A SLENDER BODY AEROELASTIC TRANSFER FUNCTION INCLUDING THE EFFECTS OF STRUCTURAL FLEXIBILITY AND NONSTATIONARY AERODYNAMICS

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ABSTRACT

On the basis of linearized nonstationary aerodynamics and a normal mode representation of the motion, the aeroelastic transfer functions of a symmetrically oscillating slender body are derived from energy principles.

In the present analysis no structural damping or aerodynamic viscous forces are included, but they are believed not to alter the principal features of the system significantly.

The aerodynamic force expressions, obtained as a side result, turn out to be exceedingly simple in the slender body case.

The final eigenfrequency-analysis and the open loop frequency response of the system chosen in Appendix D indicate that the aerodynamic coupling is low. The eigenfrequencies therefore are accurately obtained from the simple decoupled equations. This also holds for the frequency response in a fairly wide band around each particular resonance frequency.

The magnitude of the response of the elastic coordinates in comparison with rigid body responses at resonance frequencies clearly indicates the importance of elastic effects with respect to closed loop control problems.

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NOMENCLATURE

$A_{11}, \cdots A_{nn}$	=	dynamic coefficients
$A_0, \cdots A_n$ $B_0, \cdots B_n$ $\cdots\cdots$	=	aerodynamic coefficients
a_n	=	complex amplitude of n'th mode
a	=	velocity of sound
c	=	polynomial coefficients
f	=	external force distribution; reduced potential, Appendix II
E	=	Young's modulus
F	=	transfer function
F_1	=	power and control unit transfer function
G	=	weight of body
g	=	function of x determined from boundary conditions
h	=	altitude of flight
i	=	imaginary unit
I	=	cross sectional moment of inertia
I_m	=	mass moment of inertia
J	=	momentum imparted to fluid
ℓ	=	length of body
L	=	cross-force per unit length
m	=	mass per unit length of body
M	=	total mass of body - equations of motion; Mach number, Appendix II
M_{nn}	=	generalized selfmass for n'th mode

$n = 01, 02, 1, 2, \cdots$	=	index indicating mode of vibration
n_o	=	gravitational acceleration
$p = \frac{\partial}{\partial \tau}$	=	differential operator in nondimensional time
p	=	pressure, Appendix II
P_n	=	n'th "velocity polynomial"
q_n	=	n'th generalized coordinate
Q_n	=	n'th generalized force
R	=	radius of body cross section
S	=	cross sectional area
T	=	kinetic energy
t	=	physical time
U	=	flow velocity
V	=	potential energy
v	=	transverse disturbance velocity
r	=	radial coordinate
x	=	longitudinal coordinate
x_o	=	center of gravity coordinate
y	=	transverse coordinate
y_o	=	complex amplitude of translatory mode
Y	=	amplitude of control force, assumed to be real
$\bar{v} = \frac{v}{U}$	=	nondimensional disturbance velocity
$\left.\begin{array}{l} \bar{r} = \frac{r}{\ell} \\ \bar{x} = \frac{x}{\ell} \\ \bar{y} = \frac{y}{\ell} \end{array}\right\}$	=	nondimensional coordinates
ω	=	physical frequency of vibration

$\tau = \frac{Ut}{\ell}$	=	nondimensional time
$\lambda = \frac{\omega \ell}{U}$	=	reduced frequency
ρ	=	air density
ϑ	=	angular coordinate
θ_o	=	complex amplitude of rotary mode
φ	=	anti-symmetric disturbance potential
ϕ_n	=	nth mode of vibration
Δ	=	determinant
ξ	=	point of application of control force
$\delta(x-\xi)$	=	Dirac delta function
$\varkappa$	=	spectral variable of Fourier integral

I. INTRODUCTION

It has been found that the interaction between rigid body modes of vibration and elastic bending modes may play a vital role in the guidance and control problem of guided missiles. Neglecting it may easily lead to serious stability troubles, or at least introduce appreciable guidance errors.

As a first part of a more comprehensive research program centering around this question, the present investigation was undertaken in order to derive the fundamental expressions for the aeroelastic transfer functions of an oscillating slender body, and to make a first estimate of the elastic effects.

In general, an airframe in free flight, treated as an aeroelastic structure, does not lend itself very readily to an exact transfer function analysis, because of the involved nature of the aerodynamic coupling forces. In the slender body case however, it is shown that the complete, nonstationary, linearized force expressions take on an exceedingly simple form, which is readily applicable to conventional transfer function techniques. In fact, these forces also can be shown to be derivable from very simple two dimensional momentum considerations, similar to those employed by R. T. Jones[5] for the stationary case.

The investigation is based upon the technique of normal modes of vibration, which is a particularly convenient method in a case where, as later shown, the aerodynamic coupling is small and only one or two modes have to be included simultaneously.

The investigation so far includes no viscous forces, although these for a slender body in general are of great importance. It is believed however, that the general shape of the lift distribution is fairly well given by the linearized theory, and that the aerodynamic effects hence will be accurately given by the present method.

A. Functional Diagrams

To clarify the mechanism of flexibility in more detail the following functional diagram may be employed.

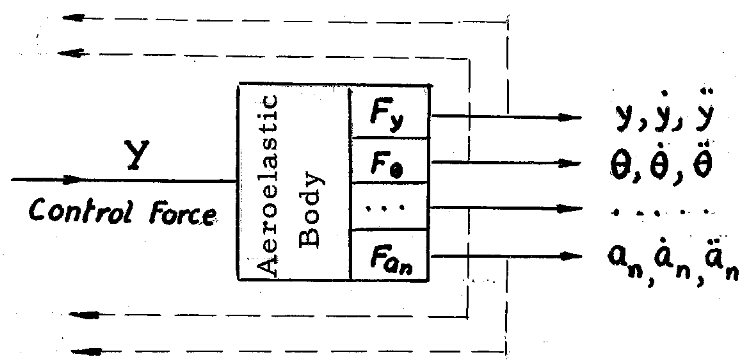

Fig. 1. Aeroelastic Transfer Diagram.

The aeroelastic body thus transforms the input to a series of rigid body and elastic outputs, the latter in general being unwanted as superpositions to controlled variables. As an example, each bending mode is seen to give an additional feedback in the highly simplified acceleration controlled system below in Fig. 2. The system transfer function in this case can easily be shown to be

$$F(p) = \frac{F_1 F_y}{1 + F_1 F_y + F_1 F_{a_1} + \cdots}$$

An indication of the seriousness of these additional feedbacks

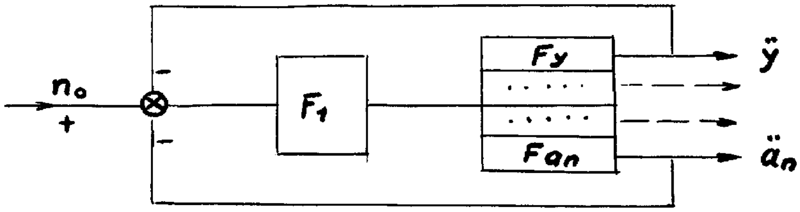

Fig. 2. Simple CG Acceleration Controlled System

is given by the frequency response of the flexible modes as compared with rigid body modes.

The aerodynamic coupling between the modes also will be a subject for investigation in the following.

II. STATEMENT OF THE PROBLEM

The problem treated in this thesis may be stated as follows: given an elastic, slender body of revolution of length ℓ and radius $R(x)$ in a flow of uniform velocity U, performing a small oscillatory motion in a symmetry plane, find the equations of motion of the body, including elastic and nonstationary aerodynamic effects.

The body is treated as a free-free beam, forced by aerodynamic and external forces, and the coordinate system found convenient for the purpose is the one given below[3].

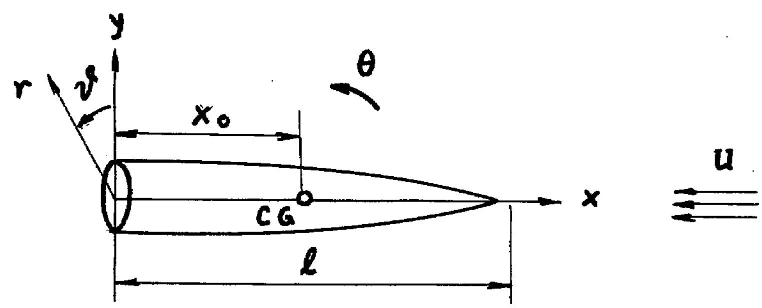

Fig. 3. Coordinate System.

The starting equation for the body motion thus is the familiar beam equation:

$$\frac{\partial^2}{\partial x^2}\left(EI\,\frac{\partial^2 y}{\partial x^2}\right) + m\,\frac{\partial^2 y}{\partial t^2} = f(x,t)$$

under the boundary conditions,

$$\frac{\partial^2 y}{\partial x^2}(0,t) = \frac{\partial^3 y}{\partial x^3}(0,t) = \frac{\partial^2 y}{\partial x^2}(\ell,t) = \frac{\partial^3 y}{\partial x^3}(\ell,t) = 0$$

$f(x,t)$ is composed of aerodynamic and external forces as presented in Part III.

The effect of rotary inertia as well as structural damping and shear have been neglected for simplicity, but may be easily included in an extended analysis.

The displacement of the body as represented by normal modes[1, 2] can be written

$$y = \sum_{n=0}^{\infty} \phi_n\left(\frac{x}{l}\right) q_n(t)$$

Taking the q_n's as generalized coordinates and writing

$$q_n = a_n e^{i\lambda\tau}$$

for a sinusoidal oscillation, Lagranges equation of motion can be easily obtained (Part IV), one equation for each coordinate.

For a body of complex shape the modes may have to be obtained by numerical or iterative methods, but this, according to Rayleigh's principle, still gives an accurate representation of the dynamic properties of the body. For the application in Appendix D a constant cross section approximation is used. This may be fairly reasonable in many cases.

In the derivation of the aerodynamic forces (Appendix D) non-stationary linearized slender body theory is employed. Exactly the same expressions however, are also shown to be derivable from very simple momentum considerations. In the aerodynamic analysis a complete nondimensional representation is used, whereas in the final equations of motion nondimensionality has been retained in time only. The generalized aerodynamic forces are obtained from the principle of virtual work in Part III.

The resulting linear frequency response equations have then been converted, without further proof, to transfer functions in non-dimensional time, by replacing $i\lambda$ with the operator $p = \frac{\partial}{\partial \tau}$.

III. AERODYNAMIC FORCES

Using the results derived in Appendix B in nondimensional form, it is easy to show that the cross-force per unit length of an oscillating slender body, in terms of physical space coordinates but nondimensional time and reduced frequency, may be written as

$$L = \rho U^2 \sum_{n=0}^{\infty} a_n \left[\frac{i\lambda}{\ell} S(x) P_n(x) - \frac{d}{dx}\left\{ S(x) P_n(x) \right\} \right] e^{i\lambda\tau},$$

where

$a_{01} = y_0$	Rigid body translation
$a_{02} = \theta_0$	Rigid body rotation
....	
a_n	Flexible modes

$$P_{01} = -\frac{i\lambda}{\ell}$$

$$P_{02} = 1 - i\lambda\left(\frac{x - x_0}{\ell}\right)$$

....

$$P_n = \frac{1}{\ell}\left(\frac{\partial \phi_n}{\partial x/\ell} - i\lambda\phi_n\right)$$

The external forcing function has been assumed in the form

$$\delta(x-\xi)\, Y e^{i\lambda\tau}$$

which represents an oscillating, concentrated force at $x = \xi$. It can be thought of, for instance, as produced by the deflection of a control surface, although nothing has been assumed about its origin in the present paper. In the same manner, any number of concentrated aerodynamic forces may be considered. In the numerical application in Appendix D, the control force has been assumed to be located at the rear end of the body.

IV. LAGRANGE'S EQUATIONS OF MOTION

As mentioned in Part II, the normal mode representation of the motion may be written

$$y = \sum_{n=0}^{\infty} \phi_n \left(\frac{x}{l}\right) q_n(t)$$

With the kinetic and potential energies expressed in terms of the q_n's as generalized coordinates, Lagrange's equations of motion are then written down.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_n}\right) + \frac{\partial V}{\partial q_n} = Q_n$$

<u>Kinetic energy</u> :

$$T = \frac{1}{2} \sum_{n=0}^{\infty} \int_0^l (\phi_n \dot{q}_n)^2 m \, dx = \frac{1}{2} \sum_{n=0}^{\infty} M_{nn} \dot{q}_n^2$$

where

$$M_{01} = \int_0^l m \, dx = M \qquad \text{total mass}$$

$$M_{02} = \int_0^l (x - x_0)^2 m \, dx = I_m \qquad \text{mass moment of inertia}$$

$$M_{nn} = \int_0^l \phi_n^2 m \, dx \qquad \text{generalized selfmass for } n^{th} \text{ mode}$$

<u>Potential energy:</u>

Neglecting gravity we have

$$V = \frac{1}{2} \sum_{n=0}^{\infty} \int_0^l EI (\phi_n'')^2 q_n^2 \, dx$$

But from Rayleigh's principle we know that

$$\omega_n^2 = \frac{\int_0^l EI (\phi_n'')^2 dx}{M_{nn}}$$

Hence the potential energy can be written as

$$V = \frac{1}{2}\sum_{n=0}^{\infty}\omega_n^2 M_{nn} q_n^2 \quad ,$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_n}\right) = M_{nn}\ddot{q}_n$$

$$\frac{\partial V}{\partial q_{01,02}} = 0 \qquad \text{neglecting gravity}$$

$$\frac{\partial V}{\partial q_n} = \omega_n^2 M_{nn} q_n$$

Assuming a sinuoidal time dependence of q_n we have, in terms of nondimensional time,

$$q_{01} = y_0 e^{i\lambda\tau} \qquad \text{Rigid body translation}$$

$$q_{02} = \theta_0 e^{i\lambda\tau} \qquad \text{Rigid body rotation}$$

$$q_n = a_n e^{i\lambda\tau} \qquad \text{Flexible modes}$$

<u>Generalized Forces:</u>

By the method of virtual work we have

$$Q_n \Delta q_n = \int_0^{\ell}\left[L(x,\tau) + \delta(x-\xi)Y(\tau)\right]\Delta q_n \phi_n \, dx$$

or

$$Q_n = \int_0^{\ell} L(x,\tau)\phi_n \, dx + Y(\tau)\phi_n(\xi)$$

Inserting the aerodynamic forces given on page 7 and using the dummy variable k in the summation gives

$$Q_n = \rho U^2 e^{i\lambda\tau}\sum_{k=0}^{\infty} a_k \int_0^{\ell}\left[\frac{i\lambda}{\ell} S(x) P_k(x) - \frac{d}{dx}\left\{S(x) P_k(x)\right\}\right]\phi_n \, dx + $$
$$+ Y\phi_n(\xi) e^{i\lambda\tau}$$

or, written out explicitly, the expression,

$$Q_n = \rho U^2 e^{i\lambda\tau} \int_0^{\ell} \left\{ \begin{array}{l} \frac{y_0}{\ell}\left[\frac{\lambda^2}{\ell} S(x) + i\lambda \frac{d}{dx} S(x)\right] + \\ +\theta_0\left[\frac{i\lambda}{\ell}\left\{1 - i\lambda\left(\frac{x-x_0}{\ell}\right)\right\} S(x) - \frac{d}{dx}\left\{S(x)\left(1 - i\lambda\frac{(x-x_0)}{\ell}\right)\right\}\right] + \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ + \frac{a_K}{\ell}\left[\frac{i\lambda}{\ell}\left(\frac{\partial\phi_K}{\partial x/\ell} - i\lambda\phi_K\right) S(x) - \frac{d}{dx}\left\{S(x)\left(\frac{\partial\phi_K}{\partial x/\ell} - i\lambda\phi_K\right)\right\}\right] \end{array} \right\} \phi_n \, dx +$$

$$+ Y\phi_n(\xi) e^{i\lambda\tau}$$

Equations of motion:

$$-\left(\frac{U}{\ell}\right)^2 M\lambda^2 y_0 \qquad -\rho U^2 \sum_{K=0}^{\infty} a_K \int_0^{\ell} \left[\frac{i\lambda}{\ell} S(x) P_K(x) - \frac{d}{dx}\left\{S(x) P_K(x)\right\}\right] dx = Y$$

$$-\left(\frac{U}{\ell}\right)^2 I_m \lambda^2 \theta_0 \qquad -\rho U^2 \sum_{K=0}^{\infty} a_K \int_0^{\ell} \left[\frac{i\lambda}{\ell} S(x) P_K(x) - \frac{d}{dx}\left\{S(x) P_K(x)\right\}\right](x - x_0)\, dx = Y(\xi - x_0)$$

$$-\left(\frac{U}{\ell}\right)^2 M_{nn} \lambda^2 a_n + \omega_n^2 M_{nn} a_n - \rho U^2 \sum_{K=0}^{\infty} a_K \int_0^{\ell} \left[\frac{i\lambda}{\ell} S(x) P_K(x) - \frac{d}{dx}\left\{S(x) P_K(x)\right\}\right] \phi_n \, dx = Y\phi_n(\xi)$$

Expanding the sum and collecting terms belonging to each coordinate the equations can be written in the form

$$A_{11} y_0 + A_{12} \theta_0 + A_{13} a_1 + \cdots\cdots A_{1,n+2} a_n = Y$$

$$A_{21} y_0 + A_{22} \theta_0 + A_{23} a_1 + \cdots\cdots A_{2,n+2} a_n = Y(\xi - x_0)$$

$$A_{n1} y_0 + A_{n2} \theta_0 + A_{n3} a_1 + \cdots\cdots A_{n,n+2} a_n = Y\phi_n(\xi)$$

where

$$A_{11} = -\left(\frac{U}{\ell}\right)^2 M\lambda^2 - A_{01}\lambda^2 + B_{01}\, i\lambda$$

$$A_{12} = -C_{01}\lambda^2 + D_{01}\, i\lambda + E_{01}$$

$$A_{13} = -F_{101}\lambda^2 + G_{101}\, i\lambda + H_{101}$$

$$A_{1n} = \cdots\cdots\cdots\cdots$$

$$A_{21} = -A_{02}\lambda^2 + B_{02}\, i\lambda$$

$$A_{22} = -\left(\frac{U}{\ell}\right)^2 I_m\lambda^2 - C_{02}\lambda^2 + D_{02}\, i\lambda + E_{02}$$

$$A_{23} = -F_{102}\lambda^2 + G_{102}\, i\lambda + H_{102}$$

$$A_{2n} = \cdots\cdots\cdots\cdots$$

$$A_{31} = -A_1\lambda^2 + B_1\, i\lambda$$

$$A_{32} = -C_1\lambda^2 + D_1\, i\lambda + E_1$$

$$A_{33} = -\left(\frac{U}{\ell}\right)^2 M_{11}\lambda^2 + \omega_1^2 M_{11} - F_{11}\lambda^2 + G_{11}\, i\lambda + H_{11}$$

$$A_{3n} = \cdots\cdots\cdots\cdots$$

$A_{01} = \frac{\varrho U^2}{\ell^2}\int_0^\ell S(x)\,dx$	$B_{01} = -\frac{\varrho U^2}{\ell}\int_0^\ell S'(x)\,dx$	
$C_{01} = \frac{\varrho U^2}{\ell^2}\int_0^\ell S(x)(x-x_0)\,dx$	$D_{01} = -\frac{\varrho U^2}{\ell}\int_0^\ell \left[S(x) + \frac{d}{dx}\left(S(x)(x-x_0)\right)\right]dx$	$E_{01} = \varrho U^2\int_0^\ell S'(x)\,dx$
$F_{01} = \frac{\varrho U^2}{\ell^2}\int_0^\ell S(x)\,\phi_1\,dx$	$G_{01} = -\frac{\varrho U^2}{\ell}\int_0^\ell \left[\ell S(x)\frac{\partial \phi_1}{\partial x/\ell} + \frac{d}{dx}\left(S(x)\phi_1\right)\right]dx$	$H_{01} = \frac{\varrho U^2}{\ell}\int_0^\ell \frac{d}{dx}\left[S(x)\frac{\partial \phi_1}{\partial x/\ell}\right]dx$
$A_{02} = \frac{\varrho U^2}{\ell^2}\int_0^\ell S(x)(x-x_0)\,dx$	$B_{02} = -\frac{\varrho U^2}{\ell}\int_0^\ell S'(x)(x-x_0)\,dx$	
$C_{02} = \frac{\varrho U^2}{\ell^2}\int_0^\ell S(x)(x-x_0)^2\,dx$	$D_{02} = -\frac{\varrho U^2}{\ell}\int_0^\ell \left[S(x) + \frac{d}{dx}\left(S(x)(x-x_0)\right)\right](x-x_0)\,dx$	$E_{02} = \varrho U^2\int_0^\ell S'(x)(x-x_0)\,dx$
$F_{02} = \frac{\varrho U^2}{\ell^2}\int_0^\ell S(x)(x-x_0)\phi_1\,dx$	$G_{02} = -\frac{\varrho U^2}{\ell}\int_0^\ell \left[\ell S(x)\frac{\partial \phi_1}{\partial x/\ell} + \frac{d}{dx}\left(S(x)\phi_1\right)\right](x-x_0)\,dx$	$H_{02} = \frac{\varrho U^2}{\ell}\int_0^\ell \frac{d}{dx}\left(S(x)\frac{\partial \phi_1}{\partial x/\ell}\right)(x-x_0)\,dx$

$A_1 = \frac{\rho U^2}{\ell^2}\int_0^\ell S(x)\phi_1\,dx$	$B_1 = -\frac{\rho U^2}{\ell}\int_0^\ell S'(x)\phi_1\,dx$	
$C_1 = \frac{\rho U^2}{\ell^2}\int_0^\ell (x-x_0)S(x)\phi_1\,dx$	$D_1 = -\frac{\rho U^2}{\ell}\int_0^\ell \left[S(x)+\frac{d}{dx}\left(S(x)(x-x_0)\right)\right]\phi_1\,dx$	$E_1 = \rho U^2\int_0^\ell S'(x)\phi_1\,dx$
$F_1 = \frac{\rho U^2}{\ell^2}\int_0^\ell S(x)\phi_1^2\,dx$	$G_1 = -\frac{\rho U^2}{\ell}\int_0^\ell \left[\ell S(x)\frac{\partial\phi_1}{\partial x/\ell}+\frac{d}{dx}\left(S(x)\phi_1\right)\right]\phi_1\,dx$	$H_1 = \frac{\rho U^2}{\ell}\int_0^\ell \frac{d}{dx}\left(S(x)\frac{\partial\phi_1}{\partial x/\ell}\right)\phi_1\,dx$
$A_n = \frac{\rho U^2}{\ell^2}\int_0^\ell S(x)\phi_n\,dx$	$B_n = -\frac{\rho U^2}{\ell}\int_0^\ell S'(x)\phi_n\,dx$	
$C_n = \frac{\rho U^2}{\ell^2}\int_0^\ell (x-x_0)S(x)\phi_n\,dx$	$D_n = -\frac{\rho U^2}{\ell}\int_0^\ell \left[S(x)+\frac{d}{dx}\left(S(x)(x-x_0)\right)\right]\phi_n\,dx$	$E_n = \rho U^2\int_0^\ell S'(x)\phi_n\,dx$
$F_{kn} = \frac{\rho U^2}{\ell^2}\int_0^\ell S(x)\phi_k\phi_n\,dx$	$G_{kn} = -\frac{\rho U^2}{\ell}\int_0^\ell \left[\ell S(x)\frac{\partial\phi_k}{\partial x/\ell}+\frac{d}{dx}\left(S(x)\phi_k\right)\right]\phi_n\,dx$	$H_{kn} = \frac{\rho U^2}{\ell}\int_0^\ell \frac{d}{dx}\left(S(x)\frac{\partial\phi_k}{\partial x/\ell}\right)\phi_n\,dx$

V. RESULTS

The set of $n+2$ nonhomogeneous linear equations in the $n+2$ unknown y_0, θ_0, $a_1 \cdots\cdots a_n$ derived in Section III may be conveniently written in matrix form

$$(A)\vec{a} = \vec{Q}$$

Solving for each particular output by means of Cramer's rule yields

$$y_0 = \frac{\Delta_{01}}{\Delta}$$

$$\theta_0 = \frac{\Delta_{02}}{\Delta}$$

$$a_1 = \frac{\Delta_1}{\Delta}$$

$$\vdots$$

$$a_n = \frac{\Delta_n}{\Delta}$$

$$\vdots$$

Here Δ is the determinant of the homogeneous left-hand side and $\Delta_{01} \cdots\cdots \Delta_n$.. are the determinants obtained by replacing the column corresponding to each particular coordinate by the right-hand forcing column.

A. Frequency Response

The above matrix equation completely determines the frequency response of the system. The solutions y_0, θ_0, $a_1, \cdots a_n$ thus are the coupled responses of the generalized coordinates, which are in general complex functions $F_n(i\lambda)$ of the reduced frequency. Expressing the frequency response in terms of magnitude and phase thus gives

$$|a_n| = |F_n(i\lambda)| \quad , \qquad \arg\, a_n = \arg\, F_n(i\lambda)$$

B. Transfer Function

The transfer function in the usual tense in nondimensional time is obtained by replacing $i\lambda$ by the operator $p = \frac{\partial}{\partial \tau}$. This is by no means an essential step in the present investigation, but gives a more compact notation, and is particularly convenient when dealing with the system as part of a complete control system.

C. Eigenfrequencies

The eigenfrequencies of the coupled system are given by the equation

$$\Delta = 0$$

This in general gives a polynomial of an order in p^2 equal to the total number of modes included. Under the assumption of no gravitation however, one pair of roots will be $p_{1,2} = 0$ for the system considered in this thesis. The order of the characteristic equation thus will be reduced by two.

An application of the Root Locus method for determining the roots of any polynomial is given in Appendix C.

D. Application to a Particular Configuration

The expressions given so far are completely general, containing any desired number of elastic modes. In applying them to a practical problem it is found that the elastic modes have very distinct frequency ranges of importance, and that the aerodynamic coupling in general is low.

Due to these facts the application given in Appendix D contains only one elastic mode, the frequency range considered being only $0.01 \leqslant \lambda \leqslant 1.0$. The particular configuration chosen is intended to resemble a typical high speed missile.

The Bode diagrams of the frequency responses are given in Figures 9 to 15.

VI. CONCLUDING REMARKS

The results of this thesis clearly show the importance of the problem as stated, and the relative ease, with which both structural features and nonstationary aerodynamics may be incorporated in the dynamic transfer functions of a slender body.

An examination of the coefficient integrals on pages 12 and 13 indicates, that very roughly one half of the aerodynamic force is contributed by nonstationary effects, the other half being the quasi-steady part. A use of quasisteady theory therefore would not give a very true picture. The expressions for the complete forces however are simple enough in the slender body case to be included completely.

From the evaluation of the characteristic equation and the frequency response of the particular configuration in Appendix D the following points are evident.

1. The aerodynamic coupling in general is low, the translatory mode in particular not influencing higher modes, and the eigenfrequencies of the coupled system are well separated. This however is due to the body being an inefficient aerodynamic force producer but possessing fairly efficient structural properties, which may not always be true in a practical case.
2. The effect of aerodynamic coupling in the case considered is to slightly increase damping and decrease the frequency of the free body roots.
3. Under the assumption of low coupling, the eigenfrequencies are accurately obtained from the "stripped down" equations for each

particular mode. Correspondingly, for forced motion, the response of each coordinate is obtained in a fairly wide band around the respective resonance frequency from the simple decoupled equations.

No attempt has been made in this thesis to make any general statements about the behavior of the system as part of a closed loop control system, this question being too sensitive to individual control system characteristics. From the response calculations in Appendix D, however, it is evident that the local 1st bending velocities and accelerations are dominating over other modes at bending resonance frequency. An angular velocity - or translatory acceleration feedback without consideration of bending, would therefore most certainly give rise to control troubles. In the numerical example given in Appendix D, the 1st bending acceleration at the CG is of the order thirty times translatory acceleration, which however (to a certain extent) also depends upon the neglected structural damping.

In accordance with the above results the closed loop analysis of the flexible, slender body system most probably would be put on a modal analysis basis, containing simple decoupled or partly coupled modes in each frequency range. This will be the subject for an extended investigation.

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APPENDIX A

Normal Modes Representation of a Free-Free Beam

The technique of expressing the deflection of a vibrating beam, governed by the equations

$$\frac{\partial^2}{\partial x^2}\left(EI\,\frac{\partial^2 y}{\partial x^2}\right) + m\,\frac{\partial^2 y}{\partial t^2} = f(x,t)$$

$$\frac{\partial^2 y}{\partial x^2}(0,t) = \frac{\partial^3 y}{\partial x^3}(0,t) = \frac{\partial^2 y}{\partial x^2}(\ell,t) = \frac{\partial^3 y}{\partial x^3}(\ell,t) = 0$$

in a series of orthogonal eigen functions is well known and will not be treated here in any detail(1).

Thus the solution of the beam equation will be written directly

$$y = \sum_{n=0}^{\infty} \phi_n\left(\frac{x}{\ell}\right) q_n(t)$$

The first two modes represent rigid body translation and rotation, the higher modes representing bending deflections.

In the present numerical example a beam of constant cross sectional properties has been used for convenience, the modes and frequencies of this being available in tabulated form(2).

The first two bending modes and corresponding frequencies are given in Fig. 7.

APPENDIX B

Nonstationary Slender Body Lift

Nondimensional Representation

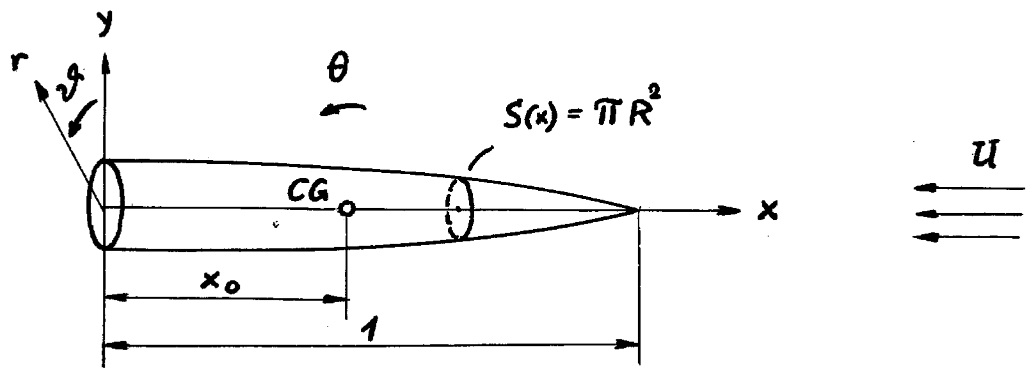

Fig. 4. Coordinate System and Nondimensional Body Dimensions.

Dimensionless Quantities	Physical Quantities
$\bar{x}$	$\ell\bar{x} = x$
$\bar{y}$	$\ell\bar{y} = y$
$\bar{r}$	$\ell\bar{r} = r$
$\overline{S(x)}$	$\ell^2\overline{S(x)} = S(x)$
$\tau = \dfrac{Ut}{\ell}$	$t = \dfrac{\tau\ell}{U}$
$\lambda = \dfrac{\omega\ell}{U}$	$\omega = \dfrac{\lambda U}{\ell}$

For simplicity, the bars are omitted during the following derivation of the lift forces.

Nonstationary, Linear Wave Equation

$$(1-M^2)\varphi_{xx} + \varphi_{rr} + \frac{1}{r}\varphi_r + \frac{1}{r^2}\varphi_{\vartheta\vartheta} = M^2(\varphi_{\tau\tau} + 2\varphi_{x\tau})$$

We are here considering only the antisymmetric part of the disturbance potential, because we are interested in the cross-force only.

Boundary Conditions

$$U v_r(x) = -\frac{1}{\ell}\left(\frac{\partial\varphi}{\partial r}\right)_{r=R}$$

body surface is impermeable.

Body Motion as Represented by Normal Modes

$$y = \sum_{n=0}^{\infty} a_n \phi_n(x) e^{i\lambda\tau}$$

$a_{01} = y_0$	Rigid body translation
$a_{02} = \theta_0$	Rigid body rotation
. . . .	
a_n	Flexible modes

$\phi_{01} = 1$

$\phi_{02} = (x - x_0)$

. . . .

ϕ_n

The transverse physical velocity is then

$$-\frac{U}{\ell}\ell\frac{\partial y}{\partial \tau} + U\frac{\partial y}{\partial x} = U\left(\frac{\partial y}{\partial x} - \frac{\partial y}{\partial \tau}\right)$$

This is equal to the y -component of the inflow which is accordingly

$$-U v_r(x) = U\left(\frac{\partial y}{\partial x} - \frac{\partial y}{\partial \tau}\right)\cos\vartheta$$

or in normal modes

$$v_r(x) = -e^{i\lambda\tau}\cos\vartheta\sum_{n=0}^{\infty} a_n\left(\frac{\partial \phi_n}{\partial x} - i\lambda\phi_n\right)$$

or

$$v_r(x) = -e^{i\lambda\tau}\cos\vartheta\sum_{n=0}^{\infty} a_n P_n(x)$$

$$P_{01} = -i\lambda$$

where
$$P_{02} = 1 - i\lambda(x - x_0)$$

$$\cdots\cdots\cdots$$

$$P_n = \frac{\partial \phi_n}{\partial x} - i\lambda\phi_n$$

From the boundary conditions it is seen, that $v_r(x)$ depends upon ϑ only as $\cos\vartheta$. Hence, assuming that

$$\varphi(x, r, \vartheta, \tau) = U\ell e^{i\lambda\tau + ikx} f(x, r)\cos\vartheta$$

and calling

$$B = \sqrt{M^2 - 1} \qquad \beta = \sqrt{1 - M^2} \quad , \quad \alpha = \frac{\lambda M}{B} \quad , \quad k = -\frac{\lambda^2 M^2}{1 - M^2}$$

the wave equation transforms into the following equation for the reduced potential,

$$(1 - M^2) f_{xx} + f_{rr} + \frac{1}{r} f_r - \frac{1}{r^2} f = -\frac{\lambda^2 M^2}{1 - M^2} f$$

This can be solved for both supersonic and subsonic flow by Fourier transform methods giving identical results. We recapitulate the first case

$$M > 1 \quad , \qquad F(\kappa, r) = \int_0^\infty e^{-i\kappa x} f(x, r)\, dx$$

Then,

$$F_{rr} + \frac{1}{r} F_r + \left(B^2\kappa^2 - \alpha^2 - \frac{1}{r^2}\right) F = 0$$

The general solution of this can be expressed in terms of Hankel functions of order 1 as follows:

$$F = C_1 H_1^{(1)}\left(r\sqrt{B^2\varkappa^2-\alpha^2}\right) + C_2 H_1^{(2)}\left(r\sqrt{B^2\varkappa^2-\alpha^2}\right)$$

Considering only out going waves, and retaining the leading term only, or observing that any integral close to $r = 0$ must have this form, one arrives at the expression

$$F(\varkappa, r) = G(\varkappa)\frac{1}{r}\cos\vartheta + O(r\ln r)\cos\vartheta$$

and hence

$$f(x,r) = g(x)\frac{1}{r}\cos\vartheta + O(r\ln r)\cos\vartheta$$

The complete potential close to $r=0$ then will be

$$\varphi = U\ell e^{i\lambda\tau + ikx}\left[g(x)\frac{1}{r} + O(r\ln r)\right]\cos\vartheta$$

$g(x)$ can be determined from the boundary conditions

$$v_{r_n}(x) = e^{i\lambda\tau + ikx}\left[g_n(x)\frac{1}{r^2} + O(\ln r)\right]\cos\vartheta$$

$$= a_n P_n(x) e^{i\lambda\tau}\cos\vartheta$$

for each mode. Therefore, retaining leading terms,

$$a_n P_n(x) \cong e^{ikx} g_n(x)\frac{1}{r^2}$$

adding over the modes and inserting in the expression for φ gives

$$\varphi = -\left(\frac{U\ell}{\pi R}\right) S(x)\sum_{n=0}^{\infty} a_n P_n(x) e^{i\lambda\tau}\cos\vartheta$$

This expression contains Munk's formula for the potential. It is observed, that under the assumptions made, φ is independent of M and λ except for the factor $e^{i\lambda\tau}$.

Pressure

$$p-p_o = -\frac{\rho U^2}{\ell}\left(\frac{\partial\varphi}{\partial\tau} - \frac{\partial\varphi}{\partial x} + \left(\frac{\partial\varphi}{\partial r}\right)^2\right)$$

Due to symmetry in the present case, the last term gives no contribution.

$$p-p_o = \frac{\rho U^2}{\pi R}\sum_{n=0}^{\infty} a_n\left[i\lambda S(x)P_n(x) - \frac{d}{dx}\left\{S(x)P_n(x)\right\}\right]e^{i\lambda\tau}\cos\vartheta$$

Integrating around the circumference finally gives the cross-force per unit length

$$L = \rho U^2\ell\sum_{n=0}^{\infty} a_n\left[i\lambda S(x)P_n(x) - \frac{d}{dx}\left\{S(x)P_n(x)\right\}\right]e^{i\lambda\tau}$$

Comparison with Simple Momentum Theory

It will now be shown that the above lift expression also may be derived from very simple momentum considerations.

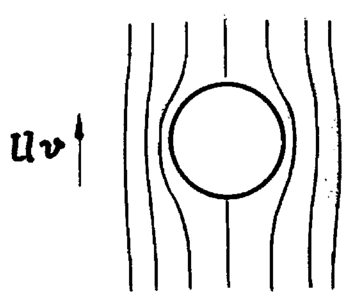

Fig. 5. Flow in a Cross Sectional Plane.

Consider a cross section of the body. The momentum imparted to the fluid is

$$J = -\rho U \ell^2 S(x)\, v(x)$$

where $\ell^2 \rho S(x)$ is the additional apparent mass. The cross force per unit length will be given by the time derivative of the momentum

$$L = -\frac{DJ}{Dt} = \rho U \ell^2 \frac{D}{Dt}\left(S(x)\, v(x)\right)$$

It is observed, that the derivative to be taken here is the total derivative, which is the sum of the local and convective derivatives.

$$\frac{D(S\,v)}{Dt} = \frac{U}{\ell}\,\frac{\partial(S v)}{\partial\tau} + U \circ \operatorname{grad}(S\,v)$$

The nondimensional local velocity v then will be expressed in terms of normal modes as

$$v = \frac{\partial y}{\partial x} - \frac{\dot{y}}{U} = \sum_{n=0}^{\infty} a_n \left(\frac{\partial \phi_n}{\partial x} - i\lambda\phi_n\right) e^{i\lambda\tau}$$

$$= \sum_{n=0}^{\infty} a_n\, P_n(x)\, e^{i\lambda\tau}$$

Inserting these expressions in the expression for L gives

$$L = \rho U \ell^2 \sum_{n=0}^{\infty} a_n \left[\frac{i\lambda U}{\ell} S(x) P_n(x) - \frac{U}{\ell}\frac{d}{dx}\left\{S(x) P_n(x)\right\}\right] e^{i\lambda\tau}$$

or

$$L = \rho U^2 \ell \sum_{n=0}^{\infty} a_n \left[i\lambda\, S(x) P_n(x) - \frac{d}{dx}\left\{ S(x) P_n(x) \right\} \right] e^{i\lambda\tau}$$

But this is exactly the expression derived from linear theory on page 25.

APPENDIX C

Factorization of Polynomials by Means of the Root Locus Method

The Root Locus Method furnishes a convenient way of factoring any polynomials with complex or real root . Given

$$c_n p^n + c_{n-1} p^{n-1} + c_{n-2} p^{n-2} + \cdots\cdots c_1 p + c_0 = 0$$

Grouping the first three terms together gives

$$c_n p^{n-2}\left(p^2 + \frac{c_{n-1}}{c_n} p + \frac{c_{n-2}}{c_n}\right) + \cdots\cdot c_1 p + c_0 = 0$$

The bracket is a quadratic and can be factored analytically

$$c_n p^{n-2}(p-\gamma_1)(p-\gamma_2) + c_{n-3} p^{n-3} + \cdots c_1 p + c_0 = 0$$

Again grouping the two first terms together gives

$$c_{n-3} p^{n-3}\left[\frac{c_n}{c_{n-3}} p(p-\gamma_1)(p-\gamma_2) + 1\right] + \cdots c_1 p + c_0 = 0$$

The bracket now can be factored by tracing the root locus in the complex plane in accordance with the usual root locus technique[6] and thus determining the roots β of the bracket expression. It gives

$$c_n p^{n-3}(p-\beta_1)(p-\beta_2)(p-\beta_3) + \cdots\cdot c_1 p + c_0 = 0$$

Continuing in the same way, each step adds one zero to the factored expression, until the final result is

$$c_n(p-\alpha_1)(p-\alpha_2)(p-\alpha_3)(\cdots\cdots\cdots)(p-\alpha_n) = 0$$

It may be pointed out that by analythical computation of all characteristic quantities of the locus — cross over points, asymptotes, etc. the accuracy of the method can be substantially improved.

In the present application, the characteristic quartic of the system of equations required the plotting of two locii, which are shown in Fig. 8.

APPENDIX D

Application of the Theory to a Particular Configuration

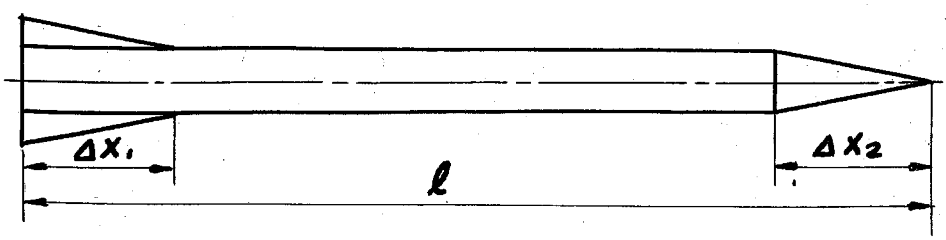

Fig. 6. Schematic Missile Configuration

Geometry

$\ell = 7.5\ m$
$\Delta x_1 = 1\ m$
$\Delta x_2 = 1\ m$
$x_0 = 3.5\ m$

$R_{2\,max} = 0.25\ m$
$R_{1\,max} = 0.5\ m$

$S_2(x) = 0.196\,(\ell - x)^2$

$S_2'(x) = -0.392\,(\ell - x)$

$S_1(x) = 0.588\,(1-x)^2 + 0.196$

$S_1'(x) = -1.176\,(1-x)$

Mass Properties

$G = 3000\ kg$
$m = 40\ kg\,m^{-2}s^2$
$\omega_1 = 31.4\ s^{-1}$

$M = 305\ kg\,m^{-1}s^2$
$I_m = 1400\ kg\,s^2 m$
$M_{11} = 270\ kg\,m^{-1}s^2$

Flight Reference Conditions

$$h = 10000 \text{ m}$$
$$\rho = 0.0371 \text{ kg s}^2 \text{m}^{-4}$$
$$a = 295 \text{ m/s}$$
$$U = 590 \text{ m/s (Mach number 2)}$$

In the present case only the fundamental elastic mode has been considered.

The constants in the equations of motion, computed from the integrals on pages 12 and 13 are

$A_{01} = 350$	$B_{01} = 1350$	
$C_{01} = -174$	$D_{01} = -7600$	$E_{01} = -10100$
$F_{01} \cong 0$	$G_{01} = 3320$	$H_{01} = 12400$

$A_{02} = -174$	$B_{02} = -2330$	$E_{02} = 17500$
$C_{02} = 1980$	$D_{02} = 16980$	
$F_{02} = -672$	$G_{02} = -16300$	$H_{02} = -40500$

$A_1 \cong 0$	$B_1 = 1950$	
$C_1 = -672$	$D_1 = -4670$	$E_1 = -14650$
$F_1 = 350$	$G_1 = 4320$	$H_1 = 8750$

Inserting these constants in the equations of motion, and changing the scale of y_0, θ_0, and a_1 by a factor of 10^3, which is the same as measuring y_0 and a_1 in *mm* and θ_0 in *rad* $\times 10^{-3}$ yields the following set of equations

$$(1885p^2+1.35p)y_0 - (0.17p^2+7.60p+10.10)\theta_0 + (3.32p+12.40)a_1 = Y$$

$$-(.17p^2+2.33p)y_0 + (8660p^2+16{,}98p+17.50)\theta_0 - (0.672p^2+16.30p+40{,}5)a_1 = -3.75Y$$

$$1.95py_0 - (0.672p^2+4{,}67p+14{,}65)\theta_0 + (1670p^2+4{,}33p+274{,}75)a_1 = 2Y$$

Frequency Range of Interest

The range λ = 0, 01 to 1.0, which is well below the natural frequency of the angular motion and well above the first bending frequency, has been considered. This immediately makes the off diagonal p^2 terms and the lower two py_0 terms negligible.

Natural Frequencies

The characteristic equation in the present case reduces to the following quartic

$$273000p^4 + 1443p^3 + 45300p^2 + 111{,}4p + 79.3 = 0$$

Factorization by the method given in Appendix III gives the roots

$$p_{1,2} = 0$$

$$p_{3,4} = -0.00124 \pm 0.042\, i$$ Rigid body rotation

$$p_{5,6} = -0.0014 \pm 0.405\, i$$ First bending mode

The locii employed are given in Fig. 8.

Frequency Response

The responses of y_o, θ_o and a_1 to a unit simusoidal force are given by the equations below

$$y_o = -\left[\frac{3.75(10.1+7.6 i\lambda)}{17.5-8660\lambda^2+16.98 i\lambda} + \frac{2(12.4+3.23 i\lambda)}{274.5-1670\lambda^2+4.32 i\lambda} - 1\right]\frac{1}{-1885\lambda^2+13.5 i\lambda}$$

$$\theta_o = -\frac{3.75(274.5-1670\lambda^2+4.32 i\lambda)-2(40.50+16.3 i\lambda)}{(17.5-8660\lambda^2+16.98 i\lambda)(274.5-1670\lambda^2+4.32 i\lambda)-(40.5+16.3 i\lambda)(14.65+4.67 i\lambda)}$$

$$a_1 = \frac{2(17.5-8660\lambda^2+16.98 i\lambda)-3.75(14.6+4.67 i\lambda)}{(17.5-8660\lambda^2+16.98 i\lambda)(274.5-1670\lambda^2+4.32 i\lambda)-(40.5+16.3 i\lambda)(14.65+4.67 i\lambda)}$$

The above equations for θ_0 and a_1 are obtained by solving for θ_0 and a_1 simultaneously but disregarding the negligible influence of y_0 on these.

It is found that the mutual coupling between θ_0 and a_1 is negligible in a wide frequency band around the particular resonance frequency. Further away from this it is important, but the magnitude of the response itself is very small. In accordance with this, the equation for y_0 contains only the simple "decoupled" responses of θ_0 and a_1, which is found to be a good approximation indeed.

By multiplying the responses with $i\lambda$ and $(i\lambda)^2$ the velocity-and acceleration responses are obtained.

The Bode diagrams of the above frequency responses are given in Figs. 9 to 16.

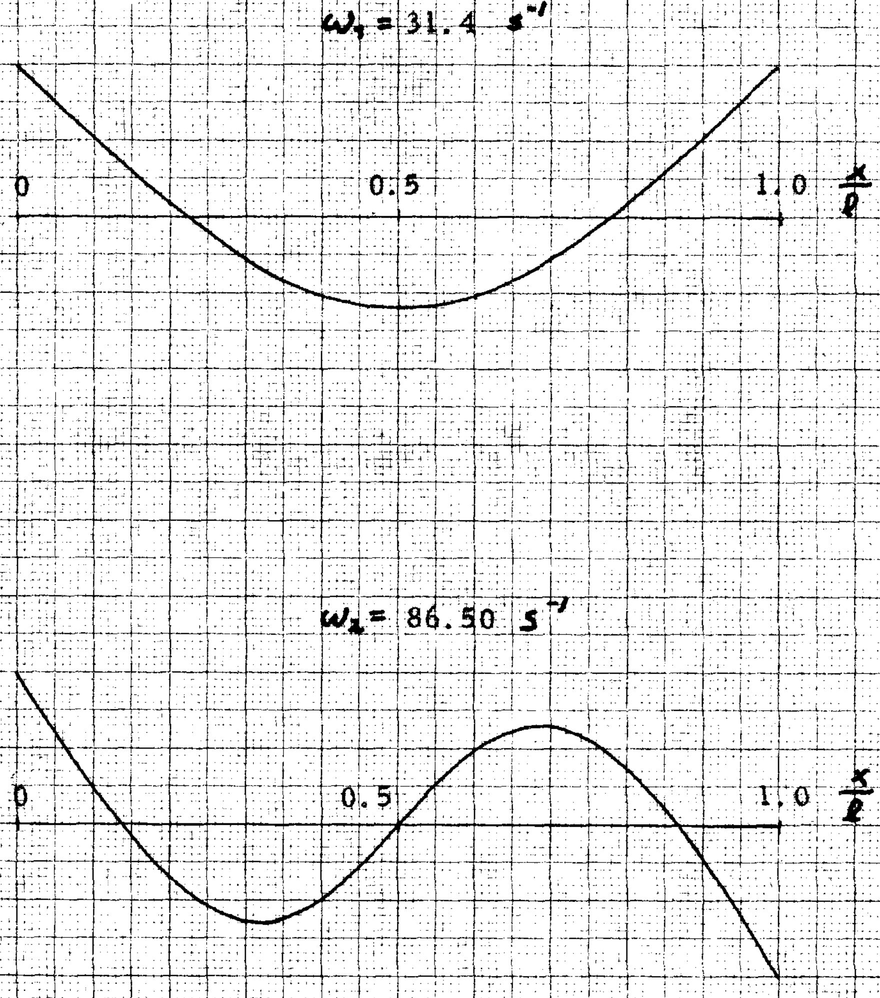

Fig. 7. First and Second Elastic Modes of Free Free Beam

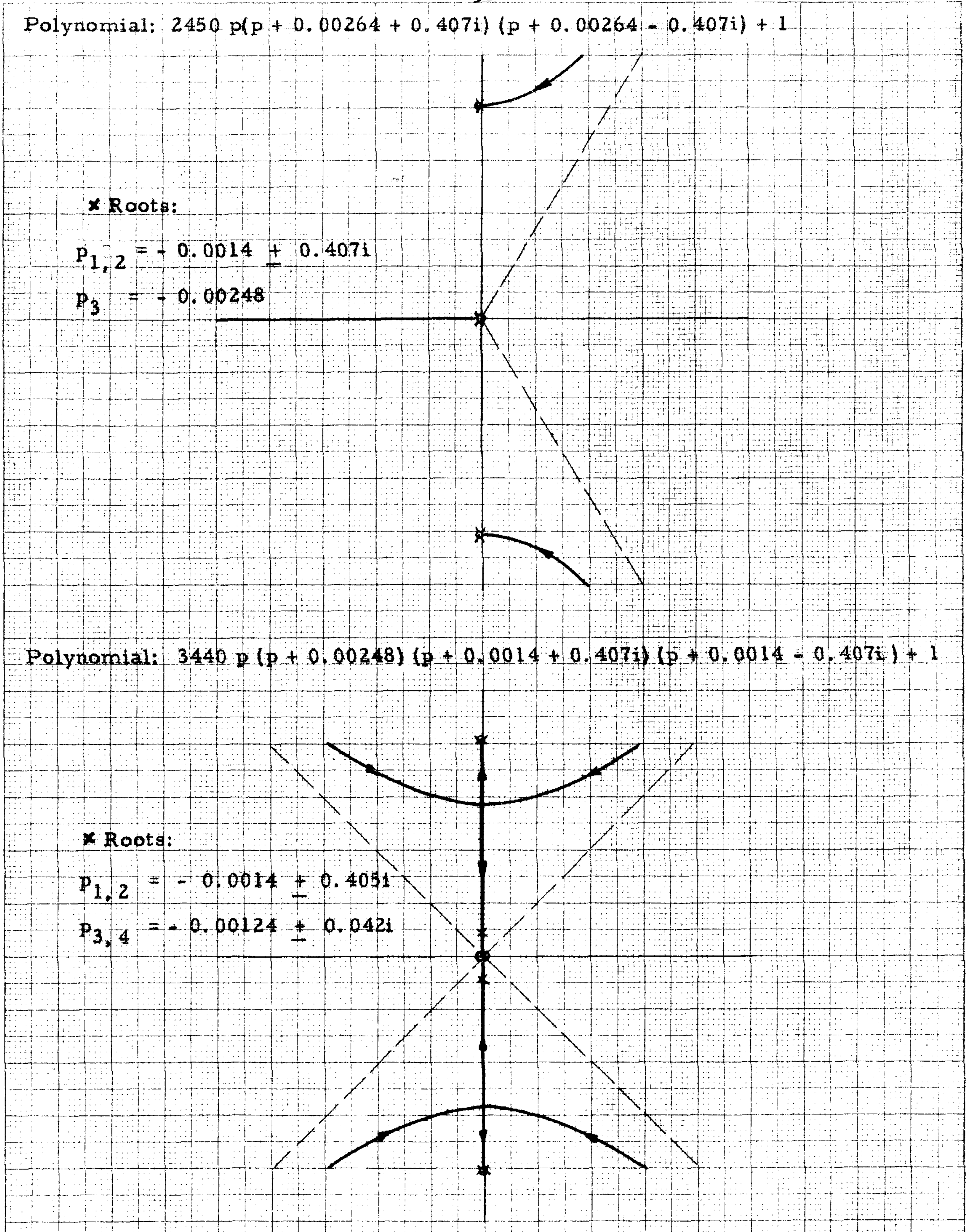

Fig. 8. Factorization of the Characteristic Polynomial by the Root Locus Method.

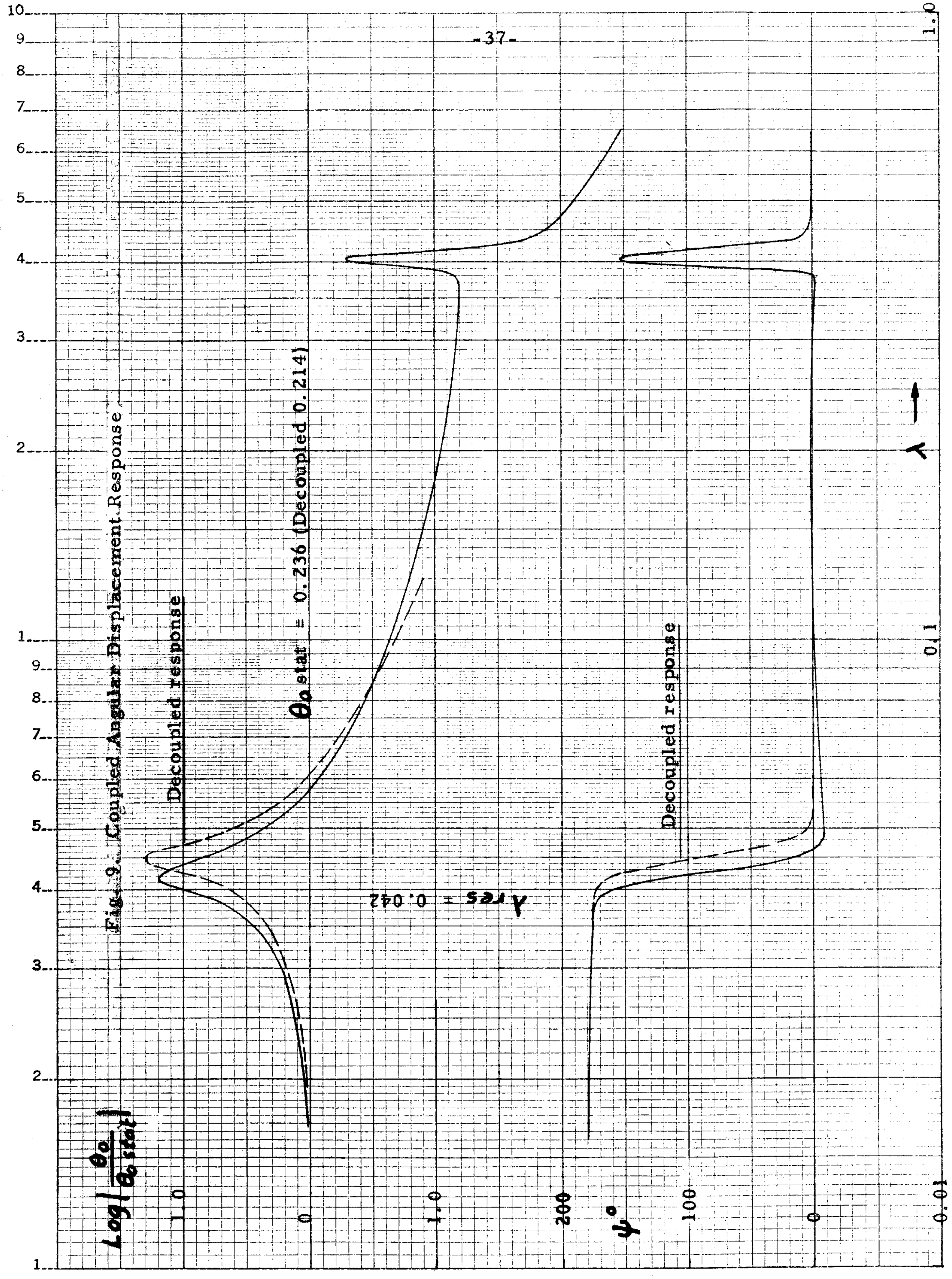

Fig. 9. Coupled Angular Displacement Response

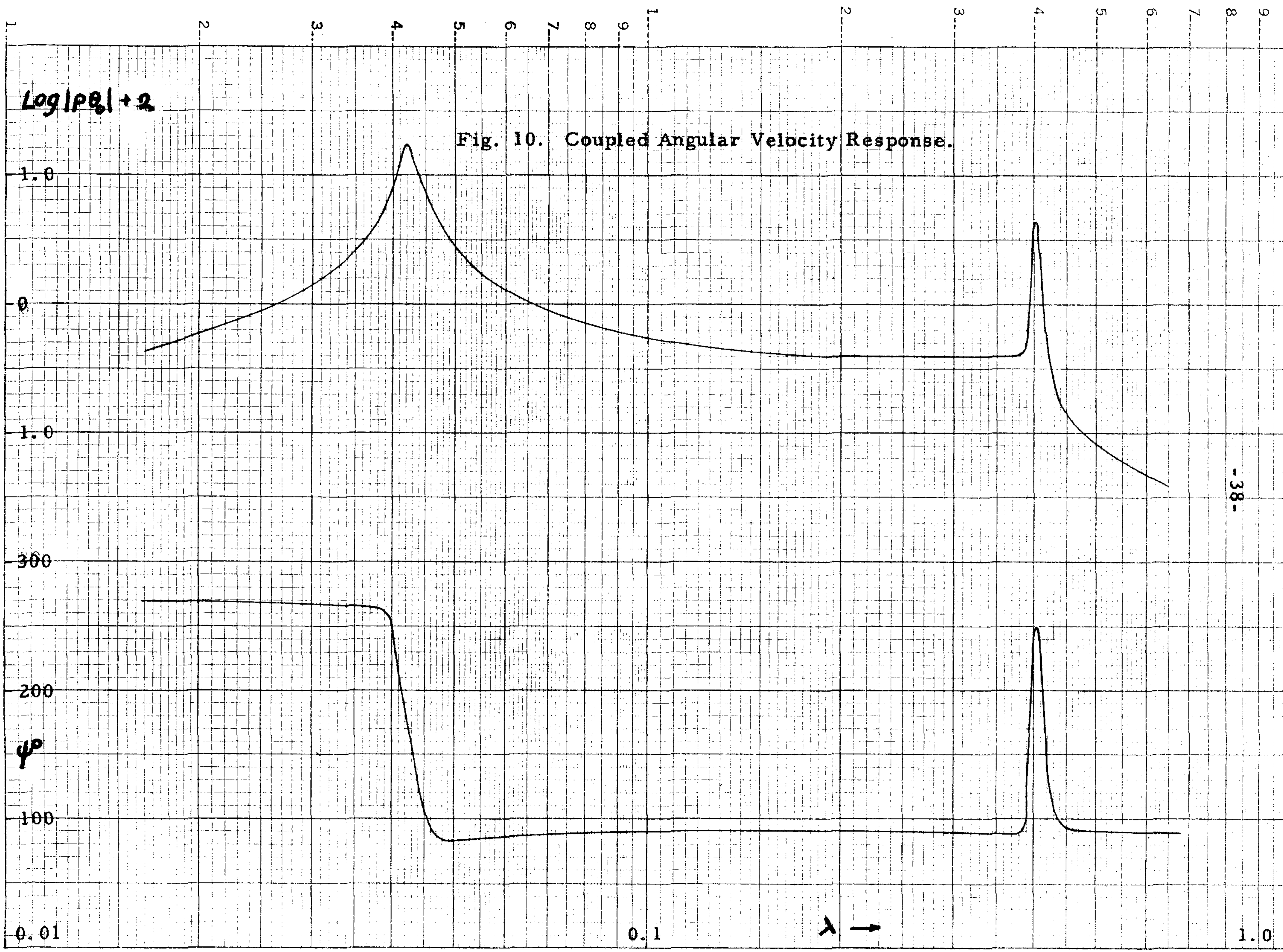

Fig. 10. Coupled Angular Velocity Response.

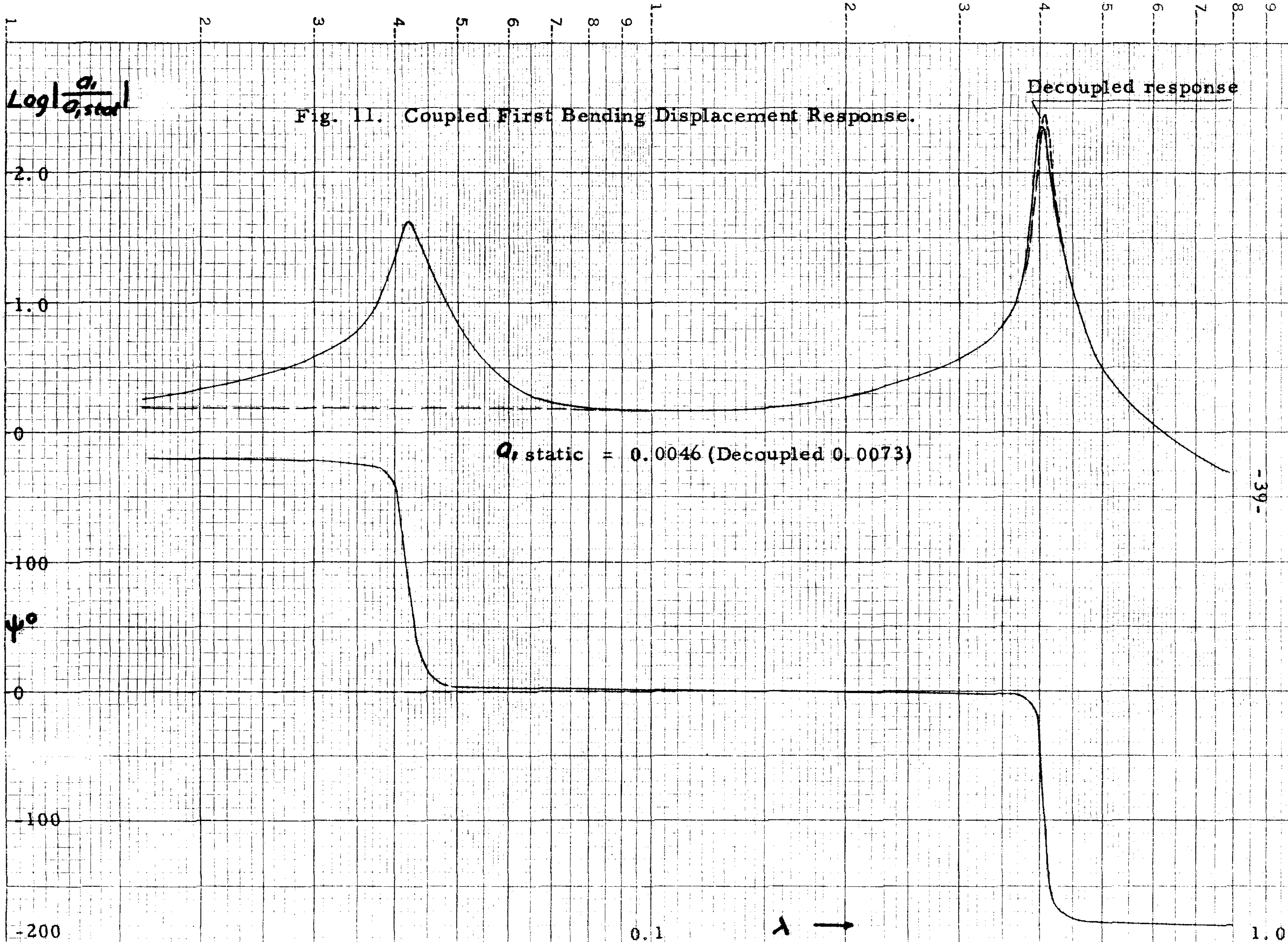

Fig. 11. Coupled First Bending Displacement Response.

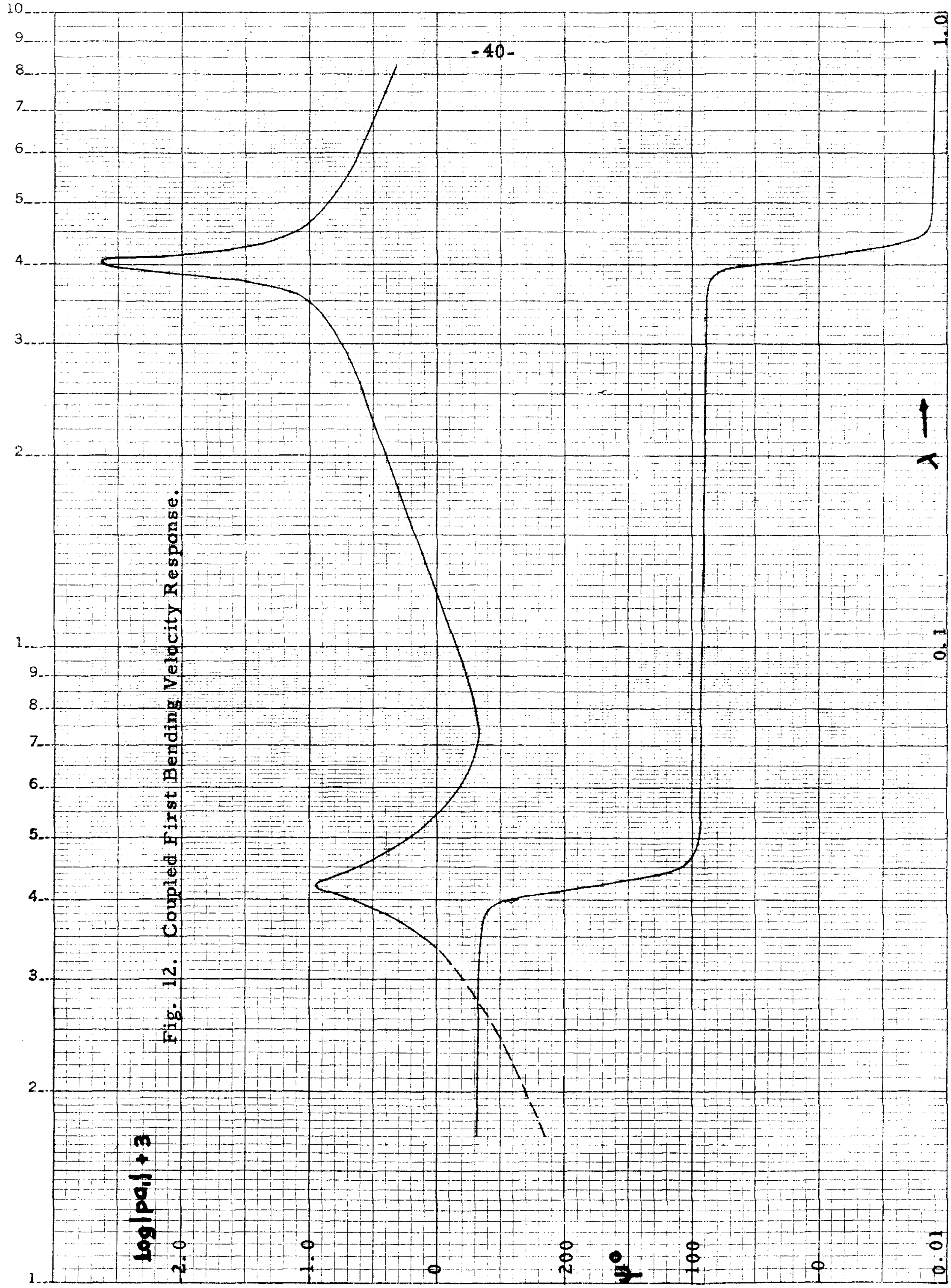

Fig. 12. Coupled First Bending Velocity Response.

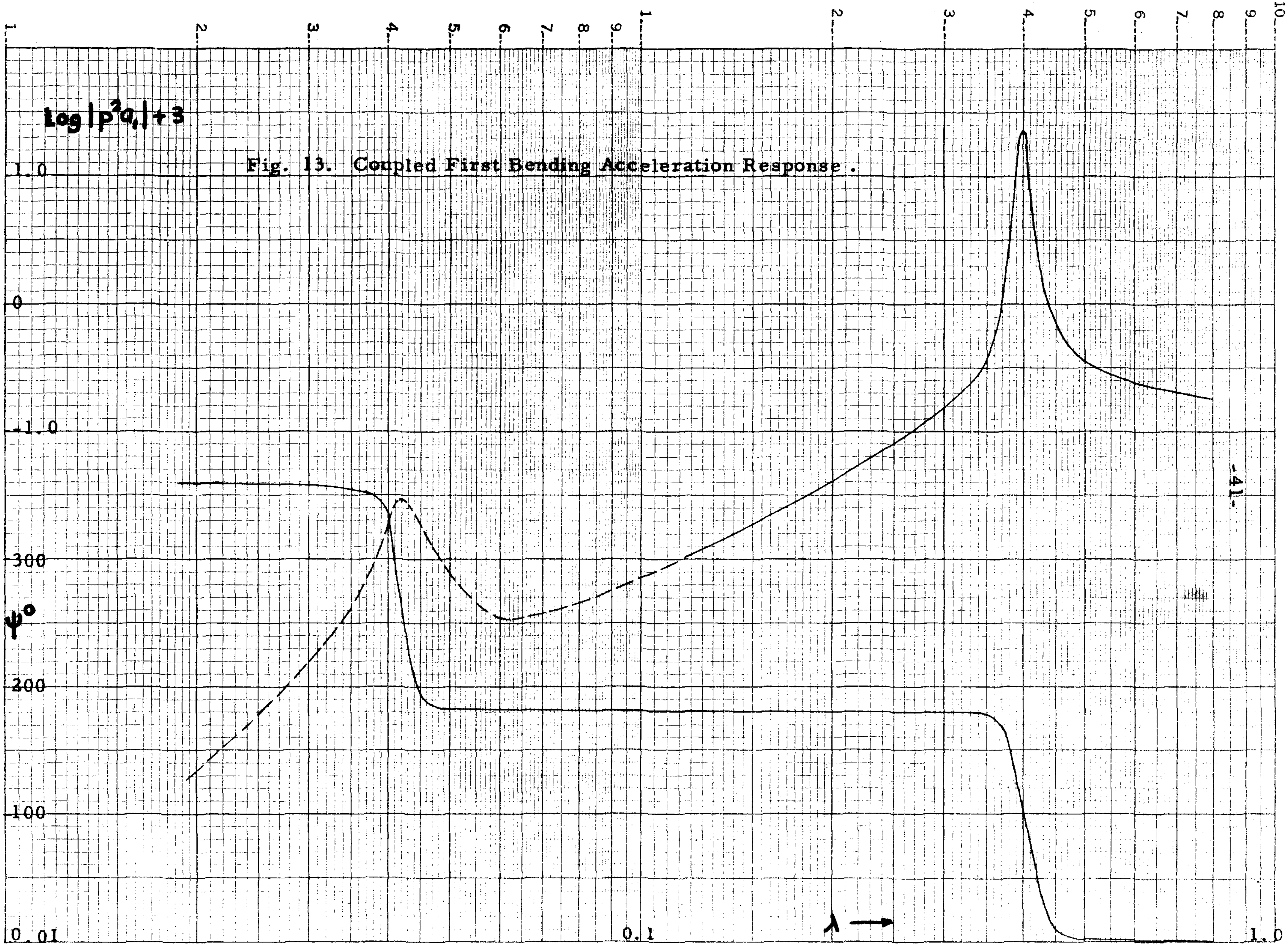

Fig. 13. Coupled First Bending Acceleration Response.

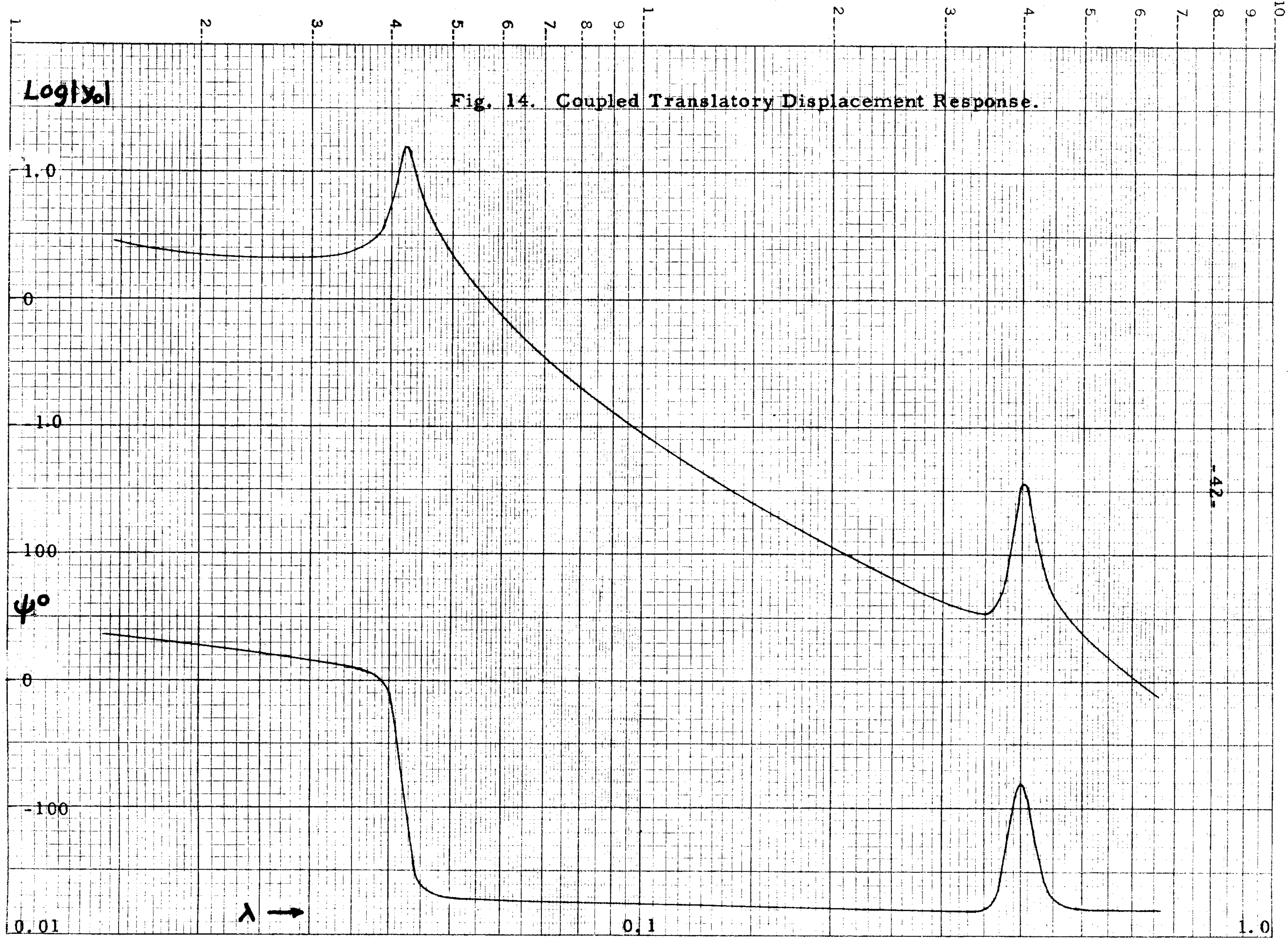

Fig. 14. Coupled Translatory Displacement Response.

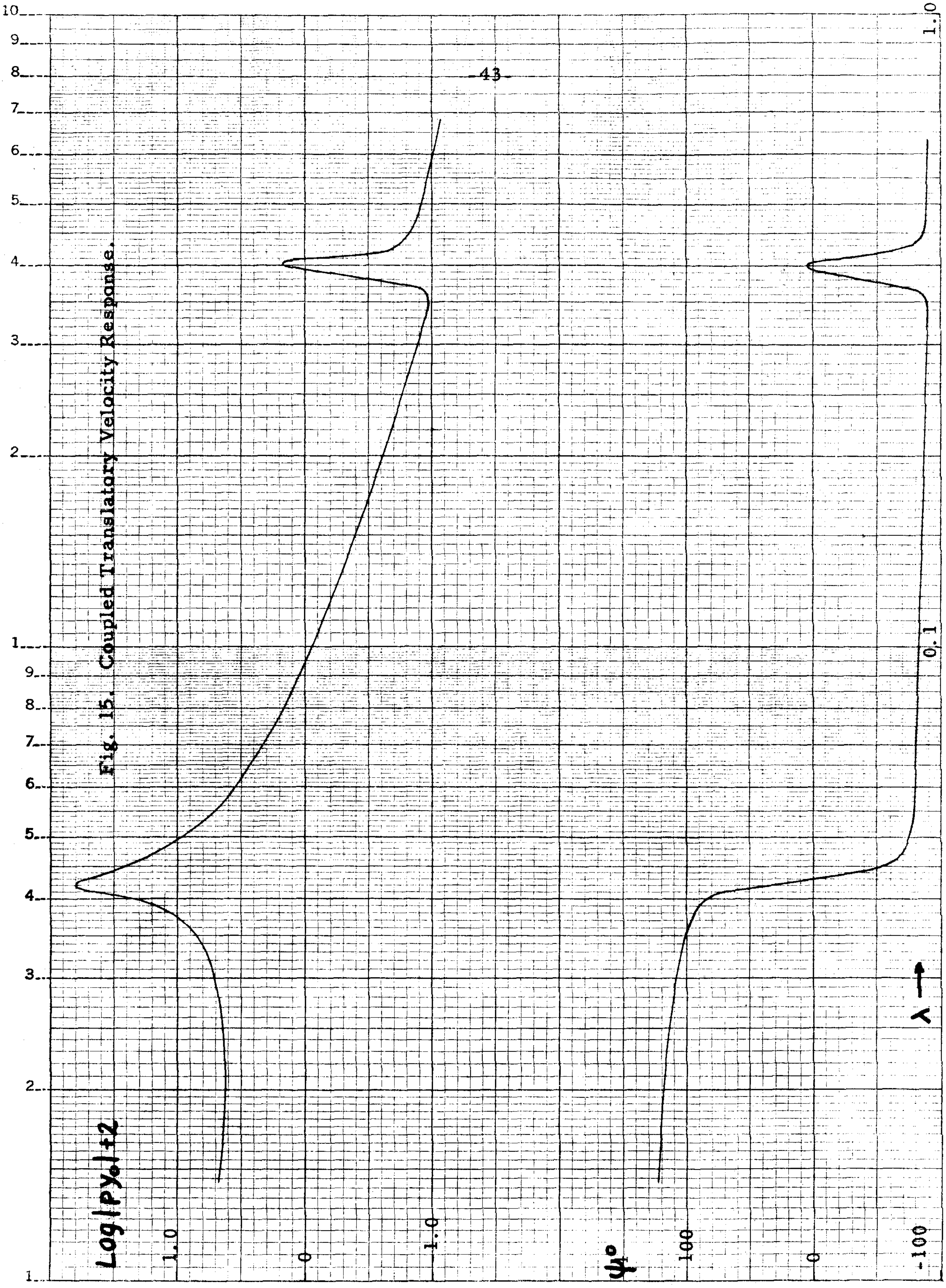

Fig. 15. Coupled Translatory Velocity Response.

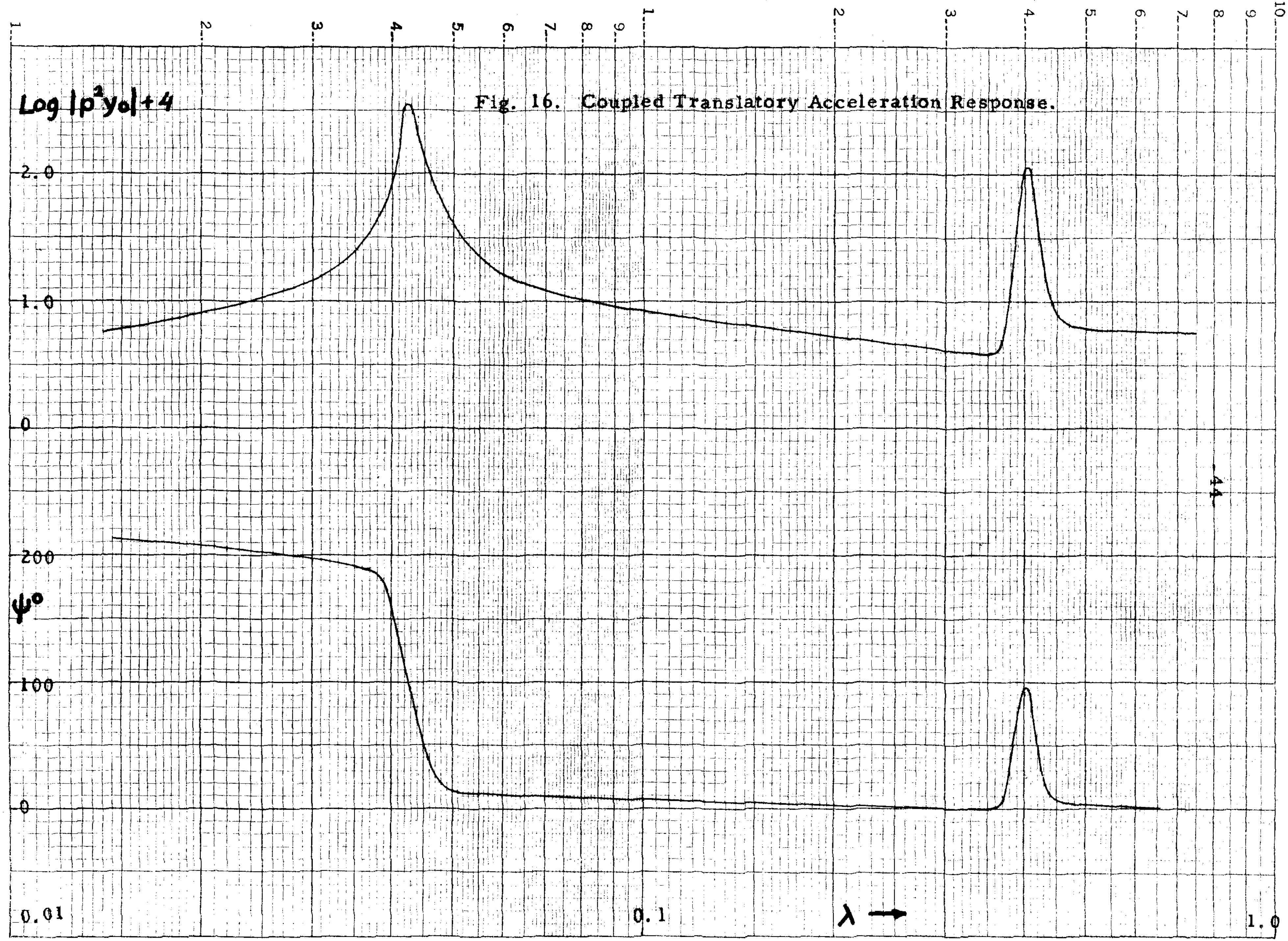

Fig. 16. Coupled Translatory Acceleration Response.