An Exact Average Formula for the Symmetric Square L-Function at the Center

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Aliekber Gürel

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In memory of my grandfather Hacı Paşa

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Abstract

In this thesis, we find an exact formula for the weighted average of the symmetric square L-values at the center. The average is taken over a Hecke eigen basis of cusp forms of $SL_2(\mathbb{Z})$ with a fixed weight $2k$. The weights are the n-th Fourier Coefficients of these functions. The terms in the formula involve quadratic Dirichlet L-values at the center, Confluent Hypergeometric functions, and some arithmetic functions.

The main ingredient, and the starting point, is a formula due Shimura, which relates the symmetric square L-function of a Hecke eigen form f to the inner product of f with the product of the theta function, θ; and a real analytic Eisenstein series of half integral weight, E. We apply Michel-Ramakrishnan's averaging technique on Shimura's formula to write the weighted average of symmetric square L-values in terms of the Fourier coefficients of the Eisenstein series.

There are two complications. First, the levels of $\theta \times E$ and f are different. Second, E is not holomorphic. That is why we first take trace of $\theta \times E$, and then we take the holomorphic projection. Computing the Fourier coefficients of the resulting function gives us the exact formula desired.

Finally, we deduce the asymptotic behavior of these formulas as $k \to \infty$.

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Chapter 1

Introduction.

1.1 Definitions and Notation

Let $k > 1$ be a positive integer and $\mathcal{H}_{2k}$ be the Hecke eigen-basis of $S_{2k}(SL_2(\mathbb{Z}))$, normalized so that the first Fourier coefficients are all 1. For $f \in \mathcal{H}_{2k}$, we will denote its Fourier coefficients by $c_n(f)$ and normalized Fourier coefficients by $\lambda_n(f)$:

$$f(z) = \sum_{n=1}^{\infty} c_n(f) e(nz) = \sum_{n=1}^{\infty} \lambda_n(f) n^{k-1/2} e(nz) \text{ (weight } 2k\text{, level 1)}\cdot$$

We will let $\theta(z)$ be the usual theta function:

$$\theta(z) = \sum_{n=-\infty}^{\infty} e(n^2 z) \text{ (weight } 1/2\text{, level 4)}\cdot$$

For a positive integer N and a complex number s, $E(z, s, 4N)$ will denote a non-holomorphic (real analytic) Eisenstein series of half integral weight, which transforms like a modular form of weight $2k - 1/2$ and level $4N$:

$$E(z, s, 4N) = y^{s/2} \sum_{g \in \Gamma_\infty \backslash \Gamma_0(4N)} \frac{j(g, z)^{1-4k}}{|j(g, z)|^{2s}} \text{ (weight } 2k - 1/2\text{, level 4N)}\cdot$$

By abuse of notation, when $N = 1$ we will simply write $E(z, s)$ for $E(z, s, 4)$, since we will mainly work with level 4. We will also apply similar abuse of notations later on with the dual Eisenstein series.

We use the standard notation

$$\Gamma = SL_2(\mathbb{Z}),\ \Gamma_\infty = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Gamma \right\}, \quad \text{and } \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \Gamma \mid c \in \mathbb{Z} \right\}.$$

$z = x + iy$ is a point in the upper half plane H, $e(z) = e^{2\pi i z}$, and

$$j(g, z) = \left(\frac{4c}{d}\right) \varepsilon_d^{-1}(4cz + d)^{1/2}, \quad \text{for } g = \begin{pmatrix} a & b \\ 4c & d \end{pmatrix} \in \Gamma_0(4),$$

where $\left(\frac{4c}{d}\right)$ is the Kronecker's symbol, $\varepsilon_d = 1$ or i depending on $d \equiv 1$ or $3 \pmod 4$, and the branch of square root is chosen so that $\sqrt{1} = +1$.

For a real analytic modular form h_1 and a real analytic cusp form h_2, both having the same weight $2k$ and level R, let

$$< h_1, h_2 >_R := \int_{\Gamma_0(R)\backslash H} h_1(z)\overline{h_2(z)} y^{2k} \frac{dxdy}{y^2},$$

where H is the upper half plane, and $z = x + iy$. When $R = 1$, we will ignore the index and simply write $< h_1, h_2 >$.

$\delta(x)$ will be the integer indicator function, i.e.

$$\delta(x) = 1 \text{ or } 0 \text{ depending on } x \text{ is an integer or not.}$$

For a non-zero integer n, $< n >$ will denote its square-free part with the sign. χ_n will denote the character defined by $\chi_n(d) = \left(\frac{n}{d}\right)$. For a character χ, and a positive integer N, we let

$$\begin{aligned} L(s,\chi) &= \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \\ L^N(s,\chi) &= \sum_{(n,N)=1} \frac{\chi(n)}{n^s} = \prod_{p\nmid N}\left(1-\frac{\chi(p)}{p^s}\right)^{-1}. \end{aligned}$$

${}_2F_1$ is the confluent hypergeometric function defined as

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n}\frac{z^n}{n!}, \quad \text{where } (r)_n = r(r+1)...(r+n-1).$$

We use the following conventions:

$$\begin{aligned} f(k) &\approx g(k) \ \text{ if } \lim_{k\to\infty}\frac{f(k)}{g(k)} = 1, \text{ and} \\ |f(k)| &<< |g(k)| \text{ if there is a positive constant } A \text{ s.t. } |f(k)| < A|g(k)|, \ \forall k. \end{aligned}$$

Finally, we let

$$e_k = \frac{(2k-2)!}{(4\pi)^{2k-1}}.$$

1.2 Shimura's Formula

By using the Euler product

$$L(f,s) := \sum_{n=1}^{\infty} c_n(f) n^{-s} = \prod_p [(1-\alpha_p p^{-s})(1-\beta_p p^{-s})]^{-1} \text{ (product over primes } p),$$

one defines the symmetric square L-function as

$$L(sym^2 f, s) := \prod_p [(1-\alpha_p^2 p^{-s})(1-\alpha_p\beta_p p^{-s})(1-\beta_p^2 p^{-s})]^{-1}.$$

By Theorem 1 in Shimura [8], $L(sym^2 f, s)$ is a meromorphic function of s, with possible simple poles at only $2k$ and $2k-1$. It also satisfies the following functional equation. The function

$$\Lambda(sym^2 f, s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-2k+1)\zeta(s-2k+1)L(sym^2 f, s)$$

is invariant under the transformation $s \leftrightarrow 4k-1-s$. We will call the center of symmetry, $s = 2k - 1/2$, *the central point.*

An analogue of this result is known over arbitrary number fields by Gelbart and Jacquet [1].

If we put

$$\Lambda_u(sym^2 f, s) = \Lambda(sym^2 f, s+2k-1),$$

then we have the unitary functional equation

$$\Lambda_u(sym^2 f, s) = \Lambda_u(sym^2 f, 1-s).$$

Moreover, by the formula (1.5) in Shimura [8], we have the following equation relating the symmetric square L-function to the Eisenstein series and theta function:

$$\frac{2\Gamma(s/2)L(sym^2 f, s)}{(4\pi)^{s/2}\zeta(2s-4k+2)} = \int_{\Gamma_0(4)\backslash H} f(z)\overline{\theta(z)E(z, s+2-4k)}y^{2k}\frac{dxdy}{y^2}\cdot \tag{1.1}$$

When $s \in \mathbb{R}$, all the terms on the LHS of Shimura's equation (1.1) are real, and so, taking complex conjugate of RHS will not affect the identity. We can rewrite it as

$$\frac{2\Gamma(s/2)(4\pi)^{-s/2}}{\zeta(2s-4k+2)} \times L(sym^2 f, s) = <\theta E(., s+2-4k), f>_4 \cdot \tag{1.2}$$

1.3 Exact Average Principle

Starting with Shimura's Integral representation formula (1.2), we will apply the averaging method, developed and used by Michel-Ramakrishnan [5], to represent the average of symmetric square L-values in terms of the function $\theta(z)E(z,s)$. However, there are a few complications. One is that the levels do not match. Another is that the function $E(z,s)$ is real analytic but not holomorphic. To resolve these problems, we will first take trace and then holomorphic cuspidal projection of $\theta(z)E(z,s)$.

Let Tr be the trace map sending (real analytic) modular forms of level 4 down to (real analytic) modular forms of level 1, and $\mathrm{Pr}^0_{\mathrm{hol}}$ be the holomorphic cuspidal projection of a real analytic modular form. Define the following function:

$$G_s(z) := \mathrm{Pr}^0_{\mathrm{hol}}(Tr(\theta(z)E(z,s))) = \sum_{n=1}^{\infty} g_n(s)e(nz).$$

Then,

$$\begin{aligned} \frac{2\Gamma(s/2)(4\pi)^{-s/2}}{\zeta(2s-4k+2)} \times L(sym^2 f, s) &= \; < \theta E(.,s+2-4k), f >_4 \\ &= \; < Tr(\theta E(.,s+2-4k)), f > \\ &= \; < G_{s+2-4k}, f > \cdot \end{aligned} \tag{1.3}$$

Averaging these equations over the orthogonal $\mathcal{H}_{2k}$, we get

$$\frac{2\Gamma(s/2)(4\pi)^{-s/2}}{\zeta(2s-4k+2)} \sum_{f\in\mathcal{H}_{2k}} \frac{L(sym^2 f, s)}{< f, f >} f \;\; = \;\; G_{s+2-4k}. \tag{1.4}$$

We can rewrite this equation in terms of the Fourier coefficients as

$$\frac{2\Gamma(s/2)(4\pi)^{-s/2} n^{k-1/2}}{\zeta(2s-4k+2)} \sum_{f\in\mathcal{H}_{2k}} \frac{L(sym^2 f, s)\lambda_n(f)}{< f, f >} = g_n(s+2-4k) \cdot \tag{1.5}$$

Note that, $\zeta(2s-4k+2)$ has a simple pole at the central value $s = 2k - 1/2$. On the other hand, $L(sym^2 f, s)$ is regular by Theorem 1 in Shimura [8]. We deduce that $g_n(s+2-4k)$ has a zero at this point, and we can rewrite the equation as

$$\frac{4\Gamma(s/2)(4\pi)^{-s/2} n^{k-1/2}}{(2s-4k+1)\zeta(2s-4k+2)} \sum_{f\in\mathcal{H}_{2k}} \frac{L(sym^2 f, s)\lambda_n(f)}{< f, f >} \;\; = \;\; \frac{g_n(s+2-4k)}{s-2k+1/2}.$$

As $s \to 2k - 1/2$, we get

$$\frac{4\Gamma(k-1/4)n^{k-1/2}}{(4\pi)^{k-1/4}} \sum_{f \in \mathcal{H}_{2k}} \frac{L(sym^2 f, 2k-1/2)\lambda_n(f)}{< f, f >} = g'_n(3/2 - 2k)\cdot \tag{1.6}$$

To find these coefficients $\{g_n\}$, we need to understand $Tr(\theta(z)E(z,s))$ first. In Chapter 2 we will find the Fourier coefficients of $Tr(\theta(z)E(z,s))$ in terms of the Fourier coefficients of the Eisenstein series.

Then, in Chapter 3 we will see how these Fourier coefficients are affected by taking the holomorphic cuspidal projection.

We will spend Chapter 4 with calculations to find $\{g'_n(s)\}$, which finally gives rise to the main formula.

Before stating the Main Theorem, recall the following notations from Section (1.1):

For a non-zero integer n, $< n >$ denotes its square-free part with the sign.

$$\delta(x) = 1 \text{ or } 0 \text{ depending on } x \text{ is an integer or not, and } e_k = \frac{(2k-2)!}{(4\pi)^{2k-1}}\cdot$$

1.4 Main Theorem: The Exact Average Formula at the Center

Main Theorem. *For integers $k > 1$ and $n > 0$, we have the following formula at the center of symmetry $s = 2k - 1/2$:*

$$\sum_{f \in \mathcal{H}_{2k}} L\left(sym^2 f, 2k - \frac{1}{2}\right) \frac{e_k \ \lambda_n(f)}{< f, f >} = \mathcal{A}_{n,k} + \sum_{m \in \mathbb{Z},\, m^2 \neq 4n} L\left(\frac{1}{2}, \chi_{4<m^2-4n>}\right) \xi_{n,k}(m).$$

$$\mathcal{A}_{n,k} = \frac{\delta(\sqrt{n})}{n^{1/4}} \left(\frac{\Gamma'(2k-1/2)}{\Gamma(2k-1/2)} + \mathcal{A} - \frac{\ln(n)}{2}\right), \quad where \ \ \mathcal{A} = \frac{\pi + 6\gamma - 6\ln\pi - 10\ln 2}{4}.$$

The weights $\xi_{n,k}(m)$ are positive for all but finitely many m. In fact, $\xi_{n,k}(m) > 0$, whenever $|m| > 2\sqrt{n}$. Explicitly,

$$\xi_{n,k}(m) = \beta(4n - m^2)\gamma(4n - m^2)\mathcal{F}_{n,k}(m).$$

For $n \neq 0$, we write $n = 2^d n_1$ with n_1 odd. Then

$$\beta(n) = \sum_{a,b>0,\ (ab)^2 | n_1} \frac{\mu(a)}{\sqrt{a}} \left(\frac{-4 < n >}{a}\right)$$

$$\gamma(n) = \begin{cases} d/2 & d \ even,\ n_1 \equiv 1 \ (mod\ 4), \\ d/2 + 2 - \sqrt{2} & d \ even,\ n_1 \equiv 3 \ (mod\ 8), \\ d/2 + 2 + \sqrt{2} & d \ even,\ n_1 \equiv 7 \ (mod\ 8), \\ d/2 - 1/2 & d \ odd. \end{cases}$$

Finally,

$$\mathcal{F}_{n,k}(m) = \begin{cases} \frac{\sqrt{2\pi}}{2n^{1/4}} \frac{(-1)^k \Gamma(k-1/4)}{\Gamma(k+1/4)} {}_2F_1\left(k - \frac{1}{4}, \frac{3}{4} - k; \frac{1}{2}; \frac{m^2}{4n}\right) & if\ m^2 < 4n, \\ \frac{1}{2n^{1/4}} \left(\frac{4n}{m^2}\right)^{k-1/4} I_k\left(\frac{m^2-4n}{m^2}\right) & if\ m^2 > 4n, \end{cases}$$

$$where \quad I_k(c) = \int_0^\infty \frac{u^{k-3/4}\, du}{(1+u)^{k+1/4}(1+cu)^{k-1/4}}.$$

In Chapter 5, we will investigate the asymptotic and deduce two Corollaries:

Corollary 1. *For a fixed square integer n, letting $k \to \infty$,*

$$\sum_{f \in \mathcal{H}_{2k}} L\left(sym^2 f, 2k - \frac{1}{2}\right) \frac{e_k \ \lambda_n(f)}{< f, f >} = \frac{1}{n^{1/4}} \left(\frac{\Gamma'(2k-1/2)}{\Gamma(2k-1/2)} + \mathcal{A} - \frac{\ln(n)}{2} \right) + O\left(\frac{1}{\sqrt{k}}\right) \cdot$$

In particular, when $n = 1$, we have

$$\sum_{f \in \mathcal{H}_{2k}} L\left(sym^2 f, 2k - \frac{1}{2}\right) \frac{e_k}{< f, f >} = \frac{\Gamma'(2k-1/2)}{\Gamma(2k-1/2)} + \mathcal{A} + O\left(\frac{1}{\sqrt{k}}\right) \cdot$$

Assuming that $L(sym^2 f, 2k - \frac{1}{2}) \geq 0$ for all $f \in \mathcal{H}_{2k}$, we get

$$\sum_{f \in \mathcal{H}_{2k}} L\left(sym^2 f, 2k - \frac{1}{2}\right) << k (\ln k)^4.$$

An analogous result for the case $n = 1$ was recently proved by R. Khan [4] with an error term of $O(k^{-1/20+\epsilon})$. Corollary 1 improves the bound on the error term to $O(k^{-1/2})$ and extends the result to arbitrary n. More importantly, ours is an *exact* formula, and the error term is also given explicitly by the Main Formula.

Corollary 2. *For a fixed non-square integer n, we have*

$$\sum_{f \in \mathcal{H}_{2k}} L\left(sym^2 f, 2k - \frac{1}{2}\right) \frac{e_k \ \lambda_n(f)}{< f, f >} = \sum_{m \in \mathbb{Z}} L\left(\frac{1}{2}, \chi_{4<m^2-4n>}\right) \xi_{n,k}(m) \cdot$$

As $k \to \infty$,

$$\sum_{f \in \mathcal{H}_{2k}} L\left(sym^2 f, 2k - \frac{1}{2}\right) \frac{e_k \ \lambda_n(f)}{< f, f >} = O\left(\frac{1}{\sqrt{k}}\right) \cdot$$

1.5 Some Comments

(A) By Shimura's formula (1.2), $L(sym^2 f, s)$ or $\Lambda(sym^2 f, s)$ is essentially given by the scalar product $< \theta E(\cdot, s+2-4k), f >$. On the other hand, J. Sturm's expression for the Fourier coefficients of the series $E(z, s)$ involves a quadratic Dirichlet L-function, $L(s+2k-1, \chi_{-4<n>})$ (see Section 2.6). So the nth Fourier coefficient of $E(z, s+2-4k)$ involves $L((s+2-4k)+2k-1, \chi_{-4<n>}) = L(s-2k+1, \chi_{-4<n>})$ which becomes $L(1/2, \chi_{-4<n>})$ at the central point $s = 2k - 1/2$.

This explains the occurrence of the central values of Dirichlet L functions in the Main Theorem.

(B) Perhaps, it is most interesting when n is non-square. Because, in this case our formula relates a finite weighted sum of Symmetric Square L-values at the center to an infinite weighted sum of quadratic Dirichlet L-values at the center. Both L-values are expected, by the Generalized Riemann Hypothesis, to be non-negative. Even though we are far from proving either, we hope to understand, in a sequel, the compatibility aspect of these two L series.

(C) Assuming the non-negativity condition in Corollary 1, the bound we get for an individual $L(sym^2 f, 2k-1/2)$ is $k^{1+\epsilon}$ which is weaker than the convexity bound $k^{1/2}$.

(D) For $k = 6$, $S_{2k}(\Gamma)$ is one dimensional and $\mathcal{H}_{12} = \{\Delta\}$ where

$$\Delta(z) = q \prod_{n \geq 1} (1-q^n)^{24} = \sum_{n=1}^{\infty} \tau_n q^n \qquad (q = e(z)).$$

In this case, the Main Theorem relates the product of τ_n and $L\left(sym^2 \Delta, \frac{11}{2}\right)$ to an infinite weighted average of quadratic Dirichlet L-values at the center.

Chapter 2

Computation of Trace.

2.1 Definition of Trace

In this chapter we will find the Fourier coefficients of the trace of $\theta(z)E(z,s)$ in terms of the Fourier coefficients of the Eisenstein series. First, let us explain what we mean by this trace map.

Let $\widetilde{M_{2k}}(\Gamma_0(N))$ be the set of modular forms of level N and weight $2k$, which are real analytic but not necessarily holomorphic.

For $N|M$, we define the trace map

$$Tr_N^M : \widetilde{M_{2k}}(\Gamma_0(M)) \to \widetilde{M_{2k}}(\Gamma_0(N))$$

as follows:

$$Tr_N^M(h)(z) := \sum_{\gamma\in\Gamma_0(M)\backslash\Gamma_0(N)} h|_\gamma(z)$$

where for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$,

$$h|_\gamma(z) = (cz+d)^{-2k}h(\gamma z).$$

Remark 2.1. Note that

1) $h|_{\gamma_1\gamma_2} = (h|_{\gamma_1})|_{\gamma_2}$.

2)$h|_\gamma$ doesn't depend on the right coset representative γ.

3) For any element $g \in \Gamma_0(N)$, as γ runs through a list of right coset representatives, γg runs through a list of right coset representatives as well. Hence, the image $Tr_N^M(h)$ is a modular form of level N.

4) It is not difficult to show that for a cusp form f with weight $2k$ and level N and a real analytic modular form g of weight $2k$ and level M where $N|M$, one has:

$$< f, Tr_N^M(g) >_N = < f, g >_M \ .$$

To proceed we need to find a set of right coset representatives of $\Gamma_0(4)\backslash\Gamma$ first.

2.2 A set of representatives for $\Gamma_0(4)\backslash\Gamma$

Lemma 2.2. *The set of matrices*

$$\left\{ g_0 = I,\ g_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},\ g_{2+j} = \begin{pmatrix} 0 & -1 \\ 1 & j \end{pmatrix},\ j = 0, 1, 2, 3 \right\}$$

is a set of representatives for the right cosets of $\Gamma_0(4)$ *in* $\Gamma = SL_2(\mathbb{Z})$*, i.e.,*

$$\Gamma = \bigsqcup_{j=0}^{5} \Gamma_0(4) g_j .$$

Proof. We need to show that any $g \in \Gamma$ is in exactly one of these cosets. Let $g = \begin{pmatrix} a & * \\ c & * \end{pmatrix} \in \Gamma$. Since $det(g) = 1$, we know that $gcd(a, c) = 1$. So $t = gcd(4, c) = gcd(4a, c)$ is a divisor of 4. Hence, $t = 1, 2,$ or 4. Also, in any case, there exists relatively prime integers m and n such that $t = 4am + cn$.

Case 1: $t = 4$**.** In this case, c is a multiple of 4, and g is already in $\Gamma_0(4) = \Gamma_0(4)g_0$. So, there is only one right coset in this form, and a representative is

$$g_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Case 2: $t = 2$**.** In this case, c is even but not divisible by 4. So

$$2 = 4am + cn$$

implies that n is odd, and since it is relatively prime with m, we conclude that $gcd(4m, n) = 1$. So, there exists a matrix $h = \begin{pmatrix} * & * \\ 4m & n \end{pmatrix} \in \Gamma_0(4)$. Note that

$$hg = \begin{pmatrix} * & * \\ 4m & n \end{pmatrix} \begin{pmatrix} a & * \\ c & * \end{pmatrix} = \begin{pmatrix} * & * \\ 2 & * \end{pmatrix} \in \Gamma_0(4)g.$$

Replacing g with hg (in the same right coset), we may assume that $c = 2$, i.e.,

$$g = \begin{pmatrix} a & * \\ 2 & * \end{pmatrix}.$$

Given two matrices g and g' in Γ,

$$\Gamma_0(4)g' = \Gamma_0(4)g \Leftrightarrow g'g^{-1} \in \Gamma_0(4).$$

Let

$$g = \begin{pmatrix} a & b \\ 2 & d \end{pmatrix} \text{ and } g' = \begin{pmatrix} a' & b' \\ 2 & d' \end{pmatrix}.$$

Observe that both d and d' are odd because $det(g) = det(g') = 1$. So,

$$g'g^{-1} = \begin{pmatrix} a' & b' \\ 2 & d' \end{pmatrix} \begin{pmatrix} d & -b \\ -2 & a \end{pmatrix} = \begin{pmatrix} * & * \\ 2(d - d') & * \end{pmatrix} \in \Gamma_0(4),$$

meaning that there is only one right coset in this form, and a coset representative, for instance, is

$$g_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Case 3: $t = 1$**.** In this case, c is odd, and $1 = 4am + cn$. Letting

$$h = \begin{pmatrix} c & -a \\ 4m & n \end{pmatrix} \in \Gamma_0(4),$$

we see that

$$hg = \begin{pmatrix} c & -a \\ 4m & n \end{pmatrix} \begin{pmatrix} a & * \\ c & * \end{pmatrix} = \begin{pmatrix} 0 & * \\ 1 & * \end{pmatrix} \in \Gamma_0(4)g.$$

Hence, we may assume that $a = 0$, $c = 1$. Then, $det(g) = 1$ implies that

$$g = \begin{pmatrix} 0 & -1 \\ 1 & * \end{pmatrix}.$$

Let

$$g = \begin{pmatrix} 0 & -1 \\ 1 & d \end{pmatrix}, \ g' = \begin{pmatrix} 0 & -1 \\ 1 & f \end{pmatrix} \in \Gamma.$$

$$g'g^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & f \end{pmatrix} \begin{pmatrix} d & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} * & * \\ d-f & * \end{pmatrix} \in \Gamma_0(4) \Leftrightarrow d \equiv f \pmod 4.$$

So,

$$\Gamma_0(4)g = \Gamma_0(4)g' \text{ if and only if } d \equiv f \pmod 4. \tag{2.1}$$

Hence, there are exactly four right cosets in this case, one for each number modulo

4, and we introduce the following four coset representatives:

$$g_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; g_3 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \; g_4 = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}, \text{ and } g_5 = \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}.$$

So far, we have proved that any element $g \in \Gamma$ is in one of the 6 right cosets, with representatives given as above. To prove the claim, it remains to check that these 6 cosets are different. Given

$$h = \begin{pmatrix} * & * \\ 4m & n \end{pmatrix} \in \Gamma_0(4), \text{ and } g = \begin{pmatrix} a & * \\ c & * \end{pmatrix} \in \Gamma,$$

we have

$$hg = \begin{pmatrix} * & * \\ 4m & n \end{pmatrix} \begin{pmatrix} a & * \\ c & * \end{pmatrix} = \begin{pmatrix} * & * \\ 4ma + nc & * \end{pmatrix}.$$

Note that n is odd because $det(h) = 1$. So,

$$gcd(4ma + nc, 4) = gcd(nc, 4) = gcd(c, 4).$$

Observe that, $gcd(4, c)$ depends only on the right coset $\Gamma_0(4)g$, but not on the coset representative g. This shows that the right cosets we have found in different cases above are different. Finally, we have already seen that the right cosets of g_2, g_3, g_4 and, g_5 are all different; thus, the lemma is proved ■

2.3 Transformation formulas for $\theta(z)$

Recall that

$$\theta(z) = \sum_{n=-\infty}^{\infty} e(n^2 z) \qquad \text{(weight 1/2, level 4)}.$$

Let

$$\theta_g(z) = (cz + d)^{-1/2}\theta(gz) \text{ for any } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

In this section we will find transformation formulas for $\theta_{g_j}(z),\ j = 0, ..., 5$.

Lemma 2.3. *We have the following transformation formulas:*

$$\begin{aligned} \theta_{g_1}(z) &= \theta(z/4) - \theta(z), \text{ and} \\ \theta_{g_{j+2}}(z) &= \frac{1-i}{2}\,\theta\left(\frac{z+j}{4}\right) \quad \text{for } j = 0, 1, 2, 3. \end{aligned}$$

Proof. We will use the following properties of the theta function repeatedly in the proof:

$$\theta(z+1) = \theta(z), \quad \theta\left(\frac{-1}{4z}\right) = \sqrt{\frac{2z}{i}}\theta(z), \text{ and } \theta\left(z + \frac{1}{2}\right) = 2\theta(4z) - \theta(z).$$

The first two are famous formulas, whereas the third one can be obtained easily by using the Fourier expansion.

Transformation formula at g_1:

$$\begin{aligned} \theta_{g_1}(z) &= \sqrt{\frac{1}{2z+1}}\,\theta\left(\frac{z}{2z+1}\right) = \sqrt{\frac{1}{2z+1}}\sqrt{\frac{-2z-1}{2iz}}\,\theta\left(\frac{-2z-1}{4z}\right) \\ &= \sqrt{\frac{-1}{2iz}}\,\theta\left(\frac{-1}{2} + \frac{-1}{4z}\right) = \sqrt{\frac{-1}{2iz}}\,\theta\left(\frac{1}{2} + \frac{-1}{4z}\right) \\ &= \sqrt{\frac{-1}{2iz}}\left(2\theta\left(\frac{-1}{z}\right) - \theta\left(\frac{-1}{4z}\right)\right) = \sqrt{\frac{-1}{2iz}}\left(2\sqrt{\frac{z}{2i}}\,\theta\left(\frac{z}{4}\right) - \sqrt{\frac{2z}{i}}\theta(z)\right) \\ &= 2\sqrt{\frac{-1}{2iz}}\sqrt{\frac{z}{2i}}\,\theta\left(\frac{z}{4}\right) - \sqrt{\frac{-1}{2iz}}\sqrt{\frac{2z}{i}}\theta(z) = \theta(z/4) - \theta(z). \end{aligned}$$

Transformation formulas at g_2, g_3, g_4, and g_5. For $j = 0, 1, 2, 3$ we have:

$$\begin{aligned} \theta_{g_{j+2}}(z) &= \sqrt{\frac{1}{z+j}}\,\theta\left(\frac{-1}{z+j}\right) = \sqrt{\frac{1}{z+j}}\sqrt{\frac{z+j}{2i}}\,\theta\left(\frac{z+j}{4}\right) \\ &= \sqrt{\frac{-i}{2}}\,\theta\left(\frac{z+j}{4}\right) = \frac{1-i}{2}\,\theta\left(\frac{z+j}{4}\right). \end{aligned}$$

2.4 Transformation formulas for $E(z,s)$

Recall that for a positive integer N,

$$E(z,s,4N) \;=\; y^{s/2}\sum_{g\in\Gamma_\infty\backslash\Gamma_0(4N)} \frac{j(g,z)^{1-4k}}{|j(g,z)|^{2s}}, \qquad \text{(weight } 2k-1/2\text{, level 4N).}$$

We also define the dual Eisenstein series F as

$$F(z,s,4N)=(\sqrt{4N}\,z)^{1/2-2k}E\left(\frac{-1}{4Nz},s,4N\right).$$

In particular,

$$F(z,s)=(2z)^{1/2-2k}E\left(\frac{-1}{4z},s\right).$$

For any $g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma$, let

$$E_g(z,s,4N)=(cz+d)^{1/2-2k}E(gz,s,4N).$$

In this section, we will find $E_{g_j}(z,s)$ for $j=0,...,5$ in terms of E and F.

Proposition 2.4. *We have the following transformation formulas:*

$$\begin{aligned} E_{g_1}(z,s) &= 2^sE(z/4,s,8)-E(z,s), \text{ and} \\ E_{g_{j+2}}(z,s) &= 2^{1/2-2k}F\left(\frac{z+j}{4},s\right), \quad \text{for } j=0,1,2,3. \end{aligned}$$

Proof. Transformation formula at g_1:

Let R_1 be a set of representatives for $\Gamma_\infty\backslash\Gamma_0(4)$,

$$W_1=\{(c,d)\in\mathbb{Z}^2\,|\,gcd(4c,d)=1,\ c_1=2c+d>0,\ d_1=d\},$$

and

$$V_1=\{(c_1,d_1)\in\mathbb{Z}^2\,|\,gcd(4c_1,d_1)=1,\ c_1>0,\ c_1\text{ odd}\}.$$

Note that there is a one to one correspondence among R_1, W_1, and V_1:

$$\begin{pmatrix} a & b \\ 4c & d \end{pmatrix} \in R_1 \leftrightarrow (c,d) \in W_1 \leftrightarrow (2c+d,d) \in V_1.$$

On one hand,

$$\begin{aligned}
E_{g_1}(z,s) &= (2z+1)^{1/2-2k} E\left(\frac{z}{2z+1}, s\right) \\
&= (2z+1)^{1/2-2k} Im\left(\frac{z}{2z+1}\right)^{s/2} \sum_{g \in R_1} \frac{j\left(g, \frac{z}{2z+1}\right)^{1-4k}}{\left|j(g, \frac{z}{2z+1})\right|^{2s}} \\
&= (2z+1)^{1/2-2k} \frac{y^{s/2}}{|2z+1|^s} \sum_{(c,d)\in W_1} \varepsilon_d^{-1} \left(\frac{4c}{d}\right) \frac{(\frac{4cz}{2z+1}+d)^{1/2-2k}}{|\frac{4cz}{2z+1}+d|^s} \\
&= y^{s/2} \sum_{(c,d)\in W_1} \varepsilon_d^{-1} \left(\frac{4c}{d}\right) \frac{((4c+2d)z+d)^{1/2-2k}}{|(4c+2d)z+d|^s} \\
&= y^{s/2} \sum_{(c_1,d_1)\in V_1} \varepsilon_{d_1}^{-1} \left(\frac{2c_1-2d_1}{d_1}\right) \frac{(2c_1z+d_1)^{1/2-2k}}{|2c_1z+d_1|^s} \\
&= y^{s/2} \sum_{c_1\,(odd)>0} \sum_{gcd(4c_1,d_1)=1} \varepsilon_{d_1}^{-1} \left(\frac{2c_1}{d_1}\right) \frac{(2c_1z+d_1)^{1/2-2k}}{|2c_1z+d_1|^s}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
E(z,s) &= y^{s/2} \left(1 + \sum_{c>0} \sum_{gcd(4c,d)=1} \varepsilon_d^{-1} \left(\frac{4c}{d}\right) \frac{(4cz+d)^{1/2-2k}}{|4cz+d|^s}\right) \\
&= y^{s/2} \left(1 + \sum_{c\,(even)>0} \sum_{gcd(4c,d)=1} \varepsilon_d^{-1} \left(\frac{2c}{d}\right) \frac{(2cz+d)^{1/2-2k}}{|2cz+d|^s}\right).
\end{aligned}$$

Adding these two equations for $E_{g_1}(z,s)$ and $E(z,s)$ we get:

$$E_{g_1}(z,s) + E(z,s) = 2^s E(z/4, s, 8)\cdot$$

Transformation formulas at g_2, g_3, g_4, and g_5. For $j = 0, 1, 2$, and 3 we have:

$$\begin{aligned} E_{g_{j+2}}(z,s) &= (z+j)^{1/2-2k} E\left(\frac{-1}{z+j}, s\right) \\ &= (z+j)^{1/2-2k} \left(\frac{z+j}{2}\right)^{2k-1/2} F\left(\frac{z+j}{4}, s\right) \\ &= 2^{1/2-2k} F\left(\frac{z+j}{4}, s\right). \end{aligned}$$

2.5 Fourier Coefficients of $Tr_1^4(\theta(z)E(z,s))$

From the definition of trace and the transformation formulas derived earlier, we get

$$\begin{aligned} \tilde{G}_s(z) := & \; Tr_1^4(\theta(z)E(z,s)) = \sum_{j=0}^{5} (\theta(z)E(z,s))|_{g_j} = \sum_{j=0}^{5} \theta_{g_j}(z) E_{g_j}(z,s) \\ = & \; \theta(z)E(z,s) + (\theta(z/4) - \theta(z))(2^s E(z/4, s, 8) - E(z,s)) \\ & + \frac{1-i}{2^{1/2+2k}} \sum_{j=0}^{3} \theta\left(\frac{z+j}{4}\right) F\left(\frac{z+j}{4}, s\right). \end{aligned}$$

Remark 2.5. $\tilde{G}_s(z)$ is $z \to z+1$ invariant. So are $\sum_{j=0}^{3} \theta\left(\frac{z+j}{4}\right) F\left(\frac{z+j}{4}, s\right)$ and $\theta(z)E(z,s)$. We conclude that $\theta_{g_1}(z)E_{g_1}(z,s)$ is also $z \to z+1$ invariant. This is not a surprise and could be derived by noticing that the right coset $\Gamma_0(4)g_1$ is invariant when we multiply with $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on the right. Using this with

$$\theta\left(\frac{z+1}{4}\right) - i\theta\left(\frac{z}{4}\right) = (1-i)\theta(z),$$

(which can be proved easily using the Fourier expansion of θ), we conclude that

$$E\left(\frac{z+1}{4}, s, 8\right) + iE\left(\frac{z}{4}, s, 8\right) = \frac{(1+i)}{2^s} E(z,s).$$

Let the Fourier expansion of $\tilde{G}_s$ be:

$$\tilde{G}_s(z) = \sum_{n\in\mathbb{Z}} \tilde{g}_n(y,s)e(nx).$$

Also, let the Fourier expansions of E and F be:

$$\begin{aligned} E(z,s,4N) &= \sum_{n\in\mathbb{Z}} A_n(y,s,4N)\, e(nx), \text{ and} \\ F(z,s,4N) &= \sum_{n\in\mathbb{Z}} B_n(y,s,4N)\, e(nx). \end{aligned}$$

By considering the contribution of each term $\theta_{g_j}(z)E_{g_j}(z,s)$ towards $\tilde{g}_n$, we get

$$\tilde{g}_n(y,s) = \left(\sum_{l\in\mathbb{Z}} e^{-2\pi l^2 y} A_{n-l^2}(y,s,4)\right) + \left(2^s \sum_{l \text{ odd}} e^{\frac{-2\pi l^2 y}{4}} A_{4n-l^2}(y/4,s,8)\right) + 4 \times \left(\frac{1-i}{2^{1/2+2k}} \sum_{l\in\mathbb{Z}} e^{\frac{-2\pi l^2 y}{4}} B_{4n-l^2}(y/4,s,4)\right).$$

Note that the term $(\theta(z/4) - \theta(z))E(z,s)$ has no contribution in $\tilde{g}_n$, and the contributions of $\theta_{g_{j+2}}(z)E_{g_{j+2}}(z,s)$ are equal for $j = 0,1,2,3$.

Next, let us see what these Fourier coefficients A_n and B_n are.

2.6 Fourier Coefficients of E and F

The formulas in this section are due Shimura [8] (Proposition 1) and Sturm [9] (Lemma 2). Note that our definition of E and F is slightly different than their definition of E and E^*. To be more precise, we have

$$\begin{aligned} E(z,s,4N) &= y^{s/2} E_{JS}(z,s,\lambda,\omega_0) \text{ and} \\ F(z,s,4N) &= (4N)^{-\frac{\lambda+s}{2}} y^{s/2} E^*_{JS}(z,s,\lambda,\omega_0), \end{aligned}$$

where ω_0 is the trivial character, $\lambda = 2k - 1/2$, and to avoid confusion we used E_{JS} and E^*_{JS} for Sturm's definition of E and E^*.

Fourier Coefficients of $F(z,s,4N)$. First we need to define b_n and τ_n.

We start with b_n:

$$b_0(s,4N) = b_0^*(s,4N) = \frac{L^{4N}(2s+4k-3)}{L^{4N}(2s+4k-2)}.$$

For a non-zero n:

$$b_n(s,4N) = \frac{L^{4N}(s+2k-1, (\frac{-4<n>N}{\cdot}))}{L^{4N}(2s+4k-2)} \beta_n(s,4N),$$

and

$$b_n^*(s,4N) = \frac{L^{4N}(s+2k-1, (\frac{-4<n>}{\cdot}))}{L^{4N}(2s+4k-2)} \beta_n^*(s,4N),$$

with $< n >$ denoting the square-free part of n when $n \neq 0$:

$$\beta_n(s,4N) = \sum \mu(a) \left(\frac{-4 < n > N}{a} \right) a^{1-2k-s} b^{3-4k-2s},$$

and

$$\beta_n^*(s,4N) = \sum \mu(a) \left(\frac{-4 < n >}{a} \right) a^{1-2k-s} b^{3-4k-2s},$$

the sums being over positive integers a, b satisfying $(ab)^2 | n$ and $gcd(ab, 4N) = 1$.

Next, τ_n is defined using the equation

$$\sum_{n=-\infty}^{\infty} (z+n)^{-\alpha} (\bar{z}+n)^{-\beta} = \sum_{n=-\infty}^{\infty} \tau_n(y,\alpha,\beta) e(nx).$$

It can be written in terms of the Whittaker function as:

$$i^{\alpha-\beta}(2\pi)^{-\alpha-\beta}\tau_n(y,\alpha,\beta) = \begin{cases} n^{\alpha+\beta-1} e^{-2\pi n y} \Gamma(\alpha)^{-1} W(4\pi n y, \alpha, \beta) & \text{if } n > 0, \\ |n|^{\alpha+\beta-1} e^{-2\pi |n| y} \Gamma(\beta)^{-1} W(4\pi |n| y, \beta, \alpha) & \text{if } n < 0, \\ \Gamma(\alpha)^{-1}\Gamma(\beta)^{-1}\Gamma(\alpha+\beta-1)(4\pi y)^{1-\alpha-\beta} & \text{if } n = 0, \end{cases}$$

where the Whittaker function is

$$W(y,\alpha,\beta) := \Gamma(\beta)^{-1}\int_0^\infty (1+u)^{\alpha-1}u^{\beta-1}e^{-yu}du.$$

It is first defined for $Re(\beta) > 0$ and then extended to all β analytically with a functional equation

$$W(y,\alpha,\beta) = y^{1-\alpha-\beta}W(y,1-\beta,1-\alpha).$$

With these definitions,

$$F(z,s,4N) = \sum_{n\in\mathbb{Z}} B_n(y,s,4N)e(nx),$$

where

$$B_n(y,s,4N) = (4N)^{1/4-k-s/2}y^{s/2}b_n(s,4N)\tau_n\left(y,\frac{s+4k-1}{2},\frac{s}{2}\right).$$

Fourier Coefficients of $E(z,s,4N)$. For A_n, we need one more definition. Let

$$c_n(s,4N) = \sum_{4N|M|(4N)^\infty}\left(\sum_{p=1}^{M}\left(\frac{M}{p}\right)\varepsilon_p^{-1}e(np/M)M^{1/2-2k}\right)M^{-s}.$$

Then

$$E(z,s,4N) = \sum_{n\in\mathbb{Z}} A_n(y,s,4N)e(nx),$$

where

$$\begin{aligned} A_n(y,s,4N) &= y^{s/2}b_n^*(s,4N)c_n(s,4N)\tau_n\left(y,\frac{s+4k-1}{2},\frac{s}{2}\right) \quad \text{if } n\neq 0,\\ A_0(y,s,4N) &= y^{s/2}\left(1+b_0^*(s,4N)c_0(s,4N)\tau_0\left(y,\frac{s+4k-1}{2},\frac{s}{2}\right)\right). \end{aligned}$$

Now that we have a Fourier expansion for the trace, and the levels are set, we are ready to take the holomorphic projection.

Chapter 3

Holomorphic Cuspidal Projection.

3.1 Definition of holomorphic cuspidal projection

Gross-Zagier [2] (Section IV.5) summarizes the definition of holomorphic projection in the following proposition:

Proposition 3.1. *Let $\tilde{F}$ be a complex valued C^∞ function on H which transforms like a modular form of weight $2k$ for Γ with the Fourier expansion:*

$$\tilde{F}(z) = \sum_{n=-\infty}^{\infty} \tilde{a}_n(y)e(nx).$$

If $\tilde{F}(z) = O(y^{-\varepsilon})$ as $y \to \infty$ for some $\varepsilon > 0$, then

$$F(z) = \mathrm{Pr}^0_{\mathrm{hol}}(\tilde{F}(z)) = \sum_{n=1}^{\infty} a_n e(nz)$$

is a holomorphic cusp form of the same weight $2k$ for Γ, where

$$a_n = \mathrm{proj}_n(\tilde{a}_n(y)) = \frac{(4\pi n)^{2k-1}}{(2k-2)!}\int_0^\infty \tilde{a}_n(y)e^{-2\pi n y}y^{2k-2}\,dy.$$

Moreover $< g, F > = < g, \tilde{F} >$ for all $g \in S_{2k}(\Gamma)$.

To apply the proposition to $Tr(\theta(z)E(z,s))$, first we need to check the growth condition required by the proposition.

3.2 Growth Condition

We would like to take the holomorphic projection of $Tr(\theta(z)E(z,s))$ when $s > s_0 = 3/2-2k$ is a real number in some small neighborhood of s_0. The reason for considering not a single value but infinitely many s values near s_0 is to be able to take the derivative of the Fourier coefficients later on.

Lemma 3.2. *When $s > s_0$ is a real number close to s_0, $Tr(\theta(z)E(z,s))$ satisfies the growth condition stated in the proposition above. More precisely, there exists some $r > 0$ such that for any $s \in (s_0, s_0 + r)$,*

$$Tr(\theta(z)E(z,s)) = O(y^{-u}), \text{ as } y \to \infty, \text{ where } u = s - s_0 > 0.$$

Proof. We will prove the lemma by checking the growth condition for each summand of trace separately, starting with $\theta(z)E(z,s)$ itself. To simplify the notation, let

$$u = s - s_0, \qquad \alpha = s/2 + 2k - 1/2, \qquad \beta = s/2.$$

$\underline{\theta(z)E(z,s):}$

$$E(z,s) = y^{s/2} + \sum_{n\in\mathbb{Z}} b_n(s,4)c_n(s,4)\tau_n(y,\alpha,\beta)e(nx).$$

The first term in the product is

$$b_n(s,4) = \frac{L^4(s+2k-1, \chi_{-4<n>})}{L^4(2s+4k-2)}\beta_n(s,4),$$

where

$$\beta_n(s,4) = \sum \mu(a)\left(\frac{-4<n>}{a}\right)\frac{a^{-u}b^{-2u}}{\sqrt{a}},$$

the sum being over positive odd integers a, b such that $(ab)^2$ divides n. Since each term in absolute value is less than or equal to 1, and the number of terms is no more than n^2, we deduce that $|\beta_n(s,4)| \leq n^2$. Also, it is not difficult to prove that

$$|L^4(s+2k-1, \chi_{-4<n>})| \leq 4 < n > \leq 4n.$$

On the other hand, as $s \to s_0$, $2s+4k-2 \to 1$, so the denominator $L^4(2s+4k-1)$ goes to infinity. Hence, its reciprocal is bounded, say by 1. We conclude that, in some s-neighborhood of s_0, $|b_n(s,4)| \le n^3$ for non-zero n. When $n=0$, note that

$$b_0(s,4) = \frac{L^4(2u)}{L^4(2u+1)} = \frac{(1-2^{-2u})\zeta(2u)}{(1-2^{-1-2u})\zeta(2u+1)}$$

has a double-zero at $u=0$ because of the $(1-2^{-2u})$ and $\zeta(2u+1)$ terms. So, $|b_0(s,4)| << u^2$.

The second term, $c_n(s,4)$, is given as an infinite summation:

$$c_n(s,4) = \sum_{4|M|4^\infty} \left(\sum_{p=1}^{M} \left(\frac{M}{p}\right) \varepsilon_p^{-1} e(np/M) M^{1/2-2k} \right) M^{-s},$$

which (by letting $M = 2^j$) can be written as

$$\sum_{j=2}^{\infty} \left(\frac{\sum_{p=1}^{2^j} \left(\frac{2}{p}\right)^j \varepsilon_p^{-1} e(np/2^j)}{2^j} \right) 2^{-ju}.$$

For non-zero n, let $n = 2^d n_1$ where n_1 is odd. When $j > d+3$, the summation over $1 \le p \le 2^j$ can be split into four parts, one for each odd p-congruence class modulo 8. The terms $\left(\frac{2}{p}\right)^j \varepsilon_p^{-1}$, depend only on p modulo 8, and in each congruence class we are summing the roots $e(np/2^j)$, which form powers of a nontrivial root of unity, $e(p/2^j)$, each power counted with the same multiplicity. We conclude that the sum is zero, and the only contribution is when $j \le d+3$. Hence,

$$|c_n(s,4)| \le \sum_{j=2}^{d+3} 2^{-ju} \le d+2 << n.$$

For $n=0$, we have

$$|c_0(s,4)| = \left| \sum_{j=2}^{\infty} \left(\frac{\sum_{p=1}^{2^j} \left(\frac{2}{p}\right)^j \varepsilon_p^{-1}}{2^j} \right) 2^{-ju} \right| \le \sum_{j=0}^{\infty} 2^{-ju} = \frac{1}{1-2^{-u}} << \frac{1}{u}.$$

Next, we bound the τ_n terms. By Lemma 4 in Shimura [8],

$$y^\beta W(y,\alpha,\beta) \text{ and } y^\alpha W(y,\beta,\alpha)$$

are both bounded when $y > 1$ as $s \to s_0$. Since $\alpha > k$ and $\beta > -k$, we have

$$|W(y,\alpha,\beta)| << y^{-k} \text{ and } |W(y,\beta,\alpha)| << y^k.$$

Using the relation between the Whittaker function and τ_n we get the following bound for $n \neq 0$:

$$|\tau_n(y,\alpha,\beta)| << |n|^{k+1} y^k e^{-2\pi|n|y}.$$

For $n = 0$, $\tau_0(y,\alpha,\beta)$ has a simple pole at $u = 0$, and from

$$i^{\alpha-\beta}(2\pi)^{-\alpha-\beta}\tau_0(y,\alpha,\beta) = \frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)\Gamma(\beta)}(4\pi y)^{1-\alpha-\beta},$$

we deduce that

$$|\tau_0(y,\alpha,\beta)| << \frac{y^{-u}}{u}.$$

Combining these bounds and summing over $n \in \mathbb{Z}$, we get

$$E(z,s) = O(y^{-u}) \text{ as } y \to \infty.$$

Also, it is easy to see that $\theta(z) = 1 + O(e^{-y})$. These two bounds combine to give

$$\theta(z)E(z,s) = O(y^{-u}).$$

<u>$(\theta(z/4) - \theta(z))(2^s E(z/4,s,8) - E(z,s))$:</u>

$$E(z,s,8) = y^{s/2} + \sum_{n\in\mathbb{Z}} b_n^*(s,8)c_n(s,8)\tau_n(y,\alpha,\beta)e(nx).$$

Applying the same ideas as before, we note that $b_n^*(s,8) = b_n(s,4)$ is bounded by n^3 and $c_n(s,8)$ is bounded by a multiple of n. So, the Eisenstein terms are bounded by y^{-u} as before. In $\theta(z/4)-\theta(z)$, the constant terms cancel each other. Hence, it decays exponentially as $y \to \infty$. We conclude that

$$(\theta(z/4)-\theta(z))(2^s E(z/4,s,8)-E(z,s)) = O(e^{-y/4}) \text{ as } y \to \infty.$$

$\underline{\sum_{j=0}^{3} \theta\left(\frac{z+j}{4}\right) F\left(\frac{z+j}{4}, s\right)}$:

Let us check $\theta(z)F(z,s)$ first. The rest follows by the substitution $z \to (z+j)/4$.

$$F(z,s) = y^{s/2} \sum_{n\in\mathbb{Z}} 4^{-s/2-k+1/4} b_n(s,4)\tau_n(y,\alpha,\beta)e(nx).$$

Using the same bounds from before for $b_n(s,4)$ and $\tau_n(y,\alpha,\beta)$, we also get that $F(z,s)$ is bounded by a constant times y^{-u}, so is the product $\theta(z/4)F(z/4,s)$. Similarly, the other terms are bounded by y^{-u} as well, and we conclude that

$$\sum_{j=0}^{3} \theta\left(\frac{z+j}{4}\right) F\left(\frac{z+j}{4}, s\right) = O(y^{-u}) \text{ as } y \to \infty$$

This finishes checking the growth condition, since $u = s - s_0 > 0$ when $s > s_0$. Although we are restricted to take holomorphic projection only for $s \in (s_0, s_0+r)$, it is not a problem for our purposes. Our goal is to find the derivative of the Fourier coefficients of the holomorphic projection at $s = s_0$ and for this, knowing these coefficients for an infinite sequence of s values converging to s_0 suffices.

Now, we are ready to take holomorphic cuspidal projection of $Tr(\theta(z)E(z,s))$ for s values larger and close to s_0.

3.3 Holomorphic Cuspidal Projection of $Tr(\theta(z)E(z,s))$

Recall that

$$Tr(\theta(z)E(z,s)) = \sum_{n\in\mathbb{Z}} \tilde{g}_n(y,s)e(nx) \in \widetilde{M}_{2k}(\Gamma).$$

So far, we have computed these coefficients $\tilde{g}_n$ in terms of the fourier coefficients of E and F:

$$\begin{aligned}\tilde{g}_n(y,s,4) &= \left(\sum_{l\in\mathbb{Z}} e^{-2\pi l^2 y} A_{n-l^2}(y,s,4)\right) + \left(2^s \sum_{l\ odd} e^{\frac{-2\pi l^2 y}{4}} A_{4n-l^2}(y/4,s,8)\right)\\ &+4\times\left(\frac{1-i}{2^{1/2+2k}}\sum_{l\in\mathbb{Z}} e^{\frac{-2\pi l^2 y}{4}} B_{4n-l^2}(y/4,s,4)\right).\end{aligned}$$

Let the holomorphic cuspidal projection of $Tr(\theta(z)E(z,s))$ be

$$\sum_{n=1}^{\infty} g_n(s)e(nz) \in S_{2k}(\Gamma).$$

For a positive integer n, we write these coefficients g_n using the definition of holomorphic cuspidal projection:

$$g_n(s) = proj_n(\tilde{g}_n(y,s)) = \frac{(4\pi n)^{2k-1}}{(2k-2)!}\int_0^\infty \tilde{g}_n(y,s)e^{-2\pi n y}y^{2k-2}dy.$$

Using the formula we derived earlier for $\tilde{g}_n(y,s)$, we get

$$\begin{aligned}g_n(s) &= 2\delta(\sqrt{n})\,proj_n\left(e^{-2\pi n y}y^{s/2}\right)\\ &+\sum_{l\in\mathbb{Z}} b^*_{n-l^2}(s,4)c_{n-l^2}(s,4)proj_n\left(e^{-2\pi l^2 y}y^{s/2}\tau_{n-l^2}(y,\alpha,\beta)\right)\\ &+\sum_{l\ odd} b^*_{4n-l^2}(s,8)c_{4n-l^2}(s,8)proj_n\left(e^{\frac{-2\pi l^2 y}{4}}y^{s/2}\tau_{4n-l^2}(y/4,\alpha,\beta)\right)\\ &+2^{2-4k-2s}(1-i)\sum_{l\in\mathbb{Z}} b_{4n-l^2}(s,4)proj_n\left(e^{\frac{-2\pi l^2 y}{4}}y^{s/2}\tau_{4n-l^2}(y/4,\alpha,\beta)\right).\end{aligned}$$

From the definition of $proj_n$, it is immediate that

$$proj_n(\tilde{a}(y/4)) = proj_{4n}(\tilde{a}(y)).$$

Using this formula in the above equation for g_n and letting

$$P_n(m,s) = proj_n\left(e^{-2\pi m y} y^{s/2} \tau_{n-m}(y,\alpha,\beta)\right) \text{ (for integers } m \geq 0,\ n > 0)\,,$$

we get

$$\begin{aligned} g_n(s) \quad = \quad & 2\delta(\sqrt{n})\, proj_n\left(e^{-2\pi n y} y^{s/2}\right) \\ & + \sum_{l\in\mathbb{Z}} b^*_{n-l^2}(s,4) c_{n-l^2}(s,4) P_n(l^2,s) \\ & + 2^s \sum_{l\ odd} b^*_{4n-l^2}(s,8) c_{4n-l^2}(s,8) P_{4n}(l^2,s) \\ & + 2^{2-4k-s}(1-i) \sum_{l\in\mathbb{Z}} b_{4n-l^2}(s,4) P_{4n}(l^2,s). \end{aligned}$$

We will spend the next chapter with calculations to evaluate $g'_n(s_0)$.

Chapter 4

Calculations for the Main Formula.

4.1 Calculations for b_n, c_n, and $P_n(m,s)$

In this section, we will make calculations for the terms b_n, c_n and $P_n(m,s)$ at the point $s_0 = 3/2 - 2k$. Let us start with $P_n(m,s)$.

4.1.1 Calculations for $P_n(m,s)$.

Recall that

$$P_n(m,s) = proj_n\left(e^{-2\pi my}y^{s/2}\tau_{n-m}\left(y,\frac{s+L}{2},\frac{s}{2}\right)\right) \text{ (for integers } m \geq 0,\ n > 0\text{)},$$

where τ_n and $proj_n$ are as defined in the previous chapters.

Let

$$\begin{aligned}
s_0 &= 3/2 - 2k, \\
u &= s - s_0, \\
\alpha &= s/2 + 2k - 1/2, \quad \alpha_0 = s_0/2 + 2k - 1 = 1/4 + k, \\
\beta &= s/2, \qquad\qquad\quad\ \beta_0 = s_0/2 = 3/4 - k.
\end{aligned}$$

♣ For $0 \le m < n$, let $c = \frac{n-m}{n}\cdot$ Then we have

$$\begin{aligned}
P_n(m, s_0) &= proj_n\left(e^{-2\pi m y} y^{3/4-k} \tau_{n-m}(y, \alpha_0, \beta_0)\right) \\
&= \frac{(4\pi n)^{2k-1}}{(2k-2)!} \int_0^\infty e^{-2\pi(m+n)y} y^{k-1/4} \tau_{n-m}(y, \alpha_0, \beta_0) \frac{dy}{y} \\
&= \frac{2\pi\sqrt{i}(-1)^k (4\pi n)^{2k-1}}{\Gamma(k+1/4)(2k-2)!} \int_0^\infty e^{-4\pi n y} y^{k-1/4} W(4\pi(n-m)y, \alpha_0, \beta_0) \frac{dy}{y} \\
&= \frac{2\pi\sqrt{i}(-1)^k (4\pi n)^{k-3/4}}{\Gamma(k+1/4)(2k-2)!} J_k(c),
\end{aligned}$$

where

$$J_k(c) = \int_0^\infty e^{-y} y^{k-1/4} W(cy, \alpha_0, \beta_0) \frac{dy}{y}\cdot$$

There are a few definitions of confluent hypergeometric functions which are closely related:

$$\begin{aligned}
W(y, \alpha, \beta) &= \frac{1}{\Gamma(\beta)} \int_0^\infty (1+u)^{\alpha-1} u^{\beta-1} e^{-uy}\, du, \\
U(a, b, z) &= \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1} (1+t)^{b-a-1} e^{-zt}\, dt, \\
M_{k,\mu}(z) &= z^{1/2+\mu} e^{-z/2} {}_1F_1(1/2+\mu-k, 2\mu+1; z), \\
W_{k,\mu}(z) &= \frac{\Gamma(-2\mu) M_{k,\mu}(z)}{\Gamma(1/2-\mu-k)} + \frac{\Gamma(2\mu) M_{k,-\mu(z)}}{\Gamma(1/2+\mu-k)}\cdot
\end{aligned}$$

The relations being

$$\begin{aligned}
W(y, \alpha, \beta) &= U(\beta, \alpha+\beta, y) \quad \text{and} \\
W_{k,m}(z) &= e^{-z/2} z^{m+1/2} U(1/2+m-k, 1+2m; z).
\end{aligned}$$

Combining them, we get

$$W(y, \alpha_0, \beta_0) = W(y, 1-\beta_0, \beta_0) = U(\beta_0, 1, y) = y^{-1/2} e^{y/2} W_{1/2-\beta_0, 0}(y).$$

Let us rewrite $J_k(c)$ using this relation:

$$J_k(c) = \frac{1}{\sqrt{c}} \int_0^\infty e^{-(1-c/2)y} y^{k-3/4} W_{k-1/4,0}(cy) \frac{dy}{y}.$$

The following formula (Bateman [7], page 216) relates the Laplace transform of the Whittaker function $W_{k,m}$ to the Hypergeometric function ${}_2F_1$:

$$\int_0^\infty e^{-pt} t^v W_{k,\mu}(at) \frac{dt}{t} = \frac{\Gamma(\mu+v+1/2)\Gamma(v-\mu+1/2)a^{\mu+1/2}}{\Gamma(v-k+1)(p+a/2)^{\mu+v+1/2}} \times {}_2F_1\left(\mu+v+1/2, \mu-k+1/2; v-k+1; \frac{p-a/2}{p+a/2}\right).$$

With the following substitutions:

$$p \to 1-c/2, \quad a \to c, \quad k \to k-1/4, \quad v \to k-3/4, \quad \mu \to 0,$$

the formula reduces to

$$J_k(c) = \frac{\Gamma(k-1/4)^2}{\sqrt{\pi}} \times {}_2F_1\left(k-\frac{1}{4}, \frac{3}{4}-k; \frac{1}{2}; 1-c\right).$$

Hence,

$$P_n(m, s_0) = (1+i)\sqrt{2\pi}(-1)^k (4\pi n)^{k-3/4} \frac{\Gamma(k-1/4)^2}{\Gamma(k+1/4)(2k-2)!} \times {}_2F_1\left(k-\frac{1}{4}, \frac{3}{4}-k; \frac{1}{2}; 1-c\right).$$

♣ For $0 < m = n$ we have

$$\begin{aligned}
P_n(n,s) &= proj_n\left(e^{-2\pi ny}y^\beta\tau_0(y,\alpha,\beta)\right)\\
&= proj_n\left(e^{-2\pi ny}y^\beta i^{\beta-\alpha}(2\pi)^{\alpha+\beta}\frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)\Gamma(\beta)}(4\pi y)^{1-\alpha-\beta}\right)\\
&= i^{1/2-2k}2^{2-\alpha-\beta}\pi\frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)\Gamma(\beta)}proj_n\left(y^{1-\alpha}e^{-2\pi ny}\right)\\
&= i^{1/2-2k}2^{2-\alpha-\beta}\pi\frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)\Gamma(\beta)}\frac{(4\pi n)^{2k-1}}{(2k-2)!}\int_0^\infty y^{2k-\alpha}e^{-4\pi ny}\frac{dy}{y}\\
&= i^{1/2-2k}2^{2-\alpha-\beta}\pi\frac{\Gamma(\alpha+\beta-1)\Gamma(1/2-\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{(4\pi n)^{\alpha-1}}{(2k-2)!}\\
&= \frac{(1+i)(-1)^k\sqrt{2}\pi(4\pi n)^{k-3/4}}{(2k-2)!}\frac{(\pi n)^{u/2}\Gamma(u)\Gamma(k-1/4-u/2)}{\Gamma(k+1/4+u/2)\Gamma(3/4-k+u/2)}.
\end{aligned}$$

♣ For $0 < n < m$, let $c = (m-n)/m$. We have

$$\begin{aligned}
&P_n(m,s) = proj_n\left(e^{-2\pi my}y^\beta\tau_{n-m}(y,\alpha,\beta)\right)\\
=\ & proj_n\left(e^{-2\pi my}y^\beta i^{\beta-\alpha}(2\pi)^{\alpha+\beta}(m-n)^{\alpha+\beta-1}e^{-2\pi(m-n)y}\Gamma(\beta)^{-1}W(4\pi(m-n)y,\beta,\alpha)\right)\\
=\ & \frac{i^{1/2-2k}(2\pi)^{\alpha+\beta}(m-n)^{\alpha+\beta-1}}{\Gamma(\beta)}proj_n\left(y^\beta e^{-2\pi(2m-n)y}W(4\pi(m-n)y,\beta,\alpha)\right)\\
=\ & \frac{i^{1/2-2k}(2\pi)^{\alpha+\beta}(m-n)^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}proj_n\left(y^\beta e^{-2\pi(2m-n)y}\int_0^\infty\frac{(1+u)^{\beta-1}u^\alpha}{e^{4\pi(m-n)yu}}\frac{du}{u}\right)\\
=\ & \frac{i^{1/2-2k}(2\pi)^{\alpha+\beta}(m-n)^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}\frac{(4\pi n)^{2k-1}}{(2k-2)!}\int_0^\infty\left(\int_0^\infty\frac{y^{2k-1+\beta}}{e^{4\pi m(1+cu)y}}\frac{dy}{y}\right)(1+u)^{\beta-1}u^\alpha\frac{du}{u}\\
=\ & \frac{i^{1/2-2k}2^{2k-1/2}\pi^\alpha(m-n)^{\alpha+\beta-1}}{m^{2k-1+\beta}}\frac{n^{2k-1}}{(2k-2)!}\frac{\Gamma(2k-1+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty\frac{(1+u)^{\beta-1}u^\alpha}{(1+cu)^{2k-1+\beta}}\frac{du}{u}.
\end{aligned}$$

At the center, $s_0 = 3/2-2k$, we get

$$P_n(m,s_0) = \frac{(1+i)(4\pi)^{k-3/4}\Gamma(k-1/4)}{(2k-2)!}\frac{n^{2k-1}}{m^{k-1/4}}I_k(c),$$

where

$$I_k(c) = \int_0^\infty\frac{u^{k-3/4}\,du}{(1+u)^{k+1/4}(1+cu)^{k-1/4}}.$$

Remark 4.1. Note that, in all the cases, we have the relation

$$P_n(m,s) = 4^{1-\alpha} P_{4n}(4m,s).$$

We summarize these calculations in the following lemma.

Lemma 4.2. *We have the following values for* $P_n(m,s_0)$*, where* $n > 0$ *and* $m \geq 0$*:*
(1) If $0 \leq m < n$*, letting* $c = \frac{n-m}{n}$*,*

$$P_n(m,s_0) = (1+i)\sqrt{2\pi}(-1)^k(4\pi n)^{k-3/4}\frac{\Gamma(k-1/4)^2}{\Gamma(k+1/4)(2k-2)!}\times {}_2F_1\left(k-\frac{1}{4},\frac{3}{4}-k;\frac{1}{2};1-c\right).$$

(2) When $0 < m = n$*,* $P_n(n,s)$ *has a simple pole at* $s = s_0$ *and,*

$$P_n(n,s) = \frac{(1+i)(-1)^k\sqrt{2}\pi(4\pi n)^{k-3/4}}{(2k-2)!}\frac{(\pi n)^{u/2}\Gamma(u)\Gamma(k-1/4-u/2)}{\Gamma(k+1/4+u/2)\Gamma(3/4-k+u/2)}.$$

(3) If $0 < n < m$*, letting* $c = \frac{m-n}{m}$*,*

$$P_n(m,s_0) = \frac{(1+i)(4\pi)^{k-3/4}\Gamma(k-1/4)}{(2k-2)!}\frac{n^{2k-1}}{m^{k-1/4}}I_k(c),$$

where for $c \in (0,1)$*,*

$$I_k(c) = \int_0^\infty \frac{u^{k-3/4}\,du}{(1+u)^{k+1/4}(1+cu)^{k-1/4}}.$$

4.1.2 Calculations for $b_n(s)$

Recall that for non-zero n,

$$b_n(s,4N) = \frac{L^{4N}(s+2k-1,(\frac{-4<n>N}{\cdot}))}{L^{4N}(2s+4k-2)}\beta_n(s,4N),$$

and

$$b_n^*(s,4N) = \frac{L^{4N}(s+2k-1,(\frac{-4<n>}{\cdot}))}{L^{4N}(2s+4k-2)}\beta_n^*(s,4N),$$

with $< n >$ denoting the square-free part of n,

$$\beta_n(s,4N) = \sum \mu(a) \left(\frac{-4 < n > N}{a} \right) a^{1-2k-s} b^{3-4k-2s},$$

and

$$\beta^*_n(s,4N) = \sum \mu(a) \left(\frac{-4 < n >}{a} \right) a^{1-2k-s} b^{3-4k-2s}.$$

The sums are over positive integers a, b satisfying $(ab)^2 | n$ and $gcd(ab, 4N) = 1$.

Also,

$$b_0(s,4N) = b^*_0(s,4N) = \frac{L^{4N}(2s+4k-3)}{L^{4N}(2s+4k-2)}.$$

To simplify the notation, let us introduce

$$b_n(s) := b_n(s,4), \quad \beta_n(s) := \beta_n(s,4).$$

Note that

$$b_n(s,4) = b^*_n(s,4) = b^*_n(s,8),$$

and that they all vanish at the center, $s_0 = 3/2 - 2k$, because the denominators, $L^{4N}(2s+4k-2)$, have a pole at this point. For our computations we still need to know their derivatives.

For a non-zero $n =< n > m^2$, where $< n >$ is a square-free integer, we have

$$\begin{aligned}
b'_n(s_0) &= \left(\frac{L^4(s+2k-1, (\frac{-4<n>}{\cdot}))}{L^4(2s+4k-2)} \beta_n(s) \right)' \Big|_{s=s_0} \\
&= \lim_{s \to s_0} \left(\frac{L^4(s+2k-1, (\frac{-4<n>}{\cdot}))}{(s+2k-3/2)L^4(2s+4k-2)} \beta_n(s) \right) \\
&= \frac{L^4(1/2, (\frac{-4<n>}{\cdot}))}{\lim_{s \to s_0}(s-s_0)L^4(1+2(s-s_0))} \beta_n(s_0) \\
&= \frac{2L^4(1/2, (\frac{-4<n>}{\cdot}))}{\lim_{v \to 1}(v-1)L^4(v)} \beta_n(s_0) \quad (v = 1+2(s-s_0)) \\
&= \frac{2L^4(1/2, (\frac{-4<n>}{\cdot}))}{(1-2^{-1})\lim_{v \to 1}(v-1)\zeta(v)} \beta_n(s_0) \\
&= 4\, L\,(1/2, \chi_{-4<n>})\, \beta_n(s_0) \quad (\text{using } \ Res_{s=1}\zeta(s) = 1) \cdot
\end{aligned}$$

Here,

$$\beta_n(s_0) = \sum \left(\frac{-4 < n >}{a} \right) \frac{\mu(a)}{\sqrt{a}},$$

the sum being over positive odd integers a, b satisfying $ab|m$.

Next, let us see what happens when $n = 0$:

$$b_0(s) = \frac{L^4(2s+4k-3)}{L^4(2s+4k-2)} = \frac{(1-2^{-2s-4k+3})\zeta(2s+4k-3)}{(1-2^{-2s-4k+2})\zeta(2s+4k-2)}.$$

Note that, as $s \to s_0 = 3/2 - 2k,\ \ 2s+4k \to 3$. The numerator has a simple zero, and the denominator has a simple pole. Hence, $b_0(s)$ has a double zero at $s = s_0$.

4.1.3 Calculations for $c_n(s)$

Let $u = s - s_0 = s + 2k - 3/2$. Recall that

$$c_n(s, 4N) = \sum_{4N|M|(4N)^\infty} \left(\frac{\sum_{p=1}^{M} \left(\frac{M}{p}\right) \varepsilon_p^{-1} e(np/M)}{M^{1+u}} \right),$$

where $M|(4N)^\infty$ means that all prime divisors of M are also divisors of $4N$, and

$$\varepsilon_p = 1 \text{ or } i \text{ depending on } p \equiv\ 1 \text{ or } 3 \pmod 4).$$

We will find the values of $c_n(s, 4)$ and $c_n(s, 8)$. From the definition, note that they only differ by $M = 4$ terms. This simple observation gives us the following relation:

$$c_n(s, 4) = c_n(s, 8) + (-1)^{\frac{n(n+1)}{2}} \frac{1-i}{4^{1+u}}.$$

We will first calculate $c_n(s, 8)$ and then use this relation to find $c_n(s, 4)$.

When $n = 0$,

$$
\begin{aligned}
c_0(s,8) &= \sum_{8|M|8^\infty} \frac{\sum_{p=1}^{M} \left(\frac{M}{p}\right) \varepsilon_p^{-1}}{M^{1+u}} = \sum_{j=3}^{\infty} \frac{\sum_{p=1}^{2^j} \left(\frac{2}{p}\right)^j \varepsilon_p^{-1}}{2^{j(1+u)}} \\
&= \sum_{j=3}^{\infty} \frac{\left(1+(-1)^j(-i)+(-1)^j+(-i)\right) 2^{j-3}}{2^{j(1+u)}} \\
&= \sum_{j=3}^{\infty} \frac{(1+(-1)^j)(1-i)}{2^{3+ju}} = \frac{1-i}{4} \sum_{j\geq 4,\, even} \frac{1}{2^{ju}} = \frac{1-i}{4^{u+1}(4^u-1)} \quad \text{and} \\
c_0(s,4) &= \frac{1-i}{4(4^u-1)}.
\end{aligned}
$$

Note that $c_0(s,8)$ and $c_0(s,4)$ both have a simple pole at $s = s_0$ with residue:

$$
\lim_{s\to s_0} (s-s_0) c_n(s,8) = \lim_{s\to s_0} (s-s_0) c_n(s,4) = \frac{1-i}{4\ln 4}.
$$

Next, for non-zero n,

$$
c_n(s_0,8) = \sum_{8|M|8^\infty} \frac{\sum_{p=1}^{M} \left(\frac{M}{p}\right) \varepsilon_p^{-1} e\left(\frac{np}{M}\right)}{M} = \sum_{j=3}^{\infty} \frac{S_n(j)}{2^j},
$$

where

$$
S_n(j) = \sum_{p=1}^{2^j} \left(\frac{2}{p}\right)^j \varepsilon_p^{-1} e\left(\frac{np}{2^j}\right).
$$

To compute $S_n(j)$, let us first investigate the product $\left(\frac{2}{p}\right)^j \varepsilon_p^{-1}$. Using

$$
\left(\frac{2}{p}\right) = \begin{cases} +1, & \text{if } p \equiv 1,7 \pmod 8 \\ -1, & \text{if } p \equiv 3,5 \pmod 8 \end{cases}
$$

and the definition of ϵ_p, we get

$$\left(\frac{2}{p}\right)^j \varepsilon_p^{-1} = \begin{cases} 1, & \text{if } p \equiv 1 \pmod 8, \\ (-1)^j(-i), & \text{if } p \equiv 3 \pmod 8, \\ (-1)^j, & \text{if } p \equiv 5 \pmod 8, \\ (-i), & \text{if } p \equiv 7 \pmod 8. \end{cases}$$

By grouping p terms (ranging from 1 to 2^j) that are congruent modulo 8, we get

$$\begin{aligned} S_n(j) &= \sum_{p\equiv 1(8)} e\left(\frac{np}{2^j}\right) + (-1)^j(-i) \sum_{p\equiv 3(8)} e\left(\frac{np}{2^j}\right) + (-1)^j \sum_{p\equiv 5(8)} e\left(\frac{np}{2^j}\right) + (-i) \sum_{p\equiv 7(8)} e\left(\frac{np}{2^j}\right) \\ &= \left(e\left(\frac{n}{2^j}\right) + (-1)^j(-i)e\left(\frac{3n}{2^j}\right) + (-1)^j e\left(\frac{5n}{2^j}\right) + (-i)e\left(\frac{7n}{2^j}\right)\right) \\ &\quad \times \left(1 + e\left(\frac{8n}{2^j}\right) + \ldots + e\left(\frac{(2^j-8)n}{2^j}\right)\right) \\ &= e\left(\frac{n}{2^j}\right)\left(1 + (-1)^j(-i)e\left(\frac{2n}{2^j}\right)\right)\left(1 + (-1)^j e\left(\frac{4n}{2^j}\right)\right) \\ &\quad \times \left(1 + e\left(\frac{8n}{2^j}\right) + \ldots + e\left(\frac{(2^j-8)n}{2^j}\right)\right). \end{aligned}$$

Note that the last factor

$$1 + e\left(\frac{8n}{2^j}\right) + \ldots + e\left(\frac{(2^j-8)n}{2^j}\right) = \begin{cases} 2^{j-3}, & \text{if } 2^j \mid 8n, \\ \\ 0, & \text{if } 2^j \nmid 8n. \end{cases}$$

So, we have

$$S_n(j) = \begin{cases} 2^{j-3} e\left(\frac{n}{2^j}\right)\left(1 + (-1)^j(-i)e\left(\frac{2n}{2^j}\right)\right)\left(1 + (-1)^j e\left(\frac{4n}{2^j}\right)\right), & \text{if } 2^j \mid 8n \\ \\ 0, & \text{if } 2^j \nmid 8n. \end{cases}$$

This last condition encourages us to write n as $2^d n_1$, where n_1 is an odd integer. We will consider cases separately depending on how large j is compared to d.

♢ For $3 \le j \le d$; $e(n/2^j) = e(2n/2^j) = e(4n/2^j) = 1$ and

$$S_n(j) = \begin{cases} 0, \text{ for odd } j \\ \\ 2^{j-2}(1-i), \text{ for even } j. \end{cases}$$

♢ When $j = d+1$; $e(n/2^j) = -1$, $e(2n/2^j) = e(4n/2^j) = 1$ and

$$S_n(d+1) = \begin{cases} 0, \text{ for even } d \\ \\ 2^{d-1}(i-1), \text{ for odd } d. \end{cases}$$

♢ When $j = d+2$; $e(n/2^j) = i^{n_1}$, $e(2n/2^j) = -1$, $e(4n/2^j) = 1$, and

$$S_n(d+2) = \begin{cases} 0, \text{ for odd } d \\ \\ 2^d i^{n_1}(1+i), \text{ for even } d. \end{cases}$$

♢ When $j = d+3$; $e(n/2^j) = (\sqrt{i})^{n_1}$, $e(2n/2^j) = i^{n_1}$, $e(4n/2^j) = -1$, and

$$S_n(d+3) = \begin{cases} 0, \text{ for odd } d \\ \\ 2^{d+1}(\sqrt{i})^{n_1}(1+i^{n_1+1}), \text{ for even } d. \end{cases}$$

♢ Finally, for $j > d+3$ as we have seen earlier $S_n(j) = 0$.

We now have all the information we need to calculate

$$c_n(s_0, 8) = \sum_{j=3}^{\infty} \frac{S_n(j)}{2^j}.$$

We will look case by case as d varies.

♠ For $d = 0$,

$$c_n(s_0, 8) = \frac{S_n(3)}{8} = \frac{(\sqrt{i})^n(1 + i^{n+1})}{4} = \begin{cases} 0, \ \text{if } n \equiv 1 \text{ (modulo 4)} \\ \\ (\sqrt{i})^n/2 \text{ if } n \equiv 3 \text{ (modulo 4)}. \end{cases}$$

♠ For $d = 1$,

$$c_n(s_0, 8) = 0.$$

♠ For $d \geq 2$ even,

$$\begin{aligned} c_n(s_0, 8) &= \left(\sum_{3 \leq j \text{ even} \leq d} \frac{1-i}{4} \right) + \frac{(1+i)i^{n_1}}{4} + \frac{(\sqrt{i})^{n_1}(1 + i^{n_1+1})}{4} \\ &= \frac{d-2}{2} \times \frac{(1-i)}{4} + \frac{(1+i)i^{n_1}}{4} + \frac{(\sqrt{i})^{n_1}(1 + i^{n_1+1})}{4}. \end{aligned}$$

♠ And for $d \geq 3$ odd,

$$c_n(s_0, 8) = \left(\sum_{3 \leq j \text{ even} \leq d} \frac{1-i}{4} \right) + \frac{i-1}{4} = \frac{d-5}{2} \times \frac{1-i}{4}.$$

Finally, to find the values $c_n(s_0, 4)$, we use the formula

$$c_n(s_0, 4) = c_n(s_0, 8) + (-1)^{\frac{n(n+1)}{2}} \frac{1-i}{4}.$$

Let us summarize these formulas in the following lemma.

Lemma 4.3. *For a non-zero integer $n = 2^d n_1$ where n_1 is odd, we have*

$$c_n(s_0,8) = \begin{cases} 0 & d=0,\ n\equiv 1\ (4) \\ -\sqrt{2}\times\left(\frac{1-i}{4}\right) & d=0,\ n\equiv 3\ (8) \\ \sqrt{2}\times\left(\frac{1-i}{4}\right) & d=0,\ n\equiv 7\ (8) \\ 0 & d=1 \\ \frac{d-4}{2}\times\left(\frac{1-i}{4}\right) & d\geq 2\ even,\ n_1\equiv 1\ (4) \\ \frac{d-2\sqrt{2}}{2}\times\left(\frac{1-i}{4}\right) & d\geq 2\ even,\ n_1\equiv 3\ (8) \\ \frac{d+2\sqrt{2}}{2}\times\left(\frac{1-i}{4}\right) & d\geq 2\ even,\ n_1\equiv 7\ (8) \\ \frac{d-5}{2}\times\left(\frac{1-i}{4}\right) & d\geq 3\ odd \end{cases}$$

and

$$c_n(s_0,4) = \begin{cases} \frac{d-2}{2}\times\left(\frac{1-i}{4}\right) & d\ even,\ n_1\equiv 1\ (4) \\ \frac{d+2-2\sqrt{2}}{2}\times\left(\frac{1-i}{4}\right) & d\ even,\ n_1\equiv 3\ (8) \\ \frac{d+2+2\sqrt{2}}{2}\times\left(\frac{1-i}{4}\right) & d\ even,\ n_1\equiv 7\ (8) \\ \frac{d-3}{2}\times\left(\frac{1-i}{4}\right) & d\ odd \end{cases}$$

Also, letting $\gamma(n) = 2(1+i)c_n(s_0,4)+1$, we have

$$\gamma(n) = \begin{cases} d/2 & d\ even,\ n_1\equiv 1\ (4) \\ d/2+2-\sqrt{2} & d\ even,\ n_1\equiv 3\ (8) \\ d/2+2+\sqrt{2} & d\ even,\ n_1\equiv 7\ (8) \\ d/2-1/2 & d\ odd. \end{cases}$$

4.2 Calculations for $g_n'(s_0)$

Recall that

$$g_n(s) = 2\delta(\sqrt{n})proj_n\left(e^{-2\pi ny}y^{s/2}\right) + \sum_{m\in\mathbb{Z}} b_{n-m^2}(s)c_{n-m^2}(s,4)P_n(m^2,s)$$
$$+\ 2^s\sum_{m\ odd} b_{4n-m^2}(s)c_{4n-m^2}(s,8)P_{4n}(m^2,s) +\ 2^{2-4k-s}(1-i)\sum_{m\in\mathbb{Z}} b_{4n-m^2}(s)P_{4n}(m^2,s).$$

Taking the derivatives at $s = s_0$, we get

$$\begin{aligned} g_n'(s_0) &= 2\delta(\sqrt{n})\left(proj_n\left(e^{-2\pi ny}y^{s/2}\right)\right)'|_{s=s_0} + 2\delta(\sqrt{n})\left(b_0(s)c_0(s,4)P_n(n,s)\right)'|_{s=s_0} \\ &\quad +2\delta(\sqrt{n})\left(2^{2-4k-s}(1-i)b_0(s)P_{4n}(4n,s)\right)'|_{s=s_0} \\ &\quad + \sum_{m\in\mathbb{Z},\ m^2\neq n} b_{n-m^2}'(s_0)c_{n-m^2}(s_0,4)P_n(m^2,s_0) \\ &\quad + 2^{s_0}\sum_{m\ odd} b_{4n-m^2}'(s_0)c_{4n-m^2}(s_0,8)P_{4n}(m^2,s_0) \\ &\quad + 2^{s_0-1}(1-i)\sum_{m\in\mathbb{Z},\, m^2\neq 4n} b_{4n-m^2}'(s_0)P_{4n}(m^2,s_0). \end{aligned}$$

Recall the following equations we derived in the previous section:

$$\begin{aligned} P_n(m,s) &= 4^{1-\alpha}P_{4n}(4m,s) \\ b_n(s) &= b_{4n}(s) \\ c_n(s_0,8) &= c_n(s_0,4) - (-1)^{\frac{n(n+1)}{2}}\frac{1-i}{4} \\ c_n(s_0,4) &= c_{4n}(s_0,4) - \frac{1-i}{4} \\ b_n'(s_0) &= 4L(1/2,\chi_{-4<n>})\beta_n(s_0). \end{aligned}$$

Using these equations above, and letting $\beta(n) = \beta_n(s_0)$, we get

$$\begin{aligned} g_n'(s_0) &= 2\delta(\sqrt{n})\left(proj_n\left(e^{-2\pi ny}y^{s/2}\right)\right)'|_{s=s_0} \\ &\quad +2\delta(\sqrt{n})\left(b_0(s)\left(c_0(s,4)+\frac{1-i}{2}\right)P_n(n,s)\right)'|_{s=s_0} \\ &\quad + 2^{s_0-2}(1-i)\sum_{m\in\mathbb{Z},\, m^2\neq 4n} b_{4n-m^2}'(s_0)\gamma(4n-m^2)P_{4n}(m^2,s_0) \\ &= 2\delta(\sqrt{n})\left(proj_n\left(e^{-2\pi ny}y^{s/2}\right)\right)'|_{s=s_0} \\ &\quad +2\delta(\sqrt{n})\left(b_0(s)\left(c_0(s,4)+\frac{1-i}{2}\right)P_n(n,s)\right)'|_{s=s_0} \\ &\quad + 2^{s_0}(1-i)\sum_{m\in\mathbb{Z},\, m^2\neq 4n} L(1/2,\chi_{-4<4n-m^2>})\beta(4n-m^2)\gamma(4n-m^2)P_{4n}(m^2,s_0). \end{aligned}$$

Rewriting formula (1.6) after scaling, we get

$$\sum_{f\in\mathcal{H}_{2k}} L\left(sym^2 f, 2k-\frac{1}{2}\right)\frac{e_k\ \lambda_n(f)}{< f,f >} = \frac{(2k-2)!}{4(4\pi)^{k-3/4}n^{k-1/2}\Gamma(k-1/4)} g_n'(s_0).$$

Now we plug in $g_n'(s_0)$ above with the values of $P_n(m,s_0)$ we found earlier and deduce the following proposition.

Proposition 4.4.

$$\sum_{f\in\mathcal{H}_{2k}} L\left(sym^2 f, 2k-\frac{1}{2}\right)\frac{e_k\ \lambda_n(f)}{< f,f >} = \mathcal{A}_{n,k}+\mathcal{B}_{n,k},$$

where

$$\begin{aligned}\mathcal{A}_{n,k} &= \frac{(2k-2)!}{4(4\pi)^{k-3/4}n^{k-1/2}\Gamma(k-1/4)}\times 2\delta(\sqrt{n})\left(proj_n\left(e^{-2\pi n y}y^{s/2}\right)\right)'|_{s=s_0}\\ &\quad+\frac{(2k-2)!}{4(4\pi)^{k-3/4}n^{k-1/2}\Gamma(k-1/4)}\times 2\delta(\sqrt{n})\left(b_0(s)\left(c_0(s,4)+\frac{1-i}{2}\right)P_n(n,s)\right)'|_{s=s_0}\end{aligned}$$

$$\mathcal{B}_{n,k} = \sum_{m\in\mathbb{Z},\, m^2\neq 4n} L\left(1/2, \chi_{4<m^2-4n>}\right)\xi_{n,k}(m).$$

The coefficients $\xi_{n,k}(m)$ are given by

$$\xi_{n,k}(m) = \beta(4n-m^2)\gamma(4n-m^2)\mathcal{F}_{n,k}(m)$$

with

$$\mathcal{F}_{n,k}(m) = \begin{cases} \frac{\sqrt{2\pi}}{2n^{1/4}}\frac{(-1)^k\Gamma(k-1/4)}{\Gamma(k+1/4)}\,{}_2F_1\left(k-\frac{1}{4},\frac{3}{4}-k;\frac{1}{2};\frac{m^2}{4n}\right) & \text{if } m^2<4n\\ \\ \frac{1}{2n^{1/4}}\left(\frac{4n}{m^2}\right)^{k-1/4} I_k\left(\frac{m^2-4n}{m^2}\right) & \text{if } m^2>4n.\end{cases}$$

Lemma 4.5. $\xi_{n,k}(m)>0$ *whenever* $m^2>4n$.

Proof. First, note that $\mathcal{F}_{n,k}(m)>0$ for $m^2>4n$. Also, from Lemma 4.3, it is easy to see that $\gamma(n)\geq 0$ for all n. Moreover $\gamma(4n-m^2)>0$ for all m,n.

Finally, for positivity of $\beta(n)$, let $n = 4^r m^2 t$ where m is odd and t is square-free. Then

$$\beta(n) = \sum_{ab|m} \frac{\mu(a)}{\sqrt{a}} \left(\frac{-4t}{a}\right).$$

Note that we are summing over the terms $\frac{\mu(a)}{\sqrt{a}} \left(\frac{-4t}{a}\right)$ which is a multiplicative function. Hence, it suffices to find the last sum for a prime power:

$$\sum_{ab|p^\alpha} \mu(a) \left(\frac{-4t}{a}\right) a^{-1/2} \quad = \quad 1 + \alpha - \frac{\alpha}{\sqrt{p}} \left(\frac{-4t}{p}\right) > 0.$$

We conclude that $\beta(n)$, being a product of these over $p|m$, is positive as well ■

4.3 Simplifying $\mathcal{A}_{n,k}$

- $\underline{\left(proj_n\left(e^{-2\pi n y} y^{s/2}\right)\right)' |_{s=s_0}}$

$$\begin{aligned} proj_n(e^{-2\pi n y} y^{s/2}) &= \frac{(4\pi n)^{2k-1}}{(2k-2)!} \int_0^\infty e^{-4\pi n y} y^{k-1/4+u/2} \frac{dy}{y} \\ &= \frac{(4\pi n)^{k-3/4} \Gamma(k-1/4+u/2)}{(4\pi n)^{u/2}(2k-2)!}. \end{aligned}$$

Taking the derivative at $u = 0$, we get

$$\left(proj_n(e^{-2\pi n y} y^{s/2})\right)' \Big|_{s=s_0} = \frac{(4\pi n)^{k-3/4} \Gamma(k-1/4)}{2(2k-2)!} \times \left(\frac{\Gamma'(k-1/4)}{\Gamma(k-1/4)} - \ln(4\pi n)\right).$$

- $\underline{\left(b_0(s)\left(c_0(s,4) + \frac{1-i}{2}\right) P_n(n,s)\right)' |_{s=s_0}}$

First, we will deal with each term, finding their first two coefficients of the Laurant expansions, around the point $s = s_0$ using the standard little-o notation.

$$\begin{aligned} b_0(s) &= \frac{L^4(2s+4k-3)}{L^4(2s+4k-2)} = \frac{(1-2^{3-2s-4k})}{(1-2^{2-2s-4k})} \frac{\zeta(2s+4k-3)}{\zeta(2s+4k-2)} \\ &= \frac{1-2^{-2u}}{1-2^{-1-2u}} \times \frac{\zeta(2u)}{\zeta(1+2u)}. \end{aligned}$$

We have already computed $c_0(s,4)$ earlier as $c_0(s,4) = \frac{1-i}{4^{u+1}-4}\cdot$ So,

$$c_0(s,4) + \frac{1-i}{2} = \frac{1-i}{4} \times \frac{2^{2u+1}-1}{2^{2u}-1}\cdot$$

Multiplying these two, we get

$$b_0(s)\left(c_0(s,4) + \frac{1-i}{2}\right) = \frac{1-i}{2} \times \frac{\zeta(2u)}{\zeta(1+2u)}\cdot$$

As $u \to 0$,

$$\begin{aligned} \zeta(2u) &= -\frac{1}{2}(1 + 2\ln(2\pi)u + o(u)), \quad (\text{using } \zeta(0) = -1/2,\ \zeta'(0) = -\ln(2\pi)/2) \\ \zeta(1+2u) &= \frac{1}{2u} + \gamma + o(1) = \frac{1}{2u}(1 + 2\gamma u + o(u)) \quad \left(\text{using } \lim_{s\to 1}\left(\zeta(s) - \frac{1}{s-1}\right) = \gamma\right), \end{aligned}$$

$$\text{and so}\ \ b_0(s)\left(c_0(s,4) + \frac{1-i}{2}\right) = \frac{-u(1-i)}{2}\,(1 + (2\ln(2\pi) - 2\gamma)u + o(u)).$$

Next, we use the calculations for $P_n(n,s)$ with $n = 1$ to get

$$P_n(n,s) = \frac{(1+i)(-1)^k\sqrt{2}\pi(4\pi n)^{k-3/4}}{(2k-2)!}\,\frac{(n\pi)^{u/2}\Gamma(u)\Gamma(k-\frac{1}{4}-\frac{u}{2})}{\Gamma(k+\frac{1}{4}+\frac{u}{2})\Gamma(\frac{3}{4}-k+\frac{u}{2})}.$$

As $u \to 0$, we have

$$\begin{aligned} &\frac{\Gamma(k-\frac{1}{4}-\frac{u}{2})}{\Gamma(k+\frac{1}{4}+\frac{u}{2})\Gamma(\frac{3}{4}-k+\frac{u}{2})} \\ =\ &\frac{\big(\Gamma(k-\frac{1}{4}) - \frac{1}{2}\Gamma'(k-\frac{1}{4})u + o(u)\big)}{\big(\Gamma(k+\frac{1}{4}) + \frac{1}{2}\Gamma'(k+\frac{1}{4})u + o(u)\big)\big(\Gamma(\frac{3}{4}-k) + \frac{1}{2}\Gamma'(\frac{3}{4}-k)u + o(u)\big)} \\ =\ &\frac{\Gamma(k-1/4)}{\Gamma(k+1/4)\Gamma(3/4-k)} \times \left(1 - \left(\frac{\Gamma'(k-1/4)}{\Gamma(k-1/4)} + \frac{\Gamma'(k+1/4)}{\Gamma(k+1/4)} + \frac{\Gamma'(3/4-k)}{\Gamma(3/4-k)}\right)\frac{u}{2} + o(u)\right). \end{aligned}$$

Gamma function has the property that $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)\cdot$

Taking logarithmic derivatives of both sides, we get

$$\frac{\Gamma'(1-x)}{\Gamma(1-x)} = \frac{\Gamma'(x)}{\Gamma(x)} + \pi\cot(\pi x).$$

In particular, letting $x = k + 1/4$, we have the two relations

$$\Gamma(k+1/4)\Gamma(3/4-k) = (-1)^k \pi\sqrt{2} \text{ and}$$

$$\frac{\Gamma'(3/4-k)}{\Gamma(3/4-k)} = \frac{\Gamma'(k+1/4)}{\Gamma(k+1/4)} + \pi\cdot$$

Also, $\Gamma(z)$ has the following expansion at 0:

$$\Gamma(u) = \frac{1}{u}\left(1 - \gamma u + o(u)\right)\cdot$$

Combining these, we get

$$\begin{aligned} P_n(n,s) &= \frac{(1+i)(-1)^k\sqrt{2}\pi(4\pi n)^{k-3/4}}{(2k-2)!}\frac{(n\pi)^{u/2}\Gamma(u)\Gamma(k-\frac{1}{4}-\frac{u}{2})}{\Gamma(k+\frac{1}{4}+\frac{u}{2})\Gamma(\frac{3}{4}-k+\frac{u}{2})} \\ &= \frac{1}{u}\times\frac{(1+i)(4\pi n)^{k-3/4}\Gamma(k-1/4)}{(2k-2)!} \\ &\quad\times\left(1+\left(\frac{\ln(\pi n)}{2}-\gamma-\frac{\Gamma'(k-1/4)}{2\Gamma(k-1/4)}-\frac{\Gamma'(k+1/4)}{\Gamma(k+1/4)}-\frac{\pi}{2}\right)u+o(u)\right). \end{aligned}$$

Finally, we add the pieces together and get

$$\begin{aligned} b_0(s)\left(c_0(s,4)+\frac{1-i}{2}\right)P_n(n,s) &= \frac{-(4\pi n)^{k-3/4}\Gamma(k-1/4)}{(2k-2)!} \\ \times\left(1+\left(2\ln(2\pi)-3\gamma+\frac{\ln(\pi n)}{2}-\frac{\Gamma'(k-1/4)}{2\Gamma(k-1/4)}-\frac{\Gamma'(k+1/4)}{\Gamma(k+1/4)}-\frac{\pi}{2}\right)u+o(u)\right)\cdot \end{aligned}$$

After simplifications, we end up with

$$\begin{aligned} \left(b_0(s)\left(c_0(s,4)+\frac{1-i}{2}\right)P_n(n,s)\right)'|_{s=s_0} &= \frac{(4\pi n)^{k-3/4}\Gamma(k-1/4)}{(2k-2)!} \\ \times\left(-2\ln(2\pi)+3\gamma-\frac{\ln(\pi n)}{2}+\frac{\Gamma'(k-1/4)}{2\Gamma(k-1/4)}+\frac{\Gamma'(k+1/4)}{\Gamma(k+1/4)}+\frac{\pi}{2}\right). \end{aligned}$$

Combining these two calculations gives

$$\mathcal{A}_{n,k} = \frac{\delta(\sqrt{n})}{2n^{1/4}}\left(\frac{\Gamma'(k-1/4)}{\Gamma(k-1/4)}+\frac{\Gamma'(k+1/4)}{\Gamma(k+1/4)}+\frac{\pi}{2}+3\gamma-3\ln(2\pi)-\ln(n)\right).$$

Now, starting with the identity $\Gamma(x)\Gamma(x+1/2) = \sqrt{2\pi}\ 2^{1/2-2x}\Gamma(2x)$, and taking logarithmic derivatives, we derive the relation:

$$\frac{\Gamma'(x)}{\Gamma(x)} + \frac{\Gamma'(x+1/2)}{\Gamma(x+1/2)} = \frac{2\Gamma'(2x)}{\Gamma(2x)} - 2\ln 2.$$

Using this identity with $x = k - 1/4$ above, we simplify $\mathcal{A}_{n,k}$ to

Proposition 4.6.

$$\mathcal{A}_{n,k} = \frac{\delta(\sqrt{n})}{n^{1/4}} \left(\frac{\Gamma'(2k-1/2)}{\Gamma(2k-1/2)} + \frac{\pi + 6\gamma - 6\ln\pi - 10\ln 2}{4} - \frac{\ln(n)}{2} \right).$$

By combining Propositions 4.4, 4.6, and Lemma 4.5 we arrive at our destination.

4.4 Main Theorem: The Exact Average Formula at the Center

Main Theorem. *For integers $k > 1$ and $n > 0$, we have the following formula at the center of symmetry $s = 2k - 1/2$:*

$$\sum_{f\in\mathcal{H}_{2k}} L\left(sym^2 f, 2k - \frac{1}{2}\right) \frac{e_k\ \lambda_n(f)}{< f, f >} = \mathcal{A}_{n,k} + \sum_{m\in\mathbb{Z},\, m^2\neq 4n} L\left(\frac{1}{2}, \chi_{4<m^2-4n>}\right) \xi_{n,k}(m).$$

$$\mathcal{A}_{n,k} = \frac{\delta(\sqrt{n})}{n^{1/4}}\left(\frac{\Gamma'(2k-1/2)}{\Gamma(2k-1/2)} + \mathcal{A} - \frac{\ln(n)}{2}\right), \text{ where } \mathcal{A} = \frac{\pi + 6\gamma - 6\ln\pi - 10\ln 2}{4}.$$

The weights $\xi_{n,k}(m)$ are positive for all but finitely many m. In fact, $\xi_{n,k}(m) > 0$, whenever $|m| > 2\sqrt{n}$. Explicitly,

$$\xi_{n,k}(m) = \beta(4n - m^2)\gamma(4n - m^2)\mathcal{F}_{n,k}(m).$$

For $n \neq 0$, we write $n = 2^d n_1$ with n_1 odd. Then

$$\beta(n) = \sum_{a,b>0,\ (ab)^2|n_1} \frac{\mu(a)}{\sqrt{a}}\left(\frac{-4<n>}{a}\right)$$

$$\gamma(n) = \begin{cases} d/2 & d \text{ even, } n_1 \equiv 1 \text{ (mod 4)},\\ d/2 + 2 - \sqrt{2} & d \text{ even, } n_1 \equiv 3 \text{ (mod 8)},\\ d/2 + 2 + \sqrt{2} & d \text{ even, } n_1 \equiv 7 \text{ (mod 8)},\\ d/2 - 1/2 & d \text{ odd.} \end{cases}$$

Finally,

$$\mathcal{F}_{n,k}(m) = \begin{cases} \frac{\sqrt{2\pi}}{2n^{1/4}} \frac{(-1)^k\Gamma(k-1/4)}{\Gamma(k+1/4)} {}_2F_1\left(k - \frac{1}{4}, \frac{3}{4} - k; \frac{1}{2}; \frac{m^2}{4n}\right) & \text{if } m^2 < 4n,\\ \frac{1}{2n^{1/4}}\left(\frac{4n}{m^2}\right)^{k-1/4} I_k\left(\frac{m^2-4n}{m^2}\right) & \text{if } m^2 > 4n, \end{cases}$$

$$\text{where} \quad I_k(c) = \int_0^\infty \frac{u^{k-3/4}\, du}{(1+u)^{k+1/4}(1+cu)^{k-1/4}}.$$

Chapter 5

Asymptotic Behavior as $k \to \infty$.

From the main formula we have

$$\sum_{f \in \mathcal{H}_{2k}} L\left(sym^2 f, 2k - \frac{1}{2}\right) \frac{e_k \ \lambda_n(f)}{< f, f >} = \mathcal{A}_{n,k} + \mathcal{B}_{n,k}.$$

We will find the asymptotic of this formula as $k \to \infty$ by finding the asymptotic of the terms $\mathcal{A}_{n,k}$, $\mathcal{B}_{n,k}$.

5.1 Asymptotic of $\mathcal{A}_{n,k}$

Recall that

$$\mathcal{A}_{n,k} = \frac{\delta(\sqrt{n})}{n^{1/4}} \left(\frac{\Gamma'(2k - 1/2)}{\Gamma(2k - 1/2)} + \mathcal{A} - \frac{\ln(n)}{2} \right).$$

By taking logarithmic derivatives of the identity $\Gamma(x+1) = x\Gamma(x)$, we get

$$\frac{\Gamma'(x+1)}{\Gamma(x+1)} = \frac{1}{x} + \frac{\Gamma'(x)}{\Gamma(x)}.$$

It follows that $\frac{\Gamma'(2k-1/2)}{\Gamma(2k-1/2)} \approx \ln k$, and we conclude

Lemma 5.1. *As $k \to \infty$, we have*

$$\mathcal{A}_{n,k} \approx \frac{\delta(\sqrt{n})}{n^{1/4}} \ln k.$$

5.2 Asymptotic of $\mathcal{B}_{n,k}$

We will first bound the $\mathcal{F}_{n,k}(m)$ terms appearing in $\mathcal{B}_{n,k}$. Recall that

$$\mathcal{B}_{n,k} = \sum_{m\in\mathbb{Z},\, m^2\neq 4n} L\left(1/2, \chi_{4<m^2-4n>}\right) \beta(4n-m^2)\gamma(4n-m^2)\mathcal{F}_{n,k}(m)$$

Let $m_0 > 0$ be the smallest integer such that $m_0^2 > 4n$.

- **Case** 1: $m^2 < m_0^2$**.** In this case, $m^2 < 4n$ as well, and we have

$$\mathcal{F}_{n,k}(m) = \frac{\sqrt{2\pi}}{2n^{1/4}} \frac{(-1)^k\Gamma(k-1/4)}{\Gamma(k+1/4)} {}_2F_1\left(k-\frac{1}{4}, \frac{3}{4}-k; \frac{1}{2}; \frac{m^2}{4n}\right).$$

From Stirling's formula, we have

$$\frac{\Gamma(k-1/4)}{\Gamma(k+1/4)} = O\left(\frac{1}{\sqrt{k}}\right).$$

It remains to bound the Gauss hypergeometric function ${}_2F_1$, which is defined as

$$F(a,b;c;z) = {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},$$

where $(r)_n = r(r+1)...(r+n-1)$.

The series converge when $|z| < 1$, assuming that c is not a non-positive integer.

For $0 < x < 1$, we can write $F\left(k-\frac{1}{4}, \frac{3}{4}-k; \frac{1}{2}; x\right)$ in terms of the Legendre functions P_ν^μ (formula 15.4.23 in Handbook of Mathematical Functions [6]).

$$\begin{aligned} F\left(a,b;\frac{1}{2};x\right) &= \pi^{-\frac{1}{2}}2^{a+b-\frac{3}{2}}\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(b+\frac{1}{2}\right)(1-x)^{\frac{1}{2}(\frac{1}{2}-a-b)} \\ &\quad \times\left(P_{a-b-\frac{1}{2}}^{\frac{1}{2}-a-b}(\sqrt{x}) + P_{a-b-\frac{1}{2}}^{\frac{1}{2}-a-b}(-\sqrt{x})\right). \end{aligned}$$

Letting $a = k - 1/4$, $b = 3/4 - k$ gives

$$F\left(k-\frac{1}{4}, \frac{3}{4}-k; \frac{1}{2}; x\right) = \frac{\Gamma\left(k+1/4\right)\Gamma\left(5/4-k\right)}{2\sqrt{\pi}}\left(P_{2k-3/2}^0(\sqrt{x}) + P_{2k-3/2}^0(-\sqrt{x})\right).$$

Using the relation

$$\Gamma(5/4-k)\Gamma(k-1/4)=\frac{\pi}{\sin(\pi k-\pi/4)}=-\sqrt{2}\pi(-1)^k,$$

we get

$$F\left(k-\frac{1}{4},\frac{3}{4}-k;\frac{1}{2};x\right)=\frac{(-1)^{k+1}\sqrt{\pi}}{\sqrt{2}}\frac{\Gamma\left(k+1/4\right)}{\Gamma(k-1/4)}\left(P^0_{2k-3/2}(\sqrt{x})+P^0_{2k-3/2}(-\sqrt{x})\right)\cdot$$

By formula 8.10.7 of Handbook of Mathematical Functions[6], we have the following asymptotic for the Legendre function. As $\nu\to\infty$, we have

$$P^\mu_\nu(\cos\theta)=\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+3/2)}\left(\frac{\pi\sin\theta}{2}\right)^{-1/2}\cos\left(\left(\nu+\frac{1}{2}\right)\theta-\frac{\pi}{4}+\frac{\mu\pi}{2}\right)+O\left(\frac{1}{\nu}\right)\cdot$$

Letting $\nu=2k-3/2$, $\mu=0$, we get (as $k\to\infty$)

$$P^0_{2k-3/2}\left(\cos\theta\right)=\frac{\Gamma(2k-1/2)}{\Gamma(2k)}\left(\frac{\sqrt{3}\pi}{4}\right)^{-1/2}\cos\left((2k-1)\theta-\frac{\pi}{4}\right)+O\left(\frac{1}{k}\right)=O\left(\frac{1}{\sqrt{k}}\right)\cdot$$

Hence,

$$F\left(k-\frac{1}{4},\frac{3}{4}-k;\frac{1}{2};x\right)=O(1)\cdot$$

We deduce that

$$\mathcal{F}_{n,k}(m)=O\left(\frac{1}{\sqrt{k}}\right)\cdot$$

Since we have finitely many m such that $m^2<4n$, the L,β,γ terms are all bounded by some constant, depending on n but not k. Thus we proved that for any fixed n, letting $k\to\infty$, we have

$$\sum_{m\in\mathbb{Z},\,m^2<4n}L\left(1/2,\chi_{4<m^2-4n>}\right)\beta(4n-m^2)\gamma(4n-m^2)\mathcal{F}_{n,k}(m)=O\left(\frac{1}{\sqrt{k}}\right)\cdot$$

- **Case 2, $m^2>m_0^2$:** In this case, $m^2>4n$ as well, and we have

$$\mathcal{F}_{n,k}(m)=\frac{1}{2n^{1/4}}\left(\frac{4n}{m^2}\right)^{k-1/4}I_k\left(\frac{m^2-4n}{m^2}\right),$$

where

$$I_k(c) = \int_0^\infty \frac{u^{k-3/4}\,du}{(1+u)^{k+1/4}(1+cu)^{k-1/4}}.$$

Note that, for $m^2 > 4n$ we have the following inequalities:

$$\begin{aligned} 0 < \gamma(4n-m^2) &< 4(m^2-4n) < 4m^2 \\ |L(1/2, \chi_{4<m^2-4n>})| &< 4(m^2-4n) < 4m^2 \\ 0 < \beta(4n-m^2) &< (m^2-4n)^2 < m^4. \end{aligned}$$

On the other hand,

$$I_k\left(\frac{m^2-4n}{m^2}\right) = \int_0^\infty \frac{u^{k-3/4}\,du}{(1+u)^{k+1/4}(1+\frac{m^2-4n}{m^2}u)^{k-1/4}}$$
$$\leq \int_0^\infty \frac{du}{(1+u)(1+\frac{m^2-4n}{m^2}u)} \leq \frac{m^2}{m^2-4n}\int_0^\infty \frac{du}{(1+u)^2} = \frac{m^2}{m^2-4n} \leq m^2.$$

Then we get

$$\left|\sum_{m\in\mathbb{Z},\, m^2>m_0^2} L\left(1/2, \chi_{4<m^2-4n>}\right)\beta(4n-m^2)\gamma(4n-m^2)\mathcal{F}_{n,k}(m)\right|$$
$$<< 2^{2k}n^k \sum_{m>m_0} \frac{1}{m^{2k-21/2}} < 2^{2k}n^k \int_{m_0}^\infty \frac{dx}{x^{2k-21/2}}$$
$$= \frac{(4n)^k}{(2k-23/2)m_0^{2k-23/2}} << \frac{1}{k}\left(\frac{4n}{m_0^2}\right)^k.$$

- **Case** 3**,** $m^2 = m_0^2$**:**

$$\mathcal{F}_{n,k}(m) = \frac{1}{2n^{1/4}}\left(\frac{4n}{m_0^2}\right)^{k-1/4} I_k\left(\frac{m_0^2-4n}{m_0^2}\right).$$

We have

$$\left|L\left(1/2, \chi_{4<m_0^2-4n>}\right)\beta(4n-m_0^2)\gamma(4n-m_0^2)\mathcal{F}_{n,k}(m_0)\right|$$
$$<< \mathcal{F}_{n,k}(m_0) < \frac{1}{2n^{1/4}}\left(\frac{4n}{m_0^2}\right)^{k-1/4} m_0^2 << \left(\frac{4n}{m_0^2}\right)^k.$$

Note that $\frac{4n}{m_0^2} < 1$, so the bounds in the last two cases decay exponentially with k. Thus, by adding the three bounds we got above, we have shown

Lemma 5.2. *For fixed n, as $k \to \infty$, we have $\mathcal{B}_{n,k} = O\left(\frac{1}{\sqrt{k}}\right)$.*

5.3 Asymptotic of the Main Formula

The Main Theorem has two immediate corollaries. When n is square, the dominant term is $\mathcal{A}_{n,k}$, and we deduce

Corollary 1. *For a fixed square integer n, letting $k \to \infty$,*

$$\sum_{f\in\mathcal{H}_{2k}} L\left(sym^2 f, 2k-\frac{1}{2}\right)\frac{e_k\ \lambda_n(f)}{< f,f >} = \frac{1}{n^{1/4}}\left(\frac{\Gamma'(2k-1/2)}{\Gamma(2k-1/2)} + \mathcal{A} - \frac{\ln(n)}{2}\right) + O\left(\frac{1}{\sqrt{k}}\right).$$

In particular, when $n = 1$, we have

$$\sum_{f\in\mathcal{H}_{2k}} L\left(sym^2 f, 2k-\frac{1}{2}\right)\frac{e_k}{< f,f >} = \frac{\Gamma'(2k-1/2)}{\Gamma(2k-1/2)} + \mathcal{A} + O\left(\frac{1}{\sqrt{k}}\right).$$

Assuming that $L(sym^2 f, 2k-\frac{1}{2}) \geq 0$ for all f, we get

$$\sum_{f\in\mathcal{H}_{2k}} L\left(sym^2 f, 2k-\frac{1}{2}\right) << k(\ln k)^4.$$

In the last statement, we used the following bound for $< f,f >$ from [3], (2.3):

$$< f,f > << (4\pi)^{-2k}(2k-1)!(\ln k)^3.$$

In particular, each L-value, $L\left(sym^2 f, 2k-\frac{1}{2}\right)$, is bounded by $k^{1+\epsilon}$; whereas the convexity bound is $\sqrt{k}$.

On the other hand, when n is not a square, $\mathcal{A}_{n,k}$ vanishes because $\delta(\sqrt{n}) = 0$.

Corollary 2. *For a fixed non-square integer n, we have*

$$\sum_{f\in\mathcal{H}_{2k}} L\left(sym^2 f, 2k-\frac{1}{2}\right)\frac{e_k\ \lambda_n(f)}{< f,f >} = \sum_{m\in\mathbb{Z}} L\left(\frac{1}{2}, \chi_{4<m^2-4n>}\right)\xi_{n,k}(m) = O\left(\frac{1}{\sqrt{k}}\right).$$

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