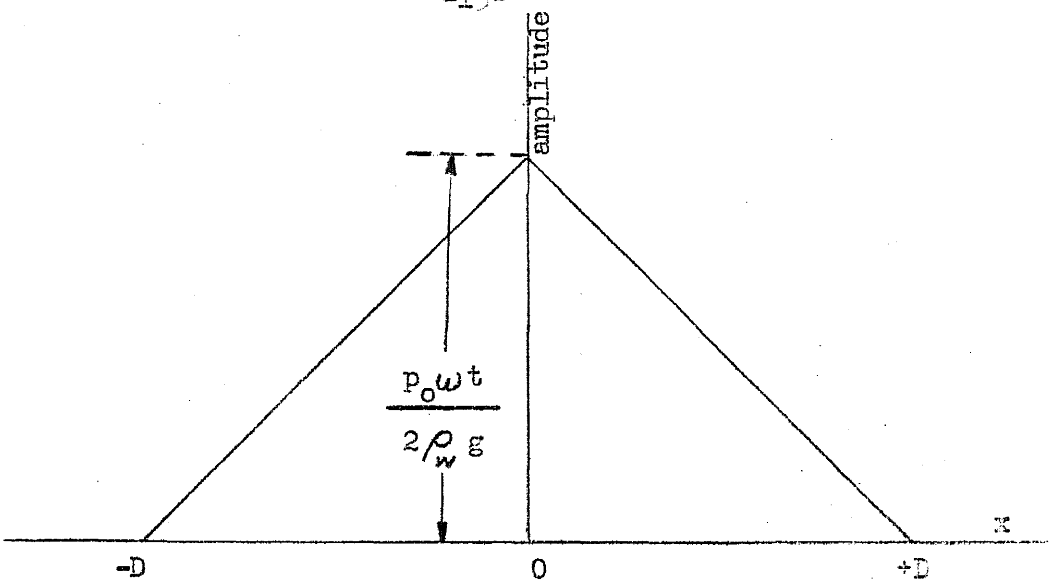


Fig. 1 Relative amplitudes of standing-wave transient (-----), traveling-wave transient (—▸---, arrow indicates direction of travel), and non-transient (——▷——) solutions for traveling sinusoidal pressure applied over a finite region of the surface. For this case, $t < 2D/C$



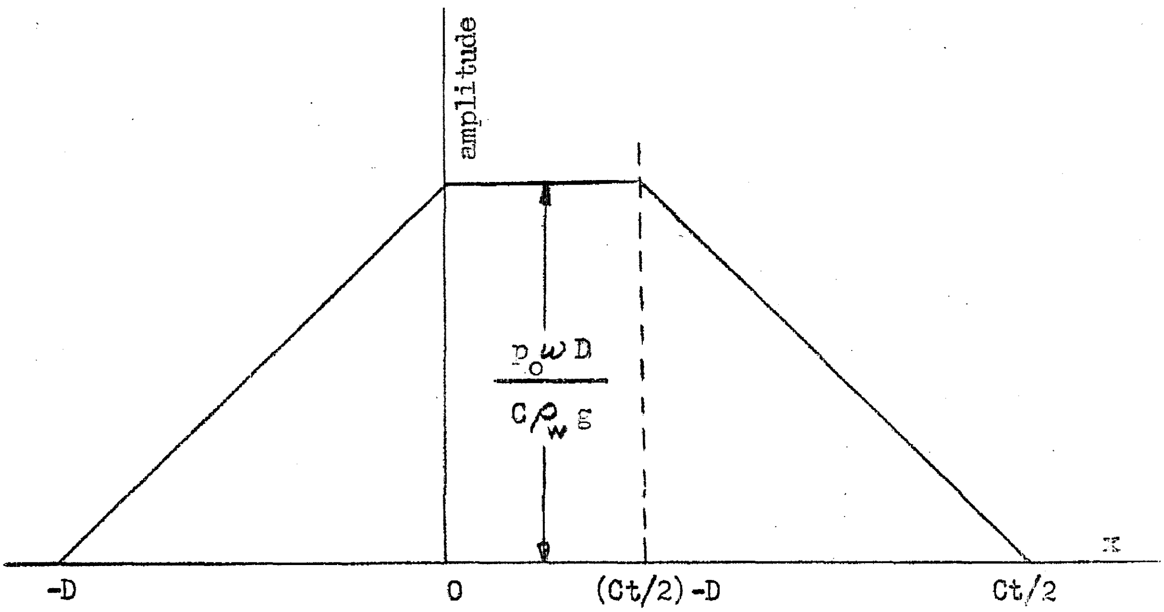


Fig. 2 Amplitude of non-transient solution for traveling sinusoidal pressure. $t = 2D/C$ (above). $t > 2D/C$ (below).

It was pointed out earlier that the tacit assumption $N > b$ was made. For the regions I through V listed above this assumption can be relaxed to $N > b/2$. However there is no difficulty in obtaining the solution when this assumption does not hold. The amplitudes of the non-transient part for $N = b/2$ and for $N < b/2$ are shown in fig. 2.

The various regions of wave development shown can be labeled in the traditional manner. Thus region II is the region of "fetch-limited" growth; region III is the region of "duration-limited" growth; and region IV is the region of propagation into the area of calm. Another well known result is to be found in this solution. It will be remembered that the applied pressure has the form of Cos (f - b) (eq. (6)). The non-transient part of the resulting surface waves have the form Sin (f - b) (eq. (23)). Thus the necessary pressure-to-surface phase relation exists, a relation pointed out by Lamb and others [2].

It was assumed (eq. (15)) that $Kg = \omega^2$. If this assumption is dropped, that is, if pressures are applied which do not advance at the same speed as gravity waves of equal wave length, a solution may still be obtained. However, the solution is trivial, and consists of nothing more than a progressing surface deformation with a magnitude approximately equal to the hydrostatic head corresponding to the applied pressure, with some transient terms.

To complete the discussion of the solution it should be mentioned that certain possibly large contributions to the integral of eq. (16) have been omitted. These are the contributions near the points of stationary phase of the exponentials in the integrand. If these terms are evaluated it will be found that they contribute terms such as those arising from the application of a local impulse (Cf. Ref. 2, sec. 239 - 240). Such terms do not contribute to the coherent wave pattern which is now of interest.

The variation in amplitude of the waves resulting from a traveling pressure disturbance is open to an interesting interpretation. Notice that for the step function type of pressure distribution here analyzed, the effect of sudden variations in the amplitude of the pressure moves across the wave system at group velocity. Thus, starting at the downwind origin of the pressure distribution (at x = -D) the wave amplitudes increase linearly from 0 to $p_o \omega t/2\rho_w g$ over the interval $\Delta x = (C/2)t$. For deep-water gravity waves the group velocity is

$$U = C/2 \tag{24}$$

Waves leaving the pressure region at the upwind end (at x = 0) decrease in amplitude from $p_o \omega t/2\rho_w g$ to 0 over an interval of the same length. It is reasonable to extend this to say that any variation in amplitude of applied pressure will cause an effect on the amplitude of the wave system which moves across the wave system at group velocity. Thus the total wave-amplitude at any point (x, t) is the sum of all "effects" which, traveling at the group velocity of the wave, arrive at the spatial point x at the time t. The magnitude of such an "effect" is $(\omega/2\rho_w g)$ times the pressure at its point of origin. This explanation can be expressed by an equation relating the wave-amplitude, a (x,t) to the amplitude of the applied pressure, p_o (x',t'). Thus

$$a(x,t) = \frac{\omega}{2\rho_w g}\int_0^t p_o(x-U\tau, t-\tau)\,d\tau \tag{25}$$

This equation certainly explains the results of the application of a step-function type of pressure distribution as analyzed in this Section. On the basis of the interpretation given above, it will be assumed to apply to all types of pressure distributions.

The result can be expressed in a more useful form. Thus, make the substitutions

$$\left.\begin{aligned} y &= x - Ut \\ \beta &= t - \tau \end{aligned}\right\} \qquad (26)$$

in eq. (25). The resulting integrand does not contain t explicitly. The expression reads

$$a(y+Ut, t) = \frac{\omega}{2\rho_w g}\int_0^t p_0(y+U\beta, \beta)\,d\beta \qquad (27)$$

This may be differentiated partially with respect to t yielding

$$\left(\frac{\partial a}{\partial t}\right)_y = \frac{\omega}{2\rho_w g}\,p_0(x,t) = U\left(\frac{\partial a}{\partial x}\right)_t + \left(\frac{\partial a}{\partial t}\right)_x \qquad (28)$$

Separation into conditions of duration-limited growth ($\frac{\partial a}{\partial x} = 0$) and fetch-limited growth ($\frac{\partial a}{\partial t} = 0$) is an obvious result of this expression.

B. <u>Forces Tangent to the Surface.</u> Earlier in this section it was pointed out that two types of forces are to be considered in explaining wave growth in a wind. These are normal pressures and tangential shear. So far the discussion has been of normal pressures only. However, similar results are obtained from considerations of tangential shear. The results differ in two ways, both of which can be seen easily from Ref. 2, pp. 629-630. First, the waves which grow from the application of periodic tangential shear forces applied to the surface are in phase with the forces, whereas the waves are 90° out of phase with normal pressures.

Second, since viscosity must be considered in order to explain energy transfer by shear forces, viscous losses are present. Of course, to make a realistic picture, viscous losses should have been included with the consideration of the effect of normal pressures. However, with justification and conditions to be presented later, viscous losses will be omitted from the present problem. Thus, if s_o is the amplitude of the shear force (per unit area), eq. (28) can be modified to read

$$U\left(\frac{\partial a}{\partial x}\right)_t + \left(\frac{\partial a}{\partial t}\right) = \frac{\omega}{2\rho_w g}\left[p_o(x,t) + s_o(x,t)\right] \qquad (29)$$

That the amplitude of the shear force can be included in the growth equation in the same manner as the amplitude of normal pressure is justified in the following manner. The analysis uses the results of Sections 349 and 350 of Ref. 2. Under the assumption that applied normal force is zero (note that since the theory being used here is linear, this assumption only simplifies the subsequent calculations, but does not limit the results, since effects of normal forces may be added into the linear theory at any point) the applied tangential force, s, is given by eq. (3), Sec. 350[2] as

$$\frac{s}{g\rho_w \eta} = \frac{in}{gk} \cdot \frac{(n+2\nu k^2)^2 + \sigma^2 - 4\nu^2 k^3 m}{n + 2\nu k^2 - 2\nu k m} \qquad (30)$$

The number n appears in the factor exp (ikx - nt) of the expression giving the surface shape (i.e. eq. (9), sec. 349 [2]). For the present case, assume $n = \beta - i\sigma$, where $\beta \ll \sigma$. Except for very minute wave-lengths, the number $\nu k^2/\sigma$ (thus also $\nu km/\sigma$) is very small. Then eq. (30) reduces, approximately, to

$$s \simeq \frac{2\rho_w \sigma}{k}\eta(\beta + 2\nu k^2) \qquad (31)$$

with the surface shape given by

$$\eta = a_o e^{\beta t} \cos(kx - \sigma t) \tag{32}$$

Suppose the applied tangential force is written in the form

$$s = s_o \cos k(x - Ct) \tag{33}$$

and the surface shape is written as in eq. (1)

$$\eta = a \cos k(x - Ct)$$

Then eqs. (31) and (32) imply

$$\frac{da}{dt} = \frac{k s_o}{2\rho_w \sigma} - 2\nu k^2 a \tag{34}$$

The second term on the right-hand side of this equation represents the viscous loss. Thus the size of the ratio

$$4\nu k^2 a \rho_w \sigma / k s_o$$

determines the importance of viscous losses in the present problem. The origin of tangential forces is considered in the next section where it is shown that

$$s_o = 9.7 \cdot 10^{-4} \rho_a W^2 / (2\pi \sqrt{ak})$$

In the c.g.s. system of units

$$\nu = 0.01 \text{ cm.}^2/\text{sec.}$$

$$g = 10^3 \text{ cm./sec.}^2$$

$$\rho_w = 1.0 \text{ gm./cm.}^3$$

$$\rho_a = 0.00125 \text{ gm./cm.}^3$$

and the ratio is approximately

$$3 \cdot 10^8 \, a^{3/2} / \lambda^2 W^2$$

where $\lambda = 2\pi/k$, the wave-length of the waves. For waves to be considered in this study the value of a/λ ranges between 0.003 and 0.07. Using a value near the upper limit of this range as a conservative estimate, the ratio becomes

$$5 \cdot 10^6 / \lambda^{1/2} W^2$$

In the range of short wave-lengths, e.g. $\lambda \lessapprox 10^3$ cm., wind speeds must be of the order of 10^3 cm./sec or greater to make this ratio small. Certainly the range of conditions wherein the ratio is large is not negligible. However, it is not the purpose of this study to make a complete analysis of all factors affecting wave growth, but rather to propose a model or air flow over a wave and to make a preliminary analysis of the predictions of such a model. With such a purpose in view, little extra understanding would be gained from the added mathematical complications involved in including the effects of viscous losses. Thus, realizing that the results must be incorrect for small winds and small wave-lengths, the viscous term will be dropped from eq. (34). (Generally, P_o S_o; Cf. part D, Sec.III.)

The resulting expression gives the change with time of the amplitude of an infinite wave train in response to a periodic shear force applied over an infinite range of space. Suppose such a shear force is applied only for a short interval of time, dt. Then from eqs. (1), (33), and (34) the resulting surface deformation is

$$d\eta = (\sigma s_o / 2\rho_w g)\, dt \, \mathrm{Cos}\, k(x - Ct) \qquad (35)$$

This result is formally identical with eq. (7), with f(k) similar to that given in eq. (12), except that p_o has been replaced by s_o and the factor i is missing. The remaining analysis could be carried through with shear forces just as it was done with normal pressures, and the only change in the result would be that the resulting surface form is in phase with the applied forces instead of 90° out of phase as in the case of normal pressures.

It must be pointed out that the results of Lamb, on which the foregoing analysis was based, were obtained under the assumption that wave amplitudes are small compared to any other characteristic length arising in the problem. One such length is the thickness of the layer of vorticity at the surface (the boundary layer), $\sqrt{2\nu/\sigma}$. The results will be of little value in the present study if they must be restricted to waves with amplitudes small compared to this thickness. A complete solution of the non-linear problem of finite amplitude waves in a rotational fluid will not be attempted here. However, it will be shown that the boundary layer thickness is independent of wave amplitude. Since the effect of shear forces applied at the surface is transmitted into the fluid by the vorticity in this boundary layer, such a result implies that the effect of shear forces is independent of amplitude. Another way to state this argument is as follows: Suppose there exists a system of finite amplitude waves. Suppose a shear force is applied at the surface of these waves which is of the right phase and wave-length to create a small incremental wave of the same wave-length and phase as the original finite amplitude wave. The resulting vorticity will be approximately the same as if the original wave did not exist. This means that the incremental wave will be the same as if the surface were originally flat. But this is to say that wave growth due to shear force will proceed at the same rate with waves of finite amplitude as it does for infinitesimal waves.

The desired result will be shown for an incompressible fluid moving in two dimensions. Suppose the vector velocity at any point and time, (x, y, t), is given by

$$\mathbf{q} = \mathbf{i}u + \mathbf{j}v \tag{36}$$

Assume the fluid motion is approximately irrotational, or that

$$\mathbf{q} = \mathbf{q}_0 + \mathbf{q}_1 \tag{37}$$

where $\mathbf{q}_0$ satisfies

$$\nabla \times \mathbf{q}_0 = 0 \tag{38}$$

and $\mathbf{q}_1 \ll \mathbf{q}_0$. The equation of motion can be written as (Cf. Ref. 2, Sec. 328)

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \circ \nabla)\mathbf{q} = -\frac{1}{\rho_w} \nabla p + \nu \nabla^2 \mathbf{q} + \mathbf{j} g \tag{39}$$

Using vector identities, the second term on the left can be written as

$$(\mathbf{q} \circ \nabla)\mathbf{q} = \frac{1}{2} \nabla |\mathbf{q}|^2 - \mathbf{q} \times \nabla \times \mathbf{q}_1 \tag{40}$$

in view of eq. (38). Substituting this into eq. (39), and taking the curl of the resulting expression results in

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{q}_1) - \nabla \times \mathbf{q} \times \nabla \times \mathbf{q}_1 = \nu \nabla^2 (\nabla \times \mathbf{q}_1) \tag{41}$$

The second term on the left of this expression can be expanded as

$$\begin{aligned} \nabla \times \mathbf{q} \times \nabla \times \mathbf{q}_1 = {} & \mathbf{q}(\nabla \circ \nabla \times \mathbf{q}_1) - \nabla \times \mathbf{q}_1 (\nabla \circ \mathbf{q}) \\ & + [(\nabla \times \mathbf{q}_1) \circ \nabla]\mathbf{q} - (\mathbf{q} \circ \nabla)(\nabla \times \mathbf{q}_1) \end{aligned} \tag{42}$$

The first term on the right-hand side of this equation is zero identically. The second term on the right vanishes since the fluid is incompressible. The factor $\nabla \times \mathbf{q}_1$ appearing in the third term has only a z component, thus only the operator $\frac{\partial}{\partial z}$ remains in the ∇ term. But $\mathbf{q}$ is a function of (x, y, t) only, so this term vanishes. Next, define the vorticity, ω, as $\omega = |\nabla \times \mathbf{q}_1|$. The remaining equation is approximately

$$\frac{\partial \omega}{\partial t} + \mathbf{q}_0 \circ \nabla \omega \simeq \nu \nabla^2 \omega \tag{43}$$

where the term $\mathbf{q}_1 \circ \nabla \omega$ has been neglected in accordance with the assumpt:

mentioned above.

Suppose the velocity q_0 corresponds to that of irrotational wave motion. Then

$$q_0 = akCe^{ky}[\mathbf{i}\,\mathrm{Cos}\,k(x-Ct) + \mathbf{j}\,\mathrm{Sin}\,k(x-Ct)] \tag{44}$$

where a, k, and C have their usual meanings. Let a new coordinate system be defined which moves with the irrotational wave, thus

$$\begin{aligned} x' &= x - Ct \\ y' &= y \\ \tau &= t \end{aligned} \tag{45}$$

In this system eq. (43) becomes

$$\frac{\partial \omega}{\partial \tau} - C\frac{\partial \omega}{\partial x'} + u_0\frac{\partial \omega}{\partial x'} + v_0\frac{\partial \omega}{\partial y'} = \nu(\nabla')^2\omega \tag{46}$$

where

$$u_0 = akCe^{ky'}\,\mathrm{Cos}\,kx' \tag{47}$$

$$v_0 = akCe^{ky'}\,\mathrm{Sin}\,kx' \tag{48}$$

In this system the potential function of the irrotational wave is

$$\varphi = \frac{C}{k}\xi = C(x' - ae^{ky'}\,\mathrm{Sin}\,kx') \tag{49}$$

and the stream function is

$$\psi = \frac{C}{k}\zeta = C(y' - ae^{ky'}\,\mathrm{Cos}\,kx') \tag{50}$$

If eq. (43) is rewritten in terms of ξ, ζ, and τ, the result is

$$\frac{\partial \omega}{\partial \tau} = \frac{k}{C}[(u_0 - C)^2 + v_0^2][\frac{\partial \omega}{\partial \xi} + \nu\frac{k}{C}(\frac{\partial^2 \omega}{\partial \xi^2} + \frac{\partial^2 \omega}{\partial \zeta^2})] \tag{51}$$

where u_0 and v_0 can, in principle, be defined in terms of ξ and ζ.

In this system the irrotational wave moves with steady motion, i.e. there is no time variation. It is reasonable, therefore, to suppose that the time variation of ω is small, or at least that the change of ω in any single period of wave motion is negligible. Then eq. (51) reduces, approximately, to

$$\frac{\partial \omega}{\partial \xi} \simeq -\nu \frac{k}{C}\left(\frac{\partial^2 \omega}{\partial \xi^2} + \frac{\partial^2 \omega}{\partial \zeta^2}\right) \tag{52}$$

which can be solved by separation of variables, giving

$$\omega = \omega_0 e^{K_1 \zeta + K_2 \xi} \tag{53}$$

with K_1 and K_2 connected by the relation

$$K_2^2 + (C/k\nu)K_2 + K_1^2 = 0 \tag{54}$$

In terms of the original variables, x, y, and t, the solution for ω is

$$\omega = \omega_0 \exp.\{kK_1[y - ae^{ky}\cos k(x-Ct)] + kK_2[x - Ct - ae^{ky}\sin k(x-Ct)]\} \tag{55}$$

The desired solution must be periodic in x with wave number k. Therefore $K_2 = i$. As already pointed out the number $k\nu/C = k^2\nu/\sigma$ is negligibly small for all but the most minute wave-lengths. Therefore

$$K_2 \simeq (1-i)\frac{1}{k}\sqrt{\frac{\sigma}{2\nu}} \tag{56}$$

where the choice of signs is determined by the condition that $\omega \rightarrow 0$ as $y \rightarrow -\infty$.

In accordance with Sec. 250[2], the surface of the wave is given by

$$y = ae^{ky}\cos K(x - Ct) \tag{57}$$

Thus the term $y - ae^{ky}\cos K(x - Ct)$, appearing in the exponent of eq. (55), measures the distance from the surface. The coefficient of this term, $\sqrt{\frac{\sigma}{2\nu}}$, determines the thickness of the boundary layer which is seen to be identical to that for infinitesimal waves. This is the desired result.

III. A MODEL OF THE AIR-WATER INTERACTION.

A. Consequences of Pure Potential Flow. As before only the two dimensional case will be considered, and the ocean will have only single frequency, sinusoidal waves. Over these waves, and in the direction of their motion, a wind is blowing with velocity W.

If the motion of the air over the water can be described as pure potential flow, then it is easy to show that the necessary normal pressure forces for wave growth are absent. Thus, consider the steady motion situation in a coordinate system moving with the waves (thus at velocity C). The boundary of the air motion is then η = a Cos kx. For $ak \ll 1$ a suitable approximation to the flow of the air is obtained from the potential

$$\varphi = (W-C)(-x + ae^{-ky} \sin kx) \qquad (58)$$

and the stream-function

$$\psi = (W-C)(-y + ae^{-ky} \cos kx) \qquad (59)$$

(The stream-function is approximately 0 for y = η and the x component of velocities approaches $W-C$ as $y \to \infty$, while the y component goes to 0.) The variable part of the pressure is the dynamic pressure, and, with the density of air = ρ_a, is given by

$$\frac{1}{2}\rho_a q^2 = \frac{1}{2}\rho_a\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right]$$
$$\simeq \frac{1}{2}(W-C)^2\rho_a[1 + 2ak \cos kx] \qquad (60)$$

As has been shown, this is the wrong phase to cause wave growth.

If the motion of both air and water follows pure potential laws right up to the boundary between the two then a shear force of the right phase for wave growth results. Thus suppose the shear force, per unit area, is given by

$$S = \frac{\rho_a}{2} C_s V^2 \qquad (61)$$

where C_s is some coefficient of shear and V is the relative velocity between air and water particles at the surface. The velocity of the water particles at the surface is (Cf. eq. (3))

$$u \simeq C(-1 + ak \cos kx) \qquad (62)$$

where C is the wave velocity of the water waves. From eq. (58) the velocity of the air particles at the surface is

$$g \simeq W(1 + ak \cos kx) \qquad (63)$$

The variable part of V^2 (still to first order in ak) is

$$V^2 \text{ (var.)} \simeq 2(W^2 - C^2) \, ak \cos kx \qquad (64)$$

which is of the right phase to cause wave growth.

There may be special conditions, perhaps when only very small amplitude waves are present, where the requirements for this force (i.e. pure potential flow of both fluids at the boundary) are met. However, under more general conditions, several observations indicate such requirements are not met. These observations will be discussed next.

B. <u>Evidence of Departure from Pure Potential Flow.</u> In a classic work on the cause of wave growth in a wind, Jeffreys[3] pointed out that a sphere moving thru air suffers a drag force because the motion of the air around the sphere is not that of potential motion. Instead the stream lines separate from the surface behind the sphere leaving a wake of turbulent motion. In this wake, or separated region, the pressures are less than they would have been had potential flow continued all the way around the sphere. Jeffreys suggested that a similar phenomenon took place in the lee of a wave, thus causing asymmetric pressures on the weather and lee sides of the wave of the right phase for wave growth. His results are described in the introduction of this paper.

The fact that such a sheltering effect actually takes place behind ocean waves was vividly pointed out by Cornish[5] who quoted old sea-captains to say "The sails are taken aback in the trough of a wave". Further, measurements by Stanton [6] over a sinusoidal wooden model gave a pressure difference between trough and crest which was only one tenth that required by pure potential flow (Cf. eq. (60)). A curve showing a typical result from Stanton's measurements is reproduced in fig. 5. The curve shows a small pressure rise initially (i.e. just after the downwind crest). This is followed by a region of fairly constant pressure as far as the center of the trough of the wave. Then the pressure rises again as the wave slopes upward to the next crest. About half way up the slope, the pressure reaches a maximum, then falls rapidly to a minimum at the crest itself.

The region of constant pressure along the lee slope probably indicates a region of separation. The pressure maximum on the weather slope may indicate the position of reattachment of the separated boundary layer. It should be pointed out that the ratio of amplitude to wave length for the models tested by Stanton was much larger than that possible for water waves. It is possible that evidence of sepatation would be even less obvious for models whose dimensions approximate those of water waves more closely than did Stanton's.

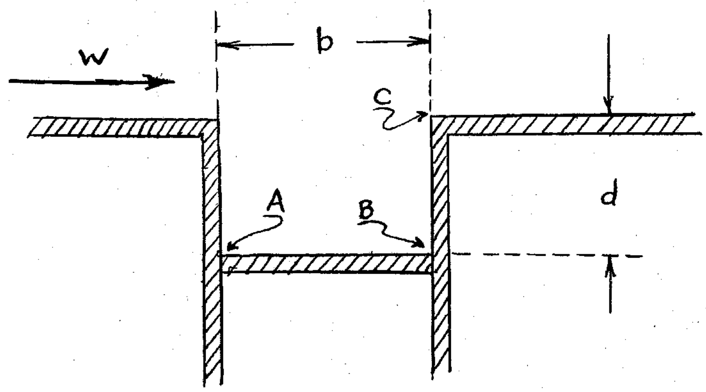


Fig. 3. Diagram of Cavity used by Roshko

Measurements by Anatol Roshko[7] in a rectangular cut-out (fig. 3) in the floor of a wind tunnel give a good indication of what takes place in the trough of a wave. His description of the results is quoted here:

> "Apparently the boundary layer, which separates at the front edge, diffuses into the cavity, so that the velocity on the streamline approaching the downstream edge has some value greater than zero. This value increases, at first, as the depth is increased, and consequently the stagnation pressures near the back edge increase at first. When the cavity depth exceeds a certain value ($d/b \simeq 0.1$), the separated boundary layer no longer reattaches to the bottom. It is probably at about this value that a general vortex system is first set up, in contrast to the shear layer diffusion at shallower depths."

His measurements of pressure along the floor of the cavity gave a low value in the region near the upstream corner (the point A in fig. 3) for $d/b \approx 0.1$. Proceeding downstream, this was followed by a fairly sharp rise in pressure, after which the presure rose more slowly to a maximum near the downstream corner (point B). The region of the sharp pressure rise moved toward the downstream corner as the ratio d/b

was increased, until, when $d/b = 0.25$, it appeared quite close to the downstream corner. The initial low pressure region probably indicated a region of separation, and the boundary layer probably reattached downstream of the sharp pressure rise.

For larger values of d/b (i.e. $0.75 \leq d/b \leq 2.5$) pressure measurements were obtained all around the wall of the cavity. In all cases measured there was a sharp increase in pressure near the higher upstream corner (point C). Other pressure maxima were found at the two corners of the base, with minima near the centers of each of the three walls (in general). The total variation of pressure between these other maxima and the minima was much smaller than the magnitude of the maximum at the point C. Thus the pressure variation near the point of reattachment of the boundary layer was much more violent than the variation due to turbulent motion below the separated boundary layer. This was true for both small and large values of d/b.

C. A Model for the Air Flow Over a Wave. On the basis of all such evidence the following model is suggested for the motion of air over water waves. The boundary layer, separating near the crest at the

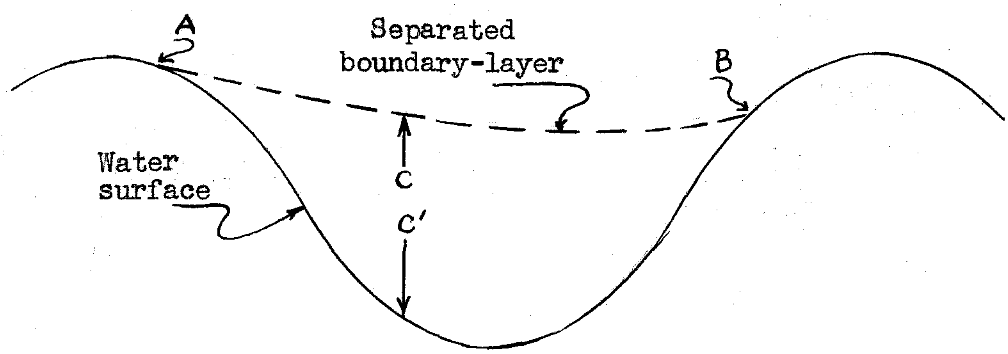


Fig. 4. Diagram of Proposed Model.

point A (in fig. 4), diffuses downward into the trough of the wave and reattaches on the weather face of the next wave at the point B. Above the boundary layer the motion of the air is adequately described by potential theory. Below the boundary layer, in the region of separation, conditions are of a type so little understood that almost nothing can be said about the air motion. However, a tentative conclusion may be drawn from pressure measurements of both Stanton and Roshko. Although the models used by these two investigators are quite dissimilar, there is a striking similarity between their pressure measurements. In the region where the boundary layer is presumed separated from the surface the pressures were small. Variation in pressure in this region was observed to be much smaller than the large increase of pressure near the point of reattachment of the boundary layer. It can be concluded that velocities in this region are too small to cause any important variations of pressure.

On the basis of such evidence, and the conclusion from it, it seems reasonable to bridge the comparatively unknown region with the following assumption. The pressure at any point of the water surface, say C' in the sketch above, is the same as that at a point immediately above it at the boundary layer, thus the point C. The question as to whether or not this assumption is critical is best investigated after the results of the assumption are completely analyzed. This assumption implies that the atmosphere, in motion over the ocean, "sees", or is affected by, only the crest of the waves, and does not "see", or does not depend on, what goes on in the troughs. Thus it is consistent with the assumption to say that the geometry of the separation phenomenon (i.e. the positions of the points of separation and reattachment, and the shape of the streamline between these points) does not depend critically upon the height of the waves. If the geometry depends on the wave size at all it depends on the wave-length only. But for the present it will be assumed independent even of the wave-length.

Such a model gives a boundary streamline for the potential flow in the region above the waves. This streamline is periodic, and can be expressed as a fourier series

$$y = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \operatorname{Cos} nkx + \sum_{n=1}^{\infty} \beta_n \operatorname{Sin} nkx \qquad (65)$$

For a sinusoidal boundary of small amplitude the pressure along the boundary in potential flow is given by eq. (60). This linearized form permits superposition of various periodic components. Thus the variable part of the dynamic pressure over a boundary given by a fourier series, as eq. (65), can be written

$$\frac{\Delta p}{\rho_a (W-C)^2} = -k \sum_{n=1}^{\infty} n\left[\alpha_n \operatorname{Cos} nkx + \beta_n \operatorname{Sin} nkx\right] \qquad (66)$$

Then if the shape of the boundary is given the pressure can be computed, or if the pressure is known the boundary may be drawn.

Using these results and the assumption that the pressure at a point of the wave surface is equal to the pressure at a point of the separated boundary layer immediately above it, the results of Stanton[6] may be analyzed. The positions of the boundary layers which would explain his measured pressures are shown in fig. 5. If the wave profile in this figure is taken as Cos kx then the important feature (for wave growth) of the shape of the separated boundary layer is given by the

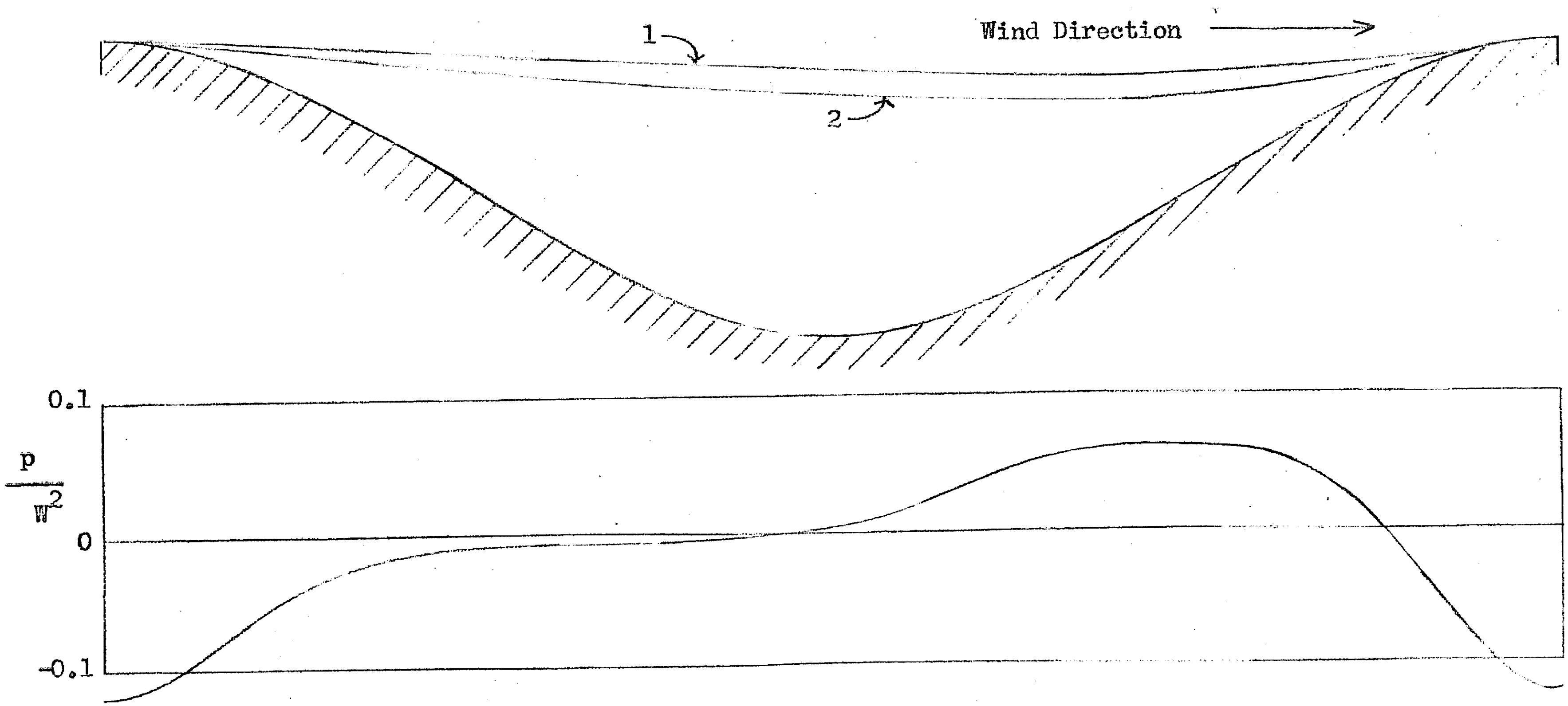


Fig. 5 Profile of test model compared to positions of the boundary of potential flow as computed from two of Stanton's measurements. Case 1 : W = 330 cm./sec. Case 2 : W = 580 cm./sec. Scale 1:1 (above). Measured variation of pressure over one wave-length for case 1 (below).

β_1 Sin kx term, which is a measure of the asymmetry of the shape. The term α_1 Cos kx, although important in determining the appearance of the profile (e.g. the "bowing-down" of the boundary layer between the crests) does not contribute a pressure term leading to wave growth.

Such observations suggest the following simplification of the model. The separated boundary layer will be taken as a straight line,

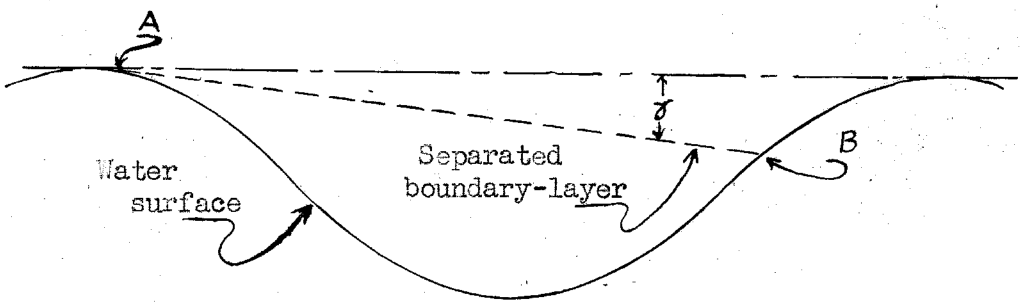


Fig. 6. Simplified Diagram of Flow Model

tangent to the wave surface at the point A (fig. 6), sloping downward at the angle γ to the horizontal, and intersecting the wave surface again at the point B. The angle, γ , will be assumed constant, independent of wave-height, wave-length, and wind speed. The size of γ will be chosen as 0.003 radians. This value is chosen so that the predictions of the theory will coincide with observations of the growth rates of water waves. The method of comparing theory and observations is discussed in Section IV.

The qualitative aspects of Stanton's observations have been used in the development of this model. It will be interesting to make a quantitative comparison between his observations and the predictions of the model. The model is supposed to give the magnitude, p_o, of that component of pressure which is 90° out of phase with the wave shape.

Eq. (74) in part D of this Section shows that $p_o / \left[1/2 \rho_a (W-C)^2 \right]$ is approximately 4γ. Table I gives the amplitude, a, and the wave-length, λ, of the sinusoidal models used by Stanton, as well as the relative air speed, (W-C), in the tunnel, and the observed values of $p_o / \left[1/2 \rho_a (W-C)^2 \right]$. The last column in the table gives the values of γ required by the model to produce the observed magnitude of the out-of-phase pressure component.

Table I. Values of Required by Stanton's Results.

Amplitude a (cm.)	Wave-length λ (cm.)	Relative air speed (W-C) (cm./sec.)	Observed pressure $p_o / \left[1/2 \rho_a (W-C)^2 \right]$	Required value of γ (radians)
1.1	10.8	325	0.046	0.012
1.1	10.8	470	0.060	0.015
2.2	21.6	330	0.086	0.022
2.2	21.6	580	0.124	0.031
1.5	7.6	1400	0.016	0.004

Except in the last case, the required values of γ are much larger than that chosen for the present study (0.003 radians).* Further, the variation among the measurements is very difficult to understand. It should be noted that the ratio of amplitude to wave-length of Stanton's models was much larger than that possible for water waves. This may be important, since measurements by Motzfeld over models of more realistic shape, described in part E of this Section, coincide reasonably well with the choice $\gamma = 0.003$ radians. Stanton's measurements imply a varia-

* See foot-note, p. 74.

tion of γ with both wind speed and wave shape, but they are insufficient for a precise description of this variation. Therefore, until more experimental information is available, only the qualitative aspects of Stanton's observations can be used.

The simplified model can be interpreted in four different ways. First, it can be taken as the position of a separated boundary layer. This is undoubtedly wrong. Second, it can be taken as a crude approximation to the position of a separated boundary layer. For the explanation of Stanton's observations, this is a reasonable interpretation. Then the principal difference between the straight line and the true boundary layer position is a sinusoidal curve, of small amplitude, in phase with the surface shape. As a matter of fact, the addition of such a sinusoidal curve of any amplitude would not change the magnitude of the out-of-phase component of pressure. This suggests the third interpretation, namely, that the straight line of the simplified model is the line which joins the approximate point of separation of the boundary layer with the approximate point of reattachment.

However, it may be undesirable to require the boundary layer to separate at all. An asymmetric pressure distribution is possible without actual separation. For instance, the boundary layer might be thicker on one side of the wave than on the other. Then the fourth, and most flexible, interpretation of the model is that it is merely a means of computing the magnitude of the asymmetry in the pressure distribution resulting from turbulent flow over a wave.

Regardless of which interpretation is used, the important feature of the model is its lack of dependence on the amplitude of the wave.

D. Pressure and Shear Stress Over the Wave Surface. The magnitude of the Sin kx component of pressure over the boundary layer (and thus over the wave surface) is obtained by evaluating the coefficient β_1 in the fourier series of eq. (66). Thus suppose the water surface is given by

$$\eta = a \cos kx \tag{67}$$

For small values of γ the detachment point may be taken at the crest. Let the point of reattachment be at x = c. Then the straight-line boundary layer (again for small γ) may be written

$$y = a - \gamma x \tag{68}$$

and

$$\gamma c = a(1 - \cos kc) \tag{69}$$

The coefficient β_1 is given by

$$\beta_1 = (2/\lambda)\left[\int_0^c (a-\gamma x)\sin kx + \int_c^\lambda a \cos kx \sin kx \, dx\right]$$

$$= (1/\pi)\left[a(1-\cos kc) - (\gamma/k)(\sin kc - kc\cos kc) - \tfrac{a}{2}\sin^2 kc\right] \tag{70}$$

Approximating for small values of γ by letting

$$kc = 2\pi - \delta$$

$$\cos kc = 1 - \delta^2/2 + \cdots \tag{71}$$

$$\sin kc = \delta - \cdots$$

$$\text{Thus} \quad \gamma \simeq (a/\lambda)(\delta^2/2) \tag{72}$$

permits eq. (70) to be reduced to

$$\beta_1 = \frac{\gamma\lambda}{\pi}\left[1 - \frac{1}{2}\left(\frac{\gamma\lambda}{a}\right)\right] \tag{73}$$

Now define $h = \dfrac{a}{\gamma\lambda}$ and the magnitude of the Sin kx component of pressure is

$$p_0 \simeq 2\rho_a (W-c)^2 \gamma \left(1-\frac{1}{2h}\right) \tag{74}$$

One thing is clear immediately from this equation. There is a lower limit for h, namely h = 1/2, below which the equation makes no sense. It is easy to understand this difficulty, for if h = 1/2 then $2a/\lambda = \gamma$. But $2a/\lambda$ is the slope of a line joining the crest of a wave to the trough, and γ is (approximately) the slope of the boundary layer. Thus in this limiting condition, the boundary layer, presumably going from crest to trough, actually never separates. This picture of limiting conditions is oversimplified, but it makes the difficulty understandable. Since only small values of γ will be used in future computations, the limit can be taken as a lower limit to the steepness of waves which can be treated by the present model. Thus this model can explain only the growth of waves whose steepness, $2a/\lambda$, is greater than γ. Clearly the model cannot explain the origin of waves from a perfectly flat ocean. A theory such as that suggested by Eckart(4) might explain such an origin.

Shear forces are next to consider. It has already been assumed that air motion in the region below the separated boundary layer makes only a negligible contribution to the variable part of the dynamic pressure. This assumption will be extended to cover the contributions to shear force. Thus the shear force is assumed different from zero only over that portion of the wave surface where the free air stream (or the boundary layer of the free air stream) actually touches the surface. This can be expressed as

$$\left.\begin{aligned} s &= (\rho_a/2)\, C_s V^2 \quad \text{for} \quad c \leq x \leq \lambda \\ s &= 0 \qquad\qquad\qquad \text{otherwise.} \end{aligned}\right\} \qquad (75)$$

As before, V is the difference between the velocities of air and water particles at the boundary, and x = c at the point of reattachment of the boundary layer.

The velocity of the water particles at the surface is approximately akC near the crest. The velocity of the air particles at the boundary is variable since the boundary is a periodic function of x. Thus $V = W + \Delta W - akC$, where ΔW is a correction due to the periodic structure of the air boundary. The amplitude of this periodic structure is so small (for small γ) that ΔW may be neglected in comparison to W. Further the term akC will be neglected for the present. This means that for large amplitude waves whose speeds approach that of the wind the results will require correction. Further, it is true that the very existence of shear forces implies that the speed of air particles near the boundary is less than the speed of the free stream. However this effect can be absorbed into the definition of C_s. That is, the magnitude of C_s depends on the height above the surface at which W (or V of eq. (75)) is measured. The free stream, or "anemometer", wind speed will be used here, and the value of C_s taken as 0.005 * . With W in place of V in eq. (75), s is a simple periodic step function which can be written as a fourier series. The term of interest in such a series representation is the Cos kx term. The coefficient of this term is

* Cf. Ref. 8, and discussions with Dr. W. H. Munk of the Scripps Institution of Oceanography, University of California.

$$s_o = \frac{\rho_a}{2} C_s W^2 \frac{2}{\lambda} \int_c^{\lambda} \cos kx\,dx$$

$$= \frac{\rho_a}{2\pi} C_s W^2 (-\sin kc) \qquad (76)$$

With the same approximations that were used for the normal pressure (Cf. eqs. (71) and (72)) this is

$$s_o = \frac{\rho_a}{2\pi} C_s W^2 \sqrt{\frac{2}{h}}$$

Substituting s_o and p_o into eq. (29), the resulting expression for the growth of the amplitude of a wave train is

$$U \frac{\partial a}{\partial x} + \frac{\partial a}{\partial t} = \frac{\omega}{2g} \cdot \frac{\rho_a}{\rho_w} \left[2\gamma (W-c)^2 \left(1 - \frac{1}{2h}\right) + \frac{C_s W^2}{2\pi} \sqrt{\frac{2}{h}} \right] \qquad (77)$$

Next, define a number R as

$$R = \frac{\sqrt{2}\, C_s W^2}{4\pi \gamma (W-c)^2} \qquad (78)$$

This is a measure of the ratio of the effect of shear to that of pressure. Further, new variables are defined as

$$\tau = \frac{\omega}{\lambda g} \cdot \frac{\rho_a}{\rho_w} (W-c)^2 t \qquad (79)$$

and

$$\xi = \frac{\omega}{\lambda g} \cdot \frac{\rho_a}{\rho_w} (W-c)^2 \cdot \frac{2x}{c} \qquad (80)$$

Then the growth equation can be written in the simplified form

$$\frac{\partial h}{\partial \xi} + \frac{\partial h}{\partial \tau} = 1 - \frac{1}{2h} + \frac{R}{\sqrt{h}} \qquad (81)$$

Solutions of this equation for various values of the parameter R are given in Fig. 7 for an initial value of h = 1. It must be remem-

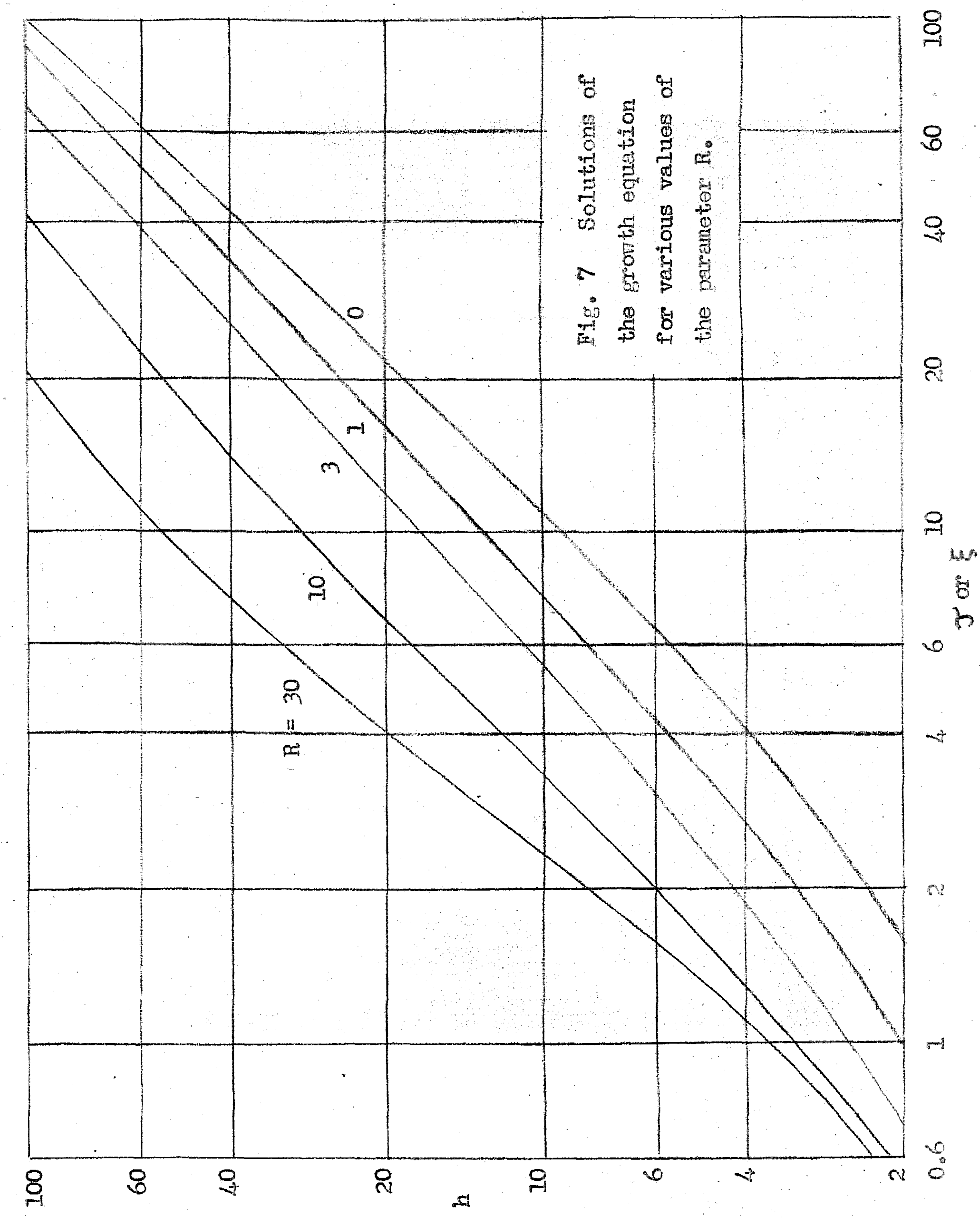


Fig. 7 Solutions of the growth equation for various values of the parameter R.

bered that the wave length, λ , (and thus the wave speed, C, and the frequency, ω) is a constant in the growth equation.

The analysis of the response to a shear force, carried out in Section II, is based on the assumption of laminar flow in the vorticity-layer just below the surface of the wave. It is reasonable to inquire whether or not this assumption is valid. For steady motion, the transition from laminar to turbulent flow will occur when the critical value of the Reynolds number

$$\mathcal{R} = u_s \delta / \nu$$

is reached; where u_s is the magnitude of the velocity induced by shear at the surface, and δ is the thickness of the vorticity-layer. A very rough estimate of the critical value of this number is 10^2.

As already mentioned (Section II, part B) the value of δ is $\sqrt{\frac{\nu}{\sigma}}$. The value of u_s is obtained as follows: The shear in a fluid per unit area is given in terms of the gradients of the velocity components as

$$s = \rho_w \nu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)$$

Changes in a direction parallel to the surface (here, the surface will be taken as y = 0) are small, so $\frac{\partial v}{\partial x}$ is neglected. The x-component of the velocity induced by surface shear is approximately

$$u \simeq u_o e^{y/\delta}$$

where u_o = x-component of velocity at the surface = u_s. So

$$s = \rho_w \nu u_s / \delta$$

at the surface.

Using eq. (75) with W replacing V

$$u_s \simeq \frac{\delta}{\rho_w \nu} \cdot \frac{\rho_a}{2} C_s W^2$$

Then

$$\mathcal{R} \simeq \frac{\rho_a C_s W^2}{2\rho_w \sigma \nu}$$

In c.g.s. units, this becomes

$$\mathcal{R} \simeq 3 \cdot 10^{-4} W^2/\sigma$$

A conservative estimate for σ is 1 sec.$^{-1}$ for waves on the open sea (this corresponds to a wave-length of about 10^4 cm.). Thus the rough estimate of the critical value for $\mathcal{R}$ will be exceeded by winds greater than 600 cm./sec.

This estimate of the critical Reynolds number is based on the case of steady motion. In the present study, the shear is applied periodically. Now it takes a certain amount of time (or distance) for the transition from laminar to turbulent flow to take place. If the shear is applied periodically instead of steadily, the available time is limited to the period, or, as in the present case, a fraction of the period. Little experimental information is available on such a situation, but it is reasonable to suppose that the critical value is much higher in this case. If it is larger by a factor of 100, which is possible, then only winds above 60 meters/sec. would cause transition. Even if the flow in the vorticity-layer were turbulent there would still be a contribution to wave growth from shear forces, but the magnitude of the contribution would be different.

This is not the only problem with regard to shear forces. Computation of the magnitude of the shear force, as carried out in this section, is based on the interpretation of the model as an approximation to a separation phenomenon. The fourth of the possible interpretations listed in part C of this section dropped the idea of separation. Such an interpretation would require a modification of the calculation of shear force. Fortunately for the present study, the effect of shear force is small compared to the effect of normal pressure for all waves except those whose speed approaches that of the wind. Therefore the problems concerning the shear force can be safely disregarded for the present.

E. Additional Experimental Information. Since the model was first devised, the author has been referred to a paper by Motzfeld[9] containing information on measurements taken in a wind tunnel using models of waves. These measurements differ in three important ways from those of Stanton, which have been discussed previously in this Section. First, the measurements were more complete. They included pitot-tube measurements of velocity on a very complete set of points above the wave surface. Pressure measurements along the surface were obtained also, as they were by Stanton. Second, the models used were much more realistic. Four different modes were used. Model 1 was a sine wave with steepness ($2a/\lambda$) equal to 1/20. Model 2, also a sine wave, had a steepness of 1/10. Model 3 was a trochoid with steepness equal to 0.0967. Model 4 had the shape of the steepest wave possible in water (Cf. Ref. 2, p. 418), and thus had a steepness of 1/7. For comparison, the steepness of the models used by Stanton ranged between 0.395 and 0.204. Third, the Reynolds numbers (Cf. Section VI) for Motzfeld's tests were about 10 times as large as those used by Stanton; however they were about 10 times smaller than those expected on the open sea.

Only in one case did Motzfeld's measurements give clear indication of separation. That was the case of flow over model 4. In the other cases, both the pressure measurements and the velocity measurements (from which the position of the streamlines was computed) indicated an asymmetry in the flow, but no region of separation. The computed position of the streamlines indicated that the boundary-layer on the lee side of the wave was thicker than on the weather side. This would account for the observed asymmetry of the pressures.

The asymmetry of pressures leads to a component of pressure 90° out of phase with the wave shape. The magnitude of this component as measured by Motzfeld can be compared with that predicted with the the present model. The results (with $\gamma = 0.003$) are presented in Table II, below. For test models 1, 2, and 3 the differences between the two values are well within the accuracy intended for the present study. The large difference for model 4 indicates the present theory fails for this case. This may be due to the fact that model 4 was a wave with pointed crests. If this is the only reason for the failure of the theory in this case, then the theory can be expected to work for most real water waves, for the crests of these are seldom as sharp as those of the highest possible wave. This question certainly needs more investigation.

Motzfeld's findings indicate that the last of the four interpretations given earlier for the present model is the one which must be used to explain the pressures over waves of realistic steepness. Thus, rather than describing any separation phenomenon, the model is a means of computing the out-of-phase component of the pressure distribution in flow over a wavy surface.

Table II. Comparison of Motzfeld's Results and the Predictions of the Model.

Test Model Used	Observed Sine Component of Pressure $2p_o/\ (W-C)^2$	Computed Sine Component of Pressure 4
1	0.011	0.012
2	0.015	0.012
3	0.018	0.012
4	0.167	0.012

F. Investigation of an Assumption. In the determination of the pressure over the wave surface it has been assumed that, in the separated region, the pressure at any point on the wave surface is the same as the pressure at a point immediately above on the boundary layer. It is possible to show that this assumption is not critical. Differences between the assumed pressure distribution and the actual distribution would cause two types of error. First, the magnitude of the pressure might be incorrect, and second, the phase of the pressure component of the same wave-length as the wave might be wrong (higher frequency components are unimportant). That is, the relative magnitude of the Cos kx and Sin kx terms in the expansion of eq. (66) might be different than assumed.

If the magnitude of the whole pressure distribution is wrong, this amounts to no more than an error in the choice of γ, and certainly would not imply any difficulty with the model. The most critical possibility is that of an error in phase. This would be dangerous only if the magnitude of the Sin kx term is much smaller than that of the Cos kx term. In that case a small change in phase would make a large change in the important Sin kx term. The magnitude of the Sin kx term is given by β_1 in eq. (73). The magnitude of the Cos kx term, α_1, can be computed in a similar manner. Using the approximations implied in eqs. (71) and (72), the ratio of α_1 to β_1 is

$$\frac{\alpha_1}{\beta_1} = \frac{5}{3}\sqrt{\frac{2\gamma\lambda}{a}} \qquad (82)$$

Thus the assumption could be critical only for waves of very small steepness (i.e. $2a/\lambda < \gamma$). However, as has already been pointed out, the model cannot be extended to cover waves of very small steepness.

If a more realistic shape is used for the separated boundary layer, the results might be similar to the results measured by Stanton[(6)]. These results give the Cos kx and the Sin kx components as the same order of magnitude, with the Sin kx component slightly larger in most cases.

On the other hand, Motzfeld's measurements give a very large value for the Cos kx term relative to the Sin kx term. However, his computation of the position of streamlines showed that the boundary of potential flow was quite near the model surface. The intervening space was so small that little error could result from the assumption now being examined.

Therefore it must be concluded that the assumption determining pressure along the wave surface is not critical.

IV. INTERPRETATION OF RESULTS OBTAINED FROM THE MODEL.

The growth law (eq. (77)) shows that the height to which any given wave-length wave grows in a given time (or distance) depends on both wind speed and wave-length. The effect of wind speed can be treated in a convenient manner through the use of certain dimensionless variables. Thus let a measure of wave-height be

$$A = ga/W^2 \tag{83}$$

and a measure of wave-length be

$$L = g\lambda/(2\pi W^2) \tag{84}$$

The measure of fetch is

$$F = gx/W^2 \tag{85}$$

while the measure of duration of the wind is

$$T = gt/W \tag{86}$$

Using F as a parameter for the fetch-limited case, or T for the duration limited case (Cf. Section II) the dimensionless wave height may be plotted against the dimensionless wave length from the results of eq. (81). Examples of such plots are given in figs. 8 and 9. In both cases, the initial conditions on h are h = 1, while γ was chosen to be 0.003. Thus initially

$$A = 0.0376L \tag{87}$$

To make these calculations, the results of linearized deep-water gravity wave theory have been employed, that is

$$kC^2 = \omega^2/k = g \tag{88}$$

The ratio ρ_a/ρ_w has been set at 0.00125.

A more exact, and non-linear, theory of gravity waves is discussed by Lamb (Cf. Ref. 2, pp. 417-420). Most of the results of this theory can be considered as negligible corrections for the present model. However, one result must be considered in full. That is the fact that a limiting height exists for any given wave-length. Waves of this limiting height have an approximately trochoidal profile between crests with points at the crests. The angle of the pointed crest is 120° and ratio of the height (crest to trough) to wave length is approximately 1/7. If the approximation to the actual wave shape which has so far been used, namely η = a Cos kx, is extended to this limiting condition, then in the limit ak = A/L $\simeq$ 0.45. The idea of a limiting size will be incorporated into the present theory by assuming the growth to be sharply cut off when the limiting height is reached. The line A = 0.45 L which represents this limit, or cut-off, is shown in both figs. 8 and 9.

The results presented in figs. 8 and 9 can now be interpreted in the following way. Suppose an infinite number of oceans are available. Each ocean has a single frequency wave system on its surface with a wave-length differing from that on any other ocean. Thus an ocean is available for every wave-length. A wind is blowing over each ocean, and assume, for example, that fetch-limited conditions apply. Then a particular value of the fetch parameter, say F_1 on fig. 8, is choosen. On each ocean, at the corresponding value of fetch, the wave-height is measured. That ocean whose waves are the

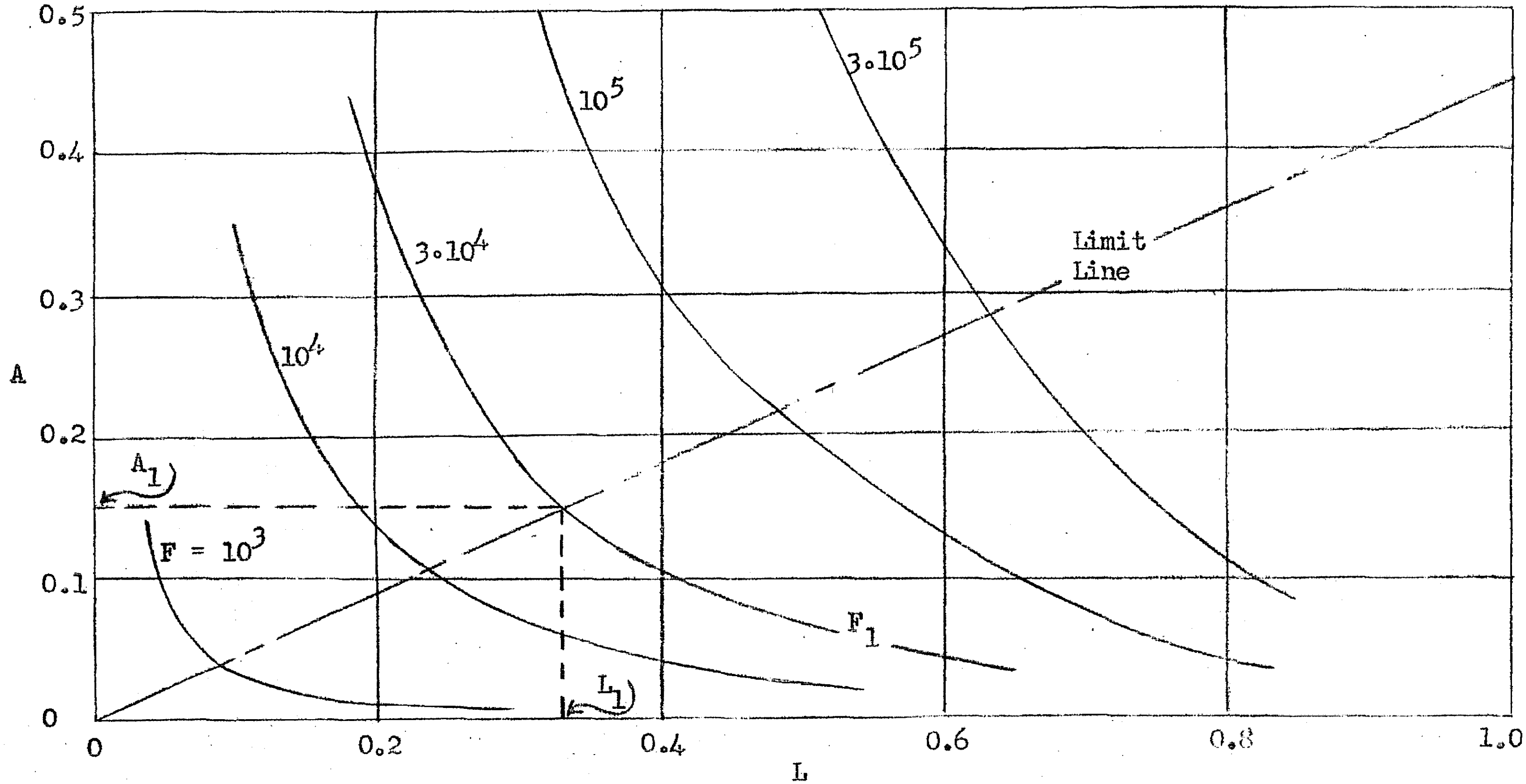


Fig. 8 Wave growth under fetch-limited conditions.

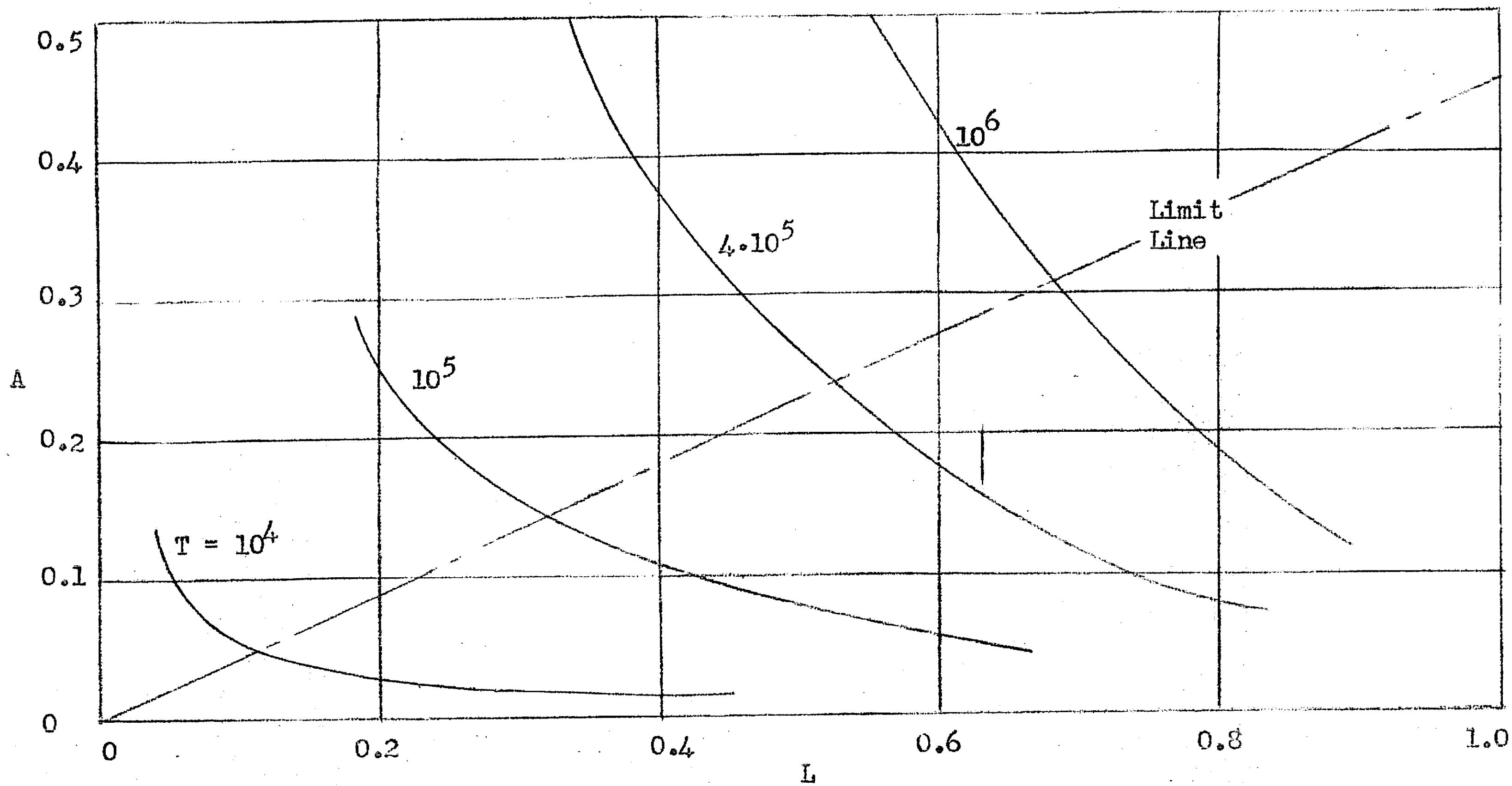


Fig. 9 Wave growth under duration-limited conditions.

highest (height A_1 on fig. 8) is the ocean with waves of length L_1 corresponding to the intersection of the constant fetch line and the limit line. In this manner the height and wave-length of the highest waves for any value of fetch (or duration) can be determined.

Extension of the present theory to the realistic situation of a bread spectrum of wave-lengths is quite difficult, and will not be attempted here. However, it is tempting to compare the results of this theory with observations. Observations for fetch-limited conditions which have been collected by Bretschneider(10) cover a wide range of values of F, and contain a very large number of separate observations. Values for small fetch were taken in various wind tunnels. In the middle region the data comes from observations in a lagoon on the shore of northern California, and data from the region of large fetch comes from observations in lakes and the open ocean. A few observations corresponding to duration-limited conditions have been compiled by Sverdrup and Munk(1), and are presented in fig.11.

There is some confusion as to what the various data points actually represent. In most cases the average height and average wave-length of what are termed the "highest" waves are reported. It has been suggested that this process, which amounts to filtering by eye, is averaging over the "1/3 highest waves present"(1). In other cases where the height has been measured by instrumentation and recorded, the waves (down to the smallest which the instrument was capable of measuring) were counted, and the 1/3 highest waves picked out for averaging(11). In any case, the data represents the characteristics of waves which were among the highest present at the time of measuring.

Curves giving the predictions of the present theory are given in figs. 10 and 11. There is surprisingly close agreement between theory and observation with regard to wave-height; but poor agreement, except at low values of F, with regard to wave-length. Another way to state the difficulty with wave-length predictions is to say the following. At large values of F the observed ratio between wave-height and wave-length (the steepness) is much smaller (by a factor of four at least) than the theoretical limit of 1/7. This implies that some other limiting condition controls wave growth in a real situation, i.e. a situation where a broad spectrum exists rather than a single frequency component. This question will be investigated more in the next Section.

For large values of F the data gives $\lambda g/2\pi W^2 > 1$, i.e. the wave speed is greater than the wind speed. Possibilities of extending the present model to such a situation are very uncertain. When $\lambda g/2\pi = W^2$, thus when there is no air motion relative to the wave shape, the model has no meaning. When the waves are moving faster than the wind it is doubtful how the model is to be interpreted. At first glance, it would seem that all that is required is to reverse the relative positions of the points A and B in fig. 6, since the relative wind is now moving from right to left. Presumably the angle, γ, would be unchanged. However, it must be remembered that it is the behavior of the boundary-layer, either its thickening or its separation, which is being described by the model. This behavior must depend on conditions at the air-water surface. The water particles

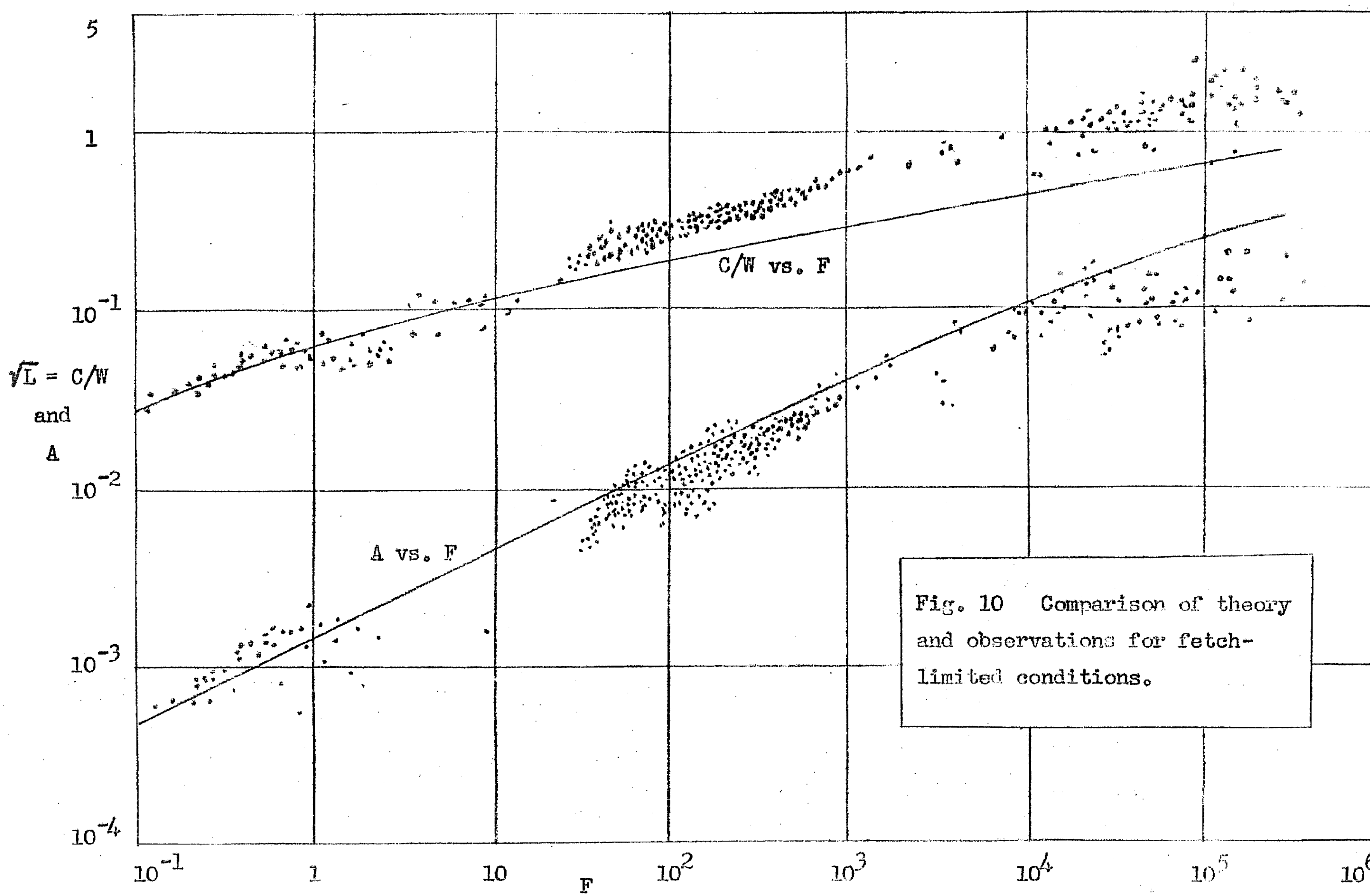


Fig. 10 Comparison of theory and observations for fetch-limited conditions.

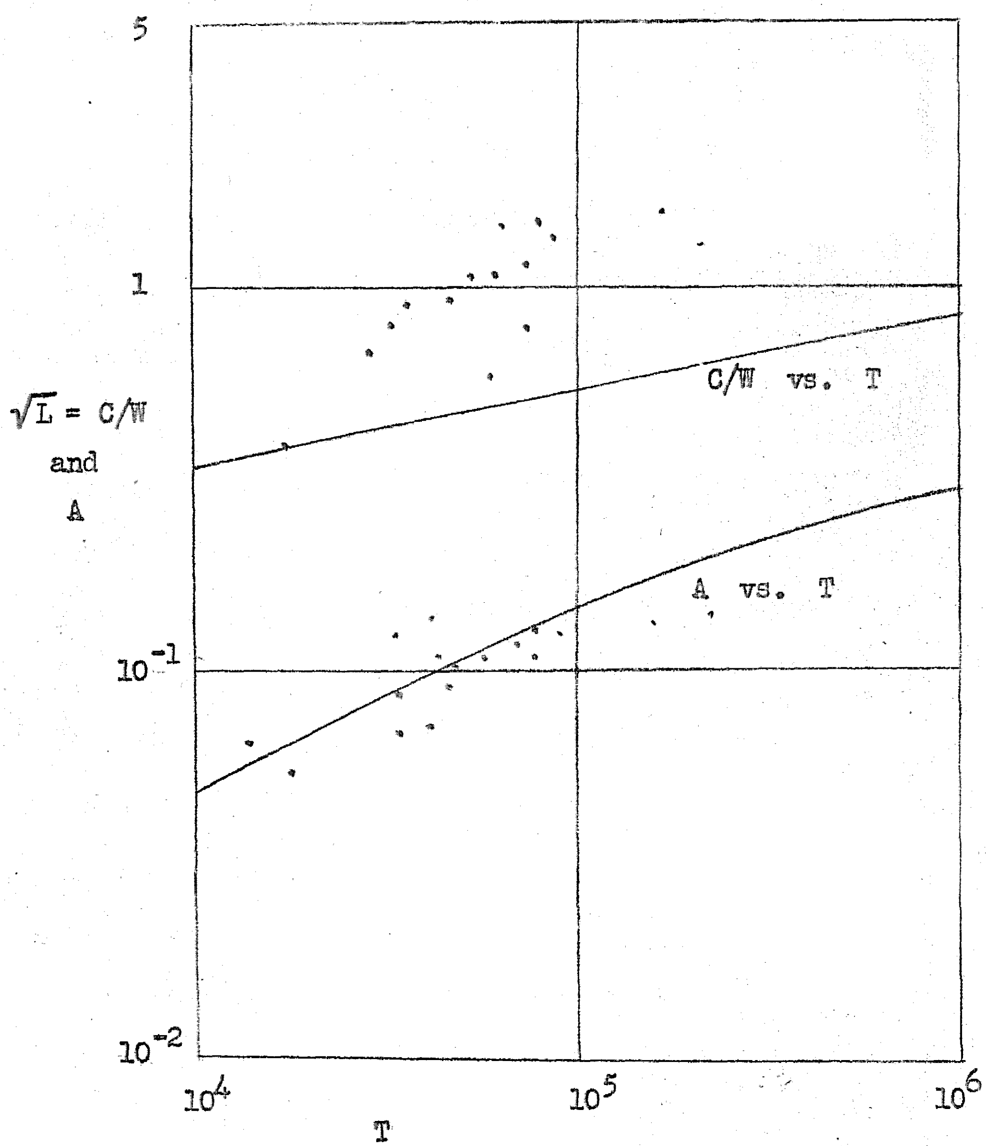


Fig. 11 Comparison of theory and observations for duration-limited conditions.

at this surface are, of course, in motion. Neglecting vorticity induced by shear at the surface, the velocity of the water particles in the critical region near the crests is in the same direction as the wave velocity and of smaller magnitude. If the wind is moving faster than the waves, its motion relative to the water particles is faster yet. In such a situation, it is reasonable to extend the results of wind tunnel observations over solid surfaces to conditions over a liquid surface. But if the waves are moving faster than the wind, the relative motion of air and water particles at the surface may be positive (in the sense of the direction of wave advance), negative, or zero. The magnitude of the relative velocity may be either greater or less than the relative velocity between wind and wave. The extension of wind tunnel results to cover such situations is very questionable.

Thus it is doubtful that the theory in its present form can explain the observed wave-lengths. For the same reason, the theory will not be applied to conditions in the area of wave decay (area of relative calm), where the wind velocity is much smaller than the wave velocity.

Wave-length data for waves moving slower than the wind might be reproduced modifying the simple theory as follows: The observed maximum steepness (as a function of F) is used, instead of the theoretical 1/7, as the limiting height curve in figs. 8 and 9. The value of γ is then taken to vary with the Reynolds number based on the wave-length, $(W - C) \lambda / \nu$. The variation is so adjusted, and the resulting curves in figs. 8 and 9 are so modified that the resulting wave-height predictions coincide with the data. The wave-length data will then be checked. This is a possible scheme, but until the observed steepness of the highest waves is explained no greater understanding of wave growth can be gained by such arithmetic.

V. INTERACTIONS IN A WAVE SYSTEM OF MORE THAN ONE FREQUENCY.

A. Possible Types of Interactions. In a realistic situation a broad spectrum of waves are present. The water surface cannot be described as a single cosine curve. If the implications of the present model are extended to such a case one would expect that the smaller, short length waves would be sheltered from the wind during the time they are in the trough of the larger waves. In such a situation the growth rate of the smaller waves would certainly not be given by the simple expression of eq. (81). Interactions of a less easily understandable nature would occur between waves of similar frequency. The analysis of such interactions will not be attempted here.

Apart from modifications of the growth rate due to the simultaneous presence of several different frequencies, another modification might occur. The theoretical condition limiting the steepness might be affected. Such a possibility can be analyzed by a perturbation method under the assumption that waves of two different frequencies are present. Such an analysis will be presented here.

B. Non-Linear Interactions of a Two Component Wave System. The unperturbed motion will be that of a finite amplitude-wave. Added to this, as a perturbation, will be a wave of small amplitude. The resulting free-surface boundary condition is then linearized in the amplitude of the perturbing wave. The analysis will be carried out without restriction on the relative size of the wave-lengths of the two waves.

The results of the analysis are then applied to the special case when the perturbing wave has a much smaller wave-length than the original wave. For this condition it is found that the perturbing wave becomes steeper near the crest of the original wave, and less steep near the trough. The extent of this effect depends on the steepness of the original wave. From this it is concluded that, if the perturbing wave is steep enough, the growth of either wave will cause the perturbing wave to break near the crests of the original wave. Further analysis is carried out which indicates that the shape of the highest possible wave is unaffected by the presence of more than one wave train. This analysis is needed to justify the conclusion just mentioned. The analysis will be carried out in two dimensions with an incompressible, irrotational fluid. The potential function is

$$\varphi = \varphi_0 + \varphi_1 \tag{89}$$

and the profile of the surface is

$$\eta(x,t) = \eta_0(x,t) + \eta_1(x,t) \tag{90}$$

Functions with the subscript $(\ )_0$ describe characteristics of a wave of large amplitude, and are assumed known exactly. In calculations the results of Lord Rayleigh (Cf. Ref. 2, p. 417) will be taken as an adequate description of such a wave. Functions with the subscript $(\ )_1$ describe the characteristics of a wave of small amplitude which will be taken as a perturbation applied to the larger wave. The resulting perturbed free-surface boundary condition will then be linearized in the amplitude of the small wave, whereas terms involving the properties of the larger wave will be carried to two orders in the amplitude of the large wave. Laplace's equation for the potential function is unaffected by the perturbation treatment, since it is already linear.

The boundary condition at the free surface is that the pressure be a constant. The pressure in a fluid is given by eq. (5) of Section 20[2] as

$$\frac{p}{\rho} = \frac{\partial \varphi}{\partial t} - \Omega - \tfrac{1}{2} q^2 \tag{91}$$

Here p is the pressure, ρ the density of the fluid, Ω the potential energy per unit mass of the fluid at the point in question, and q the magnitude of the velocity of the fluid. Let a rectangular coordinate system be established with x parallel to the undisturbed surface, and positive in the direction of motion of the large waves, and y vertical and positive upwards. Suppose the system is moving with velocity C_o in the direction of x positive. In this system

$$\varphi_o = C_o(x - \beta e^{k_o y} \sin k_o x) \tag{92}$$

$$\eta_o = a_o \cos k_o x + a_o^2 k_o \cos^2 k_o x + \cdots \tag{93}$$

$$\beta[1 + (9/8) k_o^2 \beta^2] = a_o \tag{94}$$

Next define

$$\xi = x - \beta e^{k_o y} \sin k_o x \tag{95}$$

and

$$\zeta = y - \beta e^{k_o y} \cos k_o x \tag{96}$$

The analysis will be carried out in the ξ, ζ coordinate system. The ξ, ζ network corresponds to the equipotentials and streamlines of the large wave. In such a coordinate system Laplace's equation is unchanged, that is

$$\frac{\partial^2 \varphi}{\partial \xi^2} + \frac{\partial^2 \varphi}{\partial \zeta^2} = 0 \iff \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \tag{97}$$

Now suppose the surface of the large wave, which is given by $\zeta = 0$, is also given by $y_o = y_o(\xi, 0)$. Suppose also that the deviation of the surface from y_o caused by the presence of the small wave is given by $f' = f'(\xi, t)$, and measured along the lines ξ = const. Then if θ is the angle between the vertical and a normal to the y_o surface (which is the same as a tangent to the line ξ = const. at the surface) and if the curvature of the line ξ = const. is neglected, then the height of a point on the surface is

$$y = y_o + f' \cos\theta \tag{98}$$

as indicated in fig. 12. The potential energy of a point on the surface is then

$$\Omega = g\,(y_o + f' \cos\theta) \tag{99}$$

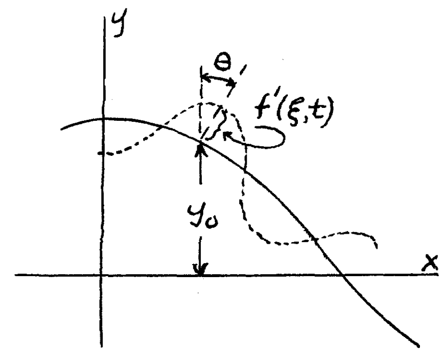


Fig. 12.

In this expression it is assumed that f' is measured in absolute units. It will be more convenient to measure distances in ξ, ζ units. To convert from one system of measure to the other, the factor h is used where

$$\begin{aligned} h &= \frac{\Delta\xi}{\sqrt{(\Delta y)^2 + (\Delta x)^2}} = \frac{\Delta\zeta}{\sqrt{(\Delta y)^2 + (\Delta x)^2}} \\ &= \left[\frac{\partial \xi}{\partial y}\left(\frac{dy}{dx}\right)_{\zeta = \text{Const.}} + \frac{\partial \xi}{\partial x}\right]\left[\left(\frac{dy}{dx}\right)^2_{\zeta=\text{Const.}} + 1\right]^{-\frac{1}{2}} \\ &= \sqrt{\left(\frac{\partial\xi}{\partial x}\right)^2 + \left(\frac{\partial\xi}{\partial y}\right)^2} = \sqrt{\left(\frac{\partial\zeta}{\partial x}\right)^2 + \left(\frac{\partial\zeta}{\partial y}\right)^2} \end{aligned} \tag{100}$$

Then if f is the deviation of the perturbed surface from the surface of the large wave $\zeta = 0$, and is measured in ξ, ζ units

$$\Omega = g\left[y_o + (f/h)\cos\theta\right] \tag{101}$$

Here h is to be evaluated on the line $\zeta = 0$; thus the expression has been linearized in f, i.e. variations in h over the distance $0 \leq \zeta \leq f$ are neglected.

Next the expression for velocities in ξ,ζ space will be derived.

$$u = -\frac{\partial\varphi(x,y)}{\partial x} = -\frac{\partial\varphi(\xi,\zeta)}{\partial\xi}\frac{\partial\xi}{\partial x} - \frac{\partial\varphi(\xi,\zeta)}{\partial\zeta}\frac{\partial\zeta}{\partial x} \tag{102}$$

$$v = -\frac{\partial\varphi(x,y)}{\partial y} = -\frac{\partial\varphi(\xi,\zeta)}{\partial\xi}\frac{\partial\xi}{\partial y} - \frac{\partial\varphi(\xi,\zeta)}{\partial\zeta}\frac{\partial\zeta}{\partial y} \tag{103}$$

Let v_ξ be the component of velocity in the direction of increasing ξ and v_ζ be the velocity in the direction of increasing ζ. Let α be the angle between the x axis and the line ζ = constant. Then

$$\operatorname{Tan}\alpha = \left(\frac{dy}{dx}\right)_{\zeta=\text{Const.}} = \frac{1-\beta k_o e^{k_o y}\operatorname{Cos} k_o x}{\beta k_o e^{k_o y}\operatorname{Sin} k_o x} = \frac{\frac{\partial\xi}{\partial x}}{\frac{\partial\zeta}{\partial x}} \tag{104}$$

and

$$\operatorname{Cos}\alpha = \frac{\partial\zeta}{\partial x}\Big/\sqrt{\left(\frac{\partial\xi}{\partial x}\right)^2+\left(\frac{\partial\zeta}{\partial x}\right)^2} \tag{105}$$

$$\operatorname{Sin}\alpha = \frac{\partial\xi}{\partial x}\Big/\sqrt{\left(\frac{\partial\xi}{\partial x}\right)^2+\left(\frac{\partial\zeta}{\partial x}\right)^2} \tag{106}$$

Then v_ζ can be written as

$$v_\zeta = u\operatorname{Cos}\alpha + v\operatorname{Sin}\alpha \tag{107}$$

$$= -\left[\left(\frac{\partial\varphi}{\partial\xi}\frac{\partial\xi}{\partial x}+\frac{\partial\varphi}{\partial\zeta}\frac{\partial\zeta}{\partial x}\right)\frac{\partial\zeta}{\partial x} + \left(\frac{\partial\varphi}{\partial\xi}\frac{\partial\xi}{\partial y}+\frac{\partial\varphi}{\partial\zeta}\frac{\partial\zeta}{\partial y}\right)\frac{\partial\xi}{\partial x}\right] \cdot\left[\left(\frac{\partial\xi}{\partial x}\right)^2+\left(\frac{\partial\zeta}{\partial x}\right)^2\right]^{-1/2} \tag{108}$$

Using the facts

$$\frac{\partial\xi}{\partial x} = \frac{\partial\zeta}{\partial y}, \quad \frac{\partial\xi}{\partial y} = -\frac{\partial\zeta}{\partial x} \tag{109}$$

this becomes

$$v_\zeta = -\sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial x}\right)^2}\left(\frac{\partial \varphi}{\partial \zeta}\right) = -h\frac{\partial \varphi}{\partial \zeta} \tag{110}$$

In a similar manner

$$v_\xi = -h\frac{\partial \varphi}{\partial \xi} \tag{111}$$

So now the free-surface boundary condition can be written as

$$Const. = \frac{\partial \varphi}{\partial t} - g\left(y_0 + \frac{f}{h}\cos\theta\right) - \frac{1}{2}h^2\left[\left(\frac{\partial \varphi}{\partial \xi}\right)^2 + \left(\frac{\partial \varphi}{\partial \zeta}\right)^2\right] \tag{112}$$

where the right hand side is evaluated at $\zeta = f$. If $\varphi = \varphi_0 = C_0\xi$ and $f = 0$ this equation is satisfied through order $(a_0k_0)^3$. To verify this write out h^2 in x, y notation

$$h^2 = 1 - 2a_0k_0e^{k_0y}\cos k_0x + a_0^2k_0^2e^{2k_0y} \tag{113}$$

The boundary condition becomes

$$Const. = -gy_0 - \frac{1}{2}C_0^2\left[1 - 2a_0k_0e^{k_0y}\cos k_0x + a_0^2k_0^2e^{2k_0y}\right] \tag{114}$$

Comparison with Sec. 250[2] shows that the choice of y_0 given in eq. (93) satisfies the equation through third order in a_0k_0.

Subtracting eq. (114) from eq. (112) and linearizing the result in f and φ_1, there remains

$$\left(\frac{\partial \varphi_1}{\partial t}\right)_{\zeta=0} - f\left[(g/h)\cos\theta + (C_0^2/2)\left(\frac{\partial h^2}{\partial \zeta}\right)_{\zeta=0} - (y_0/2h)\left(\frac{\partial h^2}{\partial \zeta}\right)_{\zeta=0}\right] - h^2C_0\left(\frac{\partial \varphi_1}{\partial \xi}\right)_{\zeta=0} = Const. \tag{115}$$

This is the free-surface perturbation boundary condition. The boundary condition applied at the bottom of an infinitely deep ocean is $\varphi_1 \to 0$ as $y \to -\infty$. Since $\zeta \to -\infty$ as $y \to -\infty$, this becomes $\varphi_1 \to 0$ as $\zeta \to -\infty$.

A kinematic relation between the surface shape, f, and the velocities, or potential function at the surface is still needed to completely define the problem. In the present coordinate system the profile of the large wave, y_0, does not change with time. Inspection of fig. 13 shows that the change in f with time can be written as

$$\Delta f = v_\zeta \Delta t - \frac{\partial f}{\partial \xi} v_\xi \Delta t \qquad (116)$$

Here all length measurements, including those contained in v_ζ and v_ξ, are in ξ, ζ units. If absolute units are used for the velocities, the factor h must be introduced on the right hand side. Then the equation becomes

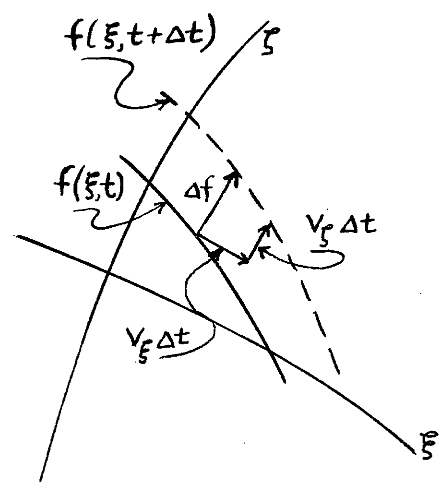


Fig. 13

$$\frac{\partial f}{\partial t} = h^2 \left[\frac{\partial f}{\partial \xi}\frac{\partial \varphi}{\partial \xi} - \frac{\partial \varphi}{\partial \zeta} \right]_{\zeta = f} \qquad (117)$$

If this equation is linearized in f and φ_1, the remaining perturbation equation reads

$$\frac{\partial f}{\partial t} = h^2 \left[C_0 \frac{\partial f}{\partial \xi} - \frac{\partial \varphi_1}{\partial \zeta} \right]_{\zeta = 0} \qquad (118)$$

The coefficients appearing in eqs. (115) and (118) must be written in terms of ξ and ζ. By successive approximations

$$x \simeq \xi + \beta e^{k_0(\zeta + \beta e^{k_0 \zeta} \cos k_0 \xi)} \cdot \sin k_0(\xi + \beta e^{k_0 \zeta} \sin k_0 \xi) \qquad (119)$$

$$y \simeq \zeta + \beta e^{k_0[\zeta + \beta e^{k_0\zeta}\operatorname{Cos} k_0\xi)} \cdot \operatorname{Cos} k_0[\xi + \beta e^{k_0\zeta}\operatorname{Sin} k_0\xi] \qquad (120)$$

For the present analysis it will be sufficient to consider only terms of order 0 and 1 in a_0k_0. To this order, Cos θ = 1. Then after making the necessary substitutions the perturbation equations are

$$\left(\frac{\partial\varphi_1}{\partial t}\right)_{\zeta=0} - gf - C_0(1 - 2a_0k_0\operatorname{Cos} k_0\xi)\left(\frac{\partial\varphi_1}{\partial\xi}\right)_{\zeta=0} = Const. \qquad (121)$$

$$\frac{\partial f}{\partial t} = (1 - 2a_0k_0\operatorname{Cos} k_0\xi)\left[C_0\frac{\partial f}{\partial\xi} - \left(\frac{\partial\varphi_1}{\partial\zeta}\right)_{\zeta=0}\right] \qquad (122)$$

Since φ_1 is a solution to Laplace's equation it can be written as

$$\varphi_1 = \varphi_+(\zeta + i\xi) + \varphi_-(\zeta - i\xi) \qquad (123)$$

First, suppose φ_- is 0. Then

$$\frac{\partial\varphi}{\partial\zeta} = -i\,\frac{\partial\varphi}{\partial\xi} \qquad (124)$$

and such a substitution may be made in eq. (122). Next new variables are defined as

$$z = C_0t + \xi + 2a_0\operatorname{Sin} k_0\xi \qquad (125)$$

$$\tau = t \qquad (126)$$

If these variables are substituted into eq. (121) and (122) and terms of order higher than the first in a_0k_0 are neglected there results

$$\left(\frac{\partial\varphi_1}{\partial\tau}\right)_{\zeta=0} - gf = Const. \qquad (127)$$

$$\frac{\partial f_1}{\partial \tau} = i\left(\frac{\partial \varphi_1}{\partial z}\right)_{\zeta=0} \tag{128}$$

These two equations are formally identical with the equations of the simplest linear gravity-wave theory (Cf. Ref. 2, Sec. 227). The resulting form of the perturbed surface is

$$f = -i a_1 e^{i k_1 (z - c_1 t)} \tag{129}$$

$$k_1 c_1^2 = g \tag{130}$$

Substitutions may be made to obtain an expression in terms of x, y, and t (again to first order in $a_0 k_0$)

$$\eta_1 = a_1 (1 + a_0 k_0 \cos k_0 x) \sin k_1 \left[x + a_0 \sin k_0 x - (c_1 - c_0) t\right] \tag{131}$$

Such a function defines a wave whose amplitude and wave number are periodic functions of position. At any particular value of x the equivalent amplitude is

$$a_1' = a_1 (1 + a_0 k_0 \cos k_0 x) \tag{132}$$

The equivalent wave number is obtained by differentiating the phase of the sine term partially with respect to x, thus

$$k_1' = k_1 (1 + a_0 k_0 \cos k_0 x) \tag{133}$$

Suppose that $k_1 \gg k_0$. Then the variation of the term $\cos k_0 x$ can be neglected over one wave length of the smaller wave. In this case it is reasonable to use eqs. (132) and (133) to write the steepness of the small wave (to first order in $a_o k_o$) as

$$\frac{a_1' k_1'}{\pi} = \frac{a_1 k_1}{\pi} \left(1 + 2 a_o k_o \cos k_o x\right) \tag{134}$$

These results are in agreement with those of P. J. H. Unna(12), and imply that the steepness of the smaller wave may be locally increased beyond the theoretical maximum value of 1/7 by the growth of the larger wave. This would cause the smaller waves to break in the vicinity of the crests of the larger waves if the theoretical limiting condition is not effected by the presence of the larger wave. Before investigating this question it should be pointed out that if eq. (124) is replaced by

$$\frac{\partial \varphi}{\partial \zeta} = +i \frac{\partial \varphi}{\partial \xi} \tag{135}$$

the results of the analysis are unchanged, so that the arbitrary selection involved in eq. (124) is not critical for the solution.

C. Limiting Height Conditions. In Section IV two important characteristics of the limiting wave shape in a single frequency wave system were pointed out. The ratio of height to length is 1/7 and the included angle at the sharp crests is 120°. It would be impractical in the case of a multiple component wave system to attempt to determine a complete description of the shape of the water surface, and thus verify directly that the ratio of height to length is still 1/7. However, it is possible to show that the angle at the crests is still 120°, and this result leads to the conclusion that the complete shape of the limiting wave must be practically the same in both cases.

An outline of the method used to obtain the value of 120° for the single component system is given in section 250[2]. The artifice of steady motion is employed. In the present case the steady motion form of the equations cannot be used. Otherwise the approach is similar. Thus suppose a polar coordinate system is established with an origin instantaneously coincident with the crest of the limiting wave (which is assumed pointed). Angles are measured by θ counter-clockwise from the downward vertical. In a sufficiently small region of the crest the boundary is given by $\theta = -\alpha_1, \alpha_2$. The free surface boundary condition to be satisfied at those values of θ is

$$Const. = \frac{\partial \varphi}{\partial t} + g\, r \cos\theta - \frac{1}{2}\left(v_r^2 + v_\theta^2\right) \tag{136}$$

Let the potential function be represented by

$$\begin{aligned} \varphi = {} & r^{m_0}(a_0 \sin m_0\theta + b_0 \cos m_0\theta) + r^{n_0}(c_0 \sin n_0\theta \\ & + d_0 \cos n_0\theta) + r^{k_0}(\cdots) \cdots + t\left[r^{m_1}(a_1 \sin m_1\theta \right. \\ & \left. + b_1 \cos m_1\theta) + r^{n_1}(\ldots) + \cdots\right] + t^2\left[r^{m_2}(\ldots) + \cdots\right] + \cdots \end{aligned} \tag{137}$$

where $m_i < n_i < p_i \ldots\ldots$

The velocity components are

$$v_r = -\frac{\partial \varphi}{\partial r} \tag{138}$$

$$v_\theta = -\frac{1}{r}\frac{\partial \varphi}{\partial \theta} \tag{139}$$

Only the lowest order terms in t and r need be considered, so the boundary condition becomes

$$r^{m_1}(a_1 \sin m_1\theta + b_1 \cos m_1\theta) + gr\cos\theta - \frac{m_0^2}{2} r^{2(m_0-1)}(a_0^2 + b_0^2) = \text{Const.} \quad (\theta = -\alpha_1, \alpha_2) \tag{140}$$

First, suppose $m_1 < 1$. By eq. (138) this implies infinite acceleration at the origin, since both $\dot{v}_r$ and $\dot{v}_\theta$ are proportional to $r^{m_1 - 1}$ at $t = 0$. Clearly this is not the type of solution being sought.

Second, suppose $m_1 = 1$, and $2(m_0 - 1) > 1$ (note that for $m_1 \geqslant 1$, the boundary condition can be satisfied only for $2(m_0 - 1) \geqslant 1$). As r approaches 0 the boundary condition becomes

$$a_1 \sin\theta + (b_1 + g)\cos\theta = 0 \quad \text{at } \theta = -\alpha_1, \alpha_2 \tag{141}$$

Obviously any choice of α_1 and α_2 can fit this equation with appropriate choices of a_1 and b_1. Thus for example, let a_1 be 0 and $b_1 = -g$. The resulting motion is that of free fall, since the acceleration of the fluid is of magnitude g, and directed vertically downward. Thus it is not surprising that any values of α_1 and α_2 are possible. Again, the solution is not that being sought.

Third, suppose $2(m_0 - 1) = 1$, thus $m_0 = 3/2$. The value of m_1 can be either equal to or greater than 1 without affecting the following analysis. The boundary condition is (with $m_1 = 1$)

$$a_1 \sin\theta + (b_1 + g)\cos\theta = \frac{m_0^2}{2}(a_0^2 + b_0^2) \tag{142}$$

near $r = 0$. Again, any values of α_1 and α_2 are possible, but consider the values of v_θ at the surface. This is

$$v_\theta = -\frac{3}{2} r^{\frac{1}{2}} \left(a_o \cos\frac{3}{2}\theta - b_o \sin\frac{3}{2}\theta \right) \qquad (143)$$

$$(\text{with } \theta = -\alpha_1, \alpha_2)$$

Assuming the term in parentheses is not 0, then at an instant of time either earlier or later than t = 0 the surface shape will be like one of the four in fig. 14.

Fig. 14

Only shape b is clearly impossible, although shape a is certainly unlikely, and must either come from or go into shape b. Shapes c and d are not impossible, and might occur under special circumstances, but they do not represent the more or less stable crest shapes which are desired in the present study. It must be then that v_θ is 0 for small r and t = 0. Thus

$$a_o \cos\frac{3}{2}\theta - b_o \sin\frac{3}{2}\theta = 0 \quad at \quad \theta = -\alpha_1, \alpha_2 \qquad (144)$$

Choose the angle β such that

$$\alpha_1 + \beta = \alpha_2 - \beta = \alpha \qquad (145)$$

Then the crest shape is symmetric about the line $\theta = \beta$. Eq. (144) can be written in terms of α and β as

$$a_0' \cos\tfrac{3}{2}\alpha \pm b_0' \sin\tfrac{3}{2}\alpha = 0 \tag{146}$$

where

$$a_0' = a_0 \cos\tfrac{3}{2}\beta - b_0 \sin\tfrac{3}{2}\beta \tag{147}$$

$$b_0' = a_0 \sin\tfrac{3}{2}\beta + b_0 \cos\tfrac{3}{2}\beta \tag{148}$$

The smallest value of α which gives a non-trivial solution to this equation is $\alpha = \pi/3$, with $b_0' = 0$, or

$$\frac{b_0}{a_0} = -\tan\tfrac{3}{2}\beta \tag{149}$$

The included angle at the crest is then

$$2\alpha = 2\pi/3 = 120^\circ \tag{150}$$

There are no other values of m_0 and m_1 which may be choosen such that the boundary condition is satisfied; therefore the desired result is proved.

D. Implications of the Interaction. The effect of the interaction on the smaller wave is clear. Although the average steepness of the smaller wave system is not critical, it may be made to exceed the critical value in the neighborhood of the crests of a coexistant larger wave system. When this happens the smaller waves will presumably break, resulting in the common phenomenon of "white-horses". This in turn represents a loss of energy to the two-wave system (the energy goes into still smaller waves, or turbulence). The limiting effect of the larger wave on the height of the smaller wave is clear. But it is possible that the larger waves are also limited. The energy loss is a periodic phenomenon, with a period the same as that of the larger waves, since it happens always at the crest of the larger wave. This may imply a limiting condition on the heights of the larger waves, but it is not clear how such a condition could be expressed numerically.

When the two wave systems have approximately equal wave-lengths, the interpretation of eq. (131) is no longer simple. Still more complex is the question of non-linear interactions in a system with a continuous spectrum. With all such problems still unanswered, the present study can be taken only as a beginning to the understanding of non-linear interactions of gravity waves.

VI. CONCLUSIONS.

It can be concluded that turbulence in the flow of air over a wavy surface is responsible for the growth of water waves in a wind. As soon as it is realized that the atmosphere is not an ideal, irrotational fluid, there is no longer any mystery to the problem of wave growth. If turbulent motion were well enough understood, the flow over an irregular surface, and the resulting pressures, could be completely described, and the problem of wave growth completely solved. Unfortunately, with the present state of knowledge, this is impossible. The best that can be done at present is to approximate the more important features of turbulent flow, and use such approximations to understand the growth of waves.

Two such approximations were discussed in the introduction of this paper. One of these, suggested by Eckart, neglects the turbulence, and resulting pressure distribution, caused by the presence of waves on the water surface. Since the available experiments indicate that the pressures from this source are important in building waves, Eckart's "gust" mechanism can be only a fraction of the total cause of wave growth. As already mentioned, preliminary measurements indicate that this fraction is 1/10.

The other approximation is due to Jeffreys. It leads to a mathematical expression which gives the pressure as proportional to the slope of the surface. It is doubtful that Jeffreys intended this expression to be applied to any type of surface deformation other than a pure sine wave. Of course, it would be possible to extend the expression given by eq. (5), to any surface, or even to make some numerical modification to permit the explanation of more recent experimental results. However, it is difficult to see any physical justification for such ideas.

It would seem better to use the more recent experimental information on the most fundamental level; namely, to develop a new approximation for the turbulent flow of air over an irregular surface. A new expression for the pressure would follow from this approximation. It has been the purpose of this study to develop such an approximation. The approximation consists of a simple geometrical model (fig. 6) which permits the computation of the out-of-phase component of pressure in air moving over a wavy surface. The model has been applied to some simple cases, and the results of these applications yield the following conclusions:

For waves of moderate steepness, the model, as described in part C of Sec. III, with the angle equal to 0.003 radians, is an adequate first approximation. This follows from the comparison of the predictions of the model with the measurements of Motzfeld, discussed in part E of Sec. III, as well as from the application of the model to the growth of water waves, as discussed in Sec. IV.

The model, in its present form, cannot describe the flow of air over waves of large steepness, or waves with sharply pointed crests. This follows from the comparison of the predictions of the model with the measurements of Stanton*, who used models of large steepness, and with one of the tests of Motzfeld, using a wave with sharp crests.

* In several talks which the author has given on this study, he reported an agreement between the predictions of the model and the measurements of Stanton. Later it was discovered that this was incorrect. It should be noted further that amplitudes of the waves in the test-models used by Stanton (6) seem to be incorrectly reported in Ref. (1).

It is possible that the difficulty in explaining, quantitatively, the results of Stanton's measurements does not depend upon the steepness of the waves, but rather depends on the low Reynold's numbers obtained by Stanton. It is to be expected that the characteristics of a turbulent phenomenon, such as that now under study, will vary with the dimensionless Reynolds number

$$R = Ul/\nu \tag{151}$$

where U is the velocity, and ν the kinematic viscosity of the fluid (in this case air) and l is a characteristic length associated with the phenomena. For the present case, $U = (W-C)$, and, in accordance with the assumptions of the model, $l = \lambda$. For Stanton's tests the Reynolds numbers were of the order of 10^5, while for Motzfeld's tests they were around 10^6. For water waves under realistic conditions, the Reynolds numbers, defined as above, are of the order of 10^7.

A variation with Reynolds number is implied not only by the measurements of Stanton and Motzfeld, but also by the fact that the model fails to reproduce the observed wave-lengths of water waves at large fetch (or duration). Even though there is no direct connection between the dimensionless fetch parameter, F (Cf. eq. (85)), and the Reynolds number defined in eq. (151), nevertheless the data at small values of F was taken, generally, for short waves and low wind-velocities, while the observations at large values of F were, generally, of long waves and high winds. Thus there is a general increase of Reynolds number with F.

There is insufficient experimental information, at present, to permit any precise representation of variation with Reynolds number, or even to check the idea that the choice of U and l for eq. (151) is correct.

Many other features of the model require more experimental information than is now available. For example, it would be desirable to reach a definite conclusion as to how the model is to be interpreted physically. The purpose of the model is, after all, to understand the growth of water waves under realistic conditions, when a broad band of frequencies exists. Before the model can be applied to such a problem it must be decided whether the straight line of fig. 6 represents the position of a separated boundary-layer, a line joining the points of separation and reattachment of a boundary-layer, or a method of computing pressures from an asymmetric, but non-separated boundary-layer. A small wave in the trough of a larger wave might or might not be subjected to the force of a high wind-velocity depending on which of these interpretations is correct. Such information might be obtained from a wind tunnel test-program.

There is another question which could be investigated in a wind tunnel, and that is the dependence of separation phenomena on the shape of the wave crest. Actual waves do not have a sinusoidal shape, but are usually more pointed at the crests. Intuitively it would seem that sharper crests would make separation even more certain, and the resulting pressure asymmetries more pronounced. Motzfeld's observations indicate this is true. Little experimental evidence on this question is available.

Although such wind tunnel work is important it can never give the complete picture. Conditions at the surface of a water wave can not be duplicated by a solid model. For one thing, the motion of water particles near the surface relative to the wave shape is, on the average over a wave length, equal to the wave speed (i.e. the wave moves through the water but does not carry the water with it). Further, at high wind speeds the motion of the water near the surface may be turbulent. Since the behavior of the boundary layer must depend on conditions at the boundary between air and water this water particle motion is quite important. It is hard to imagine how the velocity field in the air over an actual water wave could be explored, but such an experiment is certainly necessary.

In addition to the experimental work, much more theoretical work is needed. The observed ratio of height to length for large waves on the open sea is about 1/30, or four times smaller than the theoretical limit of 1/7. This is completely unexplained. In order to understand this fact, some mechanism limiting wave growth must be devised. It seems likely that such a mechanism would depend upon the existence of the broad band of frequencies present in any real wave pattern. This conclusion is an extension of one of the results of the present study. It was shown in Sec. V that non-linear interactions between waves of different wave-lengths can cause limitations on the maximum average steepness attainable by a growing wave.

Further problems exist in the extension of the present theory to the three-dimensional case where finite crest lengths must be considered. Clearly, the present theory represents only a beginning to the understanding of the growth of water waves in a wind.

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