# APPLICATION OF ASYMPTOTIC EXPANSION PROCEDURES TO LOW REYNOLDS NUMBER FLOWS ABOUT INFINITE BODIES

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## ABSTRACT

Several limiting cases for viscous incompressible flow are considered for two examples. The first example considered is that of the flow past an expanding infinite cylinder at an angle of attack. The time dependence of the radius of the cylinder is given by the power law $R = At^n$. The second example considered is the flow past a semi-infinite power law body of revolution (i. e. $R = Ax^n$) at zero angle of attack. Both examples are considered for the limiting case of small Reynolds number. The Reynolds number is based on a characteristic length obtained from the parameters in the expression for the radius. The second example is also considered for the limiting case of the flow far down stream.

Asymptotic expansions of the solution valid for the limiting cases considered (i. e. low Reynolds number or flow far down stream) are obtained by applying singular perturbation procedures. These expansions are obtained for $0 \leq n < 1$ for the first example and for $0 \leq n \leq 1/2$ for the second example. For the second example the terms in the low Reynolds number expansion are not obtained in closed form except for $n = 1/2$. For $n < 1/2$ the low Reynolds number expansion of the Navier-Stokes equations is expressed in terms of the solution of the corresponding Stokes flow problem. The expansions obtained for the flow far down stream on the power law body of revolution have the character of a very viscous flow although they are valid for any fixed Reynolds number.

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# I. INTRODUCTION

## I. 1. Definition of Purpose

This paper will consider the low Reynolds number flow about infinite and semi-infinite bodies of revolution. It is hoped that the consideration of this problem will lead to a better understanding of low Reynolds number flow in general. The peculiarities of these bodies bring out many of the fundamental ideas behind low Reynolds number flow. For instance it is not unusual to find boundary layer regions of flow along with Stokes-like flow in the same problem. In this study we shall try to make use of such situations occurring in the examples considered to clarify the meaning of low Reynolds number flow. In solving the examples major emphasis is placed upon illustrations of the various systematic expansion procedures which can be used to solve the problems. For this reason the examples are often solved by several different methods. The present study also gives a good opportunity to study the differences between parameter and coordinate-type expansions.

## I. 2. Method of Presentation

In the study of the low Reynolds number flow two examples are considered in detail. In preparation for the consideration of these examples the expansion procedures to be used are reviewed and some general aspects of low Reynolds number flow are discussed in Section 2. Section 2 is concluded with the introduction of the coordinate system and definitions of the variables to be used for the rest of the study. In

Section 3 the first example is solved. The problem is that of the viscous incompressible flow about an infinite circular cylinder having a radius proportional to $t^n$, $0 \leqq n < 1$ ($t$ = time). The cylinder simultaneously moves both normal and parallel to the axis and grows in size. Due to the simple geometry of the problem the flow may be separated into a cross flow and an axial flow such that the cross flow is entirely independent of the axial flow. The cross flow problem is the flow normal to the axis of an expanding cylinder. With the cross flow solution known the axial flow problem reduces to the solution of a single linear second order partial differential equation with variable coefficients. The solution is obtained both for the cross flow and axial flow by means of a parameter type expansion for small overall Reynolds number.

In Section 4 the second example, namely the viscous incompressible flow about a body of revolution, whose radius grows like $x^n$, $0 \leqq n < 1$ ($x$ = distance along axis), moving parallel to its axis, is considered. The uniformly valid expansion is only obtained for the case $n = 1/2$ where the first term in the expansion is the well-known Oseen flow about a paraboloid of revolution. The solution for $n \neq 1/2$ is only obtained in terms of the Stokes flow about the same body for $n < 1/2$. Thus the major difficulty in obtaining the solution for arbitrary $n$ is that the Stokes flow about the body for arbitrary $n$ is not known. Several modifications of the standard expansion procedure are pointed out and discussed.

In Section 5 the study of the second example is specialized to the flow far downstream. For this case a solution is obtained for arbitrary $n$ by use of a coordinate-type expansion. To make it possible to use the procedures established for parameter-type expansions an artificial or eliminatable parameter is introduced. In principle the solution could also have been obtained without introducing the artificial parameter.

## I. 3. Previous Work

The first example, flow about an expanding cylinder at angle of attack, is an extension of the study presented in reference 1 where the problem is studied for zero angle of attack, that is, flow parallel to the axis of the cylinder. In reference 1 this problem has been solved by use of both the parameter and coordinate-type expansions. The exact solution is also obtained for the special cases of a cylinder of constant radius and a cylinder growing parabolically in time. In the present paper this solution is extended to the case of non-zero angle of attack. It is found that for the case of non-zero angle of attack the problem may be split into an axial flow and a cross flow. The axial flow is solved in a manner similar to that used in reference 1 and much of the discussion of reference 1 applies equally well to this problem. The cross flow solution may be obtained in a manner similar to that used in reference 2 where a parameter-type expansion for the low Reynolds number flow normal to the axis of a circular cylinder is obtained.

In the second example we consider the low Reynolds number flow about a power law body of revolution moving parallel to its axis. For this reason we shall see that for bodies growing at a parabolic rate or less the Oseen equations represent a low Reynolds number limit of the Navier-Stokes equations. The solution of the Oseen equations for flow about a paraboloid of revolution is given in reference 3 and is used in the present paper in the development of an expansion of the Navier-Stokes equations for the flow about a paraboloid of revolution.

The flow far downstream on the body of revolution is solved by use of a coordinate-type expansion and is very similar to the problem considered in reference 4 for the flow at large distances from finite bodies. The problems are similar in that they both may be solved conveniently by a coordinate-type expansion. In reference 4 this led to a non-uniqueness which was partially eliminated by the use of conservation laws, that is, an external source of information. In the present study this non-uniqueness occurred only in the transcendental terms. These points will be discussed further in Section 5.

# II. REVIEW OF EXPANSION PROCEDURES AND LOW REYNOLDS NUMBER FLOW

## II. 1. Expansion Procedures

Before considering low Reynolds number flow in detail it will be useful to review some of the fundamentals of the expansion procedures which will be used. There are two types of expansions which we shall use, namely a coordinate-type and a parameter-type. The difference between these two types, as their names imply, is that they are expansions for small (or large) values of a coordinate or a parameter respectively. One often has an intuitive feel for the difference between a parameter and a coordinate from their physical interpretation, but in order to understand the differences between coordinate-type expansions and parameter-type expansions it is helpful to understand the mathematical difference between a coordinate and a parameter.

This difference between a coordinate and a parameter is dependent on the role which the variables play in certain implicit definitions of the function. If the function is given explicitly there is no mathematical reason for distinguishing between the variables and calling one a coordinate and another a parameter. For a detailed discussion of the differences between coordinates and parameters see reference 4. In reference 4 it is shown that a coordinate-type expansion must either be non-uniform or include the exact solution as the first term. Despite the fact that the coordinate-type expansions are in general not uniformly valid for the entire flow field they are very useful since one is often only interested in a certain portion of the flow field.

These expansions are sub-divided again into regular and singular perturbation problems. A regular perturbation may be defined as a perturbation which is everywhere small compared to the undisturbed system but a singular perturbation, while having a small integrated effect, may not be small in some local region. In dealing with singular perturbation problems it is often necessary to obtain different expansions valid in the different regions of the flow. Since these expansions are not valid for the entire flow there will in general be insufficient boundary conditions for determining them. This difficulty is overcome by matching the expansions. Thus matching plays a role analogous to the boundary conditions. For a detailed discussion of matching see references 1, 2, 3 and 4. For the present problem the matching conditions are:

$$\lim_{\epsilon \downarrow 0} \frac{u(x,\epsilon) - v(x,\epsilon)}{\Gamma(\epsilon)} \longrightarrow 0 \tag{2.1}$$

where $x$ is in the overlap domain, $u$ and $v$ are expansions which have overlapping domains of validity and $\Gamma(\epsilon)$ is a gage factor determining the order to which the matching is valid. Having obtained two properly matched solutions we then wish to construct a uniformly valid solution. At this point it is useful to introduce the nomenclature of "outer" and "inner" expansions for the expansions valid near the origin and infinity respectively. Taking $f_I$ as the inner expansion, $f_O$ as the outer expansion and $f_{un}$ as the uniformly valid expansion it is clear that in general

$$f_{un} \neq f_I + f_o \qquad (2.2)$$

Since the inner solution may include some of the same terms as the outer solution there is a duplication of some portions of the solution in equation 2.2. Thus we conclude:

$$f_{un} = f_I + f_o - G \qquad (2.3)$$

where $G$ is the portion of $f_{un}$ common to both the inner and outer solutions. For the problems considered in this paper it will be sufficient to take:

$$G = f_{(0)(I)} = LIM_I \; LIM_O \; f \qquad (2.4)$$

However, we should remark that equation 2.4 is not true in general and must be verified for each problem to which it is applied. A more complete discussion of the difference between the parameter and coordinate-type expansions including illustrative examples is given in reference 4 and the fundamentals of singular perturbation theory including matching are discussed extensively in references 1, 2 and 5.

In order to define the inner and outer limits in an operational form it is first necessary to choose the inner and outer variables. The choice of these variables is more of an art than a science since their choice depends on the specific problem being considered.

One method of selecting these variables which might be called the intuitive approach is first to determine the undisturbed system.

Clearly the equations governing the undisturbed system will be an important limit. For the problems considered here this limit corresponds to the outer limit. Then at least one additional limit is required to examine the singular region of the undisturbed limit. One then selects the non-dimensional variables such that the equations governing these limits are obtained from the limit process. A slightly more rigorous procedure which may help in choosing these variables is to choose trial variables in terms of one or more undetermined parameters. By substituting these trial variables into the original equations one can determine at which value of the undetermined parameter the equations take on a significant form. In general the equations will take on the same form for a large number of values of the parameter. Then one tries to choose those reduced equations which simplify the problem as much as possible but can still be matched. It is also desirable that the inner problem retain at least the leading terms of the boundary conditions near the body and the outer problem the leading term of the boundary condition at infinity. One also expects that the inner limit implies that the observer is relatively near the origin compared to the outer limit. Thus if we designate the inner limit by $\lim_I$ and the outer limit by $\lim_o$ the inner and outer limit processes are related to their respective variables by the relations:

$$\begin{aligned} \mathrm{LIM}_o\, U &= \lim_{\epsilon \to 0} U(\tilde{X}_i, \epsilon) \\ \mathrm{LIM}_I\, U &= \lim_{\epsilon \to 0} U(X_i^+, \epsilon) \end{aligned} \qquad (2.5)$$

where $\tilde{X}_i$ are the outer variables and $X_i^+$ the inner variables. At this point we shall recall the difference between a limit and an approximate solution. A limit is obtained by letting $\epsilon \to 0$ in the solution and is thus independent of $\epsilon$ while an approximate solution is an approximation valid for small $\epsilon$ which may contain $\epsilon$. The inner expansion of a solution is defined as the series associated with the solutions $f(X,\epsilon)$ obtained by repeated application of the inner limit; that is:

$$f(X,\epsilon) = \sum_{i=0} \delta_i(\epsilon) f_i(X_k^+) \tag{2.6}$$

is the inner expansion of $f(X,\epsilon)$ where:

$$f_0 = \mathrm{LIM}_I f \quad ; \quad f_j = \mathrm{LIM}_I \frac{f - f_0 - \delta_1 f_1 - \cdots - \delta_{j-1} f_{j-1}}{\delta_j} \tag{2.7}$$

$$X_k^+ = \text{inner variables}$$

The outer expansion is defined in an analogous manner. Throughout this paper we shall make the assumption: the limit of the solution of the equation considered is equal to the solution of the equation obtained by taking the limit of the original equation.

We are now able to define the following process as the standard expansion procedure for a singular perturbation problem. First the inner and outer variables are determined and the equations for the inner and outer limits (and any other limit necessary) are derived by taking the limit of the equations of motion (written in terms of the appropriate variables) as the parameter goes to zero (or infinity). One then assumes a solution of the same form as equation 2.6 where the first term, $f_o$, is the solution of the equations for the inner or outer limit, as the case may be. The solutions are then matched and substituted back into the original equations written in the proper variables which leads to the equations for the next term in the expansion. The procedure is continued until the desired accuracy is obtained. All of the singular perturbation problems considered here are solved by this standard procedure, however, certain modifications of this procedure are pointed out and examples of these modifications are given. It should be pointed out that in the majority of the problems encountered it is easiest to obtain the solution by the standard procedure. For certain problems it is not possible to obtain the solution by the standard procedure and then certain modifications to the procedure may be useful. Quite often one of the limits will include the other limit and then in principle it is only necessary to obtain the one limit; however, it is usually less accurate and a much more complex procedure. A second objection to this modification is that it hides some of the physical aspects of the problem.

Throughout this study we shall wish to discuss the relative order of magnitudes of two quantities. We say that the order of $\delta_1(\epsilon)$ is smaller than the order of $\delta_2(\epsilon)$, $o\{\delta_1\} < o\{\delta_2\}$, if:

$$\lim_{\epsilon \to 0} \frac{\delta_1(\epsilon)}{\delta_2(\epsilon)} = 0 \tag{2.8}$$

and that the two quantities are of the same order, $o\{\delta_1\} = o\{\delta_2\}$, if :

$$\lim_{\epsilon \to 0} \frac{\delta_1(\epsilon)}{\delta_2(\epsilon)} \longrightarrow \text{FINITE} \tag{2.9}$$

Note that it is possible to find $\delta(\epsilon)$'s such that $o\{\delta_1\} < o\{\delta_2^M\}$ where M is any constant. An example of such a case is $\delta_2 = \frac{1}{\ln \epsilon}$ , $\delta_1 = \epsilon$ For this case we have:

$$\lim_{\epsilon \to 0} \frac{\epsilon}{\left(\frac{1}{\ln \epsilon}\right)^M} = 0 \tag{2.10}$$

and we say that $\delta_1$ is transcendentally small with respect to an expansion in $\delta_2$ . Thus if we have an expansion in $\delta_2$ all terms of order $\delta_1$ may be neglected in computing every term in the expansion.

## II. 2. Low Reynolds Number Flow

Since we are primarily concerned with low Reynolds number flows in this study, it will be worthwhile to obtain an intuitive meaning for low Reynolds number flow. First we define the overall Reynolds number as:

$$R_e = \frac{UL}{\nu} \tag{2.11}$$

where V and L are a constant characteristic velocity and length respectively. In all of the problems considered here there is a characteristic length, and unlike the problem considered in reference 1 the Reynolds number based on this length cannot be eliminated from the problem. Thus we may deal with parameter-type expansions.

We now consider low Reynolds number flow as the flow about a very small object, that is, the flow obtained when the characteristic length of the body tends to zero with the length $\nu/U$ fixed, and the observed fixed in space. Note that this intuitive definition of low Reynolds number flow only has meaning in connection with a specific problem, but then it has the advantage that the resulting limit is a unique solution of the full equations and uniform at infinity. We also consider very viscous flow as flow in which the region where the viscous terms are significant is much larger than the local geometric length. By setting $\nu = \infty$ in the Navier-Stokes equation we see that the region of very viscous flow corresponds to the classical Stokes flow; however, low Reynolds number flow often has regions which are <u>not</u> very viscous flows.

The elimination of the apparent contradition which is implied by this last statement when one uses the historical concept of low Reynolds number flow (i. e. that the low Reynolds number flow is the limit as $\nu \to \infty$ ) is one of the advantages of the present concept of low Reynolds number flow. From the above definition it is easily seen that low Reynolds number flow cannot be the same as very viscous flow since low Reynolds number flow applies to the entire problem and is not a local phenomenon. This may also be seen by the use of the following principle: Any equation which governs a uniformly valid approximation to the Navier-Stokes equations for the flow about a body in an infinite fluid must contain at least one non-zero term which approximates the dominant inertial or transport term far from the body. This is in essence Oseen's criticism of the Stokes equations and thus we shall refer to this principle as the Oseen criticism. This principle may be argued from the fact that the vorticity is produced at the body. The vorticity produced per unit surface area is finite but this finite vorticity must diffuse into an infinite fluid and thus it is clear that far from the body the diffusion of vorticity cannot be important and some other term in the equation must be dominant unless all of the other terms are identically zero. Thus any approximate equation which neglects all of the terms but the diffusion terms cannot be valid at infinity and thus the approximation cannot be uniformly valid. It follows from the Oseen criticism that very viscous flow as defined here cannot be a uniform approximation, while the low Reynolds number approximation as defined here is a uniform approximation and thus low Reynolds number flow cannot be the same as very viscous flow. It also follows from the Oseen criticism that there are some problems where the dominant transport terms are non-linear such as steady flow past an infinite plate for which there does not exist any uniformly valid linear approximation to the Navier-Stokes equations. For these problems one might expect great difficulty in obtaining uniformly valid approximations to the Navier-Stokes equations even for very simple geometries. We also note the similarity between the present definition of the low Reynolds number limit and the outer limit. It is clear that they both represent the undisturbed system. The main difference being that the outer limit need not be valid near the body while the low Reynolds number limit must be valid everywhere. Thus it is not surprising that the same equations often govern both the outer and the low Reynolds number limits.

Since very viscous flow is a local phenomenon it is perfectly consistent to have a boundary layer type of flow region when considering the low Reynolds number limit of the Navier-Stokes equations. For semi-infinite bodies this boundary layer flow may occur far downstream simply because given any fixed overall Reynolds number one can always proceed sufficiently far downstream and find a large local or cross sectional Reynolds number provided the viscous layer grows slower than the local geometric length. Similarly if the viscous layer grows faster than the local geometric length one can always go sufficiently far downstream and find a place where the viscous layer is larger than the local characteristic length regardless of the overall Reynolds number. Thus for a body which grows rapidly enough it is always possible to find a flow which has the character of a boundary layer type flow. It is clear that these arguments apply in an analogous manner to the case where the geometric and viscous lengths grow in time rather than in distance. Thus we conclude that for a body growing at less than a parabolic rate the flow for large time is always of a very viscous or Stokes type near the body and for a body growing faster than a parabolic rate the flow for large time is always a boundary layer type flow. For finite bodies it is always possible to choose the overall Reynolds number sufficiently small (large) to insure very viscous flow (boundary layer type flow) near the entire body and thus the Stokes equations govern the inner limit for low Reynolds number flows.

The Oseen equations have also been considered historically as the low Reynolds number limit. In general this need not be the case. The Oseen equations may be considered as a linearization about the free stream. Thus they are valid for the flow at large distances about any body having zero arresting power. Clearly if we get far enough away from a finite body it will produce a negligible disturbance on the free stream and in this region the Oseen equations will be valid for <u>all</u> Reynolds numbers. There are certain semi-infinite and infinite bodies such as semi-infinite and infinite needles which possess this same property (i. e. negligible disturbance of free stream at infinity) and thus produce flow fields governed by the Oseen equations at large distances. However there are other semi-infinite bodies which do not possess this property. For example a semi-infinite flat plate will produce a finite disturbance at very large distances and thus the Oseen equations do not govern the flow at large distances from a semi-infinite flat plate.

For the case of a body which reduces to a semi-infinite flat plate when the characteristic length goes to zero, such as a two-dimensional parabolic cylinder, the proper outer limit of the Navier-Stokes equations is the still unknown flow past a semi-infinite flat plate. Clearly problems such as this will be much more complex than those which reduce to a body with zero arresting power and thus have the constant free stream velocity as an outer limit. All of the problems considered in this study are of the first type, that is have zero arresting power in the limit of $L \to 0$.

It is easily seen that at best the Oseen limit must have a non-uniformity at the body since the Oseen limit gives the free stream velocity everywhere but the no-slip boundary condition gives zero velocity on the body. The Oseen equations govern the outer limit for the flow about any body having zero arresting power. If they include the inner equations they will be valid for the entire flow and thus represent a low Reynolds number limit of the Navier-Stokes equations for the problem. For example since the Oseen equations include the Stokes equations we expect that they will be a uniform low Reynolds number approximation to the Navier-Stokes equations for the entire flow field for very viscous flow. If the flow is not very viscous flow the inner equations will be a boundary-layer type equation and since in general they are not included in the Oseen equations we only expect the Oseen equations to be a valid approximation to the Navier-Stokes equations at large distances.

Thus we arrive at the following three conditions which are necessary and sufficient for the Oseen equation to be a low Reynolds number approximation to the Navier-Stokes equations in a given region:

1. Either the Oseen equations must include the inner equations valid in the region or the region must be far from the body.
2. The limiting body obtained by letting $L \to 0$ must have zero arresting power.
3. Linearization about the free stream must lead to the Oseen equations.

If these conditions are satisfied over the entire flow the Oseen equations are a uniform low Reynolds number approximation to the Navier-Stokes equations. We note that if conditions 2 and 3 are valid they will hold for the entire flow field; however, condition 1 may be satisfied only in a certain region of the flow field. If this is true the Oseen equations will be a valid approximation to the Navier-Stokes equations in the region where condition 1 is satisfied. However, it may not be possible to determine the higher order terms uniquely without additional information since these terms will depend on the flow region for which the approximation is not valid. This is the situation which occurs in the coordinate-type expansions where some of the indeterminancy may be removed by use of the conservation laws. Finally we note that the Oseen equations include the Stokes equations and thus condition 1 is always satisfied for very viscous flow but there are cases for which condition 1 is satisfied which are not very viscous flows. An example of such a case is the axial flow problem in the first example.

## II. 3. Preliminary Considerations

Before we apply the preceding general considerations to the specific examples we shall consider some definitions that are applicable to both examples. We shall use the coordinate system defined in fig. 1 below:

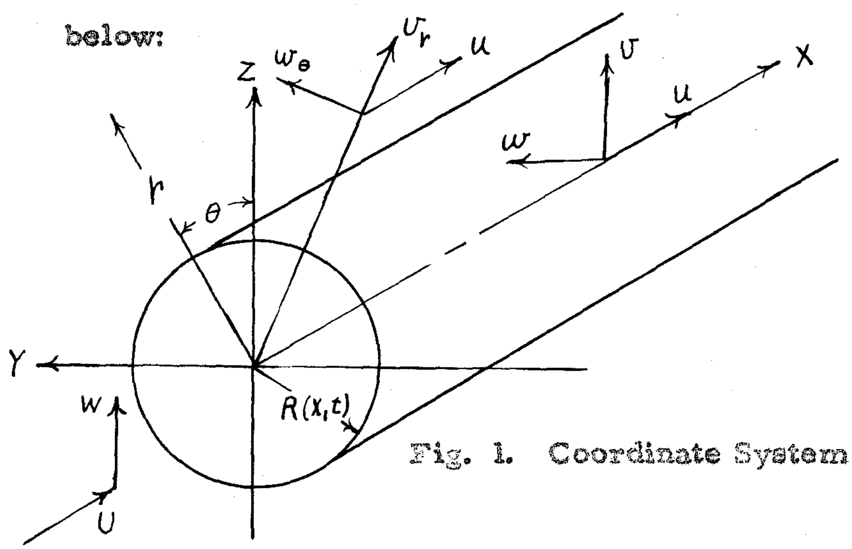


Fig. 1. Coordinate System

It has been stated previously that a characteristic length exists for both of the examples considered. Since the characteristic length may be defined in a very similar manner for both cases we shall consider them together. Consider the first example. Restricting our attention to a cylinder whose radius is given as a function of time by:

$$R(t) = At^n \qquad 0 \leq n < 1 \tag{2.12}$$

Clearly the ratio of R(t) to L, where L is the characteristic length of the problem, is dimensionless. If U is the characteristic velocity of the problem it is clear that one can define L through the relation:

$$\frac{R(t)}{L} = \frac{At^n}{L} = \left(\frac{Ut}{L}\right)^n \tag{2.13}$$

Since $n \neq 1$ equation 2.13 can be solved for L giving:

$$L = \left(\frac{A}{U^n}\right)^{\frac{1}{1-n}} \tag{2.14}$$

A similar process may also be used for the body of revolution. For this case the radius is a function of the length coordinate rather than the time and thus the "rate of growth" refers to the rate of growth in x, rather than in time. Thus using the preceding problem as a model we only consider bodies defined by:

$$R(x) = Ax^n \qquad 0 \leq n < 1 \tag{2.15}$$

Following a similar procedure as before one forms:

$$\frac{R(x)}{L} = \frac{AX^n}{L} = \left(\frac{X}{L}\right)^n \tag{2.16}$$

which, since $n \neq 1$, can again be solved for L giving:

$$L = A^{\frac{1}{1-n}} \tag{2.17}$$

For the second case, $r = Ax^n$, the characteristic length given by equation 2.17 can be shown to be equal to the altitude of the inscribed right circular cone. Note that for $n = 1$ there is no characteristic length for the problem and the low Reynolds number limit of the problem is identical to the exact solution!

In general we shall find two non-dimensionalizations of the variables important throughout this study. We shall define these variables here. Suppose the problem has a characteristic velocity, $\overline{U}$, and characteristic length, L, and a small parameter, $\epsilon$. Then we define one set of variables by:

$$\begin{aligned} u_i^* &= \frac{u_i}{\overline{U}} \\ X_i^* &= \frac{x_i}{L} \\ t^* &= \frac{\overline{U}t}{L} \\ P^* &= \frac{P-P_\infty}{\rho \overline{U}^2} R_e \\ \vec{q} &= \frac{\vec{i}u_x + \vec{j}u_Y + \vec{k}u_z}{\overline{U}} = \vec{i}u^* + \vec{V} \end{aligned} \tag{2.18}$$

and a second set of variables by:

$$\tilde{X}_i = \epsilon X_i^*$$

$$\tilde{t} = \epsilon t^* \tag{2.19}$$

$$\tilde{P} = \frac{P - P_\infty}{\rho \bar{U}^2}$$

In terms of these variables the incompressible Navier-Stokes equations may be written:

$$\operatorname{div}^* \vec{q} = 0 \tag{2.20}$$

$$\frac{\partial \vec{q}}{\partial t^*} + \vec{q}\, \operatorname{grad}^* \vec{q} + \frac{1}{R_e} \operatorname{grad}^* P^* = \frac{1}{R_e} \nabla^{*2} \vec{q} \tag{2.21}$$

or:

$$\widetilde{\operatorname{div}}\, \vec{q} = 0 \tag{2.22}$$

$$\frac{\partial \vec{q}}{\partial \tilde{t}} + \vec{q}\, \widetilde{\operatorname{grad}}\, \vec{q} + \widetilde{\operatorname{grad}}\, \tilde{P} = \frac{\epsilon}{R_e} \tilde{\nabla}^2 \vec{q} \tag{2.23}$$

There are additional variables which will be useful for certain portions of the examples which will be introduced as they are needed.

# III. EXPANDING SLIDING CYLINDER AT ANGLE OF ATTACK

## III. 1. Discussion of Problem

The first example considered is the flow about an infinite cylinder at an angle of attack where the radius is a function of time. Thus $R(x, t)$ (see fig. 1) is chosen independent of "x" and in fact we shall restrict our attention to the case for which $R(x, t) = At^n$. We shall also restrict $n$ such that $0 \leq n < 1$. This problem is particularly useful since it includes many features of interest, such as the effects of angle of attack and non-steady effects and yet has sufficiently simple geometry to allow us to obtain the solution to the problem. Because of its simplicity we shall use this example to illustrate several modifications of the standard expansion procedure.

Because the flow is at an angle of attack the present problem does not have axial symmetry but it does still have a most important simplification, namely, the separation of the axial and cross flows. This simplification results from the fact that by the conical flow argument there is no variation of any quantity in the "x" direction (parallel to the axis of the cylinder) which implies that $\frac{\partial}{\partial x} = 0$. It is easy to see that this will lead to a separation of the axial and cross flow, since the axial velocity only enters the cross flow momentum equation in terms of the form $u\frac{\partial}{\partial x}$ and the continuity equation in the term $\frac{\partial u}{\partial x}$ both of which are identically zero. Thus the continuity equation and the cross flow momentum equations form a complete system for the pressure and cross flow velocity.

Three interesting dimensionless parameters may be formed from the quantities appearing in the present problem, namely the ratio of the axial to cross flow free stream velocities and the two Reynolds numbers based on these velocities. Clearly only two of these parameters are independent. Thus the present problem has two parameters which cannot be eliminated from the problem. We shall only consider parameter-type expansions for this problem. We shall select one Reynolds number and the ratio of the two velocities as the two independent parameters of the problem. For this problem the choice is quite clear from the fact that the cross flow which is to be solved independently has only one parameter, the Reynolds number based on the cross flow velocity. Thus we choose the cross flow Reynolds number, $Re_w$, and the ratio of the cross flow to the axial flow velocities, $\alpha$ , as the two independent parameters for the problem. The ratio of velocities, $\alpha$ , is the inverse tangent of the angle of attack and is a measure of the effect of the cross flow solution on the axial flow. These two independent parameters lead to the following four possibilities for single parameter expansions: 1) $Re_w \to \infty$, $\alpha$ fixed; 2) $\alpha \to 0$, $Re_w$ fixed; 3) $\alpha \to \infty$, $Re_w$ fixed; and 4) $Re_w \to 0$, $\alpha$ fixed.

The first possible expansion (i.e. $Re_w \to \infty$, $\alpha$ fixed) is a high Reynolds number limit and will not be considered here. The second possibility, expansion for small angle of attack, ( $\alpha \to 0$, $Re_w$ fixed) is of interest because it shows the relation between the present solution and that obtained in reference 1. The first method that one thinks of to compare the present result with reference 1 is simply to set $\alpha = 0$ in the present low Reynolds number expansion. It will only be possible to obtain the approximation for $\alpha$ small in this manner if one can show that all of the higher order terms in the expansion remain small as $\alpha \rightarrow 0$. This is not true for the present case. Thus taking the limit as $\alpha \rightarrow 0$ of the low Reynolds number expansion does not lead to the low Reynolds number expansion of the solution for $\alpha = 0$. If one wants to obtain a solution which can be compared with the solution presented in reference 1 they should obtain the expansion for $\alpha \rightarrow 0$, $Re_w$ fixed. This solution would be valid for $\alpha = 0$ and a small but fixed Reynolds number and thus a comparison would be possible. However we will not obtain the expansions for $\alpha$ small or infinite angle of attack here. Since this is a study of low Reynolds number flow the last of the possibilities, ($Re_w \rightarrow 0$, $\alpha$ fixed), is of primary interest to us. To be consistent with our concept of low Reynolds number flow we visualize $L \rightarrow 0$ and $\nu/w$ fixed as the limiting process corresponding to $Re_w \rightarrow 0$. This is then the low Reynolds number limit of the problem for which it will be shown that the outer solution should be governed by the Oseen equations and the inner by the Stokes equations.

The cross flow is identical to the problem of viscous incompressible flow normal to the axis of an expanding circular cylinder. We can only expect to find a solution of this problem for the case of low Reynolds number flow. For $n = 0$ the steady low Reynolds number solution is presented in reference 2. In Section III. 4 we shall extend this solution to the non-steady case with arbitrary $n$. First we note that when the body is finite we may always choose the overall Reynolds number sufficiently small to insure very viscous flow near the body. It is clear from equations 2.12 and 2.14 that the limit $L \to 0$ reduces the body to a point (in the cross flow plane) which has zero arresting power. Thus for the cross flow problem we see that all the conditions necessary for the Oseen equations to represent the low Reynolds number limit of the Navier-Stokes equations are satisfied everywhere except perhaps for very large time. We therefore expect the Oseen equations to govern the outer solutions and the Stokes equations the inner solution for the cross flow problem.

Since the cross flow solution is a function of the characteristic length, $L$, and the equations of motion for the axial flow involve the cross flow solution and thus $L$ it is possible to determine the proper low Reynolds number limit for the axial flow directly by taking the low Reynolds number limit of the axial flow equations. This is done by holding $\nu/w$ fixed and taking the limit of the axial equations as $L \to 0$. Note that this procedure does not work for the Navier-Stokes equations since they are independent of $L$ and thus the low Reynolds number limit (as we have visualized it) applied directly to the Navier-Stokes equations does not change them. It is easy to see what the axial equation will reduce to in the inner and outer limits. The outer limit of the cross flow velocity is the constant cross flow free stream velocity. The property that $\frac{\partial}{\partial x} = 0$ implies that the transport terms are all of the form: $\vec{V} \frac{\partial u}{\partial y}$ and thus: $\vec{V}$ = const., corresponds to an Oseen type linearization about the constant cross flow velocity for the outer equation. This agrees with the argument that the outer flow is of an Oseen type since the body has zero arresting power. Similarly since the inner limit of the cross flow solution gives $\vec{V} = 0$ we see that for the present problem the inner limit is the corresponding Stokes equations. Thus the inner equation will be included in the outer equations and for this problem the outer equations represent a low Reynolds number limit of the Navier-Stokes equations. Note that this statement is true for all values of $n$ between zero and unity.

## III. 2. Equations and Boundary Conditions

In this section we shall apply the ideas discussed in Part II to the present example and obtain the equations and boundary conditions for the inner and outer solutions. Since $Re_w$ has been chosen as the Reynolds number it is consistent to take $w$ as the characteristic velocity of the problem and thus it follows from equation 2.14 that the characteristic length is given by:

$$L = \left(\frac{A}{W^n}\right)^{\frac{1}{1-n}} \tag{3.1}$$

where:

$$R(x,t) = R(t) = At^n \qquad 0 \leq n < 1 \tag{3.2}$$

From the definition of the Reynolds number we obtain:

$$R_{e_w} = \frac{A^{\frac{1}{1-n}} \, w^{\frac{1-2n}{1-n}}}{\nu} \tag{3.3}$$

We try the variables defined by equation 2.19 as the outer variables for the problem where:

$$\epsilon = R_{e_w} \quad , \qquad \bar{U} = w \tag{3.4}$$

In terms of the variables defined by equation 2.19 the boundary conditions for this problem are:

$$\text{AT} \quad r = \infty \qquad \vec{q} = \vec{k} + \frac{1}{\alpha}\vec{i} \tag{3.5}$$

Since the body has zero arresting power as $L \to 0$ the outer limit must be the free stream velocity and thus we assume an outer expansion of the form:

$$\vec{q}_o = \vec{i}\, u_o^* + \vec{V}_o \tag{3.6}$$

where:

$$u_o^* = \frac{1}{\alpha} + \sum_{m=1}^{M} \delta_m(R_{e_w})\, q_m(\tilde{x}_i, \tilde{t}) \tag{3.7}$$

$$\vec{V}_o = \vec{k} + \sum_{m=1}^{M} \bar{\delta}_m(R_{e_w})\, \vec{q}_m(\tilde{x}_i, \tilde{t})$$

$$P = \sum_{m=1}^{M} \bar{\delta}_m(R_{e_w})\, p_m(\tilde{x}_i, \tilde{t}) \tag{3.8}$$

$\vec{q}_o$ = solution of equations 2.22 and 2.23 with $\frac{\partial}{\partial x} = 0$

This choice of outer variables and expansions will result in a linearization of the equations without modifying the outer boundary conditions.

The limit of $Re_w \to 0$ holding these variables fixed is equivalent to making the physical variables very large.

Although we could use the variables defined by equations 2.18 as the inner variables this would introduce needless complication. This follows from the fact that near the body we have very viscous flow and thus only the diffusion terms are important. The derivative with respect to time only enters the problem in the inertial terms and thus near the body we may consider the flow as quasi-steady. This means that it is perfectly consistent to consider the inner limit process as an approach to the body at $t$ large thus we shall choose the outer and inner time variables equal. Replacing the time variable in equation 2.18 by $\tilde{t}$ the inner boundary condition becomes:

$$\text{AT} \qquad r^+ = \tilde{t}^{\,n} \qquad \vec{q} = \vec{1}_r \, \frac{n \, R_{e_w}^{1-n}}{\tilde{t}^{\,1-n}} \tag{3.9}$$

In the limit as $Re_w \to 0$ "r" must be well-behaved and it is clear that the proper choice for the inner "r" is:

$$r^+ = R_{e_w}^{n} \, r^* \tag{3.10}$$

And thus we choose as inner variables, $\tilde{t}$, $x_i^+$, $r^+$, $P^+$, and $\vec{q}$ where:

$$P^+ = R_{e_w}^{n} \, P^* \tag{3.11}$$

In terms of these variables remembering that $\frac{\partial}{\partial x} = 0$ the incompressible Navier-Stokes equations become:

$$\operatorname{div}^{+}\vec{q} = 0$$
$$R_{e_w}^{2(1-n)}\frac{\partial \vec{q}}{\partial \tilde{t}} + R_{e_w}^{1-n}\,\vec{q}\,\operatorname{grad}^{+}\vec{q} + \operatorname{grad}^{+}P^{+} = \nabla^{+2}\vec{q} \tag{3.12}$$

We assume an inner expansion of the form:

$$\vec{q}_I = \vec{i}\,u_I^{*} + \vec{V}_I \tag{3.13}$$

where:

$$u_I^{*} = \sum_{m=1}^{M} \delta_m(R_{e_w})\,\eta_m(x_i^{+},\tilde{t}) \tag{3.14}$$

$$\vec{V}_I = \sum_{m=1}^{M} \bar{\delta}_m(R_{e_w})\,\vec{\eta}_m(x_i^{+},\tilde{t}) \tag{3.15}$$

## III. 3. Initial Conditions

From equation 3. 9 we see that there is some difficulty at $\tilde{t} = 0$. If one computes the source strength, $Q$, due to the velocity of expansion of the body one finds:

$$Q = \oint \vec{q}\cdot\vec{dn} = 2\pi n\,R_{e_w}^{1-n}\,\tilde{t}^{\,2n-1} \tag{3.16}$$

Thus we see that the source strength is only finite for all time if $n = 1/2$. For $n < 1/2$ the source is infinite for $\tilde{t} = 0$ and for $n > 1/2$ the source strength is infinite for infinite time.

At $\tilde{t} = \infty$ the infinite source does not lead to a non-uniformity in the expansion because the velocities in the flow region remain finite, but the inner equations change character completely when:

$$\tilde{t} < O\left\{\frac{1}{Re_w^{\frac{1-n}{n}}}\right\} \qquad (3.17)$$

The essential reason for this phenomenon, which is due to the presence of the cross flow and does not occur in reference 1, is that the local cross flow Reynolds number, $\frac{R(t)L}{\nu}$, ceases to be small. Note that omission of the problem for $\tilde{t}$ large does not invalidate the present expansion because of the parabolic nature of the equations. Since the "exact" problem is governed by the incompressible Navier-Stokes equations an infinite source strength at $\tilde{t} = 0$ implies infinite velocities over the entire flow field at that time. Thus neglecting the inertial terms is not valid for all time and in particular the expansion considered here is non-uniform at $\tilde{t} = 0$ for $n < 1/2$. For this case, $n < 1/2$, the singularity at $\tilde{t} = 0$ affects the initial condition for the problem. Thus we prescribe as the initial conditions for the "exact" problem:

$$\text{As} \quad t \downarrow 0 \qquad \vec{q} \rightarrow \frac{n\, Re_w^{1-n}}{\tilde{t}^{1-2n}} \frac{\vec{I}_r}{r^+} + g(r^+) \qquad (3.18)$$

That is, we are looking for an expansion of the solution of the incompressible Navier-Stokes equations for the flow past an expanding cylinder moving at an angle of attack through an infinite fluid such that the expansion obtained is uniformly valid in space and time except for $\tilde{t}$ small and $n < 1/2$ or $\tilde{t}$ large. Thus there are two questions to be answered before proceeding with the present expansions:

1. What is the order of the effect of the singular term in the initial conditions for the times when the expansion is valid.
2. What is the order of the effect of the term $g(r)$ in the initial conditions for the times when the expansion is valid.

The second question is easily answered; since the initial conditions may select which of several possible solutions are obtained it is clear that the effects of $g(r)$ may be of order (1) for all time. Thus we must prescribe $g(r)$ when obtaining the present expansion. For the problems considered here we shall choose $g(r)$ as zero which corresponds to considering uniform rectilinear motion for $\tilde{t} < 0$. Since the first term in equation 3.17 is only singular for $n < 1/2$ and of order $R_e^{1-n}$ for $n = 1/2$ it can only affect the present expansion for $n < 1/2$. Since the first order terms of the present inner expansion are governed by a steady linear equation the effect of an initial condition on the leading term may be considered as a term added to the existing solution. For the present case this term would be:

$$\vec{q} = \frac{n\, Re_w^{1-n}}{\tilde{t}^{\,1-2n}} \; \frac{\vec{I_r}}{r^+} \tag{3.19}$$

The effect of this singular term is of order $Re_w^{1-n} / \tilde{t}^{\,n}$. Thus we would expect the effects of the singular term to be less than order $\delta_1^M$ if:

$$o\{\tilde{t}\} > o\left\{ \left( \frac{Re_w^{1-n}}{\delta_1^M} \right)^{\frac{1}{1-2n}} \right\} \tag{3.20}$$

A similar argument applied to the outer equations leads to a similar but less restrictive condition on $\tilde{t}$. Note that although the argument is the same for the non-steady outer equations the solution corresponding to equation 3.19 is slightly more complex. The result that the conditions imposed on $\tilde{t}$ by the outer solution are less restrictive is not surprising because of the $1/r$ behavior of the singular term. Equation 3.20 could also be obtained by considering a "nose expansion" for $\tilde{t}$ small. The application of the nose expansion will be illustrated on the second example. Thus we conclude that for the present expansion, which will only be valid for times satisfying equation 3.17 for all $n$ and also equation 3.20 for $n < 1/2$, the effects of the singular terms in the source strength will be transcendental. Thus in terms of coordinates fixed in the fluid the initial condition for the present expansion may be taken as:

$$\text{As} \quad \tilde{t} \downarrow 0 \qquad \vec{q} \rightarrow 0 \tag{3.21}$$

or in body coordinates:

$$\text{As} \quad \tilde{t} \downarrow 0 \qquad \vec{q} \rightarrow \vec{k} + \frac{\vec{i}}{\alpha} \tag{3.22}$$

III. 4. Cross Flow Solution

We have pointed out that the cross flow problem may be solved independently of the axial flow problem. To solve this cross flow problem we first substitute $\frac{\partial}{\partial x} = 0$ into the outer (equations 2.22 - 2.23) and inner (equations 3.12) equations and note that the axial velocity only appears in the axial momentum equation. Considering only the two cross flow momentum equations and the continuity equation the outer and inner equations become respectively:

$$\widetilde{div}\ \vec{V} = 0$$
$$\frac{\partial \vec{V}}{\partial \tilde{t}} + \vec{V}\ \widetilde{grad}\ \vec{V} + \widetilde{grad}\ P = \tilde{\nabla}^2 \vec{V} \tag{3.23}$$

$$div^+ \vec{V} = 0$$
$$R_{e_w}^{2(1-n)} \frac{\partial \vec{V}}{\partial \tilde{t}} + R_{e_w}^{1-n} \vec{V}\ grad^+ \vec{V} + grad^+ P^+ = \nabla^{+2} \vec{V} \tag{3.24}$$

and the corresponding boundary conditions become:

for the outer solution:

$$\text{at } \tilde{r} = \infty \qquad \vec{V} = \vec{K}\ , \quad P = 0 \tag{3.25}$$

for the inner solution:

$$\text{at } r^+ = \tilde{t}^{\,n} \qquad \vec{V} = \vec{i}_r\ \frac{n\, R_{e_w}^{1-n}}{\tilde{t}^{\,1-n}} \tag{3.26}$$

We note that the boundary conditions are incomplete. The resulting indeterminacy will be removed through the matching of the inner and outer solutions. We have already shown that the outer limit is:

$$\vec{V} = \vec{K} \tag{3.27}$$

and it is clear that the inner limit $\vec{V} = 0$ does not overlap with it. Thus we must show that there is an overlap domain in which both the inner and outer solutions are valid. Note that it is not always possible to find such a domain. However, in the present problem the existence of an overlap domain follows immediately from the fact that the outer equations include the inner equations. Thus the inner limit of the outer equations yields the inner equations, but since the solution of the limit of the equations yields the same solution as the limit of the solution it follows that the inner limit of the outer solution must be the inner solution. Since the inner solution must be included in the outer solution there must be some overlap domain where both solutions are valid to some as yet unspecified order $\Gamma(\epsilon)$.* This overlap domain will include every point in the fluid where the inner solution is valid since for these points the additional terms in the outer solution must be negligible. This discussion applies equally well to the asymptotic expansion of these solutions with some minor modifications. Since there are some terms in the solution which may be of transcendental order with respect to the outer solution, but not of transcendental order with respect to the inner solution, it is clear that it is not always possible to obtain all of the terms in inner expansion by taking the inner limit of the outer expansion. However, if both the expansions are valid to order $\Gamma(\epsilon)$ in the overlap domain the expansions may be matched in the domain.

It is easily verified that if a set of intermediate limits is defined by:

$$\mathrm{LIM}_f\, g = \lim_{R_e \to 0} g(r_f, \tilde{t}^n, R_e) \tag{3.28}$$

where:

$$r_f = \frac{\tilde{r}}{f(R_e)}\,, \qquad o\{R_e^{1-n}\} < o\{f\} < o\{1\}$$

*This same conclusion can be reached in a more rigorous manner by applying the extension theorem which is discussed in reference 3, p. 589.

the application of any of these limits to the present problem yields the Stokes equations. Thus the inner solution is valid for any of these limits and they must all be in the overlap domain. Thus the matching condition, equation 2.1, may be written:

$$\mathrm{LIM}_f \; \frac{\vec{V}_0 - \vec{V}_I}{\Gamma(R_{e_w})} = 0 \tag{3.29}$$

To find the inner expansion we substitute equation 3.15 into equation 3.24 and solve for each succeeding higher order. It will be seen that $o\{\delta\} = o\left\{\frac{1}{\ln R_{e_w}^{1-n}}\right\}$ and thus all terms of order $Re_w^{1-n}$ are transcendentally small and thus the equations for the inner solutions are:

$$\nabla^{+2}\vec{\eta}_m - \mathrm{grad}^+ p_m = 0 \tag{3.30}$$

$$\mathrm{div}^+ \vec{\eta}_m = 0 \tag{3.31}$$

with the boundary conditions:

$$\text{AT} \quad r^+ = \tilde{t}^{\,n} \qquad \vec{\eta}_m = 0$$

Thus equations 3.30-3.31 are satisfied if we write:

$$\vec{\eta}_m = \bar{C}_m(\tilde{t})\,\vec{\eta}_0 \tag{3.32}$$

$$p_m = \bar{C}_m(\tilde{t})\,p_0 \tag{3.33}$$

where $\vec{\eta}_0$ and $p_0$ are the homogenous solution for the Stokes equations which match the outer limit. We note that the inner problem is quasi-steady and thus independent of the initial conditions except through matching which only effects $\tilde{C}_m(\tilde{t})$ and thus the solution given for $\vec{\eta}_0$ in reference 2 is valid for the present problem for n = 0 and it is easily verified that for arbitrary n this solution may be generalized to:

$$\vec{\eta}_0 = \vec{K}\left(\ln r^+/\tilde{t}^{\,n} + \tfrac{1}{2}\right) - \frac{z^+}{r^+}\,\mathrm{grad}^+ r^+ - \frac{\tilde{t}^{\,2n}}{2}\,\mathrm{grad}^+ \frac{z^+}{r^{+2}} \qquad (3.34)$$

$$p_0 = \frac{\tilde{t}^{\,2n}}{2}\,\nabla^{+2}\frac{z^+}{r^{+2}} - \frac{2z^+}{r^{+2}} \qquad (3.35)$$

The equations for the terms in the outer expansion are obtained by substituting equation 3. 8 into equations 3. 23 which gives:

$$\widetilde{\mathrm{div}}\;\vec{q}_m = 0 \qquad (3.36)$$

$$\left(\frac{\partial}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{z}} - \tilde{\nabla}^2\right)\vec{q}_m + \widetilde{\mathrm{grad}}\;p_m = \vec{f}_m(\tilde{x}_i, \tilde{t}) \qquad (3.37)$$

$$\vec{f}_m = -\sum_{i=1}^{m-1} \vec{q}_i\;\widetilde{\mathrm{grad}}\;\vec{q}_{m-i} \qquad (3.38)$$

$$\vec{q}_m(\infty) = 0 \qquad (3.39)$$

We must now consider the matching condition. The matching condition is given by equation 3. 29 which gives (since $\Gamma(R_e) = 1$ ):

$$\lim_{Re_w \to 0}\left(1 + \delta_1 C_1 \ln Re_w^{\,1-n}\right) = 0 \qquad (3.40)$$

Equation 3. 40 implies:

$$\delta_1 = \frac{1}{\ln Re_w^{1-n} + \bar{b}_1} \qquad , \qquad C_1 = -1 \tag{3.41}$$

We shall refer to the above matching process as step-1 matching. Now we proceed to what we shall refer to as step-2 matching which will determine $\vec{g}_1$. For this we use the condition that $\vec{\eta}_2$ must be bounded in the overlap domain. This implies that if we leave out the term $\delta_2 \vec{\eta}_2$ in the matching condition:

$$\text{LIM}_f \; \frac{(\vec{K} + \delta_1 \vec{g}_1) - \delta_1 \vec{\eta}_1 - \delta_2 \vec{\eta}_2}{\delta_1} = 0 \tag{3.42}$$

the expression must be bounded at the origin, thus we have:

$$\lim_{\tilde{r} \to 0} \frac{\vec{K} + \delta_1 \vec{g}_1 - \vec{K}(\delta_1 \ln Re_w^{1-n} - \delta_1 \ln \tilde{r}) + \cdots}{\delta_1} \longrightarrow \text{FINITE} \tag{3.43}$$

which means that $\vec{g}_1$ must be the solution of the outer equations which takes on the value of $-\vec{K} \ln \tilde{r}$ as $\tilde{r} \to 0$. Although the above statement combined with the initial conditions is sufficient to determine $\vec{g}_1$ it is easy to see by intuitive considerations what $\vec{g}_1$ is. Since $\vec{g}_1$ is the response of the outer solution to the first-order drag of the body, which must be a constant since the effect of body growth is not included, it is clear that $\vec{g}_1$ must be the solution produced by a constant distribution of the fundamental singularities from time $t = 0$ to $\tilde{t}$ which may be expressed as:

$$\vec{g}_1 = 4\pi \int_0^{\tilde{t}} \underline{\underline{\Gamma_t}} \, \vec{K} \, d\tau \tag{3.44}$$

$$\mathcal{P}_1 = 4\pi \int_0^{\tilde{t}} \vec{P}_t \cdot \vec{K}\, d\tau \tag{3.45}$$

where $\underline{\underline{\Gamma_t}}$ and $\vec{P}_t$ are given by:

$$\underline{\underline{\Gamma_t}} = \underline{\underline{I}}\, \nabla^2 \Phi_t - grad\ grad\ \Phi_t \tag{3.46}$$

$$\vec{P}_t = \frac{\delta(t)}{2\pi}\, grad(\ln r) \tag{3.47}$$

$$\Phi_t(X_i, t) = \frac{1}{4\pi}\, \Gamma\left(0, \frac{r^2 - 2zt + t^2}{4t}\right) \tag{3.48}$$

where $\Gamma(\alpha, x)$ is the incomplete gamma function, and the $4\pi$ is chosen so that $\vec{g}_1$ has the correct behavior for $\tilde{r} \to 0$. It follows from equations 3. 47 and 3. 45 that:

$$\mathcal{P}_1 = 2\frac{\partial}{\partial \tilde{z}}(\ln \tilde{r}) \tag{3.49}$$

Combining equations 3. 44 and 3. 46 one obtains:

$$\vec{g}_1 = 4\pi\left[\vec{K}\int_0^{\tilde{t}} \nabla^2 \Phi_t\, d\tau - \tilde{grad}\int_0^{\tilde{t}} \frac{\partial \Phi_t}{\partial \tilde{z}}\, d\tau\right] \tag{3.50}$$

To evaluate the first integral in equation 3. 50 we substitute equations 3. 46 and 3. 48 into the equation for the fundamental solution of the non-steady two-dimensional Oseen equations which gives:

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial z} - \nabla^2\right)\nabla^2 \Phi_t = \delta(X_j)\,\delta(t - \tau) \tag{3.51a}$$

and thus $\nabla^2 \Phi_t$ is equal to the fundamental solution of the heat equation or:

$$\nabla^2 \Phi_t = \frac{e^{-\frac{[r^2 - 2z(t-\tau) + (t-\tau)^2]}{4(t-\tau)}}}{4\pi(t-\tau)} \tag{3.52}$$

and thus letting $t-\tau = \frac{t}{\eta}$ :

$$\int_0^{\tilde{t}} \nabla^2 \Phi_t \, d\tau = \frac{e^{\tilde{z}/2}}{4\pi} \int_1^{\infty} \frac{e^{-\frac{1}{2}\left(\frac{r^2}{2t}\right)\left(\eta + \frac{t^2}{r^2}\eta^{-1}\right)}}{\eta} \, d\eta \tag{3.53}$$

$$= \frac{e^{\tilde{z}/2}}{4\pi}\left[2K_0\left(\frac{r}{2}\right) + 2E\left(\frac{t}{4}, \frac{r}{4}\right)\right]$$

However equation 3.51a may also be written as:

$$\nabla^2\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial z} - \nabla^2\right)\Phi_t = \delta(x_j)\,\delta(t-\tau) \tag{3.51b}$$

which implies:

$$\frac{\partial \Phi_t}{\partial z} = \frac{\delta(\bar{\eta})}{2\pi}\ln r - \frac{\partial \Phi_t}{\partial \bar{\eta}} + \nabla^2 \Phi_t \;; \qquad \bar{\eta} = t - \tau \tag{3.54}$$

Thus substituting equations 3.52 and 3.53 into equation 3.50 we get:

$$\vec{q}_1 = 2\Big\{\vec{K}\, e^{\tilde{z}/2}\left[K_0\left(\frac{\tilde{r}}{2}\right) + E\left(\frac{\tilde{t}}{4}, \frac{\tilde{r}}{4}\right)\right] - \widetilde{\mathrm{grad}}\Big\{H(\tilde{t})\ln\tilde{r}$$

$$-\frac{1}{2}\Gamma\left(0, \frac{r^2 - 2\tilde{z}\tilde{t} + \tilde{t}^2}{4\tilde{t}}\right) + e^{\tilde{z}/2}\left[K_0\left(\frac{\tilde{r}}{2}\right) + E\left(\frac{\tilde{t}}{4}, \frac{\tilde{r}}{4}\right)\right]\Big\}\Big\} \tag{3.55a}$$

where:

$$H(\tilde{t}) = \begin{matrix} 0 & \tilde{t} < 0 \\ 1 & \tilde{t} > 0 \end{matrix} \tag{3.56}$$

$$E(t, r) = -\frac{1}{2}\int_0^1 \frac{e^{-r^2/t\left(\eta + \frac{t^2}{r^2}\eta^{-1}\right)}}{\eta}\, d\eta$$

The properties of E(t, r) are discussed in Appendix 2 where it is shown that for r small we may write:

$$E(t,r) = -\frac{1}{2}\Gamma(0,t) + o\{r^2\} \tag{3.57}$$

and thus for $\tilde{r}$ small equation 3.55a becomes:

$$\vec{q}_1 = -\vec{K}\left[\ln\tilde{r} + \gamma - \ln 4 + \frac{1}{2}\Gamma(0,\frac{\tilde{t}}{4})\right] + \frac{\tilde{z}}{\tilde{r}}\,\mathrm{grad}\,\tilde{r} + o\{\tilde{r}\ln\tilde{r}\} \tag{3.55b}$$

Applying the second order step-1 matching by substituting equation 3.55 into equation 3.42 one obtains:

$$\lim_{R_e\to 0} \frac{\vec{K} + \delta_1\{-\vec{K}[\ln\tilde{r} - \gamma + \ln 4 - \frac{1}{2}\Gamma(0,\frac{\tilde{t}}{4})] + \frac{\tilde{z}}{\tilde{r}}\,\widetilde{\mathrm{grad}}\,\tilde{r}\}}{\delta_1}$$
$$- \frac{\delta_1[\vec{K}(\ln Re_w^{1-n} - \ln\tilde{r} + \ln\tilde{t}^{\,n} - \frac{1}{2}) + \frac{\tilde{z}}{\tilde{r}}\,\widetilde{\mathrm{grad}}\,\tilde{r}]}{\delta_1} \tag{3.58}$$
$$- \frac{\delta_2\,\bar{C}_2\,\vec{K}\ln Re_w^{1-n}}{\delta_1} = 0$$

which together with equation 3.41 implies:

$$-\bar{C}_2 = -b_1 - \gamma + \ln 4 + \frac{1}{2} - n\ln\tilde{t} - \frac{1}{2}\Gamma(0,\frac{\tilde{t}}{4}) \tag{3.59}$$

In reference 2 by an appropriate choice of $b_1$ it was possible to choose $c_2 = 0$ and thus $\vec{\eta}_2 = 0$. For the present example this is not convenient for two reasons. First it would be necessary to choose $b_1$ and then $\delta_1$ as functions of $\tilde{t}$ which would introduce considerable complications. Second we would like to choose $b_1$ such that $\delta$ for the cross flow is equal to $\delta$ for the axial flow. Thus for the present example it seems most convenient to choose $b_1$ and all similar constants in the $\delta_i$ as zero. Thus $\delta_i$ are given by:

$$\delta_i = (\delta_1)^i$$
$$\delta_1 = \frac{1}{\ln R_{e_w}^{1-n}} \tag{3.60}$$

We then define:

$$\overline{C_m} = \overline{C_{1,m}}(\tilde{t}) + \overline{C_{2,m}} \tag{3.61}$$

Thus from equation 3.59:

$$\overline{C_{1,2}} = n \ln \tilde{t} + \frac{1}{2}\Gamma(0, \frac{\tilde{t}}{4})$$
$$\overline{C_{2,2}} = -\frac{1}{2} - \ln 4 + \gamma \tag{3.62}$$

and thus

$$\vec{\eta}_2 = [\overline{C_{1,2}}(\tilde{t}) + \overline{C_{2,2}}]\, \vec{\eta}_0 \tag{3.63}$$

We now carry out the second order step-2 matching which gives:

$$\lim_{\tilde{r} \to 0} \left\{ \frac{\vec{K} + \delta_1 \vec{g}_1 + \delta_1^2 \vec{g}_2 - \delta_1(\vec{\eta}_1 + \delta_1 \vec{\eta}_2)}{\delta_1^2} \right\} \rightarrow \text{FINITE} \tag{3.64}$$

which implies that $\vec{g}_2$ is the solution of equations 3.36 through 3.39 which matches $[\overline{C_{1,2}}(\tilde{t}) + \overline{C_{2,2}}]\ln \tilde{r}$ as $\tilde{r} \to 0$. Thus we may write $\vec{g}_2$ as:

$$\vec{g}_2 = \vec{\beta}_2(\tilde{x}_i, \tilde{t}) + \overline{C_{2,2}}\, \vec{g}_1 + \vec{\Omega}_2(\tilde{x}_i, \tilde{t}) \tag{3.65}$$

where:

$$\vec{\beta}_2(\tilde{X}_i,\tilde{t}) = \int_V\int_0^{\tilde{t}} \underline{\underline{\Gamma_t}}\, \vec{f}_2\, d\tau\, d\xi_i$$

$$\vec{f}_2 = -\vec{g}_1\, \widetilde{\text{grad}}\, \vec{g}_1 \qquad (3.66)$$

$$\overline{C_{2,2}} = \gamma - \frac{1}{2} - \ln 4$$

$\vec{\Omega}_2(\tilde{X}_i,\tilde{t})$ is the solution of the homogeneous non-steady Oseen equations matching $-[n \ln \tilde{t} + \frac{1}{2}\Gamma(0,\frac{\tilde{t}}{4})] \ln \tilde{r}$ as $\tilde{r} \to 0$

Following the same procedure one finds the general solution for $\vec{g}_m$ :

$$\vec{g}_m = \vec{\beta}_m(\tilde{X}_i,\tilde{t}) + \overline{C_{m,1}}\,\vec{g}_1 + \vec{\Omega}_m(\tilde{X}_i,\tilde{t}) \qquad (3.67)$$

where :

$$\vec{\beta}_m(\tilde{X}_i,\tilde{t}) = \int_V\int_0^{\tilde{t}} \underline{\underline{\Gamma_t}}\, \vec{f}_m\, d\tau\, d\xi_i$$

$$\vec{f}_m = -\sum_{i=1}^{m-1} \vec{g}_i\, \widetilde{\text{grad}}\, \vec{g}_{m-i} \qquad (3.68)$$

$\vec{\Omega}_m(\tilde{X}_i,\tilde{t})$ is the solution of the homogeneous Oseen equations matching $\overline{C_{m,2}}(\tilde{t}) \ln \tilde{r}$ As $\tilde{r} \to 0$

$\overline{C_{m,1}}$ ; $\overline{C_{m,2}}$ are determined by matching

Thus the outer solution is given by equation 3.8 with the $\vec{g}_m$ given by equation 3.67. The uniformly valid solution is now obtained by applying:

$$\vec{V}_{un} = \vec{V}_o + (\vec{V}_I - \vec{V}_{(o)(I)}) = \vec{V}_o + \vec{\ell}_m(X_i^+, \tilde{t}) \tag{3.69}$$

to equations 3.15 and 3.67. If we recall that the expansions in equation 3.69 are defined by equation 2.6 and define $\vec{\ell}_m(X_i^+, \tilde{t})$ as the part of the inner solution which is not included in the outer solution we may write for this problem:

$$\vec{\ell}_m(X_i^+, \tilde{t}) = \sum_{j=0}^{m} \left[ \vec{\eta}_j - LIM_I\, \vec{g}_j \right] \delta^i(R_{e_w}) \tag{3.70}$$

Thus for $m = 1$:

$$\vec{\ell}_1(X_i^+, \tilde{t}) = -\vec{K} + \delta_1 [\vec{\eta}_1 - LIM_I\, \vec{g}_1] = \delta_1 \frac{\tilde{t}^{2n}}{2} \operatorname{grad}^+ \frac{z^+}{r^{+2}} \tag{3.71}$$

And thus the uniformly valid solution for the cross flow may be written:

$$\vec{V}_{un} = \vec{K} + \delta_1 \vec{g}_1 + \vec{\ell}_m(X_i^+, \tilde{t}) + \sum_{i=2}^{m} \delta_i \left[ \vec{\beta}_i(\tilde{X}_j, \tilde{t}) + \bar{C}_{i,1}\, \vec{g}_1 + \vec{\Omega}_i(\tilde{X}_j, \tilde{t}) \right] \tag{3.72}$$

where $\vec{g}_1$, $\vec{\ell}_m$, $\vec{\beta}_i$, $\bar{C}_{i,1}$, and $\vec{\Omega}_i$ are given by equations 3.67a, 3.83 and 3.81,

$$\delta_i = \left( \frac{1}{\ln R_{e_w}^{1-n}} \right)^i \tag{3.73}$$

and similarly the uniformly valid expression for the pressure is:

$$P_{un} = \sum_{i=1}^{m} P_i\, \delta_i \tag{3.74}$$

where:

$$p_1 = 2\frac{\partial}{\partial \bar{z}}(\ln \tilde{r}) - \frac{\tilde{t}^{2n}}{2}\nabla^{+2}\frac{z^+}{r^{+2}}$$

$$p_i = \overline{C_{i,1}}\,p_1 + \int_v\int_0^{\tilde{t}} \vec{P}_t \cdot \vec{f}_i \, d\tau \, d\xi_j + p_\Omega \qquad (3.75)$$

$p_\Omega$ = pressure corresponding to $\vec{\Omega}(\tilde{x}_j, \tilde{t})$

III. 5. Axial Flow Solution

Setting $\frac{\partial}{\partial x} = 0$ in the outer equations (equations 2.22-2.23) and inner equations (eqs. 3.12) the outer and inner axial momentum equations are respectively:

$$\frac{\partial u^*}{\partial \tilde{t}} + \vec{V}\,\widetilde{\text{grad}}\,u^* = \tilde{\nabla}^2 u^* \qquad (3.76)$$

$$R_{e_w}^{2(1-n)}\frac{\partial u^*}{\partial \tilde{t}} + R_{e_w}^{1-n}\vec{V}\,\text{grad}^+ u^* = \nabla^{+2} u^* \qquad (3.77)$$

where $\vec{V}$ is the solution of the cross flow problem. It will be shown that it is possible and most convenient to choose $\delta(R_{e_w})$ for the axial flow equal to $\delta(R_{e_w})$ for the cross flow. Thus the terms on the left side of equation 3.77 will be transcendentally small and we need only consider the outer expansion for $\vec{V}$ which is given by equation 3.72 and may be written as:

$$\vec{V} = \vec{K} + \sum_{i=1}^{m} \delta^i \vec{g}_i(\tilde{x}_j, \tilde{t}) \qquad (3.78)$$

where :

$$\delta(R_{e_w}) = \frac{1}{\ln R_{e_w}^{1-n}} \qquad (3.79)$$

$$\vec{g}_i(\tilde{x}_j, \tilde{t}) = \vec{\beta}_i(\tilde{x}_j, \tilde{t}) + \overline{C}_{i,1}\, \vec{g}_1 + \Omega_i(\tilde{x}_j, \tilde{t}) \tag{3.80}$$

and the rest of the terms appearing in equation 3.80 are given by equations 3.55a and 3.68. The boundary conditions for the problem are:

for the inner solution:

$$u^* = 0 \qquad \text{AT} \quad r^+ = \tilde{t}^{\,n} \tag{3.81}$$

for the outer solution:

$$u^* = \frac{1}{\alpha} \qquad \text{As} \quad \tilde{r} \rightarrow \infty \tag{3.82}$$

Since terms of order $Re_w^{1-n}$ are transcendentally small for the present expansions we neglect them and the inner equation becomes:

$$\nabla^{+2} u^* = 0 \tag{3.83}$$

Substituting equation 3.14 into equations 3.81 and 3.83 we get the following problem for the terms in the inner expansion:

$$\nabla^{+2} \eta_m = 0 \tag{3.84}$$

$$\eta_m = 0 \qquad \text{AT} \quad r^+ = \tilde{t}^{\,n} \tag{3.85}$$

The solution to equations 3.84 and 3.85 is:

$$\begin{aligned} \eta_1 &= C_1(\tilde{t}) \ln r^+/\tilde{t}^{\,n} \\ \eta_i &= \frac{C_i(\tilde{t})}{C_1(\tilde{t})}\, \eta_1 \end{aligned} \tag{3.86}$$

where $C_i(\tilde{t})$ are determined by matching. Thus we may write equation 3.14 as:

$$u_I^* = \sum_{i=1}^{M} \delta_i(R_{e_w})\, C_i(\tilde{t}) \ln r^+/\tilde{t}^n \tag{3.87}$$

We have already shown that the outer limit is given by:

$$u_o^* = \frac{1}{\alpha} \tag{3.88}$$

Thus the first order step-1 matching condition* to determine $\eta_1$ is:

$$\lim_{R_{e_w}\to 0} \left(\frac{1}{\alpha} - \delta_1 \eta_1\right) = 0 \tag{3.89}$$

or:

$$\frac{1}{\alpha} - \delta_1 b_1 + \delta_1 C_1(\tilde{t}) \ln R_{e_w}^{1-n} = 0$$

Thus:

$$\delta_1 = \frac{1}{\ln R_{e_w}^{1-n} + b_1}$$

$$C_1(\tilde{t}) = -\frac{1}{\alpha} \tag{3.90}$$

The equations for the rest of the terms in the outer expansion are determined by substituting equation 3.7 into equation 3.76 which gives:

$$\frac{\partial g_i}{\partial \tilde{t}} + \frac{\partial g_i}{\partial \tilde{z}} - \nabla^2 g_i = f_i \tag{3.91}$$

*The existence of the necessary overlap domain can be shown by an analysis analogous to that used in showing the existence of the overlap domain in the cross flow problem.

$$f_i = -\sum_{j=1}^{i-1} \vec{g}_{i-j}\ \text{grad}\ g_j \tag{3.92}$$

$$g_i(\infty) = 0 \tag{3.93}$$

We note several important differences between the present problem and the cross flow problem. First the axial flow is completely independent of the pressure and thus the pressure for the entire problem is given by the cross flow solution. Second the axial flow velocity may be treated as a scalar and we shall make use of this property to distinguish between the axial and cross flow terms.

To determine the behavior of $g_1$ for $\tilde{r}$ small we apply the first order step-2 matching condition:

$$\lim_{\tilde{r}\to 0} \frac{\frac{1}{\alpha} + \delta_1 g_1 - \delta_1\left[\frac{1}{\alpha}\ln Re_w^{1-n} + \frac{1}{\alpha}\ln \tilde{E}^n - \frac{1}{\alpha}\ln \tilde{r} + \cdots\right]}{\delta_1} \rightarrow \text{FINITE} \tag{3.94}$$

Thus $g_1$ is the solution of equations 3.91-3.93 which for $\tilde{r}$ small behaves like $-\frac{1}{\alpha}\ln\tilde{r}$. The work is considerably simplified if we realize that the first order outer solution $g_1$ must be the response to a constant distribution of the fundamental singularities of the outer equations. This means we are observing the body at such a large distance that it appears as a singular force acting on the fluid. From this physical interpretation it follows that for semi-infinite bodies one would expect the body to look like a distribution of singular forces over the entire length of the body which would depend on the axial variation of the drag of the body. Since the present problem is an infinite body of constant cross section we shall try the fundamental solution of the outer equation for $g_1$ integrated from $t = 0$ to $\tilde{t}$. Thus we try:

$$g_1 = \frac{2\pi}{\alpha} \int_0^{\tilde{t}} S_t(\tilde{x}_i, \tilde{t} - \tau)\, d\tau \tag{3.95}$$

where $S_t$ is the fundamental solution of equation 3.91 and given by*

$$S_t(x, t) = \frac{e^{-\frac{r^2 - 2zt + t^2}{4t}}}{4\pi t} \tag{3.96}$$

Substituting equation 3.96 into equation 3.95 and integrating one obtains:

$$g_1 = \frac{e^{\tilde{z}/2}}{\alpha}\left[K_0\left(\frac{\tilde{r}}{2}\right) + E\left(\frac{\tilde{t}}{4}, \frac{\tilde{r}}{4}\right)\right] \tag{3.97a}$$

which for $\tilde{r}$ small becomes:

$$g_1 \simeq \frac{1}{\alpha}\left[-\ln \tilde{r} - \gamma + \ln 4 - \frac{1}{2}\Gamma\left(0, \frac{\tilde{t}}{4}\right)\right] + o\left\{\tilde{r} \ln \tilde{r}\right\} \tag{3.97b}$$

Now applying the second order step-1 matching one obtains:

$$\lim_{Re_w \to 0} \frac{\frac{1}{\alpha} + \frac{\delta_1}{\alpha}\left[-\ln \tilde{r} - \gamma + \ln 4 - \frac{1}{2}\Gamma(0, \tilde{t}/4)\right]}{\delta_1} - \frac{1}{\alpha}\left[\ln Re_w^{1-n} - \ln \tilde{r} + \ln \tilde{t}^{\,n}\right] + \delta_1 C_2 \ln Re_w^{1-n} = 0 \tag{3.98}$$

which together with equation 3.90 implies:

$$-\alpha C_2 = -b_1 - \gamma + \ln 4 - \frac{1}{2}\Gamma\left(0, \frac{\tilde{t}}{4}\right) - n \ln \tilde{t} \tag{3.99}$$

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*See Appendix 1.

Again it is most convenient to choose $b_1 = 0$ which gives:

$$\delta_1 = \frac{1}{\ln R_{e_w}^{1-n}} \tag{3.100}$$

$$C_2 = \frac{1}{\alpha}\left[\gamma - \ln 4 + \frac{1}{2}\Gamma(0, \tilde{t}/4) + n \ln \tilde{t}\right]$$

and choosing all of the succeeding constants corresponding to $b_1$ as zero gives $\delta_i = \delta_1^{\,i}$. $\eta_2$ is completely defined by equations 3.86 and 3.100. For convenience we shall again split $C_i$ as follows:

$$C_i = C_{1,i}(\tilde{t}) + C_{2,i} \tag{3.101}$$

For example it follows from this and equation 3.100 that:

$$C_{1,2} = \frac{1}{\alpha}\left[\frac{1}{2}\Gamma(0, \frac{\tilde{t}}{4}) + n \ln \tilde{t}\right] \tag{3.102}$$

$$C_{2,2} = \frac{1}{\alpha}\left[\gamma - \ln 4\right]$$

The second order step-2 matching gives that $g_2$ is the solution of equations 3.91-3.93 which satisfies the condition that as $\tilde{r} \to 0$, $g_2 \to -(C_{1,2} + C_{2,2}) \ln \tilde{r}$ Thus we may write $g_2$ as:

$$g_2 = \beta_2(\tilde{x}_j, \tilde{t}) + C_{2,2}\, g_1 + \Omega_2(\tilde{x}_j, \tilde{t}) \tag{3.103}$$

where:

$$\beta_2(\tilde{x}_j, \tilde{t}) = \int_V \int_0^{\tilde{t}} S_t\, f_2\, d\tau\, d\xi_j$$

$$f_2 = \vec{g}_1 \cdot \operatorname{grad} g_1 \tag{3.104}$$

$$C_{2,2} = \frac{1}{\alpha}(\gamma - \ln 4)$$

$\Omega_2(\tilde{x}_j, \tilde{t})$ is the solution of the homogeneous part of equation 3.91 which behaves like $-\frac{1}{\alpha}\left[\frac{1}{2}\Gamma(0, \frac{\tilde{t}}{4}) + n \ln \tilde{t}\right] \ln \tilde{r}$ as $\tilde{r} \to 0$

Following the same procedure one finds that in general $g_m$ may be written:

$$g_m = \beta_m(\tilde{x}_j, \tilde{t}) + c_{m,1}\, g_1 + \Omega_m(\tilde{x}_j, \tilde{t}) \tag{3.105}$$

where:

$$\beta_m(\tilde{x}_j, \tilde{t}) = \int_V \int_0^{\tilde{t}} S_t\, f_m\, d\tau\, d\xi_j$$

$f_m$ is given by equation 3.92

$\vec{g}_m$ are given by equation 3.67 (3.106)

$\Omega_m$ is the solution of the homogeneous part of equation 3.91 which behaves like $c_{m,2}(\tilde{t}) \ln \tilde{r}$ as $\tilde{r} \to 0$

$c_{m,1}; c_{m,2}$ are determined by matching

and:

$$u_0^* = \frac{1}{\alpha} + \sum_{m=1}^{M} g_m\, \delta_1^{\,m} \tag{3.107}$$

In principle the uniformly valid solution may be determined in the same manner as for the cross flow. However, one would find that

$\ell_m$ is zero. That is:

$$\ell_m = \sum_{i=0}^{m} \left[ \eta_i - \mathrm{LIM}_i \, g_i \right] \delta_1^i (R_{e_w}) = 0 \tag{3.108}$$

This may be easily verified for specific values of i by direct substitution. From equation 3.86 it is clear that:

$$O\left\{ \delta_1^i \, \eta_i (\tilde{x}_j , \tilde{t} , R_{e_w}) \right\} \leq O\left\{ \delta_1^{i+1} \right\} \tag{3.109}$$

and thus the inner solution not only does not contain terms which are of transcendental order when written in outer variables but also it does not contain any term in the ith order term which does not appear in the $i + 1^{th}$ order term of the outer solution. From this it is clear that equation 3.108 must be valid for all values of "i". Thus if we apply the equivalent of equation 3.69 to the present case we find that $u_{un}^* = u_o^*$ and thus that the uniformly valid solution is given by equations 3.105 and 3.107.

## III. 6. Modification of the Expansion Procedure

It has been shown that $u_o^*$ contains all of the terms in $u_I^*$ and thus:

$$u_{un}^* = u_o^* \tag{3.110}$$

It follows from equation 3.110 that we could have obtained the results given by equations 3.105-3.107 without introducing the inner limit by applying the boundary condition at the body directly to the outer solution. However, it was seen that in the cross flow there were terms in the inner expansion which were transcendentally small in the outer expansions. Thus if one omitted the inner limit in computing the cross flow the resulting solution would not include those terms which are of order $\delta$ near the body and it would not be uniformly valid to order $\delta$.

Thus for problems where the entire inner solution is contained in the outer solution one might modify the standard expansion procedure and only consider the outer solution. This could be carried out almost identical to the standard procedure except one would not define the inner solution. The matching conditions would be replaced by the appropriate approximate boundary conditions at the body. Thus as each succeeding term in the expansion was considered one would hope to increase the order to which the boundary conditions were satisfied. However, it is easily seen from the preceding discussion that this will not always be possible. The fact that one can not determine whether this method will allow one to obtain a solution valid to the same order as that obtained by the standard procedure shows the inferiority of this method compared to the standard procedure. Although this modification is not in general recommended it is a good illustration of the relation between matching conditions and boundary conditions and thus we shall apply this modified procedure to the axial flow. The problem is defined by equations 3.76 and 3.82 plus the additional boundary condition:

$$\text{AT} \qquad \tilde{r} = Re_w^{1-n}\,\tilde{t}^{\,n} \qquad\qquad u^* = 0 \tag{3.111}$$

Assuming an expansion of the form given by equation 3.7 we substitute equation 3.7 into equations 3.76, 3.82 and 3.111 obtaining equations 3.91 through 3.93 plus:

$$\lim_{Re_w \to 0} \sum_{i=1}^{M} \frac{\delta_i(Re_w)\, g_i(Re_w^{1-n}\tilde{t}^n, \tilde{t}) + \frac{1}{\alpha}}{\delta_{M-1}(Re_w)} = 0 \tag{3.112}$$

at $\tilde{t} = 0 \qquad g_i = 0$

Since $f_1 = 0$ the problem for $g_1$ is simply the homogeneous solution of the above equations which takes on the value of $-\dfrac{1}{\alpha\,\delta_1(Re_w)}$ at the body. Again, making use of the fact that $g_1$ must be the response to a constant distribution of the fundamental singularities of the outer equation along the axis of the cylinder from $t = 0$ to $\tilde{t}$ we get:

$$g_1 = A e^{\tilde{z}/2}\left[K_0(\tilde{r}/2) + E\left(\frac{\tilde{t}}{4}, \frac{\tilde{r}}{4}\right)\right] \tag{3.113}$$

Substituting equation 3.113 into equation 3.112 we obtain:

$$-\frac{1}{\alpha} = \lim_{Re_w \to 0} A\,\delta_1(Re_w)\left(-\ln Re_w - \ln \tilde{t}/4 - \gamma\right) \tag{3.114}$$

from which it follows that:

$$A = \frac{1}{\alpha}$$

$$\delta_1 = \frac{1}{\ln Re_w^{1-n}} \tag{3.115}$$

and thus $g_1$ is again given by equation 3.97. Thus the problem for $g_2$ is given by equations 3.91-3.93 with the boundary conditions:

at $\tilde{r} = \infty \qquad g_2 = 0$

at $\tilde{r} = Re_w^{1-n}\,\tilde{t}^n$ (3.116)

$$\lim_{Re_w \to 0} \frac{\frac{1}{\alpha} + \delta_1(Re_w)\,\frac{e^{Re_w^{1-n}\tilde{t}^n \cos\theta}}{2}\, K_0\left(\frac{Re_w^{1-n}\tilde{t}^n}{2}\right) + \delta_2(Re_w)\, g_2}{\delta_1(Re_w)} = 0$$

or:

$$g_2\left(R_{e_w}^{1-n}\tilde{t}^{\,n}, \theta, \tilde{t}\right) = \frac{1}{\alpha\,\delta_1}\left[c_{1,2}(\tilde{t}) + c_{2,2}\right]$$

and:

$$\delta_2 = \left(\frac{1}{\ln R_{e_w}^{1-n}}\right)^2 = \delta_1^{\,2} \tag{3.117}$$

We see that the solution of this problem consists of three parts. First the inhomogeneous solution which is still given by $\beta_2$, see equation 3.104, and is uniformly valid to order $\delta_2$. The solution satisfying the steady portion of the boundary condition which is clearly given by $c_{2,2}g_1$ and finally a solution which is equal to $\frac{c_{1,2}(\tilde{t})}{\alpha}\ln \tilde{t}^{\,n} \ln R_{e_w}^{1-n}$ at the body. $\Omega_2(\tilde{X}_i, \tilde{t})$ as defined by equation 3.104 clearly satisfies this condition and thus $g_2$ is still given by equation 3.103. The solution thus far is:

$$u_0^* = \frac{1}{\alpha} + \delta_1 g_1 + \delta_1^{\,2}\left[\beta_2(\tilde{X}_i, \tilde{t}) + c_{2,2}\, g_1 + \Omega_2(\tilde{X}_i, \tilde{t})\right] \tag{3.118}$$

However, equation 3.118 satisfies the boundary condition only to order $\delta_1$ and thus we must construct a third order term which again has two parts one the nonhomogeneous solution and the other the homogeneous solution which cancels the terms in the first three terms of $u_0^*$ of order $\delta_1^{\,2}$ at the body. Dividing these terms into a constant part $c_{2,3}$ and a time dependent part $c_{1,3}(t)$ we see that $g_3$ is of the form:

$$g_3 = \beta_3(\tilde{X}_i, \tilde{t}) + c_{2,3}\, g_1 + \Omega_3(\tilde{X}_i, \tilde{t}) \tag{3.119}$$

and in general:

$$g_i = \beta_i(\tilde{X}_j, \tilde{t}) + c_{2,i}\, g_1 + \Omega_i(\tilde{X}_j, \tilde{t}) \tag{3.120}$$

where $\beta_i$, $C_{2,i}$ and $\Omega_i$ are defined by equation 3.106. From equation 3.120 it is clear that the present result is identical to that obtained by the standard procedure, but it must be emphasized that this is an exception and not the general rule. In general the solution as obtained above will be less accurate than that obtained by the standard procedure.

A second possible modification of the expansion procedure is to construct a properly matched outer solution by making use of the fact that the fundamental solution of the Oseen equations is the response to a singular force. This method will only work if the outer equations are valid throughout the fluid. If the first order outer equations are homogeneous one begins the same as in the standard expansion procedure and obtains both the inner and outer equations. Then one obtains the solution of the inner equations which matches the outer limit. The drag on the body is then computed from the inner solution and if the outer equations are valid throughout the fluid this drag must be the distribution function for the singular forces producing the outer solution and thus the integral representation of the properly matched outer solution is obtained immediately from the fundamental solution of the outer equations. The procedure may then be continued by constructing the next term in the inner expansion so that it matches the previous terms in the outer expansion. Then the drag due to this new inner solution is computed and must be the distribution function for the corresponding outer solution. The procedure may be continued until one reaches an order where the outer equations are no longer homogeneous. Since this procedure only gives that part of the solution arising from the effect of the preceding outer solution on the drag of the body it is clear that any terms in the outer equations arising from a different source must be obtained in a different manner. For example in the cross flow problem of the present example the terms in the outer solution arise from two sources: 1) the inner boundary condition, which may be represented as a force acting on the fluid and 2) the non-linear terms. Since the non-linear terms enter in the second order terms it is necessary to add the particular solution of the outer equations to the solution obtained by this method for the second and higher order. Thus we see that the present method is actually a method of obtaining the homogeneous terms in the solution. From which it follows, that except in the unusual cases where the problem is homogeneous to all orders, the present method only yields a portion of the solution. This means that one must be very careful in applying this method to be sure that all the sources of the terms are considered. This method is particularly useful in the present problem for obtaining integral expressions for the $\Omega_i$. To illustrate the method we shall consider only the axial flow and use the previously determined cross flow as a given function and thus the problem is defined by equations 3.76-3.82. Proceeding as in section III.5 we find that the inner solution is given by equations 3.86 and 3.87:

$$u_I^* \cong \delta_1 \eta_1 = -\frac{\delta_1}{\alpha} \ln r^+/\tilde{t}^n \qquad (3.121)$$

Since for this inner solution the boundaries are all parallel to the free stream velocity the pressure does not contribute to the drag and thus the dimensionless drag is proportional to the skin friction coefficient and given by:

$$C_{D_o} \equiv -\frac{1}{R_{e_w}^{1-n}} \left.\frac{\partial u_I^*}{\partial r^+}\right|_{r^+=\tilde{t}^n} = \frac{\delta_1}{\alpha} \frac{\delta(\tilde{r} - R_{e_w}^{1-n}\tilde{t}^n)}{\tilde{r}} \tag{3.122}$$

$$\delta(x) = \begin{matrix} 0 & x \neq 0 \\ 1 & x = 0 \end{matrix}$$

We take $C_{D_o}$ as the distribution function for the singular forces producing the first order terms of the outer expansion, $g_1$, and thus assuming the forces were put in the fluid at t = 0 we get:

$$\delta_1 g_1 = \iint_V \int_0^{\tilde{t}} S_t(\tilde{t}-\tau, \tilde{x}_i - \xi_i)\, C_{D_o}(\xi_i, \tau)\, d\tau\, d\xi_i \tag{3.123}$$

where $S_t$ is given by equation 3.96 and thus in terms of polar coordinates one obtains:

$$\delta_1 g_1 = \frac{\delta_1}{4\pi\alpha} \int_0^{\tilde{t}}\int_0^{2\pi}\int_0^{\infty} \frac{\delta(\sigma - R_{e_w}^{1-n}\tau^n)}{\sigma(\tilde{t}-\tau)} e^{-\frac{(\tilde{y}-\sigma\sin\phi)^2 + (\tilde{z}-\sigma\cos\phi)^2 - 2(\tilde{z}-\sigma\cos\phi)(\tilde{t}-\tau) + (\tilde{t}-\tau)^2}{4(\tilde{t}-\tau)}} \sigma\, d\sigma\, d\phi\, d\tau \tag{3.124}$$

Integrating with respect to $\sigma$ and neglecting terms transcendentally small with respect to $(\delta_1)$ one obtains:

$$g_1 = \frac{e^{\tilde{z}/2}}{4\pi\alpha} \int_0^{\tilde{t}}\int_0^{2\pi} \frac{e^{-\frac{\tilde{r}^2 + (\tilde{t}-\tau)^2}{4(\tilde{t}-\tau)}}}{\tilde{t}-\tau}\, d\phi\, d\tau + o\left\{R_{e_w}^{1-n}\right\} \tag{3.125}$$

Integrating with respect to $\phi$ and $\tau$ we obtain:

$$g_1 = \frac{e^{\tilde{z}/2}}{\alpha} \left[K_0\left(\frac{\tilde{r}}{2}\right) + E\left(\frac{\tilde{t}}{4}, \frac{\tilde{r}}{4}\right)\right] \tag{3.126}$$

where $E(t, r)$ is given by equation 3.56. Equation 3.126 is identical with the solution obtained in section III.5 and thus following the standard procedure for constructing the next term in the inner solution will yield the same result obtained in section III.5 and thus from equation 3.86:

$$\eta_2 = \left[c_{2,2} + c_{1,2}(\tilde{t})\right] \ln r^+/\tilde{t}^n \tag{3.127}$$

At this point it should be recalled that the present modification will only give those terms in the outer solution which arise due to the drag of the body. For the axial flow we do not have non-linear terms as an additional source of terms but we do have higher order forcing functions due to the cross flow and these must be considered separately. Thus if we define $\beta_i$ as the non-homogeneous solution of the outer equation, $\Omega_i$ as the terms arising from the non-steady portion of the drag, and $J_i$ the terms arising from the steady portion of the drag one may write:

$$g_i = \beta_i(\tilde{x}_j, \tilde{t}) + J_i(\tilde{x}_j, \tilde{t}) + \Omega_i(\tilde{x}_j, \tilde{t}) \tag{3.128}$$

where the $\beta_i$ cannot be determined by the present modification of the standard expansion procedure but clearly are given by equation 3.106. Using equation 3.127 to compute the drag arising from $g_1$ one obtains:

$$C_{D_1} = \delta_1^2 \left[C_{2,2} + C_{1,2}(\tilde{t})\right] \frac{\delta(\tilde{r} - Re_w^{1-n} \tilde{t}^n)}{\tilde{r}} \tag{3.129}$$

and from equation 3.86 it follows that in general the drag due to the ith term in the outer solution is given by:

$$C_{D_i} = \delta_1^{i-1}\left[C_{2,i} + C_{1,i}(\tilde{t})\right] C_{D_0} \tag{3.130}$$

and thus in general:

$$\delta_1^i J_i(\tilde{x}_j, \tilde{t}) = C_{2,i}\, \delta_1^{i-1} \int_V \int_0^{\tilde{t}} S_t\, C_{D_0}\, d\tau\, d\xi_j \tag{3.131}$$

Comparing equations 3.123 and 3.131 it follows that:

$$J_i(\tilde{x}_j, \tilde{t}) = C_{2,i}\, g_1 \tag{3.132}$$

In a manner analogous to that used to obtain equation 3.126 one finds that the term $\delta_1^{\,i-1} C_{1,i}(\tilde{t})\, C_{D_0}$ gives rise to:

$$\Omega_i(\tilde{x}_j,\tilde{t}) = \frac{e^{\tilde{z}/2}}{2\pi}\int_0^{\tilde{t}} \frac{C_{1,i}(\tau)}{\tilde{t}-\tau}\, e^{-\frac{r^2+(\tilde{t}-\tau)^2}{4(\tilde{t}-\tau)}}\, d\tau \qquad (3.133)$$

Comparing equations 3.105 and 3.128 it is clear that the $\Omega_i$ as defined here is the same as that previously defined.

The same process could also be applied to the cross flow. The process would be identical and the main differences would be that $C_{D_0}$ and the fundamental solution would be considerably more complex and the $\beta_i$ would be due to the non-linear terms rather than a prescribed forcing function. This calculation will not be carried out here. It is interesting to note that this modification of the expansion procedure illustrates a manner by which one can construct an approximate solution of the Oseen equations if the corresponding Stokes flow is known.

# IV. POWER LAW BODY OF REVOLUTION

## IV.1. Discussion of Problem

The second example we shall consider is the steady incompressible low Reynolds number flow about a power law body of revolution at zero angle of attack. Thus $R(x,t)$ (see fig. 1) is chosen independent of "t" and is of the form $R(x,t) = Ax^n$. We again require that $0 \leq n < 1$. We shall see that this problem is more difficult than the first. Since this problem is quite complex we wish to introduce as much simplification as possible without destroying the essential features of the problem. Thus we only consider the problem for steady flow and zero angle of attack. The specification of zero angle of attack insures axial symmetry. Even with these two restrictions we no longer have the very useful simplification, which occurred in the preceding problem, of being able to separate the axial and cross flow problems.

One simplification would be to consider the problem for the special values of n for which the body takes on a particularly simple shape. One such special value of n for which considerable simplification may be expected is $n = 1/2$. The physical reasons for expecting the solution to be less complex for this case are that both the viscous layer and the body grow at the same rate and the nose radius is finite. Since the first order outer solution is the solution of the Oseen equations and the first order nose solution a solution of the Stokes equations we shall be able to obtain these first order solutions directly from the exact solution of the Oseen equations for the flow about a paraboloid which is given in Ref. 3.

Thus we shall first apply the standard expansion procedure which was illustrated on the preceding problem to the present problem for $n = 1/2$.

For the present problem there are three limits to consider, namely

1. Nose limit
2. Inner limit
3. Outer limit

These limits may be thought of as corresponding to the inner, wake and outer limits of Ref. 4. The nose limit is the limit as we approach the origin. Since we are near the body and the condition of $Re \to 0$ (for $n = 1/2$) implies this is a very viscous region we expect the nose limit to be a solution of the Stokes equations. Since the nose is clearly three-dimensional we further expect that the full three-dimensional Stokes equations will be required. The inner limit is the limit as we approach the body with x fixed and corresponds to the inner limit in the preceding example. We are interested in the flow in a region which is much closer to the body than the nose. Since we are near the body we again expect the Stokes equations, however, since we are relatively far from the nose we do not expect any variation with x. Thus the inner equations should satisfy the Stokes equations with all the derivatives with respect to "x" equal to zero. Finally since we are dealing with the flow about a body which has zero arresting power when the characteristic length tends to zero the outer limit must be the free stream velocity, or unity in non-dimensional variables, and the outer solution governed by the Oseen equations.

There are some problems for which it is advisable to introduce more than two limits. It is clear that once one has two matched solutions they may be combined into one solution which is valid for the entire region in which either of the solutions were valid separately. Thus we may extend the standard expansion procedure to problems for which there are an arbitrary number of approximate solutions each valid in a different region and having an overlap domain in common with the approximate solution in the neighboring region. The procedure is the same as described in section II.1 except that one must apply it first to two adjacent regions. Then having obtained a single solution valid in this combined region this solution is matched and then combined with the solution in another adjacent region to form still another approximation valid over all three regions. This process is continued until all of the approximate solutions are combined to give a single uniformly valid solution. Both of the examples in the present study as well as the problems studied in references 4 and 1 should really be considered as having three limits. For the first example considered here and the problem of reference 1 these limits are: 1) the limit as one approaches the body at fixed "t" or the inner limit; 2) the limit as one approaches the time-space origin or the nose limit; and 3) the limit as one tends to infinity from the time-space origin or the outer limit. However, for both of these cases the only terms of the nose solution which differ from those included in the inner solution are of transcendental order with respect to the expansions considered and thus it is only necessary to introduce the nose limit when considering the transcendental terms or the expansion for small time. In the study of reference 4 the three limits were 1) the limit as one approaches the wake for $x$ fixed, the inner (or wake) limit; 2) the limit as one approaches the body or the nose limit; and 3) the outer limit. In this example all three limits were important but in general only the inner and outer limits had a common overlap domain and thus only these two limits were considered in detail. In the second example of the present study there is 1) the inner limit, that is the limit as the body is approached with $x$ fixed; 2) the nose limit or the limit as the origin is approached; and 3) the outer limit. For this problem the nose equations include the inner equations and thus we can dispense with the inner equations applying the inner boundary conditions directly to the nose equations. It is clear that for the linear terms this would not cause any loss of accuracy since the nose and inner boundary conditions are also identical. However, this situation should not be expected in general since eliminating the inner limit is actually the same modification of the standard procedure as only using the outer limit in the first example. Thus for the same reason discussed in the first example we would in general expect the uniform solutions to be valid to a lower order if only two of the limits in a three limit problem were used even though one of the limits considered includes the equations for the neglected limit. In general we also could not expect to be able to predict how much accuracy is lost without computing the neglected approximate solution. Although for the present problem some of these disadvantages of neglecting the inner limit do not materialize for the linear terms due to the accident that the boundary conditions are the same for the two limits this luck does not hold for the non-linear terms. For the non-linear terms the inner and nose solutions are forced by the non-linear effects in the outer solution. Only the effects at $x = 0$ influence the nose solution, but one would expect the effect of these non-linear terms on the inner solution to vary with "x". This "x" variation is not accounted for unless the inner limit is used.

Thus we shall obtain the solution of the second example by employing all three limits to serve the dual purpose of obtaining a more accurate expression for the higher order terms downstream and to illustrate the extension of the standard expansion procedure to more than two limits. From the discussion above it is clear that up to the third order terms the solutions based on using all three of the limits must be the same as that obtained using only the nose and outer limits.

After obtaining the solution for $n = 1/2$ the solution for arbitrary $n$ will be obtained in terms of the Stokes flow about the body. For arbitrary $n$ we shall consider all three limits. Since the body has zero arresting power for all values of $n$ considered, the outer solution will still be governed by the Oseen equations for all values of $n$. However the inner equations will depend on whether $n$ is greater than or smaller than $1/2$. For $n$ smaller than $1/2$ the viscous layer will grow faster than the body and thus if one chose the overall Reynolds number sufficiently small there will be a large viscous layer near the body and the inner solution will be a very viscous type solution. But if $n$ is larger than $1/2$ it will always be possible, given any fixed overall Reynolds number, to go sufficiently far downstream and find a region where the viscous layer is thin compared to the body diameter. In this region the boundary-layer equations are applicable. Since this region is far downstream for low Reynolds number flow, the boundary-layer equations reduce to those for a flat plate and will not be considered here. Therefore, we shall restrict our attention for the remainder of this chapter to the case of $n \leq 1/2$. It is clear that since the flow is very viscous and we are interested in the region near the body but far from the nose the inner solution will satisfy the Stokes equations with $\partial/\partial x = 0$ for $n \quad 1/2$. Since the flow is a very viscous type near the nose, the nose equations will again be the three-dimensional Stokes equations. After obtaining the solution for arbitrary $n$ by the standard expansion procedure we shall show that some of the terms in the expansion which could not be evaluated by the standard procedure may be evaluated by use of the modified procedure in which the inner drag is used as a forcing function for the outer solution.

IV.2. Equations and Boundary Conditions

We shall obtain the equations and boundary conditions for arbitrary n here. For this problem there is only one characteristic velocity, U. The characteristic length L is given by equation 2.17 and thus it follows that:

$$R_e = \frac{L\,U}{\nu} = \frac{A^{\frac{1}{1-n}}\,U}{\nu} \tag{4.1}$$

Following the previous example we try the variables defined by equation 2.19 as the outer variables where:

$$\begin{aligned} \epsilon &= R_e \\ \bar{U} &= U \end{aligned} \tag{4.2}$$

and thus the outer boundary conditions are:

$$\text{At} \quad r = \infty \qquad \vec{q} = \vec{\imath} \tag{4.3}$$

and the outer equations are given by setting $\partial/\partial x = 0$ in equations 2.22, 2.23, and 4.2.

The nose problem for the present example is much more difficult than the inner problem was in the preceding example. In the preceding problem it was seen that the flow near the body was quasi-steady. The corresponding simplification for the present problem would be to choose the nose variables such that the terms involving $\partial/\partial x$ were of higher order, that is for "x" to behave like a constant or to choose the nose variables equal to the inner variables. In the preceding example this simplification was justifiable for all time since near the body the inertial terms were small. We might use the same argument that near the body the transport terms are small but $\partial/\partial x$ also appears in the diffusion terms which dominate near the body. Thus we see that it is physically unrealistic to set $\partial/\partial x = 0$ near the body. Thus the nose variables for the present example are given by equations 2.19 and 4.2 and the Navier-Stokes equations in terms of the nose variables are given by equations 2.20 and 2.21, which for steady flow may be written:

$$\operatorname{div}^* \vec{q}_N = 0 \tag{4.4}$$

$$\nabla^{*2} \vec{q}_N - \operatorname{grad}^* P^* = R_e (\vec{q}_N \operatorname{grad}^* \vec{q}_N) \tag{4.5}$$

with boundary conditions:

$$\text{AT} \quad r^* = x^{*n} \qquad \vec{q}_N = 0 \tag{4.6}$$

We assume a nose expansion of the form:

$$\begin{aligned} \vec{q}_N &= \sum_{i=1}^{M} \vec{q}_i^{\,*}(x^*, r^*)\,\delta_i(R_e) \\ P^* &= \sum_{i=1}^{M} p_i^*(x^*, r^*)\,\delta_i(R_e) \end{aligned} \tag{4.7}$$

We shall see that $o\{\delta_M\} < o\{R_e\}$ where M is a finite integer. Thus substituting equation 4.7 into equations 4.4 through 4.6 and neglecting terms smaller than $\delta_M$ we obtain:

$$\begin{aligned} \operatorname{div}^* \vec{q}_i^{\,*} &= 0 \\ \nabla^{*2} \vec{q}_i^{\,*} - \operatorname{grad}^* p_i^* &= 0 \end{aligned} \tag{4.8}$$

with boundary conditions:

$$\text{at} \quad r^* = x^{*^n} \qquad \vec{q}_i^{\,*} = 0 \tag{4.9}$$

Thus solutions of equations 4. 8 and 4. 9 may be written as:

$$\vec{q}_i^{\,*} = C_i \, \vec{q}_1^{\,*} \tag{4.10}$$

$$p_i^* = B_i \, p_1^* \tag{4.11}$$

where $\vec{q}_1^{\,*}$ and $p_1^*$ are the solution of the three-dimensional Stokes equations for the flow about an arbitrary body of revolution. Thus the nose solution may be written as:

$$\begin{aligned} \vec{q}_N &= \vec{q}_1^{\,*} \sum_{i=1}^{M} C_i \, \delta_i(R_e) \\ P^* &= p_1^* \sum_{i=1}^{M} B_i \, \delta_i(R_e) \end{aligned} \tag{4.12}$$

Note that equations 4. 12 are the nose solution for all values of $n$. If we specialize to $n = 1/2$ $q_1^*$ and $p_1^*$ become the solution for the Stokes flow about a paraboloid of revolution.

For the case of the paraboloid, $n = 1/2$, the solution may be obtained immediately from the Oseen solution for the flow about a paraboloid of revolution which is reported in reference 3. Here we make use of the principles on which the present expansion procedure is based. Namely that the limit of the solution is the same as the solution of the approximate equations obtained by taking the limit of the exact equations. Thus the present derivation serves the secondary purpose of illustrating this principle. It is clear that application of the Stokes limit to the Oseen equations gives the Stokes equations. Thus if we apply the Stokes limit to the solution of the Oseen equations for the flow about the paraboloid of revolution we will obtain the solution of the Stokes equations for the flow about a paraboloid of revolution. The solution presented in reference 3 may be written as:

$$\vec{q} = \vec{i}\left(1 - \frac{\Gamma(0,\tilde{\eta}^2)}{\Gamma(0,\tilde{\eta}_b^2)}\right) + \frac{1}{\Gamma(0,\tilde{\eta}_b^2)}\,\tilde{\text{grad}}\left\{e^{-\tilde{\eta}_b^2}\ln\tilde{\eta}^2 + \Gamma(0,\tilde{\eta}^2)\right\}$$

$$\tilde{p} = \frac{e^{-\tilde{\eta}_b^2}}{\Gamma(0,\tilde{\eta}_b^2)}\frac{\partial}{\partial\tilde{x}}\ln\tilde{\eta}^2 \qquad (4.13)$$

In nose variables we have:

$$\vec{q} = \vec{i}\left(1 - \frac{\Gamma(0,\eta^{*2}R_e)}{\Gamma(0,\eta_b^{*2}R_e)}\right) + \frac{1}{R_e\Gamma(0,\eta_b^{*2}R_e)}\,\text{grad}^*\left\{\Gamma(0,R_e\eta^{*2}) + e^{-R_e\eta_b^{*2}}\ln\eta^{*2}R_e\right\}$$

$$p^* = \frac{e^{-\eta_b^{*2}R_e}}{\Gamma(0,R_e\eta_b^{*2})}\frac{\partial}{\partial x^*}\ln R_e\eta^{*2} \qquad (4.14)$$

Noting that for Re small:

$$e^{-\eta_b^{*2}R_e} \simeq 1 - R_e\eta_b^{*2}$$

$$\Gamma(0,\eta^{*2}R_e) \simeq \ln R_e - (\gamma + \ln\eta^{*2}) + \eta^{*2}R_e \qquad (4.15)$$

Thus taking the limit of $\vec{q}\,\epsilon_1$ as $Re \to 0$ we get for the nose solution:

$$\vec{q}_1^{\,*} = \vec{i}\, \ln \eta^{*2}/\eta_b^{*2} + \text{grad}^* \left[\eta^{*2} - \eta_b^{*2} \ln \eta^{*2}\right]$$

$$p_1^* = \frac{\partial}{\partial x^*} \ln \eta^{*2} \tag{4.16}$$

where:

$$\eta^{*2} = \frac{\sqrt{r^{*2} + (x^* - \eta_b^{*2})^2} - (x^* - \eta_b^{*2})}{2} \; ; \quad \eta_b^{*2} = 1/4$$

$$\delta_1 = \frac{1}{\ln Re} \tag{4.17}$$

Of course this solution can also be obtained directly from equations 4.8 and 4.9. We note that since the vorticity is normal to the velocity axial symmetry implies that the vorticity may be given by curl $(\vec{i}\,\chi)$. This implies that the velocity must be given by:

$$\vec{q}_1^{\,*} = \text{grad}^* \phi + \vec{i}\,\chi \tag{4.18}$$

Substituting equation 4.18 into the continuity equation we get:

$$\nabla^2 \phi = - \frac{\partial \chi}{\partial x^*} \tag{4.19}$$

and substitution of equation 4.18 into the momentum equation gives:

$$p_1^* = - \frac{\partial \chi}{\partial x^*} \tag{4.20}$$

$$\nabla^2 \chi = 0 \tag{4.21}$$

In parabolic coordinates equation 4. 20 takes a form which makes it possible to find solutions which are functions of $\eta^*$ only. From equation 4. 21 we get:

$$\frac{d}{d\eta^*}\,\eta^*\,\frac{d\chi}{d\eta^*} = 0 \qquad (4.22)$$

which has the solution:

$$\chi = A\left(\ln \eta^{*2} + B\right) \qquad (4.23)$$

or in parabolic coordinates equation 4. 19 gives:

$$\frac{d}{d\eta^*}\,\eta^*\,\frac{d\phi}{d\eta^*} = 2\eta^{*2}\frac{d\chi}{d\eta^*} = 4A\,\eta^* \qquad (4.24)$$

which has the solution:

$$\phi = A\left(\eta^{*2} + C\ln \eta^* + D\right) \qquad (4.25)$$

Thus from equation 4. 17:

$$\vec{q}_1^{\,*} = A\left\{\vec{i}\left(\ln \eta^{*2} + B\right) + \mathrm{grad}^*\left(\eta^{*2} + C\ln \eta^*\right)\right\} \qquad (4.26)$$

Applying the boundary conditions gives:

$$B = -\ln \eta_b^{*2} \quad ; \quad C = -2\eta_b^{*2} \qquad (4.27)$$

Absorbing the A into the $B_i$ and $C_i$ (see equation 4. 12) we obtain:

$$\vec{q}_1^{\,*} = \vec{i}\,\ln r^{*2}/r_b^{*2} + \text{grad}^*\left[r^{*2} - r_b^{*2}\ln r^{*2}\right]$$

$$p_1^* = -\frac{\partial}{\partial x^*}\ln r^{*2} \qquad (4.28)$$

which are identical with equations 4.16.

Although we can not in general neglect the derivative with respect to "x" if we consider the flow far downstream and take the inner limit we can neglect this derivative. We have called this the inner limit because of its analogy with the inner limit of reference 4 which described the flow in the wake behind a finite body. Note that the inner limit corresponds to the inner limit of the preceding example except that for the present example it is not uniformly valid in x. Thus using the inner solution of the preceding example as a guide we define the inner variables as $\tilde{x}$, $r^+$ and $p^+$ where $\tilde{x}$ is defined by equation 2.19 and

$$r^+ = R_e^n\, r^*$$
$$p^+ = R_e^n\, p^* \qquad (4.29)$$

where $r^*$ and $p^*$ are defined by equations 2.18. In terms of these co-ordinates the body is given by $r^+ = \tilde{x}^n$.

The inner solution is extremely simple even for the case of arbitrary n. The inner variables for this problem are given above and in terms of these variables the Navier-Stokes equations for axially symmetric flow are:

$$R_e^{1-n}\frac{\partial u}{\partial \tilde{x}} + \frac{1}{r^+}\frac{\partial (r^+ v)}{\partial r^+} = 0$$

$$\frac{\partial^2 u}{\partial r^{+2}} + \frac{1}{r^+}\frac{\partial u}{\partial r^+} - R_e^{2(1-n)}\left(\frac{\partial^2 u}{\partial \tilde{x}^2} + u\frac{\partial u}{\partial \tilde{x}}\right) - R_e^{1-n}\left(v\frac{\partial u}{\partial r^+} + \frac{\partial P^+}{\partial \tilde{x}}\right) = 0 \tag{4.30}$$

$$\frac{\partial^2 v}{\partial r^{+2}} + \frac{1}{r^+}\frac{\partial v}{\partial r^+} - \frac{v}{r^{+2}} - R_e^{2(1-n)}\left(\frac{\partial^2 v}{\partial \tilde{x}^2} + u\frac{\partial v}{\partial \tilde{x}}\right) - R_e^{1-n} v\frac{\partial v}{\partial r^+} - \frac{\partial P^+}{\partial r^+} = 0$$

with boundary conditions:

$$\vec{q} = \vec{i}u + \vec{l}_r v = 0 \qquad \text{AT} \quad r^+ = \tilde{x}^n \tag{4.31}$$

We assume an inner expansion of the form:

$$\vec{q} = \sum_{i=1}^{m} \vec{q}_i^{\,+}(\tilde{x}, r^+)\,\delta_i(R_e)$$

$$P = \sum_{i=1}^{m} p_i^+(\tilde{x}, r^+)\,\delta_i(R_e) \tag{4.32}$$

Again making use of the fact that $o\{\delta_m\} < o\{R_e\}$ where m is a finite integer we get after substituting equations 4. 32 into equations 4. 30 and 4. 31 and neglecting terms smaller than $\delta_m$ :

$$\frac{\partial r^+ v_i^+}{\partial r^+} = 0$$

$$\frac{\partial p_i^+}{\partial r^+} = 0 \tag{4.33}$$

$$\frac{\partial}{\partial r^+} r^+ \frac{\partial u_i^+}{\partial r^+} = 0$$

with boundary conditions:

at $r^+ = \tilde{x}^n \; ; \qquad u_i^+ = v_i^+ = 0$ (4. 34)

The solution to these equations is:

$$\vec{q} = \vec{i} \ln r^+/\tilde{x}^n \sum_{i=1}^{m} C_i^+ \delta_i(R_e)$$

$$p = 0 \tag{4. 35}$$

Note that for $n = 1/2$ if $C_i^+$ = const. equation 4. 35 could also have been obtained by taking the inner limit of equation 4. 12 where $\vec{q}_1^*$ and $p_1^*$ are given by equation 4. 28. The inner solution for arbitrary $n$ is given by equation 4. 35 where the $C_i^+(\tilde{x})$ and $\delta_i(R_e)$ are determined by matching.

IV. 3. Solution for $n = 1/2$

The inner and nose solutions are given by equations 4. 35 and 4. 12 respectively. The outer solution is valid for all $x$ and thus we begin by constructing a solution near the body which is also valid for all $x$. (This is possible since we know that the inner expansion is an asymptotic expansion of the nose expansion.) We define:

$$\mathrm{LIM}_{f_1} g \equiv \lim_{R_e \to 0} g(x_f, r_f, R_e) \tag{4. 36}$$

where:

$$r_{f_1} = \frac{r^+}{f_1^{1/2}} \; ; \qquad x_{f_1} = \frac{\tilde{x}}{f_1}$$

$$o\{R_e\} < o\{f_1(R_e)\} + o\{1\} \tag{4. 37}$$

Note that if $f_1 = 1$, $\mathrm{Lim}_{f_1} = \mathrm{Lim}_I$. It is easily verified that $\mathrm{Lim}_{f_1}$ of the Navier-Stokes equations for the present problem gives the inner equations. From which it follows that the matching condition between the inner and nose solutions may be written:

$$\mathrm{Lim}_{f_1} \frac{\vec{q}_N - \vec{q}_I}{\Gamma(R_e)} = 0 \tag{4. 38}$$

For $Re \to 0$, $r_{f_1}$ and $x_{f_1}$ fixed we have:

$$\eta^{*2} = r_{f_1}^2 / x_{f_1} + o\left\{\frac{Re}{f_1}\right\} \tag{4. 39}$$

Making use of equation 4. 39 and substituting equations 4. 12 and 4. 35 into equations 4. 38 we obtain:

$$\sum_{i=1}^{M} \frac{\delta_1^i \left[C_i \ln r_{f_1}^2/x_{f_1} - C_i^+(f_1 x_{f_1}) \ln r_f^2/x_{f_1} + o\left\{\frac{Re}{f_1}\right\}\right] + o\left\{\delta_1^{M+1}\right\}}{\Gamma_M(R_e)} = 0 \tag{4. 40}$$

Thus if $C_i^+(x)$ is regular at $x = 0$ equation 4. 40 implies:

$$\Gamma_M(R_e) = \delta_1^M(R_e) \tag{4. 41}$$

$$C_i = C_i^+(0) \tag{4. 42}$$

and thus the properly matched inner and nose solutions are:

$$\vec{q}_I = \vec{i} \ln r^{+2}/\tilde{X} \sum_{i=1}^{M} C_i^{+}(\tilde{X})\, \delta_1^{i}(R_e)$$

$$\vec{q}_N = \vec{q}_1^{*} \sum_{i=1}^{M} C_i^{+}(0)\, \delta_1^{i}(R_e) \qquad (4.43)$$

We construct the solution uniformly valid in x near the body from:

$$\vec{q}_{Bu} = \vec{q}_N + \vec{q}_I - \mathrm{LIM}_I\, \vec{q}_1^{*} \sum_{i=1}^{M} C_i^{+}(0)\, \delta_1^{i} \qquad (4.44)$$

which gives:

$$\vec{q}_{Bu} = \vec{q}_N + \vec{i} \ln r^{+2}/\tilde{X} \sum_{i=1}^{M} \left[C_i^{+}(\tilde{X}) - C_i^{+}(0)\right] \delta_i + o\{\delta_{M+1}\} \qquad (4.45)$$

From this we see that the nose solution is the uniformly valid solution near the body for all "x" to order $\delta_1^{M}$ only if $C_i^{+}(x)$ is a constant. In general we shall now match $\vec{q}_N$ with $\vec{q}_o$ to determine $C_i^{+}(0)$ and $\vec{q}_I$ with $\vec{q}_o$ to determine $C_i^{+}(x)$ and then construct the uniformly valid solutions from:

$$\vec{q}_{un} = \vec{q}_{Bu} + (\vec{q}_o - \vec{G}_{Bo}) \qquad (4.46)$$

where $\vec{G}_{Bo}$ is that portion of $\vec{q}_o$ also included in $\vec{q}_{Bu}$. Since for the present problem $C_i^{+}(0)$ can be determined from $C_i^{+}(x)$ it is sufficient to match $\vec{q}_I$ and $\vec{q}_o$ and then all three matched solutions can be combined into a uniformly valid solution. Note that when the inner limit is neglected we only determine $C_i^{+}$ and thus we also leave out part of the outer solution. The matching condition is:

$$\lim_{R_e \to 0} \left\{ \frac{\vec{q}_o(r_f, \tilde{x}, R_e) - \vec{q}_I(r_f, \tilde{x}, R_e)}{\Gamma(R_e)} \right\} = 0$$

$$r_f = \frac{r}{f(R_e)} \tag{4.47}$$

$$o\{R_e^{1-n}\} < o\{f\} < o\{1\}$$

If we drop the non-steady terms in equations 2.22 and 2.23 and set $\epsilon$ = Re we get as the outer equations for the present problem:

$$\tilde{\text{div}}\ \vec{q}_o = 0 \tag{4.48}$$

$$\vec{q}_o\ \tilde{\text{grad}}\ \vec{q}_o + \tilde{\text{grad}}\ P_o = \tilde{\nabla}^2 \vec{q}_o \tag{4.49}$$

with boundary conditions:

$$\text{as } \tilde{r} \to \infty, \quad \vec{q}_o \to \vec{i} \tag{4.50}$$

We assume an outer expansion of the form:

$$\vec{q}_o = \vec{i} + \sum_{i=1}^{M} \vec{q}_i(\tilde{x}, \tilde{r})\, \delta_i(R_e)$$

$$P_o = \sum_{i=1}^{M} \tilde{p}_i(\tilde{x}, \tilde{r})\, \delta_i(R_e) \tag{4.51}$$

Substituting equations 4.51 into equations 4.48 and 4.49 one obtains:

$$\tilde{\text{div}}\ \vec{q}_i = 0$$

$$\tilde{\nabla}^2 \vec{q}_i - \frac{\partial \vec{q}_i}{\partial \tilde{x}} - \tilde{\text{grad}}\ \tilde{p}_i = \vec{f}_i \tag{4.52}$$

where:

$$\vec{f}_i = \sum_{m=1}^{i-1} \vec{q}_m \, \tilde{grad} \, \vec{q}_{i-m}$$

with the boundary condition:

$$\text{as } \tilde{r} \to \infty \quad \vec{q}_i \to 0 \tag{4.53}$$

Note that these results are valid for arbitrary n. Thus the first order step-1 matching gives:

$$\lim_{R_e \to 0} \left[1 + C_i^+(\tilde{x})\,\delta_1(R_e) \ln R_e^{1/2}\right] = 0 \tag{4.54}$$

From which it follows that:

$$C_1^+(\tilde{x}) = -1 \tag{4.55}$$

and thus for $M = 1$: $\vec{q}_{Bu} = \vec{q}_N$. Thus the first order calculation of $\vec{q}$ proceeds exactly as if we had only used the nose limit. To determine the behavior of $\vec{g}_1$ near the origin we apply the first order step-2 matching which by the same reasoning that was used in the first example gives:

$$\lim_{\tilde{r} \to 0} \frac{(\vec{I} + \delta_1 \vec{g}_1) - \delta_1 C_1 \vec{g}_1^*}{\delta_1} \to \text{FINITE} \tag{4.56}$$

which implies that:

$$\lim_{\tilde{r} \to 0} \left\{ \vec{g}_1 + \vec{i} \ln \tilde{\eta}^2 \right\} \to \text{FINITE} \tag{4.57}$$

where:

$$\text{for } \tilde{r} \text{ small} \quad \tilde{\eta}^2 \cong \tilde{r}^2/\tilde{x} + \cdots \tag{4.58}$$

From equation 4.57 we conclude that $\vec{g_1}$ is the solution of equations 4.52 with $\vec{f_1} = 0$ which goes to $-\vec{i}\ln\tilde{\eta}^2$ as $\tilde{r} \to 0$. A solution of equations 4.52 expressed in parabolic coordinates is:*

$$\vec{g}_1 = \tilde{\text{grad}}\left[A\ln\tilde{\eta}^2 + B\Gamma(0,\tilde{\eta}^2)\right] - \vec{i}\,B\Gamma(0,\tilde{\eta}^2) \tag{4.59}$$

$$\tilde{p}_1 = A\frac{\partial}{\partial\tilde{x}}\ln\tilde{\eta}^2 \tag{4.60}$$

where:

$$\tilde{\eta} = R_e^{1/2}\eta^*$$

A, B are arbitrary constants (4.61)

Neglecting terms of order $Re^{1/2}$ and substituting equation 4.59 into equation 4.57 one obtains:

$$\tilde{\text{grad}}\left[(A-B)\ln\tilde{\eta}^2\right] - \vec{i}(B+1)\ln\tilde{\eta}^2 = 0 \tag{4.62}$$

From which we conclude:

$$A = B = -1 \tag{4.63}$$

and thus:

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* See reference 3.

$$\vec{q}_1 = \vec{i}\,\Gamma(0,\tilde{\eta}^2) - \tilde{\text{grad}}\left[\ln \tilde{\eta}^2 + \Gamma(0,\tilde{\eta}^2)\right]$$
$$p_1 = -\frac{\partial}{\partial \tilde{x}} \ln \tilde{\eta}^2 \qquad (4.64a)$$

The outer solution will not contain all of the terms in the inner solution and thus regardless of the number of terms taken the outer solution for this problem will not be uniformly valid to order 1. In this respect the present example is similar to the cross flow problem for the previous example. For $\tilde{\eta}$ small equation 4.64a becomes:

$$\vec{q}_1 = \vec{i}\,(\ln \tilde{\eta}^2 + \gamma) - \tilde{\text{grad}}\, \tilde{\eta}^2 + o\{\tilde{\eta}^2\} \qquad (4.64b)$$

Applying the second order step-1 matching by substituting equations 4.12 and 4.64 into equation 4.47 one obtains:

$$\lim_{R_e \to 0} \frac{1}{\delta_1}\Big\{\vec{i} + \delta_1\left[-\vec{i}\,(\ln \tilde{\eta}^2 + \gamma) - \tilde{\text{grad}}\, \tilde{\eta}^2\right] - \delta_1\Big[\vec{i}\,(\ln R_e^{1/2} - \ln \tilde{\eta}^2$$
$$+ \ln 1/4) - \tilde{\text{grad}}\, \tilde{\eta}^2 + \frac{R_e^{1/2}}{4\tilde{\eta}^2}\, \tilde{\text{grad}}\, \tilde{\eta}\Big] - \delta_2 C_2\, \vec{i} \ln R_e^{1/2}\Big\} = 0 \qquad (4.65)$$

which together with equation 4.55 implies:

$$C_2 = -\gamma + b_1 - \ln 1/4 \qquad (4.66)$$

It is clear from this that if we choose:

$$b_1 = \gamma - \ln 4 \equiv \ln \gamma_0/4 \qquad (4.67)$$

$C_2 = 0$. Thus we only need the first term of the inner expansion to take care of the linear portion of the solution. The second order step-2 matching is unnecessary and $\vec{g}_2$ is simply the solution due to the non-linear terms or:

$$\vec{g}_2 \equiv \vec{\beta}_2 = \int_v \underline{\underline{\Gamma_A}}(\tilde{x}_i, \xi_i)\vec{f}_2(\xi_i)\,d\xi_i \qquad (4.68)$$

$$\tilde{p}_2 = \int_v \vec{P}_A(\tilde{x}_i, \xi_i)\vec{f}_2(\xi_i)\,d\xi_i \qquad (4.69)$$

where $\underline{\underline{\Gamma_A}}$ and $\vec{P}_A$ are the fundamental solution of the steady axial symmetric Oseen equation and given by:*

$$\begin{aligned}
\underline{\underline{\Gamma_A}} &= \underline{\underline{I}}\,\nabla^2\Phi_A - \text{grad grad}\,\Phi_A \\
\vec{P}_A &= -\frac{2\sigma}{\pi}\,\text{grad}\left[F\left(\frac{\pi}{2}, \sqrt{\frac{2r\sigma}{(x-\xi_1)^2 + r^2 + \sigma^2 + 2r\sigma}}\right)\right] \\
\Phi_A &= \frac{\sigma}{\pi}\int_0^{2\pi}\int_0^{R/2 - \frac{x-\xi_1}{4}} \frac{1-e^{-\alpha}}{\alpha}\,d\alpha\,d\omega \qquad (4.70) \\
R &= \sqrt{(x_1-\xi_1)^2 + r^2 + \sigma^2 - r\sigma\cos\omega}
\end{aligned}$$

$F(\varphi, K)$ = elliptic integral of the first kind

However, the third order step-1 matching now becomes:

$$\lim_{R_e\to 0}\left\{\frac{\vec{i} + \delta_1\vec{g}_1 + \delta_1^2\vec{g}_2 - (-\delta_1\vec{g}_1^{\,+} + 0 + \delta_3 C_3^+(\tilde{x})\vec{g}_1^{\,+})}{\delta_1^2}\right\} = 0 \qquad (4.71)$$

and since for this case $\lim_{R_e\to 0}\vec{g}_2(\tilde{x}, r_f, R_e) = \vec{\beta}_2(\tilde{x}, 0) = a_2(\tilde{x})\vec{i}$ equation 4.71 becomes:

$$\lim_{R_e\to 0}\left[a_2(\tilde{x}) + C_3^+(\tilde{x})\,\frac{\delta_3}{\delta_1^2}\ln R_e^{1/2}\right] = 0 \qquad (4.72)$$

* See Appendix I.

Thus we conclude:

$$C_3^+(\tilde{X}) = -a_2(\tilde{X}) \quad ; \quad \delta_3 = \frac{\delta_1^2}{\ln R_e^{1/2} + b_3} \tag{4.73}$$

In carrying out the step-2 matching we use $\vec{q}_{Bu}$ as the inner solution and thus the third order step-2 matching becomes:

$$\lim_{\tilde{r}\to 0} \frac{1}{\delta_3}\Big\{(\vec{i} + \delta_1\vec{q}_1 + \delta_2\vec{q}_2 + \delta_3\vec{q}_3) - (-\delta_1\vec{q}_1^{\,*} + 0 - \delta_3 a_2(0)\vec{q}_1^{\,*} - \delta_3\vec{i}\,(\ln \tilde{r}^2/R_e\tilde{x})[a_2(\tilde{X}) - a_2(0)]\Big\} \to \text{FINITE} \tag{4.74}$$

which implies that:

$$\lim_{\tilde{r}\to 0}\Big\{\vec{q}_3 + \vec{i}\,a_2(0)[\ln \tilde{\eta}^2 - \ln \tilde{r}^2/\tilde{x}] + \vec{i}\,a_2(\tilde{X})\ln \tilde{r}\Big\} \to \text{FINITE} \tag{4.75}$$

Thus we conclude that $\vec{g}_3$ has a homogeneous component which goes to $-\vec{i}\,[a_2(0)\ln\tilde{\eta}^2 + (a_2(\tilde{x}) - a_2(0))\ln \tilde{r}\,]$ as $\tilde{r} \to 0$. Taking into account the non-homogeneous portion of $\vec{g}_3$ also we get:

$$\vec{q}_3 = \vec{\beta}_3(\tilde{X},\tilde{r}) + a_2(0)\vec{q}_1(\tilde{X},\tilde{r}) + \Omega_3(\tilde{X},\tilde{r}) \tag{4.76}$$

or continuing the procedure one finds in general:

$$\vec{q}_m = \vec{\beta}_m(\tilde{X},\tilde{r}) + a_{m-1}(0)\,q_1(\tilde{X},\tilde{r}) + \Omega_m(\tilde{X},\tilde{r}) \tag{4.77}$$

and thus making use of equation 4.46 we have in general for the uniform solution:

$$\vec{q}_{un} = \vec{i} + \sum_{i=1}^{M}\delta_1^i\Big[(\vec{q}_1 + \vec{\ell}_1)a_{i-1}(0) + \vec{\beta}_i + \vec{\Omega}_i\Big] + \vec{\ell}_\Omega \tag{4.78}$$

and:

$$P_{un} = \sum_{i=1}^{M} \tilde{p}_i \, \delta_1^{\,i} \qquad (4.79)$$

where:

$$\delta_1 = \frac{1}{\left(\ln \frac{\gamma_0 R_e^{1/2}}{4}\right)}$$

$$\vec{q}_1 = \vec{i}\,\Gamma(0,\tilde{\eta}^2) - \widetilde{\text{grad}}\left[\ln \tilde{\eta}^2 + \Gamma(0,\tilde{\eta}^2)\right]$$

$$\vec{\ell}_1 = -\eta_b^{*2}\,\text{grad}^*(\ln \eta^{*2})$$

$$\vec{i}\,a_i(\tilde{x}) = \vec{\beta}_i(0,\tilde{x}) \qquad (4.80)$$

$$a_0(\tilde{x}) = 1$$

$$a_1(\tilde{x}) = 0$$

$$\vec{\beta}_i = \int_v \underline{\underline{\Gamma_A}}(\tilde{x}_j, \xi_j)\,\vec{f}_i(\xi_j)\,d\xi_j$$

$$p_i = -a_i(0)\frac{\partial}{\partial \tilde{x}}\ln \tilde{\eta}^2 + p_{i\Omega} + \int_v \vec{P}_A(\tilde{x}_j, \xi_j)\cdot \vec{f}_i(\xi_j)\,d\xi_j$$

$\Omega(\tilde{x},\tilde{r})$; $p_{i\Omega}$ = solution of homogeneous Oseen equations which tends to

$$-\vec{i}\left[a_{i-1}(\tilde{x}) - a_{i-1}(0)\right]\ln \tilde{r} \quad \text{as } \tilde{r} \to 0$$

$\vec{\ell}_\Omega$ = portion of $\sum_{i=3}^{M} \delta_1^{\,i}\,\vec{\Omega}_i$ included in $\vec{q}_{Bu}$

$\underline{\underline{\Gamma_A}}$ ; $\vec{P}_A$ = fundamental solution of steady axial symmetric Oseen equations.

IV. 5. Solution for Arbitrary n.

In this section we shall discuss the extension of the previous results to arbitrary n. We showed in the previous section that the best way to carry out the expansion procedure is to consider all three limits. It has been pointed out that for arbitrary "n" the inner, nose and outer limits are still given by equations 4. 35, 4. 12 and 4. 51-4. 53 respectively. However $\vec{q}_1^*$ and $p_1^*$ are now the still unknown solutions to the three-dimensional steady Stokes equations about an arbitrary power law body of revolution. Thus the difficulty in obtaining the solutions for arbitrary "n" is not in the expansion procedure but in obtaining the appropriate solution of the Stokes equations. Since the nose equations include the inner equations and the nose and inner boundary conditions are the same at the body it follows that for arbitrary n:

$$LIM_{f_1} \; \vec{q}_1^* = \vec{i} \ln r_{f_1}^2 / x_{f_1}^{2n} \qquad (4.81)$$

Thus we see that equation 4. 45 is valid for arbitrary n provided we consider $\vec{q}_1^*$ as the Stokes flow about a body given by $r^+ = \tilde{x}^n$. Since the outer limit is still the free stream velocity the first order step-1 matching between the inner and outer limits will still give $C_1^+(\tilde{x}) = -1$. The first order step-2 matching will define $\vec{g}_1$ as the solution of the Oseen equations which cancels the unbounded terms in $\vec{q}_1^*(\tilde{r}, \tilde{x}, R_e)$ as $\tilde{r} \to 0$. At this point we can no longer follow the procedure for n = 1/2 identically since $\vec{g}_1$ and $\vec{q}_1^*$ may be of such a form that it is not possible to satisfy the second order step-1 matching with $C_2^+(\tilde{x}) = 0$. Thus for n arbitrary it may be necessary to represent $\vec{g}_2$ by:

$$\vec{q}_2 = \vec{\beta}_2 + C_2^+(0)\vec{g}_1 + \vec{\Omega}_2 \qquad (4.82)$$

where $\vec{\beta}_2$ is still given by equation 4.68 with the appropriate change in $f_2(\xi_i)$. $\Omega_2$ is the solution of the homogeneous Oseen equations which tends to $-\vec{\imath}\,[C_2(\tilde{x}) - C_2^+(0)]\ln\tilde{r}$ as $\tilde{r} \to 0$. Thus constructing the uniformly valid solution we get for arbitrary n:

$$\vec{q}_{un} = \vec{\imath} + \sum_{i=1}^{M} \delta_1^i\,[C_i^+(0)\vec{g}_1 + \vec{\beta}_i + \vec{\Omega}_i] + \vec{\ell}_m \qquad (4.83)$$

where:

$$\delta_1 = \frac{1}{\ln R_e^{1-n} + b_1}$$

$b_1$ = arbitrary constant chosen to simplify result

$C_1^+(\tilde{x}) = -1$

$C_i^+(\tilde{x})$ is a function determined by matching

$\vec{g}_1$ is the solution of the Oseen equations which cancels the unbounded terms in $\vec{q}_1^*(\tilde{r}, \tilde{x}, R_e)$ as $\tilde{r} \to 0$ (4.84)

$\vec{q}_1^*$ is the solution of the steady Stokes equation for the flow about a body given by $r^* = x^{*n}$.

$\vec{\beta}_i = \int_v \underline{\underline{\Gamma_A}}(x_j, \xi_j)\,\vec{f}_i(\xi_j)\,d\xi_j$

$\underline{\underline{\Gamma_A}}$ = fundamental solution of steady axial symmetric Oseen equation

$\vec{\Omega}_i$ = solutions of homogeneous Oseen equations which tend to $-[C_i^+(\tilde{x}) - C_i^+(0)]\ln\tilde{r}$ as $\tilde{r} \to 0$

$\vec{\ell}_M$ = portion of $\vec{q}_o$ included in $\vec{q}_{Bu}$

Examining this solution we see that there are three terms which are particularly difficult to evaluate. These terms are $\vec{q}_1^*$ , $\vec{q}_1$ and $\vec{\Omega}_i$ . However, if we recall the modification of the standard expansion procedure discussed in section (III. 6) we see that if $\vec{q}_1^*$ is obtained we may express $\vec{g}_1$ and $\Omega_i$ in an integral form. To do this we first calculate the drag on the body. The force $\vec{F}$ per unit area acting on the body is given by:

$$\vec{F} = \underline{\underline{P}}\ \vec{n} \tag{4.85}$$

where:

$$\begin{aligned} P_{ij} &= P\delta_{ij} + \epsilon_{ij} \\ \epsilon_{ij} &= \text{rate of strain tensor} \\ P &= \text{pressure} \end{aligned} \tag{4.86}$$

Thus:

$$\begin{aligned} F_{\tilde{x}} &= P \sin\theta + \epsilon_{\tilde{x}\tilde{r}} \cos\theta - \epsilon_{\tilde{x}\tilde{x}} \sin\theta \\ \theta &= \tan^{-1}\left(\frac{dr_b^*}{dx^*}\right) \end{aligned} \tag{4.87}$$

The radial forces do not contribute to the drag and thus the drag per unit length is:

$$C_D = \frac{2F_x}{\rho U^2} = 2\left[P \sin\theta + \epsilon_{\tilde{r},\tilde{x}} \cos\theta - \epsilon_{\tilde{x}\tilde{x}} \sin\theta\right]_{r^+ = \tilde{x}^n} \tag{4.88}$$

where:

$$\begin{aligned} \sin\theta &= \frac{n R_e^{1-n}}{\sqrt{\tilde{x}^{2(1-n)} + n^2 R_e^{2(1-n)}}} \\ \cos\theta &= \frac{\tilde{x}^{1-n}}{\sqrt{\tilde{x}^{2(1-n)} + n^2 R_e^{2(1-n)}}} \\ \epsilon_{\tilde{r},\tilde{x}} &= \frac{1}{2}\left[\frac{\partial u}{\partial \tilde{r}} + \frac{\partial v}{\partial \tilde{x}}\right] \quad ; \quad \epsilon_{\tilde{x}\tilde{x}} = \frac{\partial u}{\partial \tilde{x}} \end{aligned} \tag{4.89}$$

and thus neglecting transcendental terms and substituting equations 4. 45 and 4.89 into equation 4. 88 we get:

$$C_{D_i}(\tilde{r},\tilde{x}) = \delta_1^i \left\{ \left[ C_i^+(\tilde{x}) - C_i^+(0) \right] \frac{2}{\tilde{r}} + C_i^+(0) \left[ \frac{\partial u_1^*}{\partial \tilde{r}} + \frac{\partial v_1^*}{\partial \tilde{x}} \right] \right\} \delta\left(\tilde{r} - R_e^{1-n} \tilde{x}^n\right) \qquad (4.90)$$

and since this modification will only give those terms in the outer solution arising from the inner solution:

$$C_i^+(0)\vec{g}_1 + \vec{\Omega}_i = \int_0^\infty \int_0^\infty \underline{\underline{\Gamma_A}}(\tilde{r}, \sigma, \tilde{x} - \xi_1) . \vec{\lambda}\, C_{D_i}(\sigma, \xi_1)\, d\sigma\, d\xi_1 \qquad (4.91)$$

From which it follows that:

$$\vec{g}_1 = \int_0^\infty \left[ \frac{\partial u_1^*}{\partial \tilde{r}} + \frac{\partial v_1^*}{\partial \tilde{x}} \right]_{\substack{\tilde{x} = \xi_1 \\ \tilde{r} = R_e^{1-n} \xi_1^n}} \underline{\underline{\Gamma_A}}(\tilde{r}, R_e^{1-n}\xi_1^n, \tilde{x} - \xi_1) . \vec{\lambda}\, d\xi_1 \qquad (4.92)$$

$$\vec{\Omega}_i = 2 \int_0^\infty \left[ C_i^+(\xi_1) - C_i^+(0) \right] \frac{\underline{\underline{\Gamma_A}}(\tilde{r}, R_e^{1-n}\xi_1^n, \tilde{x} - \xi_1)}{\xi_1 R_e^{1-n}}\, d\xi_1 \qquad (4.93)$$

Note that equation 4. 93 is also useful for obtaining $\Omega_i$ for n = 1/2. Thus we see that the present expansion procedure has allowed us to reduce the problem for finding a solution of the Navier-Stokes equations for low Reynolds flow about the body to the problem of finding the Stokes flow about this body.

# V. BODY OF REVOLUTION FAR DOWN STREAM

## V. 1. Discussion

In the preceding section we saw that the major difficulty in obtaining the solution for the arbitrary body of revolution was due to the nose limit. Thus if one were just concerned with the flow far downstream one would expect to be able to obtain a simpler solution. We have seen that this is indeed the case since the inner limit is much simpler than the nose limit. In this section we shall discuss how this flow can be studied in a systematic way without using the nose limit. Since we wish to study the flow far downstream it seems natural to use a coordinate-type expansion. This is very similar to the problem considered in reference 4 where it was essential that the flow be studied without the benefit of the limit corresponding to the nose limit since the only "nose" limit which could be matched corresponded to the solution of the full Navier-Stokes equations. Thus we would expect the methods developed in reference 4 to apply to the present problem. In addition to expecting a simpler solution for this case we also expect that the coordinate-type expansion will be valid for all Re. This advantage occurring for infinite bodies was already pointed out in reference 1 where it was concluded that this made the coordinate-type expansion more desirable than the parameter-type expansion. The more recent studies of reference 4 have pointed out that in general both expansions may be useful since the coordinate-type expansion may also have some serious limitations, such as the existence of eigensolutions which are actually those portions of the solution which arise in the regions of the flow field where the coordinate-type expansion is not valid.

We are interested in the flow far downstream. Thus we shall expand for $x$ large. It is easily seen that for $x$ large the outer equations are still the Oseen equations since the body has zero arresting power. Restricting our attention to $n < 1/2$ the flow far downstream is very viscous and the Oseen equations are valid to the body. Thus the entire flow region being considered is governed by the Oseen equations. However, we note that the Oseen equations are actually valid at a large distance from the nose even if $x$ is small. Since the flow upstream has no solid boundary we would expect very little change if we consider the expansion for a large radial distance from the nose. This is the expansion that was considered in reference 4, and both for reference 4 and the present case the entire flow considered is governed by the Oseen equations to first order. Thus a straightforward linearization in terms of some small parameter would lead to an approximate solution of the Navier-Stokes equations valid for $x$ large. We may deal with a coordinate-type expansion. We can thus choose a proper form for the coordinate-type expansion and proceed to find the terms in this expansion. However to enable us to make use of the procedures developed for parameter-type expansions we shall use the method of the artificial parameters presented in reference 4. It should be emphasized that the artificial parameter is introduced for convenience only and in principle the expansion could be obtained without introducing this artificial parameter. In reference 4 it was shown that an artificial parameter is a parameter which can be eliminated from the problem by a suitable choice of variables. For a complete discussion of the artificial parameter the reader is referred to section 2.4 of reference 4. However, we shall summarize, without proof, some of the important properties of expansions in artificial parameters, obtained there:

1. The expansion in an artificial parameter is either non-uniform or the first term contains the exact solution.

2. Principle of eliminability--it must be possible to eliminate the artificial parameter from the solution by a proper choice of the variables.

3. The ordinary technique of parameter-type expansion leads to an indeterminacy. (That is, there exist eigensolutions which may be added without violating the boundary condition. Sometimes the proper eigensolutions may be selected by the use of integral laws.)

The first and third of these properties follow directly from the properties of a coordinate-type expansion. For the present problem we introduce some artificial length $R_o$. Our problem now has two independent parameters $\epsilon$ and Re where

$$\epsilon = \frac{L}{R_o} \tag{5.1}$$

We assume $\epsilon \ll 1$.

We have seen that the Oseen equations should govern the first order solution and thus we could in principle assume a linearization about the free stream in powers of $\epsilon$ and substitute this into equations 2.20-2.21 with $\partial/\partial t^* = 0$ and solve the resulting Oseen equations for the first order term. However, difficulty would arise in determining the eigensolutions. That is, we have seen that for an expansion in an artificial parameter the solution may not be unique and thus we must not only satisfy the equations and boundary conditions but also show that we have obtained all the possible solutions which satisfy the equations and boundary conditions. Even for the simplest case (the low Reynolds number limit) the solution for the first order term as outlined above is the still unsolved problem for the exact solution of the Oseen flow about an arbitrary power law body of revolution. Thus we see that if we are to obtain a solution it will be necessary to introduce still further simplifications. One such simplification would be to divide the problem into smaller parts as in the previous chapters by introducing inner and outer limits.

We know from the previous chapter that the Stokes equations with $\partial/\partial x = 0$ may be considered as the inner equations if we retain the Oseen equations as the outer equations. Although it should be possible in principle to obtain all of the eigensolutions by repeated differentiation of any one of the eigensolutions as was done in reference 4 the complexity of the Oseen equations makes this quite impractical. Thus we would like to simplify the outer equations still further. In reference 4 it was found sufficient to use the linearized Euler equations as the outer equations. However, the linearized Euler equations do not overlap with the Stokes equations. This is easily seen since neither the Euler or the Stokes equations are valid in that region of the flow where the diffusion and transport terms are of the same order. This difficulty may be overcome by introducing an intermediate limit. This intermediate limit is the same as the inner limit of reference 4 and is governed by the Oseen boundary-layer equations. It is clear that since the Stokes equations with $\partial/\partial x = 0$ (the inner equations) only include the diffusion normal to the free stream they must be included in the Oseen boundary-layer equations which include both the normal diffusion and the linearized transport terms. Thus the intermediate and inner limit not only have an overlap domain but we might completely dispense with the inner limit. However, the inner limit is not affected by the non-uniform region of the solution and thus none of the eigensolutions appear in it. Thus we may restrict the problem of generating the eigensolutions to the intermediate limit. There is little extra work in obtaining the inner solution and as we have seen in the preceding chapters there is always the possibility of less accuracy due to terms in the inner solution which may be transcendental in the other solutions and still appear in the "uniform" solution. Note that in the above statement "uniform" means the solution which is uniformly valid to some order in the entire region in which the coordinate-type expansion is expected to be valid! That is, we are using the methods of the preceding chapters to construct a uniform approximation to the solutions far from the nose but it is clear that this solution can only be valid in those regions where the "exact" solution is valid. The results of reference 4 indicate that the Oseen boundary layer equations should govern the intermediate limit. Since we are only considering the case of $n < 1/2$ the body grows slower than the viscous disturbance from the nose and furthermore the viscous region due to the afterbody lies within the viscous region due to the nose. Thus the outer limit for the present problem sees primarily the disturbance from the nose which is the same for a finite or semi-finite body. A second similar reasoning is that for $n < 1/2$ the entire disturbance due to the body looks like a parabolic wake for which the boundary layer approximation holds. But since we are only considering the region where the Oseen equations are valid we should apply this boundary layer approximation to the Oseen equations which gives the Oseen boundary-layer equations. However, as in reference 4 the Oseen boundary-layer equations can be matched with the linearized Euler equations. Since sufficiently far from the body the diffusion terms are not important it is clear that the linearized Euler equations do represent an outer limit. Further if we make use of the principle of rapid decay of vorticity, see reference 4, we see that the outer flow is potential.

Before obtaining the solution for $n < 1/2$ we shall consider the flow far downstream for other values of $n$. For $n > 1/2$ the viscous layer is very thin compared to the body diameter and thus the boundary-layer equations are valid near the body. It is clear that since we are far from the nose the curvature effects will be negligible and thus the boundary layer is a two-dimensional boundary layer over a flat surface. However we may have an external pressure gradient. Since the external or outer flow is the potential flow about a power law body of revolution it is also known in principle. Since this is simply an example of classical boundary layer theory we shall not consider it further here. For $n = 1/2$

both the body and the viscous region grow at the same rate and thus the character of the flow is only dependent on the Reynolds number. Thus the flow far downstream will behave similar to that for $n < 1/2$ for low Reynolds number and similar to that for $n > 1/2$ for large Reynolds numbers. This case is more suitable for a parameter-type expansion and we shall not consider the coordinate-type expansion for $n = 1/2$.

## V. 2. Solution valid far down stream.

We shall now apply the method of an expansion in terms of an artificial parameter to determining the flow far downstream about a body given by $r = Ax^n$; $0 \le n < 1/2$. We have introduced the artificial length, $R_o$, into the problem and thus the interesting dimensionless parameters are:

$$\epsilon\ ;\ R_e\ ;\ R_{e_{R_o}} = \frac{U R_o}{\nu} \tag{5.2}$$

Only two of these parameters are independent since they are related by:

$$R_e = \epsilon R_{e_{R_o}} \tag{5.3}$$

The characteristic length $L$ is defined by equation 2.17 as in the previous problem. We choose as independent variables for the inner problem $r^+$ and $x^+$ where:

$$r^+ = \tilde{r}/\epsilon^{1-n}\ ;\qquad x^+ = \tilde{x} \tag{5.4}$$

and $\tilde{r}$, $\tilde{x}$, $r^*$ and $x^*$ are still defined by equations 2.18-2.19. We shall choose $\epsilon$ and $R_{e_{R_0}}$ as the parameters for the inner problem. In reference 4 it was also found convenient to distort the dependent radial velocity so that the axial and radial velocities in the inner expansion would be of the same order. However, for the present problem we shall see that since the expansion is valid to a lower order than that obtained in reference 4 the radial velocity is of transcendental order with respect to the expansion and thus there is no advantage in distorting it in the present considerations. Thus the inner equations become:

$$\frac{1}{r^+}\frac{\partial r^+ v^*}{\partial r^+} = \epsilon^{1-n}\frac{\partial u^*}{\partial \tilde{x}}$$

$$\frac{1}{r^+}\frac{\partial}{\partial r^+} r^+ \frac{\partial u^*}{\partial r^+} = \epsilon^{2-2n}\left[-\frac{\partial^2 u^*}{\partial \tilde{x}^2} + R_{e_{R_0}}\left(u^*\frac{\partial u^*}{\partial \tilde{x}} + \frac{\partial \tilde{p}}{\partial \tilde{x}}\right)\right] + \epsilon^{1-n} v^* \frac{\partial u^*}{\partial r^+} \tag{5.5}$$

$$\frac{1}{r^+}\frac{\partial}{\partial r^+} r^+ \frac{\partial v^*}{\partial r^+} - \frac{v^*}{r^{+2}} = \epsilon^{2-2n}\left[-\frac{\partial^2 v^*}{\partial \tilde{x}^2} + R_{e_{R_0}} u^* \frac{\partial v^*}{\partial \tilde{x}}\right] + \epsilon^{1-n} R_{e_{R_0}}\left(v^*\frac{\partial v^*}{\partial r^+} + \frac{\partial \tilde{p}}{\partial r^+}\right)$$

with the boundary conditions:

$$\text{at} \quad r^+ = \tilde{x}^n \qquad \vec{q} = 0 \tag{5.6}$$

However, we shall see that terms of order $\epsilon^\alpha$, $\alpha > 0$, are of transcendental order with respect to this expansion and thus the entire inner expansion is given by:

$$\frac{1}{r^+}\frac{\partial r^+ v^*}{\partial r^+} = 0$$

$$\frac{1}{r^+}\frac{\partial}{\partial r^+} r^+ \frac{\partial u^*}{\partial r^+} = 0 \tag{5.7}$$

$$\frac{1}{r^+}\frac{\partial}{\partial r^+} r^+ \frac{\partial v^*}{\partial r^+} - \frac{v^*}{r^{+2}} = 0$$

from which we conclude that the inner expansion may be expressed as:

$$\tilde{P} = v^* = 0 \tag{5.8}$$

$$u^* = \ln \frac{r^+}{\tilde{x}^n} \sum_{i=1}^{M} \delta_i(\epsilon, \tilde{x})\, C_i(\tilde{x}) \tag{5.9}$$

We now make use of the principle of eliminability which states that since $R_o$ is artificial, equations 5.8 and 5.9 expressed in terms of $x^*$ and $r^*$ must be independent of $R_o$. This can only be satisfied if:

$$\begin{aligned} \delta_i(\epsilon, \tilde{x}) &= \delta_i(\tilde{x}/\epsilon) \\ C_i(\tilde{x}) &= \text{CONST} = C_i \end{aligned} \tag{5.10}$$

We also note the requirement that:

$$o\{\delta_{i+1}\} < o\{\delta_i\} \tag{5.11}$$

Now we consider the intermediate solution: We choose as independent variables for the intermediate problem $\bar{r}$ and $\bar{x}$ where:

$$\bar{r} = \tilde{r}/\epsilon^{1/2} \quad ; \qquad \bar{x} = \tilde{x} \tag{5.12}$$

and $\epsilon$ and Re as the parameters. Again we shall find that the radial velocity is transcendental and thus shall not distort the dependent variables. Thus the intermediate equations become:

$$\frac{1}{\bar r}\frac{\partial \bar r v^*}{\partial \bar r} + \epsilon^{1/2}\frac{\partial u^*}{\partial \tilde x} = 0$$

$$\frac{1}{\bar r}\frac{\partial}{\partial \bar r}\bar r\frac{\partial u^*}{\partial \bar r} - R_e\left(u^*\frac{\partial u^*}{\partial \tilde x} + \frac{\partial \tilde P}{\partial \tilde x} - \frac{v^*}{\epsilon^{1/2}}\frac{\partial u^*}{\partial \bar r}\right) = \epsilon\frac{\partial^2 u^*}{\partial \tilde x^2} \tag{5.13}$$

$$\frac{\partial \tilde P}{\partial \bar r} = -v^*\frac{\partial v^*}{\partial \bar r} + \epsilon^{1/2}\left[-u^*\frac{\partial v^*}{\partial \tilde x} + \frac{1}{R_e}\left(\frac{1}{\bar r}\frac{\partial}{\partial \bar r}\bar r\frac{\partial v^*}{\partial \bar r} - \frac{v^*}{\bar r^2}\right)\right] + \frac{\epsilon^{3/2}}{R_e}\frac{\partial^2 v^*}{\partial \tilde x^2}$$

Note that there are no boundary conditions on this solution since it matches with the inner solution for $\bar r$ small and with the outer solution for $\bar r$ large. Since there are no solid boundaries in this region we expect the solution to be a perturbation of the free stream velocity and thus we assume an expansion of the form:

$$u^* = 1 + \sum_{i=1}^{M}\bar u_i(\tilde x, \bar r)\delta_i(\tilde x/\epsilon)$$

$$v^* = \epsilon^{1/2}\sum_{i=1}^{M}\bar v_i(\tilde x, \bar r)\delta_i(\tilde x/\epsilon) \tag{5.14}$$

$$\tilde P = \sum_{i=1}^{M}\bar p_i(\tilde x, \bar r)\delta_i(\tilde x/\epsilon)$$

Substituting equations 5.14 into equations 5.13 and neglecting terms of transcendental order one obtains:

$$\frac{1}{\bar r}\frac{\partial \bar r \bar v_i}{\partial \bar r} + \frac{\partial \bar u_i}{\partial \tilde x} = 0 \tag{5.15}$$

$$\frac{1}{\bar r}\frac{\partial}{\partial \bar r}\bar r\frac{\partial \bar u_i}{\partial \bar r} - R_e\frac{\partial \bar u_i}{\partial \tilde x} = f_i(\bar r, \tilde x) \tag{5.16}$$

$$\frac{\partial \bar p_i}{\partial \bar r} = 0 \tag{5.17}$$

Equation 5.17 implies, as in classical boundary layer theory, that the pressure is given by the outer flow and thus is a known function of x, r at each order. Thus it follows that $f_i$ is also a known function and given by:

$$f_i(\bar{r},\tilde{x}) = R_e \frac{\partial \bar{P}_i}{\partial \tilde{x}} + R_e \sum_{j=1}^{i-1} \left[ \bar{u}_j \frac{\partial \bar{u}_{i-j}}{\partial \tilde{x}} + \bar{v}_j \frac{\partial \bar{u}_{i-j}}{\partial \bar{r}} \right] \tag{5.18}$$

Thus we see from equation 5.18 that even though the radial velocity is itself of transcendental order in the intermediate expansion its influence is not and thus we must calculate the radial velocity in order to obtain the higher order terms. Since equation 5.16 is the heat equation we may express $\bar{u}_i$ as:

$$\bar{u}_i = \bar{u}_{i,a} + \bar{u}_{i,b} \tag{5.19}$$

where:

$$\bar{u}_{i,b} = \int_{-\infty}^{\infty}\int_{0}^{\infty} S_x(x-\xi_1, \bar{r}, \sigma) f_i(\xi_1,\sigma)\, d\sigma\, d\xi_1 \tag{5.20}$$

where $\bar{u}_{ia}$ contains the eigensolutions and the homogeneous solution of equation 5.16 which matches the innersolution and $S_x$ is the fundamental solution of the axial symmetric heat equation and is given by:*

$$S_x = \frac{R_e\, e^{-R_e \frac{r^2+\sigma^2}{4(\tilde{x}-\xi_1)}}}{2(\tilde{x}-\xi_1)} I_0\left(\frac{R_e\, r\sigma}{4(\tilde{x}-\xi_1)}\right)\sigma \tag{5.21}$$

$$\sigma^2 = \xi_2^2 + \xi_3^2$$

*See Appendix 1.

The problem for $\overline{u}_{ia}$ is:

$$\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\bar{r}\frac{\partial \overline{u}_{i,a}}{\partial \bar{r}} - R_e \frac{\partial \overline{u}_{i,a}}{\partial \bar{x}} = 0 \tag{5.22}$$

But because of eliminability we know the solution must have the following similarity:

$$\overline{u}_{i,a} = \overline{u}_{i,a}(\eta) \quad ; \qquad \eta = \bar{r}/\sqrt{\bar{x}} \tag{5.23}$$

Substituting equation 5.23 into equation 5.22 we get:

$$\frac{d^2 \overline{u}_{i,a}}{d\eta^2} + \left(\frac{1}{\eta} + \frac{\eta R_e}{2}\right)\frac{d\overline{u}_{i,a}}{d\eta} = 0 \tag{5.24}$$

which has the solution:

$$\overline{u}_{i,a} = A_{1,i} - A_{2,i}\,\Gamma\left(0, \frac{R_e \bar{r}^2}{4\bar{x}}\right) \tag{5.25}$$

where $A_{1i}$ and $A_{2i}$ are arbitrary constants. Thus the solution for $\overline{u}_i$ is:

$$\overline{u}_i = A_{1,i} - A_{2,i}\,\Gamma\left(0, \frac{R_e \bar{r}^2}{4\bar{x}}\right) + \overline{u}_{i,b}\left(\bar{r}/\sqrt{\bar{x}}\right) \tag{5.26a}$$

as $\bar{r} \to 0$ it is clear from equation 5.21 that the last term of equation 5.26a is at most a function of $x$ but by the eliminability requirement it is clear that this must be a constant and thus for $\bar{r}$ small equation 5.26a may be written:

$$\overline{u}_i = A_{1,i} + A_{2,i} \ln \frac{\gamma_0 R_e \bar{r}^2}{4\tilde{x}} + \overline{u_{i,b}}(0) \tag{5.26b}$$

Noting that the last term of equation 5.26a is zero for $\bar{r} \to \infty$ we get for $\bar{r}$ large:

$$\overline{u}_i \simeq A_{1,i} \tag{5.26c}$$

It is clear that since both the inner and outer equations are included in the intermediate equations we may define an overlap domain between the inner and intermediate and the intermediate and outer solutions. Thus we may carry out the matching as in the previous sections. The first order step-1 matching between the inner and intermediate solutions gives:

$$\lim_{R_e \to 0} \left\{ 1 + \delta_1\left(\frac{\tilde{x}}{\epsilon}\right)\left(A_{1,1} + A_{2,1} \ln \frac{\gamma_0 \bar{r}^2 R_e}{4\tilde{x}} + \overline{u}_{1,b}(0)\right) - \delta_1\left(\frac{\tilde{x}}{\epsilon}\right) C_1 \ln \frac{\bar{r}}{\tilde{x}^n \epsilon^{1/2-n}} \right\} \to 0 \tag{5.27}$$

Thus:

$$\delta_1\left(\frac{\tilde{x}}{\epsilon}\right) = \frac{1}{\ln\left(\frac{\tilde{x}}{\epsilon}\right)^{1/2-n} + b_1} \quad ; \quad C_1 = 1 \tag{5.28}$$

and the step-2 matching gives:

$$\lim_{\bar{r} \to 0} \frac{1 + \delta_1\left(\frac{\tilde{x}}{\epsilon}\right) 2 A_{2,1} \ln \bar{r} - \delta_1\left(\frac{\tilde{x}}{\epsilon}\right)\left[\ln \bar{r} - \ln \epsilon^{1-2n}\right]}{\delta_1} \to \text{FINITE} \tag{5.29}$$

or:

$$A_{2,1} = \frac{1}{2} \tag{5.30}$$

From the second order step-1 matching we conclude:

$$C_2 = (A_{1,1} + \overline{u}_{1,b}(0)) \quad ; \quad b_1 = \ln \frac{\sqrt{\gamma_0 R_e}}{2} \tag{5.31}$$

and it is clear that in general:

$$C_i = \left[A_{1,i-1} - \overline{u}_{i-1,b}(0)\right] , \quad i > 1 \quad ; \delta_i = \delta_1^i = \left[\ln \frac{\sqrt{\gamma_0 R_e}}{2} \left(\frac{\tilde{x}}{\epsilon}\right)^{1/2 - n}\right]^{-1} \tag{5.32}$$

$$A_{2,i} = \frac{C_i}{2} \tag{5.33}$$

and thus the inner solution is given by equations 5.8, 5.9, 5.28 and 5.32. It is easily verified that the entire inner solution is contained within the intermediate solution to the same order and thus the intermediate solution represents a "uniformly" valid approximation clear to the body and is given by:

$$\overline{u}_i = A_{1,i} + \frac{1}{2}\left[A_{1,i} + \overline{u}_{i-1,b}(0)\right] \Gamma\left(0, \frac{R_e \bar{r}^2}{4\tilde{x}}\right) + \overline{u}_{i,b}(\bar{r}/\sqrt{\tilde{x}}) \tag{5.34}$$

and equations 5.14 and 5.32. The solution for $\overline{v}_i$ is obtained immediately from equation 5.15 and in terms of $\overline{u}_i$ is:

$$\overline{v}_i(\bar{r}/\sqrt{\tilde{x}}) - \overline{v}_i(0) = \frac{1}{\bar{r}} \int_0^{\bar{r}} \bar{r} \, \frac{\partial \overline{u}_i(\bar{r}/\sqrt{\tilde{x}})}{\partial \tilde{x}} \, d\bar{r} \tag{5.35}$$

But $v \sim o\{\epsilon^{1-n}\}$ in the inner solution and thus $\overline{v}_i(0) \sim o\{\epsilon^{1/2-n}\}$ and is transcendental with respect to the present expansion and thus we take

$\overline{v}_i(0) = 0$ in equation 5. 35.

For the outer variables we choose $\tilde{x}$ and $\tilde{r}$ and the parameters $\epsilon$ and Re. Thus the outer equations are given by equation 2. 22 and 2. 23. Assuming an expansion of the form:

$$\vec{q} = \vec{i} + \sum_{i=1}^{M} \vec{\tilde{q}}_i \delta_i(\epsilon)$$
$$\tilde{p} = \sum_{i=1}^{M} \tilde{p}_i \delta_i(\epsilon) \tag{5. 36}$$

the outer equations become:

$$\operatorname{div} \vec{\tilde{q}}_i = 0$$
$$\frac{\partial \vec{\tilde{q}}_i}{\partial \tilde{x}} + \widetilde{\operatorname{grad}}\, \tilde{p}_i = \sum_{j=1}^{i-1} \vec{\tilde{q}}_j \, \widetilde{\operatorname{grad}}\, \vec{\tilde{q}}_{i-j} \tag{5. 37}$$

However, the principle of rapid decay of vorticity leads to the conclusion that the outer flow must be irrotational and thus the outer solution is potential to the first order. However, in reference 4 we note that $\vec{q}_o = \vec{i}$ to order $\epsilon$ and since $o\{\epsilon\}$ is transcendental with respect to the present expansion and since we expect the outer solution to be similar to the problem in reference 4 we would expect $\vec{\tilde{q}}_i = 0$ and $\tilde{p}_i = 0$. This is verified by the matching between the intermediate and the outer expansions which gives:

$$\vec{q} = \vec{i} + o\{\epsilon\} \quad ; \quad \tilde{P} = o\{\epsilon\} \tag{5. 38}$$

and that:

$$A_{1,i} = 0 \tag{5. 39}$$

Equation 5. 38 is exactly the same result as obtained in reference 4. It is clear that the entire outer solution is also contained within the intermediate solution and thus the intermediate solution represents the "uniformly" valid solution for the problem. This gives as the "uniformly" valid expansion to order $\left(\frac{1}{\ln\epsilon}\right)^{M}$:

$$v = \widehat{P} = 0 \tag{5.40}$$

$$u = 1 + \sum_{i=1}^{M} \bar{u}_i\left(\bar{r}/\sqrt{\tilde{x}}\right)\left(\frac{1}{\ln\frac{\sqrt{\gamma_0}R_e}{2}\left(\frac{\tilde{x}}{\epsilon}\right)^{1/2-n}}\right)^{i}$$

where:

$$\tilde{u}_i\left(\bar{r}/\sqrt{\tilde{x}}\right) = \frac{u_{i-1,b}(0)}{2}\Gamma\left(0, \frac{R_e\bar{r}^2}{4\tilde{x}}\right) + \bar{u}_{i,b}\left(\bar{r}/\sqrt{\tilde{x}}\right); \quad i>0$$

$$\bar{u}_0 = 1$$

$$\bar{u}_{i,b}\left(\bar{r}/\sqrt{\tilde{x}}\right) = \int_{-\infty}^{\infty}\int_{0}^{\infty} S_x(\tilde{x}-\xi_1, \bar{r}, \sigma)\, f_i(\xi_1, \sigma)\, d\sigma\, d\xi_1 \tag{5.41}$$

$$f_i(\tilde{x}, \bar{r}) = R_e\sum_{j=1}^{i-1}\left[\bar{u}_j\frac{\partial\bar{u}_{i-j}}{\partial\tilde{x}} + \left(\frac{1}{\bar{r}}\int_0^{\bar{r}}\bar{r}\frac{\partial\bar{u}_j}{\partial\tilde{x}}d\bar{r}\right)\frac{\partial\bar{u}_{i-j}}{\partial\bar{r}}\right]$$

$S_x$ = fundamental solution of axial symmetric heat equation.

It is clear that since the intermediate solution completely contained all the terms of both the inner and outer solutions to the same order both the inner and outer solutions could have been omitted for this case and the matching conditions replaced by approximate boundary conditions. However, as we have pointed out in previous sections this is not true in general and can not be predicted a priori. It is clear that this problem was a relatively simple problem. We were able to predict simplicity before we started since we have avoided the most difficult portion of the flow field, namely the flow near the nose, and the only other flow near a solid boundary is of a very viscous type which is usually relatively simple. Although we would expect the present problem to be slightly more difficult than that of reference 4, due to the existence of a solid boundary in the region of interest, it is clear by comparing the order of validity of the present solution with that of reference 4 that the entire expansion obtained for the present problem is only an approximation to the first term in the expansion obtained in reference 4! Since the two significant difficulties of switch-back and indeterminacy which occurred in reference 4 occurred in terms of higher order than $\epsilon$ they do not occur in the present analysis. Thus we see that the indeterminacy which we predicted in studying the coordinate-type expansion in general does not occur in the present problem only because we have not obtained the solution to a sufficiently high order. Note that this same discussion applies equally well to the coordinate-type solution presented in reference 1. The coordinate-type expansion obtained in reference 1 is analogous to that obtained in this section. That is, it is reasonable to expect the indeterminacy to occur at a higher order than the highest order term in the expansion. This is easily seen from the discussion in section IV. 2 where it was pointed out that the effects of the "nose" region (i. e. $\tilde{t} \sim 0$) were much smaller for the problem considered in reference 1 than in the present problem. Since we have shown by comparing the present problem with that of reference 4 that we do not expect the indeterminacy due to the effects of the nose region to occur until algebraic orders it follows that these effects would not occur until at least algebraic orders in reference 1 also. We note that the coordinate-type expansion obtained in reference 1 was obtained by two different methods neither of which was the method of artificial parameters. It is clear that the same expansion could also have been obtained following the method illustrated here. This illustrates the point that the artificial parameter is introduced merely for convenience in order to allow us to use the methods developed for parameter-type expansions for obtaining coordinate-type expansions.

# VI. CONCLUDING REMARKS

In this section we shall briefly review some of the most important properties of the expansion procedure and of low Reynolds number flow which were illustrated in this study. From some of these properties we shall draw some additional conclusions which are beyond the requirements of this study. Finally we shall consider some related problems which might be solved by the methods illustrated here.

## VI. 1. Expansion Procedures

In the examples considered we have found that the standard expansion procedure is quite adequate for reducing the singular perturbation problem to a series of simpler problems. It was also found that if the inner equations were included within the outer equations one could always find an integral representation for a portion of the outer solution in terms of the fundamental solution of the outer equations by modifying the standard procedure slightly. In essence this modification amounted to replacing the step-2 matching by the condition that the distribution of fundamental singularities for the outer solution was determined by the inner solution for the homogeneous terms of the outer solution. This modification proved very useful in obtaining integral expressions for terms, where the problem in terms of the differential equation given by the standard procedure was extremely difficult to solve. It seems that this will be true in general especially for more difficult problems. This modification also showed that for any problem for which the Oseen equations represent the low Reynolds number limit of the Navier-Stokes equations one can always obtain an integral expression for a solution of the Oseen equations if the Stokes solution is known for the problem. This solution of the Oseen equations will satisfy the approximate boundary conditions consistent with the approximation to the Navier-Stokes equations which the Oseen equations represent. Thus we see that even though the Stokes solution does not represent the low Reynolds number limit of the Navier-Stokes equations if the Oseen equations do represent this limit it can be obtained from the Stokes solution.

A second modification which can be applied whenever the equations for one solution include the equations for another solution is to neglect the included solution. However it was seen that in general when this procedure was used the uniformly valid solution was less accurate and the cases where this was not true could not in general be determined a priori. This procedure is not recommended especially since the neglected problem is usually simpler than the retained problem. In addition to the gain in accuracy resulting when one retains all the possible solutions one gets a better insight into the problem and is less likely to make errors.

The comparison of the solution of the second example for the flow far downstream with the solution of reference 4 aids considerably in understanding the relationship between references 1 and 4. Here we saw that the reason that the difficulties which arise in reference 4 did not occur in reference 1 or the present study was entirely due to the fact that these latter solutions were not valid to the order at which these difficulties were to be expected. Thus one of the reasons that the coordinate-type expansion seemed much more satisfactory in reference 1 was that these difficulties which may occur for coordinate-type expansions did not materialize. However a more fundamental reason for this apparent overwhelming advantage of the coordinate-type expansion appearing in reference 1 is due to the fact that the Reynolds number could be eliminated from the problem. Thus we see from the discussion of the artificial parameter given in reference 4 that the Re was an artificial parameter in reference 1 and both the expansions obtained there were coordinate-type expansions. The principle of eliminability was not applied in obtaining the solution in the artificial parameter and the resulting expansion which was presumed typical of parameter-type expansions had the disadvantages of both the parameter and the coordinate-type expansion. This situation is clarified by the present examples from which it is clear that a coordinate-type expansion is valid for all values of the parameter, but only a certain region of the space, while a parameter-type expansion is valid for all space but only for certain values of the parameter. However the coordinate-type expansion may also have the disadvantage of indeterminacy. On the other hand in the present example the coordinate-type expansion led to a simpler problem and the terms of the expansion could be found in closed form for $0 \leq n < 1/2$ which was not true for the parameter-type solution. Thus we see that the choice between coordinate and parameter-type expansion depends on the problem being considered.

VI. 2. Low Reynolds Number Flow

In this study we have argued that the most meaningful concept of low Reynolds number flow is the limit of any given problem as the characteristic length of the problem tends to zero. Assuming that the body tends to a unique body as its characteristic length tends to zero the present concept has the advantage that the low Reynolds number flow defined has the same uniqueness and existence properties as the original flow. Further this definition assures that the low Reynolds number limit is a uniform limit. This latter statement is not true in general if either Oseen or Stokes flows are considered as the low Reynolds number limit since in general they are non-uniform at the body and infinity respectively. Because of the almost universal identification of Stokes and Oseen flows with low Reynolds number flows the major emphasis given this definition here seems justified. However, it is clear that both the Stokes and Oseen approximations are closely related to low Reynolds number flow. In particular the conditions for the Oseen equations to represent the low Reynolds number limit of the Navier-Stokes equations are given in section II. 2. From these conditions we concluded that the Oseen equations are the low Reynolds number limit for the flow about a paraboloid of revolution. It has been pointed out that another consequence of the present concept is that the low Reynolds number limit of the Navier-Stokes equations only has meaning for a specific problem. This follows from the fact that the characteristic length can only be defined if the problem is specified.

Another important concept in considering low Reynolds number flow is the Oseen criticism. This was originally given as an explanation of the failure of the Stokes equations as a low Reynolds number approximation. However, in this study we have found it very useful in trying to decide what approximation might lead to some simplification. The Oseen criticism is simply that there exists some point in the flow at which some other term is of the same order as the diffusion term. In general an approximation will lead to a significant simplification only if the dominant term exclusive of the diffusion terms is linear. One case when the Oseen criticism does not apply is when all of the transport and inertial terms are <u>identically</u> zero.

## VI. 3. Related Problems

It would be useful in obtaining the overall picture of the low Reynolds number flow to solve the present problems considering some of the features which have been removed for simplicity. For example it would be of interest to consider the effects of compressibility. The effects of compressibility at low Reynolds number are discussed in reference 6. However this discussion is not complete and there remain many unanswered questions in the area of compressible low Reynolds number flow. Another extension of the examples which might prove interesting and could be handled by the methods discussed here would be in the extension to an arbitrary rate of growth or an arbitrary body of revolution. This is discussed in considerable detail in reference 1 for the problem considered there but the discussion applies equally well to the problems considered in the present study. It would also prove interesting to consider the solution of the first example for $t$ large by the method of artificial parameters. This should be a relatively simple problem and be very similar to Chapter V of the present study. For the second example one might also consider the case of non-zero angle of attack; however, since the axial and cross-flows do not separate in this case the problem will be considerably more difficult than the first example. In either problem considered here it should not be difficult to extend the results to a non-circular cross section. For the second example the first order terms for the case of a non-circular cross section are again given in reference 3 for $n = 1/2$.

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## APPENDIX I. REVIEW OF THE FUNDAMENTAL SOLUTION

The fundamental solution, S, of the differential operator L(q) is defined as the response of the operator to an impulsive force or:

$$L(S) = \delta(P-Q) \tag{I. 1}$$

where:

$$\delta(P-Q) = \begin{matrix} 0 & P \neq Q \\ 1 & P = Q \end{matrix}$$

$$\int_{-\infty}^{\infty} \delta(P-Q)\, dQ = 1$$

From which it can be shown that if L(q) is linear the particular integral of L(q) = f(P) is:

$$q = \int_{v} f(Q)\, S(P, Q)\, dQ \tag{I. 2}$$

For a more complete discussion of the fundamental solution see reference 7. The fundamental solution of the heat equation for n dimensions is given by the solution of the problem:

$$\nabla^2 S_t - K \frac{\partial S_t}{\partial t} = -K\, \delta(r-\sigma)\, \delta(t-\tau) \tag{I. 3}$$

where n = number of dimensions. The solution of this problem is given in reference 3 and after suitable change of notation may be written:

$$S_t(x_i, \xi_i, t-\tau) = \left(\frac{K}{4\pi(t-\tau)}\right)^{\frac{n}{2}} e^{-\frac{KR^2}{4(t-\tau)}}\, H(t-\tau)$$

$$R^2 = \sum_{i=1}^{n} (x_i - \xi_i)^2 \tag{I. 4}$$

For a two-dimensional axially symmetric problem the distribution function, $f(\Omega)$, in equation I. 2 is independent of the angular coordinate and thus the integration with respect to this coordinate may be carried out immediately. Thus expressing I. 4 in polar coordinates and integrating* with respect to $\phi$ the fundamental solution of the two-dimensional heat equation for an axial symmetric problem is:

$$S_t(r,\sigma,t,\tau) = \frac{K e^{-K\frac{r^2+\sigma^2}{4(t-\tau)}}}{2(t-\tau)} I_0\left(\frac{K r\sigma}{4(t-\tau)}\right)\sigma \qquad (5.21)$$

$$\sigma^2 = \xi_2^2 + \xi_3^2$$

The fundamental solution for equation 3. 104 may also be obtained directly from equation I. 4. It is easily seen that equations I. 3 and 3. 104 are related by a Galilean transformation;thus applying the Galilean transformation to equation I. 4 and setting $n = 2$ we get as the fundamental solution of equation 3. 104

$$S_t(Y-\xi_2, Z-\xi_3, t-\tau) = \frac{e^{-\frac{R^2-2(z-\xi_3)(t-\tau)+(t-\tau)^2}{4(t-\tau)}}}{4\pi(t-\tau)} \qquad (3.109)$$

The fundamental solution of the steady three-dimensional Oseen equations is given in reference 9. With the appropriate change of notation it may be written as:

$$\begin{aligned}
&\underline{\underline{\Gamma}} = \underline{\underline{I}}\,\nabla^2\Phi - \text{grad grad}\,\Phi \\
&\vec{P} = -\frac{1}{\pi}\,\text{grad}\,\frac{1}{R} \\
&\Phi = \frac{1}{\pi}\int_0^{S/2}\frac{1-e^{-\alpha}}{\alpha}\,d\alpha \quad ; \quad S = R + \tfrac{1}{2}(x-\xi_1) \\
&R^2 = (x-\xi_1)^2 + r^2 + \sigma^2 - 2r\sigma\cos(\theta-\phi)
\end{aligned} \qquad (I.5)$$

*Reference 8, equation 337. 15b.

Considering the axial symmetric case it is clear that the distribution function is again independent of the angular coordinate. Thus carrying out the integration with respect to $\phi$ in equation I. 5 we obtain as the fundamental solution of the steady three-dimensional axial symmetric Oseen equations:

$$\underline{\underline{\Gamma_A}} = \underline{\underline{I}}\,\nabla^2\Phi_A - \text{grad grad}\,\Phi_A$$

$$\vec{P} = -\frac{2\sigma}{\pi} F\left(\frac{\pi}{2}, \sqrt{\frac{2r\sigma}{(x-\xi_1)^2 + r^2 + \sigma^2 + 2r\sigma}}\right) \tag{4.52}$$

$$\Phi_A = \frac{\sigma}{\pi}\int_0^{2\pi}\int_0^{s/2} \frac{1-e^{-\alpha}}{\alpha}\, d\alpha\, d\phi$$

$F(\varphi, K)$ = elliptic integral of the first kind

It is easily verified that all the components in the circumferential direction are zero in equation 4. 52 by noting that:

$$\int_0^{2\pi} [f(x,\sigma,r,R)] \sin(\theta-\phi)\, d(\theta-\phi) \tag{I. 6}$$

where $f$ is an arbitrary function.

We assume that the fundamental solution of the non-steady two-dimensional Oseen equations may be written as:

$$\underline{\underline{\Gamma_t}} = \underline{\underline{I}}\,\nabla^2\Phi_t - \text{grad grad}\,\Phi_t \tag{3. 46}$$

$$\vec{P_t} = \text{grad}\left(\frac{\partial\Phi_t}{\partial t} + \frac{\partial\Phi_t}{\partial z} - \nabla^2\Phi_t\right) = \frac{\delta(t)}{2\pi}\,\text{grad}(\ln r) \tag{3. 47}$$

Equations 3. 46 and 3. 47 satisfy the continuity equation identically and substituting them into the equations for the fundamental solution of the non-steady two-dimensional Oseen equations gives equation 3. 51. Thus $\Phi_t$ is given by equation 3. 52 and it is clear from the application of the Galilean transformation to the solution for the corresponding Stokes problem given in reference 9 that the particular solution of equation 3. 52 is:

$$\Phi_t = \frac{1}{4\pi} \Gamma\left(0, \frac{r^2 - 2zt + t^2}{4t}\right) \tag{3. 48}$$

Equations 3. 46-3. 48 then give the desired fundamental solution.

## APPENDIX II. PROPERTIES OF E (t, r)

The function E(t, r) is defined by

$$E(t,r) \equiv -\frac{1}{2}\int_0^1 \frac{e^{-r^2/t\,(\eta + \frac{t^2}{r^2}\eta^{-1})}}{\eta}\, d\eta \qquad (3.68)$$

To study the behavior for r small we consider the first two terms of the Taylor series expansions about r = 0 which gives:

$$E(t,r) = -\frac{1}{2}\Gamma(0,t) - r^2\,\Gamma(-1,t) + \cdots \qquad (\text{II. } 1)$$

By expanding the integrand of equation 3.68 for $\frac{r^2\eta^2}{t^2}$ small and integrating one obtains the following asymptotic expansion for t large:

$$E(t,r) \simeq -\frac{1}{2}\sum_{m=0}^{M} \frac{(-1)^m}{m!}\, r^{2m}\,\Gamma(-m,t) \qquad (\text{II. } 2)$$

Similarly expanding the integrand of equation 3.68 for $\frac{t^2\eta^2}{r^2}$ small and integrating one obtains the following asymptotic expansions:*

$$E(t,r) \simeq \frac{1}{2}\sum_{m=0}^{M} \frac{(-1)^m}{m!}\, r^{2m}\,\Gamma\!\left(-2m, \frac{r^2}{t}\right) \qquad (\text{II. } 3)$$

Making use of the property**

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* Not valid for $t \to 0$ since as $t \to 0$ both terms in the exponent become indeterminant at the lower limit. From equation 3.53 we see: $E(0,r) = -K_0(2r)$

** See reference 10, volume 2, page 137.

$$\Gamma(-m,x) = \frac{(-1)^m}{m!}\left[\Gamma(0,x) - e^{-x}\sum_{j=0}^{m-1}\frac{(-1)^j j!}{x^{j+1}}\right] \tag{II. 4}$$

one obtains from the previous expansions:

$$E(t,r) \simeq -\frac{1}{2}\sum_{m=0}^{M}\frac{r^{2m}}{(m!)^2}\left[\Gamma(0,t) - e^{-t}\sum_{j=0}^{m-1}\frac{(-1)^j j!}{t^{j+1}}\right] \tag{II. 5}$$

and:

$$E(t,r) \simeq \frac{1}{2}\sum_{m=0}^{M}\frac{(-1)^m r^{2m}}{(m!)(2m!)}\left[\Gamma(0,r^2/t) - e^{-r^2/t}\sum_{j=0}^{2m-1}(-1)^j j!\left(\frac{t}{r^2}\right)^{j+1}\right] \tag{II. 6}$$