THE RESPONSE OF AN AIRPLANE TO RANDOM ATMOSPHERIC DISTURBANCES

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ABSTRACT

The statistical approach to the gust-loads problem is extended by considering the aerodynamic forces due to the lateral and longitudinal variation of instantaneous gust intensity and using them in dynamic analyses of rigid and flexible airplanes free to move vertically, in pitch, and in roll, based on the assumptions of stationarity of the process (in the statistical sense) and of linearity of the forces involved. The effect on the wing stresses of the interaction of longitudinal, vertical and lateral gusts is considered.

The method of analyzing the rigid-body motions is similar to that used for analyses of the dynamic stability of airplanes, in that the equations of motion are referred to stability axes and expressed in terms of conventional stability derivatives. The method of analyzing the dynamic effects of structural flexibility consists of an extension of a numerical-integration approach to the static aeroelastic problem and is in a form which offers the possibility of calculating divergence and flutter speeds with relatively little additional effort.

The mean-square values, correlation functions and power spectra of some of the aerodynamic forces required in this type of analysis are calculated for certain special correlation functions of the atmospheric turbulence and certain special lift distributions. It is shown, for instance, that the mean-square lift is substantially reduced due to the difference in instantaneous intensity of the turbulent velocity along the span if the span is relatively large compared to the integral scale of turbulence, but that the mean-square pitching moment is substantially increased if the tail length is relatively large. Also, the wing stresses due to vertical, horizontal and side gusts are shown to be statistically independent under certain conditions.

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I. INTRODUCTION

The local velocity fluctuations sensed by an airplane flying through atmospheric turbulence constitute a stochastic or random process, inasmuch as they are functions of time defined only in a statistical sense. Consequently the responses of the airplane, whether they are motions (linear or angular displacements, velocities or accelerations), forces (lift, pitching moment, bending moment, and so on), stresses, or any other phenomena determined by the turbulence, can also be known as functions of time in only a statistical sense.

This paper is concerned with the statistical characteristics of the responses which have a bearing on the loads and stresses experienced by the airplane; although other problems, such as those relating to passenger comfort or to the stability of the airplane as a gun platform can be treated in the same manner, they will not be considered here.

The first approaches to the gust-loads problem from the statistical point of view appear to be those of References 1, 2 and 3. (An earlier investigation concerned with the motions of an airplane in turbulent air is reported on in Reference 4). The fundamentals of these approaches are discussed in some detail in Reference 1, and reference is made therein to investigations dealing with the problem of deducing the statistical characteristics of the output or response of a dynamic system from those of its input in other fields of engineering and physics. For instance, it is pointed out that the power spectrum of the output of a linear system is equal to the product of the absolute square of the transfer function of the system and the input spectrum, provided the input is stationary in a statistical sense and has a zero mean. The mean-square values of the output and of its time derivatives can then be calculated from the power spectrum, and the expected number of peaks of the output with a given intensity can be calculated from these mean-square values if the probability distribution of the process in question is Gaussian. By an extension of this method the expected number of loading cycles of various amplitudes and about various means can be estimated. Thus, if it is known what fraction of its time the airplane will spend in various atmospheric conditions, what the spectrum of the turbulence is in each condition, and what the appropriate transfer functions are, the probability of exceeding a given peak load and the number of loading cycles incurred in a given number of hours can be calculated on the basis of certain assumptions.

Following this approach, the mean-square normal acceleration of a rigid airplane free to move in one degree of freedom, namely vertical motion, is calculated in Reference 2. In addition to rigidity, the assumption is implied that the airplane is small enough that at any instant of time all its components experience the same gust velocity. This means that the span of the airplane must be small compared to the integral scale of atmospheric turbulence, which on the basis of the available knowledge concerning the properties of the atmosphere[4] should be in the order of several hundred to 1,000 feet; that is, the span of the airplane must be less than about 50 feet.

The purpose of the present paper is to extend this approach to large flexible airplanes free to move in all directions. As used in the following, the terms "small" and "large" airplane will refer to airplanes which are very small and not very small, respectively, compared to the integral scale of turbulence; thus, an airplane flying in a wide variety of atmospheric conditions may be "small" under certain conditions and "large" under others. Similarly, the terms "rigid airplane" and "flexible airplane" will be used in the following to designate airplanes flying, respectively, at speeds far below those at which dynamic and aeroelastic effects become important, and at speeds at which these effects have to be taken into account; the same airplane can thus be "rigid" under some conditions and "flexible" under others.

Several fundamental assumptions are inherent in the analysis contained in this paper. In the first place, all atmospheric disturbances, motions and structural deformations are assumed to be sufficiently small that the resulting forces are linear and, hence, superposable. Also, the turbulent "input" to the airplane is assumed to be stationary in a statistical sense. This assumption may be rephrased as stating that the turbulence in the plane of the flight path is homogeneous. For the large airplane the additional assumption is made that the turbulence is axisymmetric with respect to vertical axes, a condition less severe than complete isotropy. The statistical characteristics of the turbulence are thus assumed to be invariant under a translation of the space origin within the horizontal plane and under a rotation of the coordinates about the vertical. Finally, Taylor's hypothesis to the effect that time displacements are equivalent to longitudinal space displacements will be assumed to be valid.

The aerodynamic forces directly due to atmospheric turbulence, which constitute the input forces for the dynamic system represented by the airplane, are calculated in Part II of this paper for the case of the large airplane, that is, for the case where the spanwise distribution of the intensity of turbulence has to be taken into account. In Sections 2 and 6 the required lift-influence function which defines the contribution of one section of the wing to the total lift and to the local lift at another point are discussed, and the assumption is made, on the basis of available knowledge, that these functions of time and distance along the span can be represented by products of functions of time and functions of distance along the span. Using these influence functions the lift and its power spectrum are calculated for two assumed spectra of atmospheric turbulence in Sections 3 and 4. (The problem analyzed in Section 3 has been treated by a slightly different method in Reference 5.) It is shown that, although the mean-square lift may be substantially less if the differences in gust intensities along the span are taken into account, the spectrum of the lift (normalized by the mean-square value) is not affected greatly.

The rolling moment is treated in Section 5. This moment exists only by virtue of the fact that local gust intensities are taken into account, because if the gust intensity is uniform along the span no rolling moment is produced. This analysis thus furnishes the foundation for the application of the statistical approach to the analysis of the motions and loads associated with the lateral degrees of freedom. The remainder of this Part of the paper is concerned with the calculation of mean-square values of various other moments and of the local lift.

The dynamics of the rigid airplane are considered in Part III. The dynamic system is now represented by a set of three simultaneous ordinary differential equations, rather than one, as in Reference 2; nonetheless, the problem of calculating the required transfer functions is still one of simple algebra. The first two Sections are concerned with the longitudinal motion of small airplanes. One result shown is that the responses to horizontal and to vertical gusts are statistically independent if isotropy of the turbulence is assumed, so that the two contributions to the spectrum of the response can be added directly. Also, it is shown that inclusion of longitudinal motion has a negligible effect on the loads. In Sections 3 and 4 the longitudinal and lateral motions of large airplanes are considered, using the results calculated for the aerodynamic forces in Part II, and in Section 5 the manner in which the stresses calculated for the longitudinal and lateral degrees of freedom must be combined is indicated. This Part of the paper not only serves as a preliminary to the treatment of the flexible airplane in the later Parts, but also has an intrinsic interest, because it applies directly to those airplanes which fly at relatively low speeds and do not experience any significant structural deformations.

Part IV is concerned with the small flexible airplane and thus has direct application to fighter-type airplanes and guided missiles operating at relatively high speeds, in addition to serving as a preliminary to the last Part. For this case the longitudinal and lateral degrees of freedom are still separable, and only the longitudinal degrees are considered; the lateral degrees can be analyzed in the same way. Also, for this case only one half of the wing need be considered as a result of the symmetry (or antisymmetry, in the case of the lateral degrees of freedom) of the problem. The dynamic system is now represented by a partial differential equation, and the calculation of the transfer functions requires the solution of ordinary equations. Once these functions are calculated, however, the statistical techniques are the same as before, as a result of the fact that the lateral variation in gust intensity is ignored.

No work appears to have been published on methods of analyzing the dynamics of a swept-wing airplane with arbitrary stiffness and mass distributions. Either modal or numerical-integration approaches may be used for this problem; although modal approaches have usually been preferred in the past for similar problems, it was felt that in view of the highly complex nature of modern aircraft structures and the advanced type of computing machinery required and generally available for their analysis the numerical-integration approach would be preferable and has, therefore, been used. This approach has the added advantage, as pointed out in Section 4, that the calculated results include those usually obtained by a separate analysis of static aeroelastic effects and also permit the calculation of flutter speeds with little additional effort.

Part V contains the analysis of the large flexible airplane. The longitudinal and lateral degrees of freedom can still be separated, if desired, but inasmuch as the entire wing has to be considered anyway, very little additional computing time is required if they are to be treated simultaneously, and the necessity of combining the results of the two analyses is then obviated. The statistical problem is now that of a system which is characterized by a partial differential equation with time and a space coordinate as independent variables and which is subjected to a random input which varies in time and space, so that more is required than the transfer functions from the gust intensity at one point on the wing to the stresses at another. The particular statistical problem presented by this case is considered in some detail in Section 1, and it is shown that the required transfer functions are, in a sense, auto-convolutions of the other transfer functions. Several ways of computing the spectrum of the stresses from these transfer functions are indicated. In Section 2 the appropriate transfer functions are then obtained by solving, in effect, the ordinary differential equations which describe the wing deformations at any given frequency using the numerical-integration approach presented in Part IV.

II. AERODYNAMIC FORCES RESULTING DIRECTLY FROM ATMOSPHERIC TURBULENCE

The motions of a rigid airplane depend on the overall forces and moments, and the distribution of these forces is required, as well, in calculations of stresses and in analyzing the motions of a flexible airplane. This section is concerned with the calculation of the integrated and distributed forces and moments due directly to atmospheric turbulence. The forces and moments caused by the motions which result from this turbulence can be calculated by conventional methods and will not be considered here.

1. Definitions of Statistical Parameters

As pointed out in the Introduction, the intensity of the vertical component of turbulence w(t) is a random process, so that the resulting forces can also be known only in a statistical sense. The purpose of this part is to calculate certain statistical properties of these forces, namely their mean-square values, their correlation functions, and their power spectra. Inasmuch as these terms are not always defined in the same manner the forms which will be used in the following are indicated in the succeeding paragraphs.

The mean of a quantity will be considered to be its time average, designated by a bar placed over the quantity, and defined as follows:

$$\overline{f(t)} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)\, dt \tag{2.1.1}$$

The assumption will always be made that this limit exists and is invariant under a translation of the origin of time. Also, this mean will always be assumed to be zero. In dealing with processes with non-zero mean this analysis is thus pertinent only to the modified process which consists of the difference of the original process and its mean value.

Similarly, the mean-square value of a quantity is

$$\overline{f^2} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f^2(t)\,dt \tag{2.1.2}$$

The correlation function of a quantity $f(t)$ will be defined as follows:

$$\psi_f(\tau) = \overline{f(t)\,f(t+\tau)} \tag{2.1.3}$$

so that the mean-square value of a quantity is equal to its correlation function at zero time-displacement, or

$$\overline{f^2} = \psi_f(0) \tag{2.1.4}$$

The power spectrum of a quantity will be defined as the Fourier transform of the correlation function, in the form

$$\varphi_f(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{i\omega\tau}\, \psi_f(\tau)\, d\tau \tag{2.1.5}$$

so that, by virtue of the reciprocal properties of Fourier transforms and of the symmetry of $\psi_f(\tau)$,

$$\overline{f^2} = \int_0^{\infty} \varphi_f(\omega)\, d\omega \tag{2.1.6}$$

These definitions are substantially those used in Reference 1. The fundamental principles involved in statistical analyses of the type considered herein are expounded there in some detail, and references to the literature on the subject are given both there and in

Reference 2. These fundamentals will therefore not be repeated here.

2. Lift-Influence-Functions in Unsteady Flow

At any time t the lift on a wing which results directly from atmospheric disturbances can be expressed for an unswept wing as

$$L(t) = \int_{-\infty}^{\infty} dt_1 \int_{-\frac{b}{2}}^{\frac{b}{2}} dy_1 \; h(t_1, y_1) \; w(t - t_1, y_1) \tag{2.2.1}$$

where h(t, y) dy is a lift-influence function which represents the total lift caused by an impulsive vertical gust of width dy which at time t = 0 impinges on the wing at station y.

The influence functions required in Equation (2.2.1) are difficult to calculate directly; methods for obtaining lift distributions on wings of finite span in unsteady flow usually require numerical solutions which do not lend themselves readily to the analysis of angle of attack distributions represented by delta functions. However, by virtue of the reciprocity theorems of linearized lifting surface theory (Reference 6, for instance) the lift influence function for a twisted wing in indicial motion is equal to the lift distribution on that wing during indicial motion with unit angle of attack in the reverse direction. Furthermore, the lift distribution in indicial motion with unit angle of attack tends, for the few cases for which calculations have been made, to be substantially invariant in time except for over-all magnitude. For instance, the calculations of Reference 7 indicate that the lift distribution of an oscillating rectangular or elliptic wing in incompressible flow is substantially independent of frequency, so that in indicial motion it is substantially independent of time.

The lift influence function can then be written as

$$h(t, y) = \frac{1}{b} h(t) \gamma(y) \tag{2.2.2}$$

where h(t) describes the variation of the over-all magnitude of the lift in time pursuant to entry into a sharp-edged gust and may be written as $h(t) = C_{L_\alpha}(t) q S/U$ and where $\gamma(y)$ defines the steady-state lift distribution for uniform angle of attack, namely

$$\gamma(y) = \frac{c c_l}{\bar{c} C_L}$$

so that

$$\int_{-\frac{b}{2}}^{\frac{b}{2}} \gamma(y)\, dy = b \tag{2.2.3}$$

3. The Mean-Square Lift and its Spectral Resolution for the Unswept Wing

The correlation function of the lift can be expressed by virtue of Equation (2.2.1), as

$$\psi_L(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}} h(t_1, y_1)\, h(t_2, y_2)\, \overline{w(t-t_1, y_1)\, w(t+\tau-t_2, y_2)}\, dy_1\, dy_2\, dt_1\, dt_2 \tag{2.3.1}$$

where the averaged product on the right side represents a velocity correlation function. These functions depend in general on both space and time displacements. However, if Taylor's hypothesis (which according to the best available knowledge appears to be valid for flying speeds greater than about 100 feet per second) is made, the time displacements are equivalent to longitudinal space displacements. The velocity correlation functions are then functions only of longitudinal and lateral space displacements. Thus, for instance for homogeneous turbulence

$$\psi_w(\xi+U\tau,\eta) = \overline{w(x+Ut,y)\,w(x+\xi+U(t+\tau),y+\eta)} \tag{2.3.2}$$

so that

$$\psi_L(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}} h(t_1,y_1)\,h(t_2,y_2)\,\psi_w(U(\tau+t_1-t_2),y_2-y_1)\,dy_1\,dy_2\,dt_1\,dt_2 \tag{2.3.3}$$

If the assumption implicit in Equation (2.2.2) is now made, the preceding equation can be written as

$$\psi_L(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t_1)\,h(t_2)\,\psi_{w_e}(U(\tau+t_1-t_2))\,dt_1\,dt_2 \tag{2.3.4}$$

where

$$\begin{aligned}\psi_{w_e}(U\tau) &= \frac{1}{b^2}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}} \psi_w(U\tau, y_2-y_1)\,\gamma(y_1)\,\gamma(y_2)\,dy_1\,dy_2 \\ &= \frac{2}{b^2}\int_0^b \Gamma(\eta)\,\psi_w(U\tau,\eta)\,d\eta\end{aligned} \tag{2.3.5}$$

where, in turn, $\Gamma(\eta)$ is the auto-convolution of $\gamma(y)$, namely

$$\Gamma(\eta) = \int_{-\frac{b}{2}}^{\frac{b}{2}-\eta} \gamma(y)\,\gamma(y+\eta)\,dy \tag{2.3.6}$$

If, in addition to Taylor's hypothesis and the assumption of homogeneity the turbulence is assumed to be axisymmetric with respect to vertical axes,

$$\begin{aligned}\psi_w(U\tau,\eta) &= \psi_w\left(\sqrt{U^2\tau^2+\eta^2},\,0\right) \\ &= \psi_w\left(0,\,\sqrt{U^2\tau^2+\eta^2}\right)\end{aligned} \tag{2.3.7}$$

In the following the right sides of this equation will be designated simply by $\psi_w\left(\sqrt{U^2\tau^2+\eta^2}\right)$ to shorten the notation. Hence,

$$\psi_{w_e}(U\tau) = \frac{2}{b^2}\int_0^b \Gamma(\eta)\, \psi_w\left(\sqrt{U^2\tau^2+\eta^2}\right) d\eta \tag{2.3.8}$$

The quantity

$$\begin{aligned} \overline{w_e^2} &= \psi_{w_e}(0) \\ &= \frac{2}{b^2}\int_0^b \Gamma(\eta)\, \psi_w(\eta)\, d\eta \end{aligned} \tag{2.3.9}$$

may be considered to be an averaged mean-square vertical component of turbulence; $\psi_{w_e}(U\tau)$ is then the corresponding correlation function, and the Fourier transform of the latter,

$$\varphi_{w_e}(\omega) = \frac{1}{\pi U}\int_{-\infty}^{\infty} e^{i\frac{\omega}{U}(U\tau)}\, \psi_{w_e}(U\tau)\; d(U\tau) \tag{2.3.10}$$

is the corresponding power spectrum.

The power spectrum of the lift is then

$$\varphi_L(\omega) = \left|H(\omega)\right|^2 \varphi_{w_e}(\omega) \tag{2.3.11}$$

where

$$H(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}\, h(t)\, dt$$

is a transfer function of the wing relating the averaged amplitude of the vertical-velocity fluctuations to the amplitude of the resulting lift, or

$$H(\omega) = C_{L_\alpha}(k)\; q\, S/U$$

where

$$k = \frac{\omega \bar{c}}{2U}$$

The mean square of the lift can then be obtained by integrating its spectrum, as indicated in Equation (2.1.6).

The problem of calculating the mean-square lift and its power spectrum thus resolves itself into two separate problems, namely the one of calculating the appropriate function $H(\omega)$ and the one of calculating the power spectrum of the averaged turbulence. Only the second of these problems will be considered here; the first is a largely unsolved problem in unsteady lifting-surface theory and is beyond the scope of this analysis. For the present purpose suffice it to point out that this transfer function represents the ratio of the lift amplitude to the gust amplitude for flight through sinusoidal gusts. For incompressible flow it can be expressed approximately in terms of the Sears function $\phi(k)$ by

$$C_{L_\alpha}(k) = C_{L_\alpha}\, \phi(k) \tag{2.3.12}$$

where C_{L_α} is the steady-state lift-curve slope, and the required mean-square value of the Sears function can be approximated by the expression given in Reference 1,

$$\left| \phi(k) \right|^2 \approx \frac{1}{1 + 2\pi k} \tag{2.3.13}$$

The following discussion will therefore be concerned with the calculation of the power spectrum of the averaged turbulence.

Equation (2.3.11) has the same form as the corresponding equation for the case where spanwise averaging of the effects of turbulence is not taken into account (see Reference 1), except that in that case $\varphi_{w_e}(\omega)$ is replaced by $\varphi_w(\omega)$. Thus the spectrum of the averaged turbulence must approach that of the unaveraged turbulence when the span approaches zero, as may be seen to be the case from Equation (2.3.5) and the definitions of $\gamma(y)$ and $\Gamma(\eta)$.

For non-vanishing span the desired correlation function can be obtained from Equation (2.3.8). The resulting function depends not only on $U\tau$ but on the span, the spanwise lift distribution $\gamma(y)$, and on the unaveraged or point correlation function for the vertical component of turbulence. In the following, both elliptic and rectangular lift distributions will be considered, for which, respectively, $\gamma(y) = 1$ and $\gamma(y) = \frac{4}{\pi}\sqrt{1-\left(\frac{y}{b/2}\right)^2}$; also, respectively,

$$\Gamma(\eta) = b - \eta$$

and

$$\Gamma(\eta) = \frac{32}{3\pi^2}(b+\eta)\left[\left\{1+\left(\frac{\eta}{b}\right)^2\right\}E(\theta) - 2\frac{\eta}{b}K(\theta)\right]$$

the functions $K(\theta)$ and $E(\theta)$ being complete elliptic integrals of the first and second kind, respectively, of modulus $\theta = (b-\eta)/(b+\eta)$.

Also, two analytic expressions for the point correlation function will be considered, these correlation functions and their associated spectra being defined in terms of the integral scale of turbulence

$$\tilde{L} = \int_0^\infty \psi_u(r)\, dr \tag{2.3.14}$$

as follows:

<u>Case 1 (see Reference 1)</u>

$$\left.\begin{aligned}
\psi_u(r) &= \overline{u^2}\, e^{-\frac{|r|}{\tilde{L}}} \\
\psi_w(r) &= \overline{w^2}\left(1-\frac{|r|}{2\tilde{L}}\right) e^{-\frac{|r|}{\tilde{L}}} \\
\varphi_u(\omega) &= \frac{2}{\pi}\,\frac{\overline{u^2}\,\tilde{L}}{U}\,\frac{1}{1+k'^2} \\
\varphi_w(\omega) &= \frac{1}{\pi}\,\frac{\overline{w^2}\,\tilde{L}}{U}\,\frac{1+3k'^2}{(1+k'^2)^2}
\end{aligned}\right\} \tag{2.3.15}$$

Case 2

$$\left.\begin{aligned}
\psi_u(r) &= \overline{u^2}\, e^{-\frac{\pi}{4}\frac{r^2}{\tilde{L}^2}} \\
\psi_w(r) &= \overline{w^2}\left(1-\frac{\pi}{4}\frac{r^2}{\tilde{L}^2}\right)e^{-\frac{\pi}{4}\frac{r^2}{\tilde{L}^2}} \\
\varphi_u(\omega) &= \frac{2}{\pi}\,\frac{\overline{u^2}\,\tilde{L}}{U}\, e^{-\frac{k'^2}{\pi}} \\
\varphi_w(\omega) &= \frac{1}{\pi}\,\frac{\overline{w^2}\,\tilde{L}}{U}\left(1+2\frac{k'^2}{\pi}\right)e^{-\frac{k'^2}{\pi}}
\end{aligned}\right\} \tag{2.3.16}$$

where

$$\left.\begin{aligned}
k' &= \frac{\omega\tilde{L}}{U} = s\,k \\
s &= \frac{\tilde{L}}{\bar{c}/2}
\end{aligned}\right\} \tag{2.3.17}$$

For isotropic turbulence $\overline{u^2} = \overline{w^2}$, but this condition is not necessary to the following.

For Case 1 the mean-square averaged vertical component of turbulence as obtained from Equation (2.3.9) for rectangular loading is

$$\frac{\overline{w_e^2}}{\overline{w^2}} = \frac{1-e^{-\beta}}{\beta} \tag{2.3.18}$$

where $\beta = b/\tilde{L}$. This relation is shown in Figure 1. Also shown in Figure 1 are the results obtained by numerical integration for elliptic loading. Similarly, for Case 2 and rectangular loading

$$\frac{\overline{w_e^2}}{\overline{w^2}} = \frac{\beta\,\mathrm{erf}(\sqrt{\pi}\,\beta) - \frac{1}{\pi}\left(1-e^{-\pi\beta^2}\right)}{\beta^2} \tag{2.3.19}$$

This relation is also shown in Figure 1. It may be noted that as $\beta \to \infty$ the ratio $\frac{\overline{w_e^2}}{\overline{w^2}}$ becomes asymptotic to $1/\beta$ in both cases.

For Case 1 and rectangular loading the function $\psi_{w_e}(U\tau)$ is given by incomplete modified Bessel functions, namely,

$$\psi_{w_e}(U\tau) = \frac{1}{\beta}\Big[\rho\,\widetilde{K}_1\big(\rho;\sinh^{-1}\tfrac{\beta}{\rho}\big) - \rho^2\,\widetilde{K}_0\big(\rho;\sinh^{-1}\tfrac{\beta}{\rho}\big) + \frac{\rho^2}{\beta}\big(e^{-\rho} - e^{-\sqrt{\rho^2+\beta^2}}\big)\Big]\overline{w^2} \tag{2.3.20}$$

where

$$\widetilde{K}_\nu(\rho;\sigma) = \int_0^\sigma e^{-\rho\cosh\theta}\cosh\nu\theta\,d\theta$$

$$\rho = \frac{U\tau}{\widetilde{L}}$$

This function $\psi_{w_e}(U\tau)$ is shown in Figure 2, as are the results obtained numerically for elliptic loading for one span ratio ($\beta = 0.5$). For the limiting case $\beta \to \infty$ the incomplete Bessel functions become complete Bessel functions, so that

$$\lim_{\beta\to\infty}\frac{\psi_{w_e}(U\tau)}{\overline{w_e^2}} = \rho\,K_1(\rho) - \rho^2 K_0(\rho)$$

where K_1 and K_0 are modified Bessel functions of the second kind. For Case 2 the correlation function for the averaged turbulence with rectangular loading is

$$\frac{\psi_{w_e}(U\tau)}{\overline{w_e^2}} = \Big[1 - \rho^2\Big\{\frac{\pi}{2} - \frac{1 - e^{-\frac{\pi}{4}\beta^2}}{\beta\,\mathrm{erf}\big(\frac{\sqrt{\pi}}{2}\beta\big)}\Big\}\Big]e^{-\frac{\pi}{4}\rho^2} \tag{2.3.21}$$

For Case 1 and rectangular loading the power spectrum of the averaged turbulence is

$$\varphi_{w_e}(\omega) = \frac{\widetilde{L}}{\pi U}\,\frac{2\,\overline{w^2}}{\beta^2(1+k'^2)^3}\Big[3k'^2\beta\sqrt{1+k'^2}\Big\{Ki_0\big(\beta\sqrt{1+k'^2}\big) - \beta\sqrt{1+k'^2}\,K_0\big(\beta\sqrt{1+k'^2}\big)\Big\} + (1-3k'^2)\Big\{2 - 2\beta\sqrt{1+k'^2}\,K_1\big(\beta\sqrt{1+k'^2}\big) - \beta^2(1+k'^2)K_0\big(\beta\sqrt{1+k'^2}\big)\Big\}\Big]$$

where

$$Ki_0(\sigma) = \int_0^\sigma K_0(\sigma')\,d\sigma'$$

This function $\varphi_{w_e}(\omega)$ is shown in Figure 3, as are the results obtained by numerical integration for elliptic loading. For the limiting case of $\beta \to \infty$ and rectangular loading

$$\lim_{\beta \to \infty} \frac{\varphi_{w_e}(\omega)}{\overline{w_e^2}} = \frac{\widetilde{L}}{U} \frac{3k'^2}{(1+k'^2)^{5/2}}$$

Similarly, for Case 2 and rectangular loading

$$\frac{\varphi_{w_e}(\omega)}{\overline{w_e^2}} = \frac{\widetilde{L}}{\pi U}\left[\left(1+\frac{2}{\pi}k'^2\right)-\left(1-\frac{2}{\pi}k'^2\right)\left\{1-\frac{4}{\pi}\frac{1-e^{-\frac{\pi}{4}\beta^2}}{\beta\,\mathrm{erf}\left(\frac{\sqrt{\pi}}{2}\beta\right)}\right\}\right]e^{-\frac{k'^2}{\pi}} \tag{2.3.22}$$

As a result of these calculations it appears that the effect of span on the correlation function and spectrum of the averaged turbulence is relatively small, provided both are normalized with the averaged mean-square turbulent velocity; in the case of the spectrum the effect is smallest in the range of reduced frequency from 2 to 4, which contains a large share of the turbulent energy. (If the power spectrum were not normalized the averaging effect of the span would tend to reduce the intensity of the spectrum at all frequencies, but the high frequencies would be attenuated much more than the low ones, as may be expected.) On the other hand, the effect of span on the mean-square turbulent velocity is quite large, and the magnitude of this effect depends on the shape of the point correlation function and spectrum. The difference in the results obtained with elliptic and rectangular lift distribution is relatively small. Consequently, in the following calculations rectangular loadings will be considered because they are much more convenient to use.

The power spectrum of the lift is then the product of the power spectrum $\varphi_{w_e}(\omega)$ and the absolute square of the transfer function $H(\omega)$, as indicated in Equation (2.3.11). Inasmuch as this lift is not an end in itself but only one of the parameters that enter into the calculations of the motion of the airplane its mean-square intensity is of little practical significance; its spectrum is the quantity needed in further calculations. However, if the mean-square intensity is wanted for any reason it can be obtained by integrating the spectrum. Thus, for instance, using the approximate expression for $|\phi(k)|^2$ given by Equation (2.3.13) and the spectrum $\varphi_w(\omega)$ of Equation (2.3.15) the mean-square lift is

$$\overline{L^2} \approx \left(\frac{C_{L_\alpha} q S}{U}\right)^2 \overline{w^2}\, \frac{s'}{\pi}\left[\frac{1+3s'^2}{(1+s'^2)^2}\left(\frac{\pi}{2}s' - \log s'\right) - \frac{\frac{\pi}{2}s' - 1}{1+s'^2}\right] \tag{2.3.23}$$

where

$$s' = \frac{\tilde{L}}{\pi \bar{c}} = \frac{s}{\pi}$$

In view of the observation that much of the turbulent energy is contained in a region for which the span has a very small effect on the properly normalized spectrum, Equation (2.3.23) is valid approximately for non-vanishing span, provided $\overline{w_e^2}$ is used instead of $\overline{w^2}$.

4. The Mean-Square Lift of the Swept Wing

For the yawed or sideslipping unswept wing Equations (2.3.4) and (2.3.11) for the lift correlation function and spectrum are still valid if an appropriate lift influence function is used and if the correlation function for the averaged turbulence is defined by

$$\psi_{w_e}(U\tau) = \frac{2\cos^2\Lambda}{b^2}\int_0^{b\cos\Lambda}\Gamma(\eta)\,\psi_w\left(\sqrt{(U\tau+\eta\tan\Lambda)^2+\eta^2}\right)d\eta$$

where now $\Gamma(\eta)$ pertains to a lift-distribution function $\gamma(y)$ which is appropriate for yawed motion and is defined for

$$-\frac{b}{2}\cos\Lambda \leq y \leq \frac{b}{2}\cos\Lambda$$

The mean-square averaged intensity of the vertical component of turbulence is then

$$\overline{w_e^2} = \frac{2\cos^2\Lambda}{b^2}\int_0^{b\cos\Lambda}\Gamma(\eta)\,\psi_w\left(\frac{\eta}{\cos\Lambda}\right)d\eta$$

and by a change of variable to $\eta' = \eta/\cos\Lambda$ this mean-square intensity may be seen to be unaffected by the yawing process (although its spectral resolution changes in the process).

For the swept wing both $y_2 - y_1$ and $|y_2| - |y_1|$ occur in the integral, so that the reduction of the double integral for $\psi_{w_e}(U\tau)$ to a single integral (see Equation (2.3.5)) cannot be effected so simply. The double integral for the swept wing is

$$\psi_{w_e}(U\tau) = \int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\gamma(y_1)\gamma(y_2)\,\psi_w\left(\sqrt{\{U\tau+(|y_2|-|y_1|)\tan\Lambda\}^2+(y_2-y_1)^2}\right)dy_1\,dy_2 \tag{2.4.1}$$

From this integral and using rectangular lift distributions and the point correlation function of Case 1 the mean-square of the averaged turbulence as well as the corresponding correlation function and power spectrum have been calculated by numerical integration for various sweep angles Λ, maintaining the ratio $\beta/\cos\Lambda$ at 0.5. (The decision to hold $\beta/\cos\Lambda$ rather than β constant was reached as a result of the foregoing analysis of the yawed unswept wing, which indicated that the effects of sweep should be minimized in this manner.) The results for $\overline{w_e^2}$ are shown in Figure 1, and the effect of sweep on $\overline{w_e^2}$ may be seen to be quite small for this comparison. The calculated correlation functions and spectra are not shown because they agreed with those for $\Lambda = 0$ within better than 1 percent for most values of $U\tau$ and k', respectively.

5. The Mean-Square Rolling Moment

In the preceding section it has been shown that the averaging effect of the span tends, essentially, only to reduce the mean-square intensity of the turbulence; it does not introduce any new considerations. If the analysis is extended to the rolling moment, however, a new phenomenon appears, because a wing which is so small relative to the scale of turbulence that at any instant all of its points experience the same turbulent velocity experiences no rolling moment as the result of the direct action of turbulence (although it may experience a rolling moment indirectly as a result of the rolling and yawing motion caused by the lateral component of the turbulence). On the other hand, on a large wing the different intensities of the turbulence at different points on the span give rise directly to a net rolling moment, which then results in rolling motion. In this section the mean-square value of this moment is calculated.

At any instant t the rolling moment $L'(t)$ can be written in the same form as the lift $L(t)$ in Equation (2.2.1); the lift-influence function $h(t,y)$ is, however, according to the previously mentioned reciprocity theorem, the lift distribution for an indicial roll with unit helix angle at the wing tip. If the assumption of invariance of this distribution in time is made, as for the symmetric case, then the lift-influence function can now be written as

$$h'(t, y) = \frac{1}{b}\, h'(t)\, \gamma'(y) \tag{2.5.1}$$

where $h'(t) = C_{l_p}(t)\, q\, S b / U$, and where the steady-state lift distribution $\gamma'(y) = \frac{c c_l}{\bar{c} C_l}$ now pertains to a linear antisymmetric angle of attack. Hence

$$\int_{-\frac{b}{2}}^{\frac{b}{2}} \gamma'(y)\, y\, dy = b^2 \tag{2.5.2}$$

The correlation function for the moment can then be written as

$$\psi_{L'}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h'(t_1)\, h'(t_2)\, \psi'_{w_e}\left(U(\tau + t_1 - t_2)\right) dt_1\, dt_2 \tag{2.5.3}$$

where

$$\begin{aligned} \psi'_{w_e}(U\tau) &= \frac{1}{b^2} \int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}} \gamma'(y_1)\, \gamma'(y_2)\, \psi_w(U\tau, y_2 - y_1)\, dy_1\, dy_2 \\ &= \frac{2}{b^2} \int_0^b \Gamma'(\eta)\, \psi_w(U\tau, \eta)\, d\eta \end{aligned} \tag{2.5.4}$$

where, in turn

$$\Gamma'(\eta) = \int_{-\frac{b}{2}}^{\frac{b}{2} - \eta} \gamma'(y)\, \gamma'(y + \eta)\, dy \tag{2.5.5}$$

If $\psi'_{w_e}(U\tau)$ is evaluated from Equation (2.5.4) and its Fourier transform $\varphi'_{w_e}(\omega)$ obtained, the spectrum of the mean-square rolling moment is

$$\varphi_{L'}(\omega) = \left(\frac{C_{l_p} q S b}{U}\right)^2 |\phi(k)|^2 \varphi'_{w_e}(\omega) \tag{2.5.6}$$

and the mean-square value of the rolling moment can be obtained from the integral of this spectrum.

This calculation has not been performed herein; on the basis of the calculations described in the preceding sections the spectrum $\varphi'_{w_e}(\omega)$ is likely to be similar to $\varphi_w(\omega)$ if normalized with the quantity $\overline{w_e'^2}$, which is calculated in the next paragraph; if this assumption is valid, the mean-square rolling moment is given by Equation (2.3.23), with C_{L_α} replaced by $C_{l_p} b$, and $\overline{w^2}$ replaced by $\overline{w_e'^2}$.

The quantity $\overline{w_e'^2}$ is defined as

$$\overline{w_e'^2} = \psi'_{w_e}(0)$$

For a linear lift distribution (which corresponds to rectangular loading in the symmetrical case) and the point correlation function of Case 1,

$$\begin{aligned}\overline{w_e'^2} &= \frac{48\,\overline{w^2}}{\beta^4}\left[\frac{1}{4}\beta^3 - 6 + e^{-\beta}\left(6 + 6\beta + 3\beta^2 + \frac{3}{4}\beta^3\right)\right] \\ &= 2\,\overline{w^2}\left[\frac{9}{5}\beta - \beta^2 + \cdots\right]\end{aligned} \tag{2.5.7}$$

the second of these expressions being obtained by developing the exponential term in power of β. These expressions indicate that as the span tends to zero so does the mean-square rolling moment.

6. Generalized Aerodynamic Influence Functions in Unsteady Flow

The aerodynamic influence functions used in the preceding sections define the contribution of a given station of a wing to the total lift and rolling moment. In the analysis of a flexible wing, and even in the calculation of certain properties of a rigid wing, generalized aerodynamic influence functions which define the contribution of one station on the wing to the lift at another and thus represent Green's function for the unsteady aerodynamic problem, are required. No work appears to have been done on such functions. For steady flow, apart from some calculations currently in progress for supersonic speeds, which are based on a subdivision of a given wing into a number of squares, the only available results appear to be those given in References 8 and 9. The analysis in this section will therefore be based on Reference 8 and consists in a generalization of the method presented therein to unsteady flow.

This method constitutes an attempt to predict the lift distribution on the basis of knowledge concerning a few definite angle-of-attack distribution and may therefore be termed a function-interpolational method. For the present purpose a lift distribution will be found which for any angle-of-attack distribution has the correct lift and rolling moment and which for angles of attack varying linearly over the span reduces to the correct distributions as well.

Let $\gamma_D(y)$ and $\gamma_R(y)$ be the dimensionless lift distributions $\frac{c c_l}{\bar{c} C_L}$ for uniform angle of attack of the given wing in direct and reverse steady flow, respectively. A lift distribution which for any angle-of-attack distribution $\alpha(y)$ meets the foregoing conditions is then the following[8]:

$$\gamma(y) = \bar{\alpha}\,\gamma_D(y) + K\left(\alpha(y) - \bar{\alpha}\right)\gamma_R(y) \tag{2.6.1}$$

where

$$\bar{\alpha} = \frac{1}{b}\int_{-\frac{b}{2}}^{\frac{b}{2}} \gamma_R(y)\,\alpha(y)\,dy \tag{2.6.2}$$

$$K = \frac{C_{l_p}}{C_{L_\alpha}}\;\frac{b^3}{4\int_0^{\frac{b}{2}} \gamma_R(y)\;y^2\,dy} \tag{2.6.3}$$

(Values of K may be obtained from Reference 8.)

That the lift distribution given by Equation (2.6.1) does indeed satisfy these conditions may be seen with the aid of the relations (which follow from elementary definitions and from the aforementioned reciprocity theorem)

$$C_L = \frac{C_{L_\alpha}}{b}\int_{-\frac{b}{2}}^{\frac{b}{2}} \gamma(y)\,dy$$

$$= \frac{C_{L_\alpha}}{b}\int_{-\frac{b}{2}}^{\frac{b}{2}} \gamma_R(y)\,\alpha(y)\,dy$$

$$\frac{1}{b}\int_{-\frac{b}{2}}^{\frac{b}{2}} \gamma_D(y)\,dy = \frac{1}{b}\int_{-\frac{b}{2}}^{\frac{b}{2}} \gamma_R(y)\,dy = 1$$

$$C_l = \frac{C_{L_\alpha}}{b^2}\int_{-\frac{b}{2}}^{\frac{b}{2}} \gamma(y)\;y\,dy$$

$$C_{l_p} = \frac{C_{L_\alpha}}{b^2}\int_{-\frac{b}{2}}^{\frac{b}{2}} \left[\gamma(y)\right]_{\alpha = \frac{y}{b/2}}\;y\,dy$$

By substituting the value for $\bar{\alpha}$ from Equation (2.6.2) in Equation (2.6.1) the following expression is obtained for $\gamma(y)$:

$$\gamma(y) = \frac{1}{b}\int_{-\frac{b}{2}}^{\frac{b}{2}} \left[\left\{\gamma_D(y) - K\gamma_R(y) + bK\,\delta(y-\eta)\right\}\gamma_R(\eta)\right]\alpha(\eta)\,d\eta \tag{2.6.4}$$

The expression in the brackets under the integral of this equation will be designated by $\tilde{\gamma}(y,\eta)$, so that

$$\tilde{\gamma}(y,\eta) = \left[\gamma_D(y) - K\gamma_R(y) + bK\,\delta(y-\eta)\right]\gamma_R(\eta)$$

This function is the desired generalized aerodynamic influence function. Thus,

$$\gamma(y) = \frac{1}{b}\int_{-\frac{b}{2}}^{\frac{b}{2}} \tilde{\gamma}(y,\eta)\,\alpha(\eta)\,d\eta \tag{2.6.5}$$

The preceding analysis can be applied to the oscillatory case at a given reduced frequency k as well as to the steady case. If now, as before, the assumption of invariance of distributions (properly normalized) with time or frequency is made, then $\gamma_D(y)$, $\gamma_R(y)$, and K are independent of frequency, so that Equation (2.6.5) can be written as

$$\begin{aligned} l(y;k) &= C_{L_\alpha}(k)\,\bar{c}\,q\,\gamma(y) \\ &= \frac{C_{L_\alpha}(k)\,\bar{c}\,q}{b}\int_{-\frac{b}{2}}^{\frac{b}{2}} \tilde{\gamma}(y,\eta)\,\alpha(\eta)\,d\eta \end{aligned} \tag{2.6.6}$$

where $\tilde{\gamma}(y,\eta)$, defined as before, is independent of k. By applying a Fourier transformation to both sides of this equation the following relation is then obtained for flight through continuously varying turbulence (Cf. Equation (2.2.1), as modified by the assumption stated in Equation (2.2.2)):

$$l(y,t) = \int_{-\infty}^{\infty} h(t_1)\, dt_1 \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{1}{b}\, \tilde{\gamma}(y,\eta)\, w(t-t_1,\eta)\, d\eta \tag{2.6.7}$$

where $l(y,t)$ is the lift per unit span at station y and time t, and where the function $h(t)$ is now $h(t) = C_{L_\alpha}(t)\, q\, \bar{c}$.

The correlation function for this lift can thus be written as

$$\psi_{l_y}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t_1)\, h(t_2)\, dt_1 dt_2\, \frac{1}{b^2} \int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}} \tilde{\gamma}(y,\eta_1)\, \tilde{\gamma}(y,\eta_2)\, \psi_w(U(\tau+t_1-t_2), \eta_2-\eta_1)\, d\eta_1 d\eta_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t_1)\, h(t_2)\, \psi_{w_\varepsilon}(U(\tau+t_1-t_2))\, dt_1\, dt_2 \tag{2.6.8}$$

where the function

$$\psi_{w_\varepsilon}(U\tau) = \frac{1}{b^2} \int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}} \tilde{\gamma}(y,\eta_1)\, \tilde{\gamma}(y,\eta_2)\, \psi_w(U\tau, \eta_2-\eta_1)\, d\eta_1 d\eta_2$$

represents an effective correlation function, which when transformed to the equivalent power spectrum $\varphi_{w_\varepsilon}(\omega)$ can be used to obtain the power spectrum for $l(y,t)$ and hence its mean-square value. Thus,

$$\varphi_{l_y}(\omega) = \left(\frac{C_{L_\alpha} q \bar{c}}{U}\right)^2 |\phi(k)|^2\, \varphi_{w_\varepsilon}(\omega) \tag{2.6.9}$$

The remainder of this section will be concerned with the calculation of $\varphi_{w_\varepsilon}(\omega)$.

In order to anticipate future needs, the function $\psi_{w_\varepsilon}(U\tau)$ will be defined in a somewhat more general form than in the preceding paragraph, namely

$$\psi_{w_\varepsilon}(U\tau) = \frac{1}{b^2} \int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}} \tilde{\gamma}(y_1,\eta_1)\, \tilde{\gamma}(y_2,\eta_2)\, \psi_w(U\tau, \eta_2-\eta_1)\, d\eta_1 d\eta_2 \tag{2.6.10}$$

If the assumption is now made that $\gamma_R(y)$ and $\gamma_D(y)$ are the same, they can both be identified with a function $\gamma(y)$ used in Equation (2.2.3), so that

$$\tilde{\gamma}(y,\eta) = \left[(1-K)\,\gamma(y) + bK\,\delta(y-\eta)\right]\gamma(\eta) \tag{2.6.11}$$

and Equation (2.6.10) becomes

$$\begin{aligned}\psi_{w_\varepsilon}(U\tau) = \Big[(1-K)^2\psi_{w_e}(U\tau) &+ K(1-K)\left\{\tilde{\psi}_{w_e}(U\tau, y_1) + \tilde{\psi}_{w_e}(U\tau, y_2)\right\} \\ &+ K^2\,\psi_w(U\tau, y_2 - y_1)\Big]\gamma(y_1)\,\gamma(y_2)\end{aligned} \tag{2.6.12}$$

where $\psi_{w_e}(U\tau)$ is the correlation function calculated previously for the averaged vertical component of turbulence, and where

$$\tilde{\psi}_{w_e}(U\tau, y) = \frac{1}{b}\int_{-\frac{b}{2}}^{\frac{b}{2}} \gamma(\eta)\,\psi_w\left(U\tau, |y-\eta|\right) d\eta \tag{2.6.13}$$

Similarly, then,

$$\begin{aligned}\varphi_{w_\varepsilon}(\omega) = \Big[(1-K)^2\varphi_{w_e}(\omega) &+ K(1-K)\left\{\tilde{\varphi}_{w_e}(\omega, y_1) + \tilde{\varphi}(\omega, y_2)\right\} \\ &+ K^2\,\varphi_w(\omega, y_2 - y_1)\Big]\gamma(y_1)\,\gamma(y_2)\end{aligned} \tag{2.6.14}$$

where $\tilde{\varphi}_{w_e}(\omega, y)$ is the Fourier transform of $\tilde{\psi}_{w_e}(U\tau, y)$. For Case 1 and rectangular loading

$$\begin{aligned}\tilde{\varphi}_{w_e}(\omega, y) = \frac{\tilde{L}}{\pi U}\,\overline{w^2}\,\frac{1}{(1+k'^2)^{5/2}}\Big[3k'^2\left\{\beta' Ki_0(\beta') - \beta'^2 K_0(\beta')\right\} \\ + 3k'^2\left\{\beta'' Ki_0(\beta'') - \beta''^2 K_0(\beta'')\right\} + \beta'^2 K_1(\beta') + \beta''^2 K_1(\beta'')\Big]\end{aligned}$$

where

$$\beta' = \left(\frac{\beta}{2} - y^*\right)\sqrt{1+k'^2}$$

$$\beta'' = \left(\frac{\beta}{2} + y^*\right)\sqrt{1+k'^2}$$

$$y^* = \frac{y}{\tilde{L}}$$

This function is shown in Figure 4 for several values of β and $\frac{y}{b/2}$.

For Case 2 and rectangular loading

$$\tilde{\varphi}_{w_e}(\omega,y) = \frac{\tilde{L}}{\pi U}\,\frac{\overline{w^2}}{\beta}\,e^{-\frac{k'^2}{\pi}}\left[\left\{\left(\frac{\beta}{2}-y^*\right)^2 \operatorname{erf}\left(\frac{\frac{\beta}{2}-y^*}{2/\sqrt{\pi}}\right) + \left(\frac{\beta}{2}+y^*\right)^2 \operatorname{erf}\left(\frac{\frac{\beta}{2}+y^*}{2/\sqrt{\pi}}\right)\right\} + \frac{2}{\pi}k'^2\left\{\operatorname{erf}\left(\frac{\frac{\beta}{2}-y^*}{2/\sqrt{\pi}}\right) + \operatorname{erf}\left(\frac{\frac{\beta}{2}+y^*}{2/\sqrt{\pi}}\right)\right\}\right] \tag{2.6.15}$$

Also,

$$\varphi_w(\omega,y) = \frac{1}{\pi U}\int_{-\infty}^{\infty} e^{i\frac{\omega}{U}U\tau}\,\psi_w\left(\sqrt{U^2\tau^2+y^2}\right)d(U\tau) \tag{2.6.16}$$

and is, for Case 1 and rectangular loading:

$$\varphi_w(\omega,y) = \frac{\tilde{L}}{\pi U}\,\overline{w^2}\left[y^*\frac{1+3k'^2}{(1+k'^2)^2}K_1\left(y^*\sqrt{1+k'^2}\right) - y^{*2}\frac{1}{1+k'^2}K_0\left(y^*\sqrt{1+k'^2}\right)\right] \tag{2.6.17}$$

and for Case 2 and rectangular loading:

$$\varphi_w(\omega,y) = \frac{\tilde{L}}{\pi U}\,\overline{w^2}\,e^{-\frac{k'^2}{\pi}}\left[1 - \frac{\pi}{2}y^{*2} + \frac{2}{\pi}k'^2\right] \tag{2.6.18}$$

It may be noted that the spectrum $\varphi_w(\omega,y)$ given by Equations (2.6.16), (2.6.17) and (2.6.18) reduces to the ordinary point power spectrum $\varphi_w(\omega)$ when y approaches 0. (See Equations (2.1.5), (2.3.15) and (2.3.16)). Also, in order to obtain the more restricted form of $\varphi_{w_\varepsilon}(\omega)$ used in Equation (2.6.9) from the more general one given in Equation (2.6.14) it is necessary only to set $y_1 = y_2 = y$ in the latter.

7. Mean-Square Bending and Pitching Moments

When the variation of the gust intensity along the span is taken into account the mean-square lift $\overline{L^2}$ is not an adequate index of the stresses in the wing, nor is the mean-square lift distribution

$\overline{l^2(y)}$. Instead, the mean-square bending and twisting moments, as well as the mean-square vertical shear, must be calculated directly.

Each of these quantities can be expressed in terms of a certain influence function, in a manner analogous to that employed for the lift and rolling moment. However, although the reciprocity theorem can still be used to relate these influence functions to certain lift distributions, the required lift distributions cannot be obtained so readily. Therefore, in this Section the alternative approach is adopted of constructing these influence functions from the generalized lift influence functions (Green's functions) considered in the previous Section.

The bending moment at any station y, $0 \leq y \leq \frac{b}{2}$, and at any time t can be obtained from the lift distribution considered in the preceding Section, as

$$
\begin{aligned}
M_B(y,t) &= \int_y^{\frac{b}{2}} (y'-y)\, l(y',t)\, dy' \\
&= \int_y^{\frac{b}{2}} (y'-y)\, dy' \int_{-\infty}^{\infty} h(t_1)\, dt_1 \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{1}{b}\, \tilde{\gamma}(y',\eta)\, w(t-t_1,\eta)\, d\eta \\
&= \int_{-\infty}^{\infty} h(t_1)\, dt_1 \int_{-\frac{b}{2}}^{\frac{b}{2}} g(y,\eta)\, w(t-t_1,\eta)\, d\eta
\end{aligned}
\qquad (2.7.1)
$$

where the new influence function for the bending moment is

$$
g(y,\eta) = \frac{1}{b} \int_y^{\frac{b}{2}} (y'-y)\, \tilde{\gamma}(y',\eta)\, dy'
$$

so that, upon introducing the previously used function for $\tilde{\gamma}(y,\eta)$

$$M_B(y,t) = \int_{-\infty}^{\infty} h(t_1)\,dt_1 \int_{-\frac{b}{2}}^{\frac{b}{2}} \left[(1-K)\,T(y) + K\,S(y,\eta)\right]\gamma(\eta)\,w(t-t_1,\eta)\,d\eta \tag{2.7.2}$$

where

$$T(y) = \frac{1}{b}\int_y^{\frac{b}{2}} (y'-y)\,\gamma(y')\,dy'$$

$$S(y,\eta) = \begin{cases} \eta - y, & \eta > 0 \\ 0, & \eta \le 0 \end{cases}$$

so that, specifically for the root bending moment

$$T(0) = \frac{1}{b}\int_0^{\frac{b}{2}} y'\,\gamma(y')\,dy'$$

$$S(0,\eta) = \begin{cases} \eta, & \eta > 0 \\ 0, & \eta \le 0 \end{cases}$$

Hence, the correlation function for the root bending moment is

$$\psi_{M_B}(\tau) = b^4 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t_1)\,h(t_2)\,dt_1\,dt_2 \int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{1}{b^4}\Big[(1-K)^2\,T^2(y) + (1-K)K\,T(0)\{S(0,\eta_1)$$
$$+ S(0,\eta_2)\} + K^2 S(0,\eta_1)\,S(0,\eta_2)\Big]\gamma(\eta_1)\gamma(\eta_2)\,\psi_w\big(U(\tau+t_1-t_2),\,\eta_2-\eta_1\big)\,d\eta_1\,d\eta_2 \tag{2.7.3}$$

The power spectrum of the root bending moment can thus be obtained, following the approach used in the preceding sections, by evaluating the inner pair of integrals, taking the Fourier transform of the result with respect to τ, and multiplying the power spectrum obtained in this manner by $\left(\frac{C_{L\alpha}\, q\, S\, b}{U}\right)^2 |\phi(k)|^2$.

By constructing suitable influence functions the power spectrum of the torque and the vertical shear at any station can be obtained in a similar manner.

For a swept wing the variation of the gust intensity along the span results in a pitching moment which must be taken into account in calculations of the dynamic response of the airplane to continuous turbulence. This pitching moment can be obtained in substantially the same manner as the bending moment. Thus, if $\bar{y}$ is the station of the mean aerodynamic chord

$$
\begin{aligned}
M(t) &= \tan\Lambda \int_{-\frac{b}{2}}^{\frac{b}{2}} (\bar{y} - |y|)\, l(y,t)\, dy \\
&= \tan\Lambda \int_{-\infty}^{\infty} h(t_1)\, dt_1 \int_{-\frac{b}{2}}^{\frac{b}{2}} g'(\eta)\, w(t-t_1, \eta)\, d\eta
\end{aligned}
\tag{2.7.4}
$$

where

$$
g'(\eta) = \frac{1}{b} \int_{-\frac{b}{2}}^{\frac{b}{2}} (\bar{y} - |y|)\, \tilde{\gamma}(y,\eta)\, dy
$$

so that, with the previously used approximation to the Green's function

$$
g'(\eta) = \left[(1-K)\left\{\bar{y} - 2\,T(0)\right\} + K(\bar{y} - |\eta|)\right] \gamma(\eta)
$$

The correlation function, spectrum and mean-square value of the pitching moment can then be obtained in the manner used in the preceding sections.

8. Wing-Tail Correlation

The tail strikes a given gust some time after the wing does; as a result, a pitching moment arises which does not exist in steady or quasi-steady motion. This pitching moment can, for the purpose of the present paper, be analyzed either in terms of the correlation between the gusts at the wing and those at the tail or, by including a time-lag term in the indicial-response function of the tail, in terms of the correlation between the wing and tail response functions.

The somewhat artificial case of a small wing and tail separated by a relatively large distance will be analyzed first. Only the distribution of turbulence along a line (the flight path) rather than in a portion of a plane will be needed. The pitching moment due to the vertical component of atmospheric turbulence can then be written as

$$M(t) = \int_{-\infty}^{\infty} h_w(t_1)\, w\left(U(t-t_1)\right) dt_1 + \int_{-\infty}^{\infty} h_t(t_1)\, w\left(x_t + U(t-t_1)\right) dt_1 \tag{2.8.1}$$

where x_t is the tail length, and where $h_w(t)$ and $h_t(t)$ are the pitching-moment responses to indicial gusts hitting the wing and tail, respectively, at $t = 0$; both may include unsteady-lift effects, and, if downwash-effects are to be considered, $h_w(t)$ should include the contribution to the pitching moment of the tail lift caused by the downwash at the tail associated with the lift on the wing which results from the indicial gust.

The spectrum of this moment can then be written as

$$\varphi_M(\omega) = \left[\left| H_w(\omega) + H_t(\omega) \right|^2 - 2\,\Re\left\{ \left(1 - e^{i\frac{\omega}{U}x_t}\right) H_w^*(\omega)\, H_t(\omega) \right\} \right] \varphi_w(\omega) \tag{2.8.2}$$

where the symbols $\Re$ and * designate, respectively, the real part and complex conjugate of a complex number. Obviously, when x_t approaches 0 the second term in the bracket in this equation vanishes, so that the first term represents the perfect-correlation effect, and the second represents the correction for imperfect correlation.

In order to furnish an estimate of the magnitude of the effects under consideration some calculations have been made on the basis of the assumption that

$$H_w(\omega) = \mu H_t(\omega)$$

which implies that the attenuation with frequency of the contributions of the wing and tail lifts, respectively, to the pitching moment is the same. The ratio μ is -1 for neutral stability, and $\mu > -1$ for stable flight; positive values of μ represent aerodynamic-center locations (tail off) behind the center of gravity and are not likely to be incurred with normal configurations and flight conditions.

For this case

$$\varphi_M(\omega) = |H_t(\omega)|^2 \left[(1+\mu)^2 - 2\mu \left(1 - \cos \frac{\omega x_t}{U}\right)\right] \varphi_w(\omega)$$

and

$$\frac{\overline{M^2}}{(\overline{M^2})_{\text{tail alone}}} = (1+\mu)^2 + 2\mu\, \Psi\left(\frac{x_t}{L}, \frac{x_t}{\bar{c}/2}\right) \tag{2.8.3}$$

The function Ψ has been calculated for several values of its arguments using Equations (2.3.13) and (2.3.15) and is shown in Figure 5, as is the ratio of the mean-square moments on the left side of Equation (2.8.3). The effect of imperfect correlation may be seen to be very large as the condition of neutral stability is approached; the entire pitching moment is then the result of instantaneous differences in gust intensities at the wing and tail.

In general, however, the tail length and the span are of the same order of magnitude, so that an analysis of the effect of imperfect correlation between the wing and tail must take into account the averaging effect of the wing span. The pitching moment at any instant is then

$$M(t) = \int_{-\infty}^{\infty} h_w(t_1)\,dt_1 \int_{-\frac{b}{2}}^{\frac{b}{2}} \gamma(y_1)\, w\big(U(t-t_1),\, y_1\big)\, dy_1\, dt_1 + \int_{-\infty}^{\infty} h_t(t_1)\, w\big(U(t-t_1)+x_t,\, 0\big)\, dt_1 \qquad (2.8.4)$$

Hence, the power spectrum of this moment is

$$\varphi_M(\omega) = |H_w(\omega)|^2 \varphi_{w_e}(\omega) + |H_t(\omega)|^2 \varphi_w(\omega) + 2\,\mathcal{R}\left[e^{i\frac{\omega}{U}x_t} H_t(\omega) H_w^*(\omega)\right] \tilde{\varphi}_{w_e}(\omega, 0) \qquad (2.8.5)$$

where $\varphi_{w_e}(\omega)$ is the averaged spectrum of Figure 3 or Equation (2.3.22), and $\tilde{\varphi}_{w_e}(\omega, 0)$ is the spectrum of Figure 4 or Equation (2.6.15) for y = 0. (It should be noted that $h_t(t)$ is the response to an indicial response which strikes the tail at t = 0 ; if it were the indicial response to a gust which strikes the wing at t = 0 the factor $e^{i\frac{\omega}{U}x_t}$ in Equation (2.8.5) would not be required.)

III. DYNAMICS OF THE RIGID AIRPLANE

In this Part the motions of an airplane subjected to atmospheric disturbances will be considered. The assumption will be made that these motions are small enough to permit the use of linear approximations to the resulting aerodynamic forces and to permit the linear superposition of these forces. The longitudinal degrees of freedom (pitch, vertical and horizontal motion) and the lateral degrees of freedom (yaw, sideslip and roll) can therefore be considered separately.

The airplane will be considered to be in steady level flight prior to disturbance. The motions studied will be the disturbances from their mean values; for instance, the flight-path angle θ considered here will be the difference between the disturbed and the initial value of the flight-path angle. Hence, the motions and forces calculated by the method indicated here must be added to their mean values to obtain the total motions and forces.

1. The Equations of Longitudinal Motion

The equations of motion of an airplane can be expressed in coordinates referred to body-centered axes or in coordinates referred to space-centered axes.[10] Body-centered axes have the advantage that the aerodynamic forces related to them can be measured more easily. For that reason they are almost universally used in analyses of airplane stability and are usually referred to as stability axes. In view of the very close relation of such an analysis to the problem considered here these axes will be used here, but for the flexible airplane space-centered axes will be used.

because they are slightly more convenient for that purpose.

It has been pointed out previously that the mean-square value of the output of a linear system subjected to a random input can be obtained from the integral of the power spectrum of the input multiplied by the absolute square of the transfer function of the system, the transfer function being the response of the system to sinusoidal oscillations of unit amplitude. Thus, for instance,

$$\overline{\ddot{z}^2} = \int_0^\infty \left|H_{\ddot{z}}^{w}(\omega)\right|^2 \varphi_w(\omega)\, d\omega \tag{3.1.1}$$

where $H_{\ddot{z}}^{w}(\omega)$ is the transfer function from the vertical gust velocity (input) to the normal acceleration (output) of the airplane. For the purpose of a statistical analysis the airplane is thus characterized by its transfer functions, and the desired solution of the equations of motion thus consists in the required transfer functions. Hence, in this Section attention will be confined to oscillatory motions.

For this case the linearized equations of longitudinal motion can be written as follows (see Equations II-193 of Reference 10, for instance:

$$\begin{bmatrix} i\omega - Z_w & -Z_u & -i\omega U \\ -X_w & i\omega - X_u & g \\ -i\omega M_{\dot{w}} - M_w & -M_u & -\omega^2 - i\omega M_q \end{bmatrix} \begin{Bmatrix} w_p \\ u_p \\ \theta \end{Bmatrix} = \phi(k) \begin{bmatrix} Z_w & Z_u \\ X_w & X_u \\ M_w & M_u \end{bmatrix} \begin{Bmatrix} w \\ u \end{Bmatrix} \tag{3.1.2}$$

In these equations the stability derivatives are defined in terms of conventional aerodynamic coefficients as follows, with the numerical values being those of the example used in Reference 10:

$$Z_w = -\frac{qS}{mU}\left(C_{L_\alpha} + C_D\right) = -1.430$$

$$Z_u = -\frac{2qS}{mU}\left(C_{L_u} + C_L\right) = -0.0955$$

$$X_w = \frac{qS}{mU}\left(C_L - C_{D_\alpha}\right) = 0.0016$$

$$X_u = -\frac{2qS}{mU}\left(C_{D_u} + C_D\right) = -0.0097$$

$$M_w = \frac{qS\bar{c}}{I_{yy}U} C_{m_\alpha} = -0.0235$$

$$M_{\dot{w}} = \frac{qS\bar{c}^2}{2I_{yy}U^2} C_{m_{\dot{\alpha}}} = -0.0013$$

$$M_u = \frac{2qS\bar{c}}{I_{yy}U}\left(C_{m_u} + C_m\right) = 0$$

$$M_q = \frac{qS\bar{c}^2}{2I_{yy}U} C_{m_q} = -1.920$$

(Other parameters of the example that will be needed are W = 30, 500 pounds, U = 660 fps, $\bar{c}$ = 10 feet, and altitude 20, 000 feet.)

In analyses of the stability of a rigid airplane the quasi-steady approximation to unsteady-lift effects is usually made (which, in effect, retains terms up to t^2 of an expansion of the unsteady forces according to powers of t and is thus equivalent to considering the forces corresponding to a steady attitude, plus those corresponding to constant disturbance velocities, plus those corresponding to constant accelerations). This approximation is justified because the motions of concern are sufficiently slow. For the same reason this approximation can also be made in analyzing the response of an airplane to atmospheric turbulence.

However, in this problem another type of unsteady-lift effect occurs, namely that related to the forces directly attributable to the turbulence. This effect has been discussed in the preceding Section and is here taken into account by multiplying the quasi-steady values of the forces due to gusts on the right side of Equation (3.1.3) by the attenuation function $\phi(k)$. This implies the assumption that the airplane is small relative to the scale of turbulence, inasmuch as no averaging effects have been taken into account; these effects will be discussed presently. Also, this attenuation function is strictly applicable only to the normal forces. The unsteady effects on the drag are not known because of the relatively complicated nature of the mechanism which gives rise to drag. If, however, the assumption is made that upon entry into a sharp-edged gust the drag rises linearly and attains its steady-state value in the time required to travel N chord lengths, the drag equivalent of $|\phi(k)|^2$ is the expression

$$\frac{1-\cos 2Nk}{2N^2k^2}$$

which, for N equal to about 5 or 6, agrees fairly well with $|\phi(k)|^2$ in the region of interest ($k \ll 1$). The unsteady moment is also difficult to predict because of the paucity of knowledge concerning unsteady downwash effects for wings of finite span. However, inasmuch as the wing lift contributes part of the moment and, through the mechanism of downwash, determines the moment contributed by the tail to a large extent, the use of the lift attenuation function for the moment appears reasonable, and the use of the same function for the lift, drag and moment facilitates the analysis.

In Equation (3.1.2) the unknown quantities w_p and u_p are the normal and axial components of the disturbance velocities of the airplane relative to the free stream. They are related to the coordinates z and x referred to space-fixed axes (or, rather, axes translating at a velocity equal to the mean velocity of the airplane) as follows:

$$\begin{aligned} \dot{z} &= -w_p + U\theta \\ \dot{x} &= -u_p \end{aligned} \tag{3.1.3}$$

Hence, by means of Equation (3.1.3), Equation (3.1.2) can be expressed in terms of z and x. These quantities and their time derivatives have a more direct bearing on the loads experienced by the airplane and the degree of passenger discomfort, and, consequently, will be the ones considered in the following.

In studies of the longitudinal stability of airplanes Equation (3.1.2) is rarely solved in the form given here. It is usually reduced to two equations with two unknowns, either u_p and θ (the phugoid case) or w_p and θ (the short-period case), the short-period case being usually the one of primary interest. The part of the turbulent energy contained in the frequency range near the phugoid frequency is so small that the phugoid case has no significance for the analysis of loads and motions resulting from atmospheric turbulence. Hence, the short-period case, which ignores the phugoid oscillations, furnishes an excellent approximation to the longitudinal motions of an airplane in turbulent air. Another two-degree-of-freedom case, namely the one involving w_p and u_p , however, is useful in certain studies of the effects related to the interaction of horizontal and vertical components of turbulence.

Both of these two-degree cases can be reduced to the single-degree-of-freedom case involving only z (or w_p). For airplanes which have a large moment of inertia in pitch this simple case furnishes a good approximation. It has been studied in Reference 2, using substantially the same approximations to the unsteady-lift effects as the ones made here, except tor the fact that in Reference 2 apparent-mass effects were included. (These effects are not included in the stability derivatives used in Equation (3.1.2) because they are usually small (less than 1°/o of the mass of the airplane) and are different for each degree of freedom. However, if desired, the apparent mass pertaining to a given degree of freedom can easily be added to the airplane mass in calculating the stability derivatives.) In the following, attention will be confined to the short-period case, although the analysis is equally applicable to the other case and easily extended to the case of three degrees of freedom.

2. Solution of the Equations of Longitudinal Motion

If the degree of freedom pertaining to x (or u_p) is ignored, the solution of Equation (3.1.2) can be written as

$$\begin{Bmatrix} \ddot{z} \\ \ddot{\theta} \end{Bmatrix} = \begin{bmatrix} H_z^w(\omega) & H_z^u(\omega) \\ H_\theta^w(\omega) & H_\theta^u(\omega) \end{bmatrix} \begin{Bmatrix} w \\ u \end{Bmatrix} \tag{3.2.1}$$

where the transfer function $H_z^w(\omega)$ is defined by

$$H_z^w(\omega) = \phi(k) \frac{-A_z^w \omega^2 + B_z^w i\omega + C_z^w}{-A_0 \omega^2 + B_0 i\omega + C_0} \tag{3.2.2}$$

where, in turn, the coefficients are defined by

$$A_0 = 1$$

$$B_0 = -(Z_w + M_q + UM_{\dot{w}})$$

$$C_0 = M_q Z_w - UM_w$$

$$A_z^w = -Z'_w$$

$$B_z^w = Z'_w\,(U M_{\dot{w}} + M_q)$$

$$C_z^w = -U(M'_w Z_w - M_w Z'_w)$$

The transfer function $H_\theta^w(\omega)$ can be defined similarly in terms of the coefficients,

$$A_\theta^w = M'_w + Z'_w M_{\dot{w}}$$

$$B_\theta^w = -(M'_w Z_w - M_w Z'_w)$$

$$C_\theta^w = 0$$

In these equations a distinction has been made between the values of Z_w, Z_u, M_w and M_u which occur on the right side of Equation (3.1.2) and are here designated by a prime mark, and those on the left side of that equation; the reason for this distinction will be discussed in a later Section. Furthermore, the coefficients A_z^u, B_z^u, C_z^u, A_θ^u, B_θ^u, and C_θ^u are the same as the coefficients A_z^w, B_z^w, ..., except that Z_w, Z'_w, M_w and M'_w are replaced by Z_u, Z'_u, M_u, and M'_u.

With these transfer functions the mean-square values of $\ddot{z}$ and $\ddot{\theta}$ can be calculated from the spectra of w and u using Equation (3.1.1), provided that the simultaneous action of w and u is taken into account. In order to analyze this effect the vertical acceleration $\ddot{z}$ will be considered, but the analysis will be applicable to $\ddot{\theta}$ or any other characteristic of the airplane which responds to w and u. Furthermore, the transfer functions need not be those considered in Equation (3.2.1), but can be those calculated for the three-degree-of-freedom system or for a flexible airplane.

For the present purpose the indicial-response functions, which are the Fourier transforms of the transfer functions, such as

$$h_z^w(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} H_z^w(\omega)\, d\omega \tag{3.2.3}$$

are more convenient. In terms of these indicial-response functions

$$\ddot{z}(t) = \int_{-\infty}^{\infty} h_z^w(t_1)\, w(t-t_1)\, dt_1 + \int_{-\infty}^{\infty} h_z^u(t_1)\, u(t-t_1)\, dt_1 \tag{3.2.4}$$

Then, if $w(t)$ and $u(t)$ are stationary in a statistical sense, the correlation function for $\ddot{z}(t)$ can be written as

$$\begin{aligned} \psi_{\ddot{z}}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Big[& h_z^w(t_1)\, h_z^w(t_2)\, \psi_w(\tau+t_1-t_2) + h_z^w(t_1)\, h_z^u(t_2)\, \psi_{wu}(\tau+t_1-t_2) \\ & + h_z^u(t_1)\, h_z^w(t_2)\, \psi_{wu}(-\tau+t_1-t_2) + h_z^u(t_1)\, h_z^u(t_2)\, \psi_u(\tau+t_1-t_2)\Big] dt_1\, dt_2 \end{aligned} \tag{3.2.5}$$

where $\psi_{wu}(\tau)$ is a cross-correlation of w and u defined by

$$\psi_{wu}(\tau) = \overline{w(t)\; u(t+\tau)}$$

Now, if the turbulence is isotropic, two mutually perpendicular velocity components are statistically independent, and their correlation is zero. Therefore the two terms in Equation (3.2.5) involving $\psi_{wu}(\tau)$ vanish, and the power spectrum of $\ddot{z}$ is

$$\varphi_{\ddot{z}}(\omega) = \left|H_z^w(\omega)\right|^2 \varphi_w(\omega) + \left|H_z^u(\omega)\right|^2 \varphi_u(\omega) \tag{3.2.6}$$

so that, generally speaking, the power spectrum of a response which depends on both the horizontal and the vertical component of turbulence is simply the sum of the power spectra of the two contributions, provided the turbulence is isotropic. (This statement is true, even if the distribution of the gusts over the span is taken into account.)

For the short-period two-degree-of-freedom case, then

$$\overline{\ddot{z}^2} = \int_0^\infty |\phi(k)|^2 \frac{A_z^{w\,2}\omega^4 - \omega^2(2A_z^w C_z^w - B_z^{w\,2}) + C_z^{w\,2}}{\omega^4 - (2C_0 - B_0^2)\omega^2 + C_0^2} \varphi_w(\omega)\, d\omega + \int_0^\infty |\phi(k)|^2 \frac{A_z^{u\,2}\omega^4 - \omega^2(2A_z^u C_z^u - B_z^{u\,2}) + C_z^{u\,2}}{\omega^4 - (2C_0 - B_0^2)\omega^2 + C_0^2} \varphi_u(\omega)\, d\omega \tag{3.2.7}$$

where the second integral is usually negligible compared to the first and will be disregarded in the following. Using Equations (2.3.13) and (2.3.15) the first integral has been evaluated (by using the technique of partial fractions for the integrand) for the example of Reference 10, and the results are shown in Figure 6, as are the results calculated similarly for the three-degree-of-freedom case, the other two-degree-of-freedom case, (horizontal and vertical motion, and referred to in the Figure as the zero-pitch case), and the single-degree-of-freedom case (vertical motion).

The preceding treatment of the short-period case has the advantage of using readily available information concerning the characteristics of any given airplane. For the purpose of trend studies a dimensionless form of the transfer functions is preferable. Equation (3.2.7) can be written in dimensionless form as

$$\frac{\overline{\ddot{z}^2}}{g^2} = \left(\frac{U^2}{g\,\bar{c}/2}\right)^2 \frac{\overline{w^2}}{U^2} \frac{4}{K^2} \int_0^\infty |\phi(k)|^2 \frac{k^4 + 4(\nu - \frac{1}{K})^2 k^2}{k^4 - 2(k_0^2 - \nu^2)k^2 + (k_0^2 + \nu^2)^2} \frac{U\varphi_w(k')}{\tilde{L}\,\overline{w^2}}\, dk' \tag{3.2.8}$$

and similarly

$$\frac{\overline{\ddot{\theta}^2}(\frac{\bar{c}}{2})^2}{g^2} = \left(\frac{U^2}{g\,\bar{c}/2}\right)^2 \frac{\overline{w^2}}{U^2} \left[(\nu - \tfrac{1}{K})^2 + k_0^2\right]^2 \int_0^\infty |\phi(k)|^2 \frac{k^4}{k^4 - 2(k_0^2 - \nu^2)k^2 + (k_0^2 + \nu^2)^2} \frac{U\varphi_w(k')}{\tilde{L}\,\overline{w^2}}\, dk' \tag{3.2.9}$$

where **K** is the mass parameter

$$K = \frac{8m}{C_{L_\alpha}\rho S \bar{c}} \tag{3.2.10}$$

ν is the dimensionless damping coefficient

$$\nu = \frac{\bar{c}}{2U} \quad \frac{\log_e 2}{T_{1/2}}$$

$T_{1/2}$ is the time to damp to one-half amplitude, which is given by

$$\frac{\log_e 2}{T_{1/2}} = -\frac{1}{2} \left(Z_w + M_q + U M_{\dot{w}} \right)$$

and k_0 is the dimensionless frequency of the short-period oscillations,

$$k_0 = \frac{\omega_0 \bar{c}}{2U}$$

with

$$\omega_0 = \sqrt{M_q Z_w - U M_w - \frac{1}{4} \left(Z_w + M_q + U M_{\dot{w}} \right)^2}$$

It may be noted that for this two-degree-of-freedom case the dimensionless mean-square responses are functions of only two additional parameters, which are dimensionless forms of the outstanding characteristics of the short-period case (the short-period frequency and the time to damp to one-half amplitude), beyond those encountered in the single-degree-of-freedom case[2], namely the mass parameter κ and the scale parameter

$$s = \frac{\tilde{L}}{\bar{c}/2}$$

which relates the dimensionless frequencies k and k' by

$$k' = sk$$

3. Special Problems Related to the Longitudinal Motion of Large Airplanes

In the preceding Sections the airplane has tacitly been assumed to be small in the sense of this paper, inasmuch as the lateral variation of the intensity of turbulence has not been taken into account. In this Section this restriction is removed by introducing the aerodynamic forces calculated in the first Part of this paper into the dynamic analysis of the preceding Sections. The arguments advanced in the preceding Section for ignoring horizontal gusts and horizontal motions are equally valid for the large airplane; therefore these gusts and motions will not be considered here.

For the single-degree-of-freedom case involving only vertical motion the required modification is very simple. For this case the transfer function is

$$H_z^w(\omega) = -\phi(k)\ \frac{i\omega Z'_w}{i\omega - Z_w} \tag{3.3.1}$$

where Z'_w, attenuated by $\phi(k)$, is the stability derivative for vertical gusts and, hence, represents the lift per unit gust intensity. Therefore, using the result for the lift calculated in Section 3 of Part II of this paper, the mean-square normal acceleration becomes

$$\overline{\ddot{z}^2} = \int_0^\infty |\phi(k)|^2\ \frac{Z_w^2\omega^2}{\omega^2 + Z_w^2}\ \varphi_{w_e}(\omega)\, d\omega \tag{3.3.2}$$

where $\varphi_{w_e}(\omega)$ is defined by Equation (2.3.10). This expression differs from the result obtained in Reference 2 only by the fact that $\varphi_w(\omega)$ is here replaced by $\varphi_{w_e}(\omega)$. If the difference in the shape of these spectra is ignored, Equation (3.3.2) can be written as

$$\overline{\ddot{z}^2} = \frac{\overline{w_e^2}}{\overline{w^2}} \int_0^\infty |\phi(k)|^2\ \frac{Z_w^2\omega^2}{\omega^2 + Z_w^2}\ \varphi_w(\omega)\, d\omega$$

where the integral represents the mean-square normal acceleration of a small airplane and can be obtained directly from Reference 2; the ratio $\frac{\overline{w_e^2}}{\overline{w^2}}$ can, for the two types of point correlation functions considered here, be obtained from Figure 1.

For the two-degree-of-freedom (short-period) case the analysis is somewhat more complicated. As indicated in Equation (3.2.1), the transfer function for $\ddot{z}$ is now

$$H_z^w(\omega) = \left[-\frac{1}{m} \frac{\omega^2 + (UM_{\dot{w}} + M_q)i\omega + UM_w}{-\omega^2 + B_0 i\omega + C_0} \right] m\,(-Z'_w)\,\phi(k) + \left[\frac{1}{I_{yy}} \frac{U(-Z_w)}{-\omega^2 + B_0 i\omega + C_0} \right] I_{yy} M'_w\,\phi(k) \tag{3.3.3}$$

(The following analysis can be applied equally well to $\ddot{\theta}$ by using $H_\theta^w(\omega)$, as defined for Equation (3.2.1), instead of $H_z^w(\omega)$.)

If the expressions inside the brackets of Equation (3.3.3) are designated, respectively, by $H_1(\omega)$ and $H_2(\omega)$, and their Fourier transforms by $h_1(t)$ and $h_2(t)$, then, as a result of the definitions of Z'_w and M'_w,

$$\ddot{z}(t) = \int_{-\infty}^{\infty} h_1(t_1)\, L(t-t_1)\,dt_1 + \int_{-\infty}^{\infty} h_2(t_1)\, M(t-t_1)\,dt_1 \tag{3.3.4}$$

where $L(t)$ and $M(t)$ are the instantaneous lift and pitching moment due to the vertical component of atmospheric turbulence, which have been obtained in Part II of this paper. The calculation of $\varphi_{\ddot{z}}(\omega)$ or of $\overline{\ddot{z}^2}$ thus requires not only the spectra of L and M, the calculation of which has been discussed, but also of the cross-spectrum of L and M, which has to be calculated directly from Equations (2.2.1) and (2.8.4). The result is

$$
\begin{aligned}
\varphi_{\ddot{Z}}(\omega) = {} & |H_1(\omega)|^2 \, |H(\omega)|^2 \, \varphi_{w_e}(\omega) \\
& + |H_2(\omega)|^2 \Big[|H_w(\omega)|^2 \varphi_{w_e}(\omega) + |H_t(\omega)|^2 \varphi_w(\omega) \\
& \qquad + 2\,\Re\left\{ e^{\,i\frac{\omega}{U}x_t} H_t(\omega) H_w^*(\omega) \right\} \tilde{\varphi}_{w_e}(\omega, 0) \Big] \\
& + 2 \Big[\Re\left\{ H_1(\omega) H_2^*(\omega) H(\omega) H_w^*(\omega) \right\} \varphi_{w_e}(\omega) \\
& \qquad + \Re\left\{ H_1(\omega) H_2^*(\omega) H(\omega) H_t^*(\omega) e^{\,i\frac{\omega}{U}x_t} \right\} \tilde{\varphi}_{w_e}(\omega, 0) \Big]
\end{aligned}
\tag{3.3.5}
$$

where the first two terms represent the contributions of the spectra of L and M, respectively (see Equations (2.3.11) and (2.8.5)), and the third represents the contribution of the cross-spectrum of L and M. For the present purpose the functions $H(\omega)$, $H_w(\omega)$ and $H_t(\omega)$ can be expressed as

$$H(\omega) = m \, (-Z_w) \, \phi(k)$$

$$H_w(\omega) = m \, \Delta x \, (-Z_w) \, \phi(k)$$

$$H_t(\omega) = \left[I_{yy} M_w - m \, \Delta x \, (-Z_w) \right] \phi(k)$$

where Δx is the distance from the aerodynamic center (tail off) to the airplane center of gravity. This definition of the contributions of the wing and tail to the pitching moment is based on the consideration that the direct contribution of the wing can be estimated quite accurately, and that the total pitching moment is likely to be known from experiments, so that the contribution of the tail (which includes the effect of the wing lift on the downwash at the tail) can be determined as the difference of the two. The functions $\varphi_{w_e}(\omega)$ and $\tilde{\varphi}_{w_e}(\omega, 0)$ have been defined in Part II of this paper. In view of the fact that the function $\phi(k)$ contained in some of the terms of Equation (3.3.5) always appears in terms multiplied by others which contain $\phi^*(k)$ only the absolute square of this function is required, as before.

4. Special Problems Related to the Lateral Motion of Large Airplanes

The equations of motion in the lateral degrees of freedom (roll, yaw, sideslip) have the same form and can be solved in the same way as those for the longitudinal motion. (See pages 53 to 67 of Reference 10.) Again it is convenient to cast the problem in the form used in a stability analysis in order to take advantage of the results of such an analysis. For a small airplane it is necessary only to replace the terms due to rudder deflection by corresponding terms involving side gusts, namely

$$Y^*_{\delta_R}\delta_R \text{ by } Y_v \frac{v}{U} \quad , \quad L_{\delta_R}\delta_R \text{ by } L_\beta \frac{v}{U} \quad , \text{ and } \quad N_{\delta_R}\delta_R \text{ by } N_\beta \frac{v}{U}$$

(in the notation of Reference 10), and to disregard the terms corresponding to aileron deflection; in the lateral degrees of freedom the small airplane thus reacts laterally only to side gusts. On the other hand, the large airplane also reacts in the lateral degrees of freedom to vertical gusts through the rolling moment calculated in Section 5 of Part II of the present paper. If this rolling moment is to be included, it replaces the term $L_{\delta_A}\delta_A$.

Instead of treating all three degrees of freedom involved in this motion, in stability analyses two one-degree-of-freedom cases are often considered, namely the one of sideslip alone, with angle of yaw equal and opposite to angle of sideslip (the dutch-roll case), and the one of rolling alone. The dutch-roll case may be used for gust-loads purposes in connection with yawing and sideslipping motion due to lateral gusts, provided the phase of the motion is not important. (As may be noted from the preceding sections, the phase of a transfer function is important only in terms involving cross-spectra.) However, it does not appear to be as satisfactory an approximation as those for longitudinal motion; it can probably be improved by including rolling motion while still maintaining the fixed relation between sideslip and yaw.

For rolling motions due to rolling moments, the single-degree case of rolling alone appears to furnish a very good approximation; although the rolling motion causes yawing and sideslipping motions these motions do not appear to reflect on the rolling motion. Thus the rolling motion which results from the rolling moment to which large airplanes are subjected in turbulent air can be calculated without regard to the other lateral degrees of freedom. Since, furthermore, within the assumption of small motions the stresses associated with these other lateral degrees of freedom do not generally contribute appreciably to those associated with the longitudinal degrees of freedom in the parts of the structure for which the latter are critical, such as the wing (although they may be critical for other parts of the structure, such as the vertical tail), these degrees will be ignored in the treatment of the large flexible airplane in Part V. However, if chordwise bending effects are important, as they may be in some cases at speeds close to the flutter speed, these other lateral degrees of freedom may have to be included in the analysis.

For the large airplane, which responds in the lateral degrees of freedom both to vertical gusts and to side gusts, the superposition of the resulting responses, such as stresses, may be effected in the way indicated for the interaction of horizontal and vertical gusts. If the turbulence is isotropic, the vertical and lateral gusts are statistically independent, so that the spectrum of a given response is equal to the sum of the spectrum of that part of the given response which is due to vertical gusts and the spectrum of that part of the response which is due to lateral gusts.

5. Combination of the Results Obtained from Analyses of the Longitudinal and Lateral Degrees of Freedom

The instantaneous wing stresses depend both on the motions in the longitudinal degrees of freedom (primarily vertical motion and pitch) and on those in the lateral degrees of freedom (primarily rolling). The purpose of this section is to indicate how the stresses associated with vertical motion and pitch can be combined with those associated with rolling due to vertical gusts, particularly in the case of a large rigid airplane. (A small airplane, flexible or rigid, does not roll as a result of the action of vertical gusts, and for the large flexible airplane it is more convenient to consider rolling motion simultaneously with the others, so that the superposition is effected automatically in the process of obtaining the required transfer functions.) For all airplanes the effect of side gusts can then be taken into account, if isotropy is assumed, by adding the stress spectra directly.

The stress at some point of the wing structure can, for some instant of time, be written as

$$\sigma(t) = \left(\frac{\partial\sigma}{\partial\ddot{z}}\right)\ddot{z}(t) + \left(\frac{\partial\sigma}{\partial\ddot{\theta}}\right)\ddot{\theta}(t) + \left(\frac{\partial\sigma}{\partial\dot{p}}\right)\dot{p}(t) \qquad (3.5.1)$$

where the partial derivatives represent the stresses per unit acceleration. These stress derivatives include the effect of the instantaneous air-loads causing the accelerations and of the inertia-loads which result from them. The stresses due to $\ddot{\theta}$ will not be considered any further because the type of problem involved in combining them with the stresses due to $\ddot{z}$ is one of using the proper cross-correlations, and has been considered previously, and the type of problem of combining them with the stresses due to $\dot{p}$ is the same as that of combining stresses due to $\ddot{z}$ and due to $\dot{p}$, which is considered in the following.

Now, as in the preceding Section, $\ddot{z}(t)$ and $\dot{p}(t)$ can be expressed in terms of the instantaneous lift and rolling moment due directly to atmospheric turbulence by means of indicial-response functions. Thus,

$$\left.\begin{aligned} \ddot{z}(t) &= \int_{-\infty}^{\infty} h_1(t_1)\, L\,(t-t_1)\, dt_1 \\ \dot{p}(t) &= \int_{-\infty}^{\infty} h_2(t_1)\, L'\,(t-t_1)\, dt_1 \end{aligned}\right\} \qquad (3.5.2)$$

(In the equation for $\ddot{z}(t)$ the contribution of the pitching moment has been ignored, not because it is insignificant but because it can easily be included using the principles outlined in the preceding Sections and because it will not affect the conclusion arrived at in this Section.)

The lift and rolling moment can, in turn, be expressed in terms of lift influence functions $\gamma(y)$ and $\gamma'(y)$ which are symmetric and antisymmetric, respectively, as in Part II, yielding

$$\left.\begin{aligned} L(t) &= \int_{-\infty}^{\infty} h(t_1)\,dt_1 \frac{1}{b}\int_{-\frac{b}{2}}^{\frac{b}{2}} \gamma(y)\, w(t-t_1, y)\, dy \\ L'(t) &= \int_{-\infty}^{\infty} h'(t_1)\,dt_1 \frac{1}{b}\int_{-\frac{b}{2}}^{\frac{b}{2}} \gamma'(y)\, w(t-t_1, y)\, dy \end{aligned}\right\} \qquad (3.5.3)$$

Substitution of Equations (3. 5. 3) into (3. 5. 2) and thence into (3. 5. 1) permits the calculation of a correlation function for $\sigma(t)$. This correlation function involves the auto-correlations of $L(t)$ and $L'(t)$, and also the cross-correlation of $L(t)$ and $L'(t)$. This cross-correlation can be written as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t_1)\, h'(t_2)\, dt_1\, dt_2 \frac{1}{b^2}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}} \gamma(y_1)\gamma'(y_2)\, \psi_w(\tau+t_1-t_2, y_2-y_1)\, dy_1\, dy_2$$

But by virtue of the fact that ψ_w is symmetric with respect to $|y_2-y_1|$ and by virtue of the symmetry properties of $\gamma(y)$ and $\gamma'(y)$ the inner double integral vanishes. Hence the cross-correlation of the lift and rolling moment is zero, and

$$\begin{aligned} \varphi_\sigma(\omega) = {} & \left(\frac{\partial\sigma}{\partial\ddot{z}}\right)^2 |H_1(\omega)|^2\, |H(\omega)|^2\, \varphi_{w_e}(\omega) \\ & + \left(\frac{\partial\sigma}{\partial\dot{p}}\right)^2 |H_2(\omega)|^2\, |H'(\omega)|^2\, \varphi'_{w_e}(\omega) \end{aligned} \qquad (3.5.4)$$

where $\varphi_{w_e}(\omega)$ is the power spectrum obtained as part of the calculation of the lift in Part II, and $\varphi'_{w_e}(\omega)$ is a spectrum obtained in a similar manner for the rolling moment, using $\gamma'(y)$ instead of $\gamma(y)$. Therefore, the power spectrum of the stress due to vertical gusts is simply the sum of the power spectra of that part of the stress which results from the lift or normal acceleration due to those gusts and of that part of the stress which results from the rolling moment or rolling acceleration due to those gusts.

IV. DYNAMICS OF THE SMALL FLEXIBLE AIRPLANE

The purpose of this Part is to outline a method of obtaining transfer functions relating the stresses at various points of a small flexible airplane to the vertical gusts which cause them. The method consists of an extension of the numerical-integration method of aero-elastic analysis described in Reference 11 to sinusoidal motions of the airplane. This extension takes into account the facts that the aerodynamic forces now have out-of-phase as well as in-phase parts, and that the structural deformations can no longer be characterized by angle-of-attack changes but that vertical deflections must now be calculated separately. Also, the "rigid-body" degrees of freedom of vertical and pitching motion of the airplane as a whole and structural deformations of the tail are now taken into account.

The result is a set of linear algebraic equations for the airplane motions and deformations in terms of the applied aerodynamic forces due directly to gusts. The desired transfer functions can then be obtained from solutions of these equations at various frequencies, and the mean-square stresses are given by the integral of the product of the absolute square of these transfer functions and of the point power spectrum of the vertical component of turbulence.

Although only longitudinal motions are considered here, the lateral motions caused by side gusts acting on the vertical tail can be analyzed in the same manner.

1. The Loads Applied to the Wing

The loads applied to the wing stem from three sources, namely the aerodynamic loads due directly to the action of the gusts, the aerodynamic loads due to the motions of the airplane, and the inertia loads.

The lift and pitching moment (about the elastic axis) per unit span on a two-dimensional airfoil undergoing sinusoidal angle-of-attack changes and vertical motions in incompressible flow are[12]

$$l_a = 2\pi q c\left[C(k)\left\{(1+2e_4 ik)\alpha - ik\frac{z}{c/2}\right\} + \left(\frac{ik}{2} - e_3 k^2\right)\alpha + \frac{k^2}{2}\frac{z}{c/2}\right]$$

$$m_a = 2\pi q c^2\left[C(k)e_1\left\{(1+2e_4 ik)\alpha - ik\frac{z}{c/2}\right\} - \left\{\frac{e_4}{2}ik - \left(\frac{1}{32} + e_3^2\right)k^2\right\}\alpha - \frac{e_3}{2}k^2\frac{z}{c/2}\right] \qquad (4.1.1)$$

The terms multiplied by C(k) are referred to as the circulatory terms because they are calculated from the bound and shed vorticity, and the others as the potential terms. The potential terms are in the nature of additional-apparent-mass effects, and all those which involve k^2 are usually treated together with the inertia forces rather than the aerodynamic forces. For compressible flow, however, the forces are calculated in a different manner, and the division of the forces into circulatory and potential parts has then little meaning. Therefore, in order to facilitate the extension of this analysis to compressible flow, this distinction will not be made in the following.

The aerodynamic forces will therefore be written as

$$l_a = 2\pi q c\left[\tilde{C}_1(k)\alpha + \tilde{C}_2(k)\frac{z}{c/2}\right]$$

$$m_a = 2\pi q c^2\left[\tilde{C}_3(k)\alpha + \tilde{C}_4(k)\frac{z}{c/2}\right] \qquad (4.1.2)$$

so that for incompressible flow

$$\tilde{C}_1(k) = (1 + 2e_4 ik)\,C(k) + \frac{ik}{2} - e_3 k^2$$

$$\tilde{C}_2(k) = -ik\,C(k) + \frac{k^2}{2}$$

$$\widetilde{C}_3(k) = (1 + 2e_4 ik)\, e_1\, C(k) - \frac{e_4}{2} ik + \frac{1}{32} + e_3^{\,2}$$

$$\widetilde{C}_4(k) = -ik\, e_1\, C(k) - \frac{e_3}{2} k^2$$

In order to calculate the lift at a given point of a wing of finite span an appropriate Green's function is required. An approximation to this function is given in Section 6 of Part II of this paper based on a reciprocity theorem of linearized lifting surface theory; as used for the computations of that section this function implies the assumptions that the spanwise distribution of the lift for oscillations of the wing as a whole is substantially invariant with frequency and that this distribution is the same in direct as in reverse flow. Neither of these assumptions is essential to the analysis but both, and particularly the first, simplify it considerably. With these assumptions the desired lift distribution is then given by expressions of the form of Equations (2.6.6) and (2.6.11).

For the present purpose, however, a set of aerodynamic influence coefficients is required, rather than influence functions. Such a set of coefficients, based on the same ideas, can be obtained readily by the techniques used in References 8 and 9. The result may be expressed as follows:

$$\{l\}_a = C_{L_\alpha} \bar{c}\, q \left\{ \widetilde{C}_1(k)[Q]\{\alpha\} + \widetilde{C}_2(k)[Q]\left\{\frac{z}{\bar{c}/2}\right\} \right\} \tag{4.1.4}$$

where the aerodynamic influence-coefficient matrix $[Q]$ is defined by

$$[Q] = (1-K)\, \lceil \gamma \rfloor \{1\} \lfloor\, I \,\rfloor \lceil \gamma \rfloor + K \lceil \gamma \rfloor$$

where, in turn, $\{1\}$ is a unit column matrix, and $\lfloor I \rfloor$ is a row of integrating coefficients suitable for integrating a continuous function for a range of its argument from 0 to 1. Thus, for instance, if n equidistant points on the semispan are considered, and n is odd, then according to Simpson's rule

$$\lfloor I \rfloor = \frac{1}{n-1} \lfloor \tfrac{1}{3}, \tfrac{4}{3}, \tfrac{2}{3}, \cdots \qquad , \tfrac{4}{3}, \tfrac{1}{3} \rfloor$$

Very little information is available concerning the spanwise distributions of the pitching moment on wings of finite span in unsteady flow. By means of the reciprocity theorems an appropriate Green's function could be estimated if the lift distribution for wings with parabolic camber were known; however, such lift distributions do not appear to have been calculated for wings of finite span in unsteady flow. In fact, relatively little is known about moment distributions even in steady flow. However, the available information indicates that the local center of pressure does not appear to be very sensitive to the lift distribution(8). Hence, it will be assumed that this is also true in unsteady flow at a given frequency, and that, furthermore, these centers of pressure are given by two-dimensional theory. With this assumption the moment can be written as

$$\{m\}_a = C_{L_\alpha} \bar{c} q \left\{ \tilde{C}_3(k) \lceil c \rfloor [Q] \{\alpha\} + \tilde{C}_4(k) \lceil c \rfloor [Q] \left\{ \frac{z}{\bar{c}/2} \right\} \right\} \qquad (4.1.5)$$

The lift and moment distribution due to the gust can be calculated in a similar manner. In the following, the magnitude of the gust intensity will be considered to be unity, and the longitudinal reference point will be the intersection of the elastic axis and the wing root, so that the instantaneous gust intensity at any station y is

$$W = \exp\left\{ -ik \frac{|y|}{\bar{c}/2} \tan \Lambda \right\} \tag{4.1.6}$$

and, with this function w,

$$\begin{aligned} \{l\}_g &= \frac{C_{L_\alpha} \bar{c} q \ \phi(k)}{U} [\, Q \,] \{w\} \\ \{m\}_g &= \frac{C_{L_\alpha} \bar{c} q \phi(k)}{U} [e_1 c] [Q] \{w\} \end{aligned} \tag{4.1.7}$$

where $\phi(k)$ is the Sears function, as before.

Finally, the inertia loads are

$$l_i = -\tilde{m}\ddot{z} + \tilde{m} e_2 c \ddot{\alpha}$$

$$m_i = -\left(r^2 + (e_2 c)^2\right)\tilde{m}\ddot{\alpha} + \tilde{m} e_2 c \ddot{z}$$

or

$$\begin{aligned} \{l\}_i &= -\frac{q}{\frac{\rho}{2}\left(\frac{\bar{c}}{2}\right)^2} k^2 [\tilde{m} e_2 c] \{\alpha\} + \frac{q}{\frac{\rho}{2}\left(\frac{\bar{c}}{2}\right)^2} k^2 [\tilde{m}] \{z\} \\ \{m\}_i &= \frac{q}{\frac{\rho}{2}\left(\frac{\bar{c}}{2}\right)^2} k^2 [r^2 + (e_2 c)^2] [\tilde{m}] \{\alpha\} - \frac{q}{\frac{\rho}{2}\left(\frac{\bar{c}}{2}\right)^2} k^2 [\tilde{m} e_2 c] \{z\} \end{aligned} \tag{4.1.8}$$

The loads applied to the wing can thus be written, in summary, as

$$\begin{Bmatrix} l \\ m \end{Bmatrix} = q \left[\begin{array}{c:c} 2 C_{L_\alpha} \tilde{C}_2(k) [Q] + \frac{k^2}{\frac{\rho}{2}\left(\frac{\bar{c}}{2}\right)^2} [\tilde{m}] & C_{L_\alpha} \bar{c} \tilde{C}_1(k) [Q] - \frac{k^2}{\frac{\rho}{2}\left(\frac{\bar{c}}{2}\right)^2} [\tilde{m} e_2 c] \\ \hdashline 2 C_{L_\alpha} \tilde{C}_4(k) [c] [Q] - \frac{k^2}{\frac{\rho}{2}\left(\frac{\bar{c}}{2}\right)^2} [\tilde{m} e_2 c] & C_{L_\alpha} \bar{c} \tilde{C}_3(k) [c] [Q] + \frac{k^2}{\frac{\rho}{2}\left(\frac{\bar{c}}{2}\right)^2} [r^2 + (e_2 c)^2] [\tilde{m}] \end{array}\right] \begin{Bmatrix} z \\ \alpha \end{Bmatrix} + q \left[\begin{array}{c} \frac{C_{L_\alpha} \bar{c} \phi(k)}{U} [Q] \\ \hdashline \frac{C_{L_\alpha} \bar{c} \phi(k)}{U} [e_1 c] [Q] \end{array}\right] \{w\} \tag{4.1.9}$$

2. The Loads Applied to the Tail

The loads applied to the tail are similar in nature to those applied to the wing, with the exception of the fact that the tail experiences additional loads as a result of the downwash produced by the lift on the wing. Again, little is known about the downwash in unsteady flow, and even in steady flow downwash cannot be predicted accurately because of boundary-layer effects on the fuselage and the wing root. Consequently, even in steady-flow analyses experimental results are usually relied upon.

In the following, the assumption will therefore be made that for steady-flow experimental results are available, in the form of the downwash derivative $\frac{\partial \varepsilon}{\partial \alpha}$. In order to obtain the attenuation of this value with frequency the results of the analysis of Reference 13 will be used. These results indicate that the time-variation of the tail lift due to the downwash caused by the wing lift which results from a unit jump in the wing angle of attack can be approximated by an immediate jump in the tail lift of -0.16 of the steady-state value and another jump to the steady-state value after the time required to travel the distance from the 45-percent-chord point of the wing to the quarter-chord point of the tail plus another eighth chord length. Hence, for sinusoidal angle-of-attack changes the tail lift due to downwash is

$$L_{t_{\varepsilon_{\alpha,z}}} = -C_{L_{\alpha_t}} q_t S_t \frac{\partial \varepsilon}{\partial \alpha}\left[-0.16 + 1.16\, e^{-ik\left(\frac{x'_t}{\bar{c}/2} + \frac{1}{4}\frac{c_r}{\bar{c}}\right)}\right]\left[(1+2e_4 ik)\alpha_0 - ik\frac{z_0}{\bar{c}/2}\right] \tag{4.2.1}$$

where x'_t is the distance from the intersection of the elastic axis and the wing root (assumed for this purpose to be at the 45 percent point on the root chord) and the aerodynamic center of the tail. This approximation is valid only for $k < 0.35$,[13]; this range is adequate for the present purpose, however.

Similarly, the downwash associated with the wing lift due to sinusoidal gusts gives rise to a tail lift which, within this approximation, is

$$L_{t_{\varepsilon_g}} = - \frac{C_{L_{\alpha_t}} q_t S_t}{U} \frac{\partial \varepsilon}{\partial \alpha}\left[-0.16 + 1.16\, e^{-ik\left(\frac{x'_t}{\bar{c}/2} + \frac{1}{4}\frac{c_r}{\bar{c}}\right)}\right] e^{-0.6\, ik \frac{c_r}{\bar{c}}} w_0 \tag{4.2.2}$$

(The additional lag represents the time required to travel the 0.6 root semi-chords from the 45-percent chord point of the root, which is the reference point for the gusts, to the 3/4-chord point of the wing root, which is assumed to be the point governing the lift at the wing root,[13] inasmuch as it is the centroid of the chordwise pressure influence function.)

The other aerodynamic forces are those due to the motions of the airplane, those due the tail deformations, and those due directly to the gusts. On the basis of the assumptions made in the preceding section these forces are

$$\begin{aligned} L_{t_{\alpha,z}} &= C_{L_{\alpha_t}} q_t S_t \left[\tilde{C}_5(k)\, \alpha_0 + \tilde{C}_6(k) \frac{z_0}{\bar{c}/2}\right] \\ &\quad + C_{L_{\alpha_t}} q_t S_t \left[\tilde{C}_7(k)\, \Delta\alpha + \tilde{C}_6(k) \frac{\Delta z}{\bar{c}/2}\right] \end{aligned} \tag{4.2.3}$$

$$L_{t_g} = \frac{C_{L_{\alpha_t}} q_t S_t}{U} \phi\left(\frac{c_t}{\bar{c}} k\right) e^{-ik \frac{x'_t - \frac{c_t}{4}}{\bar{c}/2}} \tag{4.2.4}$$

where

$$\tilde{C}_5(k) = \left(1 + 2ik \frac{x'_t + \frac{c_t}{2}}{\bar{c}}\right) C\left(\frac{c_t}{\bar{c}} k\right) + \frac{i}{2} \frac{c_t}{\bar{c}} k - \frac{c_t}{\bar{c}} \frac{x'_t + \frac{c_t}{4}}{\bar{c}} k^2$$

$$\tilde{C}_6(k) = -ik\, C\left(\frac{c_t}{\bar{c}} k\right) + \frac{1}{2} \frac{c_t}{\bar{c}} k^2$$

$$\tilde{C}_7(k) = \left(1 + 2ik \frac{e_{4_t} c_t}{\bar{c}}\right) C\left(\frac{c_t}{\bar{c}} k\right) + \frac{i}{2} \frac{c_t}{\bar{c}} k - e_{3_t}\left(\frac{c_t}{\bar{c}}\right)^2 k^2$$

and α_0 and z_0 are the angle of attack and vertical displacement of the airplane at the wing root.

The inertia load on the tail is

$$L_{t_i} = -m_t\left(\ddot{z}_0 + \Delta\ddot{z} - x'_t\,\ddot{\alpha}_0\right)$$

$$= \frac{m_t\,q\,k^2}{\frac{\rho}{2}\left(\frac{\bar{c}}{2}\right)^2}\left(z_0 + \Delta z - x'_t\,\alpha_0\right) \tag{4.2.5}$$

Here the center of gravity of the tail has been assumed to coincide with its aerodynamic center; in order to remove this assumption it is necessary only to add or subtract the distance between the two to x'_t in the preceding equation.

The normal forces on the tail can then be summarized as follows:

$$\begin{aligned} L_t = {} & q\left[F_1(k)\,z_0 + F_2(k)\,\alpha_0 + F_3(k)\,\Delta\alpha\right] \\ & + q\left[F_4(k)\,\Delta z + F_5(k)\,w_t\right] \end{aligned} \tag{4.2.6}$$

where

$$F_1(k) = 2C_{L_{\alpha_t}}\frac{q_t}{q}\frac{S_t}{\bar{c}}\left[\tilde{C}_6(k) + ik\frac{\partial\varepsilon}{\partial\alpha}\left(-0.16 + 1.16\,e^{-ik\left(\frac{x'_t}{\bar{c}/2} + \frac{1}{4}\frac{c_t}{\bar{c}}\right)}\right)\right] + \frac{m_t}{\frac{\rho}{2}\left(\frac{\bar{c}}{2}\right)^2}k^2$$

$$F_2(k) = C_{L_{\alpha_t}}\frac{q_t}{q}S_t\left[\tilde{C}_5(k) - (1 + 2e_4 ik)\frac{\partial\varepsilon}{\partial\alpha}\left(-0.16 + 1.16\,e^{-ik\left(\frac{x'_t}{\bar{c}/2} + \frac{1}{4}\frac{c_t}{\bar{c}}\right)}\right)\right] - \frac{m_t x'_t}{\frac{\rho}{2}\left(\frac{\bar{c}}{2}\right)^2}k^2$$

$$F_3(k) = C_{L_{\alpha_t}}\frac{q_t}{q}S_t\,\tilde{C}_7(k)$$

$$F_4(k) = 2C_{L_{\alpha_t}}\frac{q_t}{q}\frac{S_t}{\bar{c}}\,\tilde{C}_6(k) + \frac{m_t}{\frac{\rho}{2}\left(\frac{\bar{c}}{2}\right)^2}k^2$$

$$F_5(k) = \frac{C_{L_{\alpha_t}}\,q_t\,S_t}{U}\left[\phi\left(\frac{c_t}{\bar{c}}k\right)e^{ik\frac{c_t}{2\bar{c}}} - \frac{\partial\varepsilon}{\partial\alpha}\left(-0.16 + 1.16\,e^{-ik\left(\frac{x'_t}{\bar{c}/2} + \frac{1}{4}\frac{c_r}{\bar{c}}\right)}\right)e^{ik\left(\frac{x'_t}{\bar{c}/2} - 0.6\frac{c_r}{\bar{c}}\right)}\right]$$

and

$$w_t = e^{-ik\frac{x'_t}{\varepsilon/2}} \tag{4.2.7}$$

The pitching moments corresponding to these normal forces can be obtained in the manner employed for the wing. However, inasmuch as the tail chord is usually small compared to the fuselage length, the travel of the tail center of pressure is small compared to the length x'_t . Hence, the center of pressure will be assumed to remain at the aerodynamic center of the tail, and the pitching moments are then $(-x'_t)$ times the corresponding forces, so that

$$M_t = -q\, x'_t \left[F_1(k)\, z_0 + F_2(k)\alpha_0 + F_3(k)\Delta\alpha + F_4(k)\,\Delta Z + F_5(k)\, w_t \right] \tag{4.2.8}$$

3. The Wing and Tail Deformations

The wing deformations may be calculated either from structural influence coefficients or from the bending and torsion stiffnesses of the wing used in conjunction with simple beam theory. The latter approach will be followed here, based on the method of Reference 11.

The bending and torsion moments on the wing structure may be obtained by integrating the applied loads. If numerical methods are employed to perform these integrations the results may be written as follows:

$$\begin{Bmatrix} M_B \\ M_T \end{Bmatrix} = \left[\begin{array}{c:c} \frac{(b/2)^2}{\cos\Lambda}\,[\mathrm{II}] & -\sin\Lambda\,\frac{b}{2}\,[\mathrm{I}] \\ \hdashline [\mathrm{O}] & \cos\Lambda\,\frac{b}{2}\,[\mathrm{I}] \end{array} \right] \begin{Bmatrix} l \\ m \end{Bmatrix} \tag{4.3.1}$$

and, similarly, the deformations are

$$\begin{Bmatrix} z-z_0 \\ \alpha-\alpha_0 \end{Bmatrix} = \left[\begin{array}{c|c} \left(\frac{b}{2\cos\Lambda}\right)^2 [\mathrm{II}''] \lceil \frac{1}{EI} \rfloor & [0] \\ \hline \frac{b}{2} [\mathrm{I}''] \lceil \frac{1}{GJ} \rfloor & -\frac{b}{2} \tan\Lambda [\mathrm{I}''] \lceil \frac{1}{EI} \rfloor \end{array}\right] \begin{Bmatrix} M_B \\ M_T \end{Bmatrix}$$

$$= \left[\begin{array}{c|c} \left(\frac{b}{2}\right)^4 \frac{1}{\cos^2\Lambda} [\mathrm{II}''] \lceil \frac{1}{EI} \rfloor [\mathrm{II}] & -\left(\frac{b}{2}\right)^3 \frac{\sin\Lambda}{\cos^2\Lambda} [\mathrm{II}''] \lceil \frac{1}{EI} \rfloor [\mathrm{I}] \\ \hline \left(\frac{b}{2}\right)^3 \frac{1}{\cos\Lambda} [\mathrm{I}''] \lceil \frac{1}{GJ} \rfloor [\mathrm{II}] & -\sin\Lambda \left(\frac{b}{2}\right)^2 [\mathrm{I}''] \lceil \frac{1}{EI} + \frac{1}{GJ} \rfloor [\mathrm{I}] \end{array}\right] \begin{Bmatrix} l \\ m \end{Bmatrix}$$

(4.3.2)

where the integrating matrices[11] perform the following operations

$$[\mathrm{I}]\{f\} \doteq \int_\theta^1 f(\theta')\,d\theta'$$

$$[\mathrm{II}]\{f\} \doteq \int_\theta^1 \int_{\theta'}^1 f(\theta'')\,d\theta''\,d\theta'$$

$$[\mathrm{I}'']\{f\} \doteq \int_0^\theta f(\theta')\,d\theta'$$

$$[\mathrm{II}'']\{f\} \doteq \int_0^\theta \int_0^{\theta'} f(\theta'')\,d\theta''\,d\theta'$$

These integrating matrices may be based on the trapezoidal rule, Simpson's rule, or any other numerical method; the intervals chosen for θ $(0 \leq \theta \leq 1)$ need not be of constant width, except if a specific rule demands a uniform spacing.

The structural deformations of the wing may then be written in terms of the applied loads as

$$\begin{Bmatrix} z-z_0 \\ \alpha-\alpha_0 \end{Bmatrix} = q \left[\begin{array}{c|c} ① & ② \\ \hline ③ & ④ \end{array}\right] \begin{Bmatrix} z \\ \alpha \end{Bmatrix} + q \left[\begin{array}{c} ⑤ \\ \hline ⑥ \end{array}\right] \{w\} \qquad (4.3.3)$$

where the sub-matrices ①, ②, ③, ④, ⑤, and ⑥ designate, respectively, the four quadrants and two halves of the matrices, obtained by post-multiplying the square matrix of Equation (4.3.2) by the square and the rectangular matrix of Equation (4.1.9), respectively.

For the purpose of the present analysis, which is concerned primarily with the wing stresses, the tail deformations will be treated by including only the vertical displacement and angle-of-attack change of the tail as a whole due to the tail load. These quantities may be obtained from a static test which consists in applying a concentrated normal load at the aerodynamic center of the tail and measuring these deformations. They may also be obtained from a vibration test in which the deflection curve of the rear part of the fuselage in the lowest vertical-bending mode is measured; in this case the desired spring constants can be deduced from the frequency relation of a simple mass oscillator in terms of the measured frequency and of the mass of the empennage (including the part of the fuselage which may be considered to move with the empennage). In the absence of such tests these constants may be calculated in an analogous manner.

The tail deformations may then be written as

$$\begin{aligned} \Delta\alpha &= -K_1 L_t \\ \Delta Z &= K_2 L_t \end{aligned} \qquad (4.3.4)$$

so that, also

$$\Delta Z = -\frac{K_2}{K_1}\Delta\alpha \qquad (4.3.5)$$

Inasmuch as these deformations are not independent of each other, only one need be retained in the analysis. In the following, ΔZ will be eliminated by means of Equation (4.3.5), and $\Delta\alpha$ can then be obtained in a form similar to that used for the wing deformations in Equation (4.3.3), namely,

$$\Delta\alpha = -q\,K_1\left[F_1(k)\,z_0 + F_2(k)\,\alpha_0 + \left(F_3(k) - \frac{K_2}{K_1}F_4(k)\right)\Delta\alpha + F_5(k)\,w_t\right] \tag{4.3.6}$$

4. The Equations of Motion

Equations (4.3.3) and (4.3.6) are equations of motion inasmuch as they describe balances of aerodynamic structural and inertia forces. In fact, if the airplane fuselage were immobile ($\alpha_0 = z_0 = 0$), they would be sufficient to calculate all unknown quantities. However, if the fuselage is free to move two additional equations are required to obtain the two additional unknown quantities α_0 and z_0 . These additional equations are those expressing the dynamic equilibrium of the forces on the fuselage, namely

$$L_w + L_t - m_f\,(\ddot{z}_0 - \Delta x\,\ddot{\alpha}_0) = 0 \tag{4.4.1}$$

$$M_w - x_t L_t - \left(r_f^2 + (\Delta x)^2\right) m_f\,\ddot{\alpha}_0 + \Delta x\;m_f\,\ddot{z}_0 = 0 \tag{4.4.2}$$

where the wing lift and pitching moment can be expressed in terms of the lift and moment distributions l and m as

$$\begin{aligned} L_w &= \left\lfloor 2\,\tfrac{b}{2}\lfloor I \rfloor,\ \lfloor 0 \rfloor \right\rfloor \begin{Bmatrix} l \\ m \end{Bmatrix} \\ M_w &= \left\lfloor -2\left(\tfrac{b}{2}\right)^2 \tan\Lambda \lfloor II \rfloor,\ 2\,\tfrac{b}{2}\lfloor I \rfloor \right\rfloor \begin{Bmatrix} l \\ m \end{Bmatrix} \end{aligned} \tag{4.4.3}$$

or,

$$L_w = q \lfloor ⑪ \rfloor \begin{Bmatrix} z \\ \alpha \end{Bmatrix} + q \lfloor ⑬ \rfloor \{w\}$$

$$M_w = q \lfloor ⑫ \rfloor \begin{Bmatrix} z \\ \alpha \end{Bmatrix} + q \lfloor ⑭ \rfloor \{w\} \tag{4.4.4}$$

where $\lfloor ⑪ \rfloor$, $\lfloor ⑫ \rfloor$, $\lfloor ⑬ \rfloor$, and $\lfloor ⑭ \rfloor$ are the rows obtained by postmultiplying the rows of Equation (4.4.3) by the square and rectangular matrices of Equation (4.1.9). In Equation (4.4.1) the fuselage lift and moment has been neglected; it can easily be expressed in terms of α_0 and z_0 and included, if desired.

The equations for the tail deformation $\Delta\alpha$ and those for the over-all normal force and pitching moment can be combined with Equation (4.3.3) as follows: For the sake of definiteness it will be assumed that n stations on the wing are considered, including the one at the root, so that there are 2n+1 unknown quantities, and that in the column matrices defining applied loads, deformations, and so on, the values at the root of the wing are written at the top so that, in the following

$$\begin{Bmatrix} z \\ \alpha \\ \Delta\alpha \end{Bmatrix} = \begin{Bmatrix} z_{root} \\ \vdots \\ z_{tip} \\ \alpha_{root} \\ \vdots \\ \alpha_{tip} \\ \Delta\alpha \end{Bmatrix}$$

The first and $(n+1)^{th}$ equations of the system defined by Equation (4.3.3) express only the trivial fact that the structural deformation at the wing root is zero. They will be replaced by Equations (4.4.4) and (4.3.6), which will be joined to the system, yielding the combined equation of motion

$$\left[\begin{array}{c:c:c}
\begin{matrix}0&&&&&\\-1&1&&&&\\-1&0&1&&&\\-1&0&0&1&&\\-1&0&0&0&1&\\ \vdots&&&&&\ddots\end{matrix} & 0 & 0\\ \hdashline
0 & \begin{matrix}0&&&&&\\-1&1&&&&\\-1&0&1&&&\\-1&0&0&1&&\\-1&0&0&0&1&\\ \vdots&&&&&\ddots\end{matrix} & 0 \\ \hdashline
0 & 0 & 1
\end{array}\right]
\left\{\begin{matrix} z\\ \alpha \\ \Delta\alpha\end{matrix}\right\}
= q \left[\begin{array}{c:c:c} (1') & (2') & \\ \hdashline (3') & (4') & \\ \hdashline & & \end{array}\right]
\left\{\begin{matrix} z\\ \alpha \\ \Delta\alpha\end{matrix}\right\}
+ q \left[\begin{array}{c:c} (5') & \\ \hdashline (6') & \\ \hdashline 0 & \end{array}\right]
\left\{\begin{matrix} w\\ w_t\end{matrix}\right\} \tag{4.4.5}$$

The square matrix on the right side of this equation will be designated by $[A]$, the rectangular matrix by $[B]$, and the quasi-unit matrix on the left side of the equation by $[1']$, so that the equation can also be written as

$$\Big[[1'] - q[A]\Big]\left\{\begin{matrix} z\\ \alpha\\ \Delta\alpha\end{matrix}\right\} = q[B]\left\{\begin{matrix} w\\ w_t\end{matrix}\right\} \tag{4.4.6}$$

The matrices (1') to (6') are the same as the matrices (1) to (6) of Equation (4.3.3), except that the first rows of the latter, which are all zero, are replaced as follows:

First row of:	Replaced by:	With the following quantity added to the leading element:
(1)	First half of $\lfloor (11) \rfloor$	$F_1(k) + \dfrac{m_f}{\frac{\rho}{2}\left(\frac{\bar{c}}{2}\right)^2} k^2$
(2)	Second half of $\lfloor (11) \rfloor$	$F_2(k) - \dfrac{m_f\ \Delta x}{\frac{\rho}{2}\left(\frac{\bar{c}}{2}\right)^2} k^2$
(3)	First half of $\lfloor (12) \rfloor$	$-x_t F_1(k) - \dfrac{m_f x_t}{\frac{\rho}{2}\left(\frac{\bar{c}}{2}\right)^2} k^2$
(4)	Second half of $\lfloor (12) \rfloor$	$-x_t F_2(k) + \dfrac{m_f\left(r_f^2 + (\Delta x)^2\right)}{\frac{\rho}{2}\left(\frac{\bar{c}}{2}\right)^2} k^2$
(5)	$\lfloor (13) \rfloor$	0
(6)	$\lfloor (14) \rfloor$	0

Also, the elements of the last rows and columns of the matrices $[A]$ and $[B]$ are zero, except for the following:

$$A_{1,2n+1} = F_3(k) - \frac{K_2}{K_1} F_4(k)$$
$$A_{n+1,2n+1} = -x_t \left(F_3(k) - \frac{K_2}{K_1} F_4(k) \right)$$
$$A_{2n+1,1} = -K_1 F_1(k)$$
$$A_{2n+1,n+1} = -K_1 F_2(k)$$
$$A_{2n+1,2n+1} = -K_1 \left(F_3(k) - \frac{K_2}{K_1} F_4(k) \right)$$

$$B_{1,n+1} = F_5(k)$$
$$B_{n+1,n+1} = -x_t F_5(k)$$
$$B_{2n+1,n+1} = -K_1 F_5(k)$$

5. Solution of the Equations of Motion

For the purpose of calculating the desired transfer functions Equation (4.4.6) may be solved directly for a given value of q as a set of linear algebraic equations with coefficients given by the matrix $[1'] - q[A]$ and with "knowns" given by the column matrix $q[B]\begin{Bmatrix} w \\ w_t \end{Bmatrix}$ (where $[A]$, $[B]$ and $\begin{Bmatrix} w \\ w_t \end{Bmatrix}$ are functions of k). The result is a column matrix of the unknown amplitudes of the motions of the airplane. If this column is calculated for several values of k in the range of interest, these amplitudes, considered as functions of k, are transfer functions from the gust to the motions.

This column matrix can be substituted in Equation (4.1.9) and the resulting column matrix $\begin{Bmatrix} l \\ m \end{Bmatrix}$ substituted in Equation (4.3.1), yielding a column matrix of bending and twisting moments which again, considered as a function of k, represents transfer functions from the gust to these moments. A set of transfer functions for the vertical shear could be calculated similarly, from the relation

$$\{V\} = \left[\tfrac{b}{2} [I] \,\vdots\, [0] \right] \begin{Bmatrix} l \\ m \end{Bmatrix}$$

The stress at any point of the structure can be assumed to be given by a linear superposition of the bending moment, twisting moment and vertical shear at the given station if elementary beam theory is used. If due to the interaction of bending and torsion stresses or due to shear lag elementary theory cannot be used, the stress at a given point can be expressed as a linear superposition of moments and shears at other stations as well as the given station. In either case, the transfer function for the given stress is then given by the same linear superposition of the transfer functions for the corresponding moments and shears.

It may be noted that at zero frequency solution of Equation (4.4.6) yields the static aeroelastic deformations and thus permits the calculation of the changes in the lift distribution and the shift of the aerodynamic center as a result of static aeroelastic action. Also, inasmuch as Equation (4.4.6) completely describes the dynamic behavior of the airplane, it is possible to calculate from it the speeds at which aeroelastic instability phenomena occur, although such calculations are beyond the scope of this paper. Suffice it to point out that for such a calculation the body degrees of freedom must be eliminated first, as a result of the way in which the problem has been set up. This elimination can be effected readily by considering the first and $(n+1)^{th}$ rows of $[A]$ but with $A_{11}, A_{1,n+1}, A_{n+1,1}$ and $A_{n+1,n+1}$ replaced by 0. If these rows are premultiplied by

$$-\begin{bmatrix} A_{11} & A_{1,n+1} \\ A_{n+1,1} & A_{n+1,n+1} \end{bmatrix}^{-1}$$

and used as the first and $(n+1)^{th}$ rows of a matrix which is otherwise a unit matrix, and if this resulting matrix is referred to as $[1'']$, then Equation (4.4.6) can be written for homogeneous case as

$$\Big[[1'][1''] - q[A][1'']\Big]\begin{Bmatrix} z \\ \alpha \\ \Delta\alpha \end{Bmatrix} = \{0\} \tag{4.5.1}$$

The products $[1'][1'']$ and $[A][1'']$ will now have two null rows and columns each, which correspond to z_0 and α_0. If these rows and columns are deleted and z_0 and α_0 are deleted in $\begin{Bmatrix} z \\ \alpha \\ \Delta\alpha \end{Bmatrix}$, the remaining matrices are non-singular, so that they can be inverted and Equation (4.4.7) can be written as

$$\Big[[E] - q[D(k)]\Big]\begin{Bmatrix} z \\ \alpha \\ \Delta\alpha \end{Bmatrix} = \{0\} \tag{4.5.2}$$

where $[E]$ is the identity matrix, and

$$[D(k)] = \Big[\widetilde{[1'][1'']}\Big]^{-1}\Big[\widetilde{[A][1'']}\Big]$$

where the tilde designates the fact that the null rows and columns have been deleted.

Equation (4.5.2) is in the canonical for the calculation of eigenvalues. If k is set equal to zero and the eigenvalues of $[D(k)]$ are calculated by iteration, expansion of the determinant, or any other suitable method, the lowest real and positive one represents the value of the dynamic pressure at divergence. Usually for swept wings the value lowest in absolute magnitude is negative and is therefore of no practical significance, although it is often used as an index of the aeroelastic behavior of the airplane.

This calculation can be repeated for various positive values of k, calculating the first few eigenvalues for each. The results, which will generally be complex, can be plotted against k. When any of the eigenvalues becomes purely real, it represents a dynamic pressure at flutter, and the corresponding value of k represents the reduced frequency at flutter. (This statement is true only if the structural damping is zero; such damping effects can easily be included, but the details of this process are beyond the scope of this paper.)

V. DYNAMICS OF THE LARGE FLEXIBLE AIRPLANE

For the large flexible airplane the dynamic and the statistical problems are interlinked, in the sense that the information required for a statistical analysis is more than the response of the airplane to sinusoidal gusts which are uniform along the span. The first Section of this part of the paper is concerned with the nature of the required transfer functions, and the second Section is concerned with the extension required for the method outlined in Part IV to make it applicable to the calculation of these transfer functions.

1. Extension of the Statistical Approach

The mean-square stress at a given point on the wing of a large flexible airplane due to flight through turbulent air can be calculated in several ways. Perhaps the most direct of these consists in writing the stress in terms of a suitable indicial-response influence function as:

$$\sigma(t) = \int_{-\infty}^{\infty}\int_{-\frac{b}{2}}^{\frac{b}{2}} h_\sigma^w(t_1, y)\, w(t - t_1, y)\, dy\, dt_1 \tag{5.1.1}$$

so that

$$\psi_\sigma(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}} h_\sigma^w(t_1, y_1)\, h_\sigma^w(t_2, y_2)\, \psi_w(\tau + t_1 - t_2,\, y_2 - y_1)\, dy_1\, dy_2\, dt_1\, dt_2 \tag{5.1.2}$$

and the only problems are the calculation of this influence function and the evaluation of the quadruple integral, which cannot be effected so readily as before, because $h_\sigma^w(t, y)$ cannot be separated into a function of time and a function of distance along the span in this case.

One way of evaluating the integral, similar to the method used previously, is to calculate the Fourier transform of $\psi_\sigma(\tau)$ yielding

$$\varphi_\sigma(\omega) = \int_0^b \mathcal{H}(\omega, s)\, \varphi_w(\omega, s)\, ds \tag{5.1.3}$$

where $\varphi_w(\omega, y)$ is defined in Equation (2.6.16), $H_\sigma^w(\omega, y)$ is the Fourier transform of $h_\sigma^w(t, y)$ and $\mathcal{H}(\omega, s)$ an auto-convolution of this transform, namely,

$$\mathcal{H}(\omega, s) = \int_{-\frac{b}{2}}^{\frac{b}{2}-s} \left[H_\sigma^w(\omega, y) H_\sigma^{w*}(\omega, y+s) + H_\sigma^w(\omega, -y) H_\sigma^{w*}(\omega, -(y+s)) \right] dy \tag{5.1.4}$$

so that, finally, $\overline{\sigma^2}$ can be calculated from

$$\overline{\sigma^2} = \int_0^\infty \varphi_\sigma(\omega)\, d\omega \tag{5.1.5}$$

An alternative method of evaluating the quadruple integral in Equation (5.1.2) is based on the fact that a convolution represents a Fourier transform of the product of the Fourier transform of the functions involved. This procedure requires the double Fourier transform of the correlation function of the vertical component of atmospheric turbulence, namely

$$\varphi_w(\lambda_1, \lambda_2) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\lambda_1 r_1 + \lambda_2 r_2)} \psi_w(r_1, r_2)\, dr_1\, dr_2 \tag{5.1.6}$$

so that, also

$$\psi_w(r_1, r_2) = \frac{1}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\lambda_1 r_1 + \lambda_2 r_2)} \varphi_w(\lambda_1, \lambda_2)\, d\lambda_1\, d\lambda_2 \tag{5.1.7}$$

Now if, as assumed herein, the turbulence is axisymmetric,

$$\psi_w(r_1, r_2) = \psi_w\left(\sqrt{r_1^2 + r_2^2}, 0\right) = \psi_w\left(0, \sqrt{r_1^2 + r_2^2}\right) = \psi_w\left(\sqrt{r_1^2 + r_2^2}\right) \tag{5.1.8}$$

then it follows that

$$\varphi_w(\lambda_1, \lambda_2) = \varphi_w\left(\sqrt{\lambda_1^2 + \lambda_2^2}, 0\right) = \varphi_w\left(0, \sqrt{\lambda_1^2 + \lambda_2^2}\right) = \varphi_w\left(\sqrt{\lambda_1^2 + \lambda_2^2}\right) \tag{5.1.9}$$

with

$$\varphi_w(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} r\, J_0(\lambda r)\, \psi_w(r)\, dr \tag{5.1.10}$$

For the correlation functions and spectra of Cases 1 and 2 considered previously (Equations (2.3.15) and (2.3.16)) this spectrum is, respectively

$$\varphi_w(\lambda) = \frac{3}{\pi}\,\tilde{L}^2\,\frac{\lambda^2\tilde{L}^2}{(1+\lambda^2\tilde{L}^2)^{5/2}}$$

and

$$\varphi_w(\lambda) = \frac{4}{\pi^2}\,\tilde{L}^2\,\frac{\lambda^2\tilde{L}^2}{\pi}\,e^{-\frac{\lambda^2\tilde{L}^2}{\pi}}$$

It may be noted that $h_\sigma^w(t,y)$ is a Green's function for the partial differential equation which relates the stress as a function of space and time coordinates to the applied loads, which are also functions of space and time coordinates. Similarly, $H_\sigma^w(\omega,y)$ is the Green's function for the ordinary differential equation which relates the stress amplitude as a function of the space coordinate y to the amplitude of the applied sinusoidal loads; the quantity ω may be considered to enter the problem as a parameter rather than as an independent variable. At the same time, $H_\sigma^w(\omega,y)$ may also be regarded as a transfer function from the sinusoidal applied loads at a given station y on the wing to the stress σ. Hence, for the large airplane, that is, for the system represented by a partial differential equation, the spectrum of the output can be expressed in terms of auto-convolutions of the transfer function or the Green's functions for the sinusoidal case, rather than in terms of transfer functions directly, as for the small airplane, whose motions can be represented by an ordinary differential equation.

Now, substituting Equation (5.1.10) into Equation (5.1.2),

$$\psi_\sigma(\tau) = \frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_\sigma^w(t_1,y_1)\, h_\sigma^w(t_2,y_2)\, e^{-i(\lambda_1 U(\tau+t_1-t_2)+\lambda_2(y_2-y_1))} \varphi_w(\lambda_1,\lambda_2)\, d\lambda_1 d\lambda_2 dy_1 dy_2 dt_1 dt_2$$

$$= \frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\lambda_1 U\tau} \left|\tilde{H}_\sigma^w(\lambda_1,\lambda_2)\right|^2 \varphi_w(\lambda_1,\lambda_2)\, d\lambda_1 d\lambda_2$$

(5.1.11)

where the function

$$\tilde{H}_\sigma^w(\lambda_1,\lambda_2) = \int_{-\infty}^{\infty}\int_{-\frac{b}{2}}^{\frac{b}{2}} h_\sigma^w(t,y)\, e^{-i(\lambda_1 U t+\lambda_2 y)}\, dy\, dt$$

(5.1.12)

represents the Fourier transform (with respect to y) of the function $H_\sigma^w(\omega, y)$, that is

$$\tilde{H}_\sigma^w(\lambda_1,\lambda_2) = \int_{-\frac{b}{2}}^{\frac{b}{2}} e^{-i\lambda_2 y}\, H_\sigma^w(\lambda_1 U, y)\, dy \tag{5.1.13}$$

Hence,

$$\varphi_\sigma(\omega) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\lambda_1 U\tau} e^{i\omega\tau} \left|\tilde{H}_\sigma^w(\lambda_1,\lambda_2)\right|^2 \varphi_w(\lambda_1,\lambda_2)\, d\lambda_1 d\lambda_2 d\tau$$

$$= \frac{1}{2U}\int_{-\infty}^{\infty} \left|\tilde{H}_\sigma^w\left(\frac{\omega}{U},\lambda\right)\right|^2 \varphi_w\left(\sqrt{\left(\frac{\omega}{U}\right)^2+\lambda^2}\right) d\lambda$$

(5.1.14)

The function $H_\sigma^w(\omega, y)$ represents the transfer function from gusts of width dy acting at station y to the given stress. Using the approach outlined in the next Section this function can be calculated either directly or indirectly, by first calculating the transfer function from that gust to the lift distribution and then the transfer function from the lift distribution to the stress. For the indirect method

$$H_\sigma^w(\omega,y) = \int_{-\frac{b}{2}}^{\frac{b}{2}} H_\sigma^l(\omega,\eta)\, H_l^w(\omega)\, \tilde{\gamma}(\eta,y)\, d\eta \tag{5.1.15}$$

where the function $H_l^w(\omega)\,\tilde{\gamma}(\eta, y)$ is the Green's function for the aerodynamic problem involving sinusoidal gusts considered in Section 6 of Part II; the symbols η and y in $\tilde{\gamma}(\eta, y)$ are interchanged, however, so that the function now defines the contribution of a gust at station y to the lift at station η. The transfer function $H_\sigma^l(\omega, \eta)$ relates the stress at the given point to a unit concentrated normal force acting at station η.

Using this indirect method, the mean-square stress at a given point can also be calculated by starting with the power spectrum for the lift distribution calculated in Section 6 of Part II. For this approach $\sigma(t)$ may be written as

$$\sigma(t) = \int_{-\infty}^{\infty} \int_{-\frac{b}{2}}^{\frac{b}{2}} h_\sigma^l(t_1, y)\; l\,(t - t_1, y)\, dy\, dt_1 \tag{5.1.16}$$

so that

$$\psi_\sigma(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} h_\sigma^l(t_1, y_1) h_\sigma^l(t_2, y_2)\, \psi_l\,(\tau + t_1 - t_2, y_1, y_2)\, dy_1\, dy_2\, dt_1\, dt_2$$

and

$$\varphi_\sigma(\omega) = \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} H_\sigma^l(\omega, y_1)\, H_\sigma^{l\,*}(\omega, y_2)\, \varphi_l(\omega, y_1, y_2)\, dy_1\, dy_2 \tag{5.1.17}$$

where

$$\psi_l(\tau, y_1, y_2) = \overline{l\,(t, y_1)\, l\,(t + \tau, y_2)}$$

(See Equations (2.6.7), (2.6.8), (2.6.10), and (2.6.12)), so that the Fourier transform of this correlation function is

$$\varphi_l(\omega, y_1, y_2) = \left| H_l^w(\omega) \right|^2 \varphi_{w_\varepsilon}(\omega) \tag{5.1.18}$$

(See Equations (2.6.9) and (2.6.14)). Hence, finally,

$$\varphi_\sigma(\omega) = |H_l^w(\omega)|^2 \int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}} H_\sigma^l(\omega,y_1)\, H_\sigma^{l\,*}(\omega,y_2)\, \varphi_{w_\varepsilon}(\omega)\, dy_1\, dy_2 \tag{5.1.19}$$

whence $\overline{\sigma^2}$ can again be obtained by integration over ω .

If $\varphi_{w_\varepsilon}(\omega)$ is given by Equation (2.6.14), the double integral can be expressed in terms of single integrals as follows:

$$\varphi_\sigma(\omega) = |H_l^w(\omega)|^2 \Big[(1-K)^2 |G(\omega)|^2 \varphi_{w_e}(\omega) + 2K(1-K)\, \mathcal{R}\{G(\omega) G'^{*}(\omega)\} + K^2 \int_0^b \varphi_w(\omega,s)\, G''(\omega,s)\, ds \Big] \tag{5.1.20}$$

where

$$G(\omega) = \int_{-\frac{b}{2}}^{\frac{b}{2}} H_\sigma^l(\omega,y)\, \gamma(y)\, dy$$

$$G'(\omega) = \int_{-\frac{b}{2}}^{\frac{b}{2}} H_\sigma^l(\omega,y)\, \tilde{\varphi}_{w_e}(\omega,y)\, \gamma(y)\, dy$$

$$G''(\omega,s) = 2 \int_{-\frac{b}{2}}^{\frac{b}{2}-s} \Big[\mathcal{R}\{H_\sigma^l(\omega,y)\}\, \mathcal{R}\{H_\sigma^l(\omega,y+s)\} + \mathcal{I}\{H_\sigma^l(\omega,y)\}\, \mathcal{I}\{H_\sigma^l(\omega,y+s)\} \Big] \gamma(y)\, \gamma(y+s)\, dy$$

Equations (5.1.3), (5.1.13) and (5.1.20) thus represent three methods of obtaining the power spectrum of the given stress. One requires a transfer function directly from the local gust intensities to the stress, an auto-convolution of this transfer function, and a spectrum of turbulence defined by Equation (2.6.16); the second requires a two-dimensional spectrum of the turbulence defined by Equation (5.1.6) and a Fourier transform of the aforementioned transform in a direction perpendicular to the plane of symmetry; the third utilizes the spectra of the variously weighted averages of the vertical component of turbulence calculated in Part II, as well as a transfer function from the local lifts to the given stress, for which an auto-convolution has to be calculated. Which of these functions is used depends to some extent on the information available, but is largely a matter of individual preference.

2. Calculation of the Required Transfer Functions

Depending on which of the methods outlined in the preceding Section is used one of two types of transfer functions is required--either the one from local gusts to the stress of interest, or from the local lift to that stress. Both of these functions differ in several respects from those considered in Part IV.

Unlike the case of the small flexible airplane, it is now necessary to consider the entire wing rather than one semispan, so that the various transfer functions are now asymmetric. With the number of the degrees of freedom considered in the analysis thus doubled, it becomes preferable to include one additional degree of freedom, namely roll, rather than to perform two separate analyses for symmetric and antisymmetric motions with, respectively, one and two less degrees of freedom, and then to combine the results. Also, for either type of transfer function the structural deformations of the wing under concentrated loads as well as under distributed loads will be required, so that the numerical-integration schemes used in Part IV have to be modified to some extent.

Before discussing the modifications required to extend the approach of Part IV to the large airplane it might be pointed out that chordwise deformations will be ignored here, as they were in Part IV. In both instances they can readily be included by a straightforward extension of the approach used here if it is felt that they may be significant in any given case. If they are included, however, yawing and possibly also sideslipping motions can no longer be ignored, because they give rise to large forces in the chordwise direction. In that case again, it is preferable to include these two additional degrees of freedom and to treat all longitudinal and lateral degrees of freedom simultaneously rather than to make two separate analyses, which would virtually duplicate each other, for the longitudinal and lateral functions, and then to combine the results.

The structural deformation due to concentrated loads can be obtained in several ways. If measured influence coefficients are used, they pertain precisely to such loads and, in fact, must be modified before they can be used for distributed loads,[11] so that it is necessary only to use the unmodified coefficients.

If the deformations are to be calculated in a manner similar to that employed in Part IV, the integrating matrices must be replaced as follows:

$$\frac{b}{2}[I] \quad \text{by} \quad \frac{b}{2}[\tilde{I}], \text{ where } \tilde{I}_{ij} = \begin{cases} 1, & j>i \\ \frac{1}{2}, & j=i \\ 0, & j<i \end{cases}$$

and

$$\left(\frac{b}{2}\right)^2[II] \quad \text{by} \quad \frac{b}{2}[\widetilde{II}], \text{ where } \widetilde{II}_{ij} = \begin{cases} y_j - y_i, & j \geq i \\ 0, & j<i \end{cases}$$

The factor of 1/2 for $\tilde{I}_{ii}$ constitutes an approximation which implies fairing through a discontinuity.

If this approximation is to be avoided, the deflections due to unit concentrated loads (the structural influence coefficients) can be calculated directly from simple beam theory, in which case the limits of integration take care of the discontinuisites. Thus, for instance, for an unswept wing, the normal deflection and twist at y_i due to a unit concentrated load and torque, respectively, at y_j are

$$z_{ij} = \int_0^{y_i}\int_0^{y'} \frac{y_j - y}{EI(y)}\, dy\, dy' , \qquad y_i \leq y_j$$

$$= \int_0^{y_j}\int_0^{y'} \frac{y_j - y}{EI(y)}\, dy\, dy' + (y_i - y_j)\int_0^{y_j} \frac{y_j - y}{EI(y)}\, dy , \quad y_i > y_j$$

$$\alpha_{ij} = \int_0^{y_i} \frac{1}{GJ(y)}\, dy , \quad y_i \leq y_j$$

$$= \int_0^{y_j} \frac{1}{GJ(y)}\, dy , \quad y_i > y_j$$

The concentrated loads under consideration arise as follows: For the transfer functions relating local lifts to the desired stress, the local lifts may be considered to be concentrated loads of unit magnitude, associated with concentrated torques of magnitude $e_1 c$. Equation (4.4.6) can then be written as (see also Equation (4.3.3)):

$$\left[[1'] - q \; [A] \right] \begin{bmatrix} z \\ \alpha \\ \Delta\alpha \end{bmatrix} = [R'] \begin{bmatrix} \lfloor w \rceil \\ \hline \lfloor e_1 c \rceil \lfloor w \rceil \end{bmatrix} \tag{5.2.1}$$

where $\lfloor w \rceil$ is a diagonal matrix of the values of w defined by Equation (4.1.6), and where the matrix $[R']$ represents either the four influence-coefficient matrices for z and α due to concentrated loads and torques, or the square matrix of Equation (4.3.2) with modified integrating matrices, as discussed in the preceding paragraphs.

It may be noted that Equation (5.2.1) now represents not one set of simultaneous equations but several, all having the same coefficients, but with different sets of knowns (as defined by the columns of the matrix on the right side) and, hence, different sets of unknowns (the columns of the matrix $\begin{bmatrix} z \\ \alpha \\ \Delta\alpha \end{bmatrix}$). This is due to the fact that the functions under consideration are, in effect, the responses of the airplane as a whole to sinusoidally varying concentrated loads, and these functions are different for each location of the applied load.

Once this equation has been modified to take into account the over-all body motions and tail deflections (as explained in Part IV) as well as the rolling motions (as explained in the following) it can be solved to yield the unknown values of $\begin{bmatrix} z \\ \alpha \\ \Delta\alpha \end{bmatrix}$. From these values and the bending and twisting moments, as well as the vertical shears, can be calculated and added to those due to the concentrated loads. When combined linearly as required for the desired stress these moments and shear yield the desired transfer function $H_\sigma^l(\omega, y)$.

Or else, if the transfer function from the local gusts to the desired stress is to be determined, the response of the airplane to the lift distribution induced by a sinusoidal gust of width dy acting at station y must be calculated. This lift distribution is the Green's function considered previously. If it is represented by the relation $H_l^w(\omega)\tilde{\gamma}(y,\eta)$, with $\tilde{\gamma}(y,\eta)$ defined by the approximation given in Equation (2.6.11), the concentrated loads arise from the delta function in that expression. The right side of Equation (5.2.1) becomes in that case

$$q(1-K)\,H_l^w(\omega)\,[R]\begin{bmatrix} [\gamma(y)\gamma(\eta)] \\ \hdashline [e,c]\,[\gamma(y)\gamma(\eta)] \end{bmatrix}[w] + q\,b\,K\,H_l^w(\omega)\,[R']\begin{bmatrix} [\gamma(y)]\,[w] \\ \hdashline [e,c]\,[\gamma(y)]\,[w] \end{bmatrix}$$

where $[R]$ represents the square matrix of Equation (4.3.2), and $[R']$ the one discussed in connection with Equation (5.2.1). Again, several sets of simultaneous equations are implied. Their solution (after modification for over-all motions and tail deflections) yields values of z and α from which the transfer functions $H_\sigma^w(\omega, y)$ can be calculated.

The extension of the method of Part IV to the calculation of the deformations on both wings is straightforward; essentially, distributed lifts and torques now have to be calculated for both wings and integrated with matrices which can be assembled from those used for one wing alone. No new problems arise in this process, so that it need not be discussed further.

The inclusion of rolling motion, however, is not so trivial. One way of doing so consists in replacing all values of z in Equation (4.4.5) or its equivalent by $z + \Theta y$ and then reducing the columns involving this quantity by the relation (which assumes that the new unknown quantity, the roll angle Θ , is listed at the end of the column)

$$\left\{\begin{array}{c} z+\Theta y \\ \hline \alpha \\ \hline \Delta\alpha \\ \Theta \end{array}\right\} = \left[\begin{array}{c|c|cc} 1 \; 1 \; 1 \ddots & & & y_{tip} \downarrow 0 \downarrow -y_{tip} \\ \hline & 1 \; 1 \; 1 \ddots & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right] \left\{\begin{array}{c} z \\ \hline \alpha \\ \hline \Delta\alpha \\ \Theta \end{array}\right\} \tag{5.7.2}$$

An additional equation must then be joined to the set, which is given by the equation of equilibrium in roll

$$\int_{-\frac{b}{2}}^{\frac{b}{2}} l(y)\, y\, dy - I_{xx}\ddot{\Theta} - M_{D_t}\dot{\Theta} = 0 \tag{5.2.3}$$

where I_{xx} is the inertia in roll of the fuselage and empennage alone, inasmuch as the inertia effects of the wing are included in $l(y)$, and M_{D_t} is the coefficient of damping in roll for the empennage. For most cases both of these contributions can be neglected, yielding simply, in matrix notation,

$$\lfloor \mathrm{II} \rfloor \{l\} = 0$$

where $\lfloor \mathrm{II} \rfloor$ is now a matrix which serves to perform the integration indicated in Equation (5.2.3). This condition can then be adjoined to the other equations in the same manner as the other equations descriptive of over-all motions were included in Part IV.

The result, again, is a set of simultaneous equations for $z, \alpha, \Delta\alpha$ and Θ from the solution of which the desired transfer functions can be obtained as outlined in the preceding paragraphs. Also, as in Part IV, once the unknowns z_0, α_0 and Θ are eliminated from the set, the divergence and flutter speeds can be calculated by conventional matrix operations; these speeds will then pertain to an airplane free to move vertically as well as in pitch and roll and, hence, will include divergence and flutter speeds in antisymmetric as well as symmetric modes.

VI. CONCLUDING REMARKS

The statistical approach to the problem of calculating the dynamic response and the stresses in an airplane subjected to atmospheric turbulence has been extended in several respects, retaining basically only the assumptions of linearity, that is, of small motions and deformations, as well as homogeneity and axisymmetry of the turbulence.

The first problem considered was that of the effect of spanwise variation of the instantaneous turbulent velocities on the lift and moments due to turbulence. It has been shown that the mean-square lift is reduced considerably if the span of the airplane is relatively large compared to the integral scale of turbulence. The spectrum of this lift, particularly that part which contains most of the energy, is affected relatively little by these variations if the decrease in the mean-square intensity is taken into account. The mean-square lift is not very sensitive to the spanwise distribution of the lift, but varies considerably for various shapes of the correlation function of atmospheric turbulence. The effect of sweep on the mean-square lift and its spectrum is very small for wings with the same value of $b / \tilde{L} \cos \Lambda$.

If the variation of the instantaneous velocities is taken into account, the instantaneous rolling moment to which the airplane is subjected can be calculated. The mean-square rolling moment is shown to be proportional to the ratio of the wing span to the integral scale of turbulence for small values of that ratio. Similarly, the mean-square values and the power spectra of the local lift, the bending moments and the pitching moment can be calculated from expressions given herein.

For some of these forces the required aerodynamic information cannot be calculated by existing methods, so that certain approximations, based on experience with steady aerodynamic forces and available knowledge concerning unsteady forces had to be made for the aerodynamic influence functions in unsteady flow.

The next problem considered was the dynamic response of a rigid airplane to random turbulence. This problem had previously been treated for the case of an airplane free to move only in the degree of freedom of vertical motion and small enough so that variation of the turbulent velocities along the span could be neglected. In the present paper the response of an airplane in three longitudinal degrees of freedom was considered; calculations were made which suggest that the inclusion of deviations from the mean horizontal motion is superfluous in gust-loads calculations. For the remaining two longitudinal degrees of freedom, the mean-square normal and angular acceleration have been shown to be functions of only two parameters other than the mass ratio and scale parameter of the single-degree-of-freedom case, namely dimensionless forms of the short-period frequency and of the time to damp to one-half amplitude. It has also been shown how the results obtained in connection with the first problem can be used to extend this dynamic analysis to the case where variation of the turbulent velocity along the span have to be taken into account.

The last problem treated was the dynamic response of a flexible airplane, including vertical motion, pitch and, when necessary (as is the case when spanwise variations in gust intensity are taken into account), roll. Horizontal and lateral motions were disregarded because they do not generally affect the wing stresses due to vertical gusts. A method which represents an extension to the dynamic case of a numerical-integration approach to the static aeroelastic problem has been outlined for the analysis of the problem at hand. Again, the modifications required in the statistical approach to treat this problem when spanwise variations of the gust intensity are to be considered have been discussed, as have been some of the modifications of the method of dynamic analysis required in that case. This method has been shown to be in a form which permits the calculation of divergence and flutter speeds with relatively small additional effort.

Although most of this analysis has been confined to the vertical component of turbulence, it has been shown that the simultaneous action of longitudinal, vertical and lateral gusts on the wing stresses (with due allowance for the fact that vertical gusts affect both the longitudinal and the lateral motions of the airplane) can be taken into account by simply adding the power spectra of the various contributions, provided the turbulence is isotropic as well as axisymmetric; the cross-correlations or spectra have been shown to vanish either by the symmetry or antisymmetry of the influence functions involved, or as a result of the statistical independence of mutually perpendicular velocity components.

The approach presented herein thus furnishes a foundation for the prediction of the statistical properties of the stress experience of a given airplane once the appropriate statistical characteristics of the atmosphere have been determined.

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APPENDIX - LIST OF SYMBOLS

b	span
c	chord, parallel to plane of symmetry
$\bar{c}$	average chord, S/b
C_L	lift coefficient, L/qS
C_{L_a}	lift-curve slope
C_l	rolling-moment coefficient, L'/qSb
C_{l_p}	coefficient of damping in roll, defined as positive for positive damping
c_l	section lift coefficient at station y, l/qc
$C(k)$	Theodorsen function
e_1	distance from section aerodynamic center to shear center, fraction of chord
e_2	distance from shear center to section center of gravity, fraction of chord
e_3	distance from shear center to the mid-chord position, fraction of chord
e_4	distance from shear center to the three-quarter chord position, fraction of chord
g	acceleration of gravity
$g(y,\eta)$	influence function for bending moment
$h(t)$	indicial or response function
$H(\omega)$	response to sinusoidal oscillation, Fourier transform of $h(t)$
I_{xx}	mass moment of inertia about x axis
I_{yy}	mass moment of inertia about y axis
k	reduced frequency, $\frac{\omega\bar{c}}{2U}$
k'	dimensionless frequency, sk
L	lift
l	distributed lift per unit distance along the span

L'	rolling moment
$\widetilde{L}$	integral scale of turbulence
M	pitching moment
M_B	bending moment
M_T	twisting moment
m	mass (of airplane, unless designated otherwise by subscripts)
$\widetilde{m}$	distributed mass, per unit distance along the span
$\mathcal{m}$	distributed twisting moment (about axes perpendicular to the plane of symmetry) per unit distance along the span
q	dynamic pressure
r	radius of gyration about center of gravity
S	wing area
s	scale parameter, $\frac{\widetilde{L}}{\bar{c}/2}$
t	time
U	mean flying speed
u	longitudinal component of gust velocity
u_p	horizontal component of disturbed motion
v	lateral component of gust velocity
W	weight of airplane
w	vertical component of gust velocity
w_p	vertical component of disturbed motion
x	coordinate along mean flight path
Δx	distance from intersection of elastic axis and root chord to airplane center of gravity
x_t	tail length, distance from airplane center of gravity to aerodynamic center of tail
x_t'	modified tail length, distance from intersection of elastic axis and root chord to aerodynamic center of tail
y	coordinate perpendicular to plane of symmetry

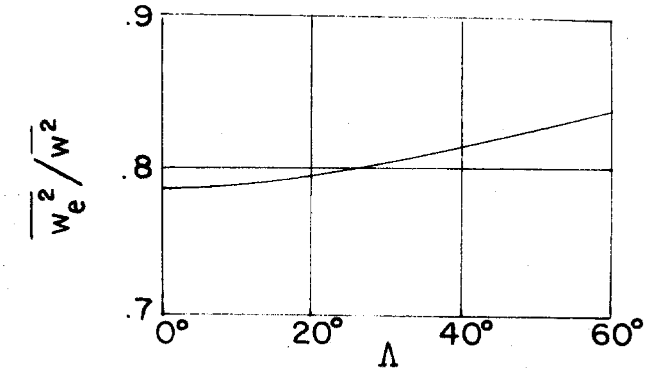
z	coordinate in plane of symmetry perpendicular to mean flight path or vertical deflection
α	inclination of chord to x axis
$\gamma(y)$	dimensionless lift distribution, $\frac{c c_l}{\bar{c} C_L}$
$\tilde{\gamma}(y,\eta)$	dimensionless lift-influence (Green's) function
η	variable of integration corresponding to y
θ	angle of attack (in Part III)
Λ	angle of sweepback
ω	frequency of oscillation
ψ	correlation function
φ	power spectrum
$\phi(k)$	Sears function
τ	time displacement, argument of correlation function

Subscripts:

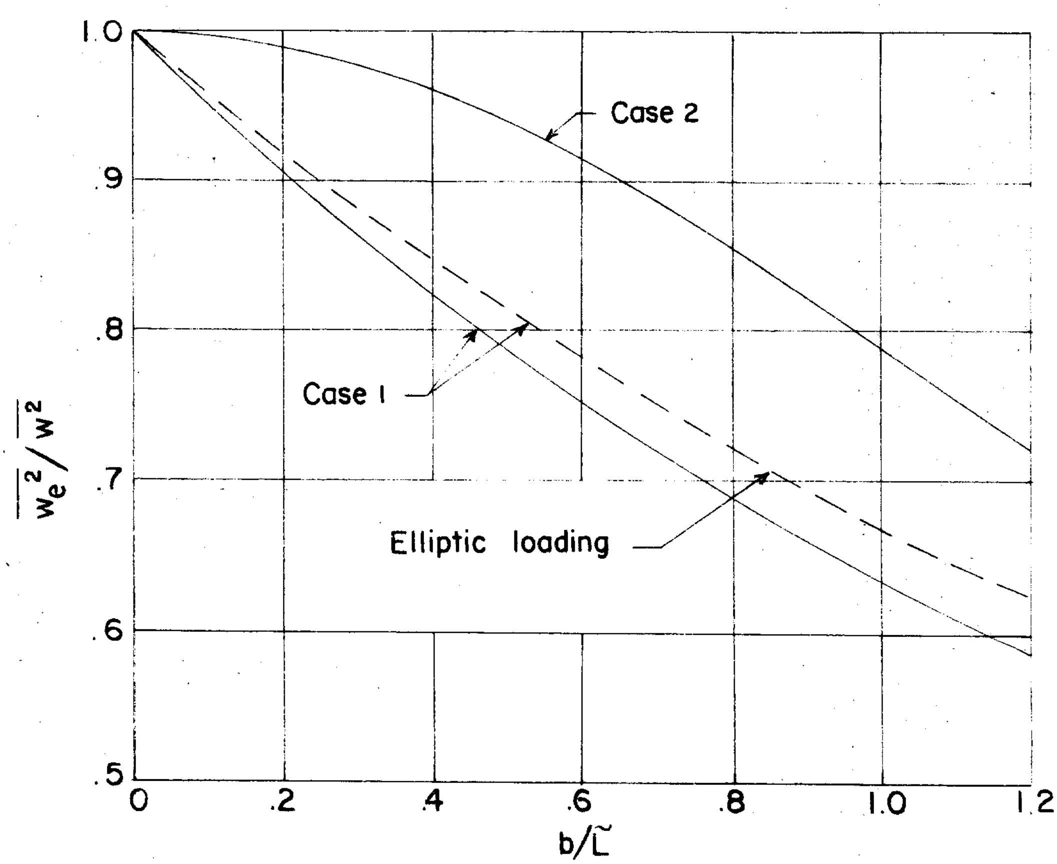
f	fuselage
r	root
t	tail
w	wing

Matrix Notation

[ ]	square or rectangular matrix
$\lceil\ \rfloor$	diagonal matrix
$\lfloor\ \rfloor$	row matrix
{ }	column matrix



(b) Swept wings
(case I, rectangular loading)



(a) Unswept wings

Figure I. - The mean-square averaged vertical component of turbulence

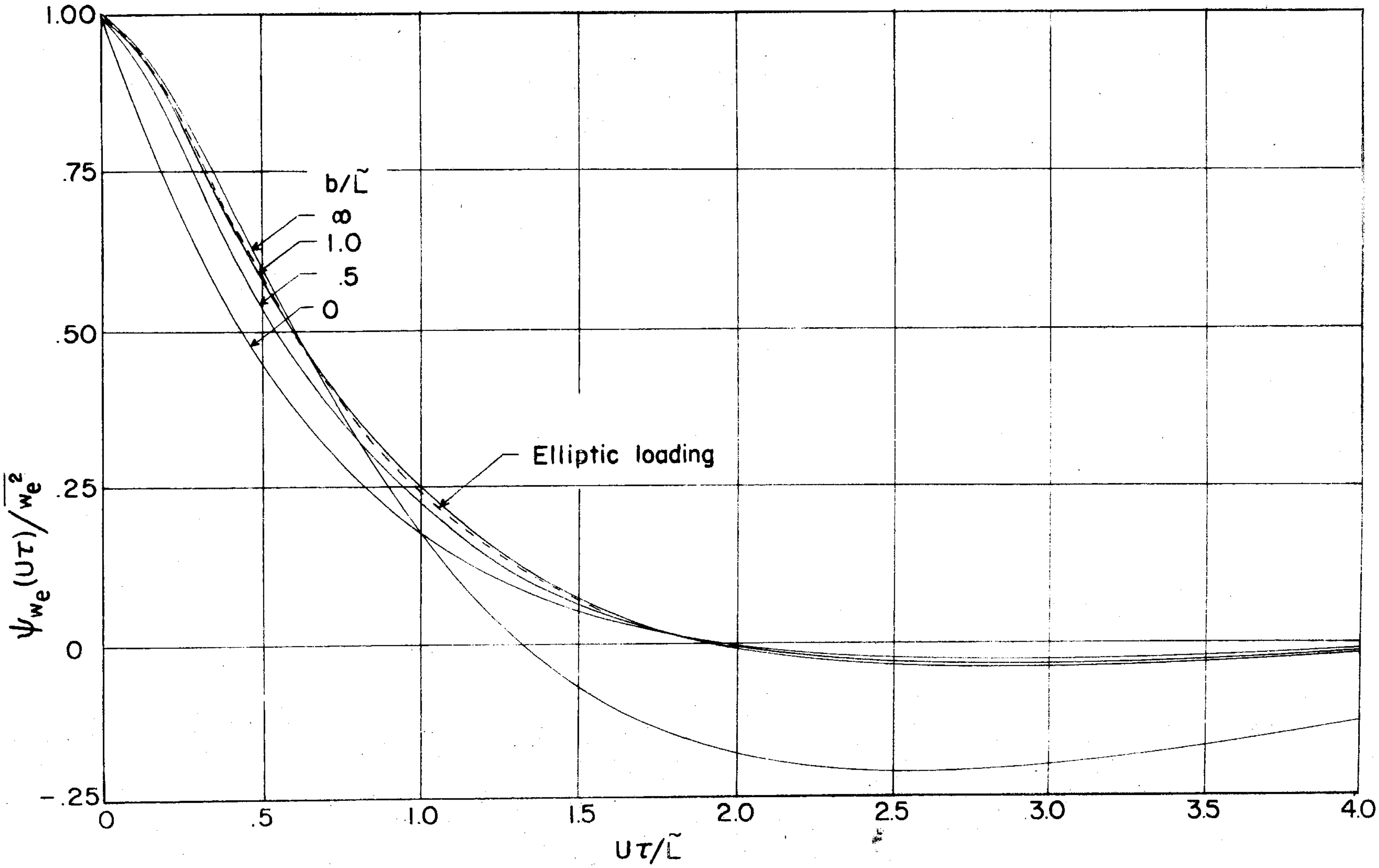


Figure 2 - The correlation function of the averaged vertical component of turbulence (case I)

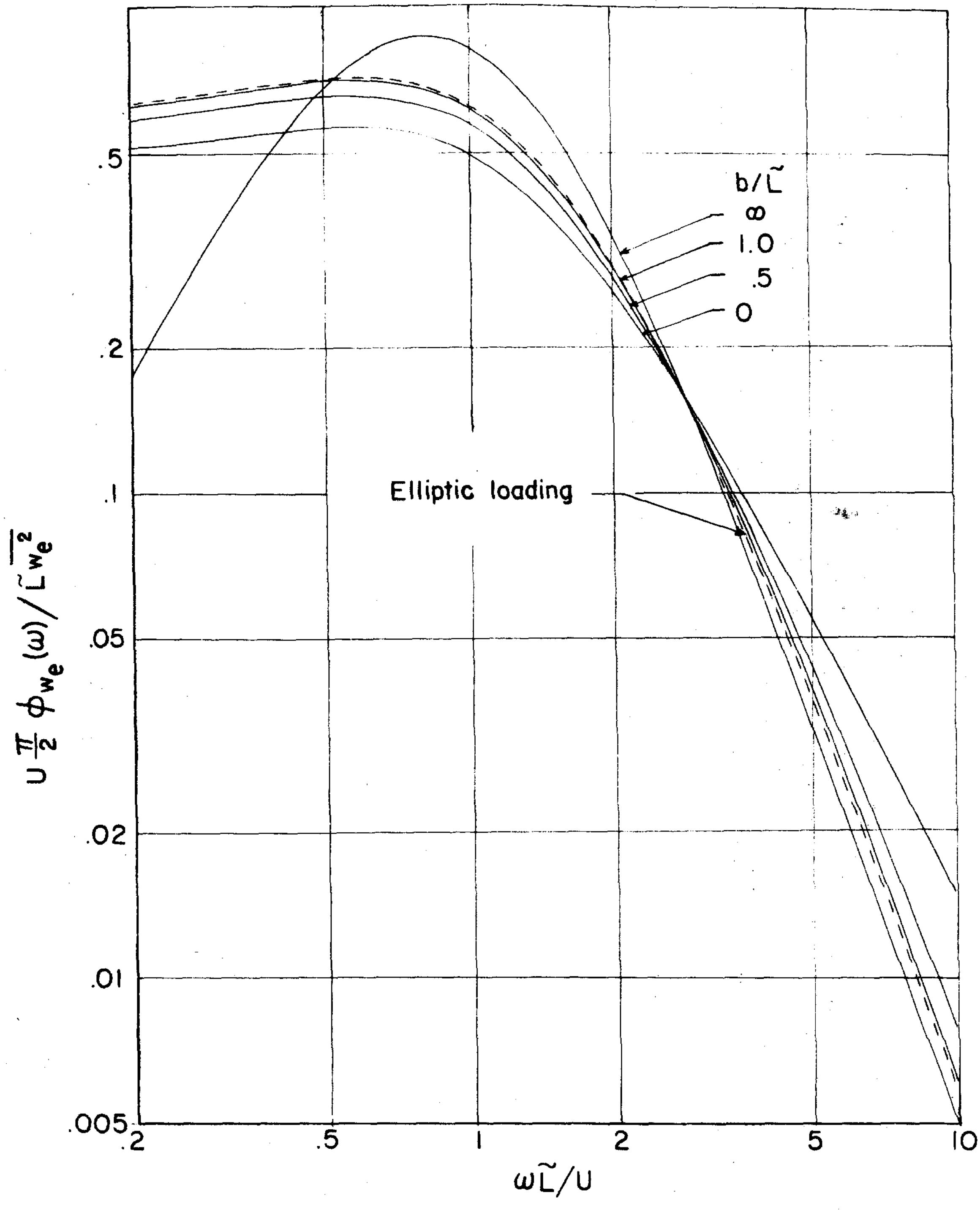


Figure 3.- The power spectrum of the averaged vertical component of turbulence (case 1)

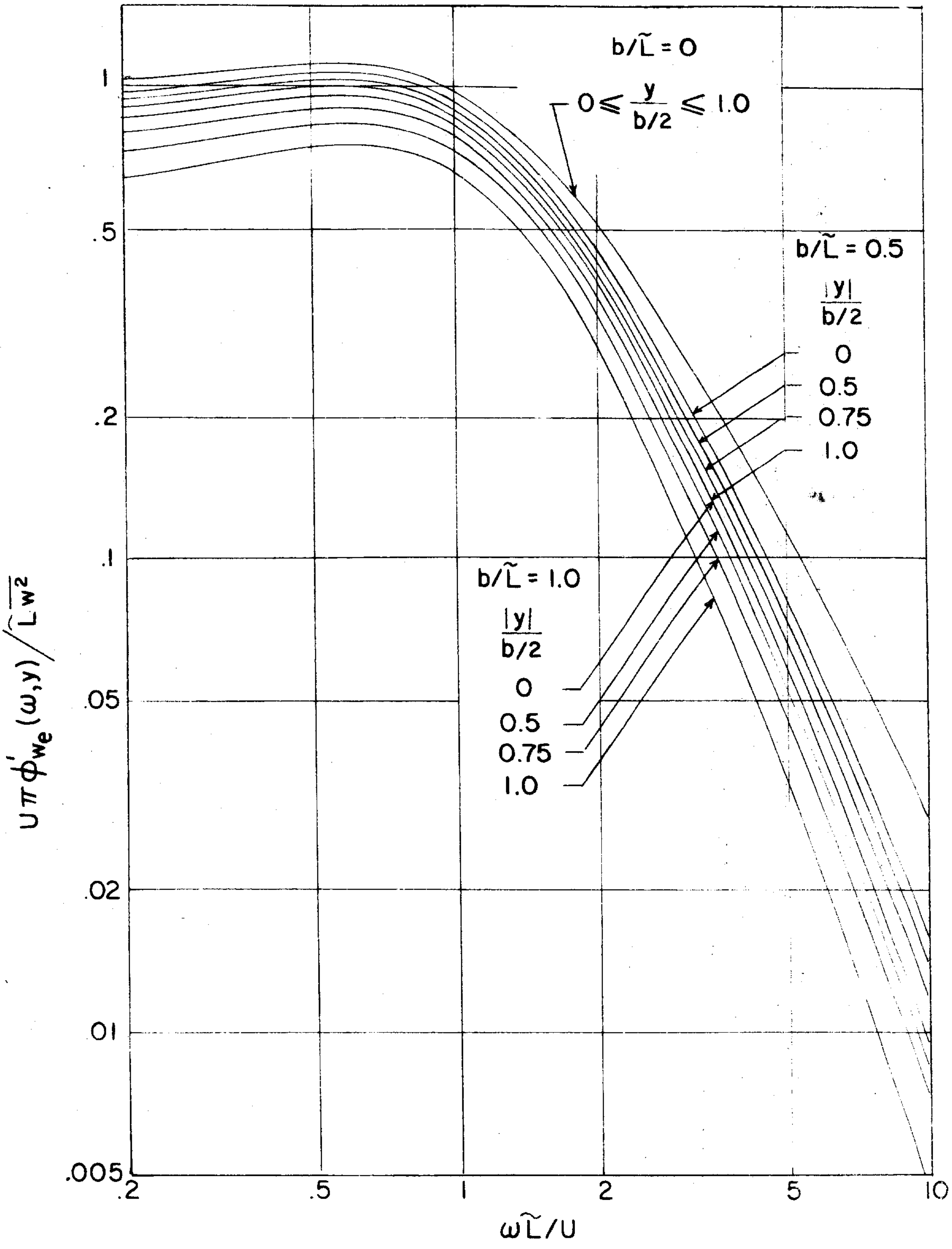
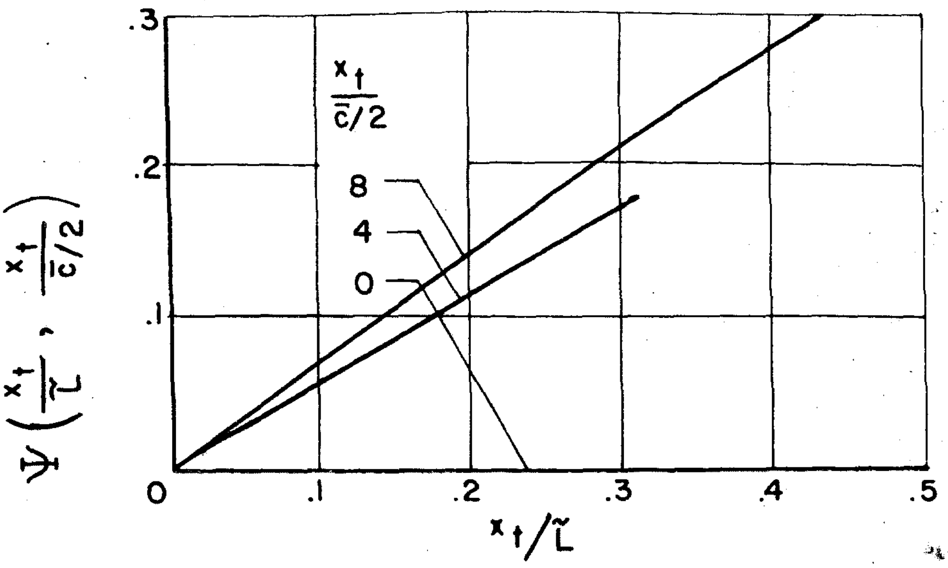


Figure 4. – The power spectrum $\phi'_{w_e}(\omega,y)$ for case 1, rectangular loading



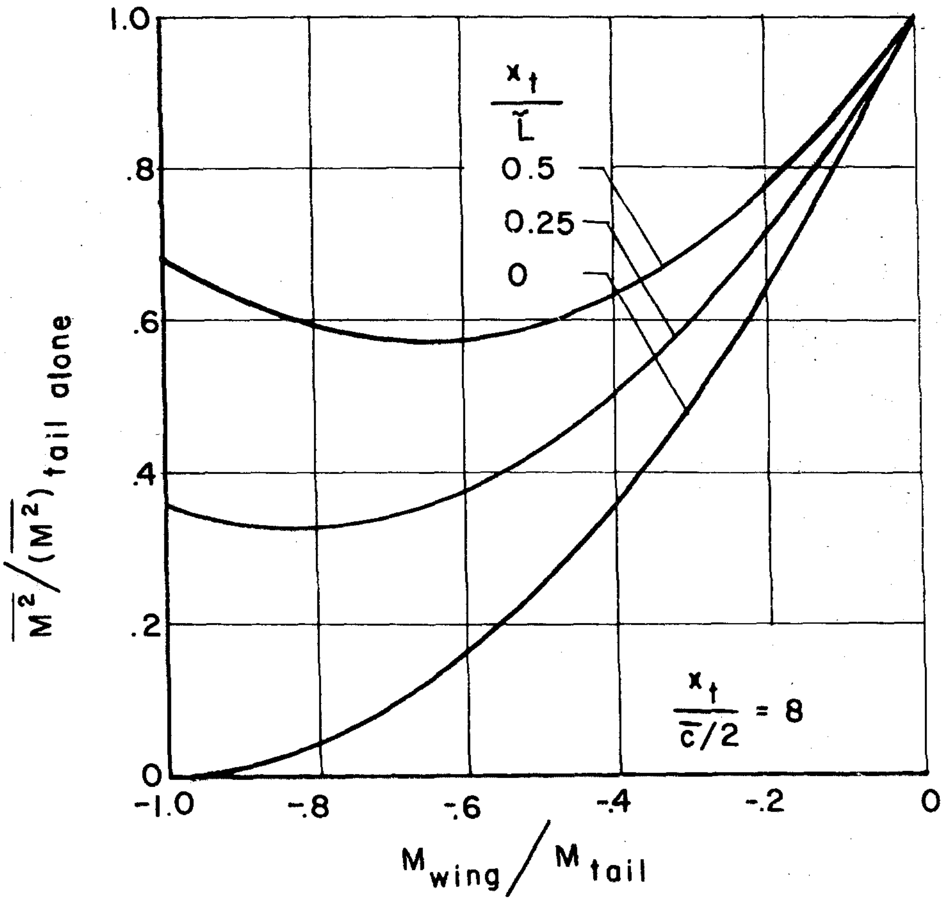


Figure 5.- Effect of tail length on the mean square pitching moment.

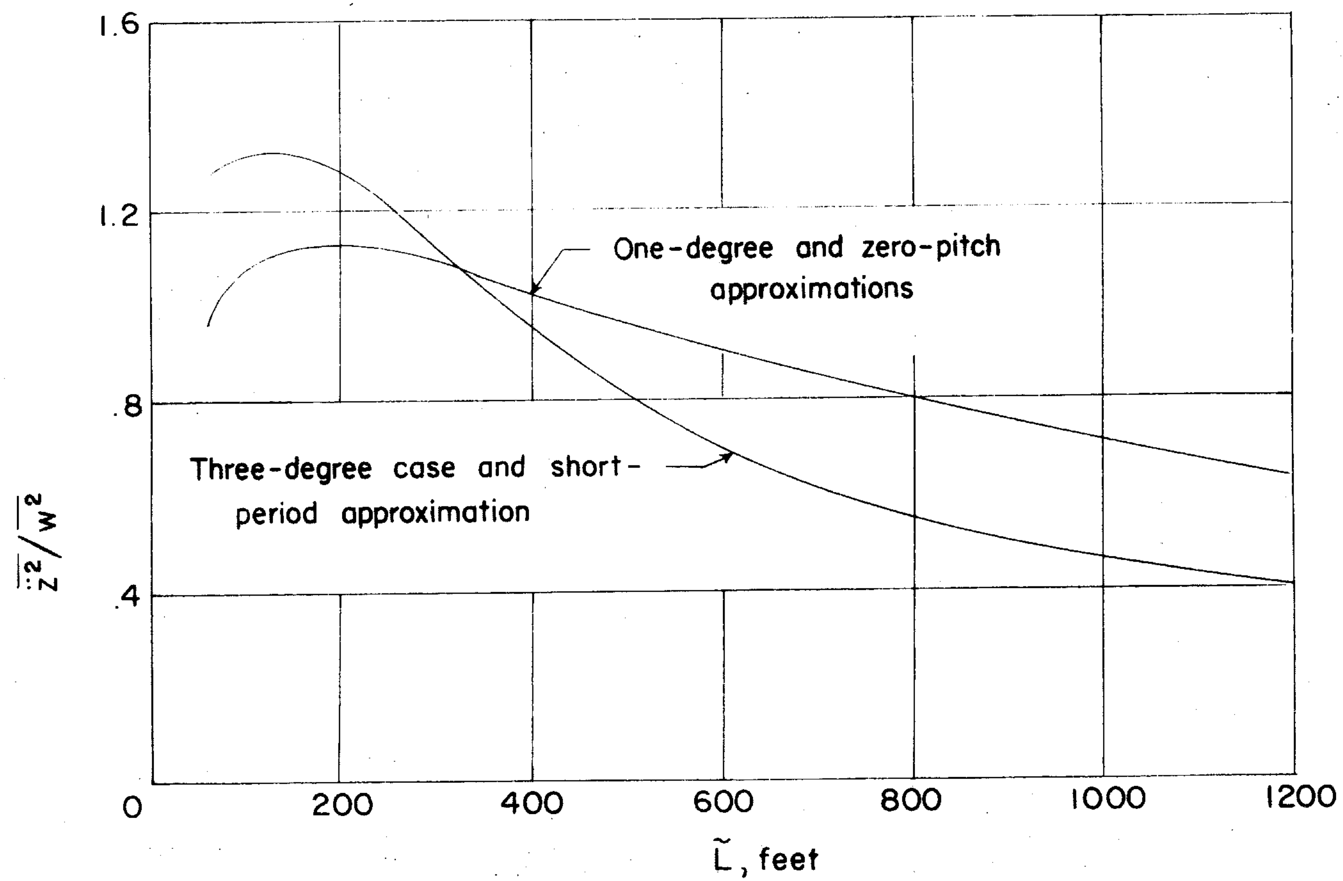


Figure 6.- Effect of scale of turbulence on the mean-square acceleration of the example airplane