A TIME CORRELATOR FOR PROBLEMS IN AERODYNAMICS

Thesis by

George Tolmie Skinner

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ABSTRACT.

An instrument, of fairly simple design, for measuring time correlation functions of two stationary random electrical signals is discussed. It is intended primarily for use in problems connected with aerodynamically produced acoustic fields, but has suitable properties for application to a rather wide range of aerodynamic problems involving turbulent fields. It has been designed and constructed with a view to economy, and simplicity of operation, and makes extensive use of the general statistical properties of the problems for which it is intended.

A few experimentally determined auto-correlation functions are given in order to indicate the degree of accuracy achieved, and the Fourier transform of the auto-correlation function of a random input is compared with the power-spectrum of the same function.

Some practical points of general interest are discussed.

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SYMBOLS.

A	Area of Output Pulse.
E	Fixed Voltage.
T	Time to fill Counter at n counts per second.
$f(\omega)$	Power Spectrum of an input.
$h(\omega)$	Square of the Response Function of an Amplifier.
k	Slope of Saw-tooth Voltage used for sampling.
m	Sub-script denoting the value of a quantity after passing through an amplifier.
n	Counting Rate used in storing a sample.
n_1	Number Stored, representing a sample.
s	Number of Amplifier Stages.
T	Interstage Coupling Time-constant of Amplifier.
$u(t)$, $v(t)$	Stationary Random Variables.
δt	Time Interval representation of a sample.
δt^*	Recovered Time Interval representation of a sample.
τ	Delay Time
$\psi(\tau)$	Time Correlation Function.
$\psi(\vec{r}_1-\vec{r}_2, \tau)$	Space-Time Correlation Function.

A few other symbols are defined where they occur in the text.

I. INTRODUCTION.

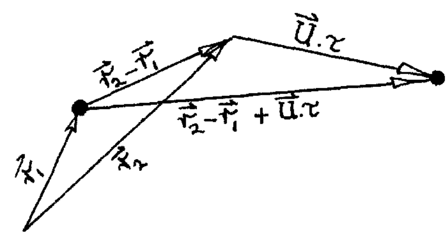
In the description of random fields, the correlation function is an important tool. If $u(\vec{r},t)$ and $v(\vec{r},t)$ are two random variables of a stationary, statistically homogeneous field then all quadratic mean values of u and v are expressible in terms of the correlation function

$$\psi(\vec{r}_2-\vec{r}_1, \tau) = \overline{u(\vec{r}_1, t)\, v(\vec{r}_2, t+\tau)} \tag{1}$$

where the bar denotes an ensemble average, which, for the types of processes to be considered here, will be equal to the time average. That is, an ergodic property will be assumed.

The space-time correlation function defined in (1) can, in many cases, be reduced to a space correlation when there exists a mean velocity which is large compared with the fluctuations, for example when turbulence in a stream of fluid is considered and the rate of decay is small. The large mean velocity, $\vec{U}$, permits the interpretation that the measuring instruments are passed rapidly through a pattern fixed in time, so that the approximation can be made that

$$\psi(\vec{r}_2-\vec{r}_1, \tau) = \psi(\vec{r}_2-\vec{r}_1 + \vec{U}\tau) \tag{2}$$



This, however, is not always possible.

Recently much interest has been expressed in aerodynamically produced acoustic fields. Here the above reduction is not possible as the acoustic disturbances are not propagated at the mean flow velocity, but at the speed of sound in the medium. The mean square pressure at a point can be expressed in terms of the space-time correlations of the turbulent field producing the noise, as is shown in reference (1). The technique of obtaining space correlations simply involves the use of two measuring probes at different locations. Introducing the time delay is the difficult part of the problem, and for the remainder of this work only the time-correlation of two random variables will be considered.

In addition to the above, it is possible that in certain cases of phenomena connected with spectra having predominantly low frequency contributions it may well be easier to measure auto-correlation than spectrum, since it is occasionally difficult to separate resonance peaks from continuous spectra with low-frequency electrical filters. Since the spectrum can be obtained by Fourier Transformation of the auto-correlation function, here lies a further possible use of the time-correlator. In fact, some work has been done in recent years on the study of heart beats by means of time correlators. In aerodynamics, some buffetting problems fall into the category of very low frequency phenomena. By the Wiener-Khinchin relations, if the power spectrum of $u(t)$ is $f(\omega)$ and the auto-correlation function is $\psi(\tau) = \overline{u(t)\,u(t+\tau)}$, then

$$f(\omega) = \frac{2}{\pi}\int_0^\infty \psi(\tau).\cos\omega\tau.d\tau$$

and
$$\psi(\tau) = \int_0^\infty f(\omega).\cos\omega\tau.d\omega. \qquad (3)$$

The first essential of a time correlator is some means of delaying an electrical signal (since the measurable quantities will ordinarily be obtained in the form of fluctuating voltages). Two fundamentally different approaches exist and may be described as follows:

a). The signal enters a "box" continuously and emerges continuously but delayed in time. This will be designated as the Continuous System.

b). Periodic samples of the signal are taken and stored in a "box" from which they are recovered after some predetermined delay time. This will be called the Sampling System.

The apparatus to be described operates on the latter principle, and the reasons for such a choice follow in the next section. Suffice it to say here that the signals to be handled will have nearly symmetrical probability distributions, and use is made of this in order to evolve as simple a device as possible for application to the current aerodynamic problems mentioned above.

II. CHOICE OF SYSTEM.

The continuous system usually takes one of two forms. In the first of these, the signal is passed through an electrical or an acoustic delay line. In the other, the signal is recorded on magnetic tape (or a magnetic drum) and recovered as the tape passes a pick-up head, so that a delay is introduced equal to the time of transit of the tape between the recording and pick-up heads.

The sampling system may also take two forms. In one, the sample of one signal is converted into a number, which is stored in some sort of counter, to be recovered later. In the other, the sample is converted into a pulse, of length proportional to the value of the sample, which is passed through a delay line.

To choose a system the short-comings of the elements involved must be discussed in the light of the expected applications. In low-speed turbulence, frequencies of a few cycles per second are to be expected, while in aerodynamic noise problems, such as those concerning acoustic radiation from boundary layers and jets, it is conceivable that frequencies approaching 100 kilocycles per second may exist. Correlation functions will then be expected to have significant contributions with maximum delays up to about 0.1 second in the former case, while in the latter case the important features may involve maximum delays of considerably less than 1 milli-second.

Clearly it is not likely that one single instrument could cover these ranges without becoming a complicated piece of electronic equipment. However, by suitable combinations of the various techniques it is possible that a relatively simple instrument can be constructed, and with inexpensive modifications can be made to handle any one of the ranges at a time.

To use a tape recorder for delay would involve an elaborate machine having many speed ranges and possibly a frequency modulation system to handle very low frequencies. For this reason, and because of the complicated compensation systems involved as a result of the frequency characteristics of magnetic tape and the associated recording and pick-up heads, the tape recorder was ruled out, although for certain purposes they have been found quite satisfactory.

The continuous delay system using an acoustic line was rejected because of size and distortion when long time delays and low frequencies are to be handled. For delays up to about 1 milli-second, the electrical delay line is certainly feasible, but it is felt that its use would be more effective in a sampling system, where amplitude distortion can be overcome, as will be explained presently.

By the use of various very high-speed electronic circuits it is possible to take a sample of almost any electrical signal. For the range of frequencies mentioned above, this presents no insurmountable problem, but, as will be shown later, for the types of signals anticipated (i.e. signals having very nearly symmetrical probability distributions) the accuracy with which each sample is measured is of little consequence, only the consistency of operation being important. Because of the statistical properties of such signals, it is possible, as will be shown, to form an accurate time-correlation function even with relatively "inaccurate" sampling.

Of importance is the fact that, once a sample is taken, it is possible by means of a pulse length-modulation system to achieve short (less than 1 milli-second) delays by the simple expedient of an electrical delay line, and yet by means of electronic counter storage to "hold" a sample as long as desired, a week being no particular problem.

This flexibility as regards available delay ranges makes the sampling system attractive. The disadvantage lies in the fact that for long delays, involving the use of a counter, only one sample can be undergoing delay at a time, consequently only a small fraction of the available information is used and time averaging of the results is more difficult.

A final consideration involves the method of forming the time average $\overline{u(t)\,v(t+\tau)}$. In a continuous system this usually takes the form of a pair of squaring circuits (e.g. thermocouples) carrying out the operation

$$\overline{(u+v)^2} - \overline{(u-v)^2} = (\overline{u^2} + \overline{v^2} + 2\overline{uv}) - (\overline{u^2} + \overline{v^2} - 2\overline{uv})$$
$$= 4\,\overline{uv}.$$

In a sampling system, on the other hand, the delayed sample arrives either as a pulse length or as a number (which, as will be discussed later, is synonymous with a pulse length). Hence the sample of $u(t)$ arrives at time $(t+\tau)$ in a form suitable for "gating" $v(t+\tau)$ for a short time which is proportional to $u(t)$, thus producing a sort of pulse whose amplitude is $v(t+\tau)$ and whose duration is proportional to $u(t)$. The area of this pulse is then directly given by $u(t)\,v(t+\tau)$. Time averaging provides the mean value.

It is hoped that the foregoing discussion indicates that the only real practical disadvantage of a sampling system for the problems on hand (i.e. where signals having nearly symmetrical probability distributions are to be correlated) arises out of the wastage of information when long delays are required, necessitating somewhat longer averaging periods. If this does not give rise to undue difficulty, it seems to be a small price to pay for the convenience of wide flexibility in delay times, and ease of multiplication.

On this basis, the sampling system was chosen and designed in principle. Perhaps not surprisingly, it was subsequently found that such an instrument had been built at M.I.T. [(2)]. This latter, however, is a very much more general and complicated sort of instrument, while the one described herein relies heavily on the statistical properties of the expected random fields to be dealt with, for its accuracy and simplicity.

III. PRINCIPLES OF OPERATION.

The overall function of the instrument is to perform the following operation on two stationary random variables, $u(t)$ and $v(t)$,

$$\psi(\tau) = \overline{u(t)\ v(t+\tau)} .$$

For the sake of clarity, an approximate treatment will be given first, followed by a more exact analysis.

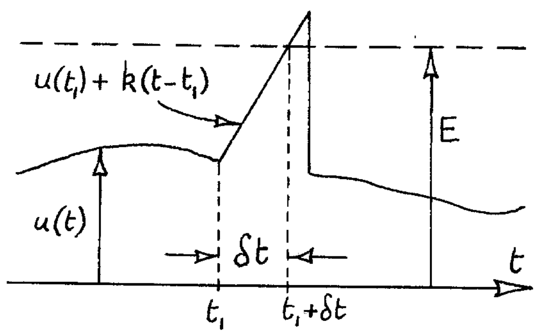
(i) Approximate Treatment.

Consider a slowly varying function of time, $u(t)$. At time t_1 let there be added to $u(t)$ a linearly increasing voltage, $k(t-t_1)$. If the variation of $u(t)$ during this addition is neglected, the sum will reach a predetermined level, E, after a time interval, δt, given by

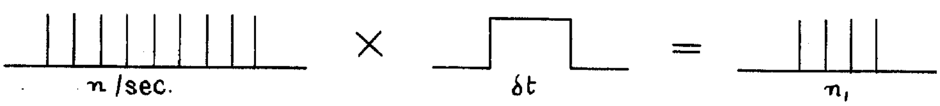
$$u(t_1) + k.\delta t = E$$

whence

$$\delta t = \frac{E}{k} - \frac{1}{k}\, u(t_1) \qquad (4)$$



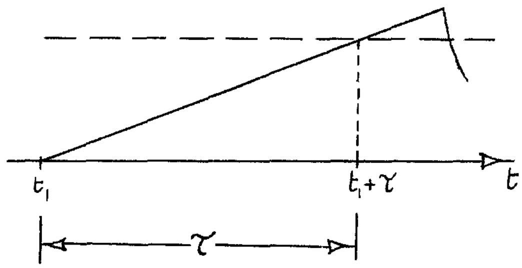
If a pulse generator, producing pulses at the rate n per second, is connected to an electronic counter of capacity N, by a "switch" which closes only during δt, the operation may be denoted by



where $n_1 = n.\delta t$ (5)

Clearly δt does not have to be an integral number of times the pulse spacing, $1/n$, but if n is assumed large enough equation (5) is very nearly exact.

The delay time, τ , may be generated by picking off a point on any monotonically rising voltage which starts at time t_1 . For example, a slow saw tooth can be used. τ can be varied by raising and lowering the pick-off voltage, and at time $(t_1+\tau)$ a signal, such as a sharp pulse, can be generated to initiate the multiplication of $u(t_1)$ by $v(t_1+\tau)$. This process is carried out as follows.



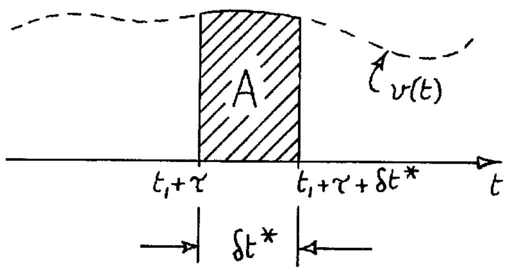
At time $(t_1+\tau)$ the "switch" between the pulse generator and counter is again closed, so that digits are fed into the counter at the rate n per second, until the counter fills up to its maximum capacity N , at which time it resets itself to zero and opens the "switch", turning off the supply of pulses. The time taken for this "filling-up" process is given by

$$\delta t^* = \frac{1}{n}(N - n_1) .$$

For convenience, a time, T , may be defined to be the time it would take to fill the counter, starting from zero, if the pulse generator were continuously connected to it. Clearly $T = N/n$, whence

$$\delta t^* = T - \delta t \tag{6}$$

Suppose, now, that there is an output terminal which is connected to $v(t)$ only during δt^*. Neglecting the variation of $v(t)$ during δt^*, this terminal will carry a pulse, of approximately rectangular shape, whose amplitude is $v(t_1+\tau)$ and whose duration is δt^*. The area of this pulse, then, is given by



$$A = \delta t^* \cdot v(t_1+\tau)$$

Using equations (6) and (4),

$$A = \left[\left(T - \frac{E}{k}\right) + \frac{1}{k}\,u(t_1)\right] v(t_1+\tau) .$$

Finally, if A is averaged over many repetitions of the process, and it is assumed that $\overline{v(t)} = 0$, then

$$A = \frac{1}{k}\,\overline{u(t)\,v(t+\tau)} = \frac{1}{k}\,\psi(\tau) . \tag{7}$$

If $u(t) \equiv v(t)$, then the auto-correlation is obtained.

(ii). More Exact Treatment.

Having established the principles of operation, a more detailed analysis will be followed. Here the essential difference will be that the variation of $u(t)$ and $v(t)$ will be taken into account by including the linear terms of the Taylor expansions of u and v about the points $t=t_1$ and $t=t_1+\tau$ respectively. Making use of the fact that the distribution functions of u and v are nearly symmetrical, it will be shown that the first correction term in equation (7) involves the second derivative of $\psi(\tau)$.

With the same numbering system for the equations as was used in part (i):-

$$u(t_1 + \delta t) \; + \; k\,\delta t \; = \; E \, ,$$

whence, writing $u(t_1+\delta t) = u(t_1) + \delta t . u'(t_1)$ and assuming $k \gg u'(t_1)$

(this is a practical requirement),

$$\delta t = \frac{E}{k} - \frac{u(t_1)}{k} - \frac{E}{k^2} u'(t_1) + \frac{u(t_1)u'(t_1)}{k^2} \qquad (4a)$$

Let $n_1 = n\,(\delta t - \epsilon)$ (5a)

where $0 \leqslant \epsilon < \frac{1}{n}$, and where n_1 is an integer.

The number registered during δt may then be

n_1 with probability $1 - n\epsilon$

$n_1 + 1$ " " $n\epsilon$

That is, there is an uncertainty of one count in the number stored during δt . This is akin to the problem of measuring the distance between two points with a ruler having very few graduations. If, for example, the points are 2.7 inches apart, and the ruler is graduated in inches, the number of graduations appearing between the points will be either 2 or 3. Now it is easy to show that if the ruler is laid across the points many times at random, the numbers 2 and 3 will occur with such probabilities that the average result will be 2.7. In fact, if the graduation marks have a width w , and a graduation is counted if any part of it appears between the points, then the average result will be $(2.7 + w)$. w corresponds to finite width pulses, and merely adds a constant to the result.

Since the counting rate, n , is very high compared with the rate of sampling, and these two are not in any way synchronized, it may be said that the interval δt is laid on the counting pulses at random.

If only linear expansions of u and v are to be considered, it may then be said that the expected value of n_1 is $n\delta t$ and the expected value of δt^* is given by

$$\delta t^* = T - \delta t \quad \text{(6a)}$$

The area under the $v(t)$ curve presented during δt^* will then be

$$A = \int_{t_1+\tau}^{t_1+\tau+\delta t^*} v(t) \, . \, dt$$

and using the linear expansion for $v(t)$,

$$A = \int_{t_1+\tau}^{t_1+\tau+\delta t^*} \left\{ v(t_1+\tau) + (t - t_1 - \tau) . v'(t_1+\tau) \right\} dt$$

$$= v(t_1+\tau) . \delta t^* + v'(t_1+\tau) . \frac{\delta t^{*2}}{2} \; .$$

Hence
$$\overline{A} = \overline{v(t_1+\tau) . \delta t^*} + \overline{v'(t_1+\tau) . \frac{\delta t^{*2}}{2}} \quad \text{(7a)}$$

which is evaluated in the appendix, for the types of functions to be considered, giving the result

$$A = \frac{1}{k} \left\{ \text{const.} + \psi(\tau) + \mathcal{O}\left[T^2 \psi''(\tau) \right] + --- \right\} \quad \text{(7b)}$$

The output circuit of the correlator is actually a balanced circuit in which the constant term is eliminated.

As a test, the correlator was checked on the auto-correlation of a square-wave, since this has discontinuities in $\psi(\tau)$. This will be discussed later.

IV. DESCRIPTION OF CIRCUIT.

The circuit consists of chains, or loops, of relatively simple elements, each of which performs one of the operations described in section III. The instrument incorporates slightly more than 40 vacuum tubes, mostly 6AK5 miniature pentodes, 12AX7, 12AU7 and 5963 twin triodes, and 6AL5 twin diodes. These are employed in such well-known circuits that little will be said about them and no circuit diagram will be attempted. The only critical adjustments are readily made with the aid of a good D.C. oscilloscope (which, incidentally is part of the equipment, being used for setting the delays). As the critical adjustments depend on the exact mode of construction, a circuit of the present set-up would be useless for any other arrangement. Considerable thought went into the arrangement of the circuit loops in order to ensure the fastest possible operation of the elements, and, for example, the diode-coupled 5963 scale-of-two counter storage operates very satisfactorily at 300 Kc/sec.

A point perhaps worth mentioning is that in this circuit, the flip-flops are driven through a separate diode for each operation, hence in the first place they cannot become confused, and in the second place their transitions follow the sharp edges of the triggering pulses and are never restrained by the tail of a pulse preventing a fast rise at the plates. Twin-triode flip-flops were found perfectly satisfactory. Voltage discriminators, on the other hand, were built from pairs of 6AK5's.

To avoid confusion, the circuit elements will be defined, and the symbols used in the schematic diagram (fig. 1) appended.

(i). ADDING NETWORK: Passive resistance network giving an output proportional to the sum of two or more inputs. (ADD.).

(ii). AVERAGER (or INTEGRATOR): A device which forms the time average (or integral) of the input signal. (AV.).

(iii). COUNTER: Chain of scales-of-two (Six in this case) which puts out a pulse when it is full (64 counts). (C).

(iv). DISCRIMINATOR: Circuit having two stable states, determined by the input voltage being above or below a predetermined value. The circuit puts out a pulse at transition. (D.).

(v). FLIP-FLOP: Circuit having two stable states, which is triggered by an input pulse. Output voltage has two possible values. As used in this set-up, the inputs are separated. (F.F.).

(vi). GATED AMPLIFIER: Amplifier to which $v(t)$ is applied and which gives an output of zero except when it is gated open during δt^*, when its output is proportional to $v(t)$. In the circuit it is actually a difference amplifier having inputs $v(t)$ and $-v(t)$, and being gated via the common cathode resistor. (G.A.).

(vii). TIME BASE (or LINEAR SWEEP GENERATOR): Circuit which produces a voltage increasing from zero linearly with time, started by an input pulse, and continuing for a time determined by a gating tube within the circuit. (T.B.).

(viii). D.C. VOLTAGE SOURCE: Potentiometer to give adjustable steady voltage for setting signal levels. (V.).

(ix). MIXER (or COINCIDENCE CIRCUIT): Circuit producing a fixed output when both inputs coincide -- acts as a switch between the continuous pulse generator and the counter. (M.).

(x). OSCILLATOR (or PULSE GENERATOR): 300 Kc/sec. oscillator followed by an amplifier which suppresses the negative halves of the waves. This is the counting frequency, n / second. (O.).

The operation of the instrument centers around the flip-flop, FF_1. (See fig. 1). An oscillator supplies the sampling frequency, and a discriminator, D_1, picks off a point on each cycle and from it produces a pulse which initiates one cycle of operation. Both time bases are triggered by this pulse, and the flip-flop, FF_1, is tripped causing the mixer, M, to feed the counting frequency from the oscillator, O, to the counter, C.

The fast time-base output is added to the input $u(t)$ and when the sum reaches a fixed value, E , the discriminator D_2 retriggers FF_1, cutting off the oscillator, O, from the counter, C, leaving C with a certain number registered.

Meanwhile, the slow time base output continues to rise, and when it reaches a preset value, determined by the triggering level of the discriminator, D_3, and the additive voltage, V , the delay pulse is generated, once again triggering FF_1 and supplying the counting frequency through M to C. FF_2 follows FF_1 through this part of the cycle.

This state continues until the counter is "full", and, as it returns to zero, it sends out a pulse to return both FF_1 and FF_2 to their initial states.

So far, FF_1 has gone through two cycles, and it will be observed that FF_2 follows it only on the second cycle. FF_2 drives a cathode follower whose load resistance is also the common cathode resistance of the difference amplifier, G.A. Ordinarily G.A. is cut off, but when FF_2 follows FF_1 on its second cycle, G.A. allows $v(t)$ to pass, thus producing an output approximately described by $\delta t^{*} \cdot v(t+\tau)$.

Finally, averaging, or integrating over a period of time, forms $\psi(\tau)$ as described in the theory.

Averaging was first accomplished by connecting a long-period ballistic galvanometer across the plates of the difference amplifier, G.A. The plate loads of the amplifier were by-passed with large capacitors in order to effect partial averaging before taking the output to the galvanometer. This arrangement proved quite satisfactory for periodic inputs, but was wholly unsatisfactory when tried on auto-correlation of a turbulence signal.

The next step taken was to construct G.A. from a pair of 6AK5 pentodes operating on 90 volts, with the load resistors removed and purely capacitive loads installed. 20 volts of bias would reduce the plate current to about 1/100 μ-Amp. so that little leakage took place during "off" periods. By charging the plate-load capacitors repeatedly for 30 seconds, discharging each time into a ballistic galvanometer, it was found that good averages could be obtained with from 5 to 10 minutes total integration time. In order to do this, 10 Meg Ω plate resistors were used with 0.1 μF capacitors by-passing them to a point held at the mean plate voltage (to reduce leakage variations). A 10^{10} Ω resistor and 0.1 μF polystyrene film capacitor were connected in series across the plates, and the voltage across the capacitor taken to an electrometer tube operating as a cathode follower and driving a micro-ammeter. This latter part is essentially the "micro-volter" described by Victoreen in their electrometer tube catalogue. The micro-volter and the battery for the galvanometer lamp are set on paraffin blocks and all connections are made with cotton-covered wire boiled in paraffin wax.

The integrating resistor and capacitor are actually incorporated in the micro-volter alongside the electrometer tube. The time-constant of the system is 15-1/2 minutes and either 5 or 10 minute integrations are used. The results presented here for random input were taken with 5 minute integrations. If the same time is used for each reading the results can be used directly, since some form of normalization will determine the scale.

The operation of the integrator was so consistent that it was used also in the spectrum measurements taken to check the auto-correlation curve discussed later. The output meter of a Hewlett-Packard hetrodyne wave analyzer was replaced by a 2.2 KΩ resistor (to increase the output voltage). The integrator was connected across the resistor via two 5 MΩ resistors and allowed to integrate the voltage for 5 minute periods, completely eliminating the dubious procedure of trying to "guess" the average position of the dancing meter needle.

Possible modifications to the correlator, as a whole, might involve the replacing of the counter storage by a delay line for very high frequency signals, at which time the sampling process can be speeded up. More accurate representation of the sample would be possible without the digital storage and more than one sample could be present in the line at one time.

For very short delays, with the instrument as it stands, (e.g. for the study of correlations near $\tau = 0$), a system of interlaced counting pulses derived from a square wave could be used to allow δt^* to overlap δt to some extent and hence remove the requirement that δt be completed before δt^* can start.

For very long delays, each individual product $u(t_i)v(t_i+\tau)$ could be integrated, and the results accumulated in a digital memory.

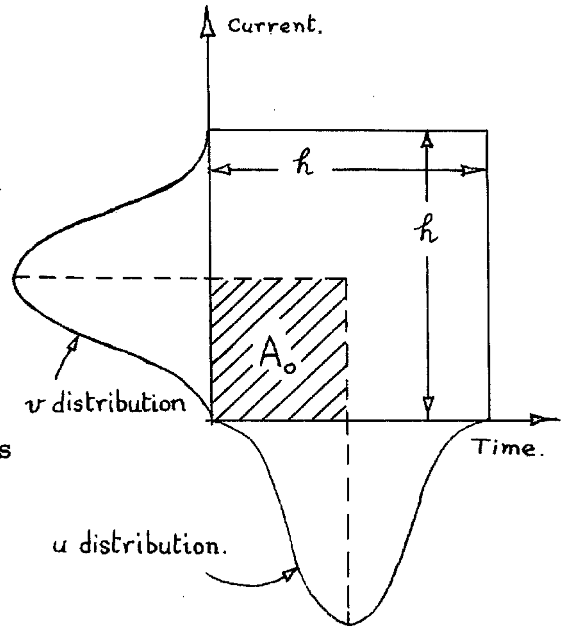
Several variations are undoubtedly possible to cover various requirements as they arise. In each case, of course, the limitations of the instrument must be considered.

The various circuit elements are discussed fully in references (9) and (10). The time-bases are of the bootstrap type, the fast one having 2nd order compensation.

V. SOME PRACTICAL DETAILS.

In section III a quantity A was defined. It is the output of the correlator per cycle of operations, and may be thought of as an essentially rectangular pulse of current. There are certain physical limitations to its size. Its average duration will be half the fill-up time, T, of the counter. With a counting rate of 300 Kc/sec. and a storage capacity of 64, $T/2$ is of the order of 10^{-4} seconds. If the rate of sampling is about 100 per second, then the "duty ratio" will be about 10^{-2}. That is, the overall average current will be about 1/100th of the average current per pulse. To give some figures without going into the details of the choice, the output pulse is drawn by a tube from a load consisting of 10 Meg Ω by-passed by 0.1 μF. For a mean voltage drop of 100 volts across this resistor, the mean current will be 10^{-5} amp. The average current per pulse, then, will be 10^{-3} amp. or 1 mA which is a reasonable figure considering that the tube must be operated at fairly low voltages in order to obtain small leakage when it is "cut-off".

By using a balanced circuit arrangement this mean drop of 100 volts is cancelled out, and it is of interest to enquire by how much it changes when the correlation coefficient goes from 0 to 1. The situation may be visualized as shown in the accompanying sketch. The solid rectangle shows the maximum size of a pulse, its minimum size being zero. With zero fluctuating input, the dotted rectangle, A_0, is obtained. The distribution of the signals



u and v are drawn on the time and current axes respectively and the meaning of the additive constants mentioned in the theory should now be apparent. Since only the fractional change in A_o is sought, no loss of generality arises from choosing equal scales for time and current.

Suppose that $v \equiv u$ and no time delay is introduced. That is the instrument is to form $\psi(0) = \overline{u(t)u(t)} = \overline{u^2(t)}$ (which, incidentally it cannot do, since it is necessary that τ be $> \delta t_{MAX}$). This would represent the maximum change in output expected.

Now in turbulence it is a fairly general result that the peaks of the signal lie within 4 times the r.m.s. value. Hence the gain controls on the inputs can be adjusted so that the peaks just cause the rectangle to run between the maximum value and zero. Hence each rectangle is given by

$$A = (\sqrt{A_o} + u)^2 \qquad \text{(when } v \equiv u \text{ and } \tau = 0 \text{).}$$

Hence $\overline{A} = A_o + \overline{u^2}$, and from the above statement, $\sqrt{\overline{u^2}} = h/8$, so that $\overline{u^2} = h^2/64$ while $A_o = h^2/4$.

Consequently $\dfrac{\overline{A} - A_o}{A_o} = \dfrac{\overline{u^2}}{A_o} = \dfrac{4}{64} = \dfrac{1}{16}$.

For linear operation it is necessary to have some extra current per pulse to ensure that $v(t)$ never causes the tube to operate too close to cut-off during the "on" period. So in practice, although the above estimates would indicate a mean change in plate voltage of 100/16 volts, it is more realistic to expect only about 3 volts. For a balanced circuit the change is doubled, with the result that unity correlation coefficient of a turbulence signal corresponds to a mean output of something like 6 volts. This is sufficient to give good accuracy with the integrating circuit.

VI. A USEFUL TRICK.

In the estimate carried out in section V it was assumed that the peak of a turbulence signal does not exceed four times its r.m.s. value, and it was assumed that the signal level can be adjusted to the point where such a peak value will just fill the counter storage. This is only approximate, and an occasional peak will cause the counter to over-fill, with the result that at a count of 64 it returns to zero and starts again. Consequently δt^*, which should be negative in such a case (an impossibility, of course) comes out rather large and positive. If many of these occur, considerable error will result and it would be more accurate to clip the signal so that such a high peak would lead to $\delta t^* = 0$.

In reference (5) it is shown that for cross-correlations of gaussian signals, if one signal is clipped, no matter how severely, the correlation remains correct except for a constant factor. Turbulence distributions do not deviate far from gaussian distributions, hence a certain amount of peak clipping should be tolerable. Such a step leads to larger outputs from the instrument for the same correlation function, and, up to a point, the accuracy is improved.

Ahead of the $u(t)$ attenuator, a box containing four 1-1/2 volt cells, two germanium diodes, and suitable resistances was inserted in the line to limit the peaks to approximately $\pm$ 3 volts. The amount of clipping is determined by the gain of the amplifier supplying the signal, and an attenuator in the correlator adjusts the result to suit the counter storage. The check result on turbulence was carried out using this crude clipper.

VII. TIME CORRELATOR IN PRACTICE.

(i). Effect of Low Frequency Cut-Off.

Here the relationship between the auto-correlation of a fluctuating velocity component in a turbulent fluid flow (e.g. downstream of a grid), and that obtained from the correlator by analyzing the signal delivered by a properly compensated a.c. hot-wire amplifier, will be discussed. The following quantities will enter the discussion:

The turbulence spectrum, $f(\omega)$, will be such that

$$\int_0^\infty f(\omega)\, d\omega = \overline{u^2} \qquad \text{and} \qquad f(0) \neq 0 .$$

The auto-correlation of the fluctuating velocity, u , will be

$$\psi(\tau) = \overline{u(t)\, u(t+\tau)} .$$

These are related by

$$f(\omega) = \frac{2}{\pi}\int_0^\infty \psi(\tau) \cos \omega\tau \, d\tau$$

and

$$\psi(\tau) = \int_0^\infty f(\omega) \cos \omega\tau \, d\omega .$$

It may be noted that $\psi(0) = \overline{u^2}$

Let it be assumed that the time-correlator is perfect. It will then form the exact auto-correlation of the signal it receives from the hot-wire amplifier. This signal will, in general, differ from the turbulent velocity in some important respects, primarily as a result of the low-frequency cut-off of the hot-wire amplifier.

If the output, u_2 , of the amplifier is related to its input u_1 by the relation

$$\left(\frac{u_2}{u_1}\right)^2 = h(\omega) \qquad \text{where} \qquad h(0) = 0$$

then the effect of the amplifier may be expressed in terms of $h(\omega)$.

Using subscript m to denote "measured" (or "entering the correlator"),

$$f_m(\omega) = h(\omega)\, f(\omega)$$

so that $\psi_m(\tau) = \int_0^\infty h(\omega)\, f(\omega) \cos \omega\tau \, d\omega$

and $\overline{u_m^2} = \psi_m(0) = \int_0^\infty h(\omega)\, f(\omega)\, d\omega.$

The amplifiers common in turbulence work have low frequency responses which are level down to less than 1 c.p.s., consequently there is not much difference between $\overline{u_m^2}$ and $\overline{u^2}$, or between $\psi_m(0)$ and $\psi(0)$.

However since $h(0) = 0$, so also $f_m(0) = 0$, with the unfortunate result that while

$$\frac{\pi}{2} f(0) = \int_0^\infty \psi(\tau)\, d\tau \neq 0 \qquad \text{for the actual turbulence,}$$

$$\frac{\pi}{2} f_m(0) = \int_0^\infty \psi_m(\tau)\, d\tau = 0 \qquad \text{for the measured signal.}$$

There is, of course, some argument as to whether $f(0) = 0$ in a wind tunnel of finite size, but certainly the cut-off occurs at fractions of 1 c.p.s. much smaller than the amplifier cut-off.

Assuming a power spectrum which approaches $f(0)$ horizontally, the effect of an amplifier whose low frequency cut-off is essentially contained within the region where $f(\omega) \doteq f(0)$ can be derived as follows.

$$\psi(0) - \psi_m(0) = \int_0^\infty \{1 - h(\omega)\} f(\omega)\, d\omega \doteq f(0) \int_0^\infty \{1 - h(\omega)\}\, d\omega$$

So that $\frac{\psi(0) - \psi_m(0)}{f(0)}$ is approximately equal to the fractional loss of energy due to the low-frequency cut-off of the amplifier. This is usually negligible in hot-wire anemometry equipment.

To obtain the effects for large values of τ the computation can be carried through fairly easily for the case of a "white" spectrum, and this shows the essential features. The amplifier may be taken to be one having s stages, each involving a resistance-capacity coupling of time-constant T . That is, the coupling time constant is the same for all stages throughout the amplifier. Then

$$f(\omega) = f(0) = \text{constant}$$

$$h(\omega) = \left(\frac{\omega^2 T^2}{1+\omega^2 T^2}\right)^s$$

$$\frac{\psi(\tau)-\psi_m(\tau)}{f(0)} = \int_0^\infty \left\{1 - \left(\frac{\omega^2 T^2}{1+\omega^2 T^2}\right)^s\right\} \cos\omega\tau \, d\omega$$

$$= \int_0^\infty \left\{1 - \left(\frac{\omega^2}{a+\omega^2}\right)^s\right\} \cos\omega\tau \, d\omega$$

where $a = 1/T^2$

Integrating by parts: -

$$\frac{\psi-\psi_m}{f(0)} = \left[\left\{1-\left(\frac{\omega^2}{a+\omega^2}\right)^s\right\} \frac{\sin\omega\tau}{\tau}\right]_0^\infty - \frac{2a}{\tau}\int_0^\infty \frac{\omega^{2s-1}}{(a+\omega^2)^{s+1}} \sin\omega\tau \, d\omega$$

Now as $\omega \rightarrow \infty$, $\frac{\omega^2}{a+\omega^2} \rightarrow 1$

and as $\omega \rightarrow 0$, $\sin\omega\tau \rightarrow 0$

so that the first term vanishes at both limits.

Therefore

$$\frac{\psi-\psi_m}{f(0)} = \frac{2a}{\tau}\int_0^\infty \frac{\omega^{2s-1}}{(a+\omega^2)^{s+1}} \sin\omega\tau \, d\omega .$$

This Fourier sine transform can be found in reference (6).

It is necessary that $0 \leq (2s-2) \leq 2s$, which is automatically satisfied, and that $|\arg a| < \pi$. Since $a = 1/T^2$ is real and positive, the latter is satisfied also.

Then

$$\frac{\psi-\psi_m}{f(0)} = \frac{2a}{\tau}\frac{(-1)^{2s-1}}{s!}\frac{\pi}{2}\frac{d^s}{da^s}\left(a^{s-1}e^{-a^{1/2}\tau}\right)$$

$$= \frac{-\pi a}{\tau s!}\frac{d^s}{da^s}\left(a^{s-1}e^{-a^{1/2}\tau}\right).$$

Using the expression for the s 'th derivative of a product, this becomes

$$\frac{\psi-\psi_m}{f(0)} = \frac{-\pi a}{\tau s!}\sum_{k=0}^{s}\frac{s!}{(s-k)!\,k!}\;\frac{d^k}{da^k}\left(e^{-a^{1/2}\tau}\right)\frac{d^{s-k}}{da^{s-k}}\left(a^{s-1}\right)$$

$$= \frac{-\pi a}{\tau}\sum_{k=1}^{s}\frac{a^{k-1}}{(s-k)!\,k!}\;\frac{(s-1)!}{(k-1)!}\;\frac{d^k}{da^k}\left(e^{-a^{1/2}\tau}\right)$$

which is of the form

$$\frac{\psi-\psi_m}{f(0)} = \frac{-\pi a}{\tau}\,e^{-a^{1/2}\tau}\left(\alpha_1\tau+\alpha_2\tau^2+\text{-----}+\alpha_s\tau^s\right)$$

$$= -\pi a\,e^{-a^{1/2}\tau}\left(\alpha_1+\alpha_2\tau+\text{-----}+\alpha_s\tau^{s-1}\right)$$

By substituting $x = a\tau^2$ it can be seen that the number of zeros of this function is the same as that of $\frac{d^s}{dx^s}\left(x^{s-1}e^{-\sqrt{x}}\right)$, which is $(s-1)$.

Hence $\frac{\psi-\psi_m}{f(0)}$ will cross the τ axis $(s-1)$ times and will approach zero as $\tau\to\infty$. This feature is noticeable in some of the results in references (7) and (8).

For the simple case of a single dominating time constant, $s = 1$, and

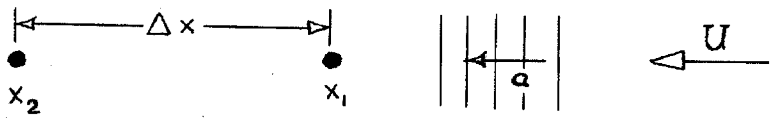
$$\frac{\psi-\psi_m}{f(0)} = \frac{-\pi a}{\tau}\frac{d}{da}\left(e^{-a^{1/2}\tau}\right) = \frac{\pi}{2}a^{\frac{1}{2}}e^{-a^{1/2}\tau} = \frac{\pi}{2T}\,e^{-\tau/T}$$

which is a monotonic decreasing function of τ. It must be remembered that this was computed for a "white" spectrum, and hence is not valid near $\tau = 0$.

(ii). <u>Separation of Acoustic Radiation from Turbulence.</u>

In a turbulent fluid motion that part of the fluctuation field which travels at the mean stream velocity may conveniently be defined as turbulence, while the disturbances which travel at the local sound velocity may be considered as noise. When this is done, the two parts can be separated, assuming the acoustic field consists of plane waves traveling in one direction, as follows.

Consider two hot-wires placed a distance Δx apart and aligned with the mean flow velocity U. Let there be plane acoustic waves traveling at velocity a also in the direction of U, and assume that the rate of attenuation of these waves is small.



The fluctuation velocity consists of two parts:

$$u(x,t) = u_T(x,t) + u_N(x,t)$$

where subscript T denotes turbulence and N denotes noise.

Then $u(x_1,t) = u_T(x_2, t+\frac{\Delta x}{U}) + u_N(x_2, t+\frac{\Delta x}{a})$,

where the decay of u_T is assumed negligible over Δx.

$$\begin{aligned}\text{Let} \quad \psi(\tau) &= \overline{u(x_2,t)\, u(x_1, t+\tau)} \\ &= \overline{u_T(x_2,t)\, u_T(x_2, t+\tau+\tfrac{\Delta x}{U})} + \overline{u_N(x_2,t)\, u_N(x_2, t+\tau+\tfrac{\Delta x}{a})} \\ &\quad + \overline{u_T(x_2,t)\, u_N(x_2, t+\tau+\tfrac{\Delta x}{a})} + \overline{u_N(x_2,t)\, u_T(x_2, t+\tau+\tfrac{\Delta x}{U})} \\ &= \psi_{TT}(\tau+\tfrac{\Delta x}{U}) + \psi_{NN}(\tau+\tfrac{\Delta x}{a}) + \psi_{TN}(\tau+\tfrac{\Delta x}{a}) + \psi_{NT}(\tau+\tfrac{\Delta x}{U})\end{aligned}$$

If u_T and u_N are independent, then $\psi_{TN} = \psi_{NT} = 0$.

If Δx is large enough so that $\psi_{TT}(\frac{\Delta x}{U}) \doteq 0$, then $\psi_{TT}(\tau+\frac{\Delta x}{U}) \doteq 0$ for all $\tau \geqslant 0$, so that, approximately,

$$\psi(\tau) = \psi_{NN}(\tau + \frac{\Delta x}{a}).$$

Alternatively it may be said that if the auto-correlation of u_T is zero for delays $\geqslant \tau_0$, Δx must be chosen such that $\Delta x = U\tau_0$. Then $\psi(\tau) = \psi_{NN}(\tau + M\tau_0)$ where M is the local Mach Number.

As an example, consider isotropic turbulence at atmospheric pressure with free-stream U = 1,000 cms./sec. Producing the turbulence with, say, a 1/2 inch grid, it would have a characteristic time of about 1/2 milli-second. So at 10 milli-seconds delay the auto-correlation should be sensibly zero. That is, Δx should be about 10 cms. Since $a = 33 \times 10^3$ cms./sec, $M = 0.03$, and $\tau_0 M = 0.3$ milli-seconds. Delaying either $u(x_1,t)$ or $u(x_2,t)$ enables the whole auto-correlation curve of the noise to be obtained.

VIII. PERFORMANCE.

The correlator, as it stands, has been tested on sine-waves, square-waves (which, of course, give discontinuities in the derivative of $\psi(\tau)$) and on turbulence behind a grid. These results are shown in figures 2, 3 and 4. Since the correlator is not designed to take square-waves, the discrepancies near the negative peaks in figure 3 are most likely due to asymmetrical distortion of the wave-form.

Figure 4 is the result after a turbulence signal has been passed through an amplifier having one dominant short time-constant. The experiment was actually carried to much larger values of τ , and, within experimental accuracy, a second crossing of the axis could not be detected. It thus bears out the features discussed in the last section.

Figure 5 shows the fourier transform of figure 4 plotted along with the measured power spectrum.

IX. CONCLUSIONS.

When the effects of auxiliary equipment are not forgotten, the instrument appears to be satisfactory for work in turbulent flows. For boundary layer studies, as discussed in reference (1), some modifications may be necessary. By defining a time scale from a characteristic length and the flow velocity, an idea of the significant time delays can be obtained. Referring to figure 4, the grid size was about 1/2 cm. and the flow velocity about 10 meters/sec. Hence the time scale may be taken as 1/2 m. second. The maximum delay required is about 10 times this value. Carrying out the same estimate for a boundary layer on a flat-plate 10 cms. long at a Reynolds' number of 5×10^5, and forming a time scale from the boundary layer thickness and the flow velocity, one obtains a figure of about 100 μ-seconds. Maximum delays of about 1 m. second might be required. If so, the rate of sampling could be increased 5 times, while the speed with which each sample is taken could be increased correspondingly, so that the details of higher frequency components could be "caught".

That is to say, as the phenomenon contains higher and higher frequencies, the time scales should go down, and the appropriate changes to the correlator fit it to the task.

Instrumentation, in the way of probes, to pick up such phenomena is under continual development.

APPENDIX.

DERIVATION OF EQUATION (7b).

$$\overline{A} = \overline{v(t+\tau)\,\delta t^*} + \overline{v'(t+\tau)\,\frac{\delta t^{2*}}{2}} \qquad (7a)$$

Substituting from (4a) into (6a),

$$\delta t^* = (T - E/k) + \tfrac{1}{k}\,u(t) + \tfrac{E}{k^2}\,u'(t) - \tfrac{1}{k^2}\,u(t)\,u'(t)$$

or

$$k\,\delta t^* = (kT - E) + u(t) + E\,\frac{u'(t)}{k} - u(t)\,\frac{u'(t)}{k}$$

Neglecting terms containing $\left(\frac{u'(t)}{k}\right)^2$,

$$\frac{1}{2}k^2\,\delta t^{2*} = \frac{1}{2}(kT-E)^2 + \frac{1}{2}u^2(t) + (kT-E)\,u(t) + (kT-E)E\,\frac{u'(t)}{k} - (kT-E)\,u(t)\frac{u'(t)}{k}$$
$$+ E\,u(t)\,\frac{u'(t)}{k} - u^2(t)\,\frac{u'(t)}{k}$$

Henceforth it will be understood that u and u' are evaluated at t, while v and v' are evaluated at $(t+\tau)$. So,

$$k\,\overline{A} = (kT-E)\,\overline{v} + \overline{uv} + E\,\frac{\overline{u'v}}{k} - \frac{\overline{uu'v}}{k}$$
$$+ \frac{1}{2}(kT-E)^2\,\frac{\overline{v'}}{k} + \frac{1}{2}\,\frac{\overline{u^2v'}}{k} + (kT-E)\,\frac{\overline{uv'}}{k}$$

where second order terms in $\frac{u'}{k}$ and $\frac{v'}{k}$ have been omitted.

From stationarity considerations,

$$\psi_{11}(\tau) = \overline{u(t)\,v(t+\tau)} = \overline{u(t-\tau)\,v(t)}$$

$$\therefore\ \psi_{11}'(\tau) = \overline{u(t)\,v'(t+\tau)} = -\overline{u'(t-\tau)\,v(t)} = -\overline{u'(t)\,v(t+\tau)}$$

and

$$\psi_{21}(\tau) = \overline{u^2(t)\,v(t+\tau)} = \overline{u^2(t-\tau)\,v(t)}$$

$$\therefore\ \psi_{21}'(\tau) = \overline{u^2(t)\,v'(t+\tau)} = -2\,\overline{u(t-\tau)\,u'(t-\tau)\,v(t)} = -2\,\overline{u(t)\,u'(t)\,v(t+\tau)}$$

That is $\overline{uv'} = -\overline{u'v}$ and $\overline{u^2v'} = -2\,\overline{uu'v}$

Also $\overline{v'} = 0$

Hence

$$k\,\overline{A} = (kT-E)\,\overline{v} + \overline{uv} + (2E - kT)\,\frac{\overline{u'v}}{k} - 2\,\frac{\overline{uu'v}}{k}$$

From the principles of operation it should be clear that u and v are always positive. The actual inputs, however have zero mean values and sufficiently large constants are added to these in the instrument. Let the actual input voltages be ξ and η, such that $\overline{\xi} = \overline{\eta} = 0$, and let the additive constants be U and V. Then

$$u = U + \xi \quad , \qquad v = V + \eta \qquad\qquad \text{and}$$

$$\begin{aligned} k\overline{A} &= (kT - E)V + UV + \overline{\xi\eta} + (2E - kT)\,\overline{\frac{\xi'}{k}\eta} \\ &\qquad - 2U\,\overline{\frac{\xi'}{k}\eta} - 2\,\overline{\xi\frac{\xi'}{k}\eta} \\ &= \text{const.} + \overline{\xi\eta} + (2E - kT - 2U)\,\overline{\frac{\xi'}{k}\eta} - 2\,\overline{\xi\frac{\xi'}{k}\eta} \end{aligned}$$

In operation, the adjustments are such that, approximately,

$$kT = E \qquad \text{and} \qquad 2U = E$$

So $(2E - kT - 2U) \approx 0$ and the presence of ξ'/k in the correlation makes this term of smaller order than $\overline{\xi\eta}$. In reference (3) it is shown that for auto-correlation $(\xi \equiv \eta)$ this term would be identically zero in the case of turbulence. Reference (4) shows that, for random variables having symmetrical probability distributions, odd order correlations are zero. Turbulence, of course, is not quite symmetrical in the derivatives of the fluctuating velocities, but the deviation is not large. Hence the last term can be assumed negligible, particularly as k appears in the denominator here also.

Hence to first order, one obtains

$$\overline{A} = \frac{1}{k}\left\{\text{const.} + \psi(\tau)\right\}$$

where $\quad \psi(\tau) = \overline{\xi(t)\,\eta(t+\tau)}$.

Now that the additive constants have been absorbed into one constant, u and v may be considered as the actual inputs and the notation

$$\psi(\tau) = \overline{u(t)\,v(t+\tau)}$$

may be resumed where it is now understood that $\overline{u} = \overline{v} = 0$.

It is not hard to see that a second order correction will give rise to terms of the form

$$\frac{\overline{uv''}}{k^2}\ ,\quad \frac{\overline{u'v'}}{k^2}\ ,\quad \frac{\overline{u''v}}{k^2}$$ and so on.

These are the second derivatives of $\psi(\tau)$, so that the expression for $\overline{A}$ may be written

$$\overline{A} = \frac{1}{k}\left\{\text{const.} + \psi(\tau) + \mathcal{O}\left[\frac{E^2}{k^2}\,\psi''(\tau)\right]\right\}$$, or since

$E/k \approx T$

$$\overline{A} = \frac{1}{k}\left\{\text{const.} + \psi(\tau) + \mathcal{O}\left[T^2\,\psi''(\tau)\right]\right\} \qquad (7b).$$

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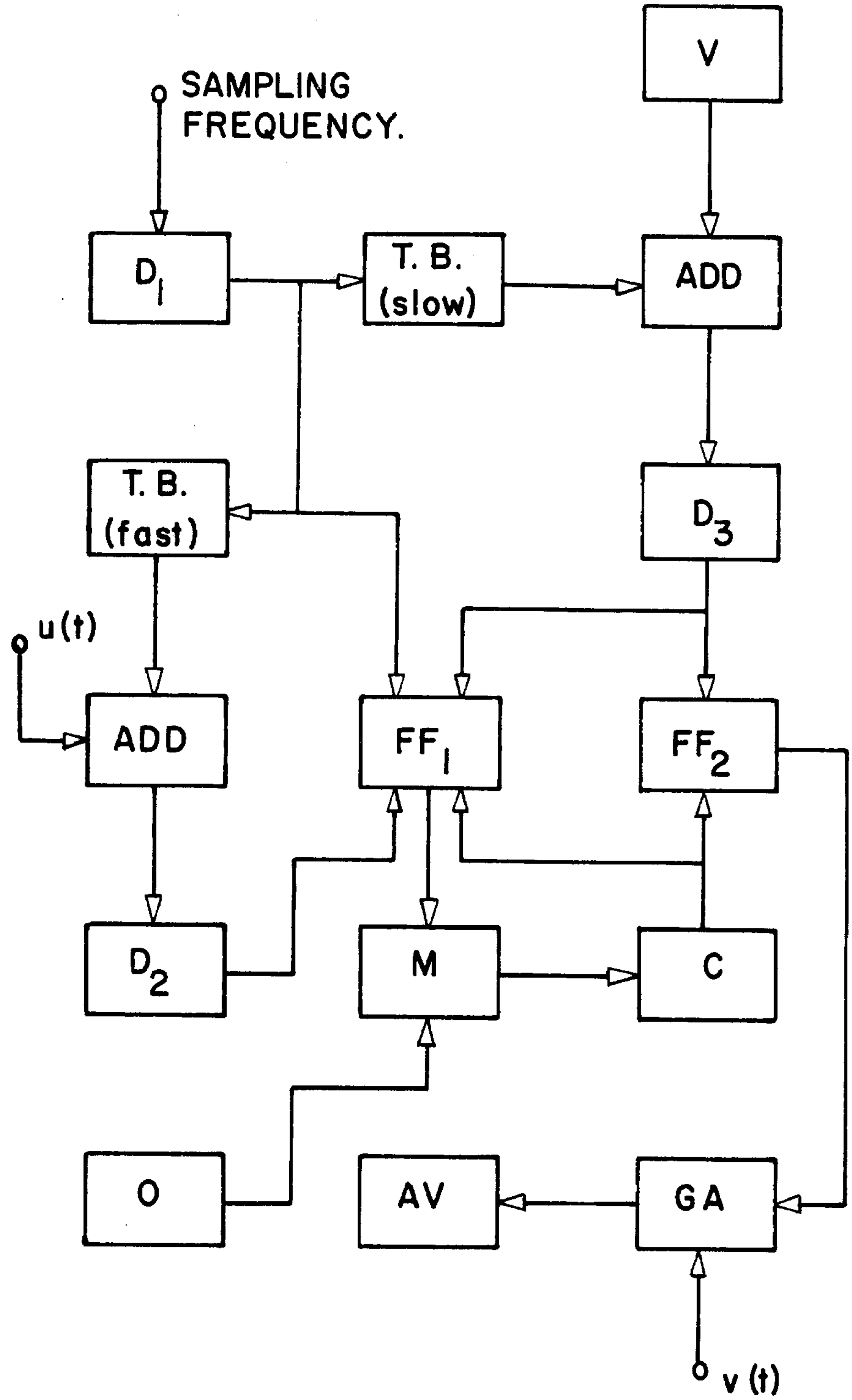

FIG. I. BLOCK DIAGRAM.

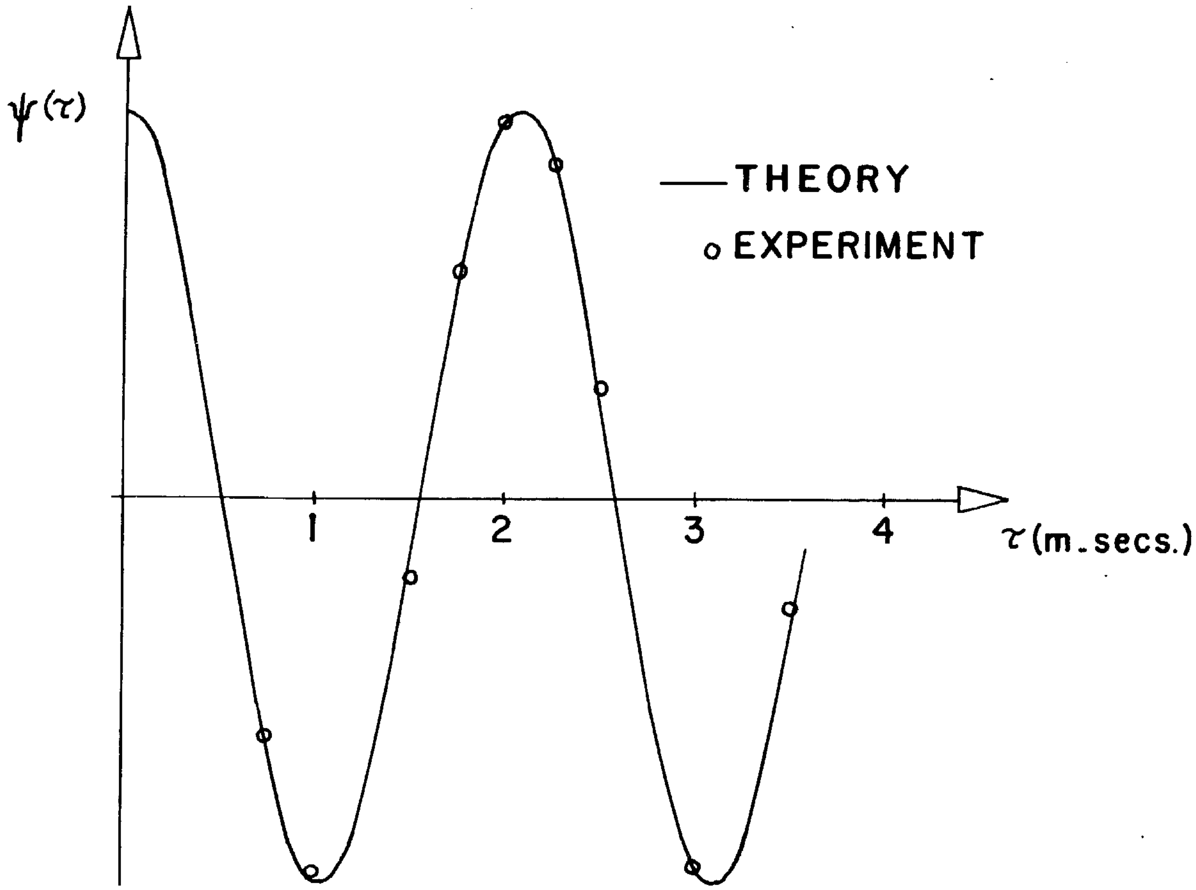

FIG. 2. AUTOCORRELATION of a SINE-WAVE.

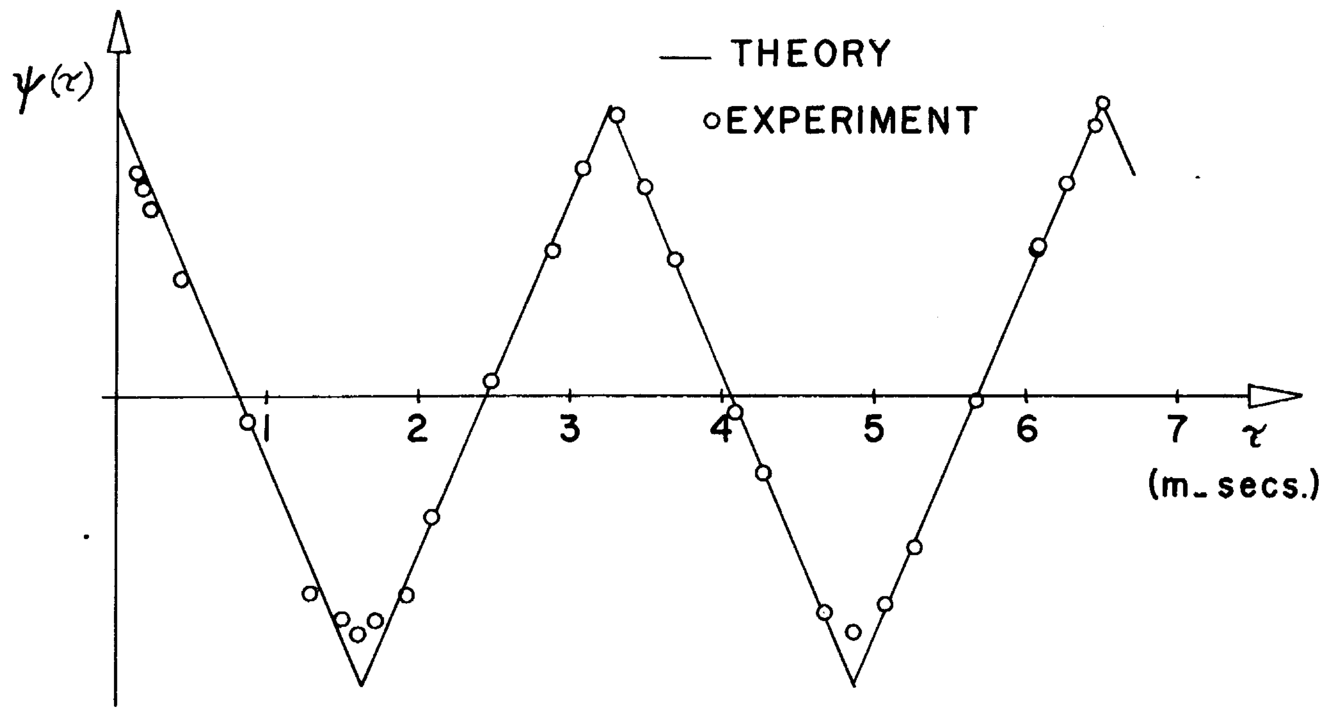

FIG. 3. AUTOCORRELATION of a SQUARE-WAVE.

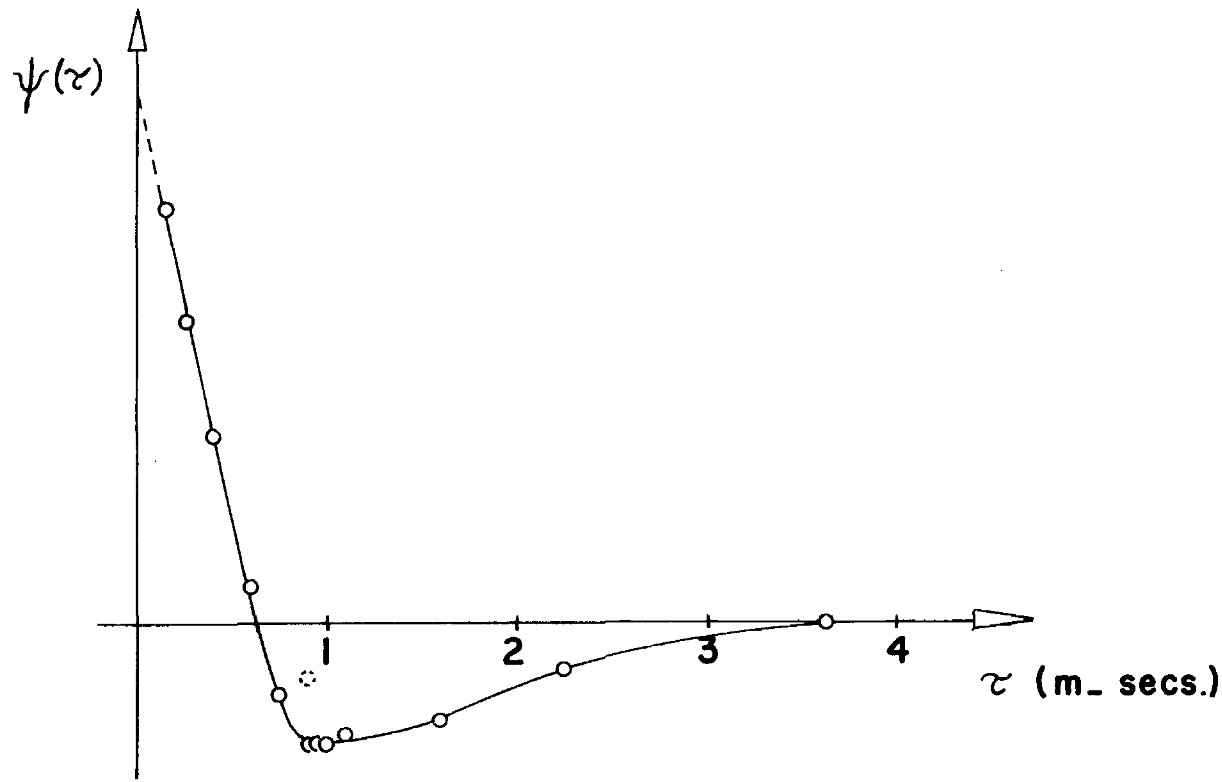

FIG. 4. AUTOCORRELATION of TURBULENCE.

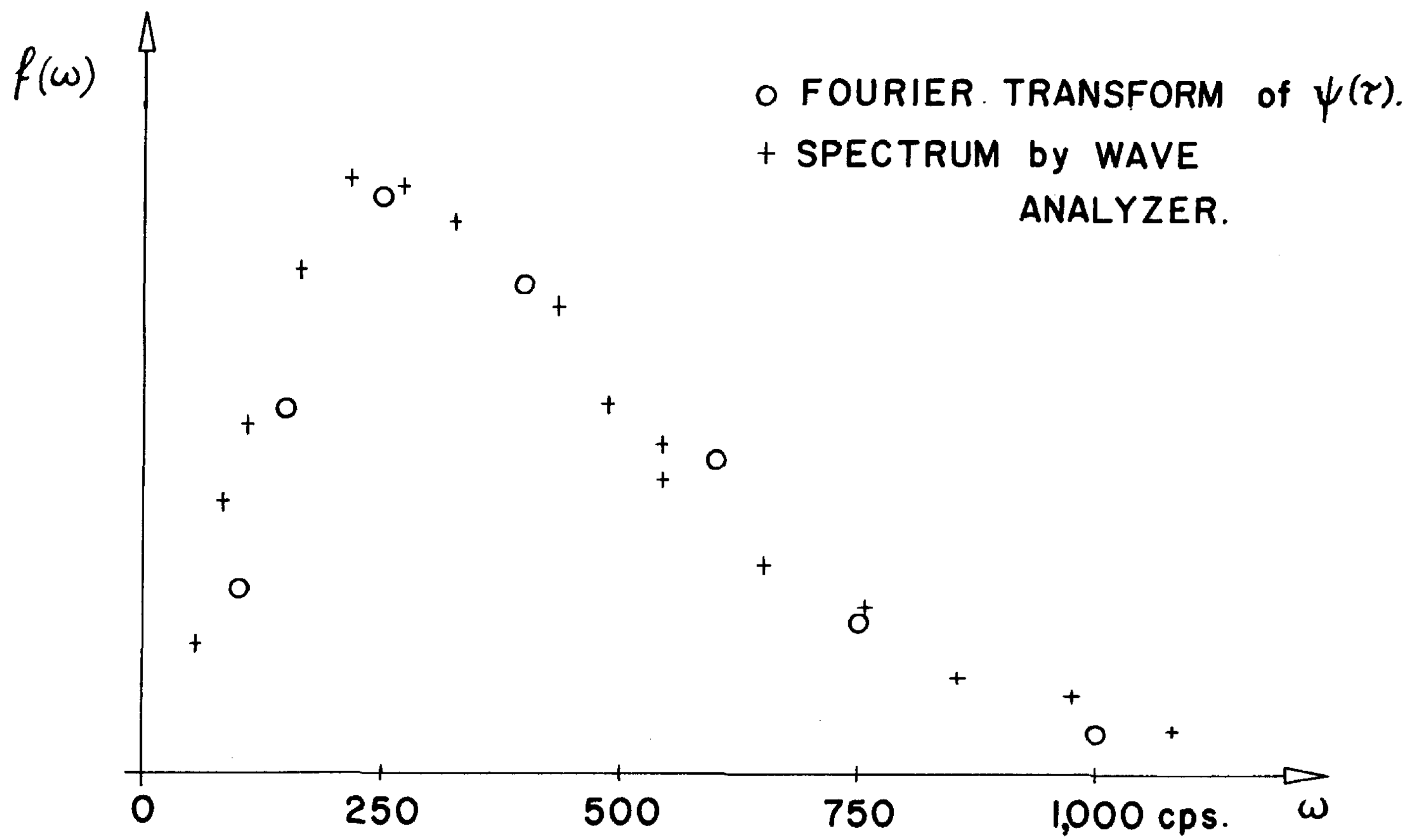

FIG. 5. CHECK against SPECTRUM.