A PROBLEM IN POTENTIAL FLOW WITH A FREE SURFACE

Thesis by

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SUMMARY

The present paper concerns itself with the determination of impact pressures and forces which act on a body when it hits a horizontal fluid surface at any angle at a high speed.

It is assumed that only the hydrodynamic force has any effect and also, that the impact is so short that the effect of the fluid splash is negligible. Under those conditions, it is possible to linearize the boundary condition of the flow and to divide the force and velocity vectors into vertical and horizontal components. The problem in the vertical direction then becomes identical with that of the motion of a body submerged in an infinite fluid, and is easily solved for bodies of simple shape. The problem in the horizontal direction resolves itself into a problem of potential flow with a symmetric discontinuity along the free surface, so that the free surface may be replaced by a symmetric configuration with velocity components opposite and equal to those in the actual fluid.

Under those conditions, two simple two-dimensional configurations are studied: an infinite elliptic semi-cylinder and an infinite flat plate. The analysis is carried out in terms of conformal transformations. Three simple three-dimensional problems are also solved: that of a sphere, an ellipsoid of revolution and a general ellipsoid. The method here is that of three dimensional harmonic analysis.

In conclusion, a specific example is given: the drag

components on a sphere which hits the surface at 45° are calculated; the results are compared to experimental data and show fair agreement with them.

A PROBLEM IN POTENTIAL FLOW WITH A FREE SURFACE

The water impact pressures on a body which enters the water at an oblique angle.

The purpose of the present study is to estimate the pressures which act upon a body when it hits the water at any angle. It is important to know the order of magnitude of those pressures and their distribution in order to solve the problems which arise in the design of sea-plane floats or in the design and ballistics of projectiles destined to naval warfare.

The general study of the mechanics of water entry of a body may conveniently be divided into three major parts. The first stage of the motion occurs as the body strikes the water surface; it may be taken to last for the short time period which precedes the formation of a cavity in the fluid. The second stage begins with the formation of a cavity; it gradually merges into the third stage, as the cavity lengthens, and an almost steady type of cavity flow develops. Of the sequence of events just described, this paper discusses only the first stage or impact stage, in which the highest pressures are encountered, and which is most important from the practical point of view.

The problem of water impact has been discussed by numerous writers, who were interested in the landing characteristics of sea-planes and the under-water trajectories of naval projectiles. Because of the great difficulty of a rigorous analysis, most papers are based on various simplifying assumptions. The earliest work of this type, by von Karman (1.1), approximates the fluid flow about the entering body by the flow about a horizontal flat plate. This approach was also taken by H. Wagner (1.3,4) who refined the Karman analysis, and by M.S. Plesset (1.5) who applied it to an elliptic disc.

A more elaborate analysis was carried out by Schiffman & Spencer (1.6,7) who approximated the flow of fluid about a sphere in the case of vertical entry by the flow about a spherical lens. P.Y. Chou (1.8), in a discussion of the same problem, replaced the sphere by a spherical bowl, the flow potential of which was discussed in a paper by A.B. Basset (1.12). While the last two investigations give very good results in the case of vertical water impact, their complication makes them unsuitable for generalization to impact at any angle.

A different method was followed by L.I. Sedoff (1.13), who investigated the impact of simple bodies (elliptic semi-cylinder and flat plate) on water, without any restriction as to the direction of the impact velocity. Sedoff's results give good agreement with experimental data obtained by A. Perelmuter (1.15) for two dimensional flow, but do not extend to three dimensional flow.

The present paper concerns itself with the developement of methods which will give an approximate solution of the problem for a three dimensional body which does not differ too much from an ellipsoid. This solution, added to that of Sedoff which is reproduced here for the sake of generality, gives a method of estimating the hydrodymanic pressures of impact in most cases of importance.

I. STATEMENT OF THE PROBLEM

Consider an infinite expanse of incompressible perfect fluid of density ρ. The fluid is assumed to be infinitely deep and perfectly at rest, so that its free surface is a horizontal plane. When a body S hits the fluid surface with velocity V, the fluid in the vicinity of the point of impact is set into motion; a small amount of the fluid rises high into the air and forms a splash; a larger quantity is set into motion by the moving body below the free surface, which, near the body, is no longer horizontal. The forces which act upon the body at this time are of four kinds: The force of gravity, the hydrostatic force of the water, a viscous drag force and the hydrodynamic force. If the velocity of the body is large, the first three forces are negligibly small, when compared to the last. As shown by H. Wagner (1.2), the hydrodynamic force may be considered to arise from the rate of change of momentum of the fluid set into motion, and as the impact stage is of very short duration (of the order of one three-hundredth of a second), the impact velocity V is taken to be constant as a first approximation. Under those circumstances, since the motion of the fluid was started from rest and is consequently irrotational, the drag force F may be written:

$$F = \frac{d}{dt}\frac{\rho}{V}\oint \Phi \frac{\partial \Phi}{\partial n}\, dS \tag{1.1}$$

where Φ is the potential of the fluid motion, t denotes a time parameter and the integration is carried out over the wetted surface of the body.

The problem is therefore one of determining the value of the potential of velocity on the wetted surface of the body at any time, as a function of time and of the geometry of the body.

The velocity potential Φ must satisfy the Laplace equation:

$$\nabla^2 \Phi = 0 \tag{1.2}$$

and a certain number of boundary conditions. It is known that any disturbance in the fluid must disappear at large distances from its origin, and this gives the boundary condition:

$$\lim_{r \to \infty} \nabla \Phi = 0 \qquad \lim_{r \to \infty} \frac{\partial \Phi}{\partial t} = 0 \tag{1.3}$$

where r denotes the distance from the origin of the disturbance. It is also known that at any point on the wetted surface of the body, no normal velocity exists:

$$-\frac{\partial \Phi}{\partial n} = V \cos \lambda \tag{1.4}$$

where n indicates the direction normal to the wetted surface of the body, and λ is the angle between that normal and the direction of motion of the body.

The last boundary condition must express the fact that the pressure at any point on the free surface of the fluid is equal to atmospheric pressure, or if atmospheric pressure is the origin of the pressure scale, the pressure on the free surface is zero. This fact can be expressed in function of known quantities by use of Euler's equation:

$$\frac{\partial \bar{u}}{\partial t} + \bar{u} \cdot \nabla \bar{u} = -\frac{\nabla p}{\rho} \tag{1.5}$$

Here $\bar{u}$ is the velocity vector at any point in the fluid and p is the corresponding pressure. When the flow is irrotational, Euler's equation is integrated as follows:

$$-\frac{\partial \Phi}{\partial t} + \frac{1}{2}(\nabla \Phi)^2 = -\frac{p}{\rho} + F(t) \tag{1.6}$$

F(t) is an arbitrary function of time independent of the position of the point where equation (1.6) is applied. At large distances from the disturbance, application of (1.3) shows that $F(t) \equiv 0$ if $P/\rho = 0$. The third boundary condition, valid on the free surface, is therefore:

$$-\frac{\partial \Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2 = 0 \qquad (1.7)$$

It is extremely difficult to solve equation (1.2) subject to boundary conditions (1.3), (1.4), (1.7) because condition (1.7) is non-linear and the surface over which it holds is unknown and must be determined from the problem itself.

In order to obtain explicit solutions which may be used in equation (1.1), simplifying assumptions must be made to linearize the boundary condition (1.7) and fix the shape of the surface over which it holds. It is noted that as the impact occurs, a small amount of water on the free surface is displaced, and very near the body, the disturbance rapidly damps out. Because of the very short time duration of the impact stage (of the order of one three-hundredth of a second), the process of impact may be compared to an instantaneous impulse with errors which are not too serious. In that case, on the free surface, the Euler equation may be integrated and the non-linear terms are small because the time interval of integration is very short, and all velocities are finite. The boundary condition obtained with that simplification is, on the free surface:

$$\Phi = 0 \qquad (1.8)$$

This can be seen as follows:

$$\lim_{\tau\to 0}\int_0^{\tau}\frac{\partial\Phi}{\partial t}\,dt+\frac{1}{2}\lim_{\tau\to 0}\int_0^{\tau}(\nabla\Phi)^2\,dt=0 \qquad (1.8a)$$

$$\lim_{\tau\to 0}\int_0^{\tau}(\nabla\Phi)^2\,dt=0 \qquad (1.8b)$$

Therefore:

$$\Phi=\text{const.} \qquad (1.8c)$$

and that constant may be selected to be zero.

This assumption is equivalent to the usual interpretation of the potential as an impulsive pressure (see ref. 2.1 etc.). Since the pressure on the free surface is constant and selected equal to zero, it follows that the potential can be chosen in the same way.

Also, since the motion of the free surface during the short duration of the impulse is very small, the equation of the free surface is given by:

$$y=0 \qquad (1.9)$$

where y refers to the normal distance from the free surface at large distances from the disturbance.

In the present study, solutions of equation (1.2) subject to boundary conditions (1.3), (1.4), (1.8), (1.9) will be submitted in explicit form, and substituted into the drag equation (1.1) so that specific values of drag force in function of dynamic pressure, time and body geometry for any angle of entry will be calculated.

It should be noted that since the impulsive pressure at any point in the fluid is $\rho\Phi$ the total pressure impulse of the fluid on the body, differentiated with respect to time, is

$$F = \frac{d}{dt}\oint \frac{\rho}{V}\Phi\frac{\partial\Phi}{\partial n}dS \tag{1.1a}$$

which is identical with the drag formula derived previously.

Since the boundary conditions and the differential equation are all linear, it is possible to superimpose various solutions, and in particular, to write the potential function:

$$\Phi = V_x\varphi_1 + V_y\varphi_2 + V_z\varphi_3 + \Omega_{yz}\varphi_4 + \Omega_{zx}\varphi_5 + \Omega_{xy}\varphi_6 \tag{1.10}$$

In this equation, $V_{x,y,z}$ denote the x,y,z components of the impact velocity V, x being a horizontal axis in the plane of motion of the body, z a horizontal axis normal to it and y a vertical axis. $\Omega_{yz,zx,xy}$ denote angular velocities about the x,y,z, axes. In the problem at hand, the velocities V_z, Ω_{ij} will be assumed to be zero; if the angle of impact of the body is α, then $V_x = V\cos\alpha$; $V_y = V\sin\alpha$ and equation (1.10) becomes:

$$\Phi = V(\varphi_1\cos\alpha + \varphi_2\sin\alpha) \tag{1.10a}$$

Consider now the system shown on Figure 1, where A represents the submerged portion of the body, and the xz plane is the free surface. Construct the symmetric image A' of A with respect to xz. Let P be a point on A, P' its image on A'. The potentials φ_1, φ_2 of the flow can be continued across xz into the upper half space so that $\varphi(xyz) = -\varphi(x{-}yz)$ since $\varphi = 0$ on xz by (1.9). If the velocities in the xyz directions at any point be designated by u,v,w, the following relations hold:

$$
\begin{aligned}
u(xyz) &= -u(x-yz) \\
v(xyz) &= +v(x-yz) \\
w(xyz) &= -w(x-yz)
\end{aligned}
\tag{1.11}
$$

or if normal components of u,v,w are taken at P and P', equations (1.11) may be combined to give:

$$\left(\frac{\partial\varphi}{\partial n}\right)_P = -\left(\frac{\partial\varphi}{\partial n}\right)_{P'} \tag{1.12}$$

a new boundary condition which, substituted into boundary condition (1.4) for φ_1 and φ_2 separately, gives for the whole space AA' the boundary condition set

$$-\frac{\partial\varphi_1}{\partial n} = \cos\lambda_1 \frac{|y|}{y} \tag{1.13a}$$

$$-\frac{\partial\varphi_2}{\partial n} = \cos\lambda_2 \tag{1.13b}$$

Here λ_1 is the angle between the normal to the surface of the body AA' and the x axis while λ_2 is the angle between the same normal and the y axis. Boundary conditions (1.13 a,b), it should be emphasized, hold over the entire space, because of the analytic continuation carried out above.

Physically, the problem has been divided into two parts, which deal respectively with horizontal and vertical components of motion. This division is possible because of the linearity of all the differential equations considered. Examination of equation (1.13b) reveals that the presence of the free surface does not affect the potential and therefore the drag integral of the body in the vertical direction. The boundary conditions are the same as those which arise in the study of the motion of a completely submerged body of shape AA'; the potential function φ_2 is therefore the same also; the drag integral (1.1) is carried out over one half of the total submerged surface AA'. This part of the problem, therefore, presents no great difficulty, and the various potential functions suitable for various shapes of the body A may generally be found in standard texts on Hydrodynamics (2.1). The same cannot be said of the other part of the problem. If the body AA' is assumed submerged in an infinite fluid, we then have a surface of discontinuity in the velocity field from u to -u, along the plane of symmetry of the figure. This is expressed by the boundary condition (1.13a). In subsequent parts of this paper, solutions of the potential function which satisfy that boundary condition, will be constructed, and integrated over the submerged portion A of the body.

Two further remarks must be made before the subject of boundary conditions is abandoned. It should first be noted that if the plane yz is a plane of symmetry of the figure, then one may expect to have:

$$\varphi_1(xyz) = -\varphi_1(-xyz) \qquad (1.14)$$

and since the impulsive pressure $p=\rho\phi V$ one obtains negative pressures on the rear of body A which exactly balance the positive pressures on its front. This is physically impossible, and at the point where the pressure is zero, one may expect the fluid to separate from the body, to remain behind as the body moves, and to create a cavity behind the body. It is known from experimental evidence that such a cavity is created. One should take this fact into account in writing the boundary condition (1.13a) and treat the problem as a three dimensional Kirchhoff-Helmholtz potential problem. The difficulty of such a treatment is well known. It is believed that during the short period of time of the impact stage, the separation has not developed to an extent sufficient to change the flow streamlines appreciably; the separation is therefore not taken into account in the construction of the potential function φ_1 except in one case where a very elegant solution of the flow with separation was developed by Sedoff. Since the rear portion of the fluid is not set into motion, all integrations are carried out over the front portion of the body A only.

It is also clear that equation (1.1) and boundary condition (1.13) can be satisfied explicitly only for bodies of very simple geometrical shapes; the submerged portion of any body during impact is too difficult to handle without approximation. In particular, when the analytic continuation process is carried out across the free surface, it is very desirable to obtain a body of such a shape that the slope of the normal to the surface be a continuous function of y. This is possible only if the body intersects the free surface at right angles. Following a suggestion of P.Y. Chou (1.9), one is led to the following type of approximation. Let the submerged portion of the body considered be replaced by one half of a given simple geometrical figure of not dissimilar shape; say for instance half an ellipsoid; as the submerged portion of the body increases in volume and changes in shape, the approximating ellipsoid also increases in volume and the relative length of its axes changes in such a manner that its shape remains related to that of the actual body. Specifically, while the ellipsoid does not change its basic properties, its axes are adjusted in such a manner that its depth of penetration b is equal to that of the actual body; its half volume is equal to the submerged volume of the body and its surface area is equal to that of the real body. Since an ellipsoid has three independent length parameters, it can always be made to satisfy three conditions as stated above. The potential function is written in terms of the parameters, and the parameters are made to vary with time according to the functions stated above. If the approximating body has two length parameters (ellipsoid of revolution or elliptic cylinder) only two of the approximating functions may be satisfied, and the sphere or the flat plate, satisfies one single condition. The potential function and the drag integral are thus functions of time while remaining functionally invariant and the differentiation in (1.1) can be carried out without excessive difficulty.

To summarize this discussion, the problem treated in this paper is to develop closed expressions for the drag of a body during water impact:

$$F_x = \frac{\rho V^2}{2} \sin 2\alpha \, \frac{d}{db} \int_w \varphi_1 \frac{\partial \varphi_1}{\partial n} \, dS \tag{1.15}$$

$$F_y = \frac{\rho V^2}{2} \, 2\sin^2\alpha \, \frac{d}{db} \int_w \varphi_2 \frac{\partial \varphi_2}{\partial n} \, dS$$

where differentiation with respect to time $\frac{d}{dt}$ is replaced by V $V \sin\alpha \frac{d}{db}$. and the integration is carried out over the wetted area of an approximating ideal body described above. The functions φ_1 and φ_2 satisfy the Laplace equation and boundary conditions given by equations (1.3) and (1.13a,b). In the detailed discussion which follows, two ideal bodies in two dimensions, an elliptic cylinder and a flat plate will be studied by means of conformal transformations; those analyses are almost entirely due to L.I. Sedoff. Three bodies in three dimensions will also be studied: The sphere, the ellipsoid of revolution and the general ellipsoid; the analysis will be based on the expression of potentials as infinite series of certain orthogonal functions. The drag integral will be calculated in each case and differentiated to determine the drag forces. A numerical example, with specific values for the approximating functions will then be presented to illustrate the working of these methods.

II. SOLUTION OF THE PROBLEM IN TWO DIMENSIONS

In order to calculate the potential of the flow about a two dimensional partially submerged body which travels at unit velocity it is convenient to transform the submerged portion of the body and its symmetric image conformally into the unit circle.

Introducing the potential function φ and the corresponding stream function ψ, one has on the boundary of the body:

$$\frac{\partial\varphi}{\partial n} = \frac{\partial\psi}{\partial s} = \cos\alpha \frac{\partial y}{\partial s}\frac{|y|}{y} - \sin\alpha \frac{\partial x}{\partial s} \tag{2.1}$$

where s is an element along the boundary and the angle of impact is α as before. Equation (2.1) is integrated as follows:

$$\psi = |y|\cos\alpha - x\sin\alpha \tag{2.2}$$

The complex potential function $F(z) = F(x+iy) = \varphi + i\psi = F_1(z)\cos\alpha + F_2(z)\sin\alpha$ is now introduced. Then:

$$\begin{aligned} F_1(z) &= \varphi_1 + i\psi_1 \\ F_2(z) &= \varphi_2 + i\psi_2 \end{aligned} \tag{2.3}$$

and the boundary conditions to be satisfied are

$$\begin{aligned} \psi_1 &= |y| \\ \psi_2 &= -x \end{aligned} \tag{2.4}$$

In order to transform the boundary of the body in the z plane into the unit circle in the ζ plane, one uses the general conformal transformation:

$$z = f(\zeta) = \frac{k}{\zeta} + k_0 + k_1\zeta + k_2\zeta^2 + \dots \tag{2.5}$$

The boundary conditions (2.4) on the unit circle become

$$\begin{aligned} \psi_1(\zeta) &= \psi_1(z) = |y| = |\operatorname{Im} z| = |\operatorname{Im} f(e^{i\theta})| = \sum_0^\infty a_n \cos n\theta \\ \psi_2(\zeta) &= \psi_2(z) = -x = -\operatorname{Re} z = -\operatorname{Re} f(e^{i\theta}) = \sum_0^\infty b_n \sin n\theta \end{aligned} \tag{2. 6}$$

and the complex potential functions become:

$$F_1(z) \sim F_1(\zeta) = i\sum_0^\infty a_n \zeta^n$$
$$F_2(z) \sim F_2(\zeta) = i\sum_0^\infty b_n \zeta^n \qquad (2.7)$$

These expansions must now be written explicitly in terms of the transformation function $f(\zeta)$.

$iF_2(\zeta)$ and $f(\zeta)$ have identical real parts on the unit circle, but $f(\zeta)$ has a simple pole at $\zeta = 0$, the principal part of which is k/ζ. On the unit circle, $Rl\, k/\zeta = Rl\, k\zeta$ so that the boundary condition (2.4) is satisfied by

$$F_2(\zeta) = -if(\zeta) + \frac{ik}{\zeta} - ik\zeta \qquad (2.8)$$

The following identities hold on the unit circle:

$$Im\ \frac{i}{2}\left[f(\zeta) - f(1/\zeta)\right] = 0$$
$$Im\ f(\zeta) = \frac{i}{2}\left[f(\zeta) - f(1/\zeta)\right] \qquad (2.9,abc)$$
$$\log\frac{1+\zeta}{1-\zeta} = \pm\frac{\pi i}{2} - \log\left|\tan\frac{\theta}{2}\right| \qquad \zeta = e^{i\theta}$$

In the last identity, the positive sign applies when $0<\theta<\pi$ and the negative when $\pi<\theta<2\pi$. If relations (2.9b and c) are now combined, a function which satisfies the boundary condition (2.4) can be obtained:

$$\psi_1 = Rl\left\{-\frac{1}{\pi}\left[f(\zeta) - f(1/\zeta)\right)\log\frac{1+\zeta}{1-\zeta}\right\} \qquad (2.10)$$

The potential function is therefore written as:

$$F_1(\zeta) = -\frac{i}{\pi}\left[f(\zeta) - f(1/\zeta)\right]\log\frac{1+\zeta}{1-\zeta} + A(\zeta) \qquad (2.11)$$

where $A(\zeta)$ is a function, real on the unit circle, added to cancel the singularities of the first term, since the functions F must be regular everywhere on and within the unit circle.

The potential functions $F_1(\zeta)$ and $F_2(\zeta)$ are thus expressed explicitly in terms of the transformation function $f(\zeta)$. When the geometry of the body and therefore the form of $f(\zeta)$ is known, equations (2.8) and (2.11) give the complex potential of the flow by direct substitution.

Two specific examples of two dimensional bodies are now discussed in some detail on the basis of the general formulas derived above: A semi-submerged infinitely long elliptic cylinder and an infinitely wide flat plate.

III. THE WATER IMPACT OF AN ELLIPTIC CYLINDER

As discussed in part I, the problem of water impact is subdivided into two parts: The former is the analysis of the steady flow about a semi-submerged elliptic cylinder of semi axes a,b (Fig. 2) and the determination of the potential and of the drag function. The latter part is the study of the impact force which results when the parameters a,b, vary as given functions of time.

The physical system studied in the problem is shown on Fig. 2 with the coordinate axes, the elliptic section BCD and its image about x, BC'D. It is known that the conformal transformation which will throw the ellipse BCDC' into the unit circle is:

$$Z = -\frac{1}{2}\left[(a-b)\zeta + (a+b)\,{}^{1}/\zeta\right] \tag{3.1}$$

If the variable in the transformed plane is $\zeta = re^{i\theta}$, the boundary conditions which correspond to conditions (2.4) are:

$$\begin{aligned} \psi_1 &= b\,|\sin\theta| \\ \psi_2 &= a\cos\theta \end{aligned} \tag{3.2}$$

Boundary condition (3.2) is expanded into a Fourier series; then, all odd terms vanish because of symmetry and the boundary condition becomes:

$$\psi_1 = \frac{2b}{\pi} - \frac{4b}{\pi}\sum_{1}^{\infty}\frac{\cos 2n\theta}{4n^2-1} \tag{3.3}$$

It follows that on the unit circle, the complex potential function is:

$$\begin{aligned} F_1 &= \frac{2bi}{\pi} + \frac{4bi}{\pi}\sum_{1}^{\infty}\frac{\zeta^{2n}}{4n^2-1} \\ F_2 &= ai\zeta \end{aligned} \tag{3.4}$$

The infinite series which occurs in the expression for $F_1(\zeta)$ can be summed as follows:

$$\frac{2\zeta^{2n}}{4n^2-1} = -\frac{\zeta^{2n+1}}{(2n+1)\zeta} + \zeta\,\frac{\zeta^{2n-1}}{(2n-1)} \tag{3.5}$$

and

$$\sum_1^\infty \zeta^{2n} = \frac{\zeta^2}{1-\zeta^2} \tag{3.6}$$

Therefore

$$\sum_1^\infty \frac{\zeta^{2n+1}}{2n+1} = \int_0^\zeta \frac{\zeta^2 d\zeta}{1-\zeta^2} = -\zeta + \frac{1}{2}\log\frac{1+\zeta}{1-\zeta} \tag{3.7}$$

$$\sum_1^\infty \frac{\zeta^{2n-1}}{2n-1} = \int_0^\zeta \frac{1}{1-\zeta^2}\,d\zeta = \frac{1}{2}\log\frac{1+\zeta}{1-\zeta} \tag{3.8}$$

Since all the series and integrals considered converge uniformly when ζ is inside and on the unit circle, the above manipulations are legitimate. Collecting terms, one obtains:

$$F_1(\zeta) = \frac{ib}{\pi}\left[\frac{1}{\zeta} - \zeta\right]\log\frac{1+\zeta}{1-\zeta} \tag{3.9}$$

and separating real and imaginary parts on the unit circle:

$$\begin{aligned} \varphi_1(\zeta) &= -\frac{2b}{\pi}\sin\theta\log\left|\tan\frac{\theta}{2}\right| \\ \psi_1(\zeta) &= b\,|\sin\theta| \end{aligned} \tag{3.10}$$

Combining (3.4) and (3.10) one obtains the potential function

$$\varphi = -\sin\theta\left[\frac{2b}{\pi}\log\left|\tan\frac{\theta}{2}\right|\cos\alpha + a\sin\alpha\right] \tag{3.11}$$

This function satisfies all the boundary conditions and is therefore a suitable description of the motion under study.

The drag integrals can now be computed:

$$\oint \varphi_1 \, d\psi_1 = \frac{b^2}{\pi}$$

$$\oint \varphi_2 \, d\psi_2 = \frac{\pi a^2}{2} \tag{3.12}$$

The approximating functions which correlate the elliptic cylinder to the actual body under study may be expressed as functions of time or also, since the velocity of vertical penetration is constant, in function of the total depth of penetration. The first approximating function is $b_a = b_c$ the depth of penetration of the actual body is equal to that of the cylinder. The second one is equivalent to expressing a as a(b). When the results from (3.12) are substituted into (1.15), one obtains the following expressions for drag force per unit length:

$$F_x = \frac{\rho V^2}{2} \sin 2\alpha \ \frac{2b}{\pi}$$

$$F_y = \frac{\rho V^2}{2} \sin^2\alpha \ 2\pi a \frac{da}{db} \tag{3.13}$$

These formulae complete the solution of the impact problem when an elliptic cylinder is taken as the approximating body.

IV. THE WATER IMPACT OF A VERTICAL FLAT PLATE

The method of solution of this problem is very similar to the one used in the case of the elliptic cylinder. Again, steady motion about a semi-submerged flat plate will be investigated, and the resulting drag integral will be differentiated with respect to the penetration depth.

Figure 3 shows the coordinate system and the flat plate BC, of depth of penetration b. In this analysis, an attempt is made to account for the separation of the fluid on the rear face of the plate, so that a separation point P is defined.

The boundary conditions in the physical plane are summarized as follows:

$$\begin{aligned} &\varphi = 0 \quad \text{on} \quad AB, PD, DE \\ &\psi = |y| \quad \text{on} \quad BCP \\ &\lim_{P} \frac{\partial \psi}{\partial y} = 1 \end{aligned} \tag{4.1}$$

If the image of the flat plate is also considered, the boundary conditions become:

$$\begin{aligned} &\varphi = 0 \quad \text{on} \quad P'DP \\ &\psi = |y| \quad \text{on} \quad PCBC'P' \\ &\lim \frac{\partial \psi}{\partial y} = 1 \end{aligned} \tag{4.2}$$

A transformation must be found to map the actual space conformally inside the upper half of the unit circle so that P and P' become the points 1, -1. Then the complex potential function is imaginary along the real diameter and in virtue of Schwartz's lemma, the space may be continued into the lower half of the unit circle so that

$$\begin{aligned} \varphi(\theta) &= -\varphi(-\theta) \\ \psi(\theta) &= +\psi(-\theta) \end{aligned} \tag{4.3}$$

The construction of such a transformation function is carried out in four steps. The vertical flat plate is first mapped on the unit circle by the well known formula:

$$Z = \frac{b}{2}\left(\zeta - 1/\zeta\right) \tag{4.4}$$

Then, a linear transformation throws the ζ plane into a $\mathfrak{z}$ plane in such a manner that the points $\zeta = 1, ie^{i\theta}, -1$ become the points $\mathfrak{z} = 1, i, -1$. This is done by the mapping function:

$$\zeta = \frac{\mathfrak{z} + \tan \theta_1/2}{\mathfrak{z}\tan \theta_1/2 + 1} \tag{4.5}$$

In this plane, the point $\mathfrak{z} = i$ corresponds to the point $z = ib\cos\theta_1' = iy_1$ Next, the plane is transformed into the plane $t = e^{i\theta}$ as follows:

$$\mathfrak{z} = \frac{2 + (t+i)^2}{2 + (t-i)^2} \tag{4.6}$$

If the tip of the flat plate is $z = ib$ or $t = e^{i\theta_1}$, then the position of the separation point is

$$y_1 = b\cos\theta_1' = \frac{2b\cos\theta_1}{1+\cos^2\theta_1} \tag{4.7}$$

The last step consists in gathering results and substituting equations (4.6), (4.5) into (4.4) to write out the required mapping function:

$$Z = i\,\frac{4b\cos\theta_1\, t(1+t^2)}{(1+t^2)^2 + 4t^2\cos^2\theta_1} \tag{4.8}$$

Real and imaginary parts are now separated to give $y(\theta)$:

$$y = \frac{2b\cos\theta\cos\theta_1}{\cos^2\theta + \cos^2\theta_1} \tag{4.9}$$

so that boundary condition (4.2) becomes on the unit circle:

$$\psi = 2b\cos\theta_1\,\frac{|\cos\theta|}{\cos^2\theta + \cos^2\theta_1} \tag{4.10}$$

with the additional condition:

$$\overline{\lim} \frac{\partial \psi}{\partial y} = 1 \tag{4.11}$$

Equations (4.10), (4.11) correspond to (4.2).

To go back to the transformation equation (4.8), it is noted that

$$Rl\, z = -iy$$
$$Im\, z = 0 \tag{4.12}$$

Let $\log \frac{i-t}{i+t}$ be defined so that $Im \log \frac{i-t}{i+t} = \pm\frac{\pi}{2}$. Since on the unit circle

$$\frac{i-t}{i+t} = e^{\pi i/2} \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \tag{4.13}$$

it follows that:

$$|\theta| < \pi/2 \qquad \log \frac{i-t}{i+t} = \frac{\pi i}{2} + \log \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$
$$|\theta| > \pi/2 \qquad \log \frac{i-t}{i+t} = -\frac{\pi i}{2} + \log \left|\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right| \tag{4.14}$$

Combining equations (4.12) and (4.14) one can construct a function

$$g(t) = \frac{8b\cos\theta_1}{\pi} \frac{t(1+t^2)}{(1+t^2)^2 + 4t^2\cos^2\theta_1} \log \frac{i-t}{i+t} \tag{4.15}$$

whose imaginary part equals $\psi(\theta)$ on the unit circle. But g(t) has two simple poles at the roots of the denominator equation:

$$(1+t^2)^2 + 4t^2 \cos^2\theta_1 = 0 \tag{4.16}$$

The values of those roots t_1, t_2, t_3, t_4 are:

$$t_1 = i\chi \qquad t_3 = i/\chi$$
$$t_2 = -i\chi \qquad t_4 = -i/\chi \tag{4.17}$$

$$\chi = \sqrt{1+\cos^2\theta_1} - \cos\theta_1$$

The roots t_1, t_2 are inside the unit circle while t_3 and t_4 are outside. To do away with the above singularities, the function Ω is introduced:

$$\Omega = \frac{t - 1/t}{i\chi - 1/i\chi} \log \frac{1-\chi}{1+\chi} \tag{4.18}$$

Ω is real on the unit circle, and has the following special values:

$$\begin{aligned} t &= t_1 & \Omega &= \log \frac{1-\chi}{1+\chi} \\ t &= t_2 & \Omega &= -\log \frac{1-\chi}{1+\chi} \end{aligned} \tag{4.19}$$

so that at the points where g(t) has simple poles inside the unit circle, Ω (t) has the same values as $\log \frac{i-t}{i+t}$. The complex potential function is therefore constructed as:

$$F(t) = \frac{8b\cos\theta_1}{\pi} \frac{t(1+t^2)}{(1+t^2)^2 + t^2\cos^2\theta_1} \left[\log \frac{i-t}{i+t} - \frac{t-1/t}{i\chi - 1/i\chi} \log \frac{1-\chi}{1+\chi}\right] \tag{4.20}$$

F(t) satisfies all the boundary conditions since on the unit circle, $\Omega(t)$ and f(t) are real and g(t) satisfies the boundary condition (4.10), and since F(t) is regular everywhere within and on the unit circle.

It remains to evaluate the parameters θ_1 and χ in order to solve the problem completely.

It is possible to write the following equation to define a parameter ε :

$$\frac{1}{i\chi - 1/i\chi} \log \frac{1-\chi}{1+\chi} = \frac{i}{2} \frac{1}{\sqrt{1+\cos^2\theta_1}} \log \frac{\sqrt{1+\cos^2\theta_1}+1}{\cos\theta_1} = \frac{i}{2}\varepsilon \tag{4.21}$$

Equation (4.20) now becomes:

$$F(t) = \frac{8b\cos\theta_1}{\pi} \frac{t(1+t^2)}{(1+t^2)^2 + t^2\cos^2\theta_1} \left[\log \frac{i-t}{i+t} - \frac{i}{2}(t - 1/t)\varepsilon\right] \tag{4.22}$$

To determine the parameter ε, use is made of the boundary condition (4.11). On the real axis within the unit circle, if $\tau = R\ell\, t$,

$$F(\tau) = i\,\frac{8b\cos\theta_1}{(1+\tau^2)^2+\tau^2\cos^2\theta_1}\,\tau(1+\tau^2)\left[2\tan^{-1}\tau - \frac{\varepsilon}{2}\left(\tau-\frac{1}{\tau}\right)\right] \tag{4.23}$$

It is known that the velocity at any point $u = \frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial\tau}\frac{\partial\tau}{\partial y}$

But one deduces from (4.9)

$$y = \frac{4b\cos\theta_1\,\tau(1+\tau^2)}{(1+\tau^2)^2+\tau^2\cos^2\theta_1} \tag{4.24}$$

This is differentiated as follows:

$$\frac{dy}{d\tau} = \frac{y}{\tau}\;\frac{(1-\tau^2)\left[(1+\tau^2)^2-4\tau^2\cos^2\theta_1\right]}{(1+\tau^2)\left[(1+\tau^2)^2+4\tau^2\cos^2\theta_1\right]} \tag{4.25}$$

Differentiating equation (4.23) and using (4.25) one obtains the velocity u:

$$u = \frac{2}{\pi}\left\{2\tan^{-1}\tau + \frac{2\tau\left\{(1+\tau^2)^2+4\tau^2\cos^2\theta_1 - \varepsilon\left[(1+\tau^2)^2+2\cos^2\theta_1(1+\tau^2)\right]\right\}}{(1-\tau^2)\left[(1+\tau^2)^2+4\tau^2\cos^2\theta_1\right]}\right\} \tag{4.26}$$

The limit as τ approaches 1 of expression (4.26) is now taken. For equation (4.8) to hold, the parameter ε must be 1. Then, the following are true:

$$u = \frac{4}{\pi}\left[\tan^{-1}\tau + \frac{2\tau(1-\tau^2)\cos^2\theta_1}{4\tau^2\cos^2\theta_1-(1+\tau^2)^2}\right]$$
$$\lim_{\tau\to 1} u = 1 \tag{4.27}$$

Equation (4.21) can now be solved for θ_1 : it becomes

$$\log\frac{\sqrt{1+\cos^2\theta_1}+1}{\cos\theta_1} = \sqrt{1+\cos^2\theta_1} \tag{4.28}$$

A few simple transformations bring it into the form:

$$\frac{1}{\sqrt{1+\cos^2\theta_1}} = \tanh\sqrt{1+\cos^2\theta_1} \tag{4.28a}$$

This is a transcendental equation which is solved graphically. The solution is: $\cos\theta_1 = 0.663$

Equation (4.7) now gives the separation point to be at $y_1 = 0.92b$

To finish the problem, it is only necessary to substitute $\cos\theta_1$ into the complex potential function F(t), to separate real and imaginary parts. The resulting potential function is:

$$\varphi_1 = \frac{2}{\pi} y \left[\log\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right) + \sin\theta\right] \tag{4.29}$$

This potential function goes to 0 as x goes to ∞ and therefore fulfills all the conditions of the problem.

The drag integral is now calculated as follows:

$$\oint \varphi \frac{\partial\varphi}{\partial n} ds = \int \varphi\, dy = \frac{2b^2}{\pi}\cos^2\theta_1 \tag{4.30}$$

Since an infinitely wide flat plate has only one parameter to correlate it to the actual body considered, the impact force has a particularly simple expression: The impact force per unit width is:

$$F_x = \frac{\rho V^2}{2}\sin 2\alpha\ \frac{4b}{\pi}\cos^2\theta_1$$

$$F_y = 0 \tag{4.31}$$

These formulas are the solution of the problem of water impact load determination on a flat plate. It may be noted that as an elliptic cylinder becomes thinner and thinner, the separation point, assumed to be at $y_1 = b$ in the previous solution, moves up to the analytically determined position $y_1 = 0.92b$. Then a slight negative force component exists near the bottom of the plate, and the factor $\frac{1}{2}$ in equation (3.13) becomes $\cos^2\theta_1 = 0.440$. In the vertical direction as a in the elliptic cylinder goes to 0, F_y also goes to 0. This agrees with the obvious result obtained here. The two two-dimensional solutions obtained here are therefore consistent. With these remarks, the field of two-dimensional analysis is left, and the investigation of several problems of three dimensional flow is undertaken.

V. SOLUTION OF THE PROBLEM IN THREE DIMENSIONS

The determination of the flow potential about a partially submerged body which travels at unit velocity is carried out in the following general manner.

A system of orthogonal coordinates X_1, X_2, X_3 is introduced, in such a manner that the surface of the body satisfies the equation $X_1 =$ constant. When the shape of the body under study and its image, is a continuous closed surface with continuous derivatives it is generally possible to express the solution of Laplace's equation in the normal form:

$$\varphi = \sum_{0}^{\infty} \sum_{1}^{2n+1} a_n^s \, H_n^s(X_1) \, H_n^s(X_2) \, H_n^s(X_3) \tag{5.1}$$

In this equation, $H_n^s(X_i)$ designates a solution of Laplace's equation in the system of coordinates (X_1, X_2, X_3); that solution is generally obtained by considering the solution of the differential equation in X_i obtained when the variables are separated in the Laplace equation expressed in terms of the coordinate system ($X_1, X_2 X_3$). Examples of such functions are spherical harmonics, Bessel harmonics, Mathieu functions, Lamé functions. Since the separation of $\nabla^2 \varphi = 0$ into total differential equations in X_1, X_2, X_3 involves the introduction of arbitrary constants, those appear as the summation of all possible solutions when the constant is n = 1,2,3... It is also known that there are s=2 n+1 harmonics of degree n. These statements explain how a solution of the form of equation (5.1) may be written down. The coefficients a_n^s are constants selected to make φ fit the boundary conditions.

The nature of the harmonics $H_n^s(x_i)$ is such that for each coordinate there are two such harmonics possible: They are the two independent solutions of a differential equation of second order. One solution generally converges as x_i has an upper bound, while the other converges when x_i has a lower bound. Such are for instance the functions P_n^s, Q_n^s in spherical coordinates, or the functions E_n^s, F_n^s in elliptic coordinates.

The system of coordinates is such that a given value of x_1 fixes a body of the same type as the fundamental body under study, and of a size determined by the value of x_1. The coordinates then vary between known bounds and sweep out all the points on the surface defined by x_1. In spherical coordinates, for instance, x_1 is the radius while x_2 and x_3 are the azimuth and elevation angles.

It follows from these remarks, that as the distance from the disturbance increases, x_1 will go to infinity while the range of variation of x_2 and x_3 which may include the value 0 is not affected. Since there are always two harmonics, one of which vanishes when x_i is small and the other when x_i is large, if those two harmonics are designated by $(H_n^s)_1$, and $(H_n^s)_2$, the boundary condition (1.3) that any disturbance must vanish at infinity is always satisfied by a potential of the type:

$$\varphi = \sum_0^\infty \sum_1^{2n+1} a_n^s \, (H_n^s)_2 (x_1) (H_n^s)_1 (x_2) (H_n^s)_1 (x_3) \qquad (5.2)$$

The second boundary condition, given by equation (1.13a,b), is best discussed in two parts. The boundary condition for the potential φ_2 is simple and can generally be satisfied by a potential function which includes only one or at most two terms of the series. For all the body shapes under investigation in the present paper, those potential functions are given explicitly and discussed in some detail in most standard texts on Hydrodynamics. Little need therefore be said about them here; they are introduced directly into the solutions of the special problems, with the necessary reference as to their origin.

The situation is quite different when the potential φ_1 is investigated. The boundary condition to be satisfied now is:

$$\frac{\partial \varphi_1}{\partial n} = -\cos\lambda \frac{|y|}{y} \tag{5.3}$$

This equation is identical to equation (1.13b) except that all velocities are referred to unity instead of $\cos\alpha$ so that the term $\cos\alpha$ is dropped. It is reintroduced into the drag integral at the end of the calculation.

If the curved orthogonal system of coordinates is still referred to as x_1, x_2, x_3 as discussed above, and if the cartesian system is defined by the axes x,y,z, it is simpler to rewrite condition (5.3) as follows:

$$\frac{\partial \varphi_1}{\partial x_1} = -\frac{\partial x}{\partial x_1} \frac{|y|}{y} \tag{5.4}$$

In this equation, $\partial x/\partial x_1$ is a known function of x_1, x_2, x_3 which depends on the geometry of the coordinate system x_1, x_2, x_3. Let it be called $-h_1^1(x_1 x_2 x_3)$. If equation (5.2) is used to determine $\partial\varphi/\partial x_1$, equation (5.4) becomes, on the boundary of the body, where $x_1 = X_1$:

$$\sum_0^\infty \sum_1^{2n+1} a_n^s \frac{\partial (H_n^s)_2(X_1)}{\partial X_1} (H_n^s)_1(x_2)(H_n^s)_1(x_3) = h_1^1(X_1 x_2 x_3)\frac{|y|}{y} \tag{5.5}$$

or after absorption of the constants X_1 and $f(X_1)$ into the coefficients,

$$\sum_0^\infty \sum_1^{2n+1} b_n^s (H_n^s)_1(x_2)(H_n^s)_1(x_3) = (h_1')'(x_2 x_3) \frac{|y|}{y} \qquad (5.6)$$

It is a general theorem of harmonic analysis that the functions $(H_n^s)_1$ are orthogonal with respect to a weight function whose form depends directly on the expression of the area element $X_1 = \text{const.}$ in terms of X_2, X_3. Let that weight function be $g(x_2, x_3)$. Then, any function $f(x_2 x_3)$ may be written as a generalized Fourier series:

$$f(x_2 x_3) = \sum_0^\infty \sum_1^{2n+1} c_n^s (H_n^s)_1(x_2)(H_n^s)_1(x_3) \qquad (5.7)$$

$$c_n^s = \frac{\int_0^{\omega_2}\int_0^{\omega_3} f(x_2 x_3)\, g(x_2 x_3)\, (H_n^s)_1(x_2)(H_n^s)_1(x_3)\, dx_2 dx_3}{\int_0^{\omega_2}\int_0^{\omega_3} g(x_2 x_3) \left[(H_n^s)_1(x_2)(H_n^s)_1(x_3)\right]^2 dx_2 dx_3} \qquad (5.8)$$

where ω_2, ω_3 are the upper bounds of the motion of X_2, X_3 on the surface $X_1 = \text{const.}$

The function $\frac{|y|}{y}$ is a step function, a change in sign occurring when y passes through the value 0. This step function is applied to quantities on the surface of the body, and therefore changes the sign of a function of $X_2 X_3$ when the coordinates $X_2 X_3$ take certain values. Because of the symmetry of the body, all values of functions of $X_2 X_3$ are symmetric about the value where the step occurs. Therefore, we may write, in the sense of (5.7) and (5.8):

$$\frac{|y|}{y}(h_1')'(x_2 x_3) = f(x_2 x_3) = \sum_0^\infty \sum_1^{2n+1} c_n^s (H_{2n+1}^s)_1(x_2)(H_{2n+1}^s)_1(x_3) \qquad (5.9)$$

and the coefficients $c_{3\,2n+1}^{s}$ are calculated from formula (5.8).

It is now possible to express the coefficients b_n^s of the potential function. Since the right side of equation (5.6) is identical to the left side of equation (5.7), one can combine the two equations, and equate coefficients of $H_n^s(x_1)$. It is seen therefore, that

$$b_n^s = c_n^s \tag{5.10}$$

so that the potential function is determined.

The general discussion may be carried one step further, to the forming of the drag integral.

$$\oint \varphi \frac{\partial \varphi}{\partial n} ds = \iint_S \Big[\sum_0^{\infty}\sum_1^{2n+1} g(x_2 x_3)\, b_n^s H_n^s(x_2) H_n^s(x_3)\Big]\Big[\sum_0^{\infty}\sum_1^{2n+1} b_n^s \frac{\partial H_n^s(x_1)}{\partial H_n^s/\partial x_1} H_n^s(x_2) H_n^s(x_3)\Big] dx_2 dx_3 \tag{5.11}$$

This form is true because of the relation of the weight function to the surface element. The orthogonal property of the harmonic functions will now be used:

$$\begin{aligned} &\iint_S g(x_2 x_3)\, H_n^s(x_2) H_m^t(x_3) H_m^t(x_2) H_n^s(x_3)\, dx_2 dx_3 = 0 \\ &\iint_S g(x_2 x_3) \left[H_n^s(x_2) H_n^s(x_3)\right]^2 dx_2 dx_3 = K(n,s) \end{aligned} \tag{5.12}$$

where K(n,s) is a parameter which depends on the order suffixes n,s and the geometrical constants of the body. When this is substituted into equation (5.11), the following remarkable formula follows:

$$\oint \varphi \frac{\partial \varphi}{\partial n} ds = \sum_0^{\infty}\sum_1^{2n+1} (b_n^s)^2 \frac{H_n^s(x_1)}{\partial H_n^s/\partial x_1} K(n,s) \tag{5.13}$$

The general three-dimensional problem is thus solved formally. It remains to apply this method to specific body shapes: In the following pages, the sphere, the ellipsoid of revolution and the general ellipsoid are analysed in this manner.

VI. WATER IMPACT OF A SPHERE

The first problem treated by the method outlined above is that of the water impact of a body which, at all times, is approximated by a sphere. The first part of the problem is the study of the potential flow about the sphere, while the next step consists in the introduction of the approximating function and the calculation of a drag force.

The physical system is represented on Figure 4 and the analysis is carried out in spherical polar coordinates: Radius r (the constant radius of the sphere under study is designated by R), azimuth angle θ , elevation angle w. The normal solution of Laplace's equation in that system valid when $r \to \infty$ is:

$$\varphi = \sum_{0}^{\infty} \sum_{1}^{n+1} \frac{1}{r^{n+1}} P_n^s(\cos\theta) \left[a_n^s \sin s\omega + b_n^s \cos s\omega \right] \qquad (6.1)$$

It is known that

$$P_n^s(\cos\theta) = \sin^s\theta \, \frac{d^s P_n(\cos\theta)}{d(\cos\theta)^s} \qquad (6.2)$$

The boundary condition at infinity is satisfied since, as r increases beyond bound, φ vanishes.

One must now find two functions φ_1 and φ_2 which satisfy the boundary conditions (1.13a,b).

It is known that the motion defined by φ_2 is similar to the translation of a sphere in an infinite fluid. In that case, one has: on the surface of the sphere:

$$\varphi_2 = \frac{R}{2} \cos\theta \qquad (6.3)$$

which, integrated over one quarter the sphere area, gives

$$\int \varphi_2 \frac{\partial \varphi_2}{\partial n} dS = \frac{\pi}{6} R^3 \qquad (6.4)$$

The determination of φ_1 is somewhat more complicated. The boundary condition (1.13b) in this case is:

$$\frac{\partial \varphi_1}{\partial n} = \frac{\partial \varphi}{\partial r} = \cos\theta \, \frac{|\sin\omega|}{\sin\omega} \tag{6.5}$$

It is possible to expand the step function into a Fourier series as follows:

$$\frac{|\sin\omega|}{\sin\omega} = \frac{4}{\pi}\sum_0^\infty \frac{1}{2s+1}\sin(2s+1)\omega \tag{6.6}$$

This is now substituted back into condition (6.5) while (6.1) is differentiated. The boundary condition becomes:

$$\sum_0^\infty\sum_0^\infty \frac{n+1}{R^{n+2}} P_n^s(\cos\theta)\left[a_n^s \sin s\omega + b_n^s \cos s\omega\right] = \frac{4\cos\theta}{\pi}\sum_0^\infty \frac{\sin(2s+1)\omega}{2s+1} \tag{6.7}$$

A comparison of the coefficients of sin sw on the two sides of this equation shows at once that $b_n^s \equiv 0$ and $a_n^{2s} \equiv 0$

To simplify the writing of the next equations, μ is defined by $\mu = \cos\theta$ and α_n^s is defined by $\alpha_n^s = \frac{\pi}{4}\frac{n+1}{R^{n+2}} a_n^s$. When the coefficients of $\sin s\omega$ are equated, one has:

$$\frac{\mu}{2s+1} = \sum_0^\infty \alpha_n^{2s+1} P_n^{2s+1}(\mu) \tag{6.8}$$

Applying the definition of the spherical harmonic given in (6.2), one obtains

$$\frac{1}{2s+1}\frac{\mu}{(1-\mu^2)^{2s+1/2}} = \sum_0^\infty \alpha_n^{2s+1}\frac{d^{2s+1}P_n(\mu)}{d\mu^{2s+1}} \tag{6.9}$$

This last step, however, is permissible only if $|\mu| < 1$. As μ approaches 1, a simple singularity of order $2s+1/2$ arises at the nose of the sphere. Physically, this implies an infinite pressure at the nose, which is a phenomenon similar to the one encountered at the leading edge of a Joukowski airfoil. Actually, the compressibility of the fluid makes the pressure high but finite. The value $\mu = \pm 1$ is reached over an infinitely small area, so that when the potential is integrated, no singularity remains, as will be seen later. However, in the argument which follows, the restriction must be made $|\mu| < 1$.

With that restriction, it is possible to calculate the coefficients α_n^{2s+1} . First the left side of equation (6.9) is expanded by use of the binomial theorem: then

$$\frac{1}{2s+1} \frac{\mu}{(1-\mu^2)^{2s+1/2}} = \sum_{K=0}^{\infty} C_{2K+1}^{2s+1} \mu^{2K+1} \tag{6.10}$$

The coefficient C_{2K+1}^{2s+1} or $(2K+1)$ coefficient in the expansion of the $(2s+1)$ term has a known definite value.

Next, we expand the derivatives of the Legendre polynomials on the right in (6.9)

$$\frac{d^{2s+1} P_n(\mu)}{d\mu^{2s+1}} = (\gamma_n^{2m+1})_0 + (\gamma_n^{2m+1})_2 \mu^2 + \dots \tag{6.11}$$

The coefficient γ_n^{2m+1} refers to the $(2m+1)$ differential of the Polynomial of order n; the postscript 0,2 etc. is an ordering index, which is 0,2,4... if n is odd and 1,3,5.... if n is even; it refers to the power of μ which the γ term multiplies.

With this notation, the right side of equation (6.9) can be written:

$$\begin{aligned}\sum_0^{\infty} \alpha_n^{2s+1} \frac{d^{2s+1} P_n(\mu)}{d\mu^{2s+1}} = {} & \alpha_{2s+1}^{2s+1} (\gamma_{2s+1}^{2s+1})_0 + \alpha_{2s+3}^{2s+1} (\gamma_{2s+3}^{2s+1})_0 + \dots \\ & + \mu \left[\alpha_{2s+2}^{2s+1} (\gamma_{2s+2}^{2s+1})_1 + \alpha_{2s+4}^{2s+1} (\gamma_{2s+4}^{2s+1})_1 + \dots \right] \\ & + \mu^2 \left[\alpha_{2s+3}^{2s+1} (\gamma_{2s+3}^{2s+1})_2 + \alpha_{2s+5}^{2s+1} (\gamma_{2s+5}^{2s+1})_2 + \dots \right] \\ & + \dots \end{aligned} \tag{6.12}$$

If equations (6.10) and (6.12) are substituted into equation (6.9) and equal powers of μ are equated, the following infinite set of linear equations with an infinite number of unknowns is obtained:

$$\begin{aligned} c_1^{2s+1} &= \alpha_{2s+2}^{2s+1}\left(\gamma_{2s+2}^{2s+1}\right)_1 + \alpha_{2s+4}^{2s+1}\left(\gamma_{2s+4}^{2s+1}\right)_1 + \alpha_{2s+6}^{2s+1}\left(\gamma_{2s+6}^{2s+1}\right)_1 + \ldots \\ c_3^{2s+1} &= \alpha_{2s+4}^{2s+1}\left(\gamma_{2s+4}^{2s+1}\right)_3 + \alpha_{2s+6}^{2s+1}\left(\gamma_{2s+6}^{2s+1}\right)_3 + \alpha_{2s+8}^{2s+1}\left(\gamma_{2s+8}^{2s+1}\right)_3 + \ldots \\ c_5^{2s+1} &= \alpha_{2s+6}^{2s+1}\left(\gamma_{2s+6}^{2s+1}\right)_5 + \alpha_{2s+8}^{2s+1}\left(\gamma_{2s+8}^{2s+1}\right)_5 + \alpha_{2s+10}^{2s+1}\left(\gamma_{2s+10}^{2s+1}\right)_5 + \ldots \\ &\cdots \end{aligned} \tag{6.13}$$

In the set of equations (6.13), the coefficients c are binomial coefficients and therefore known. Similarly, the coefficients γ are obtained by differentiating a known Legendre polynomial and therefore also known. The coefficients α are unknown.

While the equations (6.13) cannot be solved explicitly as they stand, a good approximate solution is obtained by cutting off the set at a certain point, which corresponds to a certain order Legendre polynomial. In the calculation discussed here, the set was cut off after the term $P_{20}(\mu)$. Then, the last equation (6.13) is of the form:

$$c_{19}^{2s+1} = \alpha_{20}^{2s+1}\left(\gamma_{20}^{2s+1}\right)_{19} \tag{6.14}$$

This is simply solved for α_{20}^{2s+1}, which thus becomes a known coefficient in the equations above the last.

The equation immediately above the last is:

$$c_{17}^{2s+1} = \alpha_{18}^{2s+1}\left(\gamma_{18}^{2s+1}\right)_{17} + \alpha_{20}^{2s+1}\left(\gamma_{20}^{2s+1}\right)_{17} \tag{6.15}$$

This equation, again, has only one unknown α_{18}^{2s+1}. It is noted that if this procedure is repeated, all the α's of lower indices are easily calculated. It is necessary, however, to check the accuracy of the low index coefficients when they depend on higher index coefficients and a starting point is arbitrarily chosen. This was done by comparing α_2^1 calculated by starting the calculation at P_{10}, P_{16}, P_{20}. It was found that the computed value is probably correct to four decimal places. The list of α coefficients calculated is tabulated below:

$$\begin{array}{lll}
\alpha_2^1 = .47823 & & \\
\alpha_4^1 = .09150 & \alpha_4^3 = .00596 & \\
\alpha_6^1 = .02910 & \alpha_6^3 = .00106 & \alpha_6^5 = .00004 \\
\alpha_8^1 = .00945 & \alpha_8^3 = .00026 & \alpha_8^5 = .00001 \\
\alpha_{10}^1 = .00274 & \alpha_{10}^3 = .00006 & \\
\alpha_{12}^1 = .00066 & \alpha_{12}^3 = .00001 & \\
\alpha_{14}^1 = .00013 & & \\
\alpha_{16}^1 = .00002 & &
\end{array} \tag{6.16}$$

In this calculation, all the Legendre polynomial coefficients were determined from the recurrence formulae:

The actual coefficients in the potential formula, $\alpha_n^s = \frac{\pi}{4}\frac{n+1}{R^{n+2}} a_n^s$ must be determined, and the potential function is written:

$$\varphi = \frac{4R}{\pi} \sum_0^\infty \sum_0^\infty \frac{\alpha_n^s}{n+1} P_n^s(\cos\theta) \sin s\omega \tag{6.17}$$

The drag integral is found to be:

$$\int \varphi \frac{\partial \varphi}{\partial n} dS = \frac{8R^3}{\pi} \sum_0^\infty \sum_0^\infty \frac{(\alpha_n^s)^2}{(n+1)(2n+1)} \frac{(n+s)!}{(n-s)!} \quad (n > s) \tag{6.18}$$

It should be noted that formula (6.18) may be expected to hold only if the doubly infinite series converges absolutely. It is clearly convergent if we have the inequality $\alpha_n^s < n^{-s}$ for all n and s. This cannot be proved analytically, since no recurrence formula for α_n^s is available. It is, however, true for all α_n^s listed in (6.16), and equation (6.18) is therefore held to be valid.

It is possible at this stage to introduce the approximating function, which, since a sphere has only one parameter, R, is that the depth of penetration of the sphere is equal to that of the approximated body. Then, making use of equations (1.15), (6.4), (6.18), one has the results:

$$F_x = \frac{\rho V^2}{2} \sin 2\alpha \,(0.765\, R^2)$$
$$F_y = \frac{\rho V^2}{2} 2 \sin^2\alpha \,(1.5708\, R^2) \tag{6.19}$$

VII. WATER IMPACT OF AN ELLIPSOID OF REVOLUTION

The simplest problem after that of the sphere treated above, is that of the water impact of a body approximated by an ellipsoid of revolution. The first part of the problem, the study of the steady potential flow about the semi-ellipsoid, is taken up first.

The physical system is shown on Figure 5. The analysis is carried out in ellipsoidal coordinates defined as follows: Let the basic ellipsoid have a major semi-axis a and a minor semi-axis b. First, the case is studied when a is part of the x axis. Then the foci of the ellipsoid are on the x axis, at distances $\pm k$ from the center of the ellipsoid. A set of three curvilinear coordinates is now selected as follows:

$$\begin{aligned} x &= k\mu\zeta \\ y &= k(1-\mu^2)^{1/2}(\zeta^2-1)^{1/2}\sin\omega \\ z &= k(1-\mu^2)^{1/2}(\zeta^2-1)^{1/2}\cos\omega \end{aligned} \qquad (7.1)$$

In this system, the surfaces $\zeta = \text{const}$ are confocal ellipsoids of revolution, the basic ellipsoid is defined by the focal length k and $\zeta = \zeta_0$ such that $a = k\zeta_0$. The space outside the ellipsoid is described by a variation $\zeta > \zeta_0$. The surfaces $\mu = \text{const.}$ are confocal hyperboloids of revolution of two sheets; they intersect the ellipses $\zeta = \text{const.}$ along circles whose center is on the x axis, so that to a given value of μ corresponds a given value of the azimuth angle θ. The surfaces $\omega = \text{const}$ are planes which make an angle w with the plane zOx.

It is known that the normal solution of Laplace's equation in this system of coordinates is:

$$\varphi = \sum_{0}^{\infty}\sum_{1}^{n+1} P_n^s(\mu)\, Q_n^s(\zeta)\left[A_n^s \sin s\omega + B_n^s \cos s\omega\right] \qquad (7.2)$$

Since this normal form depends on ζ as $Q_n^s(\zeta)$, it is also known that the boundary condition at infinity is satisfied.

One must now determine functions φ_1 and φ_2 which also satisfy boundary conditions (1.13a,b) on the surface of the basic ellipsoid $\zeta = \zeta_0$.

The motion characterized by φ_2 is relatively simple: It is the translation of an ovary ellipsoid parallel to an equatorial axis. In that case, the potential is given by:

$$\varphi_2 = \frac{k}{\frac{1}{2}\log\frac{\zeta_0+1}{\zeta_0-1} - \frac{\zeta_0^2-2}{\zeta_0(\zeta_0^2-1)}} \left\{(1-\mu^2)^{1/2}(\zeta_0^2-1)^{1/2}\left[\frac{1}{2}\log\frac{\zeta_0+1}{\zeta_0-1} - \frac{\zeta_0}{\zeta_0^2-1}\right]\cos\omega\right\} \tag{7.3}$$

The drag integral which corresponds to such a potential function can be written down at once:

$$\int \varphi_2 \frac{\partial \varphi_2}{\partial n} ds = \frac{\pi k^3}{2} F(\zeta_0) \tag{7.4}$$

$$F(\zeta_0) = (\zeta_0^2-1)\zeta \frac{\log\frac{\zeta_0+1}{\zeta_0-1} - \frac{2\zeta_0}{\zeta_0^2-1}}{\log\frac{\zeta_0+1}{\zeta_0-1} - \frac{2(\zeta_0^2-2)}{\zeta_0(\zeta_0^2-1)}} \tag{7.5}$$

The determination of the potential φ_1 will now be carried through. The boundary condition (1.13b) in this case may be written as

$$\frac{\partial \varphi_1}{\partial \zeta} = k\mu \frac{|\sin\omega|}{\sin\omega} \tag{7.6}$$

The right side of equation (7.6) is now expanded as a Fourier series while the right side of equation (7.2) is differentiated with respect to ζ at $\zeta = \zeta_0$. When the two resulting quantities are substituted into the boundary equation (7.6) that equation becomes:

$$\frac{4k}{\pi}\mu \sum_0^\infty \frac{\sin(2s+1)\omega}{2s+1} = \sum_0^\infty \sum_1^{n+1} \frac{d Q_n^s(\zeta_0)}{d\zeta_0} P_n^s(\mu)\left[A_n^s \sin s\omega + B_n^s \cos s\omega\right] \tag{7.7}$$

When the substitution $\alpha_n^s = \frac{\pi}{4k} \frac{dQ_n^s(\xi_0)}{d\xi_0} A_n^s$ is made on the left side of equation (7.7), and terms equal in s are equated, the following equation results:

$$\frac{\mu}{2s+1} = \sum_0^\infty \alpha_n^{2s+1} P_n^{2s+1}(\mu) \qquad (7.8)$$

That equation is seen to be identical to equation (6.8) obtained in the study of the sphere. The results obtained in that case may therefore be taken over bodily into the present analysis, the coefficients α_{2n}^{2s+1} are given by the table (6.16) and the potential function is:

$$\varphi_1 = \frac{4k}{\pi} \sum_0^\infty \sum_1^{n+1} \frac{Q_n^s(\xi_0)}{dQ_n^s(\xi_0)/d\xi_0} \alpha_n^s P_n^s(\mu) \sin sw \qquad (7.9)$$

which is integrated into the following drag integral:

$$\int \varphi_1 \frac{\partial \varphi_1}{\partial n} dS = \frac{8k^3}{\pi}(\xi^2 - 1) \sum_0^\infty \sum_1^{n+1} \frac{Q_n^s(\xi_0)}{dQ_n^s(\xi_0)/d\xi_0} (\alpha_n^s)^2 \frac{1}{(n+1)(2n+1)} \frac{(n+s)!}{(n-s)!} \qquad (7.10)$$

Up to the present, only a prolate or ovary ellipsoid of revolution was discussed, but similar methods are applicable to an oblate or planetary ellipsoid of revolution. The investigation will be focused on that problem now. The importance of such an investigation is apparent since it extends the range of permissible values of ξ to the range where $0 < \xi < 1$, and allows more complete freedom in selecting approximating functions. The boundary between the two cases is the case of the sphere for which $\xi_0 = 1$. That case was discussed separately.

The potential function in the case of the planetary ellipsoid may be written as

$$\varphi = \sum_0^\infty \sum_1^{n+1} P_n^s(\mu)\, q_n^s(\xi')\, [A_n^s \sin sw + B_n^s \cos sw] \qquad (7.11)$$

which is different from the previous case only in that $q(\xi')$ has replaced $Q(\xi)$. The analysis is carried through as before and formulas similar to (7.4) and (7.10) are obtained:

$$\int \varphi_2 \frac{\partial \varphi_2}{\partial n} ds = \frac{\pi k^3}{2} F'(\xi_0') \tag{7.12}$$

$$F'(\xi_0') = (\xi_0'^2+1)\,\xi_0' \frac{\dfrac{\xi_0'}{\xi_0'^2+1} - \cot^{-1}\xi_0'}{\dfrac{\xi_0'^2+2}{\xi_0'(\xi_0'^2+1)} - \cot^{-1}\xi_0'} \tag{7.13}$$

$$\int \varphi_1 \frac{\partial \varphi_1}{\partial n} ds = \frac{8k^3}{\pi}(\xi_0'^2+1) \sum_0^\infty \sum_1^{n+1} \frac{q_n^s(\xi_0')}{dq_n^s(\xi_0')/d\xi_0'} (\alpha_n^s)^2 \frac{1}{(n+1)(2n+1)} \frac{(n+s)!}{(n-s)!} \tag{7.14}$$

The function $q(\xi_0')$ which has been introduced into the analysis here is connected to the Legendre function of the second kind by the relation:

$$q_n^s(\xi') = Q_n^s(i\xi) \tag{7.15}$$

All the formulae necessary to calculate the drag of a body approximated by an ellipsoid of revolution are now available. The approximating functions are two in number: The equality of depth of penetration gives the equation

$$k\sqrt{\xi^2-1} = b \tag{7.16}$$

where b is the depth of penetration of the approximated body. It may be noted that ξ is to be considered a complex variable, so that when $|\xi| < 1$ it is replaced by $i\xi'$; in that manner, one set of formulas can be used to cover the whole range of ellipsoids of revolution.

Experimental investigations carried out at the Morris Dam Hydrodynamic Station and discussed by P.Y. Chou (1.9) seem to indicate that the second approximating function should be one equating the volume of the semi-ellipsoid to the submerged volume of the approximated body. That function would be

$$\frac{2\pi}{3} k^3 \xi(\xi^2-1) = V(b) \tag{7.17}$$

where V(b) is a known function for the body under investigation.

By the use of equations (7.16) and (7.17) or similar equations which may be found to be convenient, it is possible to express the ellipsoidal parameters in functional form as follows:

$$k = f_1(b) \tag{7.18}$$

$$F_1(\xi) = (\xi^2-1)\xi \frac{\log\frac{\xi+1}{\xi-1} - \frac{2\xi}{\xi^2-1}}{\log\frac{\xi+1}{\xi-1} - \frac{2(\xi^2-2)}{\xi(\xi^2-1)}} = f_2(b) \tag{7.19}$$

$$F_2(\xi) = (\xi^2-1)\sum_0^\infty \sum_1^{n+1} \frac{Q_n^s(\xi)}{dQ_n^s(\xi)/d\xi} (\alpha_n^s)^2 \frac{1}{(n+1)(2n+1)} \frac{(n+s)!}{(n-s)!} = f_3(b) \tag{7.20}$$

While analytic forms for f_1, f_2, f_3 would be extremely complicated, those functions may be plotted, particularly if, as is usual, V(b) is not analytic but graphically determined. The drag forces are now given by:

$$\begin{aligned} F_x &= \frac{\rho V^2}{2}\sin 2\alpha\left[\frac{24k^2}{\pi}\frac{df_1}{db}f_3 + \frac{8k^3}{\pi}\frac{df_3}{db}\right] \\ F_y &= \frac{\rho V^2}{2} 2\sin^2\alpha\left[\frac{3\pi}{2}k^2\frac{df_1}{db}f_2 + \frac{\pi k^3}{2}\frac{df_2}{db}\right] \end{aligned} \tag{7.21}$$

While the formulas (7.21) do not have the simplicity of formulas (4.31) or (6.19), and while they do not give as simple a picture of the variation of drag, they are much better approximations, and do not involve any more work than the graphical solution of equations

(7.16) to (7.20). They are therefore practical methods of solving the problem, as was verified at NOTS, Inyokern, by the author of this analysis. While a fair degree of approximation may be obtained by their use, a better approximation still is obtained when similar formulas are developed for an ellipsoid with three unequal axes. That problem is the next in the present study.

VIII. WATER IMPACT OF A GENERAL ELLIPSOID

The general problem of the water impact of a body which can be approximated by an ellipsoid with three unequal axes is the last one discussed in this paper. It is divided into two parts, the former of which deals with an ellipsoid, one half of which is submerged, and the latter with the differentiation of the drag integral.

The coordinate system in which this problem is discussed is introduced by E.W. Hobson (2.5), whose work "The theory of spheroidal and ellipsoidal harmonics" forms the Mathematical background of the analysis presented below. In accordance with the discussion presented in part V, the coordinate system is based on the ellipsoid whose motion is under investigation.

The following three equations are written down to define the new variables μ, ν, ρ in terms of the cartesian variables x,y,z, (see Fig. 6)

$$\begin{aligned}
&\frac{x^2}{\rho^2} + \frac{y^2}{\rho^2-h^2} + \frac{z^2}{\rho^2-k^2} = 1 &\qquad& h<k<\rho \\
&\frac{x^2}{\rho^2} + \frac{y^2}{\rho^2-h^2} - \frac{z^2}{\rho^2-k^2} = 1 && h<\mu<k \\
&\frac{x^2}{\nu^2} + \frac{y^2}{h^2-\nu^2} - \frac{z^2}{k^2-\nu^2} = 1 && \nu<h<k
\end{aligned} \tag{8.1}$$

or after a few simple transformations:

$$\begin{aligned}
x &= \frac{\rho\mu\nu}{hk} \\
y &= \frac{\sqrt{(\rho^2-h^2)(\mu^2-h^2)(h^2-\nu^2)}}{h\sqrt{k^2-h^2}} \\
z &= \frac{\sqrt{(\rho^2-k^2)(k^2-\mu^2)(k^2-\nu^2)}}{k\sqrt{k^2-h^2}}
\end{aligned} \tag{8.2}$$

Before proceeding with the discussion it is well to show the physical significance of the variables ρ, μ, ν .

It is clear that the surface $\rho = const.$ is an ellipsoid of focal lengths h,k and of major axis ρ . The actual ellipsoid under study is one whose major axis is denoted by $a = \rho_0$. Its focal lengths h,k are basic parameters of the problem.

The surface $\mu = const.$ is a hyperboloid of one sheet of axes y,x and of focal lenghts h,k. When μ is h, y is equal to 0 and when μ is k, z is equal to 0; as μ varies between h and k, the hyperboloid intersects the ellipsoid ρ_0 along two ellipses symmetrical with respect to the plane xz.

The surface $\nu = const$ is a hyperboloid of two sheets of axes z,x and of focal lengths h,k. When ν is zero, x is equal to 0 and when ν is h, y is equal to zero; as ν varies between 0 and h, the hyperboloid intersects the ellipsoid ρ_0 along two ellipses symmetrical with respect to the plane yz.

It is clear from these statements that single values of $\rho \mu \nu$ do not define one, but eight points in space; further conventions are therefore needed to define $\rho \mu \nu$ so that for each value of ρ, μ, ν one specific point is defined. This is done by defining the signs of the radicals in the expressions (8.2). Figure 6 shows eight points 1,2,3,4,5,6,7,8 defined by the intersection of curves $\rho, \mu, \nu = Const.$. It is agreed by definition that $\sqrt{\rho^2 - k^2}$ may be taken positive or negative; when $\sqrt{\rho^2 - k^2}$ is positive, it refers to the points numbered 1,2,6,7. Similarly, as the ellipsoid from $-y_0$ to $+y_0$ is described, μ varies from k to h and back to k.

It is agreed to define the sign of $\sqrt{\mu^2-h^2}$ as positive until μ reaches h and negative as μ retraces its way back to k. Thus, when $\sqrt{\mu^2-h^2}$ is positive, its value refers to the points 1,2,3,4. Finally, as ν goes from 0 to h and back to 0, the ellipsoid is described again. By convention, when ν goes from 0 to h, $\sqrt{h^2-\nu^2}$ is positive, and negative as ν goes from h back to 0; thus $\sqrt{h^2-\nu^2}$ positive refers to points 2,3,8,7. Therefore, as all three radicals are positive, only point 2 is defined; any combination of signs similarly designates one single point, and to each point corresponds a single combination of signs. With this set of definitions, it is possible to construct a one valued transformation x,y,z into ρ, μ, ν . The sign convention is summarized below:

$$\begin{array}{rl} \sqrt{\rho^2-k^2} & z>0 \\ -\sqrt{\rho^2-k^2} & z<0 \\ \sqrt{\mu^2-h^2} & y>0 \\ -\sqrt{\mu^2-h^2} & y<0 \\ \sqrt{h^2-\nu^2} & x>0 \\ -\sqrt{h^2-\nu^2} & x<0 \end{array} \tag{8.3}$$

With these definitions in mind, it is possible to write the Laplace equation in the coordinate system ρ, μ, ν as follows:

$$(\lambda^2-h^2)(\lambda^2-k^2)\frac{d^2E(\lambda)}{d\lambda^2} + \lambda(2\lambda^2-h^2-k^2)\frac{dE(\lambda)}{d\lambda} + \left[(h^2+k^2)p-n(n+1)\lambda^2\right]E(\lambda)=0$$

$$\lambda = \rho, \mu, \nu \tag{8.4}$$

The Laplacian is split into three identical total differential equations in ρ, μ, ν . The differential equation so obtained was first discussed by Lame in connection with the flow of heat in an ellipsoid and therefore bears his name. Its two solutions are one of polynomial type convergent near the origin and similar to $P_n^m(\mu)$ and one of a type convergent toward infinity similar to $Q_n^m(\mu)$ Those solutions are generally labelled E and F. n and p are constants which arise in separating variables in the Laplacian; n is an integer 1,2,3... while p is chosen so that the polynomial E will have a finite number of terms. It is further discussed later. For each value of n, there are $2n+1$ polynomials E_n^s, as follows: $\frac{1}{2}(n+1)$ polynomials of the form

$$K_n^s = a_0 + a_1\mu + a_2\mu^2 + \dots a_n\mu^n \tag{8.5a}$$

There are $\frac{1}{2}(n+1)$ polynomials of the form:

$$L_n^s = \sqrt{\mu^2 - h^2}\,(b_0 + b_1\mu + b_2\mu^2 \dots) \tag{8.5b}$$

There are $\frac{1}{2}(n+1)$ polynomials of the form:

$$M_n^s = \sqrt{k^2 - \mu^2}\,(c_0 + c_1\mu + c_2\mu^2 + \dots) \tag{8.5c}$$

There are finally $\frac{1}{2}(n-1)$ polynomials of the form:

$$N_n^s = \sqrt{(\mu^2 - h^2)(k^2 - \mu^2)}\,(d_0 + d_1\mu + d_2\mu^2 + \dots) \tag{8.5d}$$

The general solution of Laplace's equation is therefore written as:

$$\varphi = \sum_0^{\infty}\sum_1^{2n+1} A_n^s\, F_n^s(\rho)\, E_n^s(\mu)\, E_n^s(\nu) \tag{8.6}$$

The presence of the function $F_n^s(\rho)$ guarantees that the boundary condition at infinity is satisfied. The boundary condition on the surface of the body is written in two parts: equations (1.13a,b) become

$$\frac{\partial\varphi_1}{\partial n} = -\cos\lambda_1 \frac{|y|}{y} \qquad \frac{\partial\varphi_1}{\partial\rho} = -\frac{\partial x}{\partial\rho}\,\frac{|y|}{y} \tag{8.7}$$

$$\frac{\partial\varphi_2}{\partial n} = -\cos\lambda_2 \qquad \frac{\partial\varphi_2}{\partial\rho} = -\frac{\partial y}{\partial\rho}$$

The second solution is readily written out:

$$\varphi_2 = C \frac{F_1'(\rho)}{\partial F_1'(\rho)/\partial\rho} \sqrt{\mu^2-h^2}\sqrt{h^2-\nu^2} \tag{8.8}$$

or substituting for $F_1'(\rho)_0$ on the boundary of the body

$$\varphi_2 = \rho_0^2 \left[1+\sqrt{(\rho_0^2-k^2)(\rho_0^2-h^2)} \int_{\rho_0}^{\infty} \frac{d\rho}{\rho\sqrt{(\rho^2-k^2)(\rho^2-h^2)}} \right] \frac{\sqrt{(\mu^2-h^2)(h^2-\nu^2)}}{h\sqrt{(\rho^2-h^2)(k^2-h^2)}} \tag{8.9}$$

The drag integral is also readily found: In Clebsch's notation

$$\int \varphi_2 \frac{\partial \varphi_2}{\partial n} dS = \frac{1}{2} \frac{S_0'}{\Sigma_0' - S_0'} \tag{8.10}$$

where S_0' is the integral of φ_2 evaluated by a power series expansion and Σ_0' is the square root of the products of the axes. (See ref. 3.7).

The determination of φ_1 is not nearly so simple. If equation (8.6) is differentiated, the boundary condition (8.7) becomes

$$\sum_0^\infty \sum_1^{2n+1} A_n^s \frac{\partial F_n^s(\rho_0)}{\partial \rho_0} E_n^s(\mu) E_n^s(\nu) = -\frac{\mu\nu}{hk}\psi(\mu\nu) \tag{8.11}$$

$$\begin{aligned} \psi(\mu\nu) &= +1 \qquad \sqrt{\mu^2-h^2} > 0 \\ \psi(\mu\nu) &= -1 \qquad \sqrt{\mu^2-h^2} < 0 \end{aligned} \tag{8.12}$$

The function $-\frac{\mu\nu}{kh}\psi(\mu\nu)$ must be expanded into an infinite series of Lamé polynomials; then coefficinets in (8.11) may be equated. (8.11) is written as:

$$a_n^s = -\frac{\partial F_n^s(\rho_0)}{\partial \rho_0} A_n^s hk \quad ; \sum_0^\infty \sum_1^{2n+1} a_n^s E_n^s(\mu) E_n^s(\nu) = \mu\nu\psi = f(\mu\nu) \tag{8.13}$$

But, application of a very general theorem of harmonic analysis gives

$$f(\mu\nu) = \sum_0^\infty \sum_1^{2n+1} c_n^s E_n^s(\mu) E_n^s(\nu) \tag{8.14}$$

$$c_n^s = \frac{\int_0^h \int_0^K \frac{d\nu}{\sqrt{(h^2-\nu^2)(k^2-\nu^2)}} \frac{d\mu}{\sqrt{(\mu^2-h^2)(k^2-\mu^2)}} (\mu^2-\nu^2) f(\mu\nu) E_n^s(\mu) E_n^s(\nu)}{\int_0^h \int_0^K \frac{d\nu}{\sqrt{(h^2-\nu^2)(k^2-\nu^2)}} \frac{d\mu}{\sqrt{(\mu^2-h^2)(k^2-\mu^2)}} (\mu^2-\nu^2) \left[E_n^s(\mu) E_n^s(\nu)\right]^2} \tag{8.15}$$

When the present value of $f(\mu\nu)$ is substituted into the above equations, it is seen that symmetry cancels all terms for which n is even and doubles all terms for which n is odd. The form of $f(\mu\nu)$ also indicates that the expansion is in terms of K_{2n+1}^s terms only. The number of coefficients c_n^s is thus reduced by a factor of 8 just as it was in the case of the sphere and the ellipsoid of revolution. Equations (8.14) and (8.15) are now rewritten:

$$f(\mu\nu) = \sum_0^{\infty} \sum_1^{2n+1} c_{2n+1}^s \; K_{2n+1}^s(\mu) K_{2n+1}^s(\nu) \tag{8.16}$$

$$c_{2n+1}^s = \frac{\int_0^h \int_h^k \frac{d\nu}{\sqrt{(h^2-\nu^2)(k^2-\nu^2)}} \frac{d\mu}{\sqrt{(k^2-\mu^2)(\mu^2-h^2)}} (\mu^2-\nu^2)\mu\nu K_{2n+1}^s(\mu) K_{2n+1}^s(\nu)}{\int_0^h \int_h^K \frac{d\nu}{\sqrt{(h^2-\nu^2)(k^2-\nu^2)}} \frac{d\mu}{\sqrt{(k^2-\mu^2)(\mu^2-h^2)}} (\mu^2-\nu^2)\left[K_{2n+1}^s(\mu) K_{2n+1}^s(\nu)\right]^2} \tag{8.17}$$

To evaluate the coefficients c_{2n+1}^s, it is convenient to write the K functions as follows:

$$\begin{aligned} K_{2n+1}^1 &= K_{2n+1}^a = a_1\mu + a_3\mu^3 + \dots a_{2n+1}\mu^{2n+1} \\ K_{2n+1}^2 &= K_{2n+1}^b = b_1\mu + b_3\mu^3 + \dots b_{2n+1}\mu^{2n+1} \\ &- - - - - \\ K_{2n+1}^i &= K_{2n+1}^i = i_1\mu + i_3\mu^3 + \dots i_{2n+1}\mu^{2n+1} \end{aligned} \tag{8.18}$$

The coefficients a,b,... i... can be calculated in function of the index numbers and the p constants by the following recursion formulae obtained by substitution into the differential equation and equation of equal powers of λ :

$$\begin{aligned} 2(2n-1)\,a_{n-2} &= (h^2+k^2)a_n\,(p_a-n^2) \\ 4(2n-3)\,a_{n-4} &= (h^2+k^2)(p_a-(n-2)^2)\,a_{n-2}+h^2k^2n(n-1)\,a_n \\ 6(2n-5)\,a_{n-6} &= (h^2+k^2)(p_a-(n-4)^2)\,a_{n-4}+h^2k^2(n-2)(n-3)\,a_{n-2} \end{aligned} \tag{8.19}$$

where p is given by the condition that a_{n+1} , a_{n+2} , $\ldots = 0$. It can be shown that under that condition, p is determined by equating to 0 the determinant:

$$\begin{vmatrix} 0 & 0 & 2(2n-1) & -(h^2+k^2)(p-n^2) \\ & 4(2n-3) & -(h^2+k^2)[p-(n-2)^2] & -h^2k^2n(n-1) \\ & & & 0 \\ & & & \\ -(h^2+k^2)[p-(n-2r)^2] & -h^2k^2(n-2r+2)(n-2r+1) & & 0 \end{vmatrix} \tag{8.20}$$

That determinant is of degree $r+1$, thus defining $\frac{1}{2}(n+1)$ values of $p: p_a, p_b, \ldots p_i \ldots$ all of which are real and distinct for all values of h and k.

In view of the remarks just made, the coefficients a, b, ... i, ... $r+1 \ldots$ are known functions of the basic length parameters h, k.

The following integrals are now defined:

$$\begin{aligned} A_{2n} &= \int_0^h \frac{\nu^{2n}\,d\nu}{\sqrt{(h^2-\nu^2)(k^2-\nu^2)}} \\ B_{2n} &= \int_h^k \frac{\mu^{2n}\,d\mu}{\sqrt{(\mu^2-h^2)(k^2-\mu^2)}} \end{aligned} \tag{8.21a,b}$$

With the aid of the substitutions $\nu^2 = h^2\xi^2$; $\mu^2 = k^2\left[1 - \frac{k^2-h^2}{k^2}\eta^2\right]$ it is easily verified that:

$$A_0 = \frac{1}{k} F(h/k) \qquad\qquad A_2 = k\left[F(h/k) - E(h/k)\right]$$

$$B_0 = \frac{1}{k} F\left(\frac{\sqrt{k^2-h^2}}{k}\right) \qquad\qquad B_2 = k\, E\left(\frac{\sqrt{k^2-h^2}}{k}\right) \tag{8.22a,b,c,d}$$

where F,E, are complete elliptic integrals of the first and second kind, of the shown complementary moduli.

By differentiating the expressions $\xi^{2n-3}\sqrt{(h^2-\xi^2)(k^2-\xi^2)}$; $\eta^{2n-3}\sqrt{(\eta^2-h^2)(k^2-\eta^2)}$ and integrating the results between 0 and h and between h and k, it is possible to establish the recurrence formulas:

$$(2n-1)[A,B]_{2n} = -(2n-3)h^2k^2[A,B]_{2n-4} + (2n-4)(h^2+k^2)[A,B]_{2n-2} \tag{8.23}$$

The integrals A_{2n}, B_{2n} are therefore real, simply calculated functions of the metric parameters h,k, and of the complete elliptic integrals characteristic of the ellipsoid under investigation.

In terms of the calculable quantities a_{2n+1} , A,B, it is now easy to calculate both the numerator and the denominator of c^a_{2n+1} . In each case, the integral is separated into the difference of two products of two simple integrals. The result for the numerator is seen to be:

$$N^a_{2n+1} = a_1^2(B_4A_2 - B_2A_4) + a_3a_1(B_6A_2 - B_2A_6) + a_3^2(B_6A_4 - B_4A_6) + \dots \tag{8.24}$$

A similar calculation is carried out for the denominator D^a_{2n+1} It is convenient to write first an expression for $[K(\mu)K(\nu)]^2$ as follows:

$$\left[K^a_{2n+1}(\mu)\,K^a_{2n+1}(\nu)\right]^2 = a_1^4\mu^2\nu^2 + 2a_1^3a_3\,\mu^2\nu^2(\mu^2+\nu^2) + a_3^2a_1^2\,\mu^2\nu^2(\mu^2+\nu^2)^2 + \dots \tag{8.25}$$

Then the integration is carried out as above and gives:

$$D^{a}_{2n+1} = a_1^4 (B_4 A_2 - B_2 A_4) + 2a_3 a_1^3 (A_2 B_6 - A_6 B_2) \tag{8.26}$$

When results are collected, it is seen that the coefficient C^{a}_{2n+1} can be expressed in function of h and k as follows:

$$C^{a}_{2n+1} = \frac{a_1^2(B_4A_2 - B_2A_4) + a_3a_1(B_6A_2 - B_2A_6) + \ldots}{a_1^4(B_4A_2 - B_2A_4) + 2a_3a_1^3(B_6A_2 - B_2A_6) + \ldots} = C^{a}_{2n+1}(h,K) \tag{8.27}$$

When this result is substituted into (8.13), the potential function is written as:

$$\varphi_1 = -\frac{1}{hK}\sum_0^{\infty}\sum_1^{\frac{1}{2}(n+1)} \frac{F^{s}_{2n+1}(\rho_0)}{\partial F^{s}_{2n+1}(\rho_0)/\partial\rho_0}\, C^{a}_{2n+1}(h,K)\, K^{a}_{2n+1}(\mu) K^{a}_{2n+1}(\nu) \tag{8.28}$$

Because of the orthogonality property of the Lamé polynomials K, the drag integral is deduced at once as follows:

$$\oint \varphi_1 \frac{\partial\varphi_1}{\partial\rho_1} dS = \frac{1}{h^2K^2}\sum_0^{\infty}\sum_1^{\frac{1}{2}(n+1)} \frac{F^{s}_{2n+1}(\rho_0)}{\partial F^{s}_{2n+1}(\rho_0)/\partial\rho_0}\left[C^{a}_{2n+1}(h,K)\right]^2 D^{a}_{2n+1}(h,K)\sqrt{(\rho_0^2-h^2)(\rho_0^2-k^2)} \tag{8.29}$$

Before applying equation (8.29) to any specific calculation, it is convenient to calculate the coefficients C and D in function of ρ_0, h, k. It is clear from equations (8.24) and (8.26) that

$$C_1 = 1$$

$$D_1 = B_4A_2 - B_2A_4 = \frac{h^2K^2}{3}\left[FE' - F'(F-E)\right] \tag{8.30}$$

since the coefficient $a_1' = 1$ and all other coefficients vanish. The terms F and E are complete elliptic integrals of modulus h/k while F' and E' are elliptic integrals of modulus $\frac{\sqrt{K^2-h^2}}{K}$

Suprisingly enough, when the calculation of C_3^a and C_3^b was carried out in detail, up to five decimal places, for values of h and k ranging from 0 to 0.7, the coefficients vanished identically. Such a result can be proved analytically for the special cases when the ellipsoid becomes an ellipsoid of revolution (then, h $\rightarrow$ 0); the proof is rather tedious and is not reproduced in the present paper. An analytical proof of such a result in the general case, while desirable from the point of view of rigor, is probably quite involved, and was not attempted. But on the basis of the verification that the coefficients of order 3 vanish and of the proof that coefficients of order 3 and 5 vanish in the special case of the ellipsoid of revolution (h $\rightarrow$ 0), it is felt reasonable to neglect all terms of order higher than 1. This simplifies the analysis considerably.

The new expression for the drag integral, based on equations (8.29) and (8.30) and the above argument, becomes:

$$\int \varphi_1 \frac{\partial \varphi_1}{\partial n}\, dS = \frac{1}{6}\sqrt{(\rho_0^2-h^2)(\rho_0^2-k^2)}\left[FE' - F'(F-E)\right]\frac{F_1(\rho_0)}{\partial F_1(\rho)/\partial \rho_0} \tag{8.31}$$

It remains to evaluate the Lamé function of the second kind and its derivative and to put them in a form suitable for computation. The form of $F_1(\rho)$ given by Hobson is:

$$F_1(\rho) = 3\rho\int_\rho^\infty \frac{d\rho}{\rho^2\sqrt{(\rho^2-h^2)(\rho^2-k^2)}} \tag{8.32}$$

The derivative of $F_1(\rho)$ is therefore

$$\frac{dF_1(\rho)}{d\rho} = 3\left\{\int_\rho^\infty \frac{d\rho}{\rho^2\sqrt{(\rho^2-h^2)(\rho^2-k^2)}} + \frac{1}{\rho\sqrt{(\rho^2-h^2)(\rho^2-k^2)}}\right\} \tag{8.33}$$

An integration of the formula (8.32) can be carried out as follows: The integral is rewritten in terms of the differential of $\frac{1}{\rho\sqrt{(\rho^2-h^2)(\rho^2-k^2)}}$

$$\int_\rho^\infty \frac{d\rho}{\rho^2\sqrt{(\rho^2-h^2)(\rho^2-k^2)}} = \frac{-1}{\rho\sqrt{(\rho^2-h^2)(\rho^2-k^2)}} + \int_\rho^\infty \frac{d\rho}{\sqrt{(\rho^2-h^2)^3(\rho^2-k^2)}} + \int_\rho^\infty \frac{d\rho}{\sqrt{(\rho^2-h^2)(\rho^2-k^2)^3}} \qquad (8.34)$$

With the aid of equation (8.34), the drag integral is rewritten as:

$$\int \varphi_1 \frac{\partial \varphi_1}{\partial n} ds = \frac{V}{\pi}\left[FE' - F'(F-E)\right]\left[\frac{-1+\frac{3V}{4\pi}(I_1+I_2)}{\frac{3V}{4\pi}(I_1+I_2)}\right] \qquad (8.35)$$

where $V = \frac{4}{3}\pi\rho\sqrt{(\rho^2-h^2)(\rho^2-k^2)}$ is the volume of the ellipsoid under study and I_1 and I_2 are the two integrals defined in equation (8.34). Those integrals must now be calculated.

I_1 is calculated by making use of the substitution $\rho = \frac{h}{dn\,u}$ With this substitution, the limits become $u = F'$, $u = dn^{-1} h/\rho$ and the integral becomes:

$$I_1 = \int_\rho^\infty \frac{d\rho}{\sqrt{(\rho^2-h^2)^3(\rho^2-k^2)}} = \frac{k}{h^4}\int_{u_0}^{K} \frac{1 - \frac{k^2 h^2}{k^2} sn^2 u}{sn^2 u}\, du \qquad (8.36)$$

This integral can be further reduced into the following closed form:

$$I_1 = \frac{k}{h^4}\left\{\left(\frac{h^2}{k^2}F - E\right) - \frac{h^2}{k^2} dn^{-1}\frac{h}{\rho} + E(dn^{-1} h/\rho) + \frac{h}{k\rho}\frac{\sqrt{\rho^2(k^2-h^2)+h^2k^2}}{\sqrt{\rho^2-h^2}}\right\} \qquad (8.37)$$

The integral I_2 is calculated in a very similar manner. The substitution is now $\rho = k/sn\,u$, so that the limits become $u = 0$

and $u = sn^{-1}\,k/\rho$. The integral becomes:

$$I_2 = \int_\rho^\infty \frac{d\rho}{\sqrt{(\rho^2-k^2)^3(\rho^2-h^2)}} = \frac{1}{k(k^2-h^2)}\int_{sn^{-1}k/\rho}^{0} -\frac{sn^2u}{cn^2u}\,du \tag{8.38}$$

This integral can also be put into the following closed form:

$$I_2 = \frac{1}{k(k^2-h^2)}\left[E\left(sn^{-1}\frac{k}{\rho}\right) - \frac{k}{\rho}\sqrt{\frac{\rho^2-h^2}{\rho^2-k^2}}\right] \tag{8.39}$$

The results (8.37) and (8.39) may now be substituted into the drag integral (8.35) to obtain a closed calculable formula as follows:

$$\oint \varphi_1 \frac{\partial\varphi_1}{\partial n}\,ds = \frac{4}{3}\pi\rho\sqrt{(\rho^2-h^2)(\rho^2-k^2)}\left[FE'-F'(F-E)\right] \times$$

$$\times\ \frac{-1+\rho\sqrt{(\rho^2-h^2)(\rho^2-k^2)}\left\{\frac{k}{h^4}\left[\left[\frac{h^2}{k^2}F-E\right]-\frac{h^2}{k^2}dn^{-1}h/\rho+E(dn^{-1}h/\rho)+\frac{h}{k\rho}\frac{\sqrt{\rho^2(k^2-h^2)+k^2h^2}}{\sqrt{\rho^2-h^2}}\right]+ \frac{1}{k(k^2-h^2)}\left[E(sn^{-1}k/\rho)-\frac{k}{\rho}\sqrt{\frac{\rho^2-h^2}{\rho^2-k^2}}\right]\right.}{\rho\sqrt{(\rho^2-h^2)(\rho^2-k^2)}\left\{\frac{k}{h^4}\left[\left(\frac{h^2}{k^2}F-E\right)-\frac{h^2}{k^2}dn^{-1}h/\rho+E(dn^{-1}h/\rho)+\frac{h}{k\rho}\sqrt{\frac{\rho^2(k^2-h^2)+h^2k^2}{\rho^2-h^2}}\right]+\frac{1}{k(k^2-h^2)}\left[E(sn^{-1}k/\rho)-\frac{k}{\rho}\sqrt{\frac{\rho^2-h^2}{\rho^2-k^2}}\right]\right\}} \tag{8.40}$$

Equations (8.10) and (8.40) complete the solution of the first part of the problem: The analysis of the flow a bout a half submerged ellipsoid. It remains to show how a body of any shape is approximated by an ellipsoid and to differentiate the drag integrals.

Since an ellipsoid has three free parameters ρ,k,h, it is possible to write three independent approximating functions which define ρ ,k,h in function of three characteristics of the approximated body.

The first approximating function states the equality of the minor axis of the ellipsoid with the depth of penetration of the body and is written:

$$b = \sqrt{\rho^2-h^2} \tag{8.41}$$

The second function states the equality of the major axis of the ellipsoid with the length of the submerged body.

$$L = 2\rho \tag{8.42}$$

The third function states the equality of the volume of half the ellipsoid with the submerged volume of the body.

$$V = \frac{2\pi}{3}\rho\sqrt{(\rho^2-k^2)(\rho^2-h^2)} \tag{8.43}$$

These three equations are easily inverted to give ρ, h, k in terms of the body parameters b, L, V. Thus:

$$\rho = \frac{L}{2} \tag{8.44}$$

$$h = \sqrt{\frac{L^2}{4} - b^2} \tag{8.45}$$

$$k = \sqrt{\frac{L^2}{4} - \left(\frac{3V}{\pi L b}\right)^2} \tag{8.46}$$

For most bodies, the parameters V and L cannot be expressed in function of b analytically so that ρ, h, k are determined as functions of b by graphical methods. It is then possible to differentiate equations (8.10) and (8.40) to obtain the formulae for the forces exerted on the submerging body (1.15). Because of the involved form of the equations, they will be given symbolically only:

$$F_y = \frac{\rho V^2}{2} 2\sin^2\alpha \left[\frac{d\rho}{db}\frac{d}{d\rho} + \frac{dk}{db}\frac{d}{dk} + \frac{dh}{db}\frac{d}{dh}\right] \int \varphi_2 \frac{\partial \varphi_2}{\partial n} ds \tag{8.47}$$

$$F_x = \frac{\rho V^2}{2} \sin 2\alpha \left[\frac{d\rho}{db}\frac{d}{d\rho} + \frac{dk}{db}\frac{d}{dk} + \frac{dh}{db}\frac{d}{dh}\right] \int \varphi_1 \frac{\partial \varphi_1}{\partial n} ds \tag{8.48}$$

Formulae (8.47) and (8.48) in conjunction with equation (8.40) complete the investigation of the forces which act on a body approximated by a general ellipsoid. The results of that investigation are somewhat too complicated to be visualized and therefore a numerical example is carried out in detail as an illustration of the analysis. It concerns the water impact at 45° of a sphere of unit radius. The details of the calculation form the subject of the next section of this paper.

IX. NUMERICAL EXAMPLE: WATER IMPACT OF A UNIT SPHERE

The first step in discussing the specific problem of the hydrodynamic impact fordes which act on a unit sphere at an angle of entry of 45° is a discussion of the approximating functions (8.44), (8.45), (8.46). Fortunately, in this case, one is in a position to write analytical forms for them as follows:

$$\rho = \sqrt{x(2-x)} \tag{9.1}$$

$$k = \sqrt{2x(1-x)} \tag{9.2}$$

$$h = \frac{1}{2}\sqrt{\frac{x(1-x)(7-3x)}{(2-x)}} \tag{9.3}$$

$$x = b/R \qquad 0<x<1 \tag{9.4}$$

The variation of the parameters ρ, k, h as functions of x is shown on figure 7.

The calculation of the drag is now carried out in two steps. First, the integrals (8.10) and (8.40) are calculated in terms of ρ, k, h, and plotted on a scale based on x. Because of the complexity of the functions, the differentiation with respect to x is carried out graphically, and an approximate curve results. The two curves mentioned above are reproduced on Figures 8,9. It is noted that curve 8 shows a type of variation already obtained in the previous investigations by Shiffman & Spencer and P.Y. Chou, for the vertical drag component. Figure 9 shows that the horizontal drag component is much smaller than the vertical one; while its variation is of the same type as in the previous case, it is notable that the point of maximum drag is reached at a somewhat greater depth of penetration.

X. DISCUSSION OF RESULTS

The preceding developement appears to indicate that, provided certain approximating assumptions are nearly satisfied, it is possible to predict the first portion of the underwater trajectory of a submerging body. The three principal assumptions are:

(a) That the splash may be neglected; this assumption is made when boundary condition (1.7) is replaced by condition (1.8). This is essentially a linearization of the boundary condition which is necessary to allow the splitting of the velocity potential function into additive vertical and horizontal components. It is most easily justified for a small fast moving body.

(b) That the walls of the body are normal to the free surface; this assumption is made when the actual body is replaced by an approximating body of simple geometrical shape. While it may be possible to study the motion of spherical lenses subtended by any variable angle β , it is feared that the integration of the pressure function will be of extreme difficulty.

(c) That only the portion of the water in front of the principal section of the body is set into motion by the impact. This is admittedly an arbitrary assumption but the exact calculation carried out in connection with the problem of the flat plate shows it to be fairly accurate in that case, and the most acceptable simple assumption which can be made.

In view of the three serious assumptions stated above, it is necessary to justify the results of the calculations shown above by some experimental data. The difficulty of analysing water impact test results is well known; the parameters and physical phenomena which influence the results are very numerous and their individual effects are not easily separated. In a paper presented before the American Physical Society Fluid Mechanics Symposium, Dr. H.L. Wayland of the U.S. Naval Ordnance Test Station Underwater Ordnance Division has listed the more important ones. An examination of that long list is sufficient to convince the analyst that he cannot hope to account rigorously for all the effects which may manifest themselves. The present theory is therefore admittedly approximate. It does however give a good indication of the results to be expected, and some of the results obtained are given here.

An important characteristic of the water entry of projectiles is the "whip" or change in angular velocity during impact. It is well known that any projectile oscillates about its mean path in the air, so that at the instant of water entry, it has pitch and pitch velocity. The forces of water impact cause a rapid change in pitch velocity, which occurs approximately between the time of first contact and the time when the projectile is submerged to a depth of half a diameter. This whip can be measured by photographic methods, as has been done by the Morris Dam Hydrodynamic Station. Since it is due to the moment of the water impact pressures during the time interval considered in this research, it can also be calculated. It should be noted that since the whip is proportional to the integral of the forces between contact and the time when the projectile is submerged to a depth of half a diameter, at the limits of integration, the condition that the body walls are perpendicular to the fluid surface is actually satisfied, so that the assumption made in the analysis introduces no error here. It is found that for the case when the angle of pitch is 0, experimental results and computed results agree within 5%. A change in angle of pitch changes the correlation functions, since the submerged volume at a given depth of penetration depends on the pitch angle. Approximate calculations of whip for various angles of pitch also appear to agree with experimental results of the variation of whip as a function of pitch angle. Since the results obtained by the M.D.H. Group are classified, it is not possible to show the comparison at the present time, but some data are given in a paper by P.Y. Chou (0 pitch) while the calculation of whip as a function of pitch is as yet unpublished. Some experimental data on the drag coefficient of spheres which hit the water vertically are given by P.Y. Chou (Ref. 1.8) and are reproduced in the present paper on Fig. 8. They show fairly good agreement with the calculated results. It is hoped, therefore, that the method outlined in the present paper, although lengthy, gives fair estimates of the order of magnitude of the forces which may be expected to act on the submerged portion of bodies which hit a water surface.

It is noted that the greatest forces are exerted when the depth of penetration is of the order of a tenth to a sixth of the radius, that is a zone in which the present method, and the assumption on a horizontal free surface, are still not far from reality. It is known from experimental and theoretical evidence that the drag then decreases and reaches a value of $C_D \approx 0.33$ for cavity flow. The present type of calculation may therefore be found useful to estimate the critical load which will act on a structure as it passes from air into water. The more complete knowledge of the drag coefficients may also help predict the early phases of underwater trajectories of projectiles. Finally, the drag integrals may be found useful in estimating the power necessary to maneuver ships. While the present paper is most concerned over total forces, the pressure distribution is, to the degree of approximation of this analysis, given by the potential function, and may easily be calculated also.

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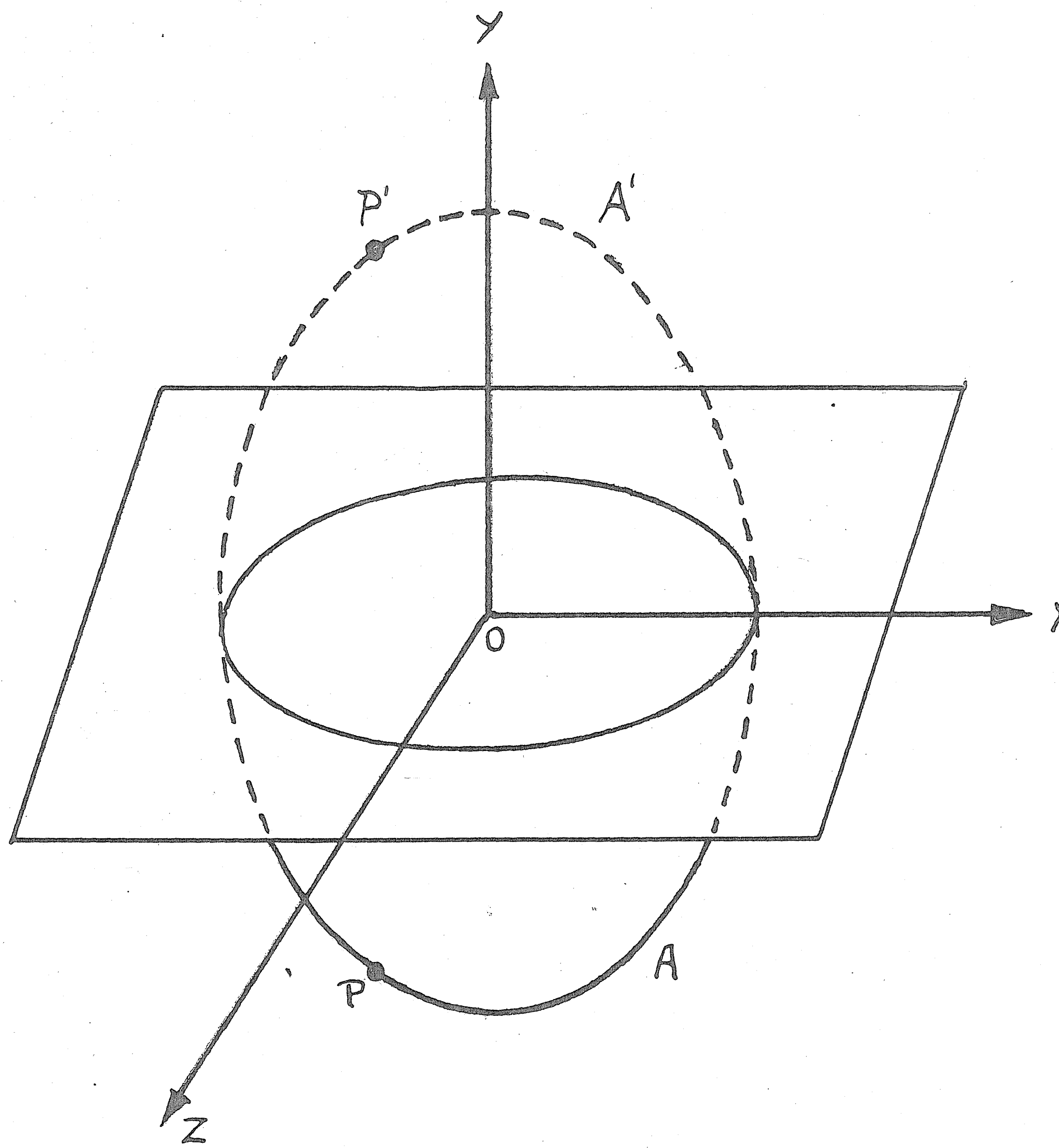
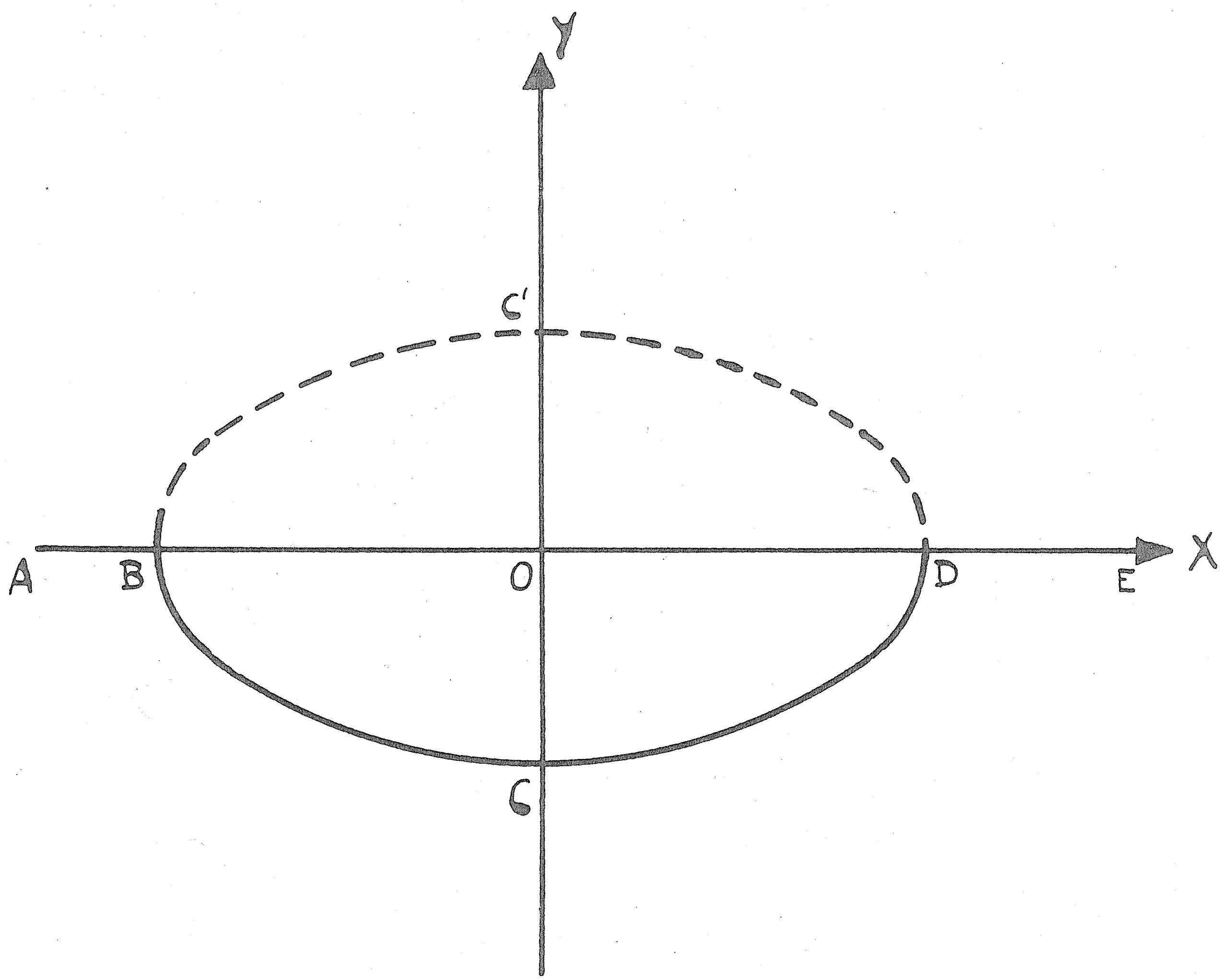
FIGURE 1



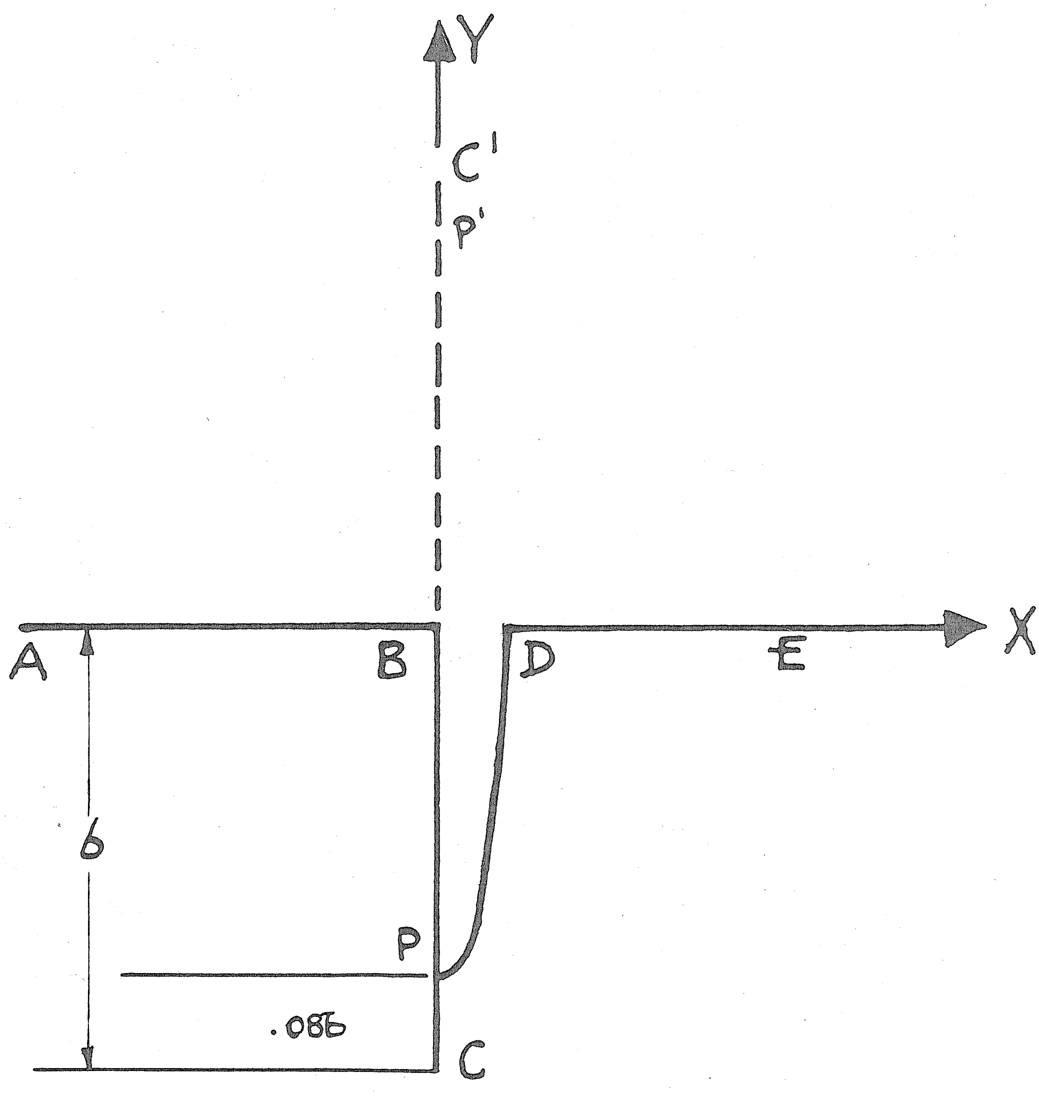
GENERAL BOUNDARY CONDITIONS

FIGURE 2



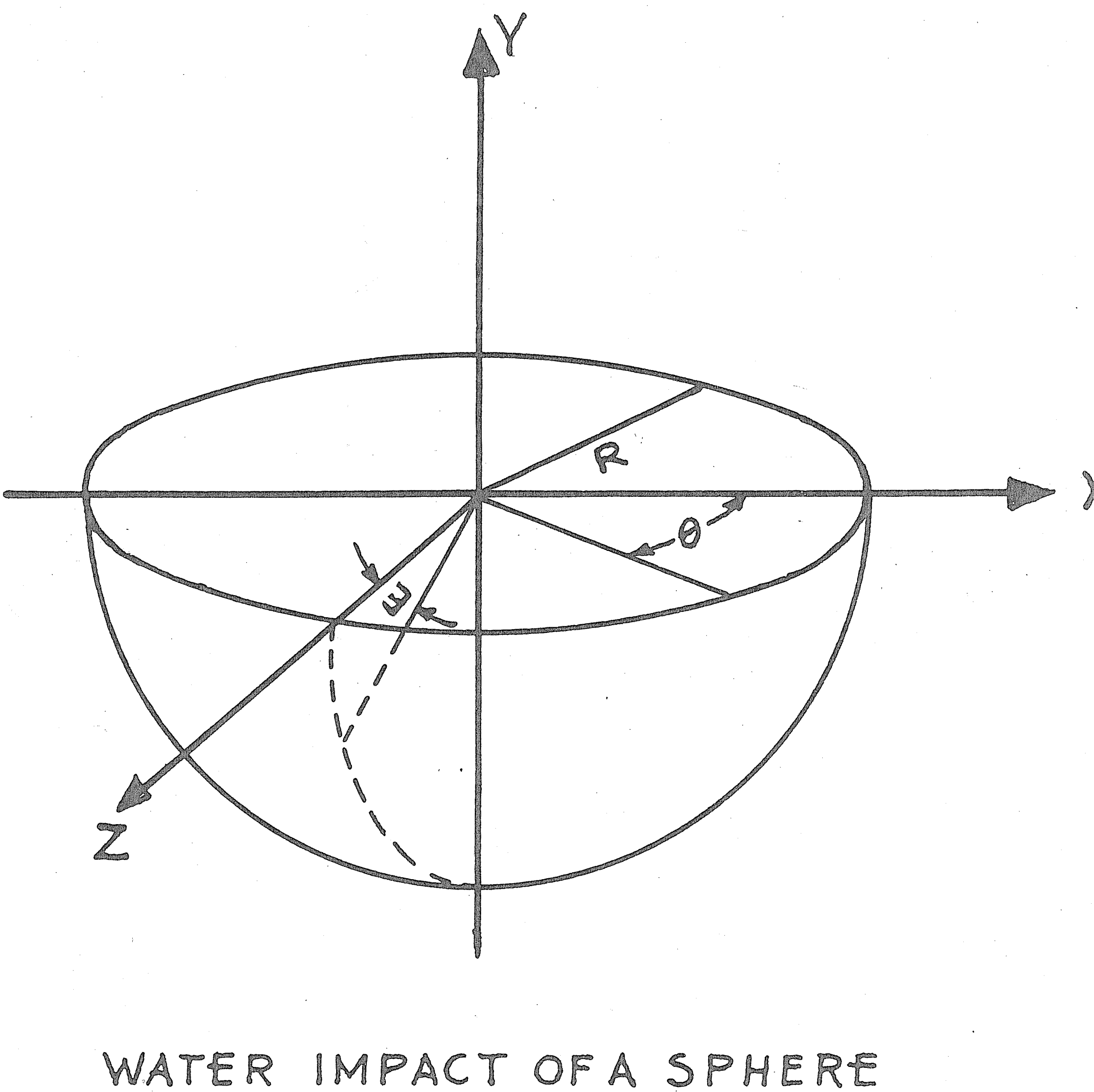
WATER IMPACT OF AN ELLIPTIC CYLINDER

FIGURE 3



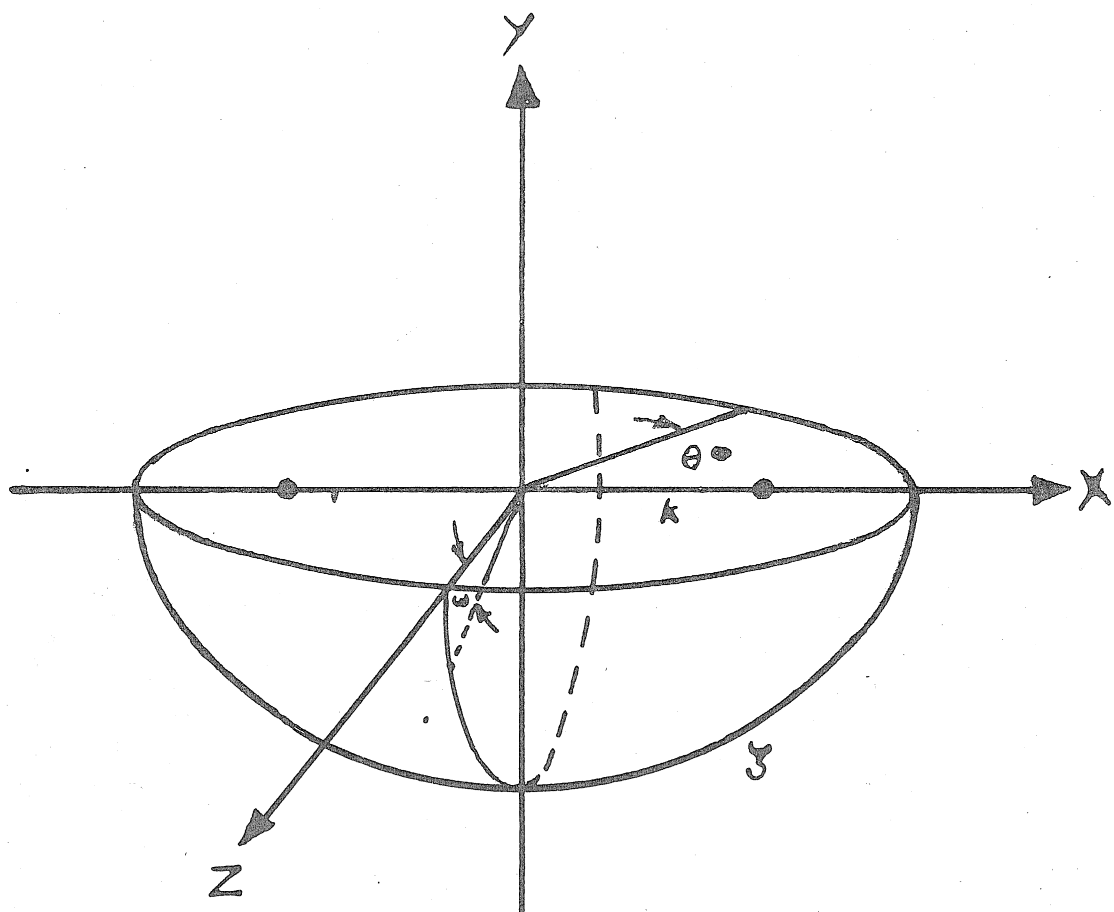
WATER IMPACT OF A FLAT PLATE

FIGURE 4



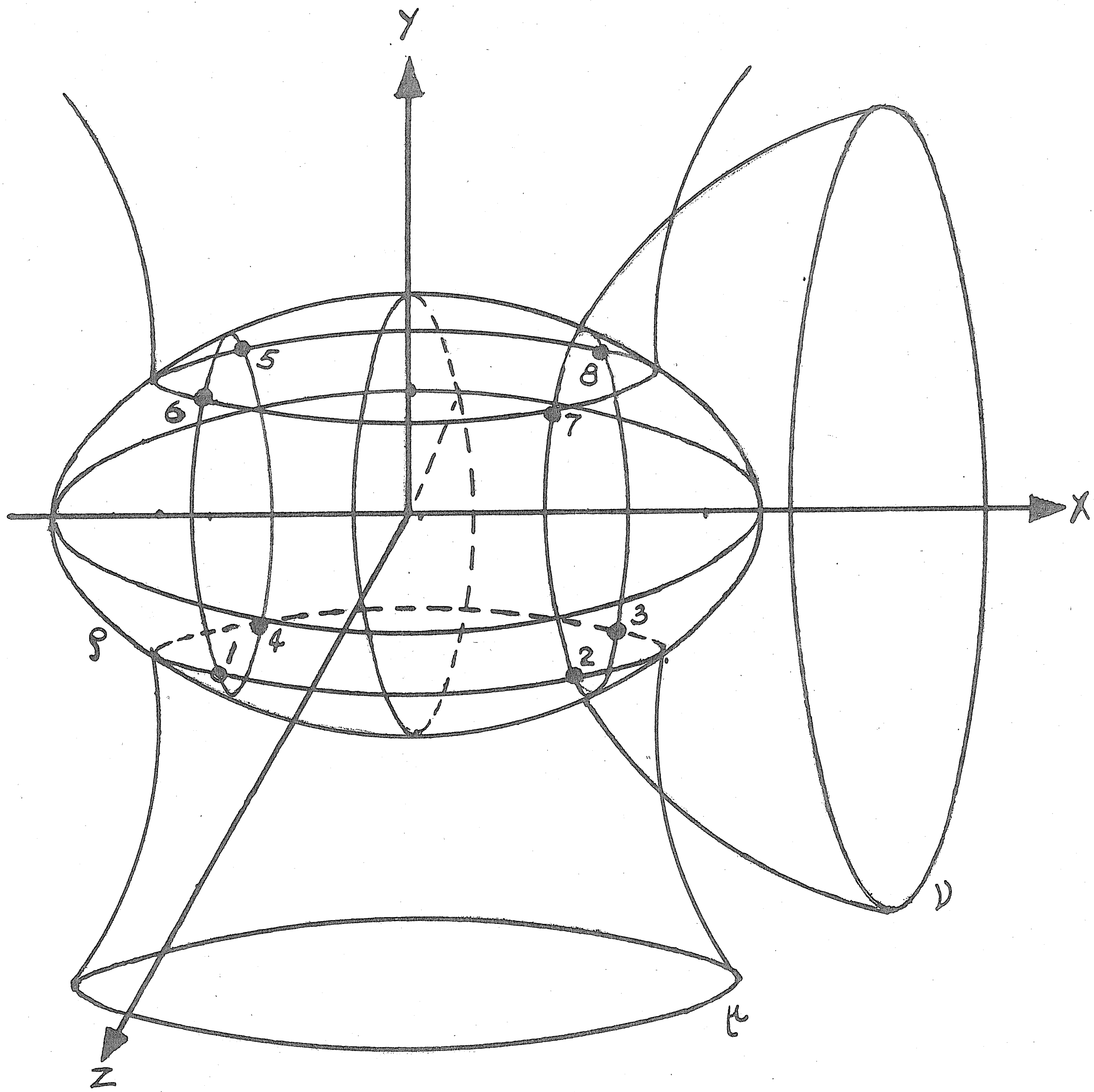
WATER IMPACT OF A SPHERE

FIGURE 5



WATER IMPACT OF AN ELLIPSOID OF REVOLUTION

FIGURE 6



COORDINATE SYSTEM FOR A GENERAL ELLIPSOID

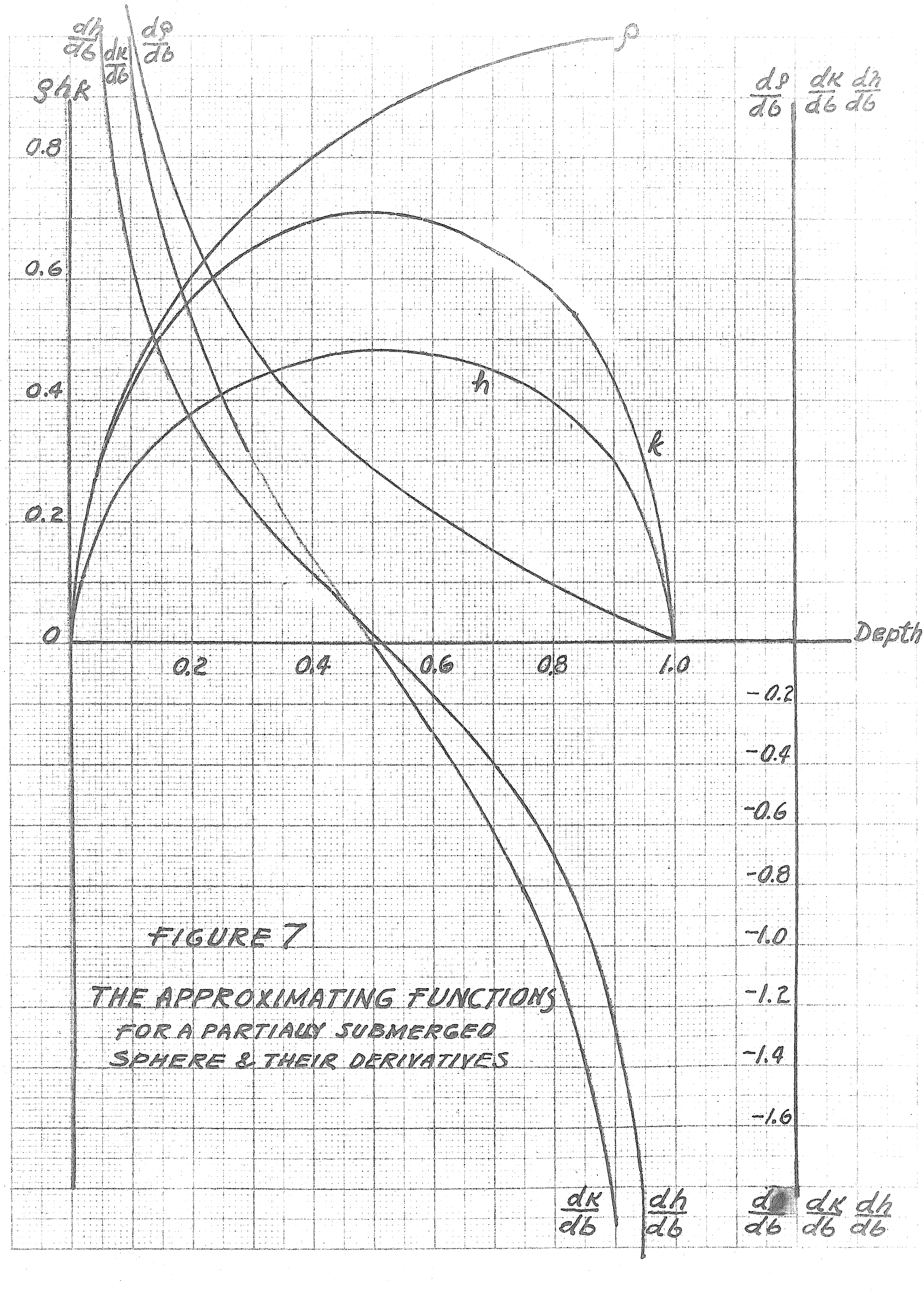


FIGURE 7

THE APPROXIMATING FUNCTIONS FOR A PARTIALLY SUBMERGED SPHERE & THEIR DERIVATIVES

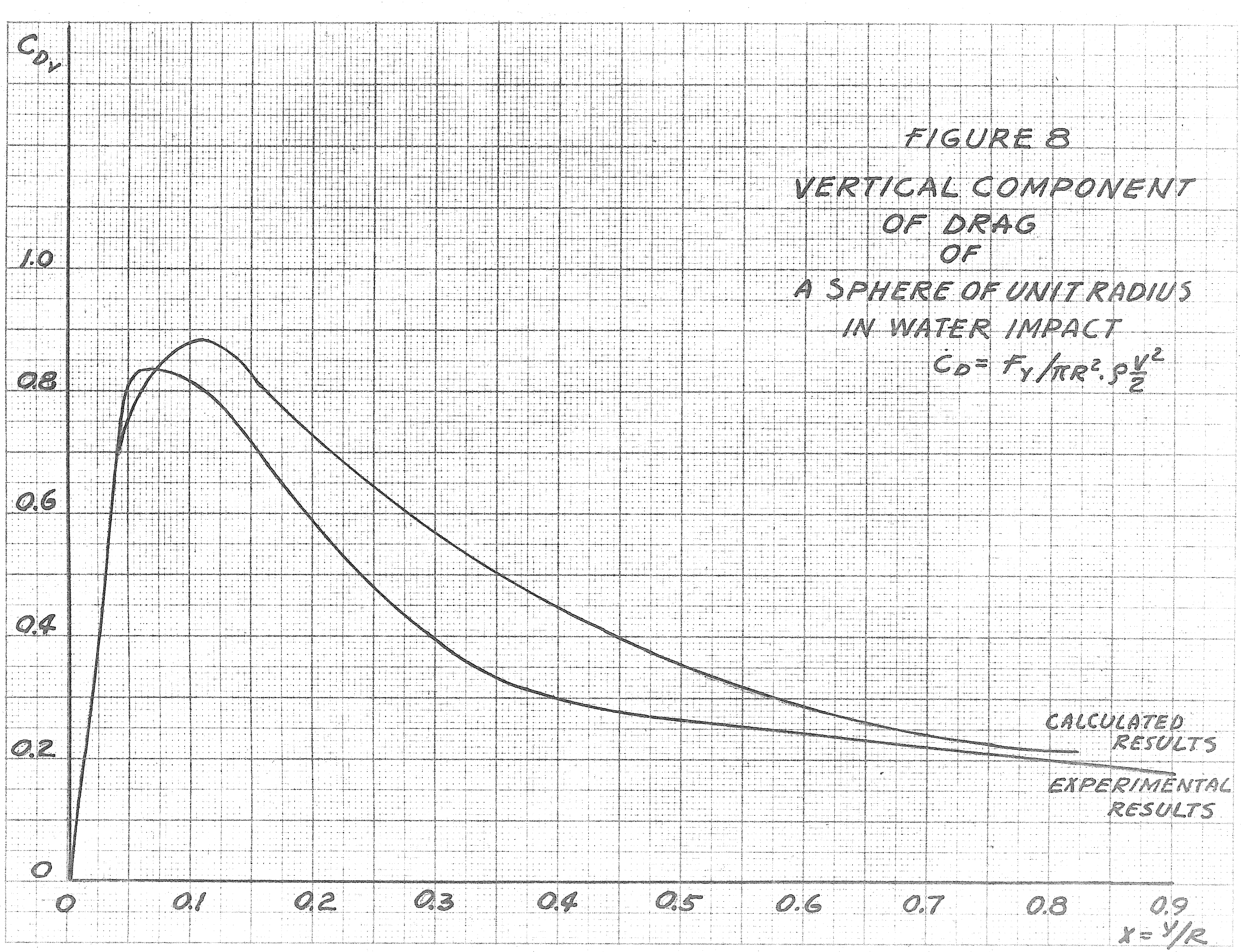


FIGURE 8
VERTICAL COMPONENT
OF DRAG
OF
A SPHERE OF UNIT RADIUS
IN WATER IMPACT
$C_D = F_Y / \pi R^2 \cdot \rho \frac{V^2}{2}$
C_{D_V}
1.0
0.8
0.6
0.4
0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
$x = y/R$
CALCULATED RESULTS
EXPERIMENTAL RESULTS

FIGURE 9

HORIZONTAL COMPONENT OF DRAG
OF
A SPHERE IN WATER IMPACT