BOUNDS FOR SOLUTIONS OF SOME NON-LINEAR PARABOLIC PROBLEMS

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Abstract

Functions $v(x,t)$ satisfying certain partial differential equations of the form $v_t = F(x,t,v,v_x,v_{xx})$ in the region R: $0 < x < 1$, $0 < t \leq T$ are studied. The principal results of Part I determine circumstances in which it can be asserted that v and v_x admit, in R, bounds which depend only on the bounds for the functions $v(x,0)$, $v(0,t)$, and $v(1,t)$, and for the derivatives of these functions. The proofs employ certain elementary comparison theorems for solutions of partial differential inequalities. Some other applications of these theorems are also included in Part I.

In Part II analogous results are obtained for the system of first order ordinary differential equations which arises when the x-derivatives in the partial differential equation are replaced by divided differences. The bounds obtained in this case hold uniformly under refinement of the discretization.

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INTRODUCTION

1. Partial Differential Equations and Physical Background. Part I of this thesis concerns some properties of solutions of special boundary value problems of the following general form:

Problem P: A function $v(x,t)$ is to be found such that

$$(1.1.1) \qquad \dot{v} = F(x,t,v,v',v'')$$

in $0 < x < 1,\ 0 < t \leq T < \infty$ and such that v assumes specified values on the left boundary line $x = 0,\ 0 \leq t \leq T$; the right boundary line $x = 1,\ 0 \leq t \leq T$; and the initial line $t = 0,\ 0 \leq x \leq 1$. Here $\dot{v} = \frac{\partial v}{\partial t},\ v' = \frac{\partial v}{\partial x},\ v'' = \frac{\partial^2 v}{\partial x^2}$ and $F(x,t,z,p,r)$ is a non-decreasing function of r. Various conditions of smoothness and boundedness are imposed on F and on the boundary and initial data. "Solution" is taken in a strong sense, including, say, the requirement that v, v' be continuous in the closure of the region. Although more general strip regions in the $x - t$ plane and other kinds of boundary conditions will not be considered in this thesis, the techniques will have obvious applications or adaptations to many such problems.

In general, no amount of smoothness on the part of data and equation will suffice to guarantee the existence of a smooth solution for more than a finite range of the "time" t. In the general situation the maximum for $x \varepsilon [0,1]$ of $|v|$, or of $|v'|$, or of $|v''|$ will approach infinity as t approaches some finite $t_o > 0$. The magnitude of t_o will depend on various moduli of smoothness and boundedness of the initial and boundary data and of the equation. The solution as conceived here will therefore cease to exist for $t \geq t_o$. One of the first objectives here is the development of a series of results which suffice to delimit a fairly broad and interesting class of equations such that a solution and certain of its derivatives admit bounds independent of t.

Certainly one of the simplest and best understood examples of a problem P is the problem of heat conduction in a homogeneous slab of unit thickness. In this case $F(x,t,z,p,r) = ar$, where a is a positive constant, v is the temperature, and the data consists of the given initial distribution of temperature and the temperatures of the faces of the slab as functions of time. The problems which will be studied here can be regarded as generalizations of this heat conduction problem. It is a widely known fact, however, that solutions of linear differential equations generally, and of the heat equation $\dot{v} = av''$ in particular, fail utterly to illustrate many of the peculiarities of behavior of solutions of non-linear partial differential equations. Moreover, many of the analytical techniques, like application of Green's theorem and fundamental solutions, which permeate studies of the linear case lose their immediate applicability in the non-linear case. Thus, in the face of added complexity of behavior, the number of analytical tools is diminished and recourse must be had to more pedestrian methods of analysis. One result of these facts is that studies of linear parabolic problems can concern themselves with more general varieties of problems than the ones treated here; for example, weaker concepts of "solution" and far less stringent assumptions on the data can be successfully dealt with. The theory of non-linear problems is presently at a relatively primitive stage and must be temporarily content with more modest aims.

A problem better than the heat conduction one for purposes of introducing the equations to be studied here is one formulated and studied by Professor J.M. Burgers. Professor Burgers' interest in the mathematical structure of hydrodynamical turbulence led him to the construction of a number of ingenious "models of turbulence" [1]. Typically, one of these models is a mathematical problem which preserves certain general features of the hydrodynamical problem while remaining sufficiently simple to admit some sort of analytical study; it need not correspond in the usual sense to any real physical problem, but is studied formally as a mathematical problem for it suggestiveness and for possible insight into the nature of its more complex parent. By skillful combination of his physical knowledge and some relatively simple analytical methods, Professor Burgers has developed this idea into a powerful heuristic tool.

One of Burgers' models is the following case of a problem P:

<u>Burgers' Problem</u>:

$$(1.1.2) \quad \dot{u} + uu' = \alpha u_{xx} + u, \qquad u = u(x,t), \qquad \alpha > 0, \text{ constant}$$

$$(1.1.3) \quad u(0,t) \equiv u(1,t) \equiv 0;$$

$$(1.1.4) \quad u(x,0) = U(x).$$

The equation (1.1.2) is a model of the Navier-Stokes equations

$$(1.1.5) \quad \frac{\partial u_k}{\partial t} + u_j \frac{\partial u_k}{\partial x_j} = \nu \frac{\partial^2 u_k}{\partial x_j \partial x_j} - \frac{1}{\rho} \frac{\partial p}{\partial x_k}$$

in which $u_k = u_k(x_1,x_2,x_3,t)$, $k = 1,2,3$, are the cartesian components of the velocity vector, ν is the viscosity, ρ is the (constant) density, and $p = p(x_1,x_2,x_3,t)$ is the pressure. The analogy is obvious except for the last term on the right; in the model no "pressure" appears but the artificial term u is added. In a sense to be described presently, this artificial term provides an "energy input" to the model system and will accordingly be called the "input term"; similarly, the term $\alpha u''$ will be called the "viscosity term" and uu' the "non-linear term".

In Burgers' treatment (1.1.2) is coupled with another equation for the "main stream velocity", while u is the superimposed "turbulent component of velocity"; thus the solution $u(x,t) \equiv 0$ is the non-turbulent or "laminar" solution. It is then natural to identify $\frac{1}{2} \int_0^1 u^2(x,t)\, dx$ as the "energy" of turbulence. If (1.1.2) is multiplied by u and integrated over $0 \leq x \leq 1$, the contribution of the non-linear term vanishes; after a partial integration one finds formally without difficulty the "energy" equation:

$$(1.1.6) \quad \frac{1}{2} \frac{d}{dt} \int_0^1 u^2(x,t)\, dx = -\alpha \int_0^1 u'^2(x,t)\, dx + \int_0^1 u^2(x,t)\, dx,$$

which shows that the viscosity term provides a dissipative influence tending to diminish the energy, while the input term provides a driving influence tending to increase it. Which influence, if either, predominates? In particular, does the input overwhelm the dissipation so that the energy increases indefinitely? As it happens it will be easy to prove that the answer is negative. The fact that this is so is strongly dependent upon the presence of the non-linearity, for if the non-linear term is dropped the equation

(1.1.7) $\dot{u} = \alpha u'' + u$

is obtained. If α is small enough, this equation has solutions which are unbounded and have unbounded energy, yet satisfy the zero boundary conditions (1.1.3)*. Nonetheless, (1.1.7) and (1.1.2) together with conditions (1.1.3) have the <u>same</u> energy equation (1.1.6).

The fluid dynamical explanation of the balancing effect of viscous dissipation is well known. Intuitively speaking, the "steepening" effect of the non-linearity and the dissipative effect of viscosity are opposing influences; as the former proceeds, the latter increases until some sort of steady or quasi-steady equilibrium is attained. The ultimate maximum velocity gradient depends on the viscosity, and tends to infinity as the viscosity tends to zero**. The crucial point is the existence of this <u>finite</u> maximum for any <u>positive</u> viscosity. If gradients greater than this maximum occur initially, the action of viscosity predominates to diminish them. For small positive viscosity the regions of large velocity gradient are localized in "transition layers" like slip or boundary layers and, in compressible fluids, shock layers. Thus <u>boundedness</u> of the velocity gradient can be interpreted as <u>positive</u> thickness of the transition layers.

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* If $\alpha = \pi^{-1}$, $u(x,t) = e^{(\pi-1)t/\pi} \sin x$ is such a solution.

** More precisely, as the "Reynolds number" tends to infinity.

Applied to Burgers' problem, arguments of this kind suggest the uniform boundedness of u' and, a fortiori, that of u. To the fluid dynamicist the correctness of such deductions is intuitively obvious and the reasoning behind them sufficient "proof". For the question of why slip or shock discontinuities do not appear, he finds a satisfactory answer in his physical understanding, which is based on empirical evidence and study of special cases of mathematical problems. To the mathematician the same question of why has different meaning; the physical arguments are at best a heuristic method which strongly suggests the truth of certain mathematical theorems. To him the problem is then to produce tools to prove these theorems and to generalize them or find their mathematical, rather than physical, rationale.

By injecting this query of why, in the mathematician's sense, at various points in a discussion of Burgers' problem, one finds the genesis of the present investigation. In Part I a class of problems generalizing Burgers' is introduced; it is proved that so long as a solution $v(x,t)$ exists, v and v' admit a priori bounds independent of t. As byproducts of the methods of proof some results on the decay of influence of the initial data and on the variation with t of the number of crossings of a stationary solution are established. Applied to Burgers' problem, the last mentioned result implies that the "turbulent fluctuation" does not increase with time, in the sense that the number of zeros of $u(x,t)$ in $0 \leq x \leq 1$ does not increase.

The same methods of proof permit the construction and discussion of some examples of problems P which illustrate some intuitively plausible possibilities of pathological behavior. In one example, for instance, suitable choice of initial data insures the boundedness of a solution for all t, while other choices of initial data suffice to guarantee that the solution does not remain bounded for more than an arbitrarily short duration of time. In another example, it can be shown that the solution v admits a uniform a priori bound, while v' admits no such bound.

In an appendix, some special features of Burgers' problem are discussed. If α, in (1.1.2), is small enough, Burgers' problem admits a (finite) multiplicity of solutions independent of time. Some features of these solutions are established and some problems connected with them are discussed. In particular, "transition layers" are observed in these solutions and the dependence of the "thickness" of these layers upon α as $\alpha \to 0$ is examined and compared with implications of the theory of Part I.

If Burgers' problem is modified by deleting the input term from equation (1.1.2), a new model of turbulence is obtained. This model can be called "decay of turbulence in a box", since the obvious modification of the energy discussion above makes it intuitively plausible that absence of the input term will result in uniform approach to the only "rest" solution, $u = 0$. Burgers has also studied this problem and variations of it obtained by using non-constant boundary data to provide energy input. It was observed independently by J.D. Cole [2] and E. Hopf [3] that the equation involved,

(1.1.8) $\quad \dot{u} + uu' = \alpha u''$,

can be reduced to the linear heat equation by a suitable non-linear transformation of the dependent variable. Cole used this fact to study problems for (1.1.8) from the point of view of the fluid dynamicist; and Hopf used it to construct an extensive and rigorous discussion of the initial value problem on the infinite line, including study of the limit of the solution as $\alpha \to 0$ and of a generalized notion of solution.

Although Burgers' problem is substantially simplified by putting (1.1.8) in place of (1.1.2), the two problems have a structural similarity important to notice here. The problem

(1.1.9) $\dot{u} + uu' = \alpha u'' + ku$ $\qquad$ k = constant

(1.1.3) $u(0,t) \equiv u(1,t) \equiv 0$

(1.1.4) $u(x,0) = U(x)$

is Burgers' if $k = 1$ and the simpler one if $k = 0$. If $v = kx - u$ is introduced in place of u, this becomes

(1.1.10) $\dot{v} = \alpha v'' + (v - kx)\, v'$

(1.1.11) $v(0,t) \equiv 0; \quad v(1,t) \equiv k.$

(1.1.12) $v(x,0) = V(x)$ $\qquad$ (where $V(x) = kx - U(x)$).

Equation (1.1.10) belongs to that class of equations (1.1.1) for which F satisfies

(1.1.13) $F(x,t,z,0,0) \equiv 0$

$F \in C^1$ for $0 \leq x \leq 1,\ 0 \leq t < \infty,\ -\infty < z,p,r < +\infty$ and

(1.1.14) $\frac{\partial}{\partial r} F(x,t,z,p,r) \geq \alpha > 0.$

For any F the following decomposition is possible (cf. Hopf [4])

$$(1.1.15)\qquad F(x,t,z,p,r) = A(x,t,z,p,r) + B(x,t,z,p) + C(x,t,z),$$

where

$$A(x,t,z,p,0) \equiv 0$$

$$B(x,t,z,0) \equiv 0,$$

and, in fact, is accomplished by defining

$$A(x,t,z,p,r) = F(x,t,z,p,r) - F(x,t,z,p,0)$$
$$B(x,t,z,p) = F(x,t,z,p,0) - F(x,t,z,0,0)$$
$$C(x,t,z) = F(x,t,z,0,0).$$

If F satisfies (1.1.14), then (1.1.15) can be put in the form

$$(1.1.16)\qquad F(x,t,z,p,r) = a(x,t,z,p,r)r + b(x,t,z,p)p + c(x,t,z)$$

where

$$(1.1.17)\qquad a(x,t,z,p,r) \geq \alpha > 0.$$

In accord with the statement of problem P, it must be assumed that $F(x,t,z,p,r)$, and therefore $a(x,t,z,p,r)r$, is a non-decreasing function of r. Of course, this will follow from (1.1.17) if a is independent of r.

If, in addition, F satisfies (1.1.13), $c(x,t,z) \equiv 0$. Motivation is thereby provided for the following generalization of problem (1.1.10 - 12):

(1.1.18) $\dot{v} = a(x,t,v,v',v'')v'' + b(x,t,v,v')v'$

(1.1.19) $v(0,t) = \phi(t); \quad v(1,t) = \Psi(t)$

(1.1.20) $v(x,0) = V(x).$

Problems of this type will be studied in Part I.

With suitable differentiability assumptions, the introduction of $\tilde{v}(x,t) = v(x,t) - (1 - x)\ \phi(t) - x\Psi(t)$ in place of v will reduce the boundary values (1.1.19) to zero, the new equation being of type (1.1.1) with F given by (1.1.16) (i.e., not satisfying (1.1.13)). As a consequence, there will be occasion to consider this more general equation.

Finally, it is noteworthy that the methods used here have application to many interesting examples of equations excluded from consideration here by the assumptions on the function a. The equation $\dot{v} = (v'')^3$ is a simple illustration.

2. Ordinary Differential Equations. If a numerical approximation to the solution of a problem like Burgers' is needed, the most common procedure is to construct an approximating discrete problem by substituting suitable divided differences for the derivatives. The differential problem is thereby replaced by an algebraic one, which can, as a rule, be solved numerically by iterative algebraic methods. Under favorable circumstances the solution of the difference problem will yield a good approximation to the solution of the differential problem.

This approach seems natural if one has a machine--like a modern digital computer--adapted to the rapid performance of algebraic operations. However, the existence of machines—like differential analyzers or analog computers—adapted to dealing directly with ordinary differential equations suggests the idea of carrying the discretiztion process only so far as is necessary to replace the partial differential equation by a system of ordinary equations. For a problem like Burgers' there are two possible choices: If "time" t is discretized, the problem can be reduced to a series of two point boundary value problems for second order ordinary differential equations; if "space" x is discretized, an initial value problem for a system of first order ordinary differential equations is obtained. Part II of this thesis is concerned with some aspects of problems of the latter kind.*

Specifically, in Part II attention is focussed on

<u>Problem Q</u>: To find $v_k(t)$, $k = 1,2,\dots,n-1$ satisfying

$$(2.1.1) \quad \dot{v}_k = a(k/n,t,v_k)n^2(v_{k+1} - 2v_k + v_{k-1}) + b(k/n,t,v_k)\,\frac{n}{2}(v_{k+1} - v_{k-1}) \qquad k = 1,2,\dots,n-1$$

$$(2.1.2) \quad v_o(t) \equiv \phi(t); \qquad v_n(t) \equiv \Psi(t)$$

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* It may be mentioned that the former approach has been employed by E. Rothe [5], not for numerical purposes but as a theoretical tool; by passage to the limit of refinement of the discretization, he obtained a local existence theorem for a certain restricted subclass of problems P. The initial value problem on the infinite line for a similar class of equations has been studied through difference equations (i.e., both variables discretized) in a recent comprehensive paper by F. John [6]. The method studied here has been mentioned in connection with numerical methods but does not seem to have been much used as a theoretical method. Some related problems treated in the Russian literature [7] have not so far been assessed by the writer.

(2.1.3) $v_k(0) = V(k/n)$ $k = 1,2,\ldots,n-1,$

where ϕ, Ψ, V are given data functions, as in (1.1.19 - 20), $a(x,t,z) \geq \alpha > 0$ and $b(x,t,z)$ are given coefficient functions, and $n \geq 2$ is a positive integer. Problem Q arises by an obvious approximation to (1.1.18 - 20) where, however, dependence of the coefficients on v' and v'' is excluded.

Briefly, the program of Part II is the investigation for problem Q of analogues of some results obtained in Part I for partial differential equations. For example, questions of uniform boundedness of $|v_k|$ and $\frac{n}{2}|v_{k+1} - v_{k-1}|$ are studied. There is, however, a basic distinction between the meaning of "uniform" as used in Part I and as used here. In Part I a uniform bound for $|v|$ or $|v'|$ is a bound independent of t and x. A uniform bound for $|v_k|$ or $\frac{n}{2}|v_{k+1} - v_{k-1}|$, however, is a bound independent of t, k, <u>and of</u> <u>n</u>. Bounds of the latter sort are clearly a prerequisite for discussion of the limiting procedure involved in indefinite refinement of the discretization. It is interesting that the analysis shows in a particularly clear light the necessity of having n large in order that the behavior of the solution of a problem Q should parallel, even qualitatively, the behavior of the solution of the analogous partial differential problem. It will be seen in one instance that n large is both necessary and sufficient for uniform boundedness of $|v_k|$.

The more restricted nature of the results in the discretized case than in the continuous case appears already in the exclusion of the divided differences from the functions a and b in (2.1.1). The degree to which such restrictions are essential and not due to weakness of the proofs is not known. Roughly speaking, the source of difficulty is the following simple fact: If the maximum of $v(x,\bar{t})$ is attained at an interior point $\bar{x}$ of $[0,1]$, then $v'(\bar{x},\bar{t}) = 0$; if, however, the maximum of v_k for $k = 0,1,\ldots,n$ is attained at k_o, $0 < k_o < n$, it cannot be inferred that $\frac{1}{2}n(v_{k+1} - v_{k-1})$ vanishes, or even that it is small.

PART I: CONTINUOUS CASE

3. Bounds for $v(x,t)$.

3.1. Notation: For $0 < T < \infty$,

$$\Gamma = [(x,t) | t = 0, 0 \leq x \leq 1; \text{ or } x = 0,1, \text{ and } 0 \leq t \leq T]$$

$$R = [(x,t) | 0 < x < 1 \text{ and } 0 < t \leq T]$$

$$S = \Gamma + R.$$

Although it is not indicated explicitly, Γ, S, and R always depend on the choice of T.

3.2. Heuristic for Boundedness. Let $v(x,t)$ satisfy $\dot{v} = av'' + bv'$ where $a \geq 0$ and a, b may depend on x,t,v,v',v''. If $\max v(x,\bar{t})$ for $x \in [0,1]$ is assumed at interior point $\bar{x}$, then $v'(\bar{x},\bar{t}) = 0$, $v''(\bar{x},\bar{t}) \leq 0$ and $\dot{v}(\bar{x},\bar{t}) \leq 0$; if assumed at an endpoint, then $v(x,\bar{t}) \leq \max\ v$ on Γ. Similarly, $\min v$ is assumed on Γ or at point where $\dot{v} \geq 0$. This suggests that the maximum and minimum of $v(x,t)$ are assumed on Γ. To precise this the following lemma proved by Westphal [8] will be useful.

3.3. Lemma 3.3.1: Let $F(x,t,z,p,r)$ be a non-decreasing function of r for any fixed values of other arguments. Let $v(x,t)$, $w(x,t)$ be two functions such that

(i) v,w are (jointly) continuous in S

(ii) $\dot{v},v',v'',\dot{w},w',w''$ exist in R

$$\begin{aligned} &\text{(iii)} \quad \dot{v} \leq F(x,t,v,v',v'') \\ &\qquad\;\; \dot{w} > F(x,t,w,w',w'') \end{aligned} \qquad \text{in } R$$

$$\text{(iv)} \quad v < w \quad \text{on } \Gamma.$$

Then $v < w$ in S.

<u>Proof</u>: Let $g = w - v$. Then $g > 0$ on Γ. If $g > 0$ in S, nothing to prove. Otherwise the subset of S for which $g \leq 0$ is closed, so compact and there exists a point $(\bar{x},\bar{t})$ such that $0 < \bar{x} < 1$, $g(\bar{x},\bar{t}) = 0$ and $g(x,t) > 0$ for $0 \leq t < \bar{t}$. Examining differences one sees that $\dot{g}(\bar{x},\bar{t}) \leq 0$, $g'(\bar{x},\bar{t}) = 0$, $g''(\bar{x},\bar{t}) \geq 0$. By the inequalities (iii) evaluated at $\bar{x},\bar{t}$

$$\begin{aligned} \dot{g} &> F(\bar{x},\bar{t},w,w',w'') - F(\bar{x},\bar{t},v,v',v'') \\ &= F(\bar{x},\bar{t},w,w',g'' + v'') - F(\bar{x},\bar{t},v,v',v'') \geq 0 \end{aligned}$$

contradicting $\dot{g} \leq 0$ at $\bar{x},\bar{t}$.

<u>Remark 3.3.1</u>: In (iii) it clearly suffices that the <u>strict</u> inequality holds in <u>at least one</u> of the two inequalities for each (x,t).

<u>Remark 3.3.2</u>: The rectangular strip region can be replaced by a region $x_1(t) \leq x \leq x_2(t)$ where $x_1(t) < x_2(t)$ are continuous functions for $0 \leq t \leq T$.

3.4. <u>Boundedness Theorems</u>.

<u>Theorem 3.4.1</u>: Let $v \in C(S) \cap D^2(R)$ satisfy

$$(3.4.1) \quad \dot{v} = a(x,t,v,v',v'')v'' + b(x,t,v,v')v' \quad \text{in } R$$

where

(3.4.2) $\quad a(x,t,z,p,r)r$ is a non-decreasing function of r.

Then

$$\min_{\Gamma} v \leq \min_{S} v \leq \max_{S} v \leq \max_{\Gamma} v.$$

This theorem will be obtained as a case of the following one.

<u>Theorem 3.4.2</u>: Let $v \in C(S) \cap D^2(R)$ satisfy

(3.4.3) $\quad \dot{v} = F(x,t,v,v',v'')$ in R

where F is non-decreasing function of r. If f,g are functions defined on $0 \leq x \leq 1$ such that

(i) $f,g \in C([0,1]) \cap D^2((0,1))$

(ii) $F(x,t,f,f',f'') \leq 0$; $F(x,t,g,g',g'') \geq 0$ in R

(iii) $F(x,t,z,f',f'')$ and $F(x,t,z,g',g'')$ are non-increasing functions of z for $(x,t) \in R$. [cf. Remark 3.4.2]

(iv) $g \leq v \leq f$ on Γ. (This means $g(x) \leq v(x,t) \leq f(x)$ for $x = 0$ or $x = 1$, $0 \leq t \leq T$; and for $t = 0$, $0 \leq x \leq 1$.)

Then $g \leq v \leq f$ in S.

<u>Proof</u>: Let $\varepsilon > 0$, $w(x,t) = g(x) - \varepsilon - \varepsilon t$. Then $w < v$ on Γ, and $\dot{w} < 0 \leq F(x,t,g,g',g'') \leq F(x,t,g-\varepsilon-\varepsilon t,g',g'') = F(x,t,w,w',w'')$. By Lemma 3.3.1, $w < v$ in S. Letting $\varepsilon \to 0$, $g \leq v$ in S. In the same way, $v \leq f$.

Remark 3.4.1: If $F(x,t,z,0,0) \equiv 0$ the conditions of Theorem 3.4.2 are fulfilled if f,g are the constant functions $f = \max_{\Gamma} v$, $g = \min_{\Gamma} v$. This proves Theorem 3.4.1.

Remark 3.4.2: Condition (iii) of Theorem 3.4.2 can be replaced by the demand that F satisfy a certain Lipschitz condition with respect to z. This will be discussed later; cf. Theorem 7.3.1.

Remark 3.4.3: A stronger "maximum principle" than Theorem 3.4.1 can be proved under suitable circumstances; cf. Nirenberg [9].

4. Asymptotic Bounds for $v(x,t)$. This section contains a result reflecting the decay of influence of initial data observed in certain problems P.

Theorem 4.1.1: Let $v \in C(S) \cap D^2(R)$ satisfy in R the equation (3.4.1), with condition (3.4.2), for every $T > 0$. Assume, in addition, that there exists a constant α such that

(4.1.1) $a(x,t,z,p,r) \geq \alpha > 0,$

for $0 \leq x \leq 1$; $0 \leq t < \infty$; $-\infty < z,p,r < +\infty$; and there exists a function $\beta(\xi) \in C^1(-\infty < \xi < +\infty)$ such that for fixed $B_0 > 0$

$$\sup |b(x,t,z,\xi e^{-\delta t})| < \beta(\xi) < \infty,$$

the supremum being taken over $0 \leq x \leq 1$; $0 \leq t < \infty$; $|z| \leq B_0$, $0 \leq \delta \leq 1$. Suppose also that the values of v on Γ are bounded independently on T for all $T > 0$.

Then, given $\varepsilon > 0$ and $T_o > 0$, there is a $T_1 \geq T_o$ such that

$$\inf_{\substack{x=0,1 \\ t\geq T_o}} v(x,t) - \varepsilon < v(x,t) < \sup_{\substack{x=0,1 \\ t\geq T_o}} v(x,t) + \varepsilon$$

holds for $0 \leq x \leq 1$, $t \geq T_1$.

<u>Proof</u>: No loss of generality goes with the assumption $T_o = 0$. Define

$$K = \sup_{\substack{x=0,1 \\ t\geq 0}} v(x,t) \qquad \text{and} \qquad w(x,t) = f(x)\, e^{-\eta t} + K$$

where f is a function and $\eta \leq 1$ a constant to be specified later. The upper bound of the theorem follows from Lemma 3.3.1 if positive $f \varepsilon C^2([0,1])$ and $\eta > 0$ can be found so that $f(x) + K > v(x,0)$ and also so that

$$\dot{w} > a(x,t,v,v',v'')w'' + b(x,t,v,w')w'$$

in R for all $T > 0$. This is equivalent to

(4.1.2) $\quad 0 > a(x,t,v,v',v'')f'' + b(x,t,v,f'\, e^{-\eta t})f' + \eta f.$

By assumption, there is a B_o independent of T such that $|v| \leq B_o$ on Γ for every $T > 0$. By Theorem 3.4.1, the same bound holds in S. Thus $b(x,t,v,f'\, e^{-\eta t}) < \beta(f')$. Suppose now that for $x \varepsilon [0,1]$, f satisfies

(4.1.3) $\quad af'' + \beta(f')f' + \eta f = 0$

(4.1.4) $\quad f'' < 0; \quad f' > 0.$

Then f also satisfies (4.1.2). Let K' be a constant such that $K + K' > \max [0, \max_x v(x,0)]$. If $f(0) = K'$, then $f' > 0$ implies $f(x) + K \geq f(0) + K = K' + K > v(x,0)$. Let $K'' > 0$ and let f be the solution of (4.1.3) satisfying the initial conditions $f(0) = K'$, $f'(0) = K''$; this f depends on the parameter η. The proof will be complete if it is shown that, for some positive η, the solution exists over $[0,1]$ and satisfies inequalities (4.1.4). This can be done by a continuity argument, which follows.

Consider first the case $\eta = 0$; then f satisfying $\alpha f'' + \beta(f')f' = 0$, $f(0) = K'$, $f'(0) = K'' > 0$ exists for some interval $I: 0 \leq x < x_0 \leq \infty$ and is unique by a classical theorem. For $x \in I$, $f'(x) > 0$ since $f'(\tilde{x}) = 0$ would imply that $f(x)$ and $g(x) \equiv f(\tilde{x})$ are two solutions which satisfy the same initial conditions at $\tilde{x}$ and thus $f(x) \equiv f(\tilde{x})$ by the uniqueness theorem; since $f'(0) > 0$, this cannot be. Because $\beta(f') > 0$ by (4.1.1), the equation then yields $f'' < 0$ in I and therefore $0 < f'(K) \leq K''$ and $K' \leq f(x) \leq K' + K''x$ in I. By the classical extension theorem, $x_0 = \infty$ may be assumed. (The same result can be obtained, incidentally, by using an explicit representation of the solution, cf. 5.2.)

Consider now the general case; a classical theorem guarantees the continuous dependence of f, f', f'', and the maximal interval of existence on the parameter η. It is readily inferred that $\eta_0 > 0$ exists such that f exists in $[0,1]$ and satisfies (4.1.4) for $0 < \eta < \eta_0$.

The proof for the lower bound of the theorem is obtained from the above applied to $-v$.

5. Bounds for $v'(x,t)$.

5.1. Some Heuristic: To study the boundedness of v', a first crude attempt might consist in differentiating the differential equation with respect to x to find an equation for $w \equiv v'$ of the form $w = G(x,t,v,w,w',w'')$ and then to try to develop some kind of maximum principle applicable to w. Even granted success this far, however, the program stumbles over the difficulty that the given data provides no a priori information about the values of w on the boundaries $x = 0$ and $x = 1$.

In the treatment of the next section following, the boundedness of v' is studied via that of the difference quotient

$$(5.1.1) \qquad \frac{v(y,t) - v(x,t)}{y - x}$$

The formal proof, however, starts from a point of view somewhat different from the one originally used. Because the latter presents some interesting aspects of its own, it seems worthwhile to sketch here the ideas involved in order to provide motivation for the later, more formal, approach.

It is proposed to show that the quotient (5.1.1) is bounded above independently of t for $x,y \in [0,1]$. It is assumed that v satisfies an equation of form $\dot{v} = av'' + bv'$ with $a \geq \alpha > 0$ and that a bound independent of t is already known for $|v|$, so that the numerator of (5.1.1) is bounded independently of t. The maximum of (5.1.1) for all $x,y \in [0,1]$, at a fixed t, is always assumed for $x = y$, by the mean value theorem, and is therefore equal to the maximum of $v'(x,t)$ for $x \in [0,1]$.

Instead of (5.1.1) consider the modified quotient

$$(5.1.2) \qquad \frac{g(v(y,t) - v(x,t))}{y - x} \qquad\qquad 0 \le x < y \le 1,$$

where $g(z)$ is a function to be determined subject to the conditions $g \in C^2([0,B])$, where $B > \sup(v(y,t) - v(x,t))$, $g(0) = 0$, $g'(0) = 1$, and $g''(z) > 0$ for $z \in (0,B)$. (It may be assumed that $g(z) \equiv 0$ for $z \le 0$, since only upper bound is under scrutiny.) Let $\theta(t)$ denote the supremum of (5.1.2) for indicated range of x,y. It is plausible to assume now that for suitable g, the supremum is attained, say for $x = \tilde{x}$, $y = \tilde{y}$ with $\tilde{x} < \tilde{y}$. Roughly speaking, introduction of g gives a greater weight to a steep "chord" than to an equally steep "tangent".

Operating formally with the (not necessarily valid) assumption that $\theta(t)$ has a derivative, one finds

$$(5.1.3) \qquad \theta(t) = \frac{g'(v(\tilde{y},t) - v(\tilde{x},t))}{\tilde{y} - \tilde{x}} [\dot{v}(\tilde{y},t) - \dot{v}(\tilde{x},t)]$$

$$(5.1.4) \qquad \theta(t) = \frac{g'}{\tilde{y} - \tilde{x}} \left[(av'' + bv')\Big|_{x=\tilde{y}} - (av'' + bv')\Big|_{x=\tilde{x}}\right].$$

In (5.1.4), and henceforth in 5.1, when g' is written, the value at $v(\tilde{y},t) - v(\tilde{x},t)$ is meant; similarly for g and g''.

Now (5.1.2) as a function of y (or of x) alone has a maximum at $y = \tilde{y}$ (or $x = \tilde{x}$). If both $\tilde{x}$ and $\tilde{y}$ are interior points, the usual necessary conditions of calculus yield

$$(5.1.5) \qquad v'(\tilde{x},t) = v'(\tilde{y},t) = \frac{\theta}{g'}$$

$$(5.1.6)\qquad v''(\tilde{y},t) \le -\frac{g''}{g'^3}\,\theta^2, \qquad -v''(\tilde{x},t) \le -\frac{g''}{g'^3}\,\theta^2,$$

θ being evaluated at t.

For simplicity here it will be assumed that the coefficient function b does not involve v' or t so that $|b(x,v)|$ is bounded independently of t by a constant β_o. Using, then, (5.1.5 - 6) in (5.1.4) one finds

$$(5.1.7)\qquad \dot{\theta} \le \frac{g'}{\tilde{y}-\tilde{x}}\left[-2\alpha\frac{g''}{g'^3}\theta^2 + 2\beta_o\frac{\theta}{g'}\right] = \frac{\theta}{\tilde{y}-\tilde{x}}\left[2\beta_o - 2\alpha\frac{g''}{g'^2}\theta\right].$$

In similar fashion, if one of $\tilde{x},\tilde{y}$ is an endpoint,

$$(5.1.8)\qquad \dot{\theta} \le \frac{\theta}{\tilde{y}-\tilde{x}}\left[D\frac{g'(B)}{\theta} + \beta_o - \alpha\frac{g''}{g'^2}\theta\right],$$

where D is a constant bounding $|\dot{v}(0,t)|$ and $|\dot{v}(1,t)|$ independently of t (additional assumption on data!).

If both $\tilde{y},\tilde{x}$ are endpoints, then $\tilde{x} = 0$, $\tilde{y} = 1$, and $\theta = g(v(1,t) - v(0,t)) \le g(B)$, so that if $\theta > g(B)$ this case does not occur.

Combining these results gives: <u>If ε is any positive number and if</u>

$$(5.1.9)\qquad \theta > g(B) \quad \underline{\text{and}} \quad \left[D\frac{g'(B)}{\theta} + \beta_o - \alpha\theta\frac{g''}{g'^2}\right] \le -\varepsilon < 0$$

<u>then</u>

$$(5.1.10)\qquad \dot{\theta} \le -\varepsilon\theta.$$

If the choice $g(z) = -(1/c)\log(1 - cz)$, $c > 0$, is made, then $g''/g'^2 = c$ and g satisfies the conditions imposed provided $c < 1/B$. Inequalities (5.1.9) are then certainly satisfied if θ is large enough, and (5.1.10) implies that θ must be rapidly decreasing. Specifically, if $\theta > g(B)$, $\theta > g'(B)$, and $\theta > (D + \varepsilon + \beta_o)/c\alpha$, then (5.1.9), (5.1.10) hold and it follows that $\theta \leq \theta(0)\, e^{-\varepsilon t}$. Thus:

$$(5.1.11) \qquad \theta(t) \leq \max\,[g(B);\ g'(B);\ (D + \varepsilon + \beta_o)/c\alpha;\ \theta(0)\, e^{-\varepsilon t}].$$

For the g chosen, $Bg'(B) \geq g(B)$. Also, since $g(z)/z$ is an increasing function,

$$(5.1.12) \qquad \theta(0) = \frac{g(v(\tilde{y},0) - v(\tilde{x},0))}{v(\tilde{y},0) - v(\tilde{x},0)} \; \frac{v(\tilde{y},0) - v(\tilde{x},0)}{\tilde{y} - \tilde{x}} \leq \frac{g(B)}{B}\, J$$

where J is an upper bound for $v'(x,0)$ (assumption on data).
As a consequence (5.1.11) may be replaced by:

$$(5.1.13) \qquad \theta(t) \leq \max\,[\frac{B}{1 - cB};\ \frac{(D + \varepsilon + \beta_o)}{c\alpha};\ \frac{g(B)}{B}\, J\, e^{-\varepsilon t}]$$

From (5.1.13) bounds of two kinds can be deduced. If t is large enough the last term in brackets is inferior to the first two and may be dropped. Of the first two terms, one is an increasing and the other a decreasing function of c, the two coinciding at a unique c^* with $0 < c^* < \frac{1}{B}$. By giving c the value c^* the inequality is optimized; in this way it is found that:

$$(5.1.14) \qquad v'(x,t) \leq \theta(t) \leq \min_{c} \max\,[\frac{B}{1 - cB};\ \frac{D + \varepsilon + \beta_o}{c\alpha}]$$

$$\leq \frac{B(\alpha + D + \varepsilon + \beta_o)}{\alpha}.$$

This bound does not depend on J (on initial data) but holds only for $t \geq t_o$ where t_o may depend on J; i.e., it is an asymptotic bound. A bound good for $t \geq 0$ can be obtained in similar fashion by retaining the last term in brackets in (5.1.13).

The inequality $\theta \leq K$ implies that

$$v(y,t) - v(x,t) \leq g^{-1}[K(y - x)] \qquad \text{for} \qquad 0 \leq x \leq y \leq 1,$$

where g^{-1} is the function inverse to g. In 5.2 the difference $u(x,y,t) = v(y,t) - v(x,t)$ will be studied as a function of x,y,t defined over a cylinder, $0 \leq x \leq y \leq 1$, $0 \leq t \leq T$, with triangular base. By finding a differential equation satisfied by $u(x,y,t)$ and applying the natural extension to a cylinder of Lemma (3.3.1), it will be shown that there exists a function f such that $f(0) = 0$, $f'(0) < \infty$ and $u(x,y,t) \leq f(y - x)$ for $0 \leq y - x < \varepsilon$ and $0 \leq x \leq y \leq 1$; i.e., in a neighborhood of the diagonal face of the cylinder. Then the constant $f'(0)$ appears as the bound for $v'(x,t)$.

5.2. Boundedness Theorems for $v'(x,t)$.

Theorem 5.2.1: Let $v \in C^1(S) \cap D^2(R)$ satisfy equation (3.4.1) in R for some $T > 0$. The coefficient a is assumed to satisfy conditions (3.4.2) and (4.1.1). Assume also the existence of a positive, even function $\beta(p) \in C^1(-\infty < p < +\infty)$ such that for fixed $B_o > 0$

$$(5.2.1) \qquad \sup|b(x,t,z,p)| < \beta(p),$$

the supremum being taken over $0 \leq x \leq 1$, $0 \leq t < \infty$, $|z| \leq B_o$, and such that

(5.2.2) $\int^{\infty} \frac{dp}{\beta(p)}$ is divergent.

Then $|v'|$ admits, in S, a bound which depends only on the constant a, the function β, and the maxima of $|v|$ on Γ, $|\dot{v}|$ on the boundary lines, and $|v'|$ on the initial line.

<u>Proof</u>: The proof has two parts. In the first the asserted bound is established for the boundary values of $|v'|$; actually, only the upper bound for $v'(0,t)$ is considered because the lower bound and the bounds for $v'(1,t)$ can be deduced from it by simple change of variables. In the second part the bound is established for interior points.

<u>First Part</u>: Let $u(x,t) = v(x,t) - v(0,t)$. Then u satisfies the differential equation

(5.2.3) $\dot{u} = a(x,t,v,v',u'')u'' + b(x,t,v,u')u' - \dot{v}(0,t)$,

and therefore the inequality

(5.2.4) $\dot{u} \leq a(x,t,v,v',u'')u'' + b(x,t,v,u')u' + D$

where D is the maximum of $|\dot{v}|$ on the boundary lines. Let B denote max $|v|$ on Γ and J, max $|v'|$ on initial line. The trivial case $B = 0$ is excluded here.

Clearly, $|u|$ vanishes on the left boundary, $|u| \leq 2B$ on right boundary, and $|u(x,0)| \leq \min[2B; Jx]$ on initial line.

Now let $w(x,t) = f(x) + \varepsilon + \varepsilon t$. If f can be chosen such that $u < w$ on Γ and such that

(5.2.5) $w > a(x,t,v,v',w'')w'' + b(x,t,v,w')w' + D$ in R, all $\varepsilon > 0$,

then $u < w$ in S by Lemma 3.3.1, and, letting $\varepsilon \to 0$, $u(x,t) \leq f(x)$.

If, in addition, $f(0) = 0$ and $f'(0) < \infty$, then $u'(0,t) \leq f'(0) < \infty$ will yield the desired bound. A slightly modified procedure, to be used here, also suffices; in the modification of Remark 3.3.2, Lemma 3.3.1 is applied to the rectangle $0 \leq x \leq X$, $0 \leq t \leq T$, instead of to S, where $0 < X \leq 1$.

The inequality (5.2.5) is equivalent to

$$(5.2.6) \qquad \varepsilon > a(x,t,v,v',f'')f'' + b(x,t,v,f')f' + D.$$

It suffices, then, to find X in $(0,1)$ and $f \in C^2(0 \leq x \leq X)$ satisfying in $(0,X)$

$$(5.2.7) \qquad \alpha f'' + \beta(f')f' + D = 0$$

$$(5.2.8) \qquad f(x) \geq \max[\min(2B;\ Jx);\ \frac{2B}{X}\,x]$$

$$(5.2.9) \qquad f(0) = 0; \quad f'(x) > 0; \quad f''(x) < 0.$$

The solution of (5.2.7) satisfying $f(0) = 0$; $f'(0) = s_o > 0$ can be expressed parametrically as follows

$$(5.2.10) \qquad f(x(s)) = \alpha \int_s^{s_o} \frac{z\,dz}{D + z\,\beta(z)}$$

$$(5.2.11) \qquad x(s) = \alpha \int_s^{s_o} \frac{dz}{D + z\beta(z)}$$

and for $0 < s \leq s_o$ this defines $f(x)$ satisfying (5.2.9) over the interval

$$(5.2.12) \qquad x(s_o) = 0 \leq x < x(0) = \alpha \int_0^{s_o} \frac{dz}{D + z\beta(z)} \leq \infty,$$

where $x(0) = \infty$ only if $D = 0$ (constant boundary data). In fact $f'(x(s)) = s$ and $f''(x(s)) = -[D + s\beta(s)]/\alpha$. Now for any choice of $s_o > 0$ and of X, $0 < X \leq \min[1; x(0)]$, the f constructed satisfies (5.2.7) and (5.2.9). It remains to show that proper choice of s_o and X will insure that (5.2.8) holds.

Define now

$$(5.2.13) \qquad X = \min \left[\alpha \int_0^{s_o} \frac{dz}{D + z\beta(z)} ; \frac{2B}{J}; 1\right],$$

for any $s_o > 0$. Then $0 < X \leq 1$ except in the (excluded) trivial case $B = 0$. Because of the concavity of $f(x)$ over $0 \leq x \leq X$, it is clear that

$$(5.2.14) \qquad f(X) \geq 2B$$

is both necessary and sufficient for (5.2.8). The First Part of the proof will be completed when it is proved that condition (5.2.2) entails

$$(5.2.15) \qquad f(X) \to \infty \quad \text{as} \quad s_o \to \infty,$$

so that (5.2.14) is assured for sufficiently large $s_o < \infty$.

To prove (5.2.15) define a function $\pi(s_o) \geq 0$ by the requirement

$$X = \alpha \int_{\pi(s_o)}^{s_o} \frac{dz}{D + z\beta(z)} .$$

It is clear from (5.2.13) and positivity of integrand that this is a valid definition. If

$$(5.2.16) \quad \int^{\infty} \frac{dz}{z\beta(z)}$$

is convergent, then $\pi(s_o)$ is bounded as $s_o \to \infty$ (because the contrary would imply $X \to 0$ as $s_o \to \infty$ while (5.2.13) shows that X is strictly positive and does not decrease as s_o increases). Condition (5.2.2) then implies

$$f(X) = \alpha \int_{\pi(s_o)}^{s_o} \frac{zdz}{D + z\beta(z)} \to \infty \quad \text{as} \quad s_o \to \infty.$$

On the other hand, (5.2.16) divergent implies $\pi(s_o) \to \infty$ as $s_o \to \infty$ (because boundedness of $\pi(s_o)$ would imply $X \to \infty$ as $s_o \to \infty$ contrary to $X \leq 1$). In this case the second mean value theorem yields

$$f(X) = \alpha \int_{\pi(s_o)}^{s_o} \frac{z\,dz}{D + z\beta(z)} = \alpha\,\tilde{s} \int_{\pi(s_o)}^{s_o} \frac{dz}{D + z\beta(z)} = \tilde{s}\,X,$$

where $\pi(s_o) < \tilde{s} < s_o$. Hence $f(X) > \pi(s_o)X$; for large enough s_o, however, X equals one of the positive constants $2B/J$ or 1 and therefore $f(X) \to \infty$ as $s_o \to \infty$.

It will be assumed that s_o is chosen large enough to insure (5.2.14).

Second Part: Let $u(x,y,t) = v(y,t) - v(x,t)$ for (x,y,t) in the cylinder S^* with base $\Delta: 0 \leq x \leq y \leq 1$ and extent $0 \leq t \leq T$. Similarly for R^* and Γ^*. Subscripts will be used for partial derivatives of u, as u_x, u_y, u_{yy}, etc.

Then u satisfies the differential equation

$$(5.2.17)\quad u_t = a(y,t,v(y,t),v'(y,t),v''(y,t))u_{yy} + a(x,t,v(x,t),v'(x,t),v''(x,t))u_{xx} + b(y,t,v(y,t),u_y)u_y + b(x,t,v(x,t),-u_x)u_x \quad \text{in } R^*,$$

which will be abbreviated

$$(5.2.18)\quad u_t = G^*(x,y,t,u,u_x,u_y,u_{xx},u_{yy}).$$

On the boundary faces of S^* the following is known about u

$$(5.2.19)\quad u(x,x,t) \equiv 0$$

$$(5.2.20)\quad \begin{aligned} &u(0,y,t) \leq \min[f(y);\ 2B], \quad f \text{ as chosen in First Part above;} \\ &u(x,1,t) \leq \min[f(1-x);\ 2B]. \end{aligned}$$

The remainder of the proof consists in showing the existence of an f^* with $f^*(0) = 0$, $f^{*\prime}(0) < \infty$, such that $u(x,y,t) < f^*(y-x) + \varepsilon + \varepsilon t$ for all $\varepsilon > 0$ and for (x,y,t) in a cylinder adjacent to the diagonal face $0 \leq x = y \leq 1$, $0 \leq t \leq T$; say the cylinder S^{**} specified by $0 \leq x \leq y \leq 1$, $y - x \leq X^*$, and $0 \leq t \leq T$. The first step is to establish an easy extension of Lemma 3.3.1 to the effect that

$$(5.2.21)\quad u_t \leq F(x,t,u,u_x,u_y,u_{xx},u_{yy})$$

$$(5.2.22)\quad w_t > F(x,t,w,w_x,w_y,w_{xx},w_{yy})$$

in R^{**} and $u < w$ on Γ^{**} implies $u < w$ in S^{**} provided F is a non-decreasing function of each of its last two arguments. Since the result, the conditions of validity, and proof are direct parallels of these of Lemma 3.3.1, a formal statement is omitted here. Letting G^* of (5.2.18) play the role of F in (5.2.21 - 22), one finds easily as in the First Part that $w(x,y,t) = f^*(y - x) + \varepsilon + \varepsilon t$ will satisfy (5.2.22) if f^* satisfies

(5.2.23) $\quad \alpha f^{*''} + \beta(f^{*'}) f^{*'} \leq 0, \quad f^{*'} \geq 0; \quad f^{*''} \leq 0 \text{ in } [0,X^*]$

From (5.2.18 - 19) one sees that $u < w$ on the boundary faces of S^{**} if

(5.2.24) $\quad X^* \leq X$ and $f^*(y - x) \geq f(y - x)$ for $0 \leq y - x \leq X^*$.

Because of (5.2.8), (5.2.24) also implies $u < w$ on the base of S^{**}.

Now choose $X^* = X$, $f^* = f$. Since f satisfies (5.2.7 - 9) and $D \geq 0$, f a fortiori satisfies (5.2.23) and (5.2.24). In this way it is found finally that $u(x,y,t) \leq f(y - x)$ for $0 \leq y - x \leq X$. Application to the function $-u$ yields $-u(x,y,t) \leq f(y - x)$ and therefore $|u(x,y,t)| \leq f(y - x)$, and the asserted bound of the theorem follows. The function (p) is assumed even so that the assumptions are invariant when u is replaced by $-u$. If theeveness is dropped, it is necessary to assume that the integral of (5.2.2) diverges at $-\infty$ as well as $+\infty$.

<u>Remark 5.2.1</u>: An example to be discussed later shows, in a sense, the necessity of a growth restriction like (5.2.2); cf. Example 8.2.1.

Remark 5.2.2: If $v(x,t)$ is a function such that the assumptions of Theorem 5.2.1 hold for every $T > 0$ and if the suprema of the maxima D, B, and J for all $T > 0$ are finite, then the theorem assures the existence of a bound for $|v'|$ which is independent of T. In certain problems P not satisfying hypotheses of Theorem 5.2.1, the existence of such a bound may be contingent upon the magnitudes of D, B, and J and not merely on their boundedness independent of T. [cf. Example 8.1.1]

Remark 5.2.3: Some smoothness assumptions on v embodied in $v \varepsilon C^1(S) \cap D^2(R)$ are redundant. Examination of the proof shows that in order to get as far as the conclusion $|v(y,t) - v(x,t)| \leq f(y - x)$ smoothness assumptions are needed for three purposes: (i) application of Lemma 3.3.1 and its analogue for a cylinder; (ii) reduction to zero boundary values, cf. (5.2.3); (iii) to obtain $|u(x,0)| = |v(x,0) - v(0,0)| \leq Jx$. The following weaker assumptions suffice: For (i): $v \varepsilon C(S)$; $\dot{v}$, v', v'' exist in R. For (ii): $v(0,t)$, $v(1,t)$ $\varepsilon D(0 \leq t \leq T)$ and $|\dot{v}(0,t)|$, $|\dot{v}(1,t)|$ bounded by D. For (iii); $v(x,0)$ absolutely continuous and $|v'(x,0)| \leq J$ for $0 \leq x \leq 1$.

From $|v(y,t) - v(x,t)| \leq f(y - x)$ one then concludes for $0 \leq t \leq T$ that $|v'(x,t)| \leq f'(0)$ for $0 < x < 1$ and that the one-sided upper and lower partial derivates with respect to x are bounded above and below for $x = 0$ and $x = 1$. If the partial derivative $v'(x,t)$ exists in S, it follows that $|v'| \leq f'(0)$ in S.

Remark 5.2.4: Theorem 5.2.1 has been formulated for equation (3.4.1), because Theorem 3.4.1 gives prior assurance that the bound for $|v|$ in S depends only on its bounds on Γ. The proof then shows that the bound for $|v'|$ depends on the bound for $|v|$ in S. This remark leads to the more general formulation following.

Theorem 5.2.2: Suppose the hypotheses of Theorem 5.2.1 modified by replacing equation (3.4.1) by the equation

$$(5.2.25) \qquad \dot{v} = a(x,t,v,v',v'')v'' + b(x,t,v,v')v' + c(x,t,v)$$

where the supremum of $|c(x,t,z)|$ for $0 \leq x \leq 1$, $0 \leq t < \infty$, $|z| \leq B$ is finite for each $B \geq 0$.

Then the conclusion remains true if <u>dependence on max $|v|$ on Γ</u> is replaced by <u>dependence on max $|v|$ on S</u>.

Proof: Presence of c in (5.2.25) is accounted for, in proof of Theorem 5.2.1, by replacing the D in (5.2.4) by a larger constant depending on $\max|v|$ on S as well as on $\max|\dot{u}|$ on $x = 0$ and 1.

Remark 5.2.5: Roughly speaking, Theorem 5.2.2 shows that for v satisfying (5.2.25) and ancillary conditions, "shocks" do not form; i.e., $|v'|$ can become infinite only if $|v|$ does. Theorem 5.2.2 provides the best mathematical rationale attained here for the physical statement "viscosity prevents shocks" [cf. Introduction].

6. <u>Asymptotic Bounds for $v'(x,t)$</u>.

In certain cases asymptotic bounds for v' can be obtained in much the same way that they were obtained for v in 4. The technique will be sketched here, omitting a formal statement and most details. The assumptions are those of Theorem 5.2.1 for <u>every</u> (instead of <u>some</u>) $T > 0$ and with the additional restriction that the function $\beta(p)$ of (5.2.1) may be assumed constant (β will denote the constant). Theorem 3.4.1 yields existence of B independent of T such that $|v| \leq B$ in S for $T > 0$ and hence $|b(x,t,v,v')| < \beta$ in S, all $T > 0$.

Proceeding as in proof of Theorem 5.2.1, u is introduced and inequality (5.2.4) found. Instead of a comparison function $w(x,t) = f(x) + \varepsilon + \varepsilon t$, however, one of form

$$(6.1.1) \qquad w(x,t) = f(x)\, e^{-\eta t} + g(x) + \varepsilon + \varepsilon t$$

is sought, where f, g, and $\eta > 0$ are to be determined such that $u < w$ on Γ and w is to satisfy in R an inequality for which

$$(6.1.2) \qquad (\alpha f'' + \beta f' + \eta f)\, e^{- \ t} + (\alpha g'' + \beta g' + D) \leq 0, \qquad 0 < x < 1$$

is sufficient provided $f(0) = g(0) = 0$ and f, g are increasing and concave on unit interval. (It is not necessary to restrict to an interval $0 \leq x \leq X < 1$ in this case.) Now let $g(x)$ be the solution of $\alpha g'' + \beta g' + D = 0$ such that $g(0) = 0$, $g'(0) = g'_0 > 0$ and $f(x)$ the solution of $\alpha f'' + \beta f' + \eta f = 0$ such that $f(0) = 0$, $f'(0) = f'_0 > 0$. An easy argument of the kind used before shows that

g'_o sufficiently large insures $g'(x) \geq 0$, $g''(x) \leq 0$ for $0 < x < 1$ and $g(1) \geq 2B'$ where B' is a bound for $|v|$ on the boundary lines (but not necessarily on the initial line, so $B' \leq B$). Let g'_o be chosen as small as is consistent with these conditions; then g'_o depends on α, B(via β), and D, but <u>not on</u> $\underline{J}$. Provided only that $f(x) \geq 0$ this choice of $g(x)$ guarantees $u < w$ on the boundary lines (but not necessarily on the initial line). Another argument of the kind used before shows that if η is taken sufficiently small but positive and if f'_o is taken sufficiently large then the conditions $f'(x) \geq 0$, $f'' \leq 0$ and $f(x) + g(x) \geq \min[Jx; 2B]$ are satisfied for $0 \leq x \leq 1$. Making appropriate choices of f'_o and η, one finds $u(x,t) \leq f(x)\, e^{-\eta t} + g(x)$ and $v'(x,0) \leq f'_o\, e^{-\eta t} + g'_o$. Thus g'_o appears as an asymptotic bound for $v'(x,0)$. The other bounds at the endpoints and interior points are treated in analogous fashion by modifying proof of Theorem 5.2.1 in the way indicated here.

It is not clear at the present writing whether the restriction to $\beta(p)$ = constant is essential for this result.

7. <u>On Uniqueness.</u>

7.1. <u>Modification of Westphal's Lemma.</u>

<u>Lemma 7.1.1</u>: Replace the three strict inequalities in the statement of Lemma 3.3.1 by non-strict inequalities of same sense and add the assumption that $w(x,t)$ and F satisfy the following condition.

(7.1.1) For any $K > 0$ there exists an M independent of (x,t) in R such that

$$|F(x,t,w+z,w',w'') - F(x,t,w,w',w'')| < M|z|$$

in R whenever $|z| < K$.

Then the lemma remains true.

<u>Proof</u>: Define $u(x,t) = w(x,t) + \Delta + \varepsilon t$ where $\Delta > 0$ and $\varepsilon > 0$ are to be determined. Then $v < u$ on Γ and $\dot{u} = \dot{w} + \varepsilon$. The inequality

(7.1.2) $u_t > F(x,t,u,u',u'') \equiv F(x,t,w + \Delta + \varepsilon t,w',w'')$

holds for $0 < x < 1$, $0 < t \leq t_o \leq T$ provided

(7.1.3) $F(x,t,w + \Delta + \varepsilon t,w',w'') - F(x,t,w,w',w'') < \varepsilon$

for same range of x and t.

In condition (7.1.1) let M be fixed by some choice of K. Then for (7.1.3) it is sufficient that $\Delta + \varepsilon t < K$ and $M(\Delta + \varepsilon t) < \varepsilon$. It is asserted now that suitable choice of $\Delta > 0$ and $\varepsilon > 0$ will guarantee these inequalities for $0 < t \leq t_o$ for any $t_o < \frac{1}{M}$. For let positive $h < 1 - Mt_o$ be chosen and put $\Delta = \frac{h\varepsilon}{M}$. Then $M(\Delta + \varepsilon t) \leq \varepsilon h + M\varepsilon t_o < \varepsilon$, all $\varepsilon > 0$, and $\Delta + \varepsilon t < (\frac{\varepsilon}{M})(h + 1) < \frac{2\varepsilon}{M} < K$, all $\varepsilon < \frac{MK}{2}$.

Hence (7.1.2) holds for any $t_o < \frac{1}{M}$ and all sufficiently small ε. Application of Lemma 3.3.1 gives $v(x,t) < u(x,t) = w(x,t) + [\frac{h}{M} + t]\,\varepsilon$ for $0 \leq x \leq 1$, $0 \leq t \leq t_o$. Letting $\varepsilon \to 0$, $v(x,t) \leq w(x,t)$. Since t_o was arbitrary in $(0,\frac{1}{M})$, this holds for $0 \leq x \leq 1$, $0 \leq t \leq \frac{1}{M}$.

By application of the above proof to the strip $0 \leq x \leq 1$ $\frac{1}{M} \leq t \leq T$, one finds $v \leq w$ for $0 \leq x \leq 1$, $0 \leq t \leq \frac{2}{M}$. By stepwise iteration it is proved after a finite number of steps that $v \leq w$ in S.

Example 7.1.1: $F(x,t,z,p,r) = z^{2/3} x^{1/3} (1 - x)^{1/3}$; $v(x,t) = x\,(1 - x)(\frac{t}{3})^3$; $w(x,t) \equiv 0$. Then $v \leq w$ on Γ, any $T > 0$, and $\dot{w} = F(x,t,w,w',w'')$, but $v \leq w$ is false.

7.2. Uniqueness Theorem.

Theorem 7.2.1: Let $T > 0$ be fixed and let v satisfy:

(i) $v \in C(S)$; $\dot{v}, v', v''$ exist in R.

(ii) v', v'' bounded in R.

(iii) $\dot{v} = F(x,t,v,v',v'')$ in R.

where (iv) $F(x,t,z,p,r)$ is a non-decreasing function of r for any fixed values of other arguments: $0 < x < 1$; $0 < t \leq T$; $-\infty < z,p,r < +\infty$.

(v) $F(x,t,z,p,r)$ satisfies a Lipschitz condition, with respect to the variable z, of the following form: For any fixed constants B, B_0, B_1, B_2, all ≥ 0, there exists an M such that

$$\sup[|F(x,t,z+z^*,p,r) - F(x,t,z,p,r)|/|z^*|] < M,$$

the supremum being taken over $0 < x < 1$, $0 < t \leq T$, $|z^*| \leq B$, $|z| \leq B_0$, $|p| \leq B_1$, $|r| \leq B_2$, $z^* \neq 0$.

Then if u also satisfies the conditions (i) - (iii) and $u = v$ on Γ, $u = v$ on S.

Proof: Because of (i), (ii), (v), both the pairs (v,F) and (u,F) satisfy the condition expressed in (7.1.1) for the pair (w,F). Lemma 7.1.1 then implies both $u \leq v$ on S and $v \leq u$ on S, so $u = v$ on S.

Remark 7.2.1: Theorem 7.2.1 removes certain obscurities in a uniqueness theorem proved by Westphal [8, Satz 2]. His "Lipschitz condition" seems to mean (7.1.1) and is therefore a condition on the solution as well as on the function F. Westphal proves his Satz 2 directly from his lemma (Lemma 3.3.1) without introducing the modified Lemma 7.1.1. More general conditions for uniqueness are also discussed in [8].

7.3. Crossings of Stationary Solutions.

Theorem 7.3.1: Theorem 3.4.2 remains true if its condition (iii) is replaced by

(iii)' The pairs (f,F) and (g,F) satisfy the condition expressed by (6.1.1) for (w,F).

Proof: Put $\varepsilon = 0$ in proof of Theorem 3.4.2 and use Lemma 7.1.1 instead of Lemma 3.3.1.

Theorem 7.3.2: Let $v \in C(S) \cap D^2(R)$ satisfy $\dot{v} = F(x,t,v,v',v'')$ in R, F being non-decreasing in last argument. Let $h(x)$ be a stationary solution of the same equation, $h \in C([0,1]) \cap D^2((0,1))$, and such that the pair (h,F) satisfies the condition expressed by (7.1.1) for the pair (w,F). Let $x_1(t)$, $x_2(t)$ be two continuous functions such that $0 \leq x_1(t) < x_2(t) \leq 1$, $v(x_1(t),t) \equiv h(x_1(t))$, and $v(x_2(t),t) \equiv h(x_2(t))$ for $0 \leq t \leq t_o \leq T$. Then $v(x,0) \geq h(x)$

[or $v(x,0) \leq h(x)$] for $x_1(0) \leq x \leq x_2(0)$ implies $v(x,t) \geq h(x)$ [or $v(x,t) \leq h(x)$] for $x_1(t) \leq x \leq x_2(t)$, $0 \leq t \leq t_o$.

Proof: For the special case $x_1(t) \equiv 0$, $x_2(t) \equiv 1$, the proof is immediate from Theorem 7.3.1. The proof for general case follows in the same way from Remark 3.3.2.

Example 7.3.1: Applied to Burgers' Problem (1.1.2 - 4) with $h(x) \equiv 0$, Theorem 7.3.2 implies that the number of zeros of $u(x,t)$ does not increase with t. For if $x = x_1(t)$, $x = x_2(t)$ are two curves in S on which u vanishes and if $u(x,0) \geq 0$ [or $u(x,0) \leq 0$] for $x_1(0) \leq x \leq x_2(0)$, then u is non-negative [non-positive] between the curves so long as they do not merge; i.e., no new zero appears.

Example 7.3.2: Let $\alpha > 0$ be a constant. The equation $\dot{v} = \alpha v'' + v^4$ has two stationary solutions vanishing on the boundary lines $x = 0$, $x = 1$. One is $v = 0$ and the other, say $v_o(x)$, is a concave function. If $v(x,t)$ is a (non-stationary) solution in R such that $v(0,t) \equiv v(1,t) \equiv 0$ then Theorem 7.3.1 shows that $0 \leq v(x,t) \leq v_o(x)$ holds for $0 \leq t \leq T$ if it holds for $t = 0$.

8. Two Examples.

8.1. Example 8.1.1: Let $v \in C(S) \cap D^2(R)$ satisfy

(8.1.1) $\dot{v} = \alpha v'' + v^4$ $\qquad \alpha > 0$, constant

(8.1.2) $v(0,t) \equiv v(1,t) \equiv 0$

(8.1.3) $v(x,0) = V(x)$

for some $T > 0$, where $V \in C^2([0,1])$. In contrast to Example 7.3.2 it will be shown here that, given a sufficiently small $\varepsilon > 0$, there exists a choice of $V \in C^2$ such that $T < \varepsilon$ is necessary. Specifically a function $u(x,t)$ will be constructed such that

(8.1.4) $\dot{u} \leq \alpha u'' + u^4$

(8.1.5) $u(0,t) \equiv u(1,t) \equiv 0$

(8.1.6) $u(\frac{1}{2}, t) \to \infty$ as $t \to \varepsilon$.

If $V \in C^2$ is chosen so $V(x) \geq u(x,0)$, Lemma 7.1.1 will then imply that $v(x,t) \geq u(x,t)$ so that v cannot remain bounded up to $t = \varepsilon$.

For brevity write $p = 4x(1-x)$ and $f = f(t) = \frac{1}{\varepsilon - t}$; i.e. f is the function determined by $f' = f^2$, $f(0) = \frac{1}{\varepsilon}$. Define $u = f^{1/2}[p]^f$. Then (8.1.5 - 6) are obvious and (8.1.4) is equivalent to

$$(8.1.7) \quad f^{1/2}\, p^{3f+2} + f[4\alpha(1-2x)^2 + p^2 \log \tfrac{1}{p}] - [\tfrac{1}{2}\, p^2 + 2\alpha p + 4\alpha(1-2x)^2] \geq 0.$$

The function in the first square brackets of (8.1.7) is positive except at $x = \frac{1}{2}$, where it vanishes. The function (polynomial) in the second square brackets is strictly positive. Hence, if f is sufficiently large, the sum of the last two terms is positive except in a small neighborhood of $x = \frac{1}{2}$. In fact, it is readily verified that the sum of these two terms is positive when $|x - \frac{1}{2}| > 1/[2\sqrt{f}]$ and bounded below by a constant if $|x - \frac{1}{2}| \leq 1/[2\sqrt{f}]$. The first term of (8.1.7) is non-negative; within the interval $|x - \frac{1}{2}| \leq 1/[2\sqrt{f}]$

it tends to infinity as $f \to \infty$ since in this interval

$$f^{1/2}\, p^{3f} \geq f^{1/2}\, p^{3f}\Big|_{x = \frac{1}{2} + \frac{1}{2\sqrt{f}}} = f^{1/2}(1 - \tfrac{1}{f})^{3f} \cong f^{1/2}\, e^{-3} \to \infty,$$

as $f \to \infty$.

Thus (8.1.7) and (8.1.4) hold if ε is sufficiently small, since $f(t) \geq f(0) = \frac{1}{\varepsilon}$.

8.2. Example 8.2.1: Let $v \in C^1(S)\quad D^2(R)$ satisfy

(8.2.1) $\dot{v} = \alpha v'' + (v')^n$ $\qquad\qquad \alpha$ = constant

(8.2.2) $v(0,t) \equiv 0; \quad v(1,t) \equiv 1$

(8.2.3) $v(x,0) \equiv x$

It will be proved here that if n is a real number, $n > 2$, and if $0 < \alpha < \varepsilon^{n-1}$, where $0 < \varepsilon = \frac{n-2}{n-1} < 1$, then a positive k can be found such that

(8.2.4) $v'(0,t) \geq \varepsilon\, e^{k(1-\varepsilon)t}$

for $0 \leq t \leq T$. Since the right member of (8.2.4) tends to infinity as $t \to \infty$, this means that no bound independent of T can exist for $|v'|$. On the other hand the bounds $0 \leq v(x,t) \leq 1$ can be asserted, independent of T, by Theorem 3.4.1; cf. Remark 5.2.5. It should be noted that for $n > 2$ the growth condition (5.2.1 - 2) of Theorem 5.2.1 is not satisfied by (8.2.1).

The proof consists of applying Lemma 7.1.1 to prove that $v(x,t) \geq w(x,t)$, where

$$(8.2.5) \qquad w(x,t) = e^{-\varepsilon kt}[(1 + e^{kt}\, x)^{\varepsilon} - 1].$$

That this function satisfies $w(0,t) = 0$, $w(1,t) < 1$, and $w(x,0) \leq x$ for $0 \leq x \leq 1$, $0 \leq t$ is easily verified. It remains to show that a sufficiently small choice of $k > 0$ will insure that

$$(8.2.6) \qquad \dot{w} \leq \alpha w'' + (w')^{n}$$

For brevity write $m = e^{kt}$ so $\dot{m} = km$ and $m \geq 1$. Then $w = m^{-\varepsilon}[(1 + mx)^{\varepsilon} - 1]$ and (8.2.6) is equivalent to

$$(8.2.7) \qquad -\varepsilon m^{-\varepsilon-1}\, \dot{m}[(1 + mx)^{\varepsilon} - 1] + m^{-\varepsilon}[\varepsilon(1 + mx)^{\varepsilon-1}\, \dot{m}x]$$
$$\leq \alpha m^{2-\varepsilon}\, \varepsilon(\varepsilon - 1)(1 + mx)^{\varepsilon-2} + [\varepsilon m^{1-\varepsilon}(1 + mx)^{\varepsilon-1}]^{n}.$$

It suffices to have (8.2.7) with the first term on the left deleted. Multiplying by $m^{\varepsilon-2}(1 + mx)^{2-\varepsilon}$ one finds

$$m^{-2}\, \dot{m}x\varepsilon(1 + mx) \leq \alpha\varepsilon(\varepsilon - 1) + \varepsilon^{n}\, m^{(1-n)\varepsilon+n-2}\, (1 + mx)^{(n-1)\varepsilon-n+2}$$

or

$$(8.2.8) \qquad km^{-1}\, x(1 + mx) \leq \alpha(\varepsilon - 1) + \varepsilon^{n-1},$$

since $(1 - n)\varepsilon + n - 2 = 0$, by definition of ε. Because $x(1 + mx) \leq 2m$, it suffices for (8.2.8) that $2k \leq -\alpha + \varepsilon^{n-1}$; hence $k = \frac{1}{2}(\varepsilon^{n-1} - \alpha) > 0$ is a suitable choice of k.

PART II. THE DISCRETE CASE

9. Existence and Boundness for Solutions of Problem Q.

9.1. Local Existence.

Theorem 9.1.1: Let n be an integer, $n \geq 2$. If

(i) $a(\bar{x},t,z)$, $b(\bar{x},t,z)$ are (jointly) continuous functions of t and z for $t \in [0, \infty]$, $z \in (-\infty, +\infty)$, and each fixed $\bar{x} \in (0,1)$;

(ii) $\phi(t)$, $\Psi(t) \in C([0,\infty])$;

(iii) $V(x)$ is defined for $x \in (0,1)$;

(iv) $a(\bar{x},t,z)$, $b(\bar{x},t,z)$ satisfy a local Lipschitz condition with respect to z, each fixed $\bar{x} \in (0,1)$.

Then there exist $T > 0$ and $v_k(t) \in C^1([0,T])$, $k = 1,2,\ldots, n-1$, such that the v_k satisfy (2.1.1 - 3) for $t \in [0,T]$. The v_k are unique; i.e., if $\tilde{v}_k$ satisfy (2.1.1 - 3) for $t \in [0,T^*]$, $T^* > 0$, then $\tilde{v}_k(t) = v_k(t)$ for $t \in [0, \min(T,T^*)]$.

Proof: The conditions (i) - (iii) suffice for application of the classical existence theorem of Peano; Kamke [10,p.126]. Condition (iv) implies that the right members of (2.1.1) satisfy a Lipschitz condition sufficient for uniqueness; [10 , p.141].

9.2. Boundedness and Global Existence.

Theorem 9.2.1: In addition to (i) - (iii) of Theorem 9.1.1 assume that

(i) $V(x)$ is bounded, $x \in (0,1)$;

(ii) $a(x,t,z) \geq \alpha > 0$, $x \in (0,1)$, $t \in [0,\infty]$, $z \in (-\infty, +\infty)$;

(iii) $b(x,t,z)$ is such that

$$\tilde{b}(z_1, z_2) = \sup_{\substack{0<x<1 \\ 0\leq t<\infty \\ z_1\leq z\leq z_2}} |b(x,t,z)| < \infty$$

for $z_1, z_2 \in (-\infty, +\infty)$, $z_1 \leq z_2$;

(iv) $n > \frac{1}{2\alpha} \tilde{b}(b_1 - \varepsilon_o, b_2 + \varepsilon_o)$, where $\varepsilon_o > 0$ and b_1, b_2 are the infemum and supremum over Γ of the function defined on Γ by $\emptyset(t)$, $\Psi(t)$, $V(x)$; (Γ is defined in 3.1);

(v) $v_k(t) \in C^1([0,T])$, $k = 1,2,\ldots, n-1$, satisfy (2.1.1 - 3), $t \in [0,T]$.

Then

$$(9.2.1) \quad b_1 \leq \min_{0\leq k\leq n} v_k(t) \leq \max_{0\leq k\leq n} v_k(t) \leq b_2, \qquad 0 \leq t \leq T.$$

<u>Proof</u>: Put $M(t) = \max_{0\leq k\leq n} v_k(t)$ and suppose, contrary to the asserted upper bound, that $M(t_o) > b_2$ for some $t_o \in [0,T]$. Choose $\varepsilon > 0$ sufficiently small to insure $M(t_o) > b_2 + \varepsilon$ and $\varepsilon_o > \varepsilon$. Then $\bar{t} = \inf[t \,|\, M(t) \geq b_2 + \varepsilon]$ exists. Since $M(0) \leq b_2$, $0 < \bar{t} \leq T$. Because $M(t)$ is continuous, $M(\bar{t}) = b_2 + \varepsilon$. In the remainder of this proof, let k be the (fixed) greatest integer such that $v_k(\bar{t}) = M(\bar{t})$; clearly, $0 < k < n$ and $\dot{v}_k(\bar{t}) \geq 0$. By choice of k, $v_k = b_2 + \varepsilon$, $v_k - v_{k-1} \geq 0$, and $v_{k+1} - v_k < 0$ at $t = \bar{t}$. Moreover,

$$|b(\tfrac{k}{n}, \bar{t}, v_k(\bar{t}))| = |b(\tfrac{k}{n}, \bar{t}, b_2 + \varepsilon)| \leq \tilde{b}(b_1 - \varepsilon_o, b_2 + \varepsilon_o)$$

$$< 2\alpha n \leq 2na(\tfrac{k}{n}, \bar{t}, v_k(\bar{t})).$$

Using the abbreviations $a_k = a(\frac{k}{n}, \bar{t}, v_k(\bar{t}))$ and $b_k = b(\frac{k}{n}, \bar{t}, v_k(\bar{t}))$, this may be written as $2na_k \pm b_k > 0$. At $t = \bar{t}$ the equation (2.1.1) gives, after a slight rearrangement,

$$\dot{v}_k = \frac{n}{2}(2a_k n + b_k)(v_{k+1} - v_k) - \frac{n}{2}(2a_k n - b_k)(v_k - v_{k-1}) < 0$$

contradicting $\dot{v}_k(\bar{t}) \geq 0$. Thus $M(t) \leq b_2$, $0 \leq t \leq T$, and proof for upper bound is complete. The same proof shows that $\max_{0 \leq k \leq n} (-v_k(t)) \leq b_1$, which is the lower bound.

Example 9.2.1: Let $n = 2$, $a(x,t,z) = \frac{1}{16}$, $b(x,t,z) = z - x$, $\phi(t) \equiv 0$, $\varpi(t) \equiv 1$, $V(x)$ such that $0 \leq V(x) \leq 1$, $V(\frac{1}{2}) = 1$. Then $b_1 = 0$, $b_2 = 1$. The only equation is $\dot{v}_1 = \frac{1}{2} v_1 - \frac{1}{4}$. All conditions other than (iv) are satisfied. The solution $v_1(t) = \frac{1}{2}(1 + e^{1/2t})$ does not satisfy the conclusion of Theorem 9.2.1 for any $t > 0$.

Remark 9.2.1: Theorem 9.2.1 does not assert that the maximum of $v_k(t)$ for $0 \leq k \leq n$, $0 \leq t \leq T$ is actually attained on Γ, i.e., at one of the points $t = 0$, $x = \frac{k}{n}$, $k = 1,2,\ldots, n - 1$ or at a point on one of the side boundary lines $x = 0$ or 1, $0 \leq t \leq T$. For any fixed n, however, a solution is not affected by changing the values of $V(x)$ at points $x \neq \frac{k}{n}$, $k = 1,2,\ldots, n - 1$. By suitable change of this kind it can be assured that the value b_2 is assumed either on the side boundaries or at one of the points $x = \frac{k}{n}$, $k = 1,2,\ldots, n - 1$, and it follows that the maximum of $v_k(t)$ is assumed on Γ. The advantage of the formulation here is that b_1, b_2 do not depend on n so that (9.2.1) holds uniformly for all n sufficiently large.

Theorem 9.2.2: To (i) - (iii) of Theorem 9.1.1 and (i) - (iii) of Theorem 9.2.1 add the assumptions that $|\phi(t)|$ and $|\Psi(t)|$ are uniformly bounded for $t \in [0,\infty)$ and that (iv) of Theorem 9.2.1 holds for every $T > 0$.

Define

$$(9.2.2) \qquad B_1 = \inf_{T>0} b_1; \qquad B_2 = \sup_{T>0} b_2.$$

Then there exist $v_k(t) \in C^1([0,\infty])$, $k = 1,2,\ldots, n-1$, satisfying (2.1.1 - 3); any such v_k satisfy

$$(9.2.3) \qquad B_1 \leq v_k(t) \leq B_2 \qquad k = 0,1,\ldots, n; \quad t \in [0,\infty).$$

Proof: By Theorem 9.1.1 a solution exists for $0 \leq t \leq T > 0$. In conjunction with the a priori bounds of Theorem 9.2.1, a classical extension theorem [10, p.135] shows that this solution can be extended to $0 \leq t < \infty$.

10. Bounds for First Differences; Preliminary Lemmas.

10.1. Implicit Assumptions and Notations. In the remainder of Part II it is assumed without further explicit mention that the hypotheses of Theorem 9.2.2 are fulfilled. It is assumed further that $\dot{\phi}(t)$ and $\dot{\Psi}(t)$ exist and are uniformly bounded for $t \in [0,\infty)$. The function $V(x)$ is extended to the closed interval by defining $V(0) = \phi(0)$, $V(1) = \Psi(0)$, and the extended function is assumed to satisfy $J < \infty$, where

$$(10.1.1) \qquad J = \sup_{0\leq x<y\leq 1} \left|\frac{V(y) - V(x)}{y - x}\right| .$$

The <u>constants</u> D, B, β are defined by

$$D = \max[\sup_{0\le t} |\dot{\phi}|; \sup_{0\le t} |\dot{\Psi}|];$$

$$B = B_2 - B_1, \quad \text{cf. } (9.2.3);$$

$$\beta = \tilde{b}(B_1, B_2), \quad \text{cf. Theorem 9.2.1 (iii).}$$

Thus $|v_j(t) - v_i(t)| \le B$ and $|b(x,t,v_k(t))| \le \beta < 2\alpha n$, $0 \le t < \infty$, $0 \le i,j,k \le n$. The trivial case $B = 0$ is excluded.

A function g and its inverse f are defined by

$$g(z) = \frac{1}{c}\log\frac{1}{1-cz}, \qquad -\infty < x < \frac{1}{c};$$

$$f(x) = \frac{1}{c}(1 - e^{-cx}), \qquad -\infty < x < +\infty,$$

where $0 < c < \frac{1}{B}$. Both f and g are increasing, f is concave, g convex, and $f'(0) = g'(0) = 1$, $f(0) = g(0) = 0$.

The following notation is used

$$H_{ij}(t) = \frac{n}{j-i}\, g(v_j(t) - v_i(t)), \quad 0 \le i < j \le n,\ 0 \le t < \infty;$$

$$\tilde{H}(t) = \max_{0\le i<j\le n} H_{ij}(t), \qquad 0 \le t < \infty.$$

For $\bar{t} \in [0,\infty)$, r and s denote any pair of integers (depending on $\bar{t}$ and not necessarily unique) such that $H_{rs}(\bar{t}) = \tilde{H}(\bar{t})$.

10.2. <u>Three Lemmas</u>.

<u>Lemma 10.2.1</u>: If $\tilde{H}(\bar{t})$ is sufficiently large so that both

(10.2.1) $\quad \tilde{H}(\bar{t}) > g(B) = \frac{1}{c}\log\frac{1}{1-cB}$

and

$$(10.2.2) \qquad \left[\frac{Dg'(B)}{\tilde{H}(\bar{t})} + \beta - 2\alpha n \tanh \frac{c\tilde{H}(\bar{t})}{2n}\right] \leq 0,$$

then

$$(10.2.3) \qquad \dot{H}_{rs}(\bar{t}) \leq H_{rs}(\bar{t}) \left[\frac{Dg'(B)}{\tilde{H}(\bar{t})} + \beta - 2\alpha n \tanh \frac{c\tilde{H}(\bar{t})}{2n}\right].$$

Proof: It is clear that (10.2.1 - 2) are satisfied if $\tilde{H}(\bar{t})$ is large enough, since $\tanh z \to 1$ as $z \to \infty$; and the assumptions insure that $2\alpha n > \beta$. To prove (10.2.3) consider the four cases implicit in $0 \leq r < s \leq n$.

Case 1: $0 = r < s = n$. $H_{rs}(t) = \frac{n}{n - 0} g(v_n(\bar{t}) - v_o(\bar{t})) \leq g(B)$.

Hence (10.2.1) excludes this case.

Case 2: $0 < r < s < n$. For brevity the argument $\bar{t}$ is understood but not written in what follows; also, a_k is written for $a(\frac{k}{n}, \bar{t}, v_k(\bar{t}))$ and similarly for b_k. The functions f and f' will occur with three arguments and corresponding abbreviations are introduced:

$$f = f[\frac{s - r}{n} \tilde{H}(\bar{t})]; \qquad f' = f'[\frac{s - r}{n} \tilde{H}(\bar{t})]$$

$$f_- = f[\frac{s - r - 1}{n} \tilde{H}(\bar{t})]; \quad f_+ = f[\frac{s - r + 1}{n} \tilde{H}(\bar{t})].$$

With these conventions, $\tilde{H} = H_{rs} = \frac{n}{s - r} g(v_s - v_r)$ and $v_s - v_r = f(\frac{s - r}{n} H_{rs}) = f$ (since $f(g(z)) \equiv z$). Noting that $g'(v_s - v_r) = g'(f) = \frac{1}{f'}$, one finds

(10.2.4) $$H_{rs} = \frac{n}{s - r} \cdot \frac{1}{f'}(\dot{v}_s - \dot{v}_r).$$

The differential equation gives, at $t = \bar{t}$,

(10.2.5) $$\dot{v}_s = \frac{n}{2}(2na_s + b_s)(v_{s+1} - v_s) + \frac{n}{2}(2na_s - b_s)(v_{s-1} - v_s),$$

provided $s < n$, and

(10.2.6) $$-\dot{v}_r = \frac{n}{2}(2na_r + b_r)(v_r - v_{r+1}) + \frac{n}{2}(2na_r - b_r)(v_r - v_{r-1})$$

provided $0 < r$.

By the maximal nature of $\tilde{H}$, $\frac{n}{s - r + 1} g(v_{s+1} - v_r) \leq \tilde{H}$ and $\frac{n}{s - r - 1} g(v_{s-1} - v_r) \leq \tilde{H}$, yielding $v_{s+1} - v_r \leq f_+$ and $v_{s-1} - v_r \leq f_-$.

Hence $v_{s+1} - v_s = (v_{s+1} - v_r) - (v_s - v_r) \leq f_+ - f$.

Similarly $v_s - v_{s-1} = (v_s - v_r) - (v_{s-1} - v_r) \geq f - f_-$.

Using these and $2na_k \pm b_k \geq 2n\alpha - \beta > 0$ in (10.2.5),

(10.2.7) $$\dot{v}_s \leq \frac{n}{2}(2na_s + b_s)(f_+ - f_-) + \frac{n}{2}(2na_s - b_s)(f_- - f),$$

provided $s < n$. Applying the same method to (10.2.6) gives

(10.2.8) $$-\dot{v}_r \leq \frac{n}{2}(2na_r + b_r)(f_- - f) + \frac{n}{2}(2na_r - b_r)(f_+ - f)$$

provided $0 < r$.

In the case (II) under consideration $0 < r < s < n$, so both (10.2.7 - 8) hold. Adding and using (10.2.4)

$$\dot{H}_{rs} \leq \frac{n}{s - r} \cdot \frac{1}{f'} [n^2(a_s + a_r)(f_+ - 2f + f_-) + \frac{n}{2}(b_s - b_r)(f_+ - f_-)].$$

Because f is concave and increasing, $f_+ - 2f + f_- < 0$ and $f_+ - f_- > 0$; hence

$$\dot{H}_{rs} \leq \frac{n}{s-r} \cdot \frac{1}{f'} \, [2\alpha n^2(f_+ - 2f + f_-) + n\beta(f_+ - f_-)]$$

or

$$(10.2.9) \qquad H_{rs} \leq \frac{n^2}{s-r} \cdot \frac{f_+ - f_-}{f'} \, [\beta - 2\alpha n \frac{f_+ - 2f + f_-}{f_+ - f_-}].$$

A straightforward calculation reveals that

$$(10.2.10) \qquad \frac{f_+ - 2f + f_-}{f_+ - f_-} = - \tanh \frac{c\tilde{H}}{2n}$$

$$(10.2.11) \qquad \frac{f_+ - f_-}{f'} = \frac{2}{c} \sinh \frac{c\tilde{H}}{n} \geq \frac{2\tilde{H}}{n} = \frac{2H_{rs}}{n} .$$

If (10.2.1 - 2) hold, $\beta - 2\alpha n \tanh \frac{c\tilde{H}}{2n} < - \frac{Dg'(B)}{\tilde{H}} \leq 0.$ Thus (10.2.9) implies

$$\dot{H}_{rs} \leq 2H_{rs} \, [\beta - 2\alpha n \tanh \frac{c\tilde{H}}{2n}]$$

and, a fortiori,(10.2.3).

<u>Case III</u>: <u>$0 < r < s = n$</u>. Putting $\dot{v}_s = \dot{\Psi}$ in (10.2.4) and using (10.2.8) one finds

$$\dot{H}_{rs} \leq \frac{n^2}{n-r} \cdot \frac{f_+ - f_-}{f'} \, [\frac{\dot{\Psi}}{n(f_+ - f_-)} + \frac{1}{2}\beta + \alpha n \frac{f_+ - 2f + f_-}{f_+ - f_-}].$$

By (10.2.10 - 11), the definition of D, and relation $\frac{1}{f'} = g'(v_s - v_r) \leq g'(B)$,

$$H_{rs} \leq \frac{n^2}{n-r} \; \frac{f_+ - f_-}{f'} \, [\frac{D}{2f'\tilde{H}} + \frac{1}{2}\beta - \alpha n \tanh \frac{c\tilde{H}}{2n}],$$

$$(10.2.12) \quad \dot{H}_{rs} \leq \frac{n^2}{n-r} \cdot \frac{f_+ - f_-}{2f'} \left[\frac{Dg'(B)}{\tilde{H}} + \beta - 2an \tanh \frac{c\tilde{H}}{2n}\right].$$

Then (10.2.1 - 2), (10.2.11), and (10.2.12) imply (10.2.3).

Case IV: $0 = r < s < n$. This case is treated in substantially the same way as Case III, using $\dot{v}_r = \emptyset$ and (10.2.7).

Lemma 10.2.2: Let $Y_m(t) \in D([0,\infty])$, each $M \in \Delta$, where Δ is a *finite* index set. Put $Y(t) = \max_{m\in\Delta} Y_m(t)$. Suppose $A(t) \in C([0,\infty])$ is a non-decreasing function such that: $Y(\bar{t}) \geq A(\bar{t})$ implies $\dot{Y}_m(\bar{t}) \leq 0$, each m such that $Y_m(\bar{t}) = Y(\bar{t})$. Then $Y(t) \leq \max[Y(0);A(t)]$, $0 \leq t < \infty$.

Proof: Since Δ is finite $Y(t)$ is continuous. For $\varepsilon > 0$ define $W_\varepsilon(t) = \max[Y(0);\ A(t)] + \varepsilon + \varepsilon t$. If $Y(t) < W_\varepsilon(t)$, all $\varepsilon > 0$, nothing to prove. In the contrary case, an ε can be chosen so that $\bar{t} = \inf[t \mid Y(t) \geq W_\varepsilon(t)]$ exists. Then $\bar{t} > 0$, since $Y(0) < W_\varepsilon(0)$; and $Y(\bar{t}) = W_\varepsilon(\bar{t})$, by continuity of $Y(t)$ and $W_\varepsilon(t)$. Let $\bar{m}$ be an index such that $Y_{\bar{m}}(\bar{t}) = Y(\bar{t})$. If $0 \leq t < \bar{t}$, then $Y_{\bar{m}}(t) \leq Y(t) < W_\varepsilon(t)$; and

$$\begin{aligned} Y_{\bar{m}}(\bar{t}) - Y_{\bar{m}}(t) &> W_\varepsilon(\bar{t}) - W_\varepsilon(t) \\ &= \max[Y(0);\ A(\bar{t})] + \varepsilon + \varepsilon\bar{t} - \max[Y(0);\ A(t)] - \varepsilon - \varepsilon t \\ &\geq \varepsilon(\bar{t} - t). \end{aligned}$$

Dividing by $(\bar{t} - t)$,

$$\frac{Y_{\bar{m}}(\bar{t}) - Y_{\bar{m}}(t)}{\bar{t} - t} > \varepsilon,$$

which implies $\dot{Y}_{\bar{m}}(\bar{t}) \geq \varepsilon$. But $Y(\bar{t}) = W_\varepsilon(\bar{t}) > A(\bar{t})$ and the hypothesis asserts $\dot{Y}_{\bar{m}}(t) \leq 0$. The contradiction proves $Y(t) < W_\varepsilon(t)$, $0 \leq t < \infty$, any $\varepsilon > 0$. Letting $\varepsilon \to 0$ proves the lemma.

Lemma 10.2.3: If $n > \frac{1}{2\alpha}(\beta + D) + \varepsilon$, $\varepsilon > 0$, then for $t \in [0,\infty)$

$$(10.2.13) \quad \tilde{H}(t) \leq \max[g(B);\ g'(B);\ \frac{2n}{c}\tanh^{-1}(\frac{D+\beta}{2\alpha n} + \frac{\varepsilon}{n});\ \tilde{H}(0)\, e^{-2\alpha\varepsilon t}]$$

Proof: Apply Lemma 10.2.2 to the collection of functions $H_{ij}(t)\, e^{2\alpha\varepsilon t}$, $0 \leq i < j \leq n$, with $Y(t) = \tilde{H}(t)\, e^{2\alpha\varepsilon t}$ and

$$(10.2.14) \quad A(t) = e^{2\alpha\varepsilon t}\max[g(B);\ g'(B);\ \frac{2n}{c}\tanh^{-1}(\frac{D+\beta}{2\alpha n} + \frac{\varepsilon}{n})].$$

To apply Lemma 10.2.2 it must be shown that

$$(10.2.15) \quad \tilde{H}(\bar{t})\, e^{2\alpha\varepsilon\bar{t}} \geq A(\bar{t})$$

implies

$$(10.2.16) \quad \frac{d}{dt}[H_{rs}(t)\, e^{2\alpha\varepsilon t}] \leq 0, \qquad t = \bar{t}.$$

Once this has been done, Lemma 10.2.2 states that $\tilde{H}\, e^{2\alpha\varepsilon t} \leq \max[\tilde{H}(0);A(t)]$ and multiplication by $e^{-2\alpha\varepsilon t}$ yields (10.2.13).

Now (10.2.14 - 15) imply

$$\tilde{H}(t) \geq g(B); \quad \tilde{H}(t) \geq g'(B); \quad \tanh\frac{c\tilde{H}(t)}{2n} \geq \frac{D+\beta}{2\alpha n} + \frac{\varepsilon}{n}.$$

The first insures (10.2.1); the second and third give

$$\frac{Dg'(B)}{\tilde{H}(\bar{t})} + \beta - 2\alpha n \tanh\frac{c\tilde{H}(\bar{t})}{2n} \leq -2\alpha\varepsilon.$$

By Lemma 10.2.1, $\dot{H}_{rs}(\bar{t}) \leq -2\alpha\varepsilon\, H_{rs}(\bar{t})$, which is equivalent to (10.2.16).

11. Bounds for First Differences.

Theorem 11.1.1: If $n > \frac{1}{2\alpha}(D+\beta)$, then

$$(11.1.1) \qquad n\frac{v_j(t)-v_i(t)}{j-i} \leq \max[1;B;J] + 2Bn\tanh^{-1}\left(\frac{D+\beta}{2\alpha n}\right)$$

for $t \in [0,\infty)$, $0 \leq i < j \leq n$.

Proof: Since $\frac{g(z)}{z}$ is increasing for $|z| < B$,

$$(11.1.2) \qquad n\frac{v_j - v_i}{j-i} = H_{ij}\frac{v_j - v_i}{g(v_j - v_i)}$$

$$\leq \max[0;\tilde{H}] \cdot \max_{|z|<B}\frac{z}{g(z)} \leq \max[0,\tilde{H}].$$

Using (11.1.2), the inequality (11.1.1) will be obtained from (10.2.13) by suitable choice of c. It is easily seen that $g(B) < Bg'(B)$ and $\tilde{H}(0)\,e^{-2\alpha ct} \leq J\frac{g(B)}{B} \leq Jg'(B)$; cf. (10.1.1), (11.1.2). For sufficiently small $\varepsilon > 0$, (10.2.13) implies

$$\tilde{H}(t) \leq \max[Bg'(B);\ g'(B);\ Jg'(B);\ \frac{2n}{c}\tanh^{-1}\left(\frac{D+\beta}{2\alpha n} + \frac{\varepsilon}{n}\right)]$$

$$(11.1.3) \qquad \tilde{H}(t) \leq \max\left[\frac{\max(B;1;J)}{1-cB};\ \frac{2n}{c}\tanh^{-1}\left(\frac{D+\beta}{2\alpha n} + \frac{\varepsilon}{n}\right)\right].$$

Of the two expressions within the square brackets, the first is an increasing function of c tending to infinity as $c \to \frac{1}{B}-$ and the second is a decreasing function tending to infinity as $c \to 0+$. The two are equal at a unique c^*; the choice $c = c^*$ minimizes the right member of (11.1.3) and therefore optimizes the estimate.

A simple computation shows

$$(11.1.4)\qquad \tilde{H}(t) \leq \frac{\max[1;B;J]}{1 - c^{*}B} \leq \max[1;B;J] + 2Bn \tanh^{-1}\left(\frac{D+\beta}{2\alpha n} + \frac{\varepsilon}{n}\right).$$

Letting $\varepsilon \to 0$ completes the proof.

<u>Corollary 11.1.1</u>: For $t \geq 0$,

$$\overline{\lim_{n\to\infty}} \max_{0\leq i<j\leq n} n\frac{v_j - v_i}{j - i} \leq \max[1;B;J] + \frac{B(D+\beta)}{\alpha}.$$

<u>Theorem 11.1.2</u>: In Theorem 11.1.1 and Corollary 11.1.1, $n\frac{v_j - v_i}{j - i}$ may be replaced by $\left|n\frac{v_j - v_i}{j - i}\right|$.

<u>Proof</u>: The maximum differences of $w_k = - v_k$ are again bounded by the right member of (11.1.1), since the constants involved do not change under $v_k, V, \phi, \Psi \to w_k, -V, -\phi, -\Psi$.

12. Asymptotic Bounds for First Differences.

<u>Theorem 12.1.1</u>: If $\varepsilon > 0$, $n > \frac{1}{2\alpha}(D+\beta) + \varepsilon$, $t \geq \frac{1}{2\alpha\varepsilon}\log\frac{J}{B}$, and $t \geq 0$, then

$$(12.1.1)\qquad \left|n\frac{v_j(t) - v_i(t)}{j - i}\right| \leq B + 2Bn \tanh^{-1}\left(\frac{D+\beta}{2\alpha n} + \frac{\varepsilon}{n}\right).$$

<u>Proof</u>: As in Theorems (11.1.1 - 2), it is a question of proving that the right member of (12.1.1) dominates $\tilde{H}(t)$. Since $\tilde{H}(0) \leq J\frac{g(B)}{B}$, the last member in the square brackets in (10.2.13) is dominated by the first for large t; specifically

$$\tilde{H}(0)\, e^{-2\alpha\varepsilon t} \leq J\, \frac{g(B)}{B}\, e^{-2\alpha\varepsilon t} \leq g(B) \qquad \text{if} \qquad \frac{J}{B}\, e^{-2\alpha\varepsilon t} \leq 1,$$

i.e., if $t \geq \frac{1}{2\alpha\varepsilon} \log \frac{J}{B}$. With this restriction on t the last member may be dropped from (10.2.13); treating the remaining members as in the proof of Theorem 11.1.1, it is found that

$$\tilde{H}(t) \leq \max\left[\frac{\max[1;B]}{1 - cB};\ \frac{2n}{c}\ \tanh^{-1}\left(\frac{D+\beta}{2\alpha n} + \frac{\varepsilon}{n}\right)\right],$$

$$\leq B + 2Bn\ \tanh^{-1}\left(\frac{D+\beta}{2\alpha n} + \frac{\varepsilon}{n}\right).$$

Corollary 12.1.1: If $\varepsilon > 0$, $t \geq \frac{1}{2\alpha\varepsilon} \log \frac{J}{B}$, and $t \geq 0$, then

$$\overline{\lim_{n\to\infty}}\ \max_{0\leq i<j\leq n} \left| n\, \frac{v_j - v_i}{j - i} \right| \leq B(1 + 2\varepsilon) + \frac{B(D+\beta)}{\alpha}.$$

13. Another Method.

No strong discrete analogue of Lemma 3.3.1 is available as yet. As a post mortem, this section is devoted to a weak analogue. Although it was recognized too late for inclusion of details in this thesis, this weaker lemma seems sufficient for proving the bounds of Part II in a way that closely parallels that used in Part I for partial differential problems.

Lemma 13.1.1: Let $A_k(t)$, $B_k(t)$, $C_k(t)$, $k = 1,2,\ldots,n-1$, be defined for $t \in (0,T]$, $T > 0$, and satisfy

(13.1.1) $A_k(t) > |B_k(t)|$, $t \in (0,T]$, $k \in [1, n-1]$.

Let $v_k(t)$ and $w_k(t) \varepsilon C([0,T]) \cap D((0,T])$, $k = 0,1,\ldots,n$. If for $k \varepsilon [1, n-1]$ and $t \varepsilon (0,T]$

$$\dot{v}_k \leq A_k(v_{k+1} - 2v_k + v_{k-1}) + B_k(v_{k+1} - v_{k-1}) + C_k$$

(13.1.2)

$$\dot{w}_k \geq A_k(w_{k+1} - 2w_k + w_{k-1}) + B_k(w_{k+1} - w_{k-1}) + C_k$$

and if

$$v_o(t) < w_o(t); \quad v_n(t) < w_n(t), \quad t \varepsilon [0,T]$$

(13.1.3)

$$v_k(0) < w_k(0), \quad k \varepsilon [1, n-1],$$

then

(13.1.4) $$v_k(t) < w_k(t), \quad k \varepsilon [0, n], \quad t \varepsilon [0,T].$$

<u>Proof</u>: Put $g_k = w_k - v_k$. If (13.1.4) holds, nothing to prove. Otherwise $\bar{t} = \inf[t| \min_{0\leq i\leq n} g_i(t) \leq 0]$ exists. In the remainder of this proof k means the greatest integer such that $g_k(\bar{t}) = \min_{0\leq i\leq n} g_i(\bar{t})$. By (13.1.3) and continuity of g_k, $\bar{t} \varepsilon (0,T]$, $k \varepsilon [1, n-1]$, and $g_k(\bar{t}) = 0$. Since g_k is differentiable at $\bar{t}$ and $g_k(t) > 0$, $t \varepsilon [0,\bar{t})$, $\dot{g}_k(\bar{t}) \leq 0$. The maximal choice of k insures $g_{k+1}(\bar{t}) > 0$. From (13.1.2) one finds

$$\dot{g}_k \geq A_k(g_{k+1} - 2g_k + g_{k-1}) + B_k(g_{k+1} - g_{k-1}).$$

At $t = \bar{t}$: $g_k = 0$, $g_{k-1} \geq 0$, $g_{k+1} > 0$, $A_k \pm B_k > 0$, and therefore

$$\dot{g}_k \geq (A_k + B_k)\, g_{k+1} + (A_k \,)\, B_k)\, g_{k-1} > 0,$$

a contradiction.

Remark 13.1.1: If for each $k \varepsilon [1, n-1]$ and $t \varepsilon (0,T]$ at least one of the inequalities (13.1.2) is strict, then (13.1.1) can be replaced by $A_k \geq |B_k|$, $k \varepsilon [1, n-1]$, $t \varepsilon (0,T)$; for then the last inequality of the proof reads

$$\dot{g}_k > (A_k + B_k)\, g_{k+1} + (A_k - B_k)\, g_{k-1} \geq 0.$$

In typical application to Problem Q, $A_k \geq n^2 \alpha > 0$ and $|B_k| \leq n\beta$, so that (13.1.1) would hold for all large n.

APPENDIX: NOTE ON BURGERS' PROBLEM

When $\alpha > 0$ is sufficiently small, the problem (1.1.2 - 3) has a finite collection, $E(\alpha)$, of non-zero stationary solutions, which can be studied by the Poincaré-Bendixson method (cf. Burgers [11]).

Define

$$M(\alpha) = \max_{u(x)\varepsilon E(\alpha)} \max_{x\varepsilon[0,1]} |u'(x)|.$$

Let $r(c) \leq 0$, $R(c) \geq 0$ be the roots of

$$\text{(A.1)} \qquad c - s + \log(1 + s) = 0, \qquad\qquad 0 \leq c < \infty;$$

and put

$$\text{(A.2)} \qquad f(c) = \int_{r(c)}^{R(c)} \frac{ds}{(1 + s)\sqrt{c - s + \log(1 + s)}}$$

From Burgers' analysis it follows that

$$M(\alpha) = R(c(\alpha))$$

if $c(\alpha)$ is determined by

$$f^2(c(\alpha)) = \frac{2}{\alpha}\,.$$

It will be shown below that

$$\text{(A.3)} \qquad f(c) \cong 2\sqrt{c} \qquad\qquad \text{as } c \to \infty,$$

It then follows that $c(\alpha) \cong \frac{1}{2\alpha}$ as $\alpha \to 0$, and, from (A.1), that

$$(A.4) \quad M(\alpha) \cong \frac{1}{2\alpha} \qquad \text{as} \quad \alpha \to 0.$$

If, on the other hand, the method of 6. of this thesis is applied to Burgers' Problem in the formulation (1.1.10 - 12) (with $k = 1$), it is readily found that

$$(A.5) \quad g'_o = \frac{1}{\alpha}\,(1 - e^{-1/\alpha})^{-1}$$

is an asymptotic (as $t \to \infty$) bound for $|v'(x,t)|$. [In fact, in the notation of 6., $D = 0$, $B' = 1$, $\beta = 1$; $g(x)$ satsfies $g'' + g' = 0$, $g(0) = 0$, $g'(0) = g'_o$, g'_o being taken large enough so $g(1) \geq 1$.] Observing that $u'(x,t) = 1 - v'(x,t)$ and, from (A.5), that $g'_o \cong \frac{1}{\alpha}$ as $\alpha \to 0$, the following statement can be made:

If $u(x,t)$ is a solution of (1.1.2 - 4), $0 \leq t$, then the theory of 6. provides a bound, say $m(\alpha)$, such that

$$|u'(x,t)| \leq m(\alpha), \quad \text{all sufficiently large } t$$

and

$$m(\alpha) \cong \frac{1}{\alpha} \qquad \text{as} \quad \alpha \to 0.$$

In comparison, the value $M(\alpha)$ is known to be <u>attained</u> for (stationary) solutions and $M(\alpha) \cong \frac{1}{2\alpha}$. This shows that the bound of the theory is sharp in the sense of its asymptotic dependence on α as $\alpha \to 0$.

Proof of (A.3): Split the integral (A.2) into the sum of an integral over $[r(c),0]$ and one over $[0,R(c)]$. Introduce a new variable t by $s = r(ct)$ in the first and by $s = R(ct)$ in the second. This yields

$$(A.6) \quad f(c) = \sqrt{c} \int_0^1 \left[\frac{1}{R(ct)} - \frac{1}{r(ct)}\right] \frac{dt}{\sqrt{1-t}} \qquad 0 < c < \infty.$$

Then (A.3) is equivalent to

$$(A.7) \quad \lim_{c\to\infty} \int_0^1 \left[\frac{1}{R(ct)} - \frac{1}{r(ct)}\right] \frac{dt}{\sqrt{1-t}} = \int_0^1 \frac{dt}{\sqrt{1-t}} = 2.$$

Choose α, $0 < \alpha < 1$, and split the integral on the left of (A.7) into four integrals, as follows:

$$I_1 = \int_{c^{-\alpha}}^1 \frac{-1}{r(ct)} \frac{dt}{\sqrt{1-t}}; \quad I_2 = \int_{c^{-\alpha}}^1 \frac{1}{R(ct)} \frac{dt}{\sqrt{1-t}};$$

$$I_3 = \int_0^{c^{-\alpha}} \frac{-1}{r(ct)} \frac{dt}{\sqrt{1-t}}; \quad I_4 = \int_0^{c^{-\alpha}} \frac{1}{R(ct)} \frac{dt}{\sqrt{1-t}}.$$

To prove $I_1 \to 2$ as $c \to \infty$ it is enough to show that

$$\lim_{c\to\infty} \left[I_1 - \int_{c^{-\alpha}}^1 \frac{dt}{\sqrt{1-t}}\right] = 0$$

or

$$\lim_{c\to\infty} \int_{c^{-\alpha}}^1 \left[1 + \frac{1}{r(ct)}\right] \frac{dt}{\sqrt{1-t}} = 0.$$

Since $r(c)$ is decreasing and tends to -1 as $c \to \infty$,

$$0 < - \int_{c^{-\alpha}}^{1} [1 + \frac{1}{r(c)}] \frac{dt}{\sqrt{1 - t}} \leq - [1 + \frac{1}{r(c^{1-\alpha})}] \int_{o}^{1} \frac{dt}{\sqrt{1 - t}} \to 0$$

as $c \to \infty$.

In the same way

$$0 < I_2 \leq \frac{1}{R(c^{1-\alpha})} \int_{o}^{1} \frac{dt}{\sqrt{1 - t}} \to 0$$

because $R(c)$ increases to infinity.

Similarly, I_3, $I_4 \to 0$ as $c \to \infty$. For example, introducing $s = tc$ as a new variable in I_4,

$$0 < I_4 = \frac{1}{\sqrt{c}} \int_{o}^{c^{1-\alpha}} \frac{1}{R(s)} \frac{ds}{\sqrt{c - s}} \leq \frac{1}{\sqrt{c}} \frac{1}{\sqrt{c - c^{1-\alpha}}} \int_{o}^{c^{1-\alpha}} \frac{ds}{R(s)} ;$$

From $R(s) \cong s$ as $s \to \infty$ it follows that the right-hand member of this inequality is asymptotically equivalent to $\frac{(1 - \alpha) \log c}{c} \to 0$ as $c \to \infty$.

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