SUPERSONIC FLOW THROUGH CASCADES, WITH APPLICATION TO DIFFUSERS

Thesis by

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ABSTRACT

Supersonic flow through a two-dimensional cascade of airfoils is discussed from the point of view of one-dimensional gas dynamics. Two generalizations to compressible flow of the usual velocity vector diagrams are discussed, namely the Mach vector and the reduced velocity vector diagram. A relative mass flow parameter is found which represents the isentropic continuity equation on the two vector diagrams, so that for a set of given cascade inlet conditions all possible outlet velocities and directions appear on the diagrams.

The largest possible tangential velocity component and the largest possible change in tangential momentum in a cascade are obtained as functions of the mass flow parameter. From these the largest obtainable stagnation pressure ratio and power of any arrangement of moving and stationary cascades may easily be obtained.

Equations are given for the ratios of relative stagnation temperatures, mass flows and Mach numbers in two systems of reference moving with respect to each other. Methods for tracing graphically on the vector diagrams the flow through a stage are presented.

The possibility of using cascades instead of shock waves for the transition from supersonic to subsonic flow in diffusers is pointed out. As an example a proposed single cascade supersonic diffuser is briefly analyzed by means of the reduced velocity vector diagram.

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I. INTRODUCTION

1. The Work of Other Authors

The earliest researches dealing with supersonic and high speed subsonic gas flow through cascades are due to the designers of steam turbines, notably Stodola and his associates (Ref. 1). The possibilities of a supersonic axial flow compressor were apparently first studied by Encke and Betz (Ref. 2), and by Weise (Ref. 3) in Germany around 1935.$^{\Delta}$ These studies led to experimental supersonic compressors which were hampered by mechanical difficulties. Weise operated successfully a compressor with subsonic rotor and transonic stator. Another unit designed by Weise with transonic rotor was destroyed by failure of the blades. A third proposed unit with a purely supersonic rotor was never built.

In 1939 Sorg published a thesis on supersonic flow in the rotors of turbo machines (Ref. 4). He studied in detail the conditions for existence of a minimum of channel area in both axial and radial flow rotors and the Mach numbers of the relative and of the absolute velocity occurring at such a "throat". He confined his considerations to stream filaments and used a one-dimensional approach. In considering the absolute velocity in a rotor he reduced the problem to one of steady flow by assuming an infinite number of blades. One of Sorg's results is briefly discussed in Section V (paragraph 2) of this paper.

Strauss (Ref. 5) studied supersonic flow patterns through two-dimensional cascades in detail. He used the Prandtl Busemaun

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$^{\Delta}$At the time of writing the author has not yet obtained the original papers describing these early German experiments.

method of characteristics to construct such flows, with the primary emphasis placed on obtaining cascades without wave drag.

In England supersonic compressors have been studied since 1938 by A. A. Griffiths, Howell and Davidson. In this country Kantrowitz and Donaldson (Ref. 6) studied supersonic diffusers in view of their application to compressors. H. S. Tsien has studied the possibilities of supersonic compressors (see note in Ref. 7). Wattendorf (Ref. 7) has recently published a study on the pressure ratios that appear possible with supersonic axial flow compressors. His calculations are based on the assumption of constant axial velocity component, which would be obtained by proper variation of the cascade area. Alperin (Ref. 8) has studied supersonic flows through two-dimensional cascades on the basis of the linearized (small deflection) theory. He developed a method for calculating the pressure ratios that could be obtained with grids of diamond shaped airfoils at the optimum chord spacing ratios. Lastly, the special solutions of two-dimensional compressible flows by use of the hodograph plane, carried out by Kraft and Dibble (Ref. 9), are of great interest in connection with turbine and compressor cascades.

A great deal of work has been done in recent years on supersonic diffusers. Some of the results obtained have immediate bearing on the efficiencies obtainable in supersonic compressors (by the methods which have so far been considered by other authors). In particular of great importance for the design of a supersonic compressor is the problem of interaction between boundary layer and shock waves in connection with the possible separation of the flow. However a thorough review of the literature on the above problems is outside the scope of this paper.

2. Assumptions, Methods and Limitations of This Paper

In order to learn the potential possibilities and limitations of axial flow machines the writer decided to study first the simplest approximation to the real problem which would still give useful, if only qualitative, information. The most desirable flow patterns in subsonic axial flow machines are usually those which in any cylindrical surface concentric with the axis approximate most closely a two-dimensional irrotational flow. For that reason two-dimensional cascades of airfoils are studied as a preliminary step. The effects of radial pressure and velocity gradients are then taken into account as well as possible. In the hope that also in the supersonic case the three-dimensional effects could be considered as "correction" terms, the writer, as well as other authors, chose two-dimensional cascades as a starting point. It is clear that especially in those cases where waves extend ahead or aft of the cascade the mere curvature of the cascade and casing would make an essential difference.

The next simplifying assumption usually made is that of irrotational flow, whereby the effects of viscosity enter merely as an area correction due to the boundary layer. The assumption of irrotationality implies that no strong, curved (transonic) shocks are assumed to occur inside the cascades.

In machines where several rows of blades are involved it is desirable that the flow leaving each row should be as nearly uniform as possible, otherwise the flow through the next row of blades will be highly nonsteady and very hard to predict. There is, however, in the case of a supersonic cascade, another reason for attempting to get as nearly as possible uniform outflow. This lies in the fact that in the supersonic case a body would have drag even in a non-viscuous fluid, this being called the wave drag. Only in case all the waves are cancelled inside, a "cascade without wave drag" is obtained.

Thus the simplest and most idealized flow (still reminiscent of the real problem to be solved in the end) is that of a two-dimensional cascade with uniform flow entering and leaving and with only isentropic processes inside. Such a case was assumed here. The assumption of uniform inflow and outflow may appear very far fetched at first. However by a suitable combination of Prandtl Meyer flows$^{\Delta}$ and straight channels one can, at least on paper, turn any uniform supersonic flow isentropically into another uniform supersonic flow. While the existence of these solutions may be taken as a justification for the assumption of uniform flow it is likely that other (possibly approximate) solutions can be constructed accomplishing the same result. It is doubtful whether blade shapes giving uniform outflow can be found for the subsonic case. However there the uniformity of the flow is not as important. No attempt is made in this paper to discuss the detailed flow inside a cascade.

$^{\Delta}$Assumed possible in either direction except near Mach number one.

All velocities used in this paper are relative to the particular cascade in which they occur.[Δ] Thus all cascades, whether moving or stationary, are treated alike. The change of state which the fluid undergoes in passing through a cascade is considered like a mathematical transformation. In this paper an isentropic cascade transformation is used (discussed in Section III), but corrections for small entropy changes due to friction and weak oblique shocks can easily be introduced.

At the transition from one cascade to the next, a velocity transformation is used in order to obtain the velocity, Mach number and stagnation temperature of the fluid relative to the new system of reference. This is discussed in Section IV. In the velocity transformation the velocity of a cascade is always taken relative to the one preceding it. In cases where the flow enters directly into a moving cascade the entering flow is referred to a stationary reference system, then the velocity transformation is used.

In this manner hypothetical machines (consisting of two-dimensional cascades) may be made up by a sequence of cascade and velocity transformations. It is clear that the inlet and outlet properties of the fluid, in order to have meaning, must be referred to the same (presumably stationary) system of reference.

While in most of this paper the strictly two-dimensional case of constant cascade area was assumed, the charts presented in

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[Δ]With the exception of a brief discussion of Sorg's results in Section V where the "absolute" velocity is introduced.

Section III can be used for any given variation of cascade area.

All curves and charts have been calculated only for a specific heat ratio γ of 1.40. The basic equations used, and hence all results of this paper, rest on the assumption of a perfect gas.

With the equations and charts given the highest obtainable pressure ratios and power (per unit area) of various "compressor" and "turbine" arrangements can be obtained as simple algebraic expressions or as numerical results from the charts. A number of such ideal performance results have been developed by the writer but these will not be presented here.

As an example a single cascade supersonic diffuser, which appears possible on the basis of this simplified theory, is discussed. As far as the writer was able to determine the possibilities of the use of cascades as supersonic diffusers have not previously been studied. All the existing or proposed supersonic compressors so far were designed with either a transonic stator or a transonic rotor, i.e. using shock diffusion. Some authors have considered only the transonic rotor type as practicable, although Weise has suggested an all supersonic rotor and transonic stator arrangement.

II. NOTATION

1. Symbols, Subscripts and Superscripts

As far as possible the same notation and symbols as in Ref. 10 were used.

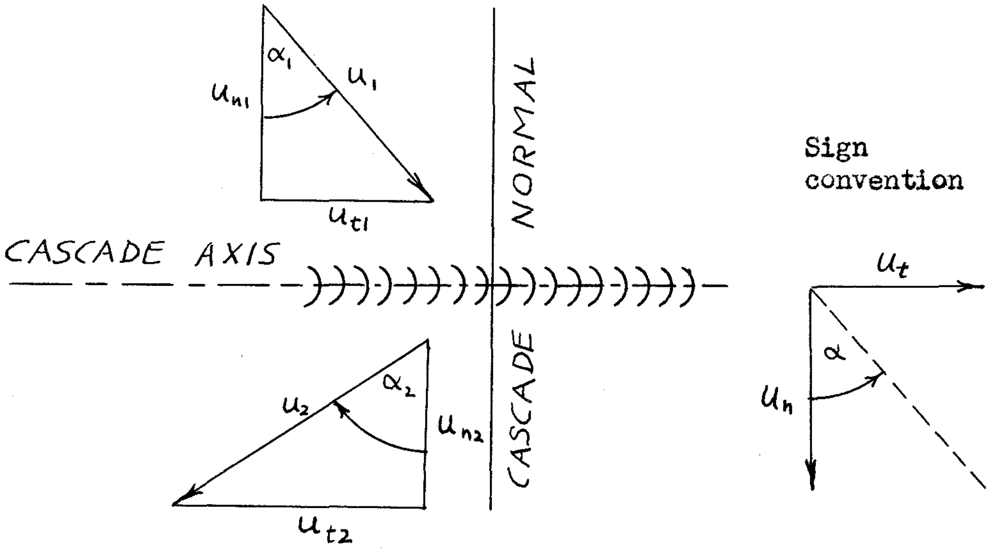
Each cascade is designated by two numbers (e.g. 1-2, 3-4,) representing the two planes which enclose the cascade. The same numbers are used as subscripts. Thus the flow enters cascade 3-4 with velocity u_3 and angle α_3.

Symbols:

A	area of channel perpendicular to flow direction.
A_c	cascade area perpendicular to cascade normal.
a	velocity of sound.
α	angle of flow relative to cascade normal. Taken positive when tangential velocity is positive.
c	absolute velocity (used only in Section V).
c_p	specific heat at constant pressure, in mechanical units.
c_v	specific heat at constant volume, in mechanical units.
f(---)	a function of ----.
G	greatest possible value of the reduced velocity $u/a*$ (corresponding to infinite Mach number. See (2) below).
g	acceleration of gravity.
γ	ratio of specific heats.
h	specific enthalpy of fluid.
J	mechanical equivalent of heat.
λ	relative mass flow ratio (see parameters).

Symbols (continued):

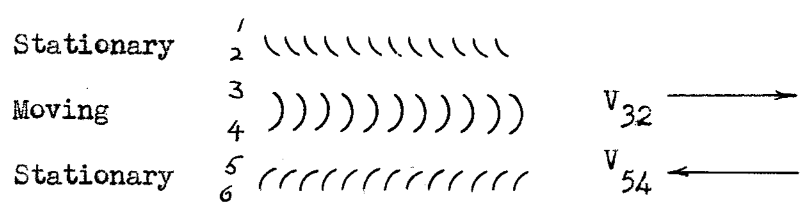
Λ	a function of λ (see parameters).
M	Mach number of the relative velocity, u/a.
$\mathcal{M}$	total mass flow.
m	mass flow per unit cascade area.
μ	momentum of fluid per unit cascade area per second.
p	absolute pressure.
R	gas constant, in mechanical units.
ρ	mass density.
T	absolute temperature of fluid.
τ	maximum value of reduced tangential velocity component, (see parameters).
u	velocity of fluid relative to the nearest cascade.
$\bar{u}$	reduced relative velocity, u/a*
V	cascade velocity relative to preceding cascade, (see following page).
$\bar{V}$	reduced cascade velocity, V/a*



Subscripts and superscripts:

*	....	pertains to a point where $M = 1$.
o	....	" " " stagnation point
D	....	" " " design condition
'	....	" " " deviation from the design condition
1,3,5	....	" " the inlet side of a cascade
2,4,6	....	" " the outlet side of a cascade
i	....	" " the inlet of an arrangement of cascades
f	....	" " the outlet of an arrangement of cascades
n	....	" " the component normal to the cascade axis
t	....	" " the component tangential to the cascade axis

Where several reference systems moving with respect to each other are involved, the velocity of each cascade or reference system is taken relative to the one preceding it, so that for each transition from one reference system to another the same velocity transformation can be applied. Thus the velocity of plane (3) relative to plane (2) is denoted by V_{32}. The cascade velocity is made dimensionless by referring it to the reference system from which it is observed. Thus V_{32}/a_2^* is written $\overline{V}_{32}$.



$$V_{54} = - V_{32}$$

2. Standard Equations

Certain well-known equations of one-dimensional gas dynamics have been taken directly from Ref. 10. They are repeated here for convenience.

The energy equation for the case of no heat transfer to a streamtube, valid with or without internal friction and shock waves, is used in the form

$$\frac{u^2}{2} + c_p T = c_p T_o = h_o \qquad \text{II-1}$$

where c_p and h are in mechanical units per unit mass. An immediate consequence of the energy equation is the relation

$$\bar{u}^2 = \left(\frac{u}{a^*}\right)^2 = \frac{(\gamma+1)M^2}{2+(\gamma-1)M^2} \qquad \text{II-2}$$

where the speed of sound at unit Mach number, a*, is given by

$$a^{*2} = \frac{2(\gamma-1)}{\gamma+1} c_p T_o \qquad \text{II-3}$$

For the case of expansion into a vacuum the reduced velocity $\bar{u}$ approaches the limiting value $\sqrt{(\gamma+1)/(\gamma-1)}$ which will be denoted by G.

The following other forms of the energy equation are also used:

$$\frac{T_o}{T} = 1 + \frac{\gamma-1}{2} M^2 = \frac{\gamma+1}{\gamma+1-(\gamma-1)\bar{u}^2} \qquad \text{II-4a}$$

$$\frac{T}{T^*} = \frac{\gamma+1}{2+(\gamma-1)M^2} = \frac{\gamma+1-(\gamma-1)M^2}{2} \qquad \text{II-4b}$$

The area-Mach number relation for <u>isentropic</u> channel flow

$$\frac{A}{A^*} = \frac{1}{M}\left(\frac{2+(\gamma-1)M^2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} = \frac{1}{M}\left(\frac{T^*}{T}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \qquad \text{II-5}$$

is of particular importance. This can also be written as a function of $\bar{u}$, by II-4b above. The density in isentropic channel flow may be obtained from

$$\frac{\rho}{\rho^*} = \left(\frac{\gamma+1}{2+(\gamma-1)M^2}\right)^{\frac{1}{\gamma-1}} \qquad \text{II-6}$$

The quantities p^*, ρ^*, T^* are related to the corresponding stagnation properties by

$$\frac{T^*}{T_0} = \left(\frac{2}{\gamma+1}\right), \qquad \frac{\rho^*}{\rho_0} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}$$

$$\frac{p^*}{p_0} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} \qquad \text{II-7}$$

For derivations and discussion of these equations the reader is referred to Ref. 10.

3. Important Parameters

Very useful in connection with cascades is the relative mass flow parameter λ discussed in section III. It is given by

$$\lambda = M \cos\alpha \left(\frac{T}{T^*} \right)^{\frac{G^2}{2}}$$

and is proportional to the mass flow ratio used by Kantrowitz and Donaldson in connection with diffusers.

The maximum value of the reduced tangential velocity $\bar{u}_t$ is of special importance. It is a function of λ and is designated by $\tau(\lambda)$:

$$\tau = G\sqrt{1 - \Lambda}$$

where

$$\Lambda = (\lambda)^{2/G^2}$$

This is discussed in section III also.

The third important physical quantity encountered in the study of cascade flows is the normal Mach number component M_n.

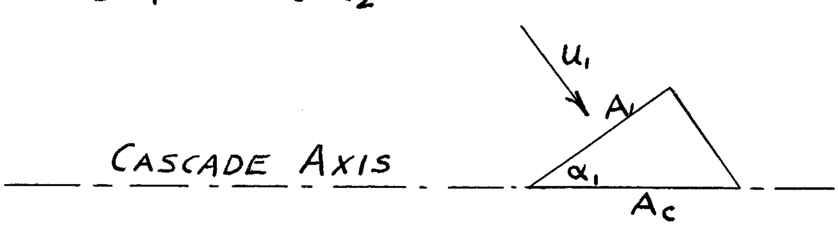
III. ISENTROPIC FLOW THROUGH ONE CASCADE

1. Generalizations of the Velocity Vector Diagram

In the following we shall not be concerned with the detailed flow pattern inside a cascade, but rather with the simple, general relations which describe the conditions of a perfect, ideal gas entering and leaving a two-dimensional cascade. At first relations are worked out for the strictly two-dimensional case, meaning that the inlet and outlet areas of the cascade (perpendicular to the cascade normal) are equal. This two-dimensional case brings out qualitatively all the properties of the more general case in which the cascade area gradually increases or decreases from one side to the other. The diagrams which are drawn for the two-dimensional case can also be used for the case of changing cascade area.

For reasons already discussed in the Introduction the cascade may be considered here as a transformation which turns a uniform flow isentropically into another uniform flow. The inflow and the outflow of the cascade are then related by the equations of one-dimensional channel flow, in particular the area-Mach number relation (equation II-5). For the strictly two-dimensional case the crossectional area A available to the fluid depends on α, the angle between the flow direction and the cascade normal, by:

$$\frac{A_1}{\cos\alpha_1} = \frac{A_2}{\cos\alpha_2} = A_c \quad (a\ constant) \qquad \text{III-1}$$



We now define A^* as that crossectional area at which the given mass flow would have to assume unit Mach number. For isentropic channel flow A^* is a constant, as a consequence of conservation of mass and energy. It then follows from III-1 that the quantity $A_i/A^*\cos\alpha_i = A_c/A^*$ is also a constant across the cascade. The inverse of this will be used throughout this paper as a parameter, λ. The isentropic cascade "transformation" for the two-dimensional case is thus distinguished by the property

$$\lambda = M\left(\frac{\gamma+1}{2+(\gamma-1)M^2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\cos\alpha = \frac{A^*}{A_c} = const. \qquad \text{III-2}$$

For the case of varying cascade area we would have instead of III-2 the relation[Δ]

$$\lambda \cdot A_c = constant \qquad \text{III-2a}$$

In terms of the dimensionless ("reduced") velocity $\bar{u} = u/a^*$ the parameter λ becomes

$$\lambda = \bar{u}\cos\alpha\left(\frac{\gamma+1-(\gamma-1)\bar{u}^2}{2}\right)^{\frac{1}{\gamma-1}} \qquad \text{III-3}$$

The parameter λ can be thought of as the percentage of the maximum possible mass flow through the area A_c (for the given

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[Δ]Throughout the remainder of this paper the cascade area A_c is considered a constant, except where the contrary is specifically stated.

inlet conditions) which is actually being used; for if the actual mass flow $\mathcal{M} = \rho u A$ and the maximum possible mass flow (without any blades) $\mathcal{M}^* = \rho^* a^* A_c$, then

$$\frac{\mathcal{M}}{\mathcal{M}^*} = \frac{\rho u A}{\rho^* a^* A_c} = \frac{A^*}{A}\frac{A}{A_c}$$

$$= \frac{A^*}{A_c} = \lambda \qquad \text{by continuity equation}$$

Thus $\mathcal{M} = \lambda \mathcal{M}^* = \lambda \rho^* a^* A_c$ and the mass flow per unit cascade area

$$m = \frac{\mathcal{M}}{A_c} = \lambda \rho^* a^* \qquad \ldots . \quad \text{III-4}$$

If the inlet conditions M_1 and α_1 are given, then λ is also given and a fixed relation exists between M_2 and α_2, the outlet Mach number and angle. Moreover for a given λ each of the quantities M_2 and α_2 is individually limited as will be evident from the following figures. In figure III-1 λ is plotted against M for various values of α. A set of inlet conditions M_1 and α_1 defines a point on the chart. If the line λ = constant is drawn through this point, the points corresponding to all possible outlet conditions lie on this line. In particular the largest and smallest possible Mach numbers M_2 are found on the line $\alpha = 0$ and the largest possible (positive or negative) value of α_2 is found where $M = 1$.

In axial flow compressor and turbine theory it is usual to refer to velocity vector diagrams. For the case of incompressible flow and constant cascade area the continuity relation is easily incorporated into the vector diagram by the relation u_n =

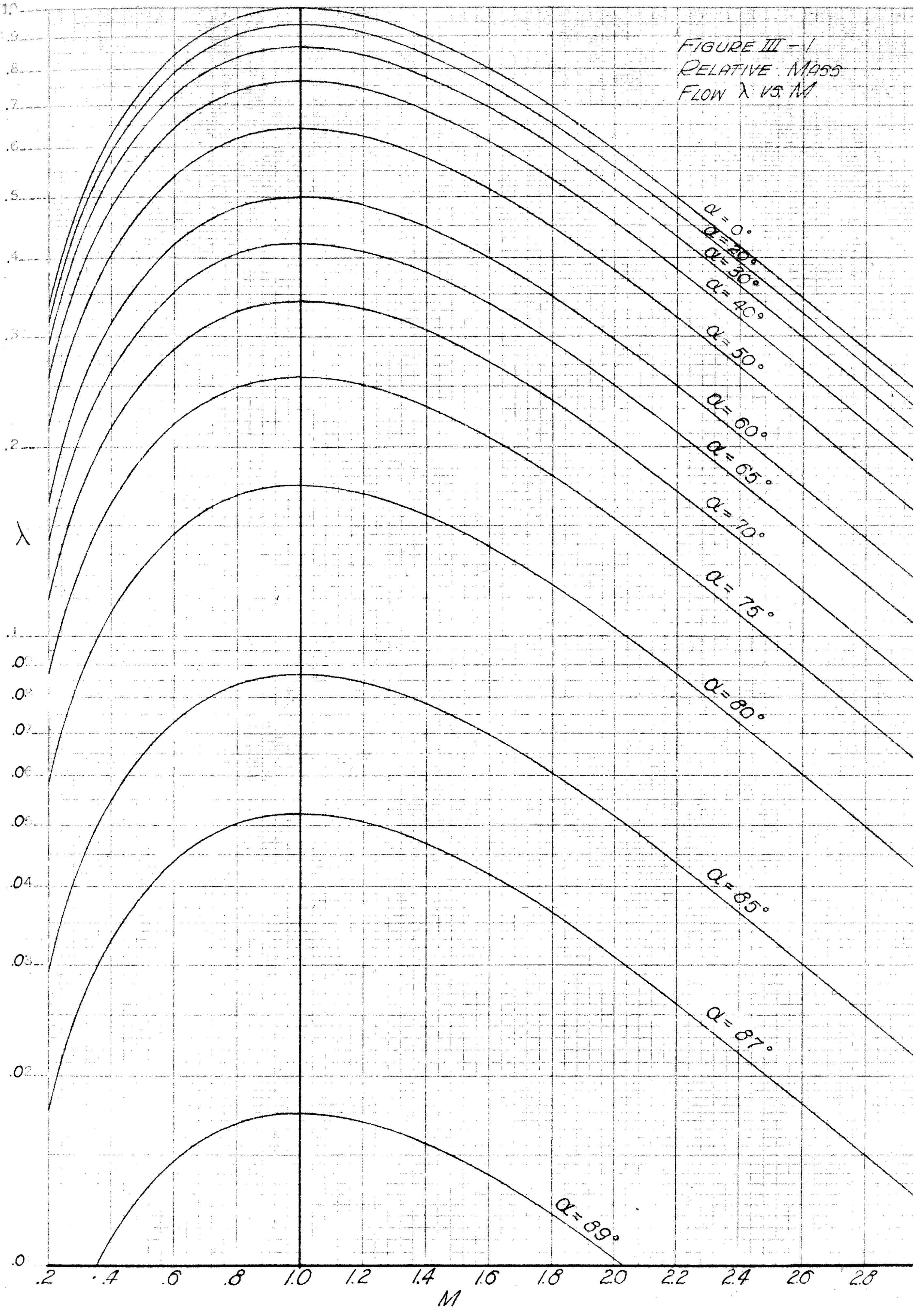


FIGURE III - 1
RELATIVE MASS
FLOW λ VS. M
λ
M
α = 0°
α = 20°
α = 30°
α = 40°
α = 50°
α = 60°
α = 65°
α = 70°
α = 75°
α = 80°
α = 85°
α = 87°
α = 89°
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
.09
.08
.07
.06
.05
.04
.03
.02
.01
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8

constant. (u_n is often called the axial velocity). Where compressibility has to be taken into account in subsonic compressor design the condition u_n = constant is often maintained and the cascade area is assumed to be adjusted accordingly. This affords appreciable simplification of the equations to be used.

For the purpose of this paper it is however advantageous to discuss the case of constant cascade area. For compressible flow two generalizations of the usual velocity vector diagram present themselves. One may be called a Mach-vector diagram (following other authors). This is shown in Figure III-2. It will be seen that λ = constant represents a family of closed curves on this diagram. A uniform flow entering a cascade at Mach number M_1 and angle α_1 is represented by a point on this diagram. According to the assumptions stated at the beginning of this chapter the flow leaving the same cascade must then be represented by a point on the same λ-curve. An increase in entropy due to friction or shock waves would force the state of the outflowing gas to a slightly higher value of λ. (This is discussed briefly in the last chapter). An increase in cascade area above that of the inlet side would of course lower the λ of the flow leaving the cascade, by equation III-2a. The direction and Mach number of the flow inside the cascade cannot be represented on this diagram since the flow is not uniform there. The mean velocity and direction in any plane parallel to the cascade axis might be entered in the diagram. Such a point, representing roughly the flow inside the cascade, would have to lie on the inside of the curve $\lambda = \lambda_1 = \lambda_2$ since the (presumably finite) thickness of the blades takes up some of the cascade area.

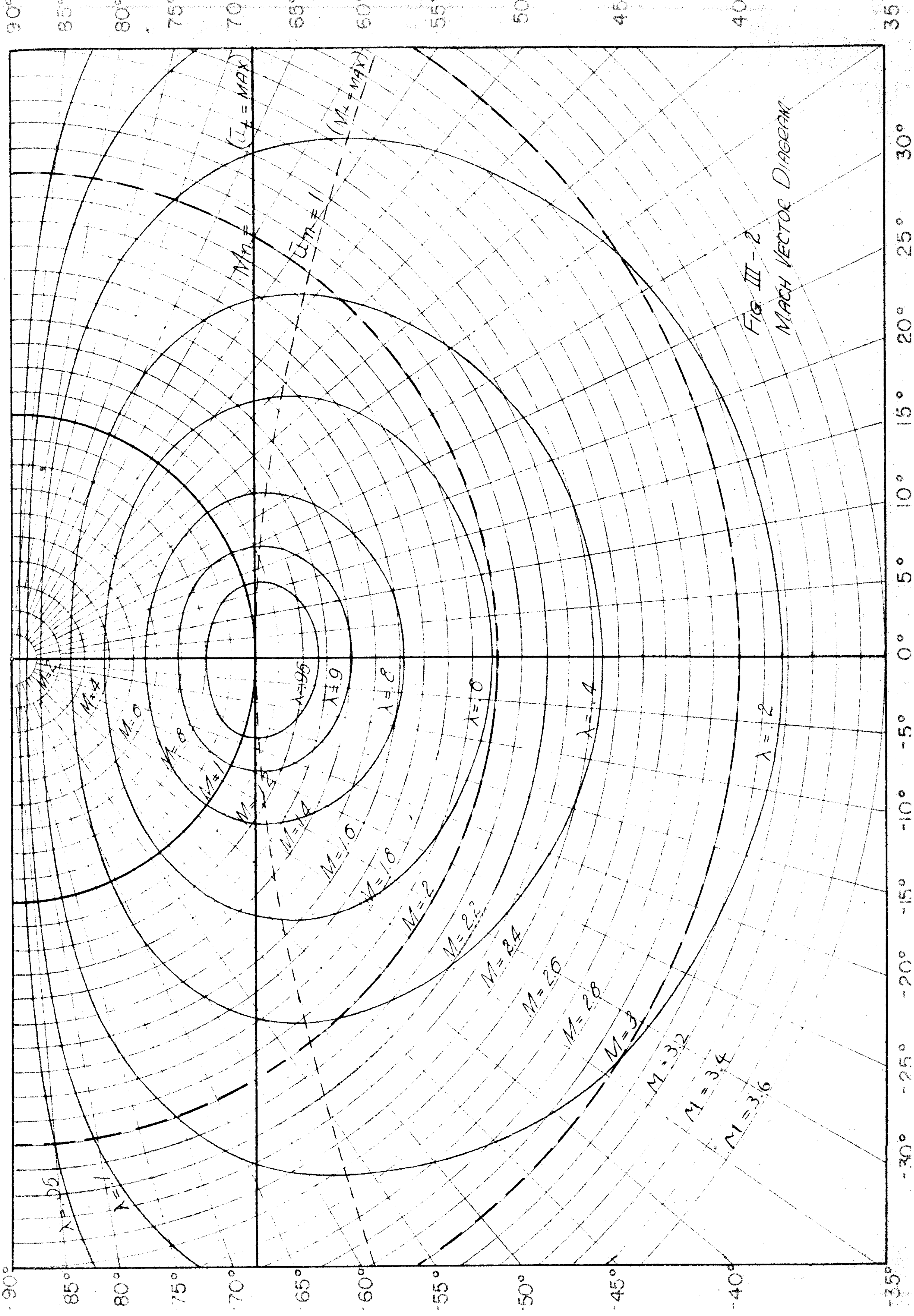


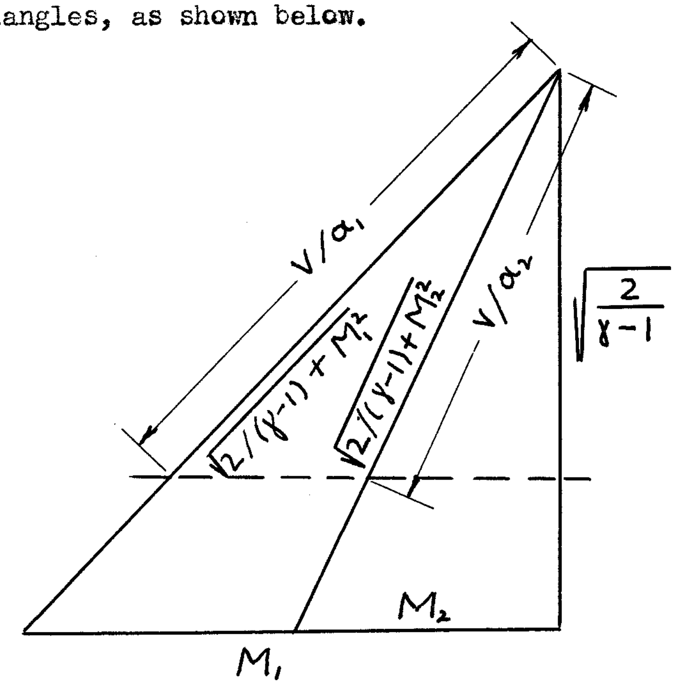
FIG. III-2
MACH VECTOR DIAGRAM

The Mach vector diagram is useful especially in illustrating the transition from one cascade to another (see Chapter IV). But since the local velocity of sound will in general be different in the inlet and outlet planes of a cascade, the Mach number of a moving cascade (to be used on the Mach vector diagram) also changes from one side to the other. If the cascade Mach number V/a_1 on the inlet side of a moving cascade 1-2 is given then a simple geometrical construction will give the corresponding Mach number V/a_2 on the outlet side:

Since $T_0/T = 1 + (\gamma - 1) M^2/2$,

$$\frac{V/a_2}{V/a_1} = \frac{a_1}{a_2} = \sqrt{\frac{2 + (\gamma-1) M_2^2}{2 + (\gamma-1) M_1^2}} \qquad \text{III-5}$$

By drawing right triangles with sides $\sqrt{2/(\gamma-1)}$ and M_1 and M_2 respectively the ratio of III-5 is easily obtained on the Mach vector diagram by similar triangles, as shown below.



The other generalization of the incompressible velocity vector triangles will be called here the "reduced velocity" diagram, figure III-3. It appears to be more instructive than the Mach vector diagram, but has, to the author's knowledge not previously been used in connection with cascades. The diagram consists of a plot of $\bar{u}$ vs. the angle of the flow α, for constant values of λ (equation III-3). Again the inlet and outlet conditions of any one (two-dimensional) cascade must be represented by points on the same $\lambda =$ constant-curve. The reduced velocity diagram has several advantages over the Mach vector diagram:

i. Since $\bar{u}$ tends toward a limit G when M tends to infinity it is possible to include all Mach numbers in the diagram.

ii. Since it is assumed that no heat transfer takes place, the stagnation temperature, and hence a*, is constant in any one cascade. Thus the reduced cascade velocity V/a* has the same magnitude throughout any one cascade.

The reduced velocity diagram is very similar to the Mach vector diagram in the subsonic regime, is identical with it along the circle $\bar{u} = M = 1$. Perhaps this similarity in the subsonic and sonic regime gives u/a* as reduced velocity a slight preference over the quantity $u/u_{max} = \bar{u}/G$. In the supersonic regime the two diagrams differ more appreciably. It will be seen that the $\bar{u}$ diagram is bordered by the semi-circle $\bar{u} = G$ and its diameter $\bar{u}_n = 0$. The semi-circle and its base together form the curve $\lambda = 0$.

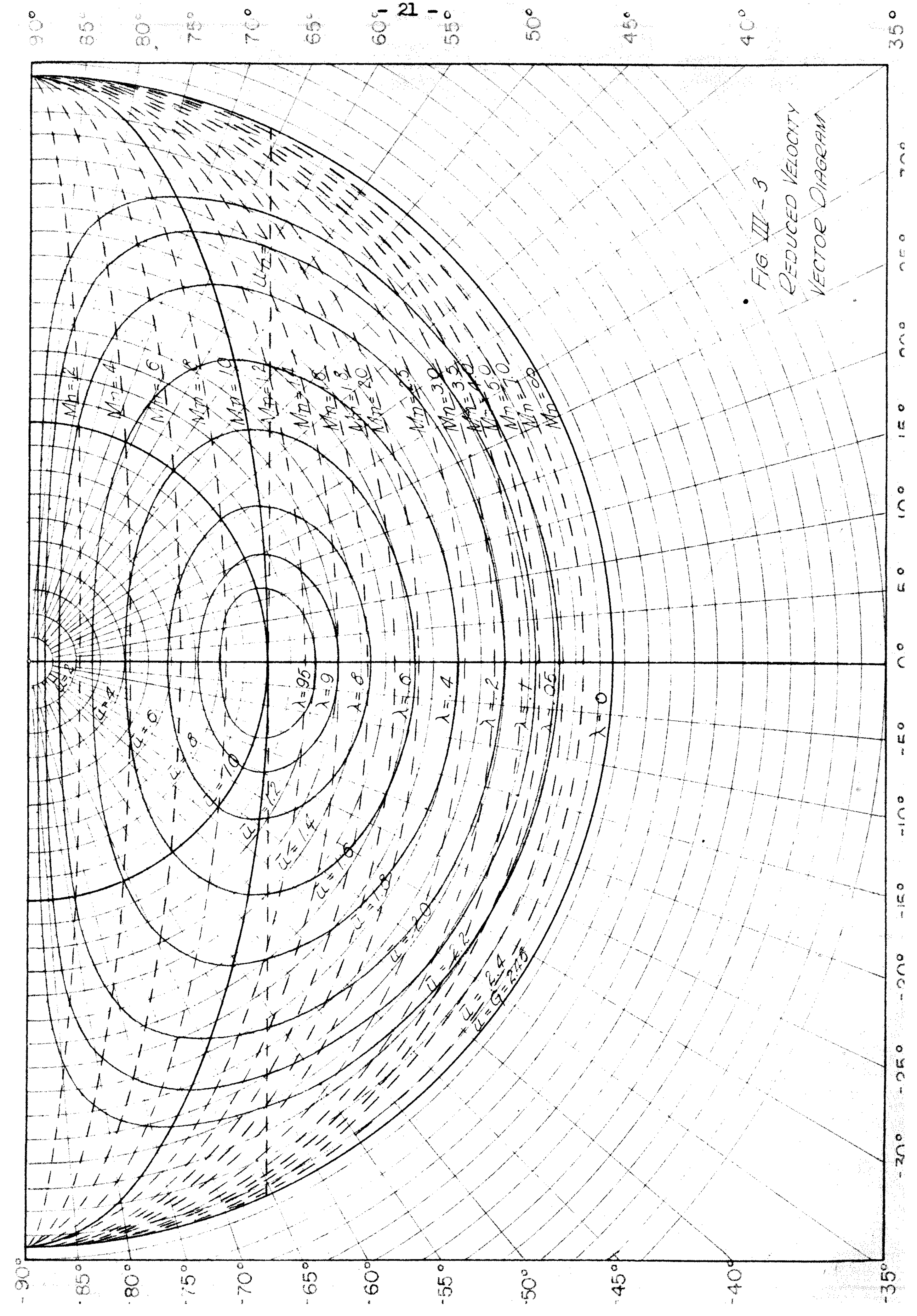


FIG. III - 3
REDUCED VELOCITY
VECTOR DIAGRAM

The continuity equation for incompressible flow through a two-dimensional cascade is u_n = constant. The $\bar{u}$ diagram shows that this is approximately satisfied by subsonic compressible flows if either the mass flow λ is very small or (for larger values of λ) if the angle α is sufficiently small. It will also be seen that for the incompressible case (small mass flow and small Mach number) a very wide range of angles, nearly $+90^\circ$ to -90°, is possible. As the relative mass flow is increased toward its maximum possible value the possible flow angles become more limited.

It may be of interest to note that the λ = constant-curves approach ellipses as λ approaches unity both in the Mach vector and in the velocity vector diagram.

2. Extrema of Various Important Quantities

i. For the following work it is important to find the maximum value of the (dimensionless) tangential velocity $\bar{u}_t = (u/a^*)\sin\alpha$, for a given value of λ . From equation III-2 we have

$$\cos^2\alpha = \lambda^2\left(\frac{A}{A^*}\right)^2 = \frac{\lambda^2}{M^2}\left(\frac{2+(\gamma-1)M^2}{\gamma+1}\right)^{G^2} \qquad \text{III-2b}$$

Using equation II-2 for $\bar{u}$ and III-2b we can express the tangential velocity component $\bar{u}_t$ as a function of M and λ :

$$\bar{u}_t^2 = (u/a^*)^2\sin^2\alpha$$

$$= \frac{(\gamma+1)M^2}{2+(\gamma-1)M^2} - \lambda^2\left(\frac{\gamma+1}{2+(\gamma-1)M^2}\right)^{-\frac{2}{\gamma-1}} \qquad \text{III-6}$$

To find the extrema of this quantity with respect to M, considering λ fixed, we set $\frac{\partial \bar{u}_t^2}{\partial M} = 0$ and get after some simplification

$$\lambda^2 = \left[\frac{\gamma+1}{2+(\gamma-1)M^2}\right]^{G^2} = \left[\frac{T}{T^*}\right]^{G^2} \qquad \text{III-7}$$

This necessary condition for an extremum may be interpreted most conveniently by comparing it with the definition of λ, equation III-2 written in the form

$$\lambda^2 = M^2 \cos^2\alpha \left[\frac{T}{T^*}\right]^{G^2}$$

This yields immediately

$$M^2 \cos^2\alpha = 1 \qquad \text{III-8a}$$

We can also solve III-7 for M^2, giving

$$M^2 = \frac{1}{\cos^2\alpha} = G^2(\lambda)^{-2/G^2} - \frac{2}{\gamma-1} \qquad \text{III-8b}$$

Equations III-8a and b are necessary conditions for an extremum of $\bar{u}_t^2$ and a little further algebra will convince us that they furnish a maximum of this quantity, and hence both maximum and minimum of $\bar{u}_t$. It is interesting to note that the reduced tangential velocity becomes a maximum when the normal Mach number is unity. Thus the conditions for maximum $\bar{u}_t$ are found easily from figure III-2 by drawing the line $M_n = 1$. On the $\bar{u}$ vs. α chart, Fig. III-3, the locus of points of maximum $\bar{u}_t$ is an ellipse with half axes G and 1 respectively (see equation IV-12).

The actual maximum value of the reduced tangential velocity is found by substituting III-8b into III-6. This yields

$$(\bar{u}_t)_{\substack{max \\ min}} = \pm G \sqrt{1 - \lambda^{2/G^2}} \qquad \text{III-9}$$

The positive value of this square root will henceforth be denoted by τ, and λ^{2/G^2} by Λ. Thus

$$\tau(\lambda) = \bar{u}_{t\,max} = G\sqrt{1-\Lambda} \qquad \text{III-9a}$$

When $\lambda = 0$, τ becomes G as would be expected. The corresponding Mach number is ∞. In figure III-6 τ has been plotted against λ for $\gamma = 1.4$. While τ varies between 0 and G the total Mach number for maximum tangential velocity is always supersonic.

ii. The tangential component of the Mach number, M_t, does not appear to have much importance for the remainder of this investigation. However it may be of interest to note briefly that M_t has its maximum when

$$(\bar{u}\cos\alpha)^2 = \bar{u}_n^2 = 1$$

The corresponding locus on the Mach vector chart is the hyperbola

$$M_n^2 - \frac{\gamma-1}{2} M_t^2 = 1$$

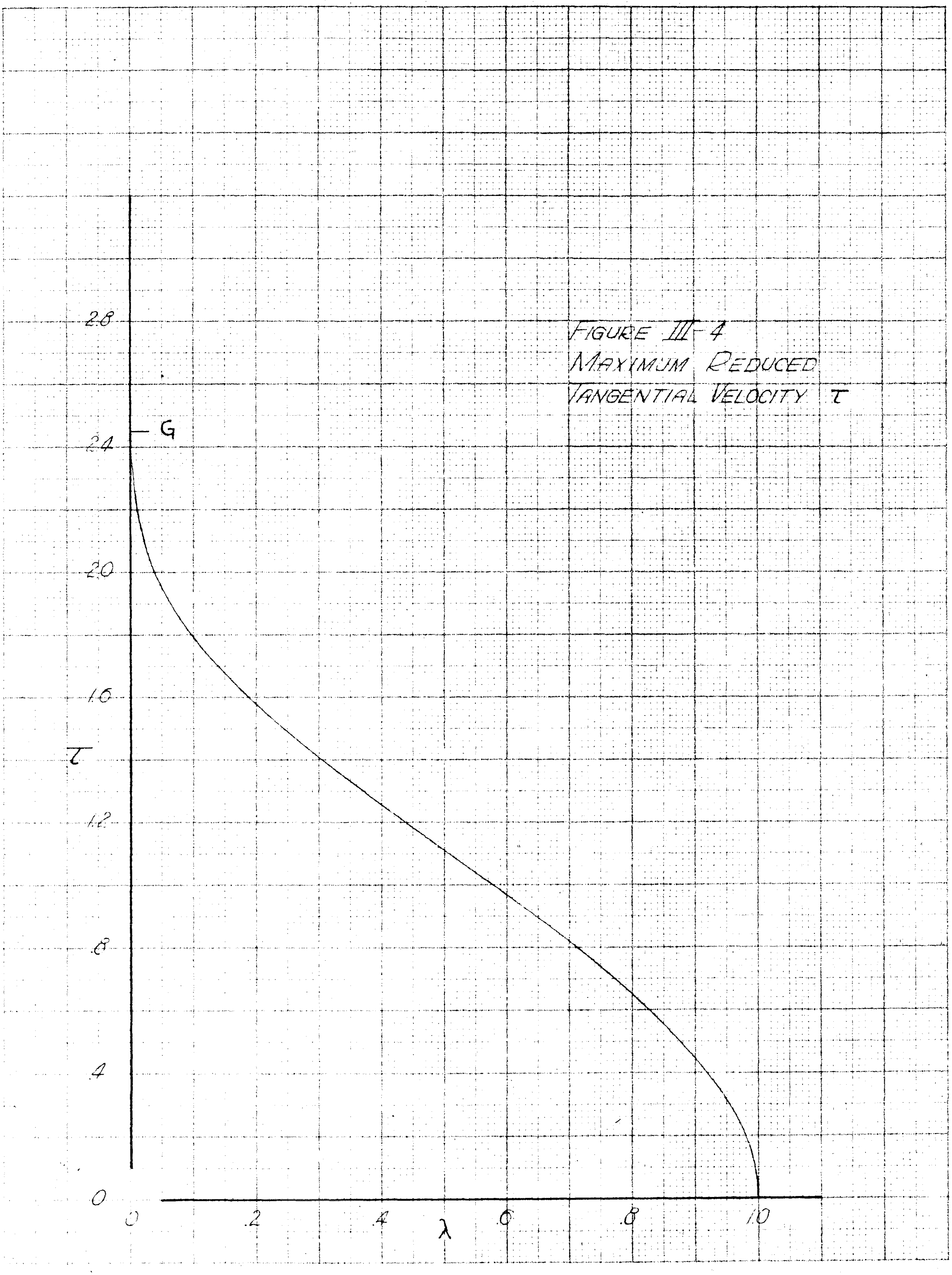


FIGURE III-4
MAXIMUM REDUCED
TANGENTIAL VELOCITY τ
2.8
G
2.4
2.0
1.6
τ
1.2
.8
.4
0
0
.2
.4
λ
.6
.8
1.0

The value of the maximum tangential Mach number is given by

$$(M_t^2)_{max} = \frac{2}{\gamma+1}(M^2-1) \qquad \text{III-10a}$$

$$= \frac{2}{\gamma-1}(\lambda^{1-\gamma}-1) \qquad \text{III-10b}$$

iii. The largest change of tangential momentum possible in a two-dimensional cascade is proportional to the quantity $\tau\cdot\lambda$. This quantity also appears when we try to find the design condition for maximum power (per unit cascade area) of a particular turbine stage.

It is fairly obvious (and can easily be proved) that the maximum tangential momentum is obtained by finding the maximum tangential velocity for a given mass flow, $\tau(\lambda)$, and then maximizing the product $\lambda\cdot\tau(\lambda)$. Differentiating with respect to λ we find as necessary and sufficient condition for a maximum of $\lambda\tau$ that

$$\Lambda = \tau^2 = \frac{\gamma+1}{2\gamma} \qquad \text{III-11}$$

and hence

$$(\lambda\tau)_{max} = \left(\frac{\gamma+1}{2\gamma}\right)^{\frac{\gamma}{\gamma-1}} \qquad \text{III-12}$$

The normal Mach number is of course again equal to unity since that was necessary for $\bar{u}_t = \tau$. By substituting III-11 into III-8b it is found that the total Mach number for maximum tangential momentum

$$M = \sqrt{2}$$

regardless of the value of γ. This result is rather surprising.

By III-8a the corresponding angle $\alpha = \pm 45^{\circ}$. This angle gives maximum tangential momentum also with an incompressible fluid.

If an incompressible fluid enters a two-dimensional cascade parallel to the cascade normal, a tangential momentum per unit cascade area per second $\mu_t = p_o \sin 2\alpha$ may be imparted to it by the cascade, where p_o is the stagnation pressure with respect to the cascade. A perfect gas entering a cascade in the same manner may leave with a tangential momentum (per unit area per second) of

$$
\begin{aligned}
(\mu_t)_{max} &= \rho^* a^* (\lambda\tau)_{max} \\
&= \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} \gamma p_o \left(\frac{\gamma+1}{2\gamma}\right)^{\frac{\gamma}{\gamma-1}} \quad \text{by III-12} \\
&= (\gamma)^{-\frac{1}{\gamma-1}} p_o \\
&= .432\, p_o \qquad \text{for } \gamma = 1.40.
\end{aligned}
$$

The maximum change of tangential momentum per second possible in a two-dimensional cascade is of course twice the figure given above.

iv. It may be of interest to mention that the quantity $\lambda\, \bar{u}_t^2$ has its maximum when $\Lambda = \frac{\gamma+1}{3\gamma-1}$. The corresponding Mach number $M = \sqrt{3}$ and $\cos\alpha = \frac{1}{\sqrt{3}}$. The quantity $\lambda\, \bar{u}_t^2 = G^2 \lambda (1-\Lambda)$ also appears in connection with the maximum power condition of certain turbine arrangements, namely two-cascade impulse turbines.

IV. THE VELOCITY TRANSFORMATION

1. Relative Stagnation Temperature, Mass Flow and Mach Number Ratios

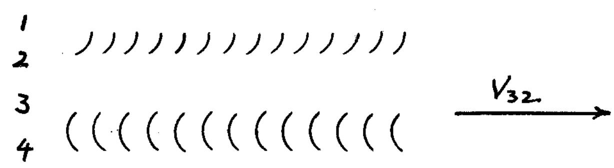
When two-dimensional cascades are discussed which are in uniform translational motion with respect to each other there is no essential difference between "moving" and "stationary" cascades since there is no preferred inertial coordinate system of reference. While in the case of rotating cascades centrifugal pressure gradients have to be considered, at least as a second order effect, in this two-dimensional case all cascades behave exactly alike with respect to their own coordinate system. Thus, when only relative velocities are used the same type of cascade transformation holds for all cascades.

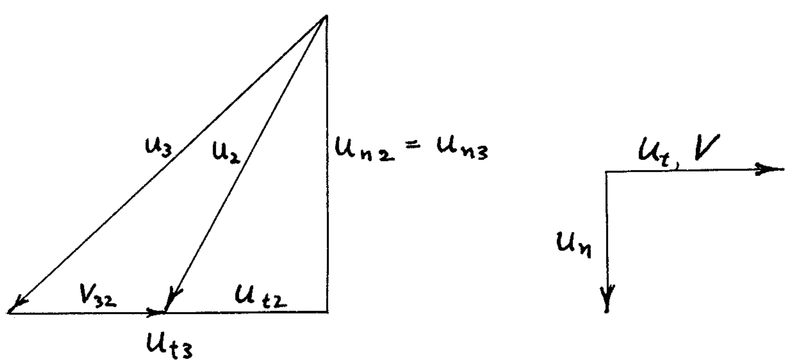
To get from one cascade to the next the flow has to be referred to a new coordinate system moving with the new cascade. The effect of this velocity transformation on the parameters describing the flow will be developed here.

Since only relative velocities are used, each cascade is considered as a "channel" in which neither heat nor mechanical energy are added (or taken out). The addition of mechanical energy to the flow is, of course, a consequence of the relative motion of several cascades.$^{\Delta}$ In (2) below it will be shown that the velocity transformation takes care of the energy equation in the proper manner.

$^{\Delta}$Compare this with Sorg's approach, section V. and Ref. 4.

To develop the velocity transformation consider two cascades, 1-2 and 3-4. Cascade 3-4 (and its coordinate system) move with a velocity V_{32} with respect to 1-2. It is assumed that no heat is added anywhere in the flow and that the flow between planes (2) and (3) is uniform and parallel (i.e. contains no waves).





The schematic sketch above of the arrangement with the corresponding velocity diagram shows the assumed positive directions of u_t, V, and u_n. We can write down the relation between the velocities immediately:

$$u_{n3} = u_{n2} \qquad \text{IV-1}$$

$$u_{t3} = u_{t2} - V_{32}$$

so that

$$u_3^2 = u_2^2 + V_{32}(V_{32} - u_{t2}) \qquad \text{IV-2}$$

To find the stagnation temperature with respect to the new coordinate system we write the energy equation (II-1) for a point in plane (2) and for one in (3):

$$c_p T_{02} = c_p T_2 + u_2^2/2$$

$$c_p T_{03} = c_p T_3 + u_3^2/2$$

If there is no difference between the cascade areas at (2) and at (3), the free stream temperature of the fluid does not change between (2) and (3), i.e. T_2 and T_3 must be equal. Using this fact and equation IV-2 we get from the two equations above

$$c_p(T_{03} - T_{02}) = V_{32}^2/2 - V_{32} u_{t2}$$

or

$$\frac{T_{03}}{T_{02}} = 1 + \frac{V_{32}}{2 c_p T_{02}}(V_{32} - 2 u_{t2}) \qquad \text{IV-3}$$

If we introduce a_2^* as reference velocity, where

$$a_2^{*2} = (2/G^2)\, c_p T_{02} \qquad \text{II-3}$$

we get the equation IV-3 with dimensionless velocities:

$$\frac{T_{03}}{T_{02}} = \frac{1}{G^2}(\bar{V}_{32}^2 - 2 \bar{V}_{32} \bar{u}_{t2}) \qquad \text{IV-4}$$

where $\overline{V}_{32}$ has been written for V_{32}/a_2^*, i.e. the cascade velocity has been made dimensionless by referring it to the inlet conditions.[Δ] It will be noticed that the relative stagnation temperature increases from (2) to (3) if the quantities $\overline{V}_{32}$ and $(\overline{V}_{32} - \overline{u}_{t2})$ have the same sign. When $u_{t3} = -u_{t2}$, $(\overline{V}_{32} - u_{t2}) = 0$ and hence $T_{03} = T_{02}$. In that case the velocity diagram (page 29) becomes symmetrical.

The effect of the velocity transformation on λ is of interest. Since the mass flow at (2) equals that at (3),

$$\rho_2^* a_2^* A_2^* = \rho_3^* a_3^* A_3^*$$

The cascade areas A_c are assumed to be equal. Therefore

$$\frac{\lambda_3}{\lambda_2} = \frac{A_3^*}{A_2^*} = \frac{\rho_2^* \, a_2^*}{\rho_3^* \, a_3^*}$$

Now we may write

$$\frac{\rho_2^*}{\rho_2} = \left(\frac{T_2^*}{T_2}\right)^{\frac{1}{\gamma-1}}$$

and a similar expression for plane (3). Again $T_2 = T_3$.

Also

$$T_2^* = \frac{2}{\gamma+1} T_{02} \qquad \text{and} \qquad a_2^* = \sqrt{\frac{2\gamma R T_{02}}{\gamma+1}}$$

[Δ]Since $T_{02} = T_{01}$, $a_2^* = a_1^*$. These equal quantities will be freely interchanged throughout this paper without further mentioning.

Using these equations we get λ_3/λ_2 as a function of the stagnation temperatures:

$$\frac{\lambda_3}{\lambda_2} = \left(\frac{T_{o2}}{T_{o3}}\right)^{G^2/2}$$

Thus an arrangement of cascades which increases the relative stagnation temperature decreases the value of λ and thereby makes a wider range of angles and Mach numbers accessible.

Equation IV-5 is not limited to a single velocity transformation but applies also to the initial and final state of a flow which has undergone any number of velocity and isentropic cascade transformations, as is evident from the derivation. The only assumptions which have been made were the isentropy and the constancy of the cascade area A_c. Actually we can easily abandon this last assumption and allow for a change in area (although we thereby depart from a strictly two-dimensional flow). Since λ is inversely proportional to A_c the more general relation would be

$$\frac{\lambda_f}{\lambda_i} = \left(\frac{T_{oi}}{T_{of}}\right)^{G^2/2} \frac{A_{ci}}{A_{cf}} \qquad \text{IV-5a}$$

where the subscripts i and f designate the initial and final states respectively. Equation IV-5a is rather useful since it enables us to predict immediately the maximum power per unit cascade area obtainable from any turbine arrangement, under the assumptions stated above, if the initial and final areas are given.

In a turbine the stagnation temperature of the fluid decreases and thus the product $\lambda\ A_c$ increases from cascade to cascade. The greatest possible expansion ratio is reached when λ_f becomes unity:

$$\left(\frac{T_{oi}}{T_{of}}\right)_{max} = \left(\frac{1}{\lambda_i}\ \frac{A_{cf}}{A_{ci}}\right)^{2/G^2}$$

The power output is proportional to

$$\lambda_i\,(T_{oi} - T_{of}) = \lambda_i\left(1 - \frac{T_{of}}{T_{oi}}\right) T_{oi}$$

$$= T_{oi}\,\lambda_i\left[1 - \left(\lambda_i\ \frac{A_{ci}}{A_{cf}}\right)^{2/G^2}\right]$$

so that for given inlet temperature and cascade area ratio the mass flow for maximum power output and the maximum power can easily be found.

The change in Mach number which the flow experiences when passing from one cascade to another is of particular interest. The Mach number used throughout this paper is that based on the relative velocity u, and should thus properly be called a relative Mach number. For design purposes this is likely to be the more important one.$^\Delta$

$^\Delta$The Mach number of the absolute velocity in both axial and radial flow machines has been discussed in some detail by Sorg, Ref. 4.

The Mach number at any point may be treated as a vector. Thus

$$M_2^2 = M_{n2}^2 + M_{t2}^2$$

In going from one cascade to the next the local velocity of sound remains unchanged: $a_2 = a_3$, so that

$$M_{n2} = M_{n3}$$

and

$$M_{t3} = \frac{u_{t3}}{a_3} = M_{t2} - \frac{V_{32}}{a_2}$$

thus

$$M_3^2 = M_2^2 + \frac{V_{32}^2 - 2V_{32}u_{t2}}{a_2^2} \qquad \text{IV-6}$$

$$= M_2^2 \left[1 + (1/u_2^2)(V_{32}^2 - 2V_{32}u_{t2})\right]$$

whence

$$\frac{M_3^2}{M_2^2} = 1 + \frac{V_{32}^2 - 2V_{32}u_{t2}}{u_2^2} \qquad \text{IV-7}$$

First we note that $M_3/M_2 \gtrless 1$ when $T_{03}/T_{02} \gtrless 1$ and vice versa. In fact

$$\frac{T_{03}/T_{02} - 1}{(M_3/M_2)^2 - 1} = \left(\frac{u_2}{a_2^*}\right)^2 \frac{1}{G^2} = \left(\frac{\bar{u}_2}{G}\right)^2$$

It is clear from equations IV-6 and IV-7 that a uniform flow may be supersonic with respect to one cascade and subsonic with respect to the next, a fact which is of course well known. In some applications it is useful to know the limiting condition which still makes possible subsonic flow entering a given cascade. If we assume $M_2 > 1$ and V_{32} positive we obtain from IV-6 after some algebra that

$$\sin\alpha_2 > \left(\frac{2}{\gamma+1}\cdot\frac{T_{02}}{T_2}\right)^{\frac{1}{2}} \frac{\bar{V}_{32}}{2M_2}\left(\frac{M_2^2-1}{T_{02}/T_2}\cdot\frac{\gamma+1}{2\bar{V}_{32}^2}+1\right) \qquad \text{IV-8}$$

if M_3 is to be smaller than unity. In figure IV-1 the minimum values of $\sin\alpha_2$ for subsonic M_3 are given with the reduced cascade velocity $\bar{V}_{32}$ as a parameter. These curves could of course be extended into the subsonic region of M_2.

The right hand side of the inequality IV-8 has a minimum with respect to $\bar{V}_{32}$ for any given value of M_2. The locus of these minima is the envelope in figure IV-1. Along the envelope

$$\bar{V}_{32}^2 = (\gamma+1)\left(\frac{M_2^2-1}{2+(\gamma-1)M_2^2}\right) \qquad \text{IV-9}$$

2. A Check On The Velocity Transformation

In the following the change in stagnation temperature will be calculated for a flow passing through three cascades, a moving

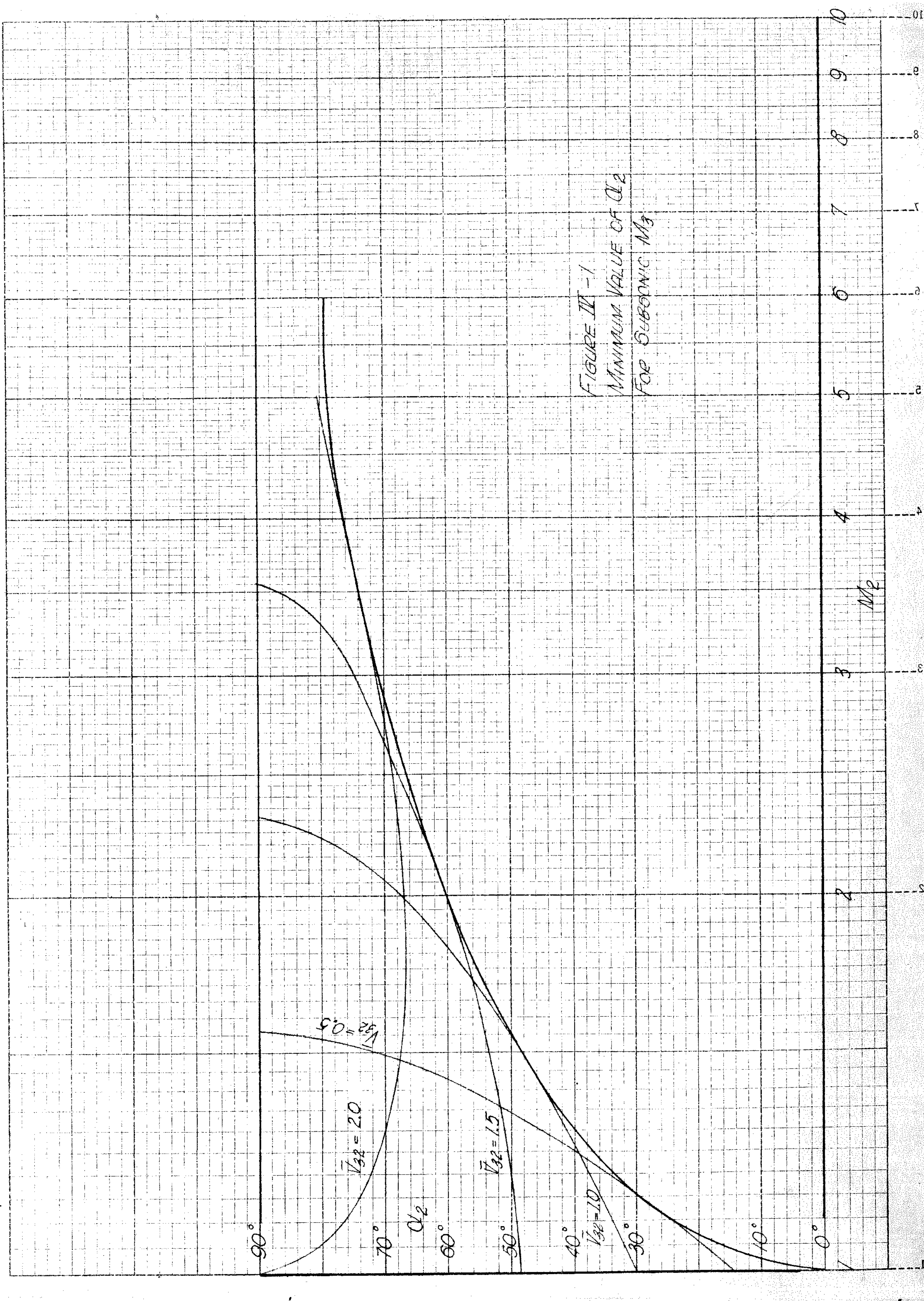
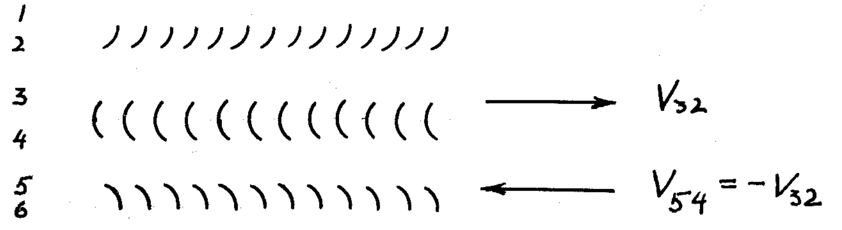


FIGURE III-1
MINIMUM VALUE OF α_2
FOR SUBSONIC M_3

cascade 3-4 between two stationary ones. The result obtained by means of two velocity transformations will be compared with the power output (or input) of the cascade. A schematic sketch of the arrangement is shown below. It is clear that all conceivable arrangements of cascades (within the general limitations of this paper) can be made up out of a number of such "stages", with possible omission of one or both stationary cascades. It is therefore sufficient to show that the energy equation is satisfied for this basic stage.



V_{54} is the velocity of (5) with respect to (4). Since 5-6 is considered stationary, $V_{54} = -V_{32}$. Equation IV-3 is written for the two velocity transformations in the form

$$T_{03} - T_{02} = \frac{1}{2c_p}\left(V_{32}^2 - 2V_{32}u_{t2}\right)$$

$$T_{05} - T_{04} = \frac{1}{2c_p}\left(V_{54}^2 - 2V_{54}u_{t4}\right)$$

Since the stagnation temperature in each cascade is constant, and since $u_{t3} = u_{t2} - V_{32}$ we get

$$c_p\left(T_{06} - T_{01}\right) = V_{32}\left(u_{t4} - u_{t3}\right) \qquad \text{IV-9}$$

$(u_{t4} - u_{t3})$ is of course the rate of change of tangential momentum per unit mass flow, or the tangential force per unit mass flow. If equation IV-9 is multiplied by an arbitrary mass flow, then the right hand side represents the power put in by the moving cascade 3-4 and the left hand side is the corresponding increase in enthalpy of the fluid (note that C_p is here in mechanical units). Thus equation IV-7 could have been written down directly from the energy equation.

For further use it is convenient to bring equation IV-7 into a form in which the velocities are made dimensionless. Using again a_2^* as reference velocity we get

$$\frac{T_{06}}{T_{01}} = 1 + \frac{2\,\bar{V}_{32}}{G^2}\left(\bar{V}_{32} + \frac{u_{t4} - u_{t2}}{a_2^{*2}}\right) \qquad \text{IV-10a}$$

and

$$\frac{T_{06}}{T_{01}} = 1 + \frac{2\,\bar{V}_{32}}{G^2} \cdot \frac{u_{t4} - u_{t3}}{a_2^{*2}} \qquad \text{IV-10b}$$

3. Graphical Representation of the Velocity Transformation

The velocity transformation is characterized by the fact that it leaves the normal component of the Mach number as well as the local velocity of sound unchanged. For those reasons it is most easily represented on the Mach vector diagram, or, of course, on the conventional velocity vector diagram.

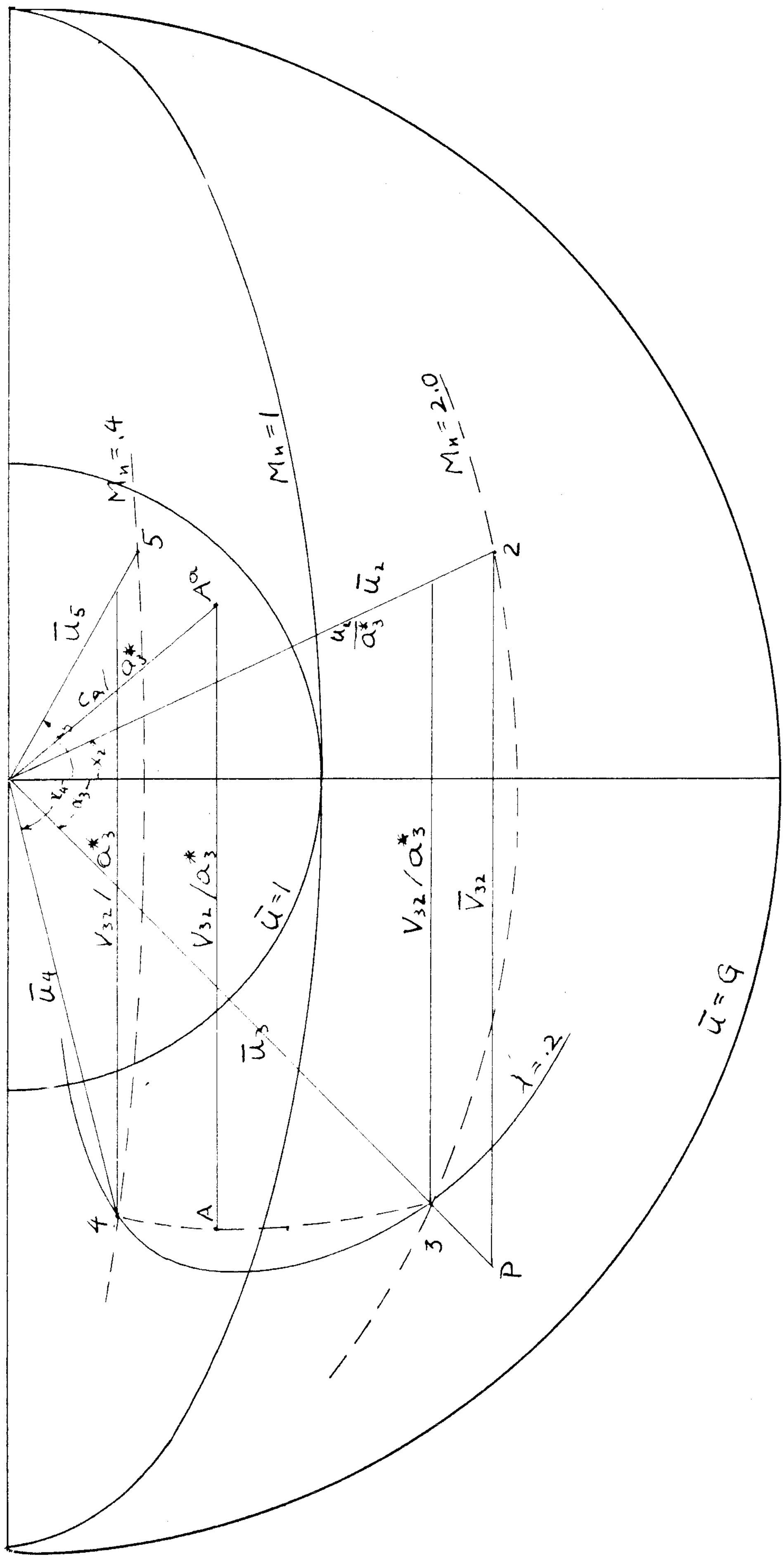


FIGURE IV-2

VELOCITY TRANSFORMATIONS ON THE $\bar{U}$ DIAGRAM

However for the purpose of tracing a complete flow the reduced velocity diagram has several advantages. Therefore the curves M_n = constant were included in figure III-3. With the aid of these curves the velocity transformation can easily be carried out graphically on the $\bar{u}$ diagram, as shown in figure IV-2. Let the vector $\bar{u}_2$ represent a uniform flow leaving cascade 1-2, and let $\bar{V}_{32}$ be the reduced cascade velocity of 3-4 (with respect to 1-2). The vector addition of $\bar{u}_2$ and $-\bar{V}_{32}$ gives a vector of length u_3/a_3^* with the correct angle α_3. The proper length of $\bar{u}_3 = u_3/a_3^*$ is then found most easily by using the fact that $M_{n3} = M_{n2}$. Having found point 3 on the diagram we then also have the value of V_{32}/a_3^* on the figure. The cascade transformation 3-4 then takes place along the curve $\lambda = \lambda_3$. The dotted line 3-4 could represent roughly the mean velocities and directions of the flow inside the cascade. The reduced "absolute" velocities c/a_3^* corresponding to any point A along 3-4 are easily found by using the vector V_{32}/a_3^* (see point A^a, figure IV-2). The transformation from (4) to (5) is carried out the same way as that from (2) to (3). Note that the point P corresponds to a velocity vector u_3/a_2^*. P thus does not necessarily fall inside the circle $\bar{u} = G$.

To obtain the curves M_n = constant in the $\bar{u}$ diagram we write

$$M_n = \frac{u_n}{a^*}\,\frac{a^*}{a} = \bar{u}\cos\alpha\sqrt{\frac{2}{\gamma+1-(\gamma-1)\bar{u}^2}} \qquad \text{IV-11}$$

from which α can be calculated as function of $\bar{u}$. We could of course obtain them graphically by drawing the lines M_n = constant on the Mach vector diagram. Equation IV-11, with M_n as parameter, represents the coaxial family of ellipses

$$\frac{2 + M_n^2(\gamma - 1)}{(\gamma + 1) M_n^2} \quad \bar{u}_n^2 + \frac{1}{G^2} \bar{u}_t^2 = 1$$

It may be of interest to show that the quantity $\sqrt{T_{03}/T_{02}} = a_3^*/a_2^*$ can be obtained graphically on the $\bar{u}$ diagram without the aid of the M_n = constant curves. We draw the vectors $\bar{u}_{t2}$ and $\bar{V}_{32}$ from the origin of the $\bar{u}$ diagram, as shown in figure IV-3. By completing the right triangle BAD with hypotenuse G and side $\bar{u}_{t2}$ we get the angle β whose cosine is $\bar{u}_{t2}/G$. The triangle BDE then has sides G and $\bar{V}_{32}$ and the included angle β. The cosine rule gives side DE as

$$(DE)^2 = G^2 + \bar{V}_{32}^2 - 2 \bar{V}_{32} G \cos\beta$$

or

$$(DE)^2 = G^2 \frac{T_{03}}{T_{02}} = G^2 + \bar{V}_{32}^2 - 2 \bar{V}_{32} \bar{u}_{t2}$$

Since

$$\frac{u}{a_2^*} : \frac{u}{a_3^*} = \sqrt{\frac{T_{03}}{T_{02}}} : 1$$

we can lay out u/a_2^* along DE and find the magnitude of u/a_3^* along BD by similar triangles.

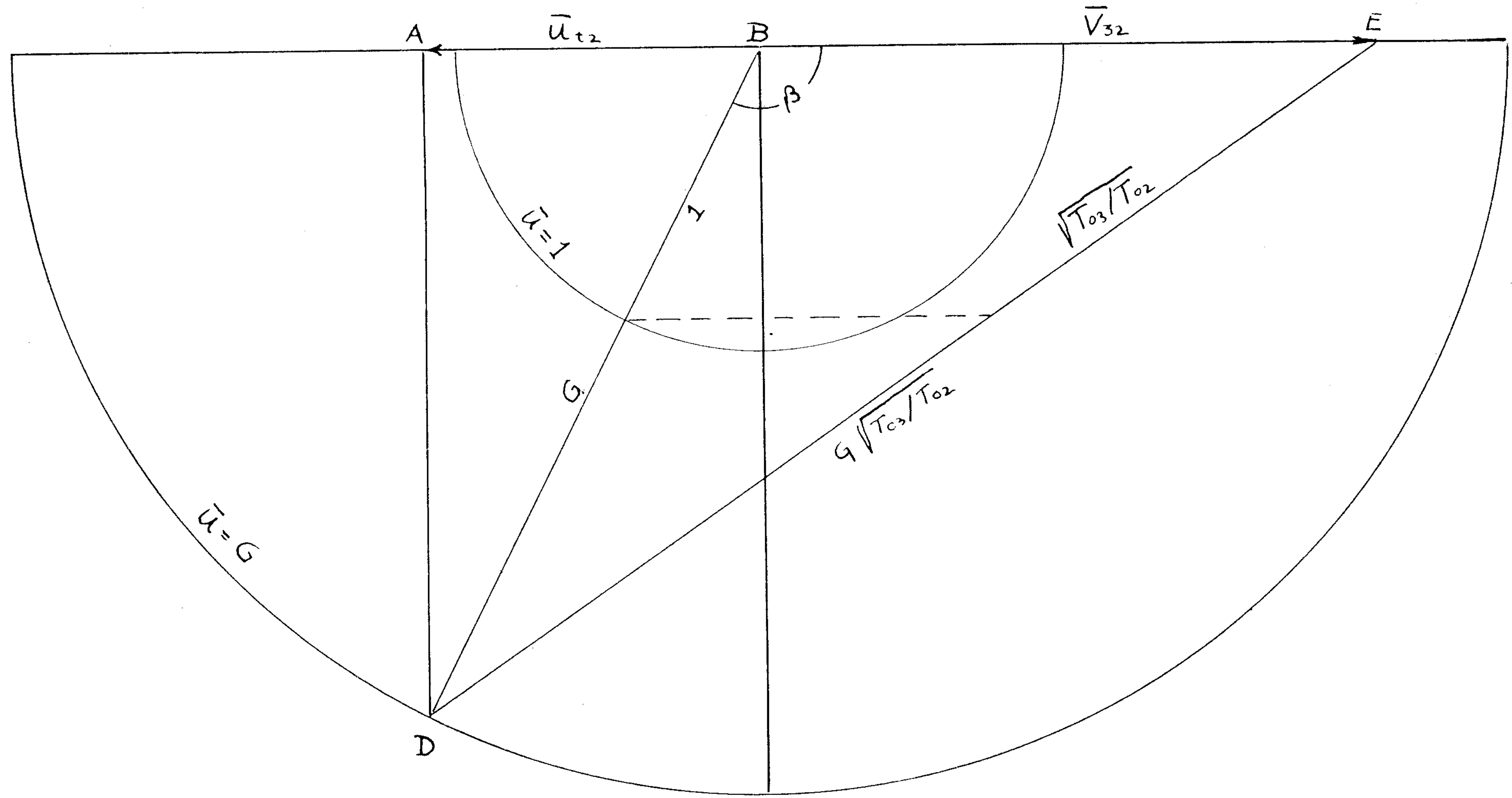


FIGURE IV-3

GRAPHICAL CONSTRUCTION
OF THE RATIO a_3^*/a_2^*

V. USE OF THE DIAGRAMS FOR PRELIMINARY ANALYSIS OF A SINGLE CASCADE SUPERSONIC DIFFUSER

1. Remarks on Supersonic Diffusers

The problem of diffusing a supersonic stream of gas with a minimum loss of stagnation pressure is one of theoretical interest as well as considerable practical importance. The existing and proposed diffusers may be grouped into two types:

i. The external shock wave diffusers, based on the ideas of Oswatitsch. (Ref. 11).

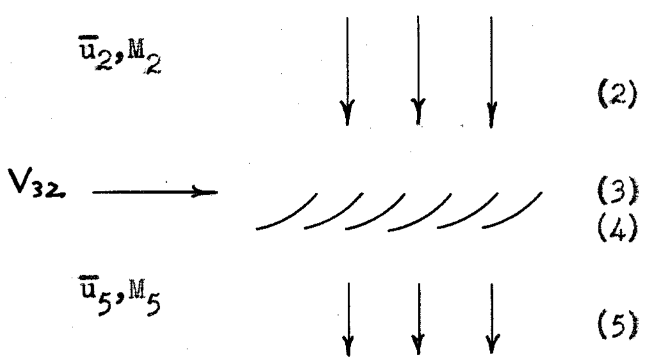
ii. Diffusers which form part of a closed duct, such as those used in supersonic wind tunnels (Ref. 6).

In some applications the two principles are combined. In all the above diffusers the transition from supersonic to subsonic flow is accomplished by normal or strong (i.e. transonic) oblique shocks, preceded by some pattern of weak oblique shocks. Because of the dominating influence of viscosity in flows near Mach number one it is hard to predict the exact flow pattern in the transonic part of these diffusers, and the flow in the subsonic part of these diffusers is usually not very advantageous. The greater part of the losses is due to the shock waves, and in general these losses increase rapidly if increased range of operating conditions is required.

Efficient diffusion of a supersonic flow in a closed channel is made difficult primarily by the transonic part of the flow. If the shock is to remain weak, only very small variations in mass flow can be tolerated. Moving cascades appear to offer the possibility of avoiding altogether the difficult transonic region in the process of diffusion, since one can get from supersonic to subsonic flow without having anywhere unit Mach number relative to the channel in which the gas is flowing. It seems then that with the aid of cascades the transition from supersonic to subsonic flow could be accomplished without a transonic shock. In the following paragraphs this possibility will be investigated from the point of view of the one-dimensional approach used in this paper.

2. <u>The Single Cascade Supersonic Diffuser</u>

One of the simplest cascade problems and at the same time one of the most interesting is that of a single cascade moving at right angles to a uniform stream. Let this moving cascade be designated by 3-4. The flow at planes 2 and 5 is described with respect to stationary coordinate axes. $\bar{V}_{32}$ is the reduced cascade velocity.



Let the flow at (2) be uniform and supersonic. On the reduced velocity diagram figure V-1 point (3) is found by use of the curves M_n = constant (as described in section IV). The figure shows that the cascade 3-4 can be designed in such a manner that the tangential velocities in planes (3) and (4) are equal, i.e. $\bar{u}_{t3} = \bar{u}_{t4}$. If that is done the points (3) and (4) must be on opposite sides of the line

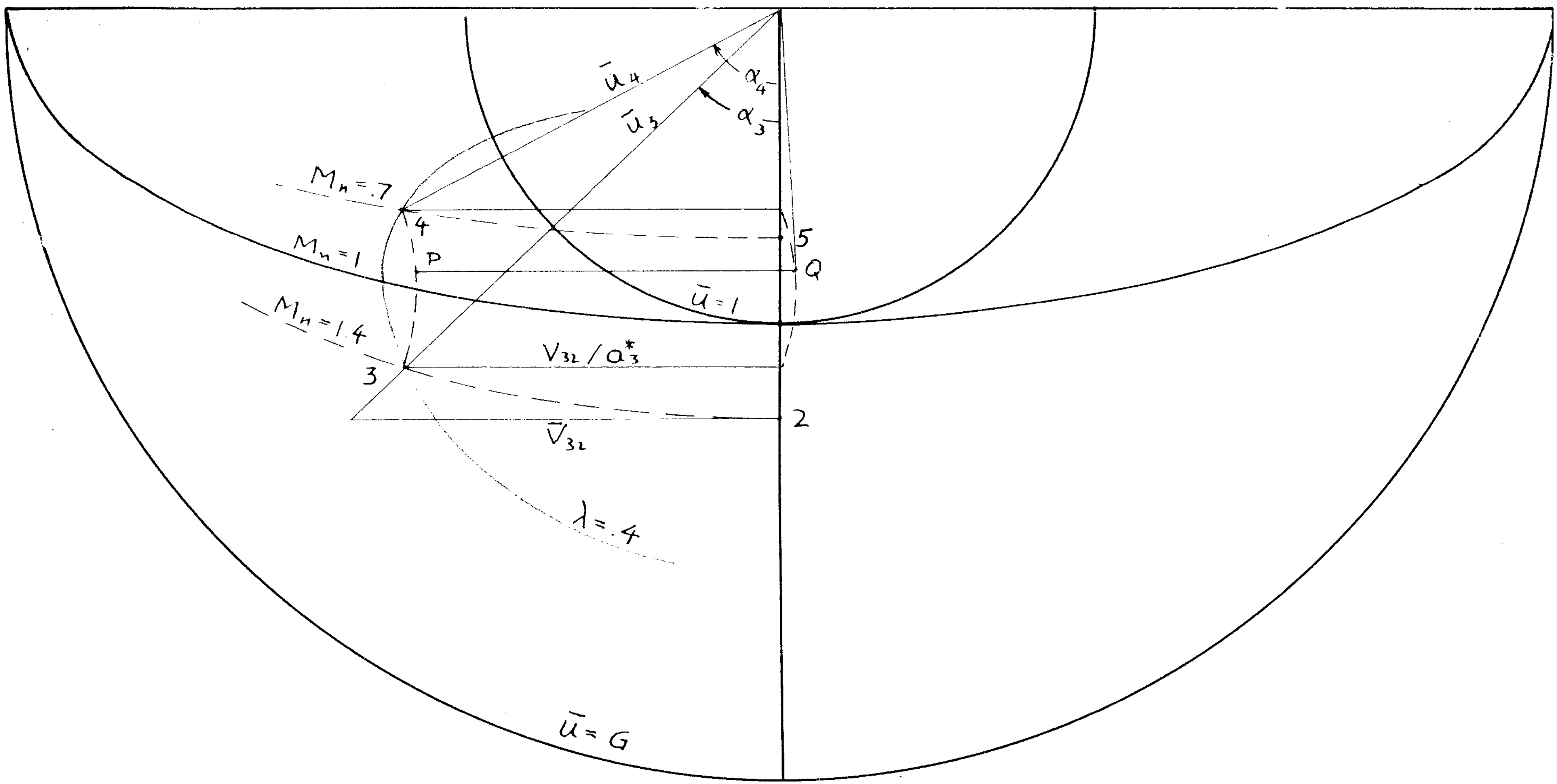


FIGURE V-1

REDUCED VELOCITY DIAGRAM OF A PARTICULAR DESIGN CONDITION

$M_n = 1$, and therefore $M_{n4} (= M_{n5})$ will be subsonic.

Since $\bar{u}_{t3} = \bar{u}_{t4}$ the flow leaving the cascade at (5) is parallel to the flow at (2). No tangential momentum has been added and of course no work has been done by the cascade 3-4.

The essential feature of this arrangement is that the mass flow nowhere reaches the maximum possible value for the crossectional channel area available to the flow. This may be associated with the fact that in the cascade the (relative) stagnation temperature is higher than in planes (2) and (5) which makes possible there a larger mass flow per unit area.

It may be of interest to discuss the above flow briefly from the point of view of the "absolute" velocity inside the cascade 3-4. The author is unable to give a rigorous justification of this approach, but merely wishes to refer the reader to Sorg's paper (Ref. 4) and repeat one of the results given there. A flow inside a moving cascade which is steady with respect to the cascade is necessarily nonsteady with respect to a reference system which is at rest.[Δ] Sorg points out that we may imagine the number of blades increased indefinitely and that thereby the flow could be made to approach, as closely as we wish, a flow which is steady with respect to a reference system at rest. The cascade is thus assumed to do work on the fluid in infinitesimal increments and in any plane parallel to the cascade axis the absolute velocity (as well as the relative velocity)

[Δ]Except for the trivial case of a cascade which does not deflect the flow in any way.

is considered steady. For such a plane inside the cascade Sorg then writes the energy equation as

$$dh + \frac{1}{gJ}\, c\,dc = \frac{1}{J}\, dL \qquad {}^{\Delta} \qquad \text{V-1}$$

where c is the absolute velocity and dL is the work done on the fluid at that point. Sorg combines this energy equation with the continuity relation for steady flow and the equation of state in order to find the necessary (and usually sufficient) condition for the existence of a minimum of crossectional area. For the case of an axial flow machine the work done on the fluid $dL = (V/g)dc_t$ where c_t is the tangential component of c. The crossectional area of the channel normal to the absolute velocity c is designated by F_c. For the axial flow machine Sorg gets as necessary and sufficient condition for the existence of a minimum of F_c the relations

$$\frac{a^2}{c^2} = 1 - \frac{V\,dc_t}{c\,dc}, \quad \text{or} \quad \frac{a^2}{c^2} = \frac{u\,du}{c\,dc} \qquad {}^{\Delta} \qquad \text{V-2}$$

Thus, if the above steady flow considerations can be applied to an actual cascade flow, we would have the result that in general F_c has a minimum at some absolute Mach number different from unity. Under some conditions a minimum of F_c is not possible at all (if the right hand sides of V-2 become zero, negative or infinite). In other cases the minimum of F_c is possible only at a Mach number which does not occur in the machine. Several examples are given

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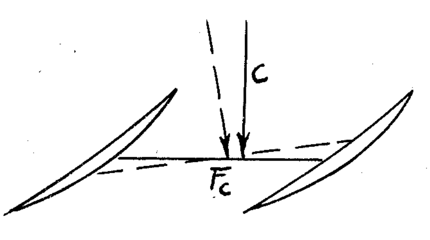
$^{\Delta}$The symbols have been changed to conform with the notation used in this paper.

in Sorg's paper.

Let us regard the single cascade diffuser from the viewpoint of Sorg's work. The cascade would have to have a very large number of very thin blades. The flow inside the cascade could then be represented by points on the $\bar{u}$ diagram, figure V-1. We can use at any point P the reduced cascade velocity V_{32}/a_3^* to find the reduced absolute velocity c/a_3^* (point Q in the figure). It will be seen that in our diffuser cascade c_t is zero in planes (3) and (4) and probably will not differ much from zero anywhere inside. Thus Sorg's equations V-2 would indicate that in this particular case there has to be a "throat" with respect to the absolute velocity, and this will occur at an absolute Mach number c/a not far from unity.

At first sight the above conclusion makes us suspect that the cascade diffuser with fixed cascade velocity has no advantages over a convergent divergent diffuser with fixed throat dimensions. However, it will be shown in the next paragraph that the single cascade diffuser can accommodate fluctuations of mass flow. To reconcile this fact with the existence of a "throat" (= minimum of F_c) it should be noted that the size of this "throat" is not fixed but varies very rapidly with the inclination of the absolute velocity c (see sketch below).



3. Deviations from the Design Condition

Any diffuser is capable of practical realization only if it can accommodate deviations from the design conditions. The following deviations should be considered:

a. Small fluctuations in mass flow (i.e. inlet Mach number) which must be accommodated without changes in cascade velocity or the geometry of the diffuser.

b. Small fluctuations in cascade velocity to be accommodated with fixed mass flow and geometry.

c. Small fluctuations in back pressure on the subsonic side of the diffuser to be accommodated without changes of the other parameters.

d. A reasonably wide range of inlet conditions which may be taken care of by some simple adjustment of the cascade velocity or the geometry of the device, or both.

Two other important considerations will determine the practicability of such a device:

e. Starting of the device must be possible by some reasonably simple procedure (depending largely on the particular application).

f. Stability of the cascade with respect to velocity is desirable though not absolutely essential. If the diffuser were coupled to some other machine a certain degree of instability could be tolerated.

Discussion of the starting procedure appears outside the purpose of this paper, since it depends largely on the particular application. The other points will be briefly discussed within the limitations of the one-dimensional approach used in this paper.

It will be seen that (a) and (b) above are closely related. Referring to figure V-2, let the design condition be designated by superscript D and the values slightly off the design point by ', ", etc. It is clear that inlet Mach numbers slightly above M_2^D do not present a problem since increased Mach number means decreased relative mass flow. Let us then focus our attention on the case of an inlet velocity $\bar{u}_2'$ slightly less than $\bar{u}_2^D$. The inlet condition to the moving cascade will be 3'. The relative mass flow λ_3' is slightly larger than λ_3^D. A weak shock and an expansion region will turn the flow to the prescribed inlet angle α_3^D. The shock will cause a slight further increase in λ. The flow in the cascade will then be not much different from the design condition except for the increased λ, but it will no longer be uniform at (4'). Presumably it will leave the cascade at approximately $\alpha_4' = \alpha_4^D$ and at an average Mach number M_4' which is somewhat closer to unity than was M_4^D. As expected the ability of the cascade to stand an increase in relative mass flow λ_3 depends on M_4^D being sufficiently far from unity.

It will be seen that the case of slightly reduced cascade velocity $\bar{V}_{32}''$ is quite similar to that of decreased inlet Mach number, except that the angle α_3'' is smaller than α_3^D, so that the wave pattern at the entrance to the moving cascade is different. While the design condition was chosen such as to make $\bar{u}_{t2}$ equal to $\bar{u}_{t3}$ this condition is not preserved off the design point. A line connecting points 3' and 4' would slope to the right, indicating

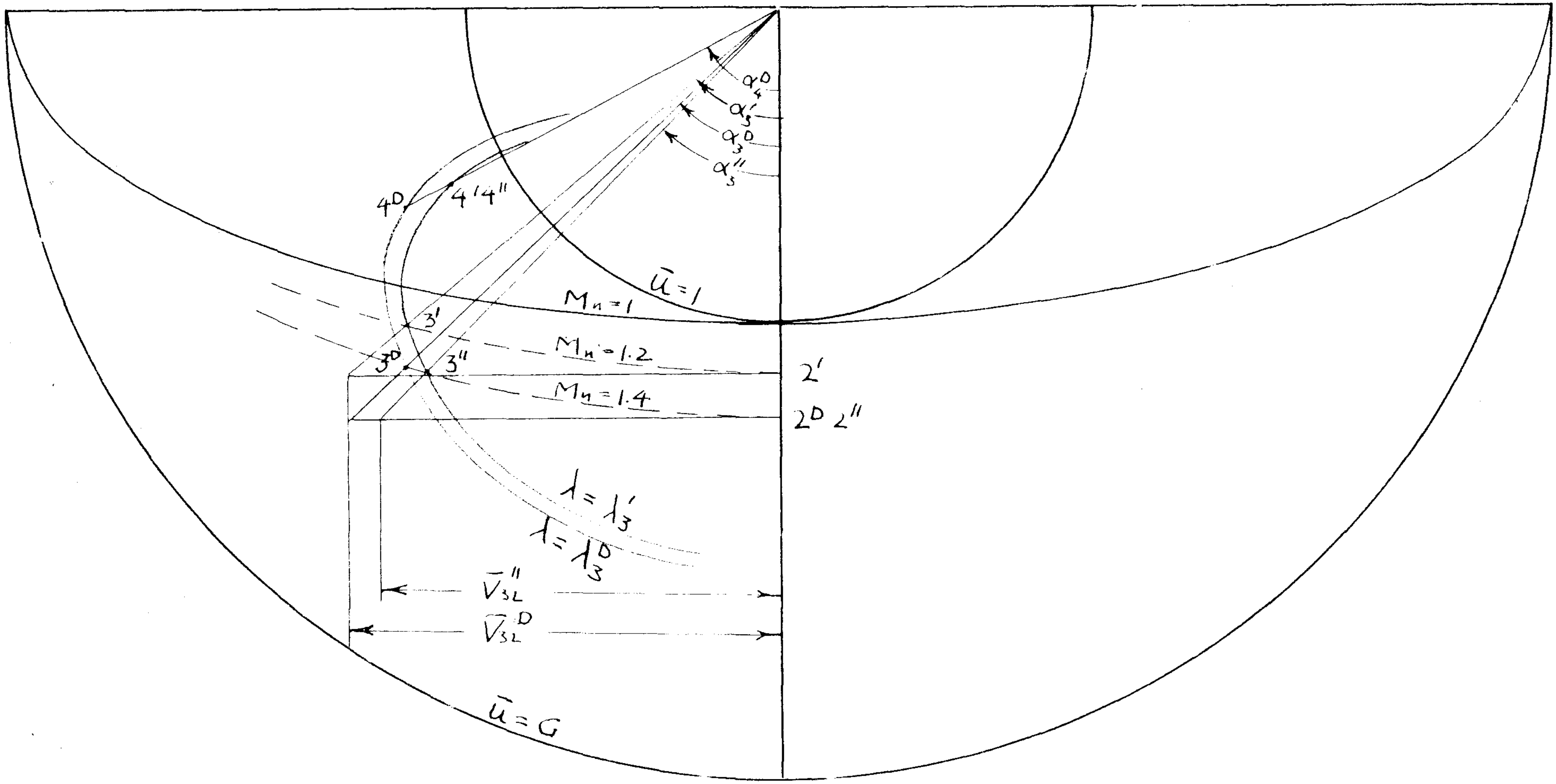
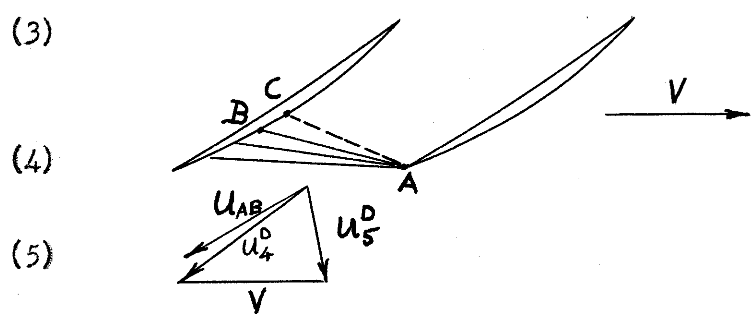


FIGURE V-2

REDUCED VELOCITY DIAGRAM FOR SMALL DEVIATIONS FROM DESIGN CONDITION that the cascade now requires power to drive it. For the case of decreased cascade velocity, points 3" and 4", this effect is less pronounced but still apparent. This is further discussed in a later paragraph in connection with the stability of the cascade in motion.

The fluctuations in back pressure must be allowed for in the design condition. The influence of the pressure at plane 5 is felt inside the cascade up to the characteristic A-B. In order to



allow for fluctuations in back pressure the design pressure at (5) should be lower than the design pressure along the characteristic A-B, so that normally there would be an expansion region following A-B. A more detailed investigation might show whether or not an oblique shock along A-C would occur and be stable if the pressure at (5) exceeded the design pressure at A-B.

From the discussion of (a) above it is clear that the cascade cannot handle a wide range of mass flows without some adjustment in either cascade velocity or diffuser geometry. Basically there are three possible ways of increasing the range of mass flows over which the device could operate:

i. The cascade velocity may be varied.

ii. A cascade of adjustable inlet vanes could be used in order to give the flow at (2) a variable tangential velocity component.

iii. The blade angle of the rotor could be made adjustable.

(i) and (ii) both would accomplish the same thing, namely a variation of the stagnation temperature in the cascade 3-4. If the stagnation temperature were properly adjusted then the cascade 3-4 could be kept at a constant value of λ for a fairly wide range of inlet mass flow values. In figure V-3 the two methods are compared. For the design condition $M_2^D = 2$ was chosen. With $\overline{V}_{32}^D = 1.62$ we get $\alpha_3^D = -45^o$. Point 2' corresponds to a Mach number $M_2' = 1.4$. If the cascade velocity is increased to $\overline{V}_{32}' = 2$ then point 3' lies on the same λ-curve as 3^D, with α_3' of about -57^o. An oblique shock and expansion region would turn the flow back to -45^o. From then on the flow would procede through the cascade roughly as in the design condition. In an actual problem the cascade velocity should be increased slightly more in order to allow for the increase in λ due to the shock and the non-uniformity of the flow at 4. In order to simplify the comparison the flow at 1" was chosen to correspond to a Mach number of about 1.54. It is assumed that the entrance vanes turn the flow nearly isentropically through 15^o to the point 2". The distance 2"Q in the figure is chosen equal to 2^DP, the design cascade velocity. The points 3" and 3' then coincide. The turning of the approaching flow by a set of inlet vanes thus would accomplish roughly the same result as the increase in cascade velocity.

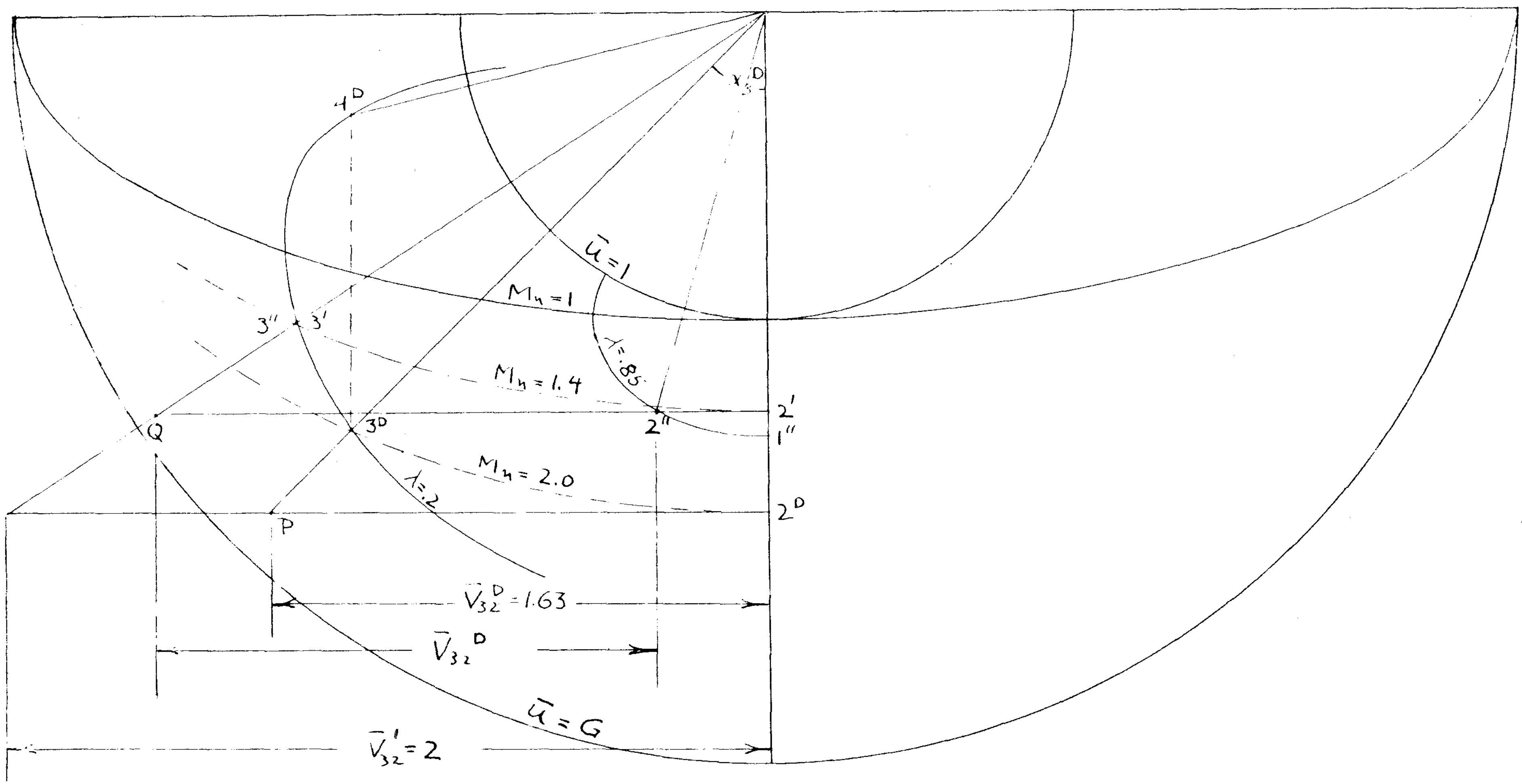


FIGURE V-3

COMPARISON OF INCREASED CASCADE VELOCITY VS. INLET TURNING VANES FOR WIDENING OPERATING RANGE

Adjustment of the rotor blade angle will in most cases be impractical because of mechanical difficulties. However a brief discussion of this possibility for widening the operating range of the device is included. The design condition in figure V-4 is the same as in figure V-3. The inlet Mach number is now lowered to $M_2' =$ 1.4 and the cascade velocity is kept constant at $\overline{V}_{32}^{D} = 1.67$. Point 3' then lies approximately on the curve $\lambda = .3$. If the blades of the rotor were rotated by ten degrees we would obtain an outlet condition approximated by point 4', so that the flow at 5' would still be approximately parallel to the entering flow at 2'. One disadvantage of this method would be in the fact that the angle at 3' now differs by some 16 degrees from the new inlet angle of the rotor, point I in the figure, thus requiring a shock wave of appreciable strength to turn the flow. It would thus be more advantageous if the rotor blades could be made in two parts so that only the outlet angle would be adjusted.

The question of rotational stability also needs considerably more detailed treatment than it can be given here, especially in view of the fact that the back pressure on the subsonic side of the cascade influences the final direction of the flow leaving the cascade. The result of the simple one-dimensional discussion given here must therefore be regarded as strictly unreliable. As an example an inlet Mach number of 1.60 and a design cascade velocity $\overline{V}_{32}^{D} = 1.24$ were chosen (figure V-5). The tangential velocities at points 3^D and 4^D were made equal, so that the cascade would not require or furnish any power. In order to bring out more clearly any

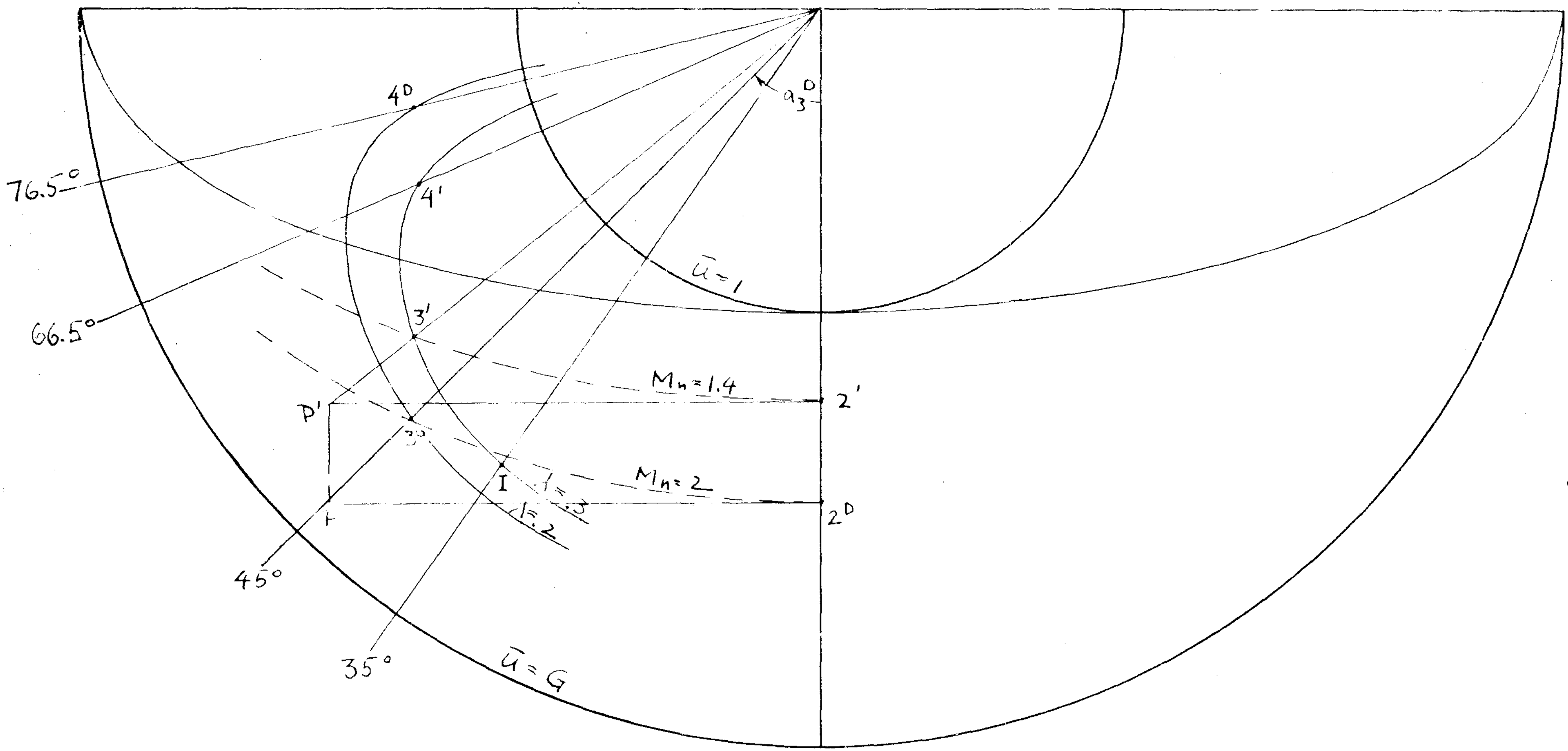


FIGURE V-4

ADJUSTMENT OF CASCADE BLADE ANGLE
FOR INCREASED INLET MASS FLOW

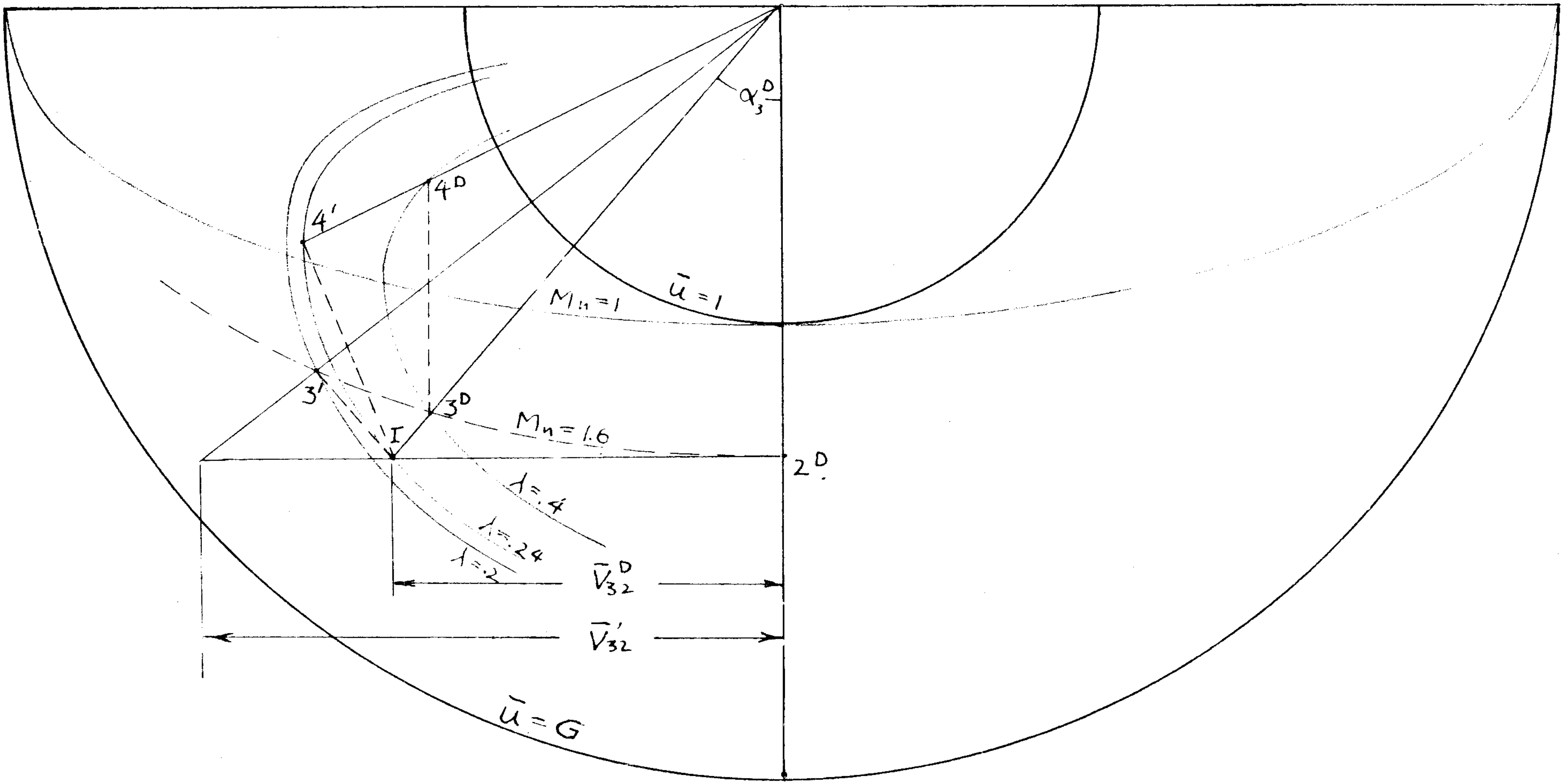


FIGURE V-5

STABILITY OF THE CASCADE IN MOTION

possible trend of stability or instability a large increase in cascade velocity, $\bar{V}'_{32} = 1.85$, was chosen, with the inlet velocity $\bar{u}^D_2$ unchanged. The flow then approaches the cascade at a velocity corresponding to point $3'_2$ and would presumably be turned in the cascade to the design angle α^D_3. The shock required to turn the flow will raise the value of λ in the cascade slightly, as indicated by point I. If it is assumed that the flow leaves the cascade at the design angle α^D_4 the point 4' in the figure is obtained. The position of point 4', i.e. the direction and Mach number of the mean flow leaving the cascade, depends of course on the detailed wave pattern in the cascade and on the back pressure. In general the greater the losses inside the cascade the further will the point 4' move to the right, i.e. toward the stable region. In figure V-5 point 4' lies slightly to the left of 3', which would tend to speed up the cascade still further. A similar graphical construction for the case of decreased cascade velocity shows a similar behaviour of slight instability (see figure V-2). As was pointed out the cascade velocity cannot be reduced very drastically with all other parameters held constant since at the lower stagnation temperature the required mass flow can no longer pass through the cascade.

It may be of interest to mention that the rotational instability of the rotor could apparently be cured by a set of adjustable inlet vanes. If simultaneously with the speeding up of the cascade to V'_{32} the entering flow were turned to the right (by some 20 degrees for the example of figure V-5), the cascade would tend to return to its original velocity, as far as can be concluded from these diagrams.

VI. CONCLUSIONS

On the basis of the one-dimensional area-Mach number relation several interesting properties of supersonic cascade flows were obtained. These should qualitatively also apply to non-uniform cascade flows. It was found that the condition of unit normal Mach number component is of particular importance in connection with maximum pressure ratio and maximum power, but that it is by no means a "critical" condition with respect to mass flow (except of course in cases where the total Mach number also approaches unity).[Δ] In connection with the one-dimensional analysis of cascade problems the reduced velocity diagram was found to be especially useful.

The use of cascades to accomplish nearly isentropic transition from supersonic to subsonic flow in diffusers appears promising. A diffuser consisting of one moving cascade with (possibly) a set of adjustable inlet vanes was discussed. The results of this preliminary analysis were:

i. The cascade would be capable of absorbing small fluctuations of mass flow, cascade velocity and back pressure without requiring any adjustments.

ii. It could be made to operate over an appreciable range of inlet Mach numbers if the cascade velocity or the geometry of the diffuser (or both) were made adjustable.

[Δ] At the entrance and exit planes of a cascade the flow does however change its type as the normal Mach number component passes through unity, because at that value the Mach waves become parallel to the cascade axis. This is discussed in papers dealing with the actual two-dimensional flow pattern in cascades.

iii. The cascade would be slightly unstable with respect to velocity changes,[Δ] but could be made stable by means of adjustable inlet vanes.

Other cascade diffusers than the one discussed appear possible. While the combination turbine and diffuser cascade is the most obvious, a combination compressor and diffuser cascade also appears feasible. However, if the single cascade diffuser proposed in this paper appears practical upon more detailed study, the most promising arrangement would probably be a subsonic or supersonic rotor followed by a diffuser cascade.

A preliminary one-dimensional analysis of cascade arrangements accomplishing both compression and diffusion should be carried out, with proper corrections for shock waves, especially in the off design conditions, being included. Special attention should be given to the problem of starting. If the results are promising, actual blade shapes for the blade root and blade tip should be studied by two-dimensional methods. Experimental verification in a cascade tunnel of the constructed flow patterns would then be in order. Lastly, the effects of radial pressure and velocity gradients should be accounted for as well as possible before an experimental compressor unit is proposed.

[Δ]Of the results given this one is the least reliable.

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