Regularization of the Amended Potential Around a Symmetric Configuration

Thesis by
Antonio Hernández Garduño

In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California

2002

(Defended October 4, 2001)

For Ivett

Acknowledgements

I want to express my gratitude to all the friends that I met at Caltech, some of whom "escaped" long before I did. In particular I would like to mention Chao Ku, Themis Mitsis, Doug Zare, Anthony Blaom, Muruhan Rathinam, Anil Hirani, Dong Eui and my officemates Jennifer, Carlos and David. Special thanks to my friend Juan Pablo Ortega from UCSC and a word of appreciation to my friend Matt Perlmutter from UCB. I thank my advisor Jerry Marsden for his academic and personal support. This thesis was finished due to the insistence of my friend Ivett A. Leyva and the supporting words of my parents and siblings. I also wish to acknowledge the DGAPA of the National University of Mexico from whom I received a partial fellowship.

Regularization of the Amended Potential Around a Symmetric Configuration

by

Antonio Hernández Garduño

In Partial Fulfillment of the
Requirements for the Degree of
Doctor of Philosophy

Abstract

Relative equilibria are periodic trajectories that, in a dynamical system with continuous symmetry, correspond to fixed points in the projected dynamics to the quotient space. In Hamiltonian systems with symmetry, it is of interest to understand the structure of relative equilibria near symmetric states. In this context, we give a method that in some cases of simple mechanical systems with compact symmetry group gives information about the relative equilibria bifurcating from a set of relative equilibria with isotropy subgroup isomorphic to S^1. This method is based on the blowing-up of the amended potential.

Contents

Chapter 1

Introduction

1.1 Generalities on Hamiltonian G-systems

In this section we briefly review some facts from Hamiltonian and Lagrangian mechanics with an emphasis on mechanical systems with symmetry. One of the key objectives of this review is to establish notation and to recall some standard results. We will start with a review of general Hamiltonian systems, quickly moving on to Hamiltonian systems on cotangent bundles and their alternative Lagrangian formulation on tangent bundles. Then we will specialize these notions to the case of simple mechanical systems. Some basics from the theory of Lie groups will also be introduced along the way.

Our presentation here follows closely the exposition found in [1, part II] and [24]. We refer the reader to these works for details.

1.1.1 Hamiltonian and Lagrangian systems

Let M be a finite dimensional manifold and ω a two-form on M. We say that (M, ω) is a **symplectic manifold** if ω is a nondegenerate closed form. That is to say, $i)$ $d\omega = 0$ and $ii)$ if $m \in M$, $v \in T_m M$ and $\omega(v, w) = 0$ for all $w \in T_m M$ then $v = 0$. A smooth function $H : M \longrightarrow \mathbb{R}$ (the **Hamiltonian**) determines a smooth vector field X_H on M defined at each $m \in M$ through the condition

$$\omega(X_H, v) = dH \cdot v \quad (v \in T_m M) \ . \tag{1.1}$$

Existence and uniqueness of X_H follows from the fact that ω is nondegenerate.

The triple (M, ω, H) is called a **Hamiltonian system**. The flow of a Hamiltonian system is given by Hamilton's equations, namely,

$$\dot{z} = X_H(z) \ .$$

From (1.1) it follows that $dH \cdot X_H = 0$ so that the Hamiltonian is invariant with respect to the flow that it induces. It can also be checked that a Hamiltonian flow $\Phi_t^{(H)}$ is a **canonical transformation**, that is to say, it preserves the symplectic form: $\omega = (\Phi_t^{(H)})^* \omega$ for all t.

If Q is a smooth manifold, then T^*Q, the cotangent bundle over Q, is endowed with a **canonical symplectic form** given by $\omega = \sum dq^i \wedge dp_i$, where the $\{q^i, p_i\}$ are the local canonical coordinates in T^*Q induced by some local coordinates $\{q^i\}$ on Q. From this definition it is clear that ω is both locally constant (and thus $d\omega = 0$) and non-degenerate. One checks that ω is independent of the choice of coordinates on Q, so it is intrinsic.

The dynamical features of a mechanical system can also be described in terms of *Lagrangian mechanics*. We now turn to a brief review of this topic and how it relates with the Hamiltonian formulation.

A Lagrangian system consists of a manifold Q and a smooth function (the **Lagrangian**) $L : TQ \longrightarrow \mathbb{R}$. As before, here we will only be concerned with the case when Q is finite dimensional. The trajectories of a Lagrangian system are the solutions to the **Euler-Lagrange equations** which in local coordinates take the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0 \ .$$

These equations arise from the application of *Hamilton's principle of critical action.* (Cf. [24, chap. 7].)

The **fiber derivative** $\mathbb{F}L$ is the map from TQ to T^*Q defined by

$$\langle \mathbb{F}L(v_q), w_q \rangle = \left.\frac{d}{dt}\right|_{t=0} L(v_q + t\,w_q)\,, \tag{1.2}$$

where $v_q, w_q \in T_qQ$. In coordinates, $\mathbb{F}L(q^i, \dot{q}^i) = (q^i, \partial L/\partial \dot{q}^i)$, or in other words, $p_i = \partial L/\partial q^i$.

If $\mathbb{F}L : TQ \longrightarrow T^*Q$ is a diffeomorphism, then we define a Hamiltonian on T^*Q as follows. Let $H := E \circ (\mathbb{F}L)^{-1}$, where $E(v) := \mathbb{F}L(v) \cdot v - L(v)$ is the **energy** of the Lagrangian L. This procedure of obtaining the Hamiltonian from a given Lagrangian is called **Legendre transform**. Of course, the Lagrangian can also be obtained from the Hamiltonian via $L(v) = \mathbb{F}L(v) \cdot v - H\left(\mathbb{F}L(v)\right)$.

The trajectories in Q obtained by base-point projection of the trajectories of the flow of the Hamiltonian vector field X_H on T^*Q correspond to the solution trajectories of the Euler-Lagrange equations introduced above. This establishes the relation between the Hamiltonian and the Lagrangian description of the dynamics in a mechanical system.

1.1.2 Lie groups

In preparation to the review of *Hamiltonian systems with symmetry* offered below, we now briefly review some basic concepts on Lie group theory. Here we follow [24, chap. 9] and [12], where the reader can look for details.

A **Lie group** is a group that is also a smooth manifold such that group multiplication is a smooth operation. We will only consider finite dimensional Lie groups. Group multiplication is variously denoted as $g_1 g_2 = L_{g_1}(g_2) = R_{g_2}(g_1)$.

A **left action** of a group G on a set M is a map $G \times M \longrightarrow M$, usually denoted $(g, m) \mapsto g \cdot m$, such that for all $g_1, g_2 \in G$ and $m \in M$, a) $e \cdot m = m$ and b) $g_1 \cdot (g_2 \cdot m) = (g_1 g_2) \cdot m$. Here e is the identity in G. Analogously, a **right action** is a map $G \times M \longrightarrow M$, usually denoted $(g, m) \mapsto m \cdot g$, such that a′)

$m \cdot e = m$ and b′) $(m \cdot g_1) \cdot g_2 = m \cdot (g_1 g_2)$. A left or right **action of a Lie group on a smooth manifold** is assumed to be a smooth operation.

The Lie algebra $\mathfrak{g}$ of a Lie group G is the tangent space at the identity. $\mathfrak{g}$ can be identified with the set of left, or right, -invariant vector fields on G via $\xi \mapsto X_\xi$ with $X_\xi(g) := T_e L_g \cdot \xi$, or $X_\xi(g) := T_e R_g \cdot \xi$, respectively. The **Lie bracket** on $\mathfrak{g}$ is defined as $[\xi, \eta] := [X_\xi, X_\eta](e)$, where the latter bracket is the usual commutator of vector fields. The Lie bracket thus defined depends on whether one chooses X_ξ to be left or right invariant but the answer differs only by a sign. We will adopt the Lie algebra induced by the left-invariant vector field.

Given $\xi \in \mathfrak{g}$, let $\gamma_\xi : \mathbb{R} \longrightarrow G$ be the unique integral curve of $\dot{\gamma}(t) = X_\xi(\gamma(t))$ starting at $\gamma_\xi(0) = e$, where X_ξ is the left-invariant vector field on G. The **exponential map** $\exp : \mathfrak{g} \longrightarrow G$ is defined as $\exp(\xi) := \gamma_\xi(1)$. One checks that $\exp(s\xi) = \gamma_\xi(s)$ and that $\exp((s+t)\xi) = \exp(s\xi)\exp(t\xi)$, where $s, t \in \mathbb{R}$. From the definition it follows that

$$\left.\frac{d}{dt}\right|_{t=0} \exp(t\xi) = \xi \, .$$

One checks that the definition of exp does not depend on whether X_ξ is chosen to be the left or right invariant vector field.

If G is a Lie group acting on a smooth manifold M then the **infinitesimal generator** at $x \in M$ is defined as

$$\xi_M(x) := \left.\frac{d}{dt}\right|_{t=0} (\exp(t\xi) \cdot x) \, .$$

In this way, every $\xi \in \mathfrak{g}$ induces a vector field ξ_M on M.

The **adjoint action** is the (linear) left action of G on its Lie algebra $\mathfrak{g}$ denoted as $g \cdot \xi = \mathrm{Ad}_g(\xi)$ and defined by

$$\mathrm{Ad}_g(\xi) := \left.\frac{d}{dt}\right|_{t=0} g \exp(t\xi) g^{-1} \, ,$$

where $g \in G$ and $\xi \in \mathfrak{g}$. If G is abelian, for example, then the adjoint action is trivial.

The **coadjoint action** Ad_g^* of G on $\mathfrak{g}^*$ is the (linear) right action denoted $\mu \cdot g$ and defined by

$$\langle \mu \cdot g, \xi \rangle = \langle \mathrm{Ad}_g^*(\mu), \xi \rangle = \langle \mu, Ad_g(\xi) \rangle \ ,$$

where $\mu \in \mathfrak{g}^*$, $\xi \in \mathfrak{g}$ and $\langle\, , \rangle$ denotes the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. There is also a (linear) left action of G on $\mathfrak{g}^*$ defined by $g \cdot \mu := \mathrm{Ad}_{g^{-1}}^*(\mu)$.

Let $\Phi : G \times M \longrightarrow M$ be an action of the Lie group G on the manifold M. Then Φ induces actions on TQ and T^*Q as follows. Let $g \in G$, $x \in M$ and $v_x \in T_xM$ be given. Let $\Phi_g(x) := \Phi(g, x)$. The **tangent lift** of Φ is defined by

$$g \cdot v_x := T_x\Phi_g \cdot v_x \ .$$

Let $p_{gx} \in T_{gx}^*M$ be given. The **cotangent lift** of Φ is defined by

$$\langle p_{gx} \cdot g, v_x \rangle := \langle p_{gx}, T_x\Phi_g \cdot v_x \rangle \ .$$

In other words, $p_{gx} \cdot g = (T_x\Phi_g)^* \cdot p_{gx}$ (where '$*$' denotes 'adjoint operator'). Notice that, while $T_x\Phi_g : T_xM \longrightarrow T_{gx}M$, we have that $(T_x\Phi_g)^* : T_{gx}^*M \longrightarrow T_x^*M$.

Let G be a group acting on a set M and let $x \in M$. The **orbit** of x is the set $\{y \in M \mid y = g \cdot x, \text{ for some } g \in G\} \subset M$. The **isotropy group** at x is the subgroup of G given by

$$G_x := \{g \in G \mid g \cdot x = x\} \subset G \ .$$

(G_x is sometimes called the **stabilizer** or **symmetry** subgroup at x.) We say that a point $x \in M$ is a **symmetric point** if the orbit of x is not trivial. Thus, x is a symmetric point iff $G_x \neq \{e\}$ (where e denotes the identity element in G).

Now, let G be a Lie group acting on a smooth manifold M. Then G_x is a closed subgroup and hence a Lie subgroup of G. The Lie algebra of G_x is denoted

$\mathfrak{g}_x$ and called the **isotropy** (or stabilizer or symmetry) **algebra** at x. One checks that

$$\mathfrak{g}_x = \{\xi \in \mathfrak{g} \mid \xi_M(x) = 0\} \ .$$

If H is a subgroup of G, we define the **set of fixed points** for H in M as

$$M^H := \{y \in M \mid h \cdot y = y, \text{ for all } h \in H\} \ .$$

If H is compact then each connected component of M^H is a closed submanifold of M.

Let $\Phi : G \times M \longrightarrow M$ be an action of a Lie group G on a smooth manifold M[1]. This action is said to be a **free action** if for every $x \in M$, $G_x = \{e\}$. It is said to be a **proper action** if the map $\tilde{\Phi} : G \times M \longrightarrow M \times M : (g, m) \mapsto (m, \Phi(g, m))$ is a proper map, i.e., the preimages of compact sets (under $\tilde{\Phi}$) are compact. Equivalently, Φ is a proper action if for every pair of sequences $\{g_i\} \subset G$, $\{m_i\} \subset M$, such that both $\{m_i\}$ and $\{\Phi(g_i, m_i)\}$ converge, we have that $\{g_i\}$ has a convergent subsequence.

It is easy to see that if the action of G on M is proper and $x \in M$ then G_x is compact. Also, it can be shown that if G acts on M freely and properly then M/G (the space of G-orbits) has the structure of a smooth manifold.

1.1.3 Hamiltonian systems with symmetry

Going back to Hamiltonian systems defined on a general symplectic manifold, we now consider the case of *Hamiltonian systems with symmetry.* In general this means that the Hamiltonian is invariant with respect to the action of some group. Here we will only be concerned with symmetries encoded by the action of a Lie group. In what follows, G will denote a Lie group and $\mathfrak{g}$ its Lie algebra.

Let (M, ω) be a symplectic manifold and let G be a Lie group acting on M. We say that the **symplectic form ω is G-invariant** (with respect to the tangent

[1]We assume that M and G are second countable.

lift of the G-action) if for any $m \in M$,

$$\omega(g \cdot v, g \cdot w) = \omega(v, w) ,$$

for all $v, w \in T_m M$.

Let $\mathcal{L}_X$ denote the Lie derivative along a vector field X. That is to say, if α is a differential form then

$$\mathcal{L}_X \alpha = \left.\frac{d}{dt}\right|_{t=0} \Phi_t^* \alpha ,$$

where Φ_t^* denotes the pull-back of the flow map of X at time t. Since the symplectic form ω is G-invariant we have that $\mathcal{L}_{\xi_M}\omega = 0$. From the fact that ω is closed and the identity $\mathcal{L}_{\xi_M}\omega = di_{\xi_M}\omega + i_{\xi_M}d\omega$ (Cartan's *magic formula*) we get that $di_{\xi_M}\omega = 0$. Poincar's lemma then says that locally $i_{\xi_M}\omega = dJ^\xi$ for some smooth real valued function J^ξ. In other words, the vector field ξ_M is locally Hamiltonian. If we can extend J^ξ globally, for every $\xi \in \mathfrak{g}$, in a way that is linear in ξ, then we can define a *momentum map* as follows.

1.1. Definition (Momentum Map). Let M be a symplectic manifold, G a Lie group acting symplectically on M and let $\{\xi^i\}$ be a basis of $\mathfrak{g}$, the Lie algebra of G. Assuming that for every i there exists a smooth real valued function J^{ξ^i} globally defined on M satisfying $i_{\xi^i_M}\omega = dJ^{\xi^i}$ we define $J : M \longrightarrow \mathfrak{g}^*$ by

$$\langle J(m), \xi \rangle = \sum_i a_i J^{\xi^i}(m) .$$

where $\xi = \sum_i a_i \xi^i$. We call J a **momentum map** for the action of G on M and say that this action is **globally Hamiltonian**.

1.2. Remark. If the symplectic manifold M is connected then any two momentum maps differ only by a constant.

1.3. Remark. This definition of momentum map generalizes the notion of angular momentum.

1.4. Definition. Let J be a momentum map for the action of G on M. We say that J is **equivariant** (or more precisely, Ad^***-equivariant**) if $J(g \cdot m) = g \cdot J(m) := \mathrm{Ad}^*_{g^{-1}} J(m)$ for all $m \in M$ and all $g \in G$.

1.5. Remark. A sufficient condition for Ad^*-equivariance of a momentum map associated with a symplectic G-action on a symplectic manifold with symplectic form ω is that there exists a globally defined G-invariant one form θ such that $\omega = d\theta$.

An important property of momentum maps is given by the following proposition.

1.6. Proposition (Noether's Theorem). *Let (M, ω, H) be a Hamiltonian system. Suppose that the Lie group G acts on M, that the action is globally Hamiltonian with momentum map J and that H is G-invariant. Then J is invariant with respect to the flow induced by H.*

Proof. (following [24, §11.4]) Since H is G-invariant, $dH \cdot \xi_M = 0$ for all $\xi \in \mathfrak{g}$. Therefore, $0 = dH \cdot \xi_M = \omega(\xi_M, X_H) = dJ^\xi \cdot X_H$, where the last equality follows from the definition of the momentum map. Therefore, the flow of X_H preserves J. □

A Hamiltonian system satisfying the hypothesis of the previous proposition plus equivariance of the momentum map provides a suitable general setting upon which the main notions in this thesis can be built. We say that:

1.7. Definition. A **Hamiltonian G-system** (M, ω, H, J, G) is a Hamiltonian system (M, ω, H) together with a Lie group G acting in a globally Hamiltonian fashion on the symplectic manifold M with a momentum map J.

In the literature, a Hamiltonian G-system is sometimes called **Hamiltonian system with symmetry** or **symplectic G-space**.

1.1.4 Simple mechanical systems with symmetry

We now specialize the notions introduced above to the case when M is the cotangent bundle of a Riemannian manifold.

Let G be a Lie group acting on a smooth manifold Q. Then the induced action of G on T^*Q given by the cotangent lift preserves the canonical symplectic form $\Sigma dq^i \wedge dp_i$. Moreover, this action has an associated equivariant momentum map $J : T^*Q \longrightarrow \mathfrak{g}^*$ given by

$$\langle J(p_q), \xi \rangle = \langle p_q, \xi_Q(q) \rangle \ , \tag{1.3}$$

where $p_q \in T_q^*Q$ and $\xi \in \mathfrak{g}$.

Given a smooth manifold Q, on which the Lie group G acts, and a Lagrangian $L : TQ \longrightarrow \mathbb{R}$ which is G-invariant with respect to the tangent lift of the action of G on Q, we define $J_L := J \circ \mathbb{F}L$, where J is the momentum map associated with the tangent lift of the action of G on Q and $\mathbb{F}L$ is the fiber derivative introduced above (see equation (1.2)). If J is equivariant then so is J_L.

If Q is a Riemannian manifold then its metric (g_{ij}) induces a metric on T^*Q defined by $\langle\langle p_i dq^i, \bar{p}_j dq^j \rangle\rangle := g^{ij} p_i \bar{p}_j$, where $g^{ij} g_{jk} = \delta^i_k$. Thus we have the following:

1.8. Definition. Let G be a Lie group acting on a Riemannian manifold $(Q, \langle\langle \, , \rangle\rangle)$, let the metric be G-invariant and let V be a smooth G-invariant real valued function defined on Q. A **simple mechanical G-system** $(Q, \langle\langle \, , \rangle\rangle, V, G)$ is the Hamiltonian G-system (T^*Q, ω, H, J, G) where $H(p_q) := \frac{1}{2}\|p_q\|^2 + V(q)$, ω is the canonical symplectic form and J is the momentum map introduced above in expression (1.3). The function V is called the **potential**.

A simple mechanical G-system has an associated *Lagrangian* which is by definition the map $L : TQ \longrightarrow \mathbb{R}$ given by $L(v_q) = \frac{1}{2}\|v_q\|^2 - V(q)$ (*kinetic* minus potential energy). It follows that, for simple mechanical G-systems, $\langle \mathbb{F}L(v_q), w_q \rangle = \langle\langle v_q, w_q \rangle\rangle$, or in coordinates, $\mathbb{F}L(v^i \, \partial/\partial q^i) = g_{ij} v^j dq^i$. (Here $\mathbb{F}L$ is the fiber derivative defined in (1.2).) One shows that the Legendre transform applied to this Lagrangian gives the Hamiltonian introduced in definition 1.8 (*kinetic* plus potential energy).

We now turn to a review of the *mechanical connection* and related notions. Here we follow closely the discussion found in [23, chap. 3].

For each $q \in Q$ that is not symmetric, the metric on Q induces an inner product on $\mathfrak{g}$ as follows.

1.9. Definition. Let Q be a Riemannian manifold on which the Lie group G acts. The **locked inertia tensor** is the map $\mathbb{I} : Q \longrightarrow L(\mathfrak{g}, \mathfrak{g}^*)$ given by

$$\langle \mathbb{I}(q)\xi, \eta \rangle = \langle\langle \xi_Q(q), \eta_Q(q) \rangle\rangle ,$$

where $q \in Q$ and $\xi, \eta \in \mathfrak{g}$.

For every $q \in Q$, $\xi \mapsto \langle \mathbb{I}(q)\xi, \xi \rangle$ is a positive quadratic form. If q is such that $\mathfrak{g}_q = \{0\}$ (in which case one says that the action at q is **locally free**) then $\langle \mathbb{I}(q)\cdot, \cdot \rangle$ is an inner product and the aforementioned quadratic form is positive definite.

1.10. Remark. In the case when G is the rotation group then the locked inertia tensor of definition 1.9 corresponds to the tensor of moments of inertia of the rigid body obtained by instantaneously locking the joints of the given mechanical system.

Let $\mathcal{A} : TQ \longrightarrow \mathfrak{g}$ be defined by

$$\mathcal{A}(v_q) := \mathbb{I}^{-1}(q) J_L(v_q) ,$$

where $v_q \in T_qQ$ and we recall that $J_L := J \circ \mathbb{F}L$ is the momentum map on TQ. $\mathcal{A}$ is called the **mechanical connection** and it generalizes the usual notion of angular velocity. It is possible to show that the mechanical connection is indeed a connection on the principal bundle $Q \longrightarrow Q/G$, that is to say, $\mathcal{A}$ is G-equivariant and $\mathcal{A}(\xi_Q(q)) = \xi$.

The **associated one-form** $\mathcal{A}_\mu$ to the mechanical connection is the one-form on Q defined by

$$\langle \mathcal{A}_\mu, v_q \rangle = \langle \mu, \mathcal{A}(v_q) \rangle .$$

It is easy to show that for every $q \in Q$, $J(\mathcal{A}_\mu(q)) = \mu$.

From G-invariance of the metric and the formula (cf. [24, lemma 9.3.7])

$$(\mathrm{Ad}_g\,\xi)_Q(q) = g\cdot\xi_Q(g^{-1}\cdot q)\;, \tag{1.4}$$

it is easy to show that for all $q\in Q$,

$$\mathbb{I}(g\cdot q) = \mathrm{Ad}^*_{g^{-1}}\circ\mathbb{I}(q)\circ\mathrm{Ad}_{g^{-1}}\;. \tag{1.5}$$

It follows that:

1.11. Proposition. *For all $q\in Q$,*

$$d\left\langle\mathbb{I}(\cdot)\xi,\eta\right\rangle(q)\cdot\zeta_Q(q) = \left\langle\mathbb{I}(q)[\xi,\zeta],\eta\right\rangle + \left\langle\mathbb{I}(q)\xi,[\eta,\zeta]\right\rangle\;.$$

For future use, we also give an infinitesimal version of equation 1.5. Multiplying both sides of 1.5 on the right by Ad_g we get that $\mathrm{Ad}^*_{g^{-1}}\,\mathbb{I}(q) = \mathbb{I}(g\cdot q)\,\mathrm{Ad}_g$. Differentiating with respect to g we obtain

$$-\,\mathrm{ad}^*_\xi\,\mathbb{I}(q) = [D\mathbb{I}(q)\cdot\xi_Q(q)] + \mathbb{I}(q)\circ\mathrm{ad}_\xi\;. \tag{1.6}$$

1.2 Relative equilibria

Generally speaking, the flow of a vector field that is equivariant with respect to the action of a group induces a flow in the quotient space. In this context, a *relative equilibrium* is simply an equilibrium in the quotient space. In Hamiltonian systems with symmetry this concept leads to the following:

1.12. Definition. Let (M,ω,H,J,G) be a Hamiltonian G-system. We say that $m_e\in M$ is a **relative equilibrium** if $X_H(m_e)\in T_{m_e}(G\cdot m_e)$.

The following proposition gives a useful characterization for relative equilibria in Hamiltonian G-systems. Here we follow [23, chap. 4].

1.13. Proposition (Characterization of Relative Equilibria). *Let $m_e \in M$, let $m_e(t)$ be the dynamic orbit of X_H with $m_e(0) = m_e$ and let $\mu = J(m_e)$. Then the following are equivalent:*

1. *m_e is a relative equilibrium.*

2. *There is a $\xi \in \mathfrak{g}$ such that $m_e(t) = \exp(t\xi) \cdot m_e$.*

3. *There is a $\xi \in \mathfrak{g}$ such that m_e is a critical point of the* ***augmented Hamiltonian***

$$H_\xi(m) := H(m) - \langle J(m) - \mu, \xi \rangle \ .$$

We say that $\xi \in \mathfrak{g}$ is the **group velocity** of a relative equilibrium m_e if $m_e(t) = \exp(t\xi) \cdot m_e$. (If $\mathfrak{g}_{m_e} \neq \{0\}$ then the group velocity is not uniquely defined.)

For simple mechanical G-systems, the criterion based on critical points of H_ξ (item 3. in the previous proposition) can be simplified in such a way that the search of relative equilibria reduces to the search of critical points of a real valued function defined on Q. Depending on whether one keeps track on the group velocity or the momentum of a relative equilibrium, this simplification yields either the *augmented* or the *amended* potential criterion, which we introduce in what follows.

Let $(Q, \langle\langle\, , \rangle\rangle, V, G)$ be a simple mechanical G-system. Given $\xi \in \mathfrak{g}$, the **augmented potential** $V_\xi : Q \longrightarrow \mathbb{R}$ is defined as

$$V_\xi(q) = V(q) - \frac{1}{2} \langle \mathbb{I}(q)\xi, \xi \rangle \ .$$

Given $\mu \in \mathfrak{g}^*$, the **amended potential** $V_\mu : Q \longrightarrow \mathbb{R}$ is defined as

$$V_\mu(q) = V(q) + \frac{1}{2} \langle \mu, \mathbb{I}^{-1}(q)\mu \rangle \ .$$

Note that the amended potential is defined at $q \in Q$ only when q is not a symmetric point ($\mathfrak{g}_q = \{0\}$).

1.14. Remark. One checks that the above definition of the amended potential is equivalent to $V_\mu(q) = H \circ \alpha_\mu(q)$, where H is the usual Hamiltonian of the form kinetic plus potential energy and α_μ is the associated one-form to the mechanical connection introduced above.

1.15. Proposition (Augmented potential criterion).
A point $z_e = (q_e, p_e) \in T^*Q$ is a relative equilibrium if and only if there exists a $\xi \in \mathfrak{g}$ such that

i) $p_e = \mathbb{F}L(\xi_Q(q_e))$ and

ii) q_e is a critical point of V_ξ.

1.16. Proposition (Amended potential criterion).
A point $z_e = (q_e, p_e) \in T^*Q$ is a relative equilibrium if and only if there exists a $\mu \in \mathfrak{g}^*$ such that

i) $p_e = \alpha_\mu(q_e)$ and

ii) q_e is a critical point of V_μ.

1.3 Some results from representation theory

Here we review some basic facts from the representation theory of compact Lie groups. In this section we follow [5] and [12, chap. 3].

Let G be a group acting on a set M. If $x \in M$ then we define the **isotropy subgroup** (or **stabilizer**) of x as

$$G_x := \{g \in G \mid g \cdot x = x\} .$$

If V is a vector space on which G acts linearly, we say that a point $v \in V$ is **regular** (for the G-action) if there is not any G-orbit in V whose dimension is strictly greater than the dimension of the G-orbit through v. The set of regular points (denoted V_{reg}) is open and dense in V. In particular, $\mathfrak{g}_{\text{reg}}$ and $\mathfrak{g}^*_{\text{reg}}$, the set of regular points in $\mathfrak{g}$ and $\mathfrak{g}^*$ with respect to the adjoint and coadjoint action, is open and dense in $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively.

Let G be a compact, connected Lie group and let $\mathfrak{g}$ be its Lie algebra. A subspace $\mathfrak{t} \subset \mathfrak{g}$ is said to be a **subalgebra** if $[\xi, \eta] \in \mathfrak{t}$ for all $\xi, \eta \in \mathfrak{t}$; it is said to be an **abelian subalgebra** if $[\xi, \eta] = 0$ for all $\xi, \eta \in \mathfrak{t}$; and it is said to be a **maximal abelian subalgebra** if it is an abelian subalgebra not properly contained in some other abelian subalgebra.

A subgroup of a Lie group is said to be a **torus** if it is isomorphic to $S^1 \times \ldots \times S^1$. Every abelian subgroup of a compact, connected Lie group is isomorphic to a torus. A subgroup of a Lie group is said to be a **maximal torus** if it is a torus not properly contained in some other torus.

Every $\xi \in \mathfrak{g}$ belongs to at least one maximal abelian subalgebra and every $\xi \in \mathfrak{g} \cap \mathfrak{g}_{\text{reg}}$ belongs to exactly one such maximal abelian subalgebra. Every maximal abelian subalgebra is the Lie algebra of some maximal torus in G.

Let $\mathfrak{t}$ be the maximal abelian subalgebra corresponding to a maximal torus T. Then, for any $\xi \in \mathfrak{t} \cap \mathfrak{g}_{\text{reg}}$, we have that $G_\xi = T$. Then the subspace $[\mathfrak{g}, \mathfrak{t}] := \{[\xi, \eta] \mid \xi \in \mathfrak{g}, \eta \in \mathfrak{t}\}$ is the orthogonal complement to $\mathfrak{t}$ in $\mathfrak{g}$ with respect to any G-invariant inner product on $\mathfrak{g}$. Such an inner product (a *Killing form*) exists by compactness of G. Therefore we have that $\mathfrak{g} = \mathfrak{t} \oplus [\mathfrak{g}, \mathfrak{t}]$.

With the same setup as in the previous paragraph, let $\underline{\mathfrak{t}} := [\mathfrak{g}, \mathfrak{t}]^\circ$ (the annihilator of $[\mathfrak{g}, \mathfrak{t}]$). Then for every $\mu \in \underline{\mathfrak{t}} \cap \mathfrak{g}^*_{\text{reg}}$ we have that $G_\mu = T$. Since $\underline{\mathfrak{t}} \cap \mathfrak{g}^*_{\text{reg}}$ is dense in $\underline{\mathfrak{t}}$ it follows (from the continuity of the coadjoint action) that for every $\mu \in \underline{\mathfrak{t}}$, $T \subset G_\mu$.

1.4 Slices

We now introduce a standard construction that simplifies the study of a neighborhood of a symmetric point in a manifold upon which a Lie group acts properly.

1.17. Definition. Let G be a Lie group acting on a smooth manifold M. A **slice** at $x_0 \in M$ (for this action) is a smooth submanifold S of M containing x_0 such that:

i) $T_{x_0}M = \mathfrak{g} \cdot x_0 \oplus T_{x_0}S$ and $T_xM = \mathfrak{g} \cdot x + T_xS$ for all $x \in S$;

ii) S is G_{x_0}-invariant;

iii) if $x \in S$, $g \in G$ and $g \cdot x \in S$ then $g \in G_{x_0}$.

1.18. Proposition (Slice Theorem). *Let G be a Lie group acting properly on a smooth manifold M and let $x_0 \in M$. Then there exists a slice at x_0.*

Proof: See section 2.3 of [12]. □

Let X, Y be smooth manifolds and let H be a Lie group acting on the left on both X and Y. Let us denote the action on X by $x \mapsto x \cdot h^{-1}$, and the action on Y by $y \mapsto h \cdot y$, where $h \in H$, $x \in X$, $y \in Y$. Assume that the action on X is proper and free, so that the diagonal action of H on $X \times Y$,

$$(h, (x, y)) \mapsto (x \cdot h^{-1}, h \cdot y) ,$$

is also proper and free. Under these conditions, define the **twisted product** $X \times_H Y$ as

$$X \times_H Y := (X \times Y)/H ,$$

so that $X \times Y \longrightarrow X \times_H Y$ is a principal fiber bundle[2] with structure group H. Then the projection $X \times Y \longrightarrow X$ onto the first factor, and the requirement that the diagram

$$\begin{array}{ccc} X \times Y & \longrightarrow & X \times_H Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & X/H \end{array}$$

[2]A smooth fibration $\pi : M \longrightarrow B$ is said to be a smooth **principal fiber bundle with structure group** G if G acts freely on M, B is equal to the orbit space M/G with $\pi : M \longrightarrow B$ its corresponding natural projection, and every local trivialization is G-equivariant, in the sense that if $\tau : x \mapsto (\pi(x), \varphi(x)) : \pi^{-1}(U) \longrightarrow U \times G$ is a local trivialization then $\varphi(g \cdot x) = g\varphi(x)$ $(g \in G)$.

commutes, induces a mapping $X \times_H Y \longrightarrow X/H$ that can be shown to be a smooth fiber bundle over X/H with fiber equal to Y. We call this the **associated fiber bundle.** (Cf. the "associated bundle construction" of [7]. See also section 2.4 of [12].)

A special case of this construction occurs when $X = G$ is a Lie group and H is a Lie subgroup of G acting on Y. Then G acts on $G \times_H Y$ via $g \cdot [g_1, y_1] := [g\, g_1, y_1]$, where $g, g_1 \in G$ and $y_1 \in Y$.

1.19. Proposition (Tube theorem). *Let G be a Lie group acting properly on a smooth manifold M and let $x_0 \in M$. Then there exists a G-invariant open neighborhood U of x_0 in M and a G_{x_0}-invariant open neighborhood B of 0 in $T_{x_0}M/(\mathfrak{g} \cdot x_0)$, such that U is G-equivariantly diffeomorphic to $G \times_{G_{x_0}} B$.*

Proof: See section 2.4 of [12]. □

1.20. Remark. G_{x_0} acts linearly on $T_{x_0}M/(\mathfrak{g} \cdot x_0)$ via the tangent action $h \cdot [v] = [T_{x_0}\Phi_h \cdot v]$. Here $\Phi_h(x) := h \cdot x$, $v \in T_{x_0}M$ and $[\ ]$ denotes equivalence class modulo $\mathfrak{g} \cdot x_0$.

1.21. Remark. If M is a Riemannian manifold then B can be chosen to be a G_{x_0}-invariant neighborhood of zero in $(\mathfrak{g} \cdot x_0)^{\perp}$, the orthogonal complement to $\mathfrak{g} \cdot x_0$ in $T_{x_0}M$. In this case $U = G \cdot \exp_{x_0}(B)$.

1.22. Remark. It is easy to check that, if H is a Lie subgroup of G acting on a manifold B, then the map

$$[g, b] \mapsto [b] : (G \times_H B)/G \longrightarrow B/H$$

is a homeomorphism. It then follows from proposition 1.19 that, given $x_0 \in M$, there exists U a G-invariant neighborhood of x_0 in M and B a G_{x_0}-invariant neighborhood of 0 in $T_{x_0}M/(\mathfrak{g} \cdot x_0)$ such that

$$U/G \sim (G \times_{G_{x_0}} B)/G \sim B/G_{x_0} \,,$$

where "$\sim$" denotes homeomorphism. In other words, *the space of G-orbits in*

$U \subset M$ is homeomorphic to the space of G_{x_0}- orbits in $B \subset T_{x_0}M/(\mathfrak{g} \cdot x_0)$.

1.5 Some basic facts

The following is an immediate consequence of the implicit function theorem.

1.23. Proposition. *Let X, Y be open sets in Banach spaces, containing the respective origins, and $I \subset \mathbb{R}$ an open interval containing zero. Let $f : X \times Y \times I \longrightarrow X$ be a smooth function. Suppose that $\forall y \in Y$, $\forall \tau \in I$ the equation $f(x, y, t) = 0$ has a unique solution for x and that the map $x(y, t)$ thus determined is smooth on $Y \times (I \setminus \{0\})$. Further, suppose that $D_x f(x, y, 0)$ is not singular for all $(x, y) \in X \times Y$. Then x extends to a smooth function on $Y \times I$.*

Proposition 1.23 follows readily from the standard implicit function theorem and compactness.

We now move on to review a standard tool in singularity theory. Recall that **Hadamard's lemma** says that if f is a smooth real valued function defined on some neighborhood U of 0 in $\mathbb{R}$ and $f(0) = 0$ then $f(x) = x\, g(x)$ for some smooth function $g(x)$ defined on U. (See, for example, [6] for a discussion of Hadamard's lemma and generalizations in the context of singularity theory.) We will use the generalization of Hadamard's lemma given in proposition 1.24 below.

The following is standard notation. Given E, F Banach spaces, $L^p_s(E, F)$ denotes the space of symmetric p-multilinear maps from E^p to F. If E_1, E_2, F are Banach spaces and $f : E_1 \times E_2 \longrightarrow F$ is of class C^k then $D^j_i f : E_1 \times E_2 \longrightarrow L^j_s(E_i, F)$ denotes the j-th partial derivative of f with respect to E_i ($i = 1, 2$, $0 \leq j \leq k$).

1.24. Proposition. *Let E_1, E_2, F be Banach spaces and $f : E_1 \times E_2 \longrightarrow F$ be of class C^k. Let n be an integer smaller than k and suppose that $D^i_1 f(0, y) = 0$ for all $0 \leq i \leq n$, $y \in E_2$. Then*

$$f(x, y) = g(x, y) \cdot x^{n+1} ,$$

for some function $g : E_1 \times E_2 \longrightarrow L^{n+1}_s(E_1, F)$ of class $C^{k-(n+1)}$.

Proof: From Taylor's theorem,

$$f(x,y) =$$
$$\frac{1}{(n+1)!}\left\{D_1^{n+1}f(0,y) + \int_0^1 \frac{(1-t)^n}{n!}\left[D_1^{n+1}f(tx,y) - D_1^{n+1}f(0,y)\right] dt\right\} \cdot x^{n+1} .$$

Since f is C^k it folows that the integral in the right-hand side is $C^{k-(n+1)}$. □

1.6 Persistence and bifurcation of relative equilibria: brief review of the literature

The search for periodic orbits is a major theme in the theory of dynamical systems. In the presence of symmetry, it is also natural to look for relative equilibria, that is to say, periodic orbits generated by the symmetry group which can be thought of equilibrium points in the quotient space. It is also natural to look for relative periodic orbits, which are periodic orbits in the quotient space. In what follows we attempt to give a brief summary of some of the literature related with the structure of the set of relative equilibria around a given equilibrium or relative equilibrium in the context of Hamiltonian systems.

In the context of Hamiltonian G-systems, the simplest situation that can occur is covered by the following proposition (cf. [3, appendix 2]):

1.25. Proposition. *Let (P,ω,H,G,J) be a Hamiltonian G-system with G acting properly on P. Let $p_e \in P$ be a relative equilibrium such that: a) the action of G is locally free at p_e; b) $\mu = J(p_e)$ is a regular point with respect to the coadjoint action; and p_e is a non-degenerate critical point of $H_\mu := H|_{P_\mu}$, where $P_\mu = J^{-1}(\mu)/G_\mu$. Then there is an open neighborhood $V \subset \mathfrak{g}^*$ around μ such that for every $\nu \in V$ there is a unique orbit of relative equilibria with momentum equal to ν.*

When we have the situation described in the previous proposition, we say that the relative equilibria **persist** to nearby momentum level sets.

Patrick ([38]) showed that in this situation (i.e., under the assumptions of proposition 1.25) the set of relative equilibria persisting from p_e form a smooth manifold. Even more, if certain assumption on the spectrum of the second variation of the augmented Hamiltonian at p_e holds (which can be interpreted as saying that the evolution in the group directions *does not resonate* with the evolution on the reduced space), then this set is a smooth $\dim G + \operatorname{rank} G$ dimensional symplectic submanifold of P.

Montaldi ([30]) extended proposition 1.25 to the case when one drops the requirement that μ_e be a regular element in $\mathfrak{g}^*$, giving a lower bound on the number of relative equilibrium orbits in nearby momentum level sets. This lower bound was given in terms of the cardinality of the Weyl group orbit of the corresponding momentum value.

The study of periodic orbits and relative equilibria around symmetric points in equivariant dynamical systems is of interest because it is usually in this context that some interesting bifurcation phenomena occur. In Hamiltonian G-systems, the structure of the singularities of the momentum map at symmetric points has been studied in [2]. With respect to relative equilibria persisting from symmetric points, Montaldi (cf. [30]) already gave an elementary point set topology argument to show that the conclusion of proposition 1.25 follows if one only assumes that the relative equilibria is a local extremum of the reduced Hamiltonian.

In a more general setting, Ortega and Ratiu ([34]) extended the persistence results of Patrick and Montaldi to cover the case of symmetric points when the action is proper. In this case the persistent surface of relative equilibria lies in a symplectic strata of $J^{-1}(\mu)/G_\mu$ (thought as a Poisson variety) corresponding to a fixed orbit type[3]. The use of a stratification point of view, however, does not seem to be the adequate one for the purpose of obtaining branches of relative equilibria that break the symmetry.

At this point, it is pertinent to give a note on terminology: given a relative equilibrium m_e in some equivariant dynamical system with symmetry group G, one

[3] An homologous result on the persistence of relative periodic orbits is also given in their paper, generalizing a result of Montaldi ([29]).

says that a branch of relative equilibria **bifurcates** from m_e if there is a connected set $\mathcal{R}_e$ of relative equilibria containing m_e such that the isotropy subgroup of points in $\mathcal{R}_e \setminus \{m_e\}$ is conjugate to a proper subgroup of G_{m_e}. In this case one usually says that the relative equilibria emanating from m_e **break the symmetry**. This is in contrast to the situation of the papers described above, where the isotropy subgroup of the relative equilibria emanating from m_e is conjugate to G_{m_e} (i.e., the points in $\mathcal{R}_e$ is of the same **orbit type** as m_e). In this case one says that the relative equilibria **persist**.

In the context of (non-Hamiltonian) equivariant dynamical systems, Krupa (cf. [19]) studied the problem of bifurcation of relative equilibria from symmetric ones, following a method that consists of the decomposition of the vector field in equivariant components, one in the direction along the group orbit and another in the (perpendicular) direction along a slice. The bifurcation analysis was then carried out by looking at the bifurcations associated to the flow induced on the slice.

In the case of Hamiltonian vector fields, the strategy followed by Krupa has been adapted to take advantage of the symplectic structure, something that is achieved by the use of Marle-Guillemin-Sternberg (**MSG**) normal form. This normal form provides again a local decomposition of the vector field in the group and slice directions, but this time the slice is further decomposed to reflect the symplectic structure and provide a convenient expression for the momentum map. The MSG normal form was introduced in [22], [16] and [17] for compact groups and generalized in [4] and [9] for proper group actions[4].

[4]The MSG normal form gives a local model for a G-invariant neighborhood around $m \in M$ of the form

$$G \times_{G_m} (\mathfrak{m}^* \times V_m)$$

where $\mathfrak{m}$ is a complement to $\mathrm{Lie}(G_m)$ in G_μ, $\mu = J(m)$, and V_m is the *symplectic normal space* at m given by

$$V_m = T_m(G \cdot m)^\omega / \left(T_m(G \cdot m)^\omega \cap T_m(G \cdot M)\right) .$$

Using the MSG normal form, Chossat and coauthors ([8]) have studied the structure of relative equilibria nearby symmetric orbits in the context of general Hamiltonian G-systems, where G is a Lie group acting properly. Their main idea is to carry out a systematic application of the implicit function theorem and the method of Lyapunov-Schmidt reduction[5] to obtain an equation that is equivalent to the relative equilibria condition $dH_\xi(p) = 0$ (the relative equilibria condition given by the augmented Hamiltonian; see proposition 1.13). This equivalent equation takes advantage of the decomposition given by the MSG normal form and also has some nice equivariance properties. The decomposition facilitates finding bifurcating branches of relative equilibria, from a symmetric relative equilibrium m_e, whose isotropy is a maximal subgroup of $G_{m_e} \cap G_x$. (Here ξ is the group velocity of m_e.)

Going back to the context of general periodic orbits in general Hamiltonian systems, we should mention the theorems of Weinstein and Moser given in [40] and [32]. These give a lower bound on the number of periodic orbits surrounding a stable equilibrium in nearby energy level sets. In their simplest version, these results say that if $m_e \in M$ is a stable equilibrium then there are at least $\frac{1}{2}\dim M$ periodic orbits in each nearby energy level set. Montaldi and coauthors ([31]) extended the theory to the case of equivariant Hamiltonian systems, obtaining a result on the existence of periodic orbits with certain prescribed symmetries.

This framework was extended by Ortega and Ratiu in [35] (see also [20] and [21]) to deal with relative equilibria instead of periodic orbits (and, in fact, recovering the Weinstein-Moser theorems as a particular case). The basic result of their paper provides a lower bound[6] on the number of relative equilibria with group velocity ξ in an energy level set surrounding a stable equilibrium m_e, when $d^2 J^\xi(m_e)$

The MSG also provides a simple expression for the equivariant momentum map, if the action of G on M admits one.

[5]For an exposition of the Lyapunov-Schmidt reduction technique in bifurcation theory, the reader can consult, e.g., [14].

[6]The lower bound of the number of relative equilibria in $h^{-1}(\epsilon)$ is expressed in terms of the Lusternik-Schnirelman category of $h^{-1}(\epsilon)/G_\xi$.

is non-degenerate. By considering the roots of det $\left(d^2(h - J^\xi)(m_e)\right) = 0$, their paper also covers the case of unstable relative equilibria. By applying the MSG normal form, it covers the case of relative equilibria surrounding genuine relative equilibria[7].

1.7 Summary of thesis

In this thesis we restrict our attention to simple mechanical G-systems, with G a compact Lie group, and consider the problem of finding the branches of relative equilibria surrounding a given set of relative equilibria of the form $\mathfrak{t}\cdot q_e$, where q_e is a critical point of the potential, $G_{q_e} \cong S^1$, and $\mathfrak{t} \subset \mathfrak{g}$ is a maximal abelian subalgebra containing $\mathfrak{g}_{q_e}$. Furthermore, we assume that G acts freely on a neighborhood around, but excluding, $G \cdot q_e$.

With these assumptions, each relative equilibrium $m_e \in \mathbb{F}L(\mathfrak{t} \cdot q_e)$ is a symmetric point in phase space and thus corresponds to the kind of relative equilibria considered in, for example, [8] and [35]. Taking advantage of the fact that in our case the symplectic manifold is a cotangent bundle and that the Hamiltonian is of the form kinetic plus potential energy, we can deal with the problem using the amended potential instead of the augmented Hamiltonian as in the mentioned papers.

In chapter 3 we give a method for predicting the existence of branches of symmetry breaking relative equilibria bifurcating from the given set of relative equilibria. That is to say, we obtain branches of relative equilibria with trivial continuous isotropy emanating from symmetric relative equilibria.

These branches can be thought of as surfaces in T^*Q/G parametrized by slice coordinates and momentum value. (Such surfaces have a singularity where they intersect the given set of symmetric relative equilibria.) This yields a method, based on counting non-degenerate critical points of a certain function, for finding

[7]These ideas and analytical tools, together with some Morse-theoretic machinery, have been applied by Ortega in [33] to study relative periodic orbits around an equilibrium or relative equilibrium m_e. In this case it is key to consider the resonant subspaces of $dh(m_e)$. See also [41].

a lower bound on the number of relative equilibria in nearby momentum level sets.

The main geometrical tool for our study is the slice construction described in §1.4. This is analogous with the strategy followed in [8] and other papers that use the MSG normal form. However, our slice decomposition is done at the level of the configuration space. This suffices because the underlying symplectic structure is already captured by the usage of the amended potential.

The idea behind our strategy is based on a simultaneous rescaling of the slice directions in configuration space and certain directions in the dual of the Lie algebra. This allows to regularize, or *blow-up*, the amended potential around q_e. This results in a decomposition of the amended potential criterion $dV_\mu(q) = 0$ into two equivalent conditions. Loosely speaking, this pair of conditions are obtained from the restriction of the blown-up version of the amended potential criterion to subspaces tangent to the group and slice directions (cf. propositions 3.22 and 3.23).

For an aternative approach to the study of the amended potential at symmetric configurations, see [18].

In this thesis we do not attempt to consider the problem of finding lower bounds on the number of relative equilibria in energy level sets. Therefore our results can probably be more easily compared with [8] than with the generalizations of the Weinstein-Moser theorem given in [35].

As it is the case with the papers mentioned above, the appearance of bifurcating branches of relative equilibria is related with the presence of a Lyapunov-Schmidt reduction in the analysis of the problem. In [8], the Lyapunov-Schmidt reduction takes place on the symplectic normal space inside the slice through the given relative equilibrium. Analogously, the rescaling of configuration coordinates and the associated Lyapunov-Schmidt reduction described in chapter 3 are done along the slice through q_e.

In chapter 2 we obtain the branches of relative equilibria emanating from a symmetric state in the double spherical pendulum. This illustrates the method of *blowing-up* the amended potential in a simple context. In chapter 3 we give a general framework in which the blowing-up of the amended potential can be used to study bifurcating branches of relative equilibria. In chapter 4 we apply the theory developed in chapter 3 to study the relative equilibria in the example of the *symmetric coupled rigid bodies*. Our conclusions appear in chapter 5, where we summarize the results obtained and future directions of research.

Chapter 2

Bifurcation of relative equilibria in the double spherical pendulum

The purpose of this chapter is to illustrate the basic idea underlying the general theory that we will present in the next chapter using the double spherical pendulum (DSP) as an example. This is a simple example that already exhibits some interesting branches of relative equilibria bifurcating from symmetric states.

The literature on the DSP is extensive. It is studied in detail in [23] (see also [26]), where the relative equilibria of the system are determined. Here we give an alternative derivation.

The strategy of this chapter is to blow-up the amended potential around the straight down configuration. This amounts to a simultaneous rescaling of the configuration coordinates and the momentum. The blowing-up approach that we discuss in this chapter extends ideas present in [10].

2.1 Description and preliminaries

The DSP consists of two point masses m_1, m_2 in three-dimensional space in the presence of a constant gravitational field pointing in the negative vertical direction. The mass m_1 is constrained to move on a sphere of radius l_1 around the origin and m_2 is constrained to move on a sphere of radius l_2 centered around m_1. See figure 2.1. Thus, the configuration space can be thought of as $Q := S^2 \times S^2$.

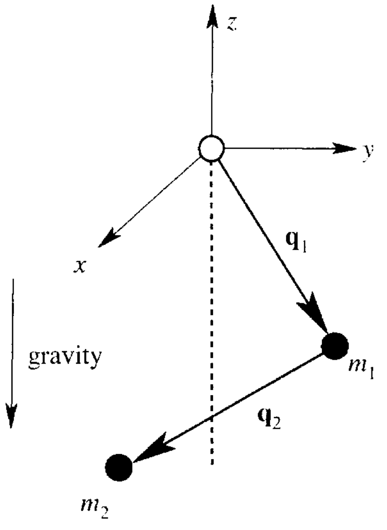

Figure 2.1: The double spherical pendulum.

The Lagrangian for the DSP is of the form kinetic minus potential energy and is given by

$$L(\mathbf{q}_1, \mathbf{q}_2; \dot{\mathbf{q}}_1, \dot{\mathbf{q}}_2) = \frac{1}{2}\left(m_1\|l_1\dot{\mathbf{q}}_1\|^2 + m_2\|l_1\dot{\mathbf{q}}_1 - l_2\dot{\mathbf{q}}_2\|^2\right) \\ - g\left(m_1 l_1 \mathbf{q}_1 + m_2(l_1\mathbf{q}_1 + l_2\mathbf{q}_2)\right)\cdot \mathbf{e}_3 \, .$$

where $(\mathbf{q}_1, \mathbf{q}_2) \in Q$, g is the gravitational constant and $\{\mathbf{e}_i\}_{i=1}^3$ is the standard orthonormal basis for $\mathbb{R}^3$.

Let S^1 act on Q through rotations around the vertical axis. It is easy to see that the Lagrangian is invariant with respect to the tangent lift of this group action and thus S^1 is the symmetry group of the system. Its Lie algebra is $\mathbb{R}$.

The infinitesimal generator for the S^1-action on Q is given by

$$\xi_Q(\mathbf{q}_1, \mathbf{q}_2) = (\xi\, l_1(-q_{1y}, q_{1x}, 0), \xi\, l_2(-q_{2y}, q_{2x}, 0))_{(\mathbf{q}_1, \mathbf{q}_2)} \, .$$

where $\mathbf{q}_i = (q_{ix}, q_{iy}, q_{iz})$ $(i = 1, 2)$ and $\xi \in \mathbb{R}$. Since the Lie algebra is one-dimensional, the locked inertia tensor (see definition 1.9) is in this case just a scalar and it is given by

$$\mathbb{I}(\mathbf{q}_1, \mathbf{q}_2) = \|(-q_{1y}, q_{1x}, 0), (-q_{2y}, q_{2x}, 0)\|_K^2$$
$$= m_1\|l_1\mathbf{q}_1^\perp\|^2 + m_2\|l_1\mathbf{q}_1^\perp + l_2\mathbf{q}_2^\perp\|^2$$

where $\| \ \|_K$ is the norm associated with the metric induced by the kinetic energy, which can be read of from the Lagrangian; '$\perp$' denotes projection onto the horizontal plane; and $\| \ \|$ denotes the usual norm in $\mathbb{R}^2 \cong \text{span}\{\mathbf{e}_1, \mathbf{e}_2\}$.

The momentum map is given by

$$J_L(\mathbf{q}_i, \dot{\mathbf{q}}_i) = \mathbf{e}_3 \cdot [m_1 l_1^2(\mathbf{q}_1 \times \dot{\mathbf{q}}_1) + m_2(l_1\mathbf{q}_1 + l_2\mathbf{q}_2) \times (l_1\dot{\mathbf{q}}_1 + l_2\dot{\mathbf{q}}_2)] \, .$$

2.2 Rescaling the amended potential

Now we want to study relative equilibria whose configuration is close to the two pendulae pointing downwards. The strategy for analyzing the relative equilibria around the other three symmetric states of the system (one arm pointing downwards and the other pointing upwards, etc.) is completely analogous.

We start by introducing polar coordinates $\{r_i, \theta_i\}$ defined by

$$\mathbf{q}_i = \frac{1}{l_i}\left(r_i\cos\theta_i, r_i\sin\theta_i, -\sqrt{l_i^2 - r_i^2}\right) \; , \qquad i = 1, 2 \, ,$$

with $0 \leq r_i \leq l_i$ and $\theta_i \in S^1$. The potential then takes the form

$$V = -g(m_1 l_1\mathbf{q}_1 + m_2(l_1\mathbf{q}_1 + l_2\mathbf{q}_2)) \cdot \mathbf{e}_3$$
$$= -m_1 g\sqrt{l_1^2 - r_1^2} - m_2 g\left(\sqrt{l_1^2 - r_1^2} + \sqrt{l_2^2 - r_2^2}\right)$$
$$= V_0 + \frac{(m_1 + m_2)g}{2l_1}r_1^2 + \frac{m_2 g}{2l_2}r_2^2 + \text{h.o.t.}$$

where $V_0 = -g(m_1 l_1 + m_2(l_1 + l_2))$ is the value of the potential at the straight down configuration. The locked inertia tensor becomes, with $\varphi := \theta_2 - \theta_1$,

$$\mathbb{I} = m_1 r_1^2 + m_2(r_1^2 + r_2^2 + 2r_1 r_2\cos\varphi) \, .$$

Therefore, the amended potential (as defined in section 1.2) is given by

$$V_\mu = V + \frac{\mu^2}{2m_1r_1^2 + m_2(r_1^2 + r_2^2 + 2r_1r_2\cos\varphi)} ,$$

which of course is not defined at $(r_1, r_2) = (0, 0)$.

We now introduce the following rescaling to blow-up the singular straight down configuration. Assuming that $\mu \geq 0$, let $\mu = \tau^2$, $r_1 = \tau s_1$, $r_2 = \tau s_2$. The variables s_1, s_2 for a fixed τ are assumed to be bounded away from zero as $\tau \to 0$. Then the amended potential takes the form

$$V_\mu = V_0 + \tau^2 W(\tau, s_1, s_2, \varphi) ,$$

where

$$W(\tau, s_1, s_2, \varphi) = \frac{1}{2}\left(\frac{g(m_1+m_2)}{l_1}s_1^2 + \frac{gm_2}{l_2}s_2^2 + \frac{1}{(m_1+m_2)s_1^2 + m_2s_2^2 + 2m_2s_1s_2\cos\varphi}\right) + O(\tau^2) .$$

Notice that W is smooth, even at $\tau = 0$.

It is clear that, for $\tau \neq 0$ fixed, the point $(\tau s_1, \tau s_2, \varphi)$ is a critical point of V_μ if and only if (s_1, s_2, φ) is a critical point of

$$W_\tau := W(\tau, \cdot) .$$

Also, if $(\tilde{s}_1, \tilde{s}_2, \tilde{\varphi})$ is a non-degenerate critical point of W_0 then, by the implicit function theorem, there are functions $s_1(\tau), s_2(\tau), \varphi(\tau)$ defined on some interval $[0, \varepsilon]$ such that, for each $\tau \in [0, \varepsilon]$, $\alpha(\tau) := (s_1(\tau), s_2(\tau), \varphi(\tau))$ is a critical point of W_τ. Therefore, to each non-degenerate critical point of W_0 we can associate a branch of relative equilibria, parametrized by τ, of the form

$$\mathcal{A}_{\tau^2}\left[\psi\left(\tau s_1(\tau), \tau s_2(\tau), \varphi(\tau)\right)\right] \in T^*Q ,$$

where $\mathcal{A}_\mu$ is the associated one-form to the mechanical connection defined in 1.1.4 and

$$\psi(r_1, r_2, \varphi) :=$$
$$\left(\frac{1}{l_1}\left(r_1, 0, -\sqrt{l_1^2 - r_1^2}\right), \frac{1}{l_2}\left(r_2 \cos\varphi, r_2 \sin\varphi, -\sqrt{l_2^2 - r_2^2}\right)\right) \in Q\ .$$

In other words, to each non-degenerate critical point of W_0 we can associate a symmetry-breaking branch of relative equilibria emanating from the straight-down configuration.

We can think of W_0 as the *blown-up* amended potential.

2.3 Critical points of the blown-up amended potential

We proceed now to obtain the critical points of W_0.

A computation shows that

$$\frac{\partial W_0}{\partial \varphi} = \frac{m_2 s_1 s_2 \sin\varphi}{(m_1 s_1^2 + 2m_2 s_1 s_2 \cos\varphi + m_2(s_1^2 + s_2^2))^2}$$

If we assume that $s_1 s_2 \neq 0$ then equating the right-hand side to zero gives $\varphi = 0$ or $\varphi = \pi$ which corresponds to $\mathbf{q}_1^\perp$ and $\mathbf{q}_2^\perp$ being colinear. By allowing s_1 and s_2 to have opposite signs, we only need to consider the case

$$\varphi = 0\ . \tag{2.1}$$

Furthermore,

$$\left.\frac{\partial W_0}{\partial s_1}\right|_{\varphi=0} = -\frac{m_1 + (1+\rho)m_2}{(m_1 + m_2(1+\rho)^2)^2 s_1^3} + \frac{g(m_1+m_2)s_1}{l_1}\ ,$$
$$\left.\frac{\partial W_0}{\partial s_2}\right|_{\varphi=0} = m_2\left(\frac{g\rho s_1}{l_2} - \frac{1+\rho}{(m_1 + m_2(1+\rho)^2)^2 s_1^3}\right)\ ,$$

where $\rho = s_2/s_1$. Equating the above expressions to zero gives

$$l\,\bar{m}(1+\rho) - \rho(\bar{m}+\rho) = 0 \tag{2.2}$$

and

$$s_1^4 = \frac{l_1}{m_2^2}\,\frac{l\,(1+\rho)}{g\,\rho(\bar{m}+\rho(2+\rho))^2} \tag{2.3}$$

where $\bar{m} := (m_1+m_2)/m_2$ and $l := l_2/l_1$. Equations 2.1, 2.2 and 2.3 give the critical points of W_0.

Since equation 2.3 is already explicit once we know ρ, it suffices to consider only equation 2.2. This is a quadratic equation thus giving two possible branches of relative equilibria. One verifies that the two roots $\rho_\pm$ of this equation lie in the ranges

$$\begin{aligned} -\bar{m} < \rho_- < -1 \qquad & \text{(cowboy)} \\ 0 < \rho_+ < \bar{m}\,l \qquad & \text{(stretched-out)} \end{aligned}$$

which correspond to the "cowboy" and "stretched-out" types of relative equilibria. These two types are illustrated in figure 2.2.

Also, one verifies that for physical values of the system parameters, that is to say, $\bar{m} > 1$, $l > 0$, the critical points of W_0 obtained above are always non-degenerate. Therefore we have obtained all the branches of relative equilibria emanating from the straight-down configuration for the double spherical pendulum.

2.1. Remark. A direct computation shows that there are no critical points of W_0 when $s_1 s_2 = 0$ but either $s_1 \neq 0$ or $s_2 \neq 0$.

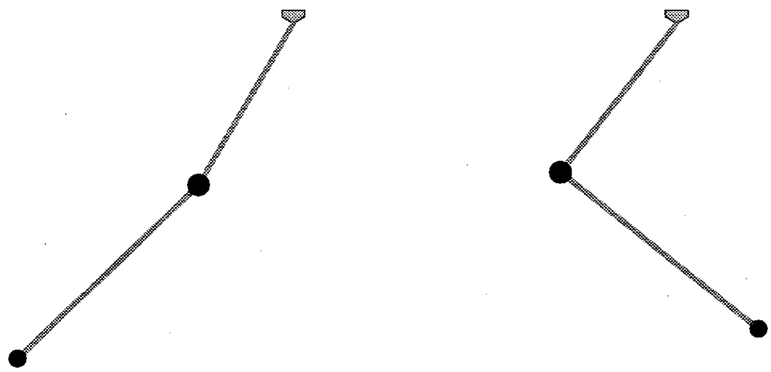

Figure 2.2: Relative equilibria configurations for the DSP with $\bar{m} = 3$, $l = 4/3$, $\rho = 1.9$ (stretched-out) and $\rho = -1.7$ (cowboy).

2.4 Stability Considerations

An equilibrium is said to be *stable* if trajectories that start near the equilibrium remain close to it for all future time[1]. We say that a relative equilibrium is S^1-stable if the corresponding class in the (Poisson) reduced space $(T^*Q)/S^1$ is stable. From the energy-momentum method of stability analysis (cf. [23, chap. 5]) we know that a relative equilibrium of the DSP is S^1-stable if $\delta^2 V_\mu$ is positive definite. It is clear that if $(\tilde{s}_1, \tilde{s}_2, 0)$ is a critical point of W_0 then the signature of $\delta^2 V_{\mu=\tau^2}[\psi(\tau\tilde{s}_1, \tau\tilde{s}_2, 0)]$ coincides with the signature of $\delta^2 W_0(\tilde{s}_1, \tilde{s}_2, 0)$, for small values of τ. Therefore, the signature of the second variation of the blown-up amended potential W_0 determines the stability type of the corresponding bifurcating branch of relative equilibria, nearby the straight-down configuration.

The second variation of the blown-up amended potential W_0 is computed to be as follows. Let $s_e = (s_1(\rho), \rho\, s_1(\rho), 0)$ with ρ equal to one of the two roots of eq. (2.2) and $s_1(\rho)$ given by eq. (2.3), so that s_e is one of the two critical points

[1]See [1, chap. 8] for a discussion on the appropriate notion of stability for Hamiltonian systems.

of W_0. A computation shows that

$$\delta^2 W_0(s_e) = \begin{pmatrix} a & b & 0 \\ b & c & 0 \\ 0 & 0 & c \end{pmatrix}$$

with

$$\begin{aligned} a &= \frac{m_2 g}{l_1} \frac{\left(\bar{m}\,\sigma + \rho\left(3\bar{m}^2 - \bar{m}(\rho-6)\rho + 4\rho^2\right)\right)}{\sigma}, \\ b &= \frac{m_2 g}{l_1} \frac{\rho\left(\bar{m}(3+4\rho) + \rho(2+3\rho)\right)}{\sigma}, \\ c &= \frac{m_2 g}{l_1} \frac{\bar{m} + \rho(6+9\rho+4\rho^2)}{\sigma}, \\ d &= \sqrt{\frac{g}{l_1}}\, l\, \frac{\rho\sqrt{\rho(1+\rho)}}{\sigma}, \end{aligned}$$

where $\sigma = l(1+\rho)(\bar{m} + \rho(2+\rho))$.

The signature of $\delta^2 W_0(s_e)$ is obtained by joining the sign of d with the signature of the block $\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$. If $ac - b^2 > 0$ then the block under consideration is definite; it is positive/negative definite according to $\operatorname{sign}(a+c)$. If $ac - b^2 < 0$ then the block has signature $\{+,-\}$.

Recall that ρ_+ and ρ_- are the two roots of eq. (2.2) (corresponding to the stretched-out and cowboy branches of relative equilibria, respectively). Numerical evidence shows that for all physical non-degenerate system parameters (i.e. $0 < l < \infty$ and $1 < \bar{m} < \infty$) we have the following:

1. $a + c > 0$ for both $\rho = \rho_\pm$;

2. $ac - b^2 \gtrless 0$ for $\rho = \rho_\pm$;

3. $d \gtrless 0$ for $\rho = \rho_\pm$.

It follows that nearby the straight-down configuration, the signature of $\delta^2 V_\mu$ is $\{+,+,+\}$ at relative equilibria in the stretched-out branch and is $\{+,-,-\}$ at relative equilibria in the cowboy branch.

We can then conclude that, nearby the straight-down configuration, the relative equilibria in the stretched-out branch are S^1-stable. (This analysis is inconclusive with respect to the stability nature of the cowboy branch.)

2.2. Remark. It is possible to obtain information about the stability nature of the cowboy branch (nearby the straight-down configuration) by studying the linearization of the dynamics in the blown-up variables. To this end one blows-up the Routhian[2] of the DSP, in a way similar to how we have blown-up the amended potential, and then one computes the eigenvalues of the linearization (at the *cowboy* critical point $s_e^- = (s_1(\rho_-), \rho_- s_1(\rho_-), 0))$ of the Euler-Lagrange equations corresponding to the blown-up Routhian.

[2]For each $\mu \in \mathfrak{g}^*$, the Routhian $R^\mu : TQ \longrightarrow \mathbb{R}$ is defined by $R^\mu(q,v) = L(q,v) - \langle \mathcal{A}_\mu(q), v \rangle$ where L is the Lagrangian and $\mathcal{A}_\mu$ is the one-form associated to the mechanical connection defined in section 1.1.4. Following the procedure of *Lagrangian reduction*, the reduced dynamics corresponding to $J = \mu$ can be expressed in terms of a (Lagrange-d'Alambert type of) variational principle associated to R^μ and the corresponding Euler-Lagrange equations. Cf. [26].

Chapter 3

Regularization of the amended potential around a symmetric group orbit

In this chapter we generalize certain features that we encountered in the double spherical pendulum. In that example the symmetric points were isolated. We now consider a more general setting in which this need not be the case, but which still includes the double spherical pendulum. This theory will be applied to the example of symmetric coupled rigid bodies (SCRB) in chapter 4.

In this general setting, instead of having an isolated symmetric point, we will consider the situation in which one has a symmetric group orbit in a deleted neighborhood of which the action is free. That is, we will assume that G acts freely in some G-invariant neighborhood of the G-orbit of a symmetric configuration q_e, excluding the orbit itself.

We will continue to restrict ourselves to the case when G is a compact Lie group and $G_{q_e} \cong S^1$.

In this setting, we are interested in the following problem. Suppose that $\mathcal{R}'_e$ is a subspace in $T_{q_e}Q$, with q_e being a symmetric point, and such that every $v_{q_e} \in \mathcal{R}'_e$ *i)* is a relative equilibrium and *ii)* is fixed by G_{q_e} (and hence $\mathcal{R}'_e$ intersects each G orbit only once). We want to *i)* give sufficient conditions that guarantee the existence of a "surface" of relative equilibria emanating from $\mathcal{R}_e = \pi_{TQ,G}(\mathcal{R}'_e)$, where $\pi_{TQ,G} : TQ \longrightarrow (TQ)/G$ is the canonical projection, and *ii)* give a criteria for enumerating distinct "surfaces" of relative equilibria emanating from $\mathcal{R}_e$.

3.1 Setting of the problem

Let $(Q, \langle\!\langle\, , \rangle\!\rangle, V, G)$ be a simple mechanical G-system, with G a compact Lie group with Lie algebra $\mathfrak{g}$. Let $q_e \in Q$ be a symmetric point with $H := G_{q_e} \cong S^1$ and $dV(q_e) = 0$. Suppose that G acts freely in some G-invariant neighborhood of the G-orbit of q_e excluding the orbit itself and that there is a maximal abelian subalgebra $\mathfrak{t} \subset \mathfrak{g}$ such that

1. every $v \in \mathfrak{t} \cdot q_e$ is a relative equilibrium,
2. $\mathfrak{g}_{q_e} \subset \mathfrak{t}$,
3. $\mathbb{I}(q_e) \cdot \mathfrak{t} \subset [\mathfrak{g}, \mathfrak{t}]^\circ$, where $\mathbb{I}$ is the locked inertia tensor.

3.1. Remark. Assumption 2 implies that every point in $\mathfrak{t} \cdot q_e$ is a fixed point of G_{q_e} and thus $G_{q_e} = G_{v_e}$ for all $v_e \in \mathfrak{t} \cdot q_e$.

3.2. Remark. Assumption 3 is generically redundant. Indeed, let us denote $\mathfrak{m} := \mathbb{I}(q_e) \cdot \mathfrak{t}$ and suppose that $\mathfrak{g}_{q_e} \cap \mathfrak{g}_{\mathrm{reg}} \neq \emptyset$. We claim that $\mathfrak{m} \cap \mathfrak{g}^*_{\mathrm{reg}} \subset [\mathfrak{g}, \mathfrak{t}]^\circ$. It follows that if $\mathfrak{m} \cap \mathfrak{g}^*_{\mathrm{reg}}$ is dense in $\mathfrak{m}$, then $\mathfrak{m} \subset [\mathfrak{g}, \mathfrak{t}]^\circ$.

To prove the claim, observe first that for all $\mu \in \mathfrak{m}$ and all $\xi \in \mathfrak{g}_{q_e}$ we have that $\xi \in \mathfrak{g}_\mu$. To see this, let $\eta \in \mathfrak{t}$ such that $\mu = \mathbb{I}(q_e)\eta$ and notice that, from equation (1.6),

$$-\operatorname{ad}^*_\xi \mu = -\operatorname{ad}^*_\xi \mathbb{I}(q_e)\eta = \mathbb{I}(q_e)[\xi, \eta] = 0 ,$$

since $\mathfrak{g}_{q_e} \subset \mathfrak{t}$. Thus $\xi \in \mathfrak{g}_\mu$ as claimed. Now, let $\xi \in \mathfrak{g}_{q_e} \cap \mathfrak{g}_{\mathrm{reg}}$. Then $\mathfrak{g}_\xi = \mathfrak{t}$. Also, for every $\mu \in \mathfrak{m} \cap \mathfrak{g}^*_{\mathrm{reg}}$, $\mathfrak{g}_\xi = \mathfrak{g}_\mu$ and thus $\mathfrak{g}_\mu = \mathfrak{t}$. It is easy to see that for all $\mu \in \mathfrak{g}^*$, $\mu \in [\mathfrak{g}, \mathfrak{g}_\mu]^\circ$. Therefore, $\mathfrak{m} \cap \mathfrak{g}^*_{\mathrm{reg}} \subset [\mathfrak{g}, \mathfrak{t}]^\circ$, as claimed.

3.2 Splitting the Lie algebra

We now introduce some notation and constructions that will be used throughout the rest of this chapter. We start by splitting the Lie algebra. For notational convenience, let $\mathfrak{k}_0 := \mathfrak{g}_{q_e} = \ker \mathbb{I}(q_e)$ and choose $\mathfrak{k}_1 \subset \mathfrak{g}$ a complementary subspace of $\mathfrak{k}_0$ in $\mathfrak{t}$. Let $\mathfrak{k}_2 = [\mathfrak{g}, \mathfrak{t}]$. Since $\mathfrak{t}$ is a maximal abelian subalgebra, from the discussion in section 1.3 it follows that $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{k}_2 = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \mathfrak{k}_2$.

The corresponding splitting in the dual of the Lie algebra is given by

$$\mathfrak{m}_i = (\mathfrak{k}_j \oplus \mathfrak{k}_k)^\circ$$

where (i, j, k) is a cyclic permutation of $(0, 1, 2)$. It follows that $\mathfrak{g}^* = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$.

3.3. Lemma. *For $i = 1, 2$, $\mathfrak{m}_i = \mathbb{I}(q_e)\mathfrak{k}_i$.*

Proof: Since $\mathfrak{k}_0 = \ker \mathbb{I}(q_e)$, it follows that $\langle \mathbb{I}(q_e)\mathfrak{g}, \mathfrak{k}_0 \rangle = \langle \mathbb{I}(q_e)\mathfrak{k}_0, \mathfrak{g} \rangle = \{0\}$, hence $\mathbb{I}(q_e)\mathfrak{g} \subset \mathfrak{k}_0^\circ$. Since $\dim \mathbb{I}(q_e)\mathfrak{g} = \dim \mathfrak{g} - \dim \mathfrak{k}_0 = \dim \mathfrak{k}_0^\circ$ we have that

$$\mathbb{I}(q_e)\mathfrak{g} = \mathfrak{k}_0^\circ \,. \tag{3.1}$$

In particular, $\mathbb{I}(q_e)\mathfrak{k}_1 \subset \mathfrak{k}_0^\circ$. From assumption 2 we have that $\mathbb{I}(q_e)\mathfrak{t} \subset \mathfrak{k}_2^\circ$. Since $\mathfrak{t} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$ then $\mathbb{I}(q_e)\mathfrak{k}_1 = \mathbb{I}(q_e)\mathfrak{t}$ and it follows that $\mathbb{I}(q_e)\mathfrak{k}_1 \subset \mathfrak{k}_2^\circ$. Therefore $\mathbb{I}(q_e)\mathfrak{k}_1 \subset \mathfrak{k}_0^\circ \cap \mathfrak{k}_2^\circ = (\mathfrak{k}_0 \oplus \mathfrak{k}_2)^\circ = \mathfrak{m}_1$. Since $\dim \mathfrak{k}_1 = \dim(\mathfrak{k}_0 \oplus \mathfrak{k}_2)^\circ$ we conclude that $\mathbb{I}(q_e)\mathfrak{k}_1 = \mathfrak{m}_1$.

From equation (3.1) we have in particular that $\mathbb{I}(q_e)\mathfrak{k}_2 \subset \mathfrak{k}_0^\circ$. Since $\mathbb{I}(q_e)\mathfrak{k}_1 \subset \mathfrak{k}_2^\circ$ and $\mathbb{I}(q_e)$ is symmetric we have that $\mathbb{I}(q_e)\mathfrak{k}_2 \subset \mathfrak{k}_1^\circ$. Therefore $\mathbb{I}(q_e)\mathfrak{k}_2 \subset \mathfrak{k}_0^\circ \cap \mathfrak{k}_1^\circ = (\mathfrak{k}_0 \oplus \mathfrak{k}_1)^\circ = \mathfrak{m}_2$. As above, a dimension count shows that the contention is indeed an equality. $\triangledown$

Finally, we define the following rescaling in $\mathfrak{g}^*$ to be used below. Let

$$\beta : \mathbb{R} \times J_L(\mathfrak{g} \cdot q_e) \longrightarrow \mathfrak{g}^* : (\tau, \mu) \mapsto \pi_1\mu + \tau\, \pi_2\mu + \tau^2 \nu_0 \tag{3.2}$$

where ν_0 is a generator of $\mathfrak{m}_0$ and $\pi_i : \mathfrak{g}^* \longrightarrow \mathfrak{m}_i$ is the projection induced by the splitting $\mathfrak{g}^* = \bigoplus_i \mathfrak{m}_i$. Notice that $J_L(\mathfrak{g} \cdot q_e) = \mathbb{I}(q_e)\mathfrak{g} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$.

3.3 Smoothness of the "angular velocity"

Let us now introduce some more notation that we will use throughout this chapter.

3.4. Definition. If U is a subset of some normed vector space and $r > 0$, let

$$(\mathbb{R}^\circ \times U)_r := \{(\tau, u) \in \mathbb{R} \times U \mid 0 < \|\tau u\| < r\} ,$$
$$(\mathbb{R} \times U)_r := \{(\tau, u) \in \mathbb{R} \times U \mid \|\tau u\| < r,\ u \neq \mathbf{0}\} .$$

Hence $(\mathbb{R} \times U)_r = (\mathbb{R}^\circ \times U)_r \cup (\{0\} \times (U \setminus \{\mathbf{0}\}))$.

We will also agree on the following **notational convention**: When we consider cartesian products of the form $(\mathbb{R}^\circ \times U)_r \times X$ (or $(\mathbb{R} \times U)_r \times X$), we will write the corresponding elements as (τ, u, x) instead of $((\tau, u), x)$.

3.5. Remark. The previous definition makes sense even if U is a subset of a set that is not a vector space, as long as scalar multiplication is defined on it.

We now consider the following Lie algebra-valued function associated to a simultaneous rescaling of both the configuration space and the dual of the Lie algebra.

To start, consider the set $N := (\mathfrak{g} \cdot q_e)^\perp$ and let $r_0 \in \mathbb{R}$ such that $\exp_{q_e}$ is a diffeomorphism when restricted to the open ball in N of radius r_0.

Let

$$\xi(\tau, v, \mu) := \mathbb{I}(\exp_{q_e}(\tau v))^{-1}\beta(\tau, \mu) ,$$

where $(\tau, v, \mu) \in (\mathbb{R}^\circ \times N)_{r_0} \times J_L(\mathfrak{g} \cdot q_e)$. Informally, we think of $\xi(\tau, v, \mu)$ as the "angular velocity" associated to a point in $Q \times \mathfrak{g}^*$ after rescaling.

In this section we will show that, although the definition of $\xi(\tau, v, \mu)$ only makes sense when $(\tau, v) \in (\mathbb{R}^\circ \times N)_{r_0}$, it can be extended to a smooth function on $(\mathbb{R} \times N)_{r_0}$. We start with the following lemmas.

Recall that G_{q_e} acts linearly on $T_{q_e}Q$. For $\xi \in \mathfrak{g}_{q_e}, v \in T_{q_e}Q$, let $\xi \cdot v$ denote the infinitesimal generator of ξ at v.

3.6. Lemma. *Let G be a Lie group acting on a Riemannian manifold Q such that the metric is G-invariant. Then, for $\xi \in \mathfrak{g}_{q_e}$, $v \in T_{q_e}Q$, $\tau \in \mathbb{R}$,*

$$\left\|\xi_Q\left(\exp_{q_e}(\tau v)\right)\right\|^2 = \|\xi \cdot v\|^2\tau^2 - \frac{1}{3}\left\langle\!\left\langle R(v, \xi \cdot v)v, \xi \cdot v\right\rangle\!\right\rangle \tau^4 + o(\tau^4)\ ,$$

where $\langle\!\langle R(v,w)v,w\rangle\!\rangle$ denotes the sectional curvature at q_e corresponding to the subspace spanned by (v,w).

Proof: Since the metric is G-invariant then, for $g \in G_{q_e}$, $\exp_{q_e}(g \cdot v) = g \cdot \exp_{q_e}(v)$. Differentiating with respect to g we see that if $\xi \in \mathfrak{g}_{q_e}$,

$$\xi_Q\left(\exp_{q_e}(v)\right) = T_v \exp_{q_e}(\xi \cdot v)\ .$$

From the theory of Jacobi fields (cfr. [11, p. 114]) one learns that $\forall q \in Q$ and $v, w \in T_qQ$,

$$\left\|T_{\tau v} \exp_q(\tau w)\right\|^2 = \|w\|^2\tau^2 - \frac{1}{3}\left\langle\!\left\langle R(v,w)v,w\right\rangle\!\right\rangle \tau^4 + o(\tau^4)\ .$$

The claim follows by letting $w = \xi \cdot v$. ∇

3.7. Lemma. *Let $\xi, \eta \in \mathfrak{g}$ and $q \in Q$. Suppose that $dV_\xi(q) = 0$, where V_ξ is the augmented potential (see section 1.1.4), and suppose that both η and $[\xi, \eta]$ belong to $\mathfrak{g}_q$. Then $d\left\langle \mathbb{I}(\cdot)\xi, \eta\right\rangle(q) = 0$.*

Proof: Since $dV_\xi(q) = 0$ it follows from section 1.2 that $\xi_Q(q)$ is a relative equilibrium, that is to say, $X_H(z_q) = \xi_P(z_q)$, where $z_q = \mathbb{F}L(\xi_Q(q))$ and $P = T^*Q$. Now, suppose that both $\eta, [\eta, \xi] \in \mathfrak{g}_q$. Then

$$\eta_P(z_q) = \left.\frac{d}{dt}\right|_{t=0} \mathbb{F}L(\exp(t\eta) \cdot \xi_Q(q)) = \mathbb{F}L([\eta, \xi]_Q(q)) = 0\ ,$$

where we have used that $g \cdot (\xi_Q(q)) = (\mathrm{Ad}_g\, \xi)_Q(g \cdot q)$. It follows that $(\xi + \eta)_P(z_q) = X_H(z_q)$ and hence that $0 = dV_{\xi+\eta}(q) = dV_\xi(q) + d\left\langle \mathbb{I}(0)\xi, \eta\right\rangle(q) + \frac{1}{2}d\|\eta_Q(\cdot)\|^2(q)$. The first term in the latter expression vanishes by assumption and last term vanishes by lemma 3.6. Therefore $d\langle \mathbb{I}(0)\xi, \eta\rangle = 0$. ∇

3.8. Lemma. *Let $q_e \in Q$, $\mathfrak{t} \subset \mathfrak{g}$ be as defined in the general setup of the problem (section 3.1). Then $d\langle \mathbb{I}(\cdot)\xi, \eta\rangle (q_e) = 0$ for all $\xi \in \mathfrak{t}$, $\eta \in \mathfrak{g}_{q_e}$.*

Proof: Since $\mathfrak{g}_{q_e} \subset \mathfrak{t}$ and $\mathfrak{t}$ is a maximal abelian subalgebra then $[\xi, \eta] = 0 \in \mathfrak{g}_{q_e}$. Therefore the claim follows by the previous lemma. ∇

For the remaining of this chapter we will sometimes use the following abuse of notation. For $v \in T_{q_e}Q$, $\|v\| < r_0$, write

$$\mathbb{I}(v) := \mathbb{I}(\exp_{q_e}(v)) \ .$$

Let $\pi_i : \mathfrak{g}^* \longrightarrow \mathfrak{m}_i$ $(i = 0, 1, 2)$ be the projection induced by the splitting introduced in section 3.2. Let $\pi_{12} : \mathfrak{g}^* \longrightarrow \mathfrak{m}_1 \oplus \mathfrak{m}_2 = J_L(\mathfrak{g} \cdot q_e)$ be given by $\pi_{12} = \pi_1 + \pi_2$. For $v \in N$, $\|v\| < r_0$, let

$$\hat{\mathbb{I}}(v) := \pi_{12} \circ \mathbb{I}(v)|_{(\mathfrak{k}_1 \oplus \mathfrak{k}_2)} \quad \text{and} \quad \tilde{\mathbb{I}}(v) := \pi_{12} \circ \mathbb{I}(v)|_{\ker \mathbb{I}(0)} \ .$$

Notice that $\hat{\mathbb{I}}(v)$ is an isomorphism even when $v = 0$.

3.9. Proposition. *Let $\beta(\tau, \mu)$ be as in equation (3.2). With η_0 a generator of $\mathfrak{g}_{q_e}$, let*

$$\Sigma := \{v \in T_{q_e}Q \mid \|\eta_0 \cdot v\|^2 - \left\langle (D\tilde{\mathbb{I}}(0) \cdot v)\eta_0, \hat{\mathbb{I}}(0)^{-1}(D\tilde{\mathbb{I}}(0) \cdot v)\eta_0 \right\rangle = 0\} \ .$$

Let $N_1 := N \setminus \Sigma$. Consider the map $\xi : (\mathbb{R}^\circ \times N_1)_{r_0} \times J_L(\mathfrak{g} \cdot q_e) \longrightarrow \mathfrak{g}$ defined by

$$\xi(\tau, v, \mu) := \mathbb{I}(\exp_{q_e}(\tau v))^{-1}\beta(\tau, \mu) \ .$$

Then ξ can be extended to a smooth function on $(\mathbb{R} \times N_1)_{r_0} \times J_L(\mathfrak{g} \cdot q_e)$.

3.10. Remark. $\xi(\tau, v, \mu)$ is well defined and smooth for $\tau \neq 0$ because the G-action is (locally) free outside $G \cdot q_e$.

Proof. Let $\Phi : (\mathbb{R} \times N_1)_{r_0} \times J_L(\mathfrak{g} \cdot q_e) \times \mathfrak{k}_0 \times (\mathfrak{k}_1 \oplus \mathfrak{k}_2) \longrightarrow \mathfrak{g}^*$ be given by

$$\Phi(\tau, v, \mu, \xi_0, \xi_1) := \mathbb{I}(\tau v)(\xi_0 + \xi_1) - \beta(\tau, \mu) \; . \tag{3.3}$$

When $\tau \neq 0$, $\xi(\tau, v, \mu)$ is determined by $\xi(\tau, v, \mu) = \xi_0 + \xi_1$, where (ξ_0, ξ_1) is the unique solution to the equation $\Phi(\tau, v, \mu, \xi_0, \xi_1) = 0$.

The equation $\Phi = 0$ is obviously equivalent to the pair of equations $\{\pi_{12}\Phi = 0, (\mathrm{Id} - \pi_{12})\Phi = 0\}$. We now consider the first one.

Clearly, if $\tau \neq 0$, the equation $\pi_{12}\Phi = 0$ determines a unique solution for ξ_1 depending smoothly on the other parameters. Let $\mu_i := \pi_i(\mu)$, $i = 1, 2$. At $\tau = 0$,

$$\pi_{12}\Phi(0, v, \mu, \xi_0, \xi_1) = \pi_{12}[\mathbb{I}(0) \cdot (\xi_0 + \xi_1) - \beta(0, \mu)] = \hat{\mathbb{I}}(0)\xi_1 - \mu_1 \; ,$$

so that $\xi_1 = \hat{\mathbb{I}}(0)^{-1}\mu_1$ is the unique solution of $\pi_{12}\Phi = 0$ at $\tau = 0$. Moreover, from the definition of Φ,

$$\frac{\partial\, \pi_{12}\Phi}{\partial \xi_1}(0, v, \mu, \xi_0, \hat{\mathbb{I}}(0)^{-1}\mu) = \pi_{12}\mathbb{I}(0)|_{(\mathfrak{k}_1 \oplus \mathfrak{k}_2)} = \hat{\mathbb{I}}(0) \; .$$

Since $\hat{\mathbb{I}}(0)$ is an isomorphism then the implicit function theorem in the form of proposition 1.23 implies that the equation $\pi_{12}\Phi = 0$ implicitly defines a smooth function $\xi_1 : (\mathbb{R} \times N_1)_{r_0} \times J_L(\mathfrak{g} \cdot q_e) \times \ker \mathbb{I}(0) \longrightarrow \mathfrak{g}$ satisfying $\xi_1(0, v, \mu, \xi_0) = \hat{\mathbb{I}}(0)^{-1}\mu_1$. In particular, $\xi_1(0, v, \mu, \xi_0)$ only depends on $\mu_1 = \pi_1(\mu)$.

Now we proceed to extract information from the equation $(\mathrm{Id} - \pi_{12})\Phi = 0$. Let $\varphi : J_L(\mathfrak{g} \cdot q_e) \times \ker \mathbb{I}(0) \longrightarrow \mathfrak{m}_0$ be given by

$$\begin{aligned} \varphi(\tau, v, \mu, \xi_0) &= (\mathrm{Id} - \pi_{12})\Phi(\tau, v, \mu, \xi_0, \xi_1(\tau, v, \mu, \xi_0)) \\ &= (\mathrm{Id} - \pi_{12}) \left[\mathbb{I}(\tau v)(\xi_0 + \xi_1(\tau, v, \mu, \xi_0)) - \beta(\tau, \mu)\right] \; . \end{aligned}$$

3.11. Lemma. *For every $(v, \mu, \xi_0) \in (N \setminus \{0\}) \times J_L(\mathfrak{g} \cdot q_e) \times \ker \mathbb{I}(0)$,*

$$\varphi(0, v, \mu, \xi_0) = D_\tau \varphi(0, v, \mu, \xi_0) = 0 \; .$$

Proof of Lemma. Firstly,

$$\begin{aligned}\varphi(0,v,\mu,\xi_0) &= (\mathrm{Id}-\pi_{12})[\mathbb{I}(0)\cdot(\xi_0+\xi_1(0,v,\mu,\xi_0))-\beta(0,\mu)]\\ &= (\mathrm{Id}-\pi_{12})[\mathbb{I}(0)\cdot(\xi_0+\hat{\mathbb{I}}(0)^{-1}\mu_1)-\mu_1]=0\end{aligned}$$

because $\operatorname{Im}\mathbb{I}(0)=J_L(\mathfrak{g}\cdot q_e)=\mathfrak{m}_1\oplus\mathfrak{m}_2$, $\mu_1\in\mathfrak{m}_1$ and $(\mathrm{Id}-\pi_{12})(\mathfrak{m}_1\oplus\mathfrak{m}_2)=0$.

Secondly,

$$\begin{aligned}D_\tau\varphi(0,v,\mu,\xi_0) &= (\mathrm{Id}-\pi_{12})[(D\mathbb{I}(0)\cdot v)(\xi_0+\hat{\mathbb{I}}(0)^{-1}\mu_1)\\ &\qquad+\mathbb{I}(0)\frac{\partial}{\partial\tau}\xi_1(0,v,\mu,\xi_0)-\frac{\partial\beta}{\partial\tau}(0,\mu)]\\ &= (\mathrm{Id}-\pi_{12})(D\mathbb{I}(0)\cdot v)(\xi_0+\hat{\mathbb{I}}(0)^{-1}\mu)\end{aligned}$$

using, as above, that $\operatorname{Im}\mathbb{I}(0)=\mathfrak{m}_1\oplus\mathfrak{m}_2$, $\partial\beta/\partial\tau=\mu_2\in\mathfrak{m}_2$ and $(\mathrm{Id}-\pi_{12})(\mathfrak{m}_1\oplus\mathfrak{m}_2)=0$.

Let η_0 be a generator of $\mathfrak{g}_{q_e}$. Since $\mathfrak{m}_0$ is the annihilator of a complement to $\mathfrak{g}_{q_e}$ in $\mathfrak{g}$, then, for all $\nu\in\mathfrak{g}^*$, $(\mathrm{Id}-\pi_{12})\cdot\nu=0$ if and only if $\langle\nu,\eta_0\rangle=0$. But now the equality

$$\big\langle(D\mathbb{I}(0)\cdot v)(\xi_0+\mathbb{I}(0)^{-1}\mu_1),\eta_0\big\rangle=0$$

follows from the fact that $\xi_0+\hat{\mathbb{I}}(0)^{-1}\mu_1\in\mathfrak{t}$ and lemma 3.8. Therefore, we have that $D_\tau\Phi(0,v,\mu,\xi_0)=0$ as claimed. This finishes the proof of lemma 3.11. ▽

We can now apply Hadamard's lemma in the form of proposition 1.24 to conclude that $\varphi(\tau,v,\mu,\xi_0)=\tau^2\varphi'(\tau,v,\mu,\xi_0)$ for some smooth function φ' defined on $(\mathbb{R}\times N_1)_{r_0}\times J_L(\mathfrak{g}\cdot q_e)\times\ker\mathbb{I}(0)$ and $\varphi'(0,v,\mu,\xi_0)=D_{\tau\tau}\varphi(0,v,\mu,\xi_0)/2$. We now want to conclude that the equation $\varphi'=0$ defines ξ_0 as a function of (v,μ), but first we need to prove the following

3.12. Lemma. *Let η_0 be a generator of $\mathfrak{g}_e$ and let $\alpha \in \mathbb{R}$. Then*

$$\frac{1}{2}\langle D_{\tau\tau}\varphi(0,v,\mu,\alpha\,\eta_0),\eta_0\rangle =$$
$$\alpha\left[\|\eta_0\cdot v\|^2 - \left\langle (D\tilde{\mathbb{I}}(0)\cdot v)\eta_0, \hat{\mathbb{I}}(0)^{-1}(D\tilde{\mathbb{I}}(0)\cdot v)\eta_0\right\rangle\right]$$
$$+\frac{1}{2}\left\langle D^2\tilde{\mathbb{I}}(0)(v,v)\eta_0, \hat{\mathbb{I}}(0)^{-1}\mu_1\right\rangle + \left\langle (D\tilde{\mathbb{I}}(0)\cdot v)\eta_0, (D\hat{\mathbb{I}}(0)^{-1}\cdot v)\mu_1\right\rangle$$
$$+\left\langle (D\tilde{\mathbb{I}}(0)\cdot v)\eta_0, \hat{\mathbb{I}}(0)^{-1}\mu_2\right\rangle - \langle \nu_0,\eta_0\rangle$$

Proof of Lemma: A direct computation using the definition of $\varphi(\tau,v,\mu,\xi_0)$ shows that (with the notation $\xi_1(\tau,\alpha) := \xi_1(\tau,v,\mu,\alpha\eta_0)$),

$$D_{\tau\tau}\varphi(0,v,\mu,\alpha\eta_0) = (\mathrm{Id}-\pi_{12})\big[\left(D^2\mathbb{I}(0)\cdot(v,v)\right)(\xi_0+\xi_1(0,\alpha)) + 2\,(D\mathbb{I}(0)\cdot v)\,\frac{\partial\xi_1}{\partial\tau}(0,\alpha) - 2\nu_0\big]\,. \quad (3.4)$$

But

$$\frac{\partial\xi_1}{\partial\tau}(\alpha,0) = -D_{\xi_1}(\pi_{12}\Phi)^{-1}\cdot D_\tau(\pi_{12}\Phi)|_{\tau=0}$$
$$= -\alpha\,\hat{\mathbb{I}}(0)^{-1}(D\tilde{\mathbb{I}}(0)\cdot v)\eta_0 + (D\hat{\mathbb{I}}(0)^{-1}\cdot v)\mu_1 + \hat{\mathbb{I}}(0)^{-1}\mu_2\,.$$

Substituting this in the expansion of $\langle D_{\tau\tau}\varphi(0,v,\mu,\alpha\eta_0,\eta_0\rangle$ obtained from equation (3.4) and using the fact that, from lemma 3.6,

$$\left\langle \left(D^2\mathbb{I}(0)\cdot(v,v)\right)\eta_0,\eta_0\right\rangle = \left.\frac{\partial^2}{\partial\tau^2}\right|_{\tau=0}\langle\!\langle(\eta_0)_Q(\tau v),(\eta_0)_Q(\tau v)\rangle\!\rangle = 2\|\eta_0\cdot v\|^2\,,$$

we get the desired result. This finishes the proof of lemma 3.12. ∇

We know that given $(\tau,v,\mu)\in(\mathbb{R}^\circ\times N_1)_{r_0}\times J_L(\mathfrak{g}\cdot q_e)$ (i.e., $\tau v\neq 0$), the equation $\varphi'(\tau,v,\mu,\xi_0)=0$ yields a unique solution for ξ_0, namely, the $\mathfrak{k}_0$-component of $\mathbb{I}(\tau v)^{-1}\beta(\tau,\mu)$. Clearly, this solution is a smooth function of the (τ,v,μ) parameters when $\tau\neq 0$. By the previous lemma and since $v\notin\Sigma$, there is also a unique $\xi_0\in\mathfrak{g}_{q_e}$ solving $\varphi'(0,v,\mu,\xi_0)=0$. Moreover, $\partial\varphi'/\partial\alpha\,(0,v,\mu,\xi_0)\neq 0$. It follows from proposition 1.23 that $\xi_0(\tau,v,\mu)$ extends to a smooth function on

$(\mathbb{R} \times N_1)_{r_0} \times J_L(\mathfrak{g} \cdot q_e)$ and therefore the same is true for $\xi(\tau, v, \mu) = \xi_0(\tau, v, \mu) + \xi_1(\tau, v, \mu, \xi_0(\tau, v, \mu))$. This completes the proof of proposition 3.9. □

For future reference, we prove

3.13. Proposition. *N_1 is H-invariant.*

Proof. It suffices to show that Σ is H-invariant. The only non-obvious fact is that $\left\langle (D\tilde{\mathbb{I}}(0) \cdot v)\xi_0, \hat{\mathbb{I}}(0)^{-1}(D\tilde{\mathbb{I}}(0) \cdot v)\xi_0 \right\rangle$ is invariant with respect to $v \mapsto h \cdot v$, $h \in H := G_{q_e}$. To see this we first show that $\tilde{\mathbb{I}}(v)\xi_0$ is H-equivariant as a function of v, so that the same is true for $(D\tilde{\mathbb{I}}(0) \cdot v)\xi_0$. Since $\left\langle \cdot, \hat{\mathbb{I}}(0)^{-1} \cdot \right\rangle$ is an H-invariant inner product on $\mathfrak{g}^*$ the claim follows.

Now we show that indeed $\tilde{\mathbb{I}}(v)\xi_0$ is H-equivariant as a function of v. Recall that $\tilde{\mathbb{I}}(v) \cdot \xi_0 = \pi_{12}\mathbb{I}(v) \cdot \xi_0$. Since $\mathfrak{t} = (\mathfrak{k}_0 = \mathfrak{g}_{q_e}) \oplus \mathfrak{k}_1$ is a maximal abelian subalgebra then H acts trivially on $\mathfrak{t}$ and in particular on $\mathfrak{k}_0$ and $\mathfrak{k}_1$. Since $\mathfrak{k}_2$ is the orthogonal complement of $\mathfrak{t}$ with respect to some G-invariant inner product on $\mathfrak{g}$ then in particular $\mathfrak{k}_2$ is H-invariant. It follows that both $\mathfrak{m}_0$ and $\mathfrak{m}_1 \oplus \mathfrak{m}_2$ are H-invariant. Therefore the projection $\pi_{12} : \mathfrak{g}^* \longrightarrow \mathfrak{m}_1 \oplus \mathfrak{m}_2$ is H-equivariant. Also, since H acts trivially on $\mathfrak{k}_0$ and using G-invariance of the metric it follows that $\mathbb{I}(h \cdot v)\xi_0 = h \cdot \mathbb{I}(v)\xi_0$. Therefore $\tilde{\mathbb{I}}(h \cdot v)\xi_0 = h \cdot \tilde{\mathbb{I}}(v)\xi_0$. □

3.4 Relative equilibria in the associated bundle

Let $J_L : TQ \longrightarrow \mathfrak{g}^*$ be the momentum map as defined in section 1.1.4.

3.14. Proposition. *The map from TQ to $Q \times \mathfrak{g}^*$ given by $(q, v) \mapsto (q, J_L(q, v))$ restricted to the set of relative equilibria is one to one.*

Proof. First observe that, since $J_L(q, \xi_Q(q)) = \mathbb{I}(q)\xi$, then $J_L(q, \xi_Q(q)) = 0 \Rightarrow \xi \in \mathfrak{g}_q$. Now, from the augmented potential criterium (proposition 1.15), we have that if (q, v) is a relative equilibrium then $dV_\xi(q) = 0$ and $v = \xi_Q(q)$. So if $(q, \xi_{1Q}(q)), (q, \xi_{2Q}(q))$ are two relative equilibria with the same momentum then $J_L(q, (\xi_1 - \xi_2)_Q(q)) = 0$, then $\xi_1 - \xi_2 \in \mathfrak{g}_q$ and then $\xi_{1Q}(q) = \xi_{2Q}(q)$. □

The previous proposition says that we can regard the set of relative equilibria in TQ as elements in $Q \times \mathfrak{g}^*$. Let $\mathcal{RE} \subset Q \times \mathfrak{g}^*$ denote the image of the set of relative equilibria in TQ under the map of proposition 3.14. From the formula for the action of G on infinitesimal generators (cf. equation 1.4) it follows that $\mathcal{RE}$ is G-invariant with respect to the diagonal action $g \cdot (q, \mu) = (g \cdot q, Ad^*_{g^{-1}}\mu)$. Therefore $\mathcal{RE}$ drops to $(Q \times \mathfrak{g}^*)/G$, the *associated bundle to* $\mathfrak{g}^*$.

3.15. Remark. The associated bundle to $\mathfrak{g}^*$ is also relevant in understanding the bundle structure of $(T^*Q)/G$ in the context of Poisson cotangent bundle reduction. See for example [25, §II].

Recall that $N := (\mathfrak{g}\cdot q_e)^\perp \subset T_{q_e}Q$ and $H := G_{q_e}$. Let $N' \subset N$ be an H-invariant open neighborhood of 0 such that $\exp_{q_e}|_{N'}$ is a diffeomorphism onto its image and such that H acts freely on $N' \setminus \{0\}$. We then have that $Q' := G \cdot \exp_{q_e}(N')$ is a G-invariant neighborhood of $G \cdot q_e$ and G acts freely on $Q' \setminus (G \cdot q_e)$.

It is easy to see that $N' \times \mathfrak{g}^*$ is a slice of $Q' \times \mathfrak{g}^* \longrightarrow (Q' \times \mathfrak{g}^*)/G$ at $(q_e, 0)$. Therefore (see for example [12, §2.3]) we have that

3.16. Proposition. *The map from $(N' \times \mathfrak{g}^*)/H$ to $(Q' \times \mathfrak{g}^*)/G$ given by*

$$[v, \mu]_H \mapsto [\exp_{q_e}(v), \mu]_G$$

is a homeomorphism. Moreover, it is a diffeomorphism when restricted to $((N' \setminus \{0\}) \times \mathfrak{g}^*)/H$.

3.17. Definition. We say that a point $[q, \mu]_G \in (Q' \times \mathfrak{g}^*)/G$ **corresponds to a class of relative equilibria** if $(q, \mu) \in \mathcal{RE}$, that is to say, $(q, \mu) = (\pi_Q \times J_L)(v_q)$ for some $v_q \in TQ$ that is a relative equilibrium. Similarly, we say that a point $[v, \mu]_H \in (N' \times \mathfrak{g}^*)/H$ **corresponds to a class of relative equilibria** if, under the homeomorphism of proposition 3.16, $[v, \mu]_H$ maps to a point in $(Q' \times \mathfrak{g}^*)/G$ that corresponds to a class of relative equilibria.

From the definitions it follows that $[v, \mu]_H$ corresponds to a class of relative equilibria if and only if $dV_\mu(\exp_{q_e}(v)) = 0$.

Since our objective is to find relative equilibria that break the symmetry, we now turn our attention to the case when $v \neq 0$.

Suppose that $U \subset (N \setminus \{0\})/H$ admits a smooth local section $\psi_1 : U \longrightarrow N \setminus \{0\}$. Since H acts freely on $N\setminus\{0\}$ then the map from $U\times\mathfrak{g}^*$ to $((N \setminus \{0\}) \times \mathfrak{g}^*)/H$ given by $(u, \mu) \mapsto [\psi_1(u), \mu]_H$ is a diffeomorphism onto its image.

Given such smooth local section (U, ψ_1), it will be useful to "pullback" the relative equilibrium condition to $U \times \mathfrak{g}^*$ for the purpose of "blowing-up" in the directions orthogonal to the group action at q_e. Since $(N \setminus \{0\})/H$ can be covered by open sets admitting smooth local sections (see for example [12, §1.11]), then there is no loss of generality in following this procedure.

Also, since what we ultimately want is to study relative equilibria whose configuration is close to being symmetric, we will need to consider an open cover for $(N \setminus \{0\})/H$ consisting of open sets with the property that if $u \in U$ then $\tau u \in U$ for all $\tau \in \mathbb{R}$, $\tau > 0$. These open sets can be constructed as follows. Let $\hat{N}$ be the intersection of N with the unit sphere in $T_{q_e}Q$. Choose an open set $U' \in \hat{N}/H$ that admits a smooth local section $\psi_1' : U' \longrightarrow \hat{N}$. (In a concrete example, it is convenient to choose U' as "big" as possible.) Since multiplication by scalars is well defined in N/H, we can define a local smooth section over $(N \setminus \{0\})/H$ given by

$$U := \mathbb{R}^+ U' = \{r\, u \in N/H \mid r \in \mathbb{R}, r > 0, u \in U'\}\ , \tag{3.5}$$

$$\psi_1 : U \longrightarrow N : [ru] \mapsto r\psi_1'(u)\ , \tag{3.6}$$

where $r \in \mathbb{R}^+$, $u \in U'$. It is clear that ψ_1 is a smooth local section of $N \setminus \{0\} \longrightarrow (N \setminus \{0\})/H$.

Let $\mathfrak{k}_0, \mathfrak{k}_1, \mathfrak{k}_2$ and $\beta(\tau, \mu)$ be as in section 3.2. We have:

3.18. Proposition. *Let $\mu \in J_L(\mathfrak{g} \cdot q_e)$ be regular. Let (ψ_1, U) be the local smooth section over $(N \setminus \{0\})/H$ given by (3.5) and (3.6). Let $u \in U$ and $\psi := \exp_{q_e} \circ \psi_1$.*

Then there exists an $\epsilon > 0$ such that for all $0 < \tau < \epsilon$,

$$T_{\psi(\tau u)}Q = \mathfrak{g}_{\beta(\tau,\mu)} \cdot \psi(\tau u) \oplus \psi_*(T_{\tau u}U) \oplus T_{\psi_1(\tau u)} \exp_{q_e}(\mathfrak{k}_2 \cdot q_e) \, .$$

For the proof of this proposition we will need the following two lemmas which are special cases of the stability of the transversality of smooth maps. See for example [15].

3.19. Lemma. *Let G be a Lie group acting on a Riemannian manifold Q, $q \in Q$ and let $\mathfrak{k} \subset \mathfrak{g}$ (subspace) such that $\mathfrak{k} \cap \mathfrak{g}_q = \{0\}$. Let $M \subset T_qQ$ (subspace) such that $\mathfrak{k} \cdot q \oplus M = T_qQ$. Then there is an $\epsilon > 0$ such that if $\|v\| < \epsilon$,*

$$T_{\exp_q(v)}Q = \mathfrak{k} \cdot \exp_q(v) \oplus T_v \exp_q(M) \, .$$

3.20. Lemma. *Let X, Y_1, Y_2 be Banach spaces and Z a topological space. Let $F : X \times Z \longrightarrow Y_1$ be a continuous map such that $F_z := F(\cdot, z)$ is linear $\forall z \in Z$ and let $L : X \longrightarrow Y_2$ be linear. Let $z_0 \in Z$ and assume that $\ker F_z$ does not change dimension for z sufficiently close to z_0. Let $W \subset Y_2$ be a subspace and suppose that $Y_2 = W \oplus L(\ker F_{z_0})$. Then there is an open neighborhood $U \subset Z$ around z_0 such that if $z \in U$ then $Y_2 = W \oplus L(\ker F_z)$.*

Proof of proposition 3.18: By definition, $T_{q_e}Q = \mathfrak{k}_1 \cdot q_e \oplus \mathfrak{k}_2 \cdot q_e \oplus N$. Using lemma 3.19 (with $\mathfrak{k} = \mathfrak{k}_1$ and $M = \mathfrak{k}_2 \cdot q_e \oplus N$) we get that there is an $\epsilon_1 > 0$ such that if $0 \leq \tau < \epsilon_1$,

$$T_{\psi(\tau u)}Q = \mathfrak{k}_1 \cdot \psi(\tau u) \oplus T_{\psi_1(\tau u)} \exp_{q_e}(N \oplus \mathfrak{k}_2 \cdot q_e) \, . \tag{3.7}$$

Since (ψ_1, U) is a smooth local section then

$$T_{\psi_1(\tau u)}N = (\psi_1)_*(T_{\tau u}U) \oplus \mathfrak{k}_0 \cdot \psi_1(\tau u) \, .$$

Since $\exp_{q_e}$ is a (local) diffeomorphism and $T_v \exp_{q_e}(\xi \cdot v) = \xi \cdot (\exp_{q_e}(v))$ for all $\xi \in \mathfrak{k}_0$, $v \in T_{q_e}Q$, we have that

$$\begin{aligned} T_{\psi_1(\tau u)} \exp_{q_e}(N) &= (\exp_{q_e})_*(\psi_1)_*(T_{\tau u}U) \oplus T_{\psi_1(\tau u)} \exp_{q_e}(\mathfrak{k}_0 \cdot (\psi_1(\tau u)) \\ &= \psi_*(T_{\tau u}U) \oplus \mathfrak{k}_0 \cdot \psi(\tau u) \ . \end{aligned}$$

Using this expression and the fact that $\mathfrak{t} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$ we get from equation (3.7) that

$$T_{\psi(\tau u)}Q = \mathfrak{t} \cdot \psi(\tau u) \oplus \psi_*(T_{\tau u}U) \oplus T_{\psi_1(\tau u)} \exp_{q_e}(\mathfrak{k}_2 \cdot q_e) \ . \tag{3.8}$$

Fix $\tau \in (0, \epsilon_1)$. Let $L : \mathfrak{g} \longrightarrow T_{\psi(\tau u)}Q : \xi \mapsto \xi \cdot \psi(\tau u)$ and $F : \mathfrak{g} \times [0, \epsilon_1) \longrightarrow \mathfrak{g}^* : (\xi, \tau') \mapsto \operatorname{ad}^*_\xi \beta(\tau', \mu)$. Then $\ker F_{\tau'} = \mathfrak{g}_{\beta(\tau',\mu)}$, where $F_{\tau'} := (F\cdot, \tau')$. Since μ is regular then $\dim \ker F_{\tau'}$ does not change for τ' sufficiently small. Since $\ker F_0 = \mathfrak{t}$ then from equation (3.8) it follows that $L(\ker F_0)$ is a complement to $W := \psi_*(T_{\tau u}U) \oplus T_{\psi_1(\tau u)} \exp_{q_e}(\mathfrak{k}_2 \cdot q_e)$ in $T_{\psi(\tau u)}Q$. Thus we can apply lemma 3.20 to conclude that there is an $\epsilon_2 > 0$ such that if $0 < \tau' < \epsilon_2$ then

$$T_{\psi(\tau u)}Q = \mathfrak{g}_{\beta(\tau',\mu)} \cdot \psi(\tau u) \oplus W \ .$$

Letting $\epsilon = \min(\epsilon_1, \epsilon_2)$ and $\tau' = \tau$ we get the desired result. □

Since $V_{\beta(\tau,\mu)}$ is $G_{\beta(\tau,\mu)}$-invariant it follows as a consequence of proposition 3.18 that

3.21. Corollary. *Let $u \in U$. Then $[\tau u, \beta(\tau, \mu)]_H$ corresponds to a class of relative equilibria if and only if both the following equations hold:*

$$d\left(\psi^* V_{\beta(\tau,\mu)}\right)(\tau u) = 0 \ , \tag{3.9}$$

$$d\left((\exp_{q_e})^* V_{\beta(\tau,\mu)}\right)(\psi_1(\tau u))\Big|_{\mathfrak{k}_2 \cdot q_e} = 0 \ . \tag{3.10}$$

We are now prepared to enumerate and prove the main propositions of this section, where we achieve the blowing-up at the amended potential. Theorem 3.22 blows-up equation (3.9) and theorem 3.23 blows-up equation (3.10).

Above (see equations (3.5) and (3.6))) we constructed a smooth local section

(U, ψ_1) of the principal bundle $(N \setminus \{0\}) \longrightarrow N \setminus \{0\}$ where U is such that if $u \in U$ then $\tau u \in U$ for all $\tau \in \mathbb{R}$, $\tau > 0$. In what follows, however, we will need (U, ψ_1) to be a smooth local section of $N_1 \longrightarrow N_1/H$, where $N_1 := N \setminus \Sigma$ was introduced in proposition 3.9. Since (by proposition 3.13) N_1 is H-invariant, the same procedure to construct (U, ψ_1) that we discussed above applies to this case.

Let $r_0 > 0$ be such that $\exp_{q_e}$ is a diffeomorphism on a ball of radius r_0 in $T_{q_e}Q$. Let $(\mathbb{R}^\circ \times U)_{r_0}$ and $(\mathbb{R} \times U)_{r_0}$ be as in definition 3.4. As before, let $\psi := \exp_{q_e} \cdot \psi_1$.

3.22. Theorem. *Let $W' : (\mathbb{R}^\circ \times U)_{r_0} \times J_L(\mathfrak{g} \cdot q_e) \longrightarrow \mathbb{R}$ be given by*

$$W'(\tau, u, \mu) = \psi^* V_{\beta(\tau,\mu)}(\tau u) .$$

Then W' can be extended to a smooth function on $(\mathbb{R} \times U)_{r_0} \times J_L(\mathfrak{g} \cdot q_e)$ and

$$W'(\tau, u, \mu) = W_0(\mu) + \tau^2 W(\tau, u, \mu)$$

for some smooth real valued functions W_0, W defined over $J_L(\mathfrak{g} \cdot q_e)$ and $(\mathbb{R} \times U)_{r_0} \times J_L(\mathfrak{g} \cdot q_e)$, respectively.

Proof: Let $v = \psi_1(u) \in N$. Abusing notation, we will write V instead of $V \circ \exp_{q_e}$ and $\mathbb{I}$ instead of $\mathbb{I} \circ \exp_{q_e}$. We have that

$$\begin{aligned} \psi^* V_{\beta(\tau,\mu)}(\tau u) &= V(\tau v) + \frac{1}{2} \left\langle \beta(\tau, \mu), \mathbb{I}^{-1}(\tau v) \beta(\tau, \mu) \right\rangle \\ &= V(\tau v) + \frac{1}{2} \left\langle \beta(\tau, \mu), \xi(\tau, v, \mu) \right\rangle \end{aligned} \tag{3.11}$$

where $\xi(\tau, v, \mu)$ is as in proposition 3.9. Since $\xi(\tau, v, \mu)$ is smooth then the same is true for the left-hand side of (3.11).

The remaining assertion follows from the following straightforward, albeit lengthy, computations.

Let η_0 be a generator of $\mathfrak{g}_{q_e}$. By the proof of proposition 3.9 we have that $\xi(\tau, v, \mu) = \alpha(\tau, v, \mu)\eta_0 + \xi_1(\tau, v, \alpha(\tau, v, \mu), \mu)$ where α and ξ_1 are smooth functions. ξ_1 was defined implicitly by equation (3.3). Fix $\mu \in J_L(\mathfrak{g} \cdot q_e)$, $v \in N \setminus \Sigma$ and for brevity write $\alpha(\tau) = \alpha(\tau, v, \mu)$, $\xi_1(\tau, \alpha) = \xi_1(\tau, v, \alpha, \mu)$. Let the projections π_i and the operators $\hat{\mathbb{I}}, \tilde{\mathbb{I}}$ be as in page 39. We will use the notation $\mu_i := \pi_i \mu$. From equation (3.3),

$$\begin{aligned}
\frac{\partial \xi_1}{\partial \tau}(0,\alpha) &= -D_{\xi_1}(\pi_{12}\Phi)^{-1} \cdot D_\tau(\pi_{12}\Phi)|_{\tau=0} \\
&= -\alpha\, \hat{\mathbb{I}}(0)^{-1}(D\tilde{\mathbb{I}}(0)\cdot v)\eta_0 + (D\hat{\mathbb{I}}(0)^{-1}\cdot v)\mu_1 + \hat{\mathbb{I}}(0)^{-1}\mu_2 \,, \\
\frac{\partial \xi_1}{\partial \alpha}(0,\alpha) &= -D_{\xi_1}(\pi_{12}\Phi)^{-1} \cdot D_\alpha(\pi_{12}\Phi)|_{\tau=0} \\
&= -\hat{\mathbb{I}}(0)^{-1}\tilde{\mathbb{I}}(0)\eta_0 = 0 \,,
\end{aligned}$$

since $\tilde{\mathbb{I}}(0)\eta_0 = 0$.

From (3.11) and since $\xi_1(0,\alpha) = \hat{\mathbb{I}}(0)^{-1}\mu_1$ and $\mathfrak{m}_1$ annihilates $\ker \mathbb{I}(0)$, we get

$$V_{\beta(\tau,\mu)}(\tau v)|_{\tau=0} = V(0) + \frac{1}{2}\left\langle \mu_1, \hat{\mathbb{I}}(0)^{-1}\mu_1 \right\rangle \,, \tag{3.12}$$

which is independent of v.

Now, differentiating (3.11) with respect to τ we get

$$\begin{aligned}
\left.\frac{\partial}{\partial \tau}\right|_{\tau=0} V_{\beta(\tau,\mu)}(\tau v) &= dV(0)\cdot v + \frac{1}{2}\langle \mu_2, \alpha\,\eta_0 + \xi_1(0,\alpha)\rangle \\
&\quad + \frac{1}{2}\left\langle \mu_1, \frac{\partial \alpha}{\partial \tau}\left(\eta_0 + \frac{\partial \xi_1}{\partial \alpha}(0,\alpha)\right) + \frac{\partial \xi_1}{\partial \tau}(0,\alpha)\right\rangle
\end{aligned}$$

because $\beta(0,\mu) = \mu_1$ and $\partial\beta/\partial\tau(0,\mu) = \mu_2$. In the right-hand side, the first term vanishes because we have assumed that $dV(0) = 0$. The second term vanishes because $\alpha\eta_0 + \xi_1(0,\alpha) \in \mathfrak{t}$ and $\mathfrak{m}_2 = \mathfrak{t}^\circ$. Using the expressions for $\partial\xi_1/\partial\alpha$ and $\partial\xi_1/\partial\tau$ obtained above, and the fact that $\mathfrak{m}_1$ annihilates η_0, we see that the third term is equal to one half of

$$\begin{aligned}
\left\langle \mu_1, \frac{\partial \xi_1}{\partial \tau}(0,\alpha)\right\rangle &= -\alpha\left\langle \mu_1, \hat{\mathbb{I}}(0)^{-1}(D\tilde{\mathbb{I}}(0)\cdot v)\eta_0\right\rangle \\
&\quad + \left\langle \mu_1, (D\hat{\mathbb{I}}(0)^{-1}\cdot v)\mu_1\right\rangle + \left\langle \mu_1, \hat{\mathbb{I}}(0)^{-1}\mu_2\right\rangle .
\end{aligned}$$

Now we check that each of the terms in the right-hand side of this expression vanishes: Let $\zeta := \hat{\mathbb{I}}(0)^{-1}\mu_1 \in \mathfrak{k}_1 \subset \mathfrak{t}$. Then

$$\left\langle \mu_1, \hat{\mathbb{I}}(0)^{-1}(D\tilde{\mathbb{I}}(0)\cdot v)\eta_0 \right\rangle = \left\langle (D\tilde{\mathbb{I}}(0)\cdot v)\eta_0, \zeta \right\rangle = \langle (D\mathbb{I}(0)\cdot v)\eta_0, \zeta\rangle = 0 \, ,$$

because of lemma 3.7.

$$\begin{aligned}\left\langle \mu_1, (D\hat{\mathbb{I}}(0)^{-1}\cdot v)\mu_1 \right\rangle &= \left\langle \mu_1, -\hat{\mathbb{I}}(0)^{-1}(D\hat{\mathbb{I}}(0)\cdot v)\hat{\mathbb{I}}(0)^{-1}\mu_1 \right\rangle \\ &= -\left\langle (D\hat{\mathbb{I}}(0)\cdot v)\zeta, \zeta \right\rangle = -\langle (D\mathbb{I}(0)\cdot v)\zeta, \zeta\rangle = 0 \, ,\end{aligned}$$

by assumption 1. (See section 3.1.) Finally,

$$\left\langle \mu_1, \hat{\mathbb{I}}(0)^{-1}\mu_2 \right\rangle = \langle \mu_2, \zeta\rangle = 0 \, ,$$

because $\mathfrak{m}_2$ annihilates $\mathfrak{t}$. We conclude that $\partial/\partial\tau|_{\tau=0}V_{\beta(\tau,\mu)}(\tau v) = 0$. Thus, by Hadamard's lemma in the form of proposition 1.24,

$$\psi^* V_{\beta(\tau,\mu)}(\tau u) = W_0(\mu) + \tau^2 W(\tau, u, \mu)$$

where $W_0(\mu)$ is equal to the right-hand side of (3.12) and W is some smooth function. □

3.23. Theorem. *Let $X' : (\mathbb{R}^\circ \times U)_{r_0} \times J_L(\mathfrak{g}\cdot q_e) \longrightarrow \mathfrak{k}_2^*$ be given by*

$$\langle X'(\tau, u, \mu), \eta\rangle = d\left((\exp_{q_e})^* V_{\beta(\tau,\mu)}\right)(\psi_1(\tau u)) \cdot \eta_Q(q_e) \, .$$

Then X' can be extended to a smooth function on $(\mathbb{R}\times U)_{r_0} \times J_L(\mathfrak{g}\cdot q_e)$ and

$$X'(\tau, u, \mu) = \tau X(\tau, u, \mu)$$

for some smooth function $X : (\mathbb{R}\times U)_{r_0} \times J_L(\mathfrak{g}\cdot q_e) \longrightarrow \mathfrak{k}_2^*$.

Proof: It suffices to show that $X'(\tau, u, \mu)$ is a smooth function at $\tau = 0$ and that $X'(0, u, \mu) = 0$. Abusing notation, we will write V instead of $V \circ \exp_{q_e}$ and $\mathbb{I}$

instead of $\mathbb{I}\circ\exp_{q_e}$. Let $v=\psi_1(u)$. Then

$$\begin{aligned}\big\langle X'(\tau,u,\mu),\eta\big\rangle &= d\left((\exp_{q_e})^* V_{\beta(\tau,\mu)}\right)(\tau v)\cdot\eta_Q(q_e)\\ &= dV(\tau v)\cdot\eta_Q(q_e)+\frac{1}{2}\left\langle\beta(\tau,\mu),(D\mathbb{I}^{-1}(\tau v)\cdot\eta_Q(q_e))\beta(\tau,\mu)\right\rangle\\ &= dV(\tau v)\cdot\eta_Q(q_e)-\frac{1}{2}\left\langle(D\mathbb{I}(\tau v)\cdot\eta_Q(q_e))\xi(\tau,v,\mu),\xi(\tau,v,\mu)\right\rangle,\end{aligned} \tag{3.13}$$

where $\xi(\tau,v,\mu)=\mathbb{I}^{-1}(\tau v)\beta(\tau,\mu)$. Since $\xi(\tau,v,\mu)$ is smooth at $\tau=0$ (by proposition 3.9) then so is $\langle X'(\tau,u,\mu),\eta\rangle$. Using proposition 1.11 we get that

$$\big\langle X'(0,u,\mu),\eta\big\rangle = dV(0)\cdot\eta_Q(0)-2\left\langle\mathbb{I}(0)[\xi(0,v,\mu),\eta],\xi(0,v,\mu)\right\rangle .$$

Since V is G-invariant then $dV(0)\cdot\eta_Q(q_e)=0$. Since $\xi(0,v,\mu)\in\mathfrak{t}$ then $[\xi(0,v,\mu),\eta]\in\mathfrak{k}_2$. Since $\mathbb{I}(0)\mathfrak{t}\subset\mathfrak{k}_2^\circ$ then $\langle X'(0,u,\mu),\eta\rangle=0$. □

The expression for $X(0,u,\mu)$ is relatively simple and it is convenient to include it here. Recall that $\xi(\tau,u,\mu):=\mathbb{I}^{-1}(\psi(\tau u))\beta(\tau,\mu)$.

3.24. Proposition. *With $u\in U$, $\mu\in J_L(\mathfrak{g}\cdot q_e)$, $\eta\in\mathfrak{k}_2$,*

$$\begin{aligned}\langle X(0,u,\mu),\eta\rangle &= D^2V(q_e)\cdot(u,\eta_Q(q_e))\\ &\qquad-\left\langle\left(D^2\mathbb{I}(q_e)\cdot(u,\eta_Q(q_e))\right)\xi,\xi\right\rangle\\ &\qquad\qquad-2\left(\left\langle\mathbb{I}(q_e)\left[\frac{\partial\xi}{\partial\tau},\eta\right],\xi\right\rangle+\left\langle\mathbb{I}(q_e)\frac{\partial\xi}{\partial\tau},[\xi,\eta]\right\rangle\right),\end{aligned}$$

where $\xi=\xi(0,u,\mu)$ and $\partial\xi/\partial\tau=\partial\xi/\partial\tau(0,u,\mu)$.

Proof: We have that $\langle X(0,u,\mu),\eta\rangle=\frac{d}{d\tau}\big|_{\tau=0}\langle X'(\tau,u,\mu),\eta\rangle$. Differentiating

(3.13) we get

$$\left.\frac{d}{d\tau}\right|_{\tau=0} \langle X'(\tau,u,\mu),\eta\rangle = D^2V(q_e)\cdot(\eta_Q(q_e),u)$$
$$-\left\langle \left(D^2\mathbb{I}(q_e)\cdot(u,\eta_Q(q_e))\right)\xi(0,v,\mu),\xi(0,v,\mu)\right\rangle$$
$$-2\left\langle (D\mathbb{I}(q_e)\cdot\eta_Q(q_e))\frac{\partial\xi}{\partial\tau}(0,v,\mu),\xi(0,v,\mu)\right\rangle .$$

Applying proposition 1.11 to the last term in the right-hand side, which can be rewritten as

$$\left\langle (D\mathbb{I}(q_e)\cdot\eta_Q(q_e))\,\frac{\partial\xi}{\partial\tau},\xi\right\rangle = d\left\langle \mathbb{I}(\cdot)\frac{\partial\xi}{\partial\tau},\xi\right\rangle(q_e)\cdot\eta_Q(q_e)\,,$$

gives the desired result. □

3.5 Bifurcating branches of relative equilibria

Recall that $J_L(\mathfrak{g}\cdot q_e)=\mathfrak{m}_1\oplus\mathfrak{m}_2$.

3.25. Definition. For $(u,\mu)\in U\times J_L(\mathfrak{g}\cdot q_e)$ let $\Delta_{(u,\mu)}\in L^2(T_uU\times\mathfrak{m}_2,\mathbb{R})$ be given by

$$\Delta_{(u,\mu)}\left((\delta u,\nu),(\bar{\delta u},\bar{\nu})\right)=\frac{\partial^2W}{\partial u^2}\cdot(\delta u,\bar{\delta u})+\frac{\partial^2W}{\partial\mu_2\partial u}\cdot(\delta u,\bar{\nu})$$
$$+\frac{\partial\left\langle X,\hat{\mathbb{I}}(0)^{-1}\nu\right\rangle}{\partial u}\cdot\bar{\delta u}+\frac{\partial\left\langle X,\hat{\mathbb{I}}(0)^{-1}\nu\right\rangle}{\partial\mu_2}\cdot\bar{\nu}$$

with the partial derivatives evaluated at $(\tau=0,u,\mu)$. Here $\partial/\partial\mu_2$ denotes partial differentiation with respect to the $\mathfrak{m}_2$ component of μ.

Let $M=TQ$ and $\pi_{M,G}:M\longrightarrow M/G$ its canonical projection. Let $\mathcal{R}_e=\pi_{M,G}(\mathfrak{t}\cdot q_e)$, where $M=TQ$ and $\mathfrak{t}$ is as in the maximal abelian subalgebra of section 3.1.

3.26. Theorem. *For every $(u,\mu_1,\mu_2)\in U\times\mathfrak{m}_1\times\mathfrak{m}_2$ such that*

1. $\dfrac{\partial W}{\partial u}(0, u, \mu_1 + \mu_2) = \mathbf{0}$,

2. $X(0, u, \mu_1 + \mu_2) = \mathbf{0}$,

3. $\Delta_{(u,\mu_1+\mu_2)}$ *is non-degenerate,*

there exists a continuous curve $\rho_{(u,\mu_1,\mu_2)} : [0,1] \longrightarrow M/G$ *consisting of classes of relative equilibria such that*

$$\operatorname{Im}\rho_{(u,\mu_1,\mu_2)} \cap \mathcal{R}_e = \{\rho_{(u,\mu_1,\mu_2)}(0)\} ,$$

and $\rho_{(u,\mu_1,\mu_2)}(0)$ *is the unique class of relative equilibria in* M/G *corresponding to* $[q_e, \mu_1]_G$ *(see definition 3.17).*

Furthermore, the images of two such curves corresponding to two triples (u, μ_1, μ_2), (u', μ_1', μ_2') *have non-empty intersection only if* $\mu_1 = \mu_1'$ *and in this case, if* $(u, \mu_2) \neq (u', \mu_2')$*, then*

$$\operatorname{Im}\rho_{(u,\mu_1,\mu_2)} \cap \operatorname{Im}\rho_{(u',\mu_1,\mu_2')} = \{\rho_{(u,\mu_1,\mu_2)}(0)\} .$$

Proof: Suppose that conditions *1.–3.* hold at $(\bar{u}, \bar{\mu}_1, \bar{\mu}_2) \in U \times \mathfrak{m}_1 \times \mathfrak{m}_2$, $\bar{u} \neq 0$. We will use the notation $\bar{\mu} = \bar{\mu}_1 + \bar{\mu}_2$. Since $\Delta_{(\bar{u},\bar{\mu})}$ is non-degenerate then by the implicit function theorem there exists $\epsilon > 0$ and functions $u : (-\epsilon, \epsilon) \longrightarrow U$, $\mu_2 : (-\epsilon, \epsilon) \longrightarrow \mathfrak{m}_2$ such that $u(0) = \bar{u}$, $\mu_2(0) = \bar{\mu}_2$ and

$$\begin{aligned} \frac{\partial W}{\partial u}(\tau, u(\tau), \bar{\mu}_1 + \mu_2(\tau)) &= 0 , \\ \frac{\partial X}{\partial u}(\tau, u(\tau), \bar{\mu}_1 + \mu_2(\tau)) &= 0 . \end{aligned}$$

Therefore, from theorems 3.22 and 3.23, it follows that if $\tau > 0$ then the relative equilibria conditions (3.9) and (3.10) are both satisfied at $(u, \mu) = (\tau u(\tau), \bar{\mu}_1 + \mu_2(\tau))$ so that $[\tau u(\tau), \beta(\tau, \bar{\mu}_1 + \mu(\tau))]_H$ corresponds to a class of relative equilibria (see definition 3.17). Define $\rho_{(\bar{u},\bar{\mu}_1,\bar{\mu}_2)}(\tau)$ to be such class of relative equilibria.

It is clear that, for all $(q, \mu) \in Q \times \mathfrak{g}^*$, $[q, \mu]_G$ corresponds to a class of relative equilibria in $\mathcal{R}_e$ only if $q = q_e$ and hence $[v, \mu]_H$ corresponds to such class only if $v = 0$. Therefore $\rho_{(\bar{u},\bar{\mu}_1,\bar{\mu}_2)}(\tau) \in \mathcal{R}_e$ only if $\tau = 0$. It is clear that $\rho_{(\bar{u},\bar{\mu}_1,\bar{\mu}_2)}(0)$ corresponds to $[q_e, \mu_1]_G$. By rescaling we can assume that $\rho_{(\bar{u},\bar{\mu}_1,\bar{\mu}_2)}$ is defined on $[0, 1]$.

Suppose that $\pi_1\mu' \neq \pi_1\mu$, where $\mu', \mu \in \mathfrak{m}_1 \oplus \mathfrak{m}_2$. Then $\beta(\tau, \mu')$ is not in the H-orbit of $\beta(\tau, \mu)$. To see this it suffices to notice that $\mathfrak{m}_1$ is H -invariant and that $\pi_1\beta(\tau, \mu) = \pi_1\mu$. It follows that $[v', \beta(\tau', \mu')]_H \neq [v, \beta(\tau, \mu)]_H$ for all $v', v \in N$ and all $\tau', \tau \in \mathbb{R}$. Thus $\rho_{(\bar{u}',\mu')}(\tau') \neq \rho_{(\bar{u},\mu)}(\tau)$ for all $\bar{u}', \bar{u} \in U$ and all $\tau', \tau \in \mathbb{R}$. This shows that the intersection of the images of $\rho_{(\bar{u}',\mu')}$ and $\rho_{(\bar{u},\mu)}$ is non-empty only if $\pi_1\mu' = \pi_1\mu$.

Observe that $\beta(\tau', \mu')$ is in the H-orbit of $\beta(\tau, \mu)$ then $\tau' = \tau$. This follows from the fact that $\pi_0\beta(\tau, \mu) = \tau^2\nu_0$ (where ν_0 is a generator of $\mathfrak{m}_0$) and that $\mathfrak{m}_0$ is H-invariant. It follows that if $[\tau'\psi_1(u'), \beta(\tau', \mu')]_H = [\tau\psi_1(u), \beta(\tau, \mu)]_H$ then $\tau' = \tau$ and $u' = u$. Therefore, if $(\bar{u}', \bar{\mu}_1, \bar{\mu}_2'), (\bar{u}, \bar{\mu}_1, \bar{\mu}_2)$ are both data satisfying conditions *1.–3.* and $\bar{u}' \neq \bar{u}$ then $\rho_{(\bar{u}',\bar{\mu}_1,\bar{\mu}_2')}(\tau') = \rho_{(\bar{u},\bar{\mu}_1,\bar{\mu}_2)}(\tau)$ only if $\tau' = \tau = 0$. In a similar fashion, one proves that this is also the case if $\bar{\mu}_2' \neq \bar{\mu}_2$. This shows that if $(\bar{u}', \bar{\mu}_2') \neq (\bar{u}, \bar{\mu}_2)$ then the images of $\rho_{(\bar{u}',\bar{\mu}_1,\bar{\mu}_2')}$ and $\rho_{(\bar{u},\bar{\mu}_1,\bar{\mu}_2)}$ intersect only at $\tau = 0$. □

Chapter 4

Example: The symmetric coupled rigid bodies

In this chapter we will illustrate the application of the theory developed in chapter 3 with a concrete mechanical example, namely, two symmetric coupled rigid bodies moving in three-dimensional space with zero potential. (The precise description is given below.) In contrast with the double spherical pendulum (the example that we studied in chapter 2), the set of symmetric states in the symmetric coupled rigid bodies, from which branches of relative equilibria bifurcate, is not discrete.

There has been an extensive mathematical study of the symmetric coupled rigid bodies. In [36] and [37], Patrick studied the relative equilibria in this example using the augmented potential criterion together with an explicit classification of all the group orbits, thus achieving a complete enumeration of the relative equilibria. With a different approach, Mittagunta (cf. [27] and [28]) gave a lower bound on the number of relative equilibria in momentum level sets based on a Morse theoretic analysis of the topology of the reduced spaces. Roberts and de Sousa Dias (cf. [39]) used the Marle-Guillemin-Sternberg slice decomposition to study the bifurcation of relative equilibria nearby symmetric states in a system consisting also of symmetric rigid bodies but requiring the presence of a potential (to ensure a certain non-degeneracy condition).

In the analysis of the symmetric coupled rigid bodies presented in this chapter we do not attempt to obtain new results. Our objective is only to illustrate how

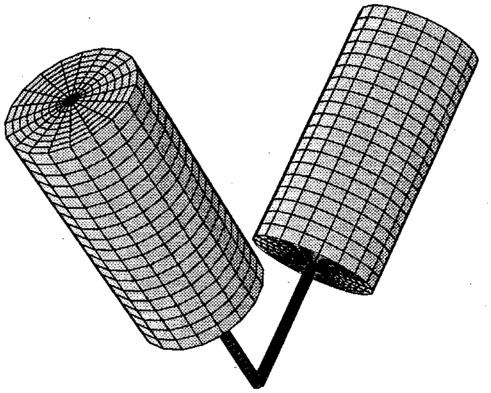

Figure 4.1: The symmetric coupled rigid bodies.

our theory can be applied to an example with symmetric states where the group is non-abelian and relatively large. Applying the technique of the previous chapter, we are able to recover the relative equilibria that bifurcate from the class of symmetric relative equilibria consisting of the states where the axis of symmetry of the two bodies are aligned, each body is rotating around its axis of symmetry with independent velocity and the total angular momentum of the system is different from zero. The branches of relative equilibria thus obtained (cf. proposition 4.3) break the symmetry.

4.1 Description and preliminaries

Consider the mechanical system formed by two symmetric rigid bodies with equal moments of inertia coupled by an ideal spherical joint along their axes of symmetry and such that the distance from the center of mass of either body to the joint is the same. (See figure 4.1.)

From this description it follows that we can attach to each rigid body a coordinate system with respect to which its inertia matrix is equal to

$$I = \begin{pmatrix} I^{xy} & & \\ & I^{xy} & \\ & & I^z \end{pmatrix} ,$$

for some positive real numbers I^{xy}, I^z. The Lagrangian of the system consists purely of kinetic energy. After reducing by translations, the configuration space becomes

$$Q := SO(3) \times SO(3) .$$

The physical interpretation of this configuration space is as follows. A given $(A_1, A_2) \in Q$ represents the configuration obtained by applying the rotation A_i to body i, where the initial *reference configuration* consists of the two bodies aligned on top of each other, the common center of mass lying at the origin and the common axis of symmetry being aligned with the $\mathbf{e}_3$-axis.

It is convenient to express elements in $TQ = T(\mathrm{SO}(3)^2)$ in terms of the **body coordinates**. These are defined by the diffeomorphism $\cap : (\mathrm{SO}(3) \times \mathbb{R}^3)^2 \longrightarrow T(\mathrm{SO}(3)^2)$ given by[1]

$$(A_i; \Omega_i)^\cap := (A_i; A_i\hat{\Omega}_i) ,$$

where $\wedge : \mathbb{R}^3 \longrightarrow \mathfrak{so}(3)$ is the standard isomorphism

$$\hat{X} = \begin{pmatrix} 0 & -X_3 & X_2 \\ X_3 & 0 & -X_1 \\ -X_2 & X_1 & 0 \end{pmatrix} .$$

[1]Here $(A_i; \Omega_i)$ represents $(A_1, A_2; \Omega_1, \Omega_2) \in \mathrm{SO}(3)^2 \times (\mathbb{R}^3)^2$ and $(A_i; \dot{A}_i)$ represents $(A_1, A_2; \dot{A}_1, \dot{A}_2) \in T((\mathrm{SO}(3))^2)$.

The inverse diffeomorphism is given by

$$(A_i; \dot{A}_i)^{\cup} := (A_i; (A_i^{-1}\dot{A}_i)^{\vee}) ,$$

where $\vee$ denotes the inverse of $\wedge$.

One verifies (cf. [37]) that the Lagrangian of the system after reduction by translations is given by

$$L(A_1, A_2; \dot{A}_1, \dot{A}_2) = \frac{1}{2}\Omega_1^t \tilde{J}\Omega_1 + \frac{1}{2}\Omega_2^t \tilde{J}\Omega_2 - \beta A_1(\mathbf{e}_3 \times \Omega_1) \cdot A_2(\mathbf{e}_3 \times \Omega_2)$$

where $\dot{A}_i = A_i\hat{\Omega}_i$, $\Omega_i \in \mathbb{R}^3$,

$$\tilde{J} := \begin{pmatrix} 1 & & \\ & 1 & \\ & & \alpha \end{pmatrix} ,$$

and the system parameters α, β are given by

$$\alpha = \frac{2I^{xy}}{I^{xy} + I^z + \epsilon}, \quad \beta = \frac{\epsilon}{I^{xy} + I^z + \epsilon} ,$$

with $\epsilon = (m_1 m_2)|S|^2/(m_1 + m_2)$, where $|S|$ denotes the distance from the center of mass of either body to the joint. β is called the *coupling constant.* Observe that $0 \leq \beta < 1$.

A straightforward computation shows that the fiber derivative corresponding to this Lagrangian is given (in body coordinates) by

$$\begin{aligned} \langle \mathbb{F}L(A_1, A_2)(\Omega_1, \Omega_2), (W_1, W_2)\rangle &= \Omega_1^T \tilde{J} W_1 + \Omega_2^T \tilde{J} W_2 \\ &-\beta \left(A_1(\mathbf{e}_3 \times W_1) \cdot A_2(\mathbf{e}_3 \times \Omega_2) + A_1(\mathbf{e}_3 \times \Omega_1) \cdot A_2(\mathbf{e}_3 \times W_2)\right) . \qquad (4.1) \end{aligned}$$

Now, consider the action of $G := SO(3) \times S^1 \times S^1$ on Q given by

$$(B, \theta_1, \theta_2) \cdot (A_1, A_2) = (BA_1 \exp(-\theta_1 \hat{\mathbf{e}}_3), BA_2 \exp(-\theta_2 \hat{\mathbf{e}}_3)) .$$

Physically this corresponds to rotating each body around its axis of symmetry by angles θ_1, θ_2 and then applying the rotation B to the system as a whole. One verifies that the Lagrangian is invariant with respect to the tangent lift of this action. Thus, G is the symmetry group of the system.

The Lie algebra $\mathfrak{g}$ of G is isomorphic to $\mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$ with the Lie bracket given by

$$[(\mathbf{x}, y_1, y_2), (\mathbf{x}', y_1', y_2')] = (\mathbf{x} \times \mathbf{x}', 0, 0) .$$

For every $\xi = (\mathbf{x}, y_1, y_2) \in \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R} \cong \mathfrak{g}$, the infinitesimal generator associated with the given action of G on Q is computed to be

$$\xi_Q(A_1, A_2) = (A_1, A_2; A_1^T \mathbf{x} - y_1 \, \mathbf{e}_3, A_2^T \mathbf{x} - y_2 \, \mathbf{e}_3)^{\cap} . \tag{4.2}$$

Identifying T^*Q with $\mathrm{SO}(3)^2 \times (\mathbb{R}^3)^2$ via the standard inner product on $(\mathbb{R}^3)^2$ and identifying $\mathfrak{g}^*$ with $\mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$ via the standard inner product on $\mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$, one computes that the *momentum map* $J : T^*Q \longrightarrow \mathfrak{g}^*$ associated with the cotangent lift of the action is given by

$$J(A_1, A_2, \Pi_1, \Pi_2) = (A_1 \Pi_1 + A_2 \Pi_2, -\Pi_1 \cdot \mathbf{e}_3, -\Pi_2 \cdot \mathbf{e}_3) .$$

4.2 Fiber over a symmetric point

We will now study the branches of relative equilibria emanating from a subspace of symmetric relative equilibria in the fiber over a symmetric point. The configurations with non-trivial isotropy are the ones in which the axis of symmetry of the two bodies are aligned, so that the two bodies lie on top of each other or they point in opposite directions. We will only treat the former case, since the latter is analogous.

Let $q_e = (\mathrm{Id}, \mathrm{Id}) \in Q$. This corresponds to a configuration consisting of the two bodies on top of each other. Let $\{\mathbf{e}_i\}_{i=1}^3$ be the canonical basis in $\mathbb{R}^3$. The isotropy subgroup of q_e is

$$G_{q_e} = \{(\exp(t\hat{\mathbf{e}}_3), t, t)\} \cong S^1 \,. \tag{4.3}$$

Its Lie algebra is $\mathfrak{g}_{q_e} = \mathrm{span}\{(\mathbf{e}_3, 1, 1)\}$.

Let $\mathcal{RE}$ denote the set of relative equilibria in TQ. Let us obtain $\mathcal{RE} \cap T_{q_e}Q$, the set of relative equilibria inside the fiber over q_e.

The augmented potential for the SCRB is given by

$$V_\xi(A_1, A_2) = \langle \mathbb{F}L(A_1, A_2) \cdot \xi_Q(A_1, A_2), \xi_Q(A_1, A_2) \rangle \ ,$$

where $\xi_Q(A_1, A_2)$ is given by 4.2. (See section 1.2 for the definition of the augmented potential.) Therefore $V_\xi(A_1, A_2)$ is given by 4.1 with

$$(\Omega_1, \Omega_2) = (W_1, W_2) = (A_1^T \mathbf{x} - y_1\, \mathbf{e}_3, A_2^T \mathbf{x} - y_2\, \mathbf{e}_3) \ .$$

A computation shows that, for $i = 1, 2$,

$$\Omega_i^T \tilde{J} \Omega_i = (A_i^T \mathbf{x}) \cdot \tilde{J}(A_i^T \mathbf{x}) - 2\alpha y_i (A_i^T \mathbf{x}) \cdot \mathbf{e}_3 + y_i^2 \alpha \ ,$$

and

$$A_1(\mathbf{e}_3 \times \Omega_1) \cdot A_2(\mathbf{e}_3 \times \Omega_2) = (A_1 \mathbf{e}_3 \times \mathbf{x}) \cdot (A_2 \mathbf{e}_3 \times \mathbf{x}) \ .$$

Collecting terms we get that

$$\begin{aligned} V_\xi(A_1, A_2) = (A_1^T \mathbf{x}) \cdot \tilde{J}(A_1^T \mathbf{x}) + (A_2^T \mathbf{x}) \cdot \tilde{J}(A_2^T \mathbf{x}) \\ - 2\alpha \left(y_1\, A_1^T \mathbf{x} + y_2\, A_2^T \mathbf{x} \right) \cdot \mathbf{e}_3 + \alpha (y_1^2 + y_2^2) \\ - 2\beta \left((A_1 \mathbf{e}_3) \times \mathbf{x} \right) \cdot \left((A_2 \mathbf{e}_3) \times \mathbf{x} \right) \ . \end{aligned}$$

For $i = 1, 2$, let $A_i = \exp(t\hat{\mathbf{w}}_i)$, $\mathbf{w}_i \in \mathbb{R}^3$. A computation shows that

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} (A_i^T \mathbf{x}) \cdot \tilde{J}(A_i^T \mathbf{x}) &= x^T [\hat{\mathbf{w}}_i, \tilde{J}]\mathbf{x} \\
&= 2(1-\alpha)(\mathbf{x}\cdot\mathbf{e}_3)\,(\mathbf{x}\times\mathbf{e}_3)\cdot\mathbf{w}_i \ , \\
\left.\frac{d}{dt}\right|_{t=0} \left[-2\alpha\left(y_1 A_1^T\mathbf{x} + y_2 A_2^T\mathbf{x}\right)\cdot\mathbf{e}_3\right] &= 2\alpha(\mathbf{x}\times\mathbf{e}_3)\cdot(y_1\mathbf{w}_1 + y_2\mathbf{w}_2) \ , \\
\left.\frac{d}{dt}\right|_{t=0} \left[2\beta\left((A_1\mathbf{e}_3)\times\mathbf{x}\right)\cdot\left((A_2\mathbf{e}_3)\times\mathbf{x}\right)\right] &= 2\beta(\mathbf{x}\cdot\mathbf{e}_3)\,(\mathbf{x}\times\mathbf{e}_3)\cdot(\mathbf{w}_1+\mathbf{w}_2) \ .
\end{aligned}$$

Collecting terms we obtain

$$\left.\frac{d}{dt}\right|_{t=0} V_\xi(A_1, A_2) = 2\sum_{i=1}^{2} \left\{\left[(1-\alpha-\beta)(\mathbf{x}\cdot\mathbf{e}_3) + \alpha y_i\right](\mathbf{x}\times\mathbf{e}_3)\right\}\cdot\mathbf{w}_i \ .$$

It follows that $dV_\xi(\mathrm{Id}, \mathrm{Id}) = 0$ if and only if either

$$(1-\alpha-\beta)(\mathbf{x}\cdot\mathbf{e}_3) + \alpha\, y_i = 0 \quad \text{(for both } i = 1 \text{ and } i = 2\text{)} ,$$

or

$$\mathbf{x}\times\mathbf{e}_3 = 0 \ .$$

From this computation and the augmented potential criterion (cf. 1.15), it follows that the relative equilibria inside $T_{q_e}Q$ are given by

$$\mathcal{RE}\cap T_{(\mathrm{Id},\mathrm{Id})}Q = \{\xi_Q(\mathrm{Id},\mathrm{Id}) \mid \xi \in \mathfrak{l}_1 \cup \mathfrak{l}_2\} \ ,$$

where

$$\begin{aligned}
\mathfrak{l}_1 &:= \mathrm{span}\{(\mathbf{e}_3, 0, 0), (\mathbf{0}, 1, 0), (\mathbf{0}, 0, 1)\} \ , \\
\mathfrak{l}_2 &:= \mathrm{span}\left\{(\mathbf{e}_1, 0, 0), (\mathbf{e}_2, 0, 0), \left(\frac{\alpha}{1-\alpha-\beta}\mathbf{e}_3, -1, -1\right)\right\} \ .
\end{aligned}$$

Notice that for every $v \in \mathfrak{l}_1 \cdot q_e$ we have that $G_v = G_{q_e}$, with G_{q_e} as in equation 4.3. In contrast, the relative equilibria corresponding to $\mathfrak{l}_2 \cdot q_e$ have trivial symmetry, i.e., $\forall v \in \mathfrak{l}_2 \cdot q$, $\mathfrak{g}_v = \{0\}$. (Notice that we can not adjust α so that $\mathfrak{g}_{q_e} \subset \mathfrak{l}_2$ because we would need $\alpha/(1-\alpha-\beta) = -1$, which in turn would imply $\beta = 1$, which is not possible unless the two bodies degenerate to point masses.) Therefore,

$$\mathcal{RE}_{q_e}^{G_{q_e}} := \mathcal{RE} \cap (TQ)^{G_{q_e}} \cap (T_{q_e}Q) = \mathfrak{l}_1 \cdot q_e \ , \tag{4.4}$$

which corresponds to the states in which the two bodies are rotating around their common axis of symmetry, each one with independent arbitrary constant angular velocity.

For the remaining of our discussion, we will only study the relative equilibria bifurcating from $\mathfrak{l}_1 \cdot q_e$.

4.3 Regularization of the amended potential

Recall that $\mathbb{I} : Q \longrightarrow L(\mathfrak{g}, \mathfrak{g}^*)$ is the locked inertia tensor induced by the metric on Q. (Cf. definition 1.9.) Consider the basis $\mathcal{B} = \{\xi_i\}_{i=1}^5$ for $\mathfrak{g}$ given by $\xi_1 = (\mathbf{e}_3, 1, 1)$, $\xi_2 = (\mathbf{0}, 1, 0)$, $\xi_3 = (\mathbf{0}, 0, 1)$, $\xi_4 = (\mathbf{e}_1, 0, 0)$ and $\xi_5 = (\mathbf{e}_2, 0, 0)$. Then $\mathfrak{g} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \mathfrak{k}_2$, where

$$\begin{aligned}
\mathfrak{k}_0 &:= \ker \mathbb{I}(q_e) = \operatorname{span}\{\xi_1\} \ , \\
\mathfrak{k}_1 &:= \operatorname{span}\{\xi_2, \xi_3\} \ , \\
\mathfrak{k}_2 &:= [\mathfrak{g}, \mathfrak{k}_0 \oplus \mathfrak{k}_2] = \operatorname{span}\{\xi_4, \xi_5\} \ .
\end{aligned}$$

Notice that $\mathfrak{l}_1 = \mathfrak{k}_0 \oplus \mathfrak{k}_1$ is a maximal abelian Lie subalgebra and that $\mathfrak{l}_1 \cdot q_e = \mathfrak{k}_1 \cdot q_e$.

Denote with $\mathcal{B}^*$ the dual basis of $\mathcal{B}$. Identify $\mathfrak{g}^*$ with $\mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$ via the natural inner product $\langle (\mathbf{x}, y_1, y_2), (\mathbf{x}', y_1', y_2') \rangle := \mathbf{x} \cdot \mathbf{x}' + y_1 y_1' + y_2 y_2'$ (where "$\cdot$" denotes the standard inner product on $\mathbb{R}^3$). Then $\mathcal{B}^* = \{\nu^i\}_{i=1}^5$, where $\nu^1 = (\mathbf{e}_3, 0, 0)$, $\nu^2 = (-\mathbf{e}_3, 1, 0)$, $\nu^3 = (-\mathbf{e}_3, 0, 1)$, $\nu^4 = (\mathbf{e}_1, 0, 0)$, $\nu^5 = (\mathbf{e}_2, 0, 0)$.

A calculation shows that the matrix of the locked inertia tensor at q_e with respect to the basis $\mathcal{B}, \mathcal{B}^*$ is given by

$$[\mathbb{I}(q_e)]_{\mathcal{B},\mathcal{B}^*} = \begin{pmatrix} 0 & & & & \\ & \alpha & & & \\ & & \alpha & & \\ & & & 2(1-\beta) & \\ & & & & 2\,(1-\beta) \end{pmatrix} . \tag{4.5}$$

Thus we see that $\mathbb{I}(q_e) \cdot \xi_i = \alpha\, \nu^i$ for $i = 2, 3$. Hence, $\mathbb{I}(q_e) \cdot (\mathfrak{k}_0 \oplus \mathfrak{k}_1) = \mathbb{I}(q_e) \cdot \mathfrak{k}_1 \subset \operatorname{span}(\nu^1, \nu^2, \nu^3) = \mathfrak{k}_2^\circ$. Therefore all the conditions of section 3.1 hold. This means that we can follow the recipe of chapter 3 for splitting and rescaling the dual of the Lie algebra in order to blow-up the amended potential.

As in section 3.2, the splitting $\mathfrak{g} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \mathfrak{k}_2$ induces the dual splitting $\mathfrak{g}^* = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$, where

$$\begin{aligned} \mathfrak{m}_0 &:= (\mathfrak{k}_1 \oplus \mathfrak{k}_2)^\circ = \operatorname{span}(\nu^1) \,, \\ \mathfrak{m}_1 &:= (\mathfrak{k}_0 \oplus \mathfrak{k}_2)^\circ = \operatorname{span}(\nu^2, \nu^3) \,, \\ \mathfrak{m}_2 &:= (\mathfrak{k}_0 \oplus \mathfrak{k}_1)^\circ = \operatorname{span}(\nu^4, \nu^5) \,. \end{aligned}$$

Notice that $\mathfrak{m}_2 = (\mathfrak{l}_1)^\circ$.

Now consider the map $\beta : \mathbb{R} \times (\mathbb{I}(q_e) \cdot \mathfrak{g}) \longrightarrow \mathfrak{g}^*$ defined in equation 3.2. For our example it is explicitly given as follows. Let $\mu = \mu_1 + \mu_2$ with $\mu_1 = x_2\nu^2 + x_3\nu^3$, $\mu_3 = x_4\nu^4 + x_5\nu^5$, so that $\mu_i \in \mathfrak{m}_i$ $(i = 1, 2)$. Then

$$\begin{aligned} \beta(\tau, \mu) = \beta(\tau; x_2, x_3, x_4, x_5) &= \mu_1 + \tau\mu_2 + \tau^2\nu^1 \\ &= \tau^2\nu^1 + x_2\nu^2 + x_3\nu^3 + \tau(x_4\nu^4 + x_5\nu^5) \,. \end{aligned}$$

Since we want to consider directions transversal to the group action at q_e, we define

$$N := (\mathfrak{g} \cdot q_e)^\perp = \operatorname{span}\{(\mathbf{e}_1, -\mathbf{e}_1)^\wedge, (\mathbf{e}_2, -\mathbf{e}_2)^\wedge\} \,.$$

Then G_{q_e} acts irreducibly on N. It is clear that

$$U_0 := \{\rho\,(\mathbf{e}_1, -\mathbf{e}_1)^\wedge \mid \rho > 0\}$$

is a (global) section of the principal bundle $(N \setminus \{0\}) \longrightarrow (N \setminus \{0\})/G_{q_e}$. Let

$$\mathbb{I}_0(\rho) := \mathbb{I}\left(\exp(\rho\,(\mathbf{e}_1, -\mathbf{e}_1)^\wedge)\right) .$$

A computation shows that the matrix representation of $\mathbb{I}_0$ with respect to the basis $\mathcal{B}, \mathcal{B}^*$ is given by

$$[\mathbb{I}_0(\rho)]_{\mathcal{B},\mathcal{B}^*} =$$

$$\begin{pmatrix} 2\left[\begin{matrix}\alpha(-1+\cos\rho)^2 \\ +(1+\beta)\sin^2\rho\end{matrix}\right] & \alpha(1-\cos\rho) & \alpha(1-\cos\rho) & 0 & 0 \\ \alpha(1-\cos\rho) & \alpha & 0 & 0 & \alpha\sin\rho \\ \alpha(1-\cos\rho) & 0 & \alpha & 0 & -\alpha\sin\rho \\ 0 & 0 & 0 & 2(1-\beta\cos(2\rho)) & 0 \\ 0 & \alpha\sin\rho & -\alpha\sin\rho & 0 & 2\left[\begin{matrix}(1-\beta)\cos^2\rho \\ +\alpha\sin^2\rho\end{matrix}\right] \end{pmatrix} .$$

As a check, the reader can verify that $\mathbb{I}_0(0)$ corresponds to the right-hand side of 4.5.

Let $\xi(\tau, \rho;\, x_i) := \mathbb{I}_0(\tau\rho)^{-1}\beta(\tau;\, x_i)$. From proposition 3.9 we know that $\xi(\tau, \rho;\, x_i)$ is a smooth function even in a neighborhood of $\tau = 0$, provided that ρ is away from zero. A computation shows that $\xi(\tau, \rho;\, x_i) = \xi_0 + \tau\,\xi_1 + O(\tau^2) \in \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$ with

$$\begin{aligned} \xi_0 &= \left(\frac{2 - (x_2 + x_3)\,\rho^2}{4\,(1+\beta)\,\rho^2}, \frac{x_2}{\alpha}, \frac{x_3}{\alpha}; 0, 0\right) , \\ \xi_1 &= \left(0, 0, 0; \frac{x_4}{2 - 2\,\beta}, \frac{x_5 - x_2\,\rho + x_3\,\rho}{2 - 2\,\beta}\right) . \end{aligned} \tag{4.6}$$

4.4 Relative equilibria bifurcating from the subspace of symmetric states in the fiber over q_e

Theorem 3.23 guarantees the existence of a $\mathfrak{k}_2^*$-valued smooth function X such that

$$dV_{\beta(\tau,\mu)}\left(\exp(\tau\rho(\mathbf{e}_1,-\mathbf{e}_1)^\wedge)\right)\cdot\eta_Q(q_0)=\tau\left\langle X(\tau,\rho,\mu),\eta\right\rangle$$

where $\eta\in\mathfrak{k}_2$ and $\mu:=(x_2,x_3,x_4,x_5)$. For the symmetric coupled rigid bodies, the formula for $\langle X,\eta\rangle|_{\tau=0}$ given in proposition 3.24 reduces to

$$\langle X(0,\rho,\mu),\eta\rangle=\langle\mathbb{I}_0(0)[\xi_1,\eta],\xi_0\rangle+\langle\mathbb{I}_0(0)\xi_1,[\xi_0,\eta]\rangle\ ,$$

where ξ_0 and ξ_1 are given in (4.6) and $\mathbb{I}_0(0)$ is equal to the right-hand side of 4.5. Using that $\mathfrak{k}_2=\operatorname{span}\{(\mathbf{e}_1,0,0),(\mathbf{e}_2,0,0)\}$ we get, after a computation, that the condition $X(0,\rho,\mu)=0$ is equivalent to the pair of equations

$$\left.\begin{aligned}\langle X(0,\rho,\mu),(\mathbf{e}_1,0,0)\rangle&=(x_5-\rho\,y_-)\,f(\rho,x_2,x_3)=0\\ \langle X(0,\rho,\mu),(\mathbf{e}_2,0,0)\rangle&=x_4\,f(\rho,x_2,x_3)=0\end{aligned}\right\}\qquad(4.7)$$

where $f(\rho,x_2,x_3)=\Delta/\left(4(\beta^2-1)\rho^2\right)$ with

$$\Delta=2(1-\beta)+(1+3\beta)\rho^2y_+\ ,\qquad(4.8)$$

and where we have introduced the linear change of variables

$$y_+:=x_2+x_3\ ,\quad y_-:=x_2-x_3\ .$$

The rescaled amended potential restricted to U_0 is given by

$$V_{\beta(\tau,\mu)}\left(\exp\left(\tau\,\rho(\mathbf{e}_1,-\mathbf{e}_1)^\wedge\right)\right)=\left\langle\beta(\tau,\mu),\mathbb{I}_0^{-1}(\tau\rho)\beta(\tau,\mu)\right\rangle\ .$$

By theorem 3.22 we know that $V_{\beta(\tau,\mu)}\left(\exp\left(\tau\,\rho(\mathbf{e}_1,-\mathbf{e}_1)^\wedge\right)\right)$ is a smooth function, even in a neighborhood of $\tau = 0$, provided that ρ is away from zero. A computation shows that its Taylor expansion with respect to τ is given by

$$V_{\beta(\tau,\mu)}\left(\exp\left(\tau\,\rho(\mathbf{e}_1, -\mathbf{e}_1)^\wedge\right)\right) = \frac{y_+^2 + y_-^2}{2\alpha} + \tau^2 W(\tau, \rho, \mu)$$

with

$$\begin{aligned} W(\tau,\rho,\mu) = \frac{(1+\beta)(x_4^2+x_5^2) - (1-\beta)y_+}{2(1-\beta^2)} + \frac{1}{2(1+\beta)\rho^2} \\ - \frac{\rho\, x_5\, y_-}{1-\beta} + \frac{\left((1-\beta)\, y_+^2 + 4\,(1+\beta)\, y_-^2\right)\rho^2}{8(1-\beta^2)} + O(\tau^2) \end{aligned}$$

Therefore,

$$\frac{\partial W}{\partial \rho}(0,\rho,\mu) = \frac{4(1-\beta) + 4(1+\beta)\rho^3 x_5\, y_- - \left((1-\beta)\, y_+^2 + 4\,(1+\beta)\, y_-^2\right)\rho^4}{4\,(\beta^2-1)\,\rho^3}\,. \tag{4.9}$$

From the pair of equations in 4.7 we see that $X(0,\rho,\mu) = 0$ if and only if either $\Delta = 0$ or

$$x_5 - \rho\, y_- = 0 \quad \text{and} \quad x_4 = 0\,. \tag{4.10}$$

If we assume that 4.10 holds then, after substituting in 4.9, we see that the equation $\partial W/\partial\rho(0,\rho,\mu) = 0$ is equivalent to

$$4 - \rho^4\, y_+^2 = 0\,, \tag{4.11}$$

and thus $\rho = \sqrt{2/|x_2 + x_3|}$.

In summary, we have shown the following

4.1. Proposition. *Given x_2, x_3 such that $x_2 + x_3 \neq 0$, let*

$$(\tilde\rho, \tilde x_4, \tilde x_5) = \sqrt{2/\left|x_2 + x_3\right|}\,\left(1, 0, x_2 - x_3\right)\,, \tag{4.12}$$

and let $\tilde{\mu} = (x_2, x_3, \tilde{x}_4, \tilde{x}_5)$. *Then*

$$X(0, \tilde{\rho}, \tilde{\mu}) = 0 \quad \textit{and} \quad \partial W/\partial\rho(0, \tilde{\rho}, \tilde{\mu}) = 0\,.$$

Eliminating $\rho^2 y_+$ from the equations 4.11 and $\Delta = 0$ (with Δ given by 4.8) we get

$$\beta(\beta^2 - 1) = 0\,.$$

Therefore:

4.2. Lemma. *If* $0 < \beta < 1$ *then the conditions* $\rho = \sqrt{2/|x_2 + x_3|}$ *and* $\Delta = 0$ *are mutually exclusive.*

Expressing $X(\tau, \rho, \mu)$ in terms of the basis of $\mathfrak{k}_2$ dual to $\{(e_1, 0, 0), (e_2, 0, 0)\}$, we compute from 4.7 and 4.9 that

$$\frac{\partial(X, \partial W/\partial\rho)}{\partial(\rho, x_4, x_5)}(0, \tilde{\rho}, \tilde{\mu}) =$$
$$\begin{pmatrix} \Delta\,\tilde{\rho}\,y_- & 0 & -\Delta\,\tilde{\rho} \\ 0 & \Delta\,\tilde{\rho} & 0 \\ -4[(1-\beta)y_+^2 + (1+\beta)y_-^2]\tilde{\rho}^3 & 0 & 4(1+\beta)\tilde{\rho}^3 y_- \end{pmatrix}.$$

The determinant is computed to be

$$\left|\frac{\partial(X, \partial W/\partial\rho)}{\partial(\rho, x_4, x_5)}(0, \tilde{\rho}, \tilde{\mu})\right| = \frac{16\sqrt{2}(\beta - 1)\Delta^2}{\sqrt{|x_2 + x_3|}}\,.$$

Therefore, if $0 < \beta < 1$ then it follows from lemma 4.2 and the implicit function theorem that the equations

$$\left.\begin{aligned} X(\tau, \rho; x_2, x_3, x_4, x_5) &= 0 \\ \frac{\partial W}{\partial\rho}(\tau, \rho; x_2, x_3, x_4, x_5) &= 0 \end{aligned}\right\} \qquad (4.13)$$

implicitly define the parameters (ρ, x_4, x_5) as smooth functions of (τ, x_2, x_3). More precisely, for every bounded open region $V \subset \mathbb{R}^2 \setminus \{(x_2, x_3) \mid x_2 + x_3 = 0\}$, there exists an $\epsilon > 0$ and smooth functions (ρ, x_4, x_5) defined on $(-\varepsilon, \varepsilon) \times V$ such that (ρ, x_4, x_5) evaluated at $(0, x_2, x_3)$ is equal to the right-hand side of 4.12 and for all $(\tau, x_2, x_3) \in (-\varepsilon, \varepsilon) \times V$,

$$(\tau, \rho(\tau, x_2, x_3);\ x_2, x_3, x_4(\tau, x_2, x_3), x_5(\tau, x_2, x_3))$$

is a solution of 4.13.

Recall that $\mathcal{RE}_{q_e}^{G_{q_e}} := \mathfrak{l}_1 \cdot q_e$ is the set of relative equilibria in $T_{q_e}Q$ with symmetry group equal to G_{q_e} (cf. equation 4.4) and that $\mathfrak{m}_2 := (\mathfrak{l}_1)^\circ \subset \mathfrak{g}^*$. Recall also that $\mathcal{B}^* = \{\nu^i\}_{i=1}^5$ is the basis for $\mathfrak{g}^*$ chosen in page 62. As a notational facilitator, let $\tilde{\mathcal{A}} : Q \times \mathfrak{g}^* \longrightarrow T^*Q$ be given by $\tilde{\mathcal{A}}(q, \mu) := \mathcal{A}_\mu(q)$, where $\mathcal{A}_\mu$ is the associated one-form of the mechanical connection introduced in section 1.1.4. From the preceding remarks we conclude the following

4.3. Proposition. *Suppose that $0 < \beta < 1$. For every $\mu_1 = x_2\nu^2 + x_3\nu^3 \in J_L(\mathcal{RE}_{q_e}^{G_{q_e}})$ such that $x_2 + x_3 \neq 0$ there exist an $\epsilon > 0$ and a curve $\left(\rho^{(\mu_1)}, \mu_2^{(\mu_1)}\right) : [0, \epsilon] \longrightarrow \mathbb{R} \times \mathfrak{m}_2$ such that*

$$\left(\rho^{(\mu_1)}(0), \mu_2^{(\mu_1)}(0)\right) = \sqrt{2/|x_2 + x_3|}\,(1, (x_2 - x_3)\nu^5)\,,$$

and such that, for $\tau \in [0, \epsilon]$, the curve

$$\alpha^{(\mu_1)}(\tau) := \left(\exp(\tau\rho^{(\mu_1)}(\tau)(\mathbf{e}_1, -\mathbf{e}_1)^\wedge), \tau^2\nu^1 + \mu_1 + \tau\mu_2^{(\mu_1)}(\tau)\right) \in Q \times \mathfrak{g}^*$$

satisfies that $\tilde{\mathcal{A}}\left(\alpha^{(\mu_1)}(\tau)\right)$ is a symmetry breaking branch of relative equilibria emanating from $\tilde{\mathcal{A}}(q_e, \mu_1)$.

Chapter 5

Conclusions

Results Obtained. We have given a method for predicting the existence of bifurcating branches of relative equilibria around a symmetric equilibrium or relative equilibrium in the context of simple mechanical G-systems, where G is a compact Lie group. The main results are contained in theorems 3.22, 3.23 and 3.26. The technical assumptions that are needed appear in section 3.1.

Most of the results on persistence or bifurcation of relative equilibria that we are aware of appear in the context of Hamiltonian G-systems on a general symplectic manifold[1]. In our thesis we have studied the problem in the special case of cotangent bundles, where we get more specific and detailed information. Our main contribution has been to prove a bifurcation result by *blowing-up* the amended potential in a neighborhood of a group orbit with non-trivial isotropy. This blowing-up consists of a simultaneous rescaling of the momentum and directions transversal to the group orbit in configuration space.

Future Directions. In the future we plan to generalize the results of this thesis by relaxing some of the technical assumptions that we made in the development of the general theory (cf. §3.1), namely:

i) allow the dimension of the isotropy subgroup to be bigger than one;

ii) generalize the theory to cover the case when the isotropy type of the points

[1] We have given a brief account of the literature in section 1.6.

in the region around, but excluding, the symmetric orbit is not trivial.

We believe that the first of these issues can be handled, at least in the case when the isotropy subgroup is a torus, by substituting the definition of Σ in proposition 3.9 with

$$\Sigma = \{v \in N \mid \Delta(v) \text{ is non-degenerate}\} ,$$

where $\Delta(v)$ is the quadratic form on $\mathfrak{g}$ given by

$$\Delta(v)(\xi,\eta) := \frac{1}{2}\left\langle D^2\mathbb{I}(0)\cdot(v,v)\xi,\eta\right\rangle - \left\langle (D\tilde{\mathbb{I}}(0)\cdot v)\xi, \hat{\mathbb{I}}(0)^{-1}(D\tilde{\mathbb{I}}(0)\cdot v)\eta\right\rangle .$$

The second issue listed above can probably be solved by generalizing our method in a way that takes into account the lattice of isotropy subgroups. Roughly, each branch in the lattice would induce a series of slice decompositions similar to the one described in this thesis. In any case, it is known in the general theory of bifurcations of systems with symmetry that the lattice of isotropy subgroups plays an important role[2].

Finally, it would be desirable to study the relationship between the blown-up amended potential and the energy-momentum method for stability analysis.

[2]See, e.g., [13].

Bibliography

[1] R. Abraham and Jerrold E. Marsden. *Foundations of Mechanics.* Addison-Wesley, second edition, 1985.

[2] Judith Arms, Jerrold E. Marsden, and Vincent Moncrief. Symmetry and bifurcations of momentum mappings. *Commun. Math Phys.*, 78:455–478, 1981.

[3] V.I. Arnold. *Mathematical Methods of Classical Mechanics.* Springer-Verlag, 1978.

[4] L. Bates and Eugene Lerman. Proper group actions and symplectic stratified spaces. *Pac. J. of Math*, 181(2):201–229, 1997.

[5] Theodor Bröcker and Tammo tom Dieck. *Representations of compact Lie groups.* Springer-Verlag, New York, 1985.

[6] J. W. Bruce and P. J. Giblin. *Curves and singularities.* Cambridge University Press, 2nd edition, 1992.

[7] Hernan Cendra, Jerrold E. Marsden, and Tudor S. Ratiu. Lagrangian reduction by stages. Technical report, UCSC, 1999.

[8] Pascal Chossat, Debra Lewis, Juan Pablo Ortega, and Tudor S. Ratiu. Bifurcation of relative equilibria in mechanical systems with symmetry. *in preparation*, 1999.

[9] R.H. Cushman and L.M. Bates. *Global Aspects of Classical Integrable Systems.* Birkhäuser Verlag, 1997.

[10] Michael Dellnitz, Jerrold E. Marsden, Ian Melbourne, and Jürgen Scheurle. Generic bifurcations of pendula. *Int. Series of Num Math*, 104:111–122, 1992.

[11] M.P. do Carmo. *Riemannian Geometry.* Birkhäuser, 1992.

[12] J.J. Duistermaat and J.A.C. Kolk. *Lie Groups.* Universitext. Springer, 1999.

[13] M. Golubitsky, D. Schaeffer, and I.N. Stewart. *Singularities and Groups in Bifurcation Theory (vol. II) (v. 69)*, volume 69 of *Applied Mathematical Sciences.* Springer-Verlag, 1988.

[14] Martin Golubitsky and David Schaeffer. *Singularities and groups in bifurcation theory (v. 51)*, volume II of *Applied mathematical sciences.* Springer-Verlag, New York, 1985.

[15] V. Guillemin and Alan Pollack. *Differential topology.* Prentice-Hall, Englewood Cliffs, N.J., 1974.

[16] V. Guillemin and S. Sternberg. A normal form for the moment map. In S. Sternberg, editor, *Differential Geometric Methods in Mathematical Physics*, volume 6 of *Mathematical Physics Studies.* D. Reidel Publishing Company, 1984.

[17] V. Guillemin and S. Sternberg. *Symplectic Techniques in Physics.* Cambridge University Press, 1984.

[18] A. V. Karapetyan. On construction of the effective potential in singular cases. *Regular and Chaotic Dynamics*, 5(2), 2000.

[19] Martin Krupa. Bifurcations of relative equilibria. *SIAM J. Math. Anal.*, 21(6):1453–86, 1990.

[20] Eugene Lerman. Normal modes in symplectic stratified spaces. *math.SG/9906007*, 1999.

[21] Eugene Lerman and T.F. Tokieda. On relative normal modes. *C. R. Acad. Sci. Paris Sér. I Math.*, 328:413–418, 1999.

[22] C. M. Marle. Modéle d'action hamiltonienne d'un groupe the Lie sur une variété symplectique. *Rend. Sem. Mat. Univers. Politecn. Torino*, 43(2):227–251, 1985.

[23] Jerrold E. Marsden. *Lectures on Mechanics*, volume 174 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1992.

[24] Jerrold E. Marsden and Tudor S. Ratiu. *Introduction to mechanics and symmetry : a basic exposition of classical mechanical systems*, volume 17 of *Texts in applied mathematics*. Springer-Verlag, New York, 1994.

[25] Jerrold E. Marsden, Tudor S. Ratiu, and Jürgen Scheurle. Reduction theory and the Lagrange-Routh equations. *Journal of Mathematical Physics*, 41(6):3379–3429, 2000.

[26] Jerrold E. Marsden and Jurgen Scheurle. Lagrangian reduction and the double spherical pendulum. *Z angew Math Phys*, 44, 1993.

[27] G. Mittagunta. Reduced spaces for coupled rigid bodies. *J. Nonlinear Sci.*, 6:293–310, 1996.

[28] Girija G. Mittagunta. *Reduced Spaces for Coupled Rigid Bodies and their Relation to Relative Equilibria*. Ph.D. Thesis, UCSC, 1994.

[29] J. Montaldi. Persistance d'orbites périodiques relatives dans les systèmes hamiltoniens symétriques. *C. R. Acad. Sci. Paris, série I*, 324:553–558, 1997.

[30] J. Montaldi. Persistence and stability of relative equilibria. *Nonlinearity*, 10:449–466, 1997.

[31] J.A. Montaldi, R.M. Roberts, and I.N. Stewart. Periodic solutions near equilibria of symmetric hamiltonian systems. *Ph.Tr. R.Soc.London*, A325:237–293, 1988.

[32] J. Moser. Periodic orbits near an equilibrium and a theorem by alan weinstein. *Comm. Pure Appl. Math.*, 29:727–747, 1976.

[33] Juan-Pablo Ortega. Relative normal modes for nonlinear hamiltonian systems. *in preparation*, 1999.

[34] Juan Pablo Ortega and Tudor S. Ratiu. Persistance and smoothness of critical relative elements in hamiltonian systems with symmetry. *C.R. Acad. Sci. Paris*, 325(l):1107–1111, 1997.

[35] Juan Pablo Ortega and Tudor S. Ratiu. The dynamics around stable and unstable hamiltonian relative equilibria. *in preparation*, 2000.

[36] George Patrick. *Two Axially Symmetric Coupled Rigid Bodies: Relative Equilibria, Stability, Bifurcations, and a Momentum Preserving Symplectic Integrator*. Ph.D. Thesis, UCB, 1990.

[37] George W. Patrick. The dynamics of two coupled rigid bodies in three space. In Jerrold E. Marsden, P.S. Krishnaprasad, and J.C. Simo, editors, *Dynamics and Control of Multivody Systems*, volume 97 of *Contemporary Mathematics*. American Mathematical Society, 1989.

[38] George W. Patrick. Relative equilibria of hamiltonian systems with symmetry: Linearization, smoothness, and drift. *Journal of Nonlinear Science*, 5:373–418, 1995.

[39] R. Mark Roberts and M.E.R. de Sousa Dias. Bifurcations from relative equilibria of hamiltonian systems. *Nonlinearity*, 10:1719–1738, 1997.

[40] Alan Weinstein. Normal modes for nonlinear hamiltonian systems. *Inventiones Math.*, 20:47–57, 1973.

[41] Claudia Wulff and Mark Roberts. Hamiltonian systems near relative periodic orbits. *in preparation*, 2001.