A TIME-OPTIMAL CONTROL PROBLEM IN DYNAMICS WITH SAMPLED DATA

Thesis by

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ABSTRACT

A two-degree-of-freedom dynamical system has been analyzed to determine an optimum control sequence which will drive the dynamical system from an arbitrary initial position and velocity to one of a prescribed set of terminal position and velocity's in minimum time. The basic complexities are:

(1) that the forcing function can change only at discrete intervals of time, and

(2) that the prescribed terminal states allow a multiplicity of solutions to prevail.

A novel but not unique force program which is dependent upon the initial state of the system has been determined. This program consists essentially of the continuous application of a force of the proper sense and maximum allowable amplitude followed by a time during which no force is applied. This is followed by a time interval in which the forcing function has a maximum amplitude but is of the opposite sign to that used in the first part of the program.

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I. INTRODUCTION

The study undertaken in this paper may be represented as an attempt at applying the results obtained by Bellman[1] and thoroughly treated by others such as Desoer,[2] Tsien,[3] and Athanassiades[4] in regards to the transfer of a dynamical system from a prescribed initial state to a prescribed final state in minimum time. This particular study will be concerned with determining a force program in which the forcing function can change only at discrete intervals of time. Such a constraint also appears in the discipline of sampled-data control systems. The results of investigators such as Desoer and Wing,[5] Mullin and Jury,[6] and Altar and Helstrom[7] provided considerable background information in regards to the effects of the sampling constraint on the response of a dynamical system.

The work of Desoer and Wing utilizing sampled data provides the most applicable parallel reference to the classical work of Bellman. Both investigators delved extensively into the fundamental mathematical proofs of the existence of a minimum time state transfer technique for two-dimensional systems - Desoer and Wing for discrete control time and Bellman for continuous control time.

In addition, this study has included a novel set of constraints on the terminal state that differ markedly from the approach taken by the above investigators. Here the usual practice of defining the terminal state as a unique point, to be reached in minimum time, will be relaxed to include a bounded set of acceptable states.

This study will be seen to draw on the results of investigations conducted by both Bellman and Desoer and Wing as limit cases to the present problem. Thus:

(1) If the discrete control time constraint of the present problem were removed and the terminal state were a unique point instead of a region, then the problem and its solution could be related directly to the Bellman study.

(2) If force program defined herein were altered to coincide with that used by Desoer and Wing, and the terminal state were a unique point instead of a region, then the problem and its solution could be related directly to these investigators' results.

II. BACKGROUND

Some of the investigations conducted by Bellman in the continuous control time case, and Desoer and Wing in the discrete control time case represent neighboring cases to the present problem where the minimum time solution has been treated thoroughly. These cases are reviewed here briefly in order to:

(1) amplify the differences between the previous works and the present study, and

(2) establish the existence of proofs to certain theorems which are utilized in the present study.

The related problem considered by Bellman was that of determining the optimum force program for a state transference of any dynamical system which can be described by a linear differential equation with constant coefficients. The roots of its characteristic equation were restricted to terms whose real parts were always non-positive.

With switching allowed to occur at any instant of time, Bellman proved that time-optimum control resulted when the maximum force available was applied at all times. This mode of control is often referred to as bang-bang control.

Bellman further proved that when the characteristic roots were real, distinct and negative the direction of the maximum force need be reversed at most $N - 1$ times. (N being the order of the differential equation describing the system.)

The related problem considered by Desoer and Wing pertained to a more restricted class of dynamical systems than those studied by Bellman. However, these investigators introduced the constraint of sampled data.

Here the dynamical system was represented by a second order differential equation whose characteristic equation contained one negative real root and one zero root. A force program was sought which would cause the system to return to an equilibrium position in the minimum possible time subject to the following conditions:

(1) The applied force must be piecewise constant for a discrete interval of T seconds.

(2) The magnitude of the applied force may assume any value between zero and its maximum value (which is determined by saturation considerations) at the beginning of any interval T.

(3) The total time for the transference must be an integer multiple of the basic sampling period T.

The solution proposed by Desoer and Wing consisted of driving the system with the maximum force available until the state variables were within one T interval of an optimum switching condition. The magnitude of the forcing function was then reduced. This action caused the state variables to lie on the optimum switching curve at the next sampling period. At this point the maximum force was reapplied but in the opposite sense to the initial application.

This action was maintained until the state variables were within one time interval T of attaining the equilibrium position. The maximum force level was then reduced a second time resulting in an interception with the desired state at the end of the terminal sampling period. Desoer and Wing proved that this technique brought about a change of state in minimum time.

On comparing the preceding optimal force programs (with and without sampled data) one can see that the control process becomes more complicated when applied for fixed intervals of time. At this point one may ask, "Can more of the simplicity of the continuous control process proposed by Bellman be retained in the discrete control regime"? This thesis attempts to answer that question.

III. STATEMENT OF THE PROBLEM

The problem to be investigated in this thesis is the following:

A dynamical system can be represented by the following normalized linear differential equation with constant coefficients

$$\ddot{c}(t) + \dot{c}(t) = f(t) \tag{1}$$

$f(t)$ represents a forcing function that is piecewise constant over a time interval T and can assume the normalized values of $+1.0$, -1.0 or zero at the beginning of each new interval.

The problem is to devise a control sequence whereby the state variables of the above dynamical system can be brought from arbitrary initial conditions at $t = 0$, which will be expressed as:

(1) $|c(0)|$ arbitrary

(2) $|\dot{c}(0)| \leq 1.0$

to a final state at time t_s where:

(1) t_s is the minimum time required to reach the final state,

(2) $|c(t_s) + e\dot{c}(t_s)| \leq K_1$,

(3) $|c(t_s) + \dot{c}(t_s)| \leq K_2$,

(4) K_1, K_2, and e are specified constants.

These final state conditions are shown in the c, $\dot{c}$ plane in Figure 1.

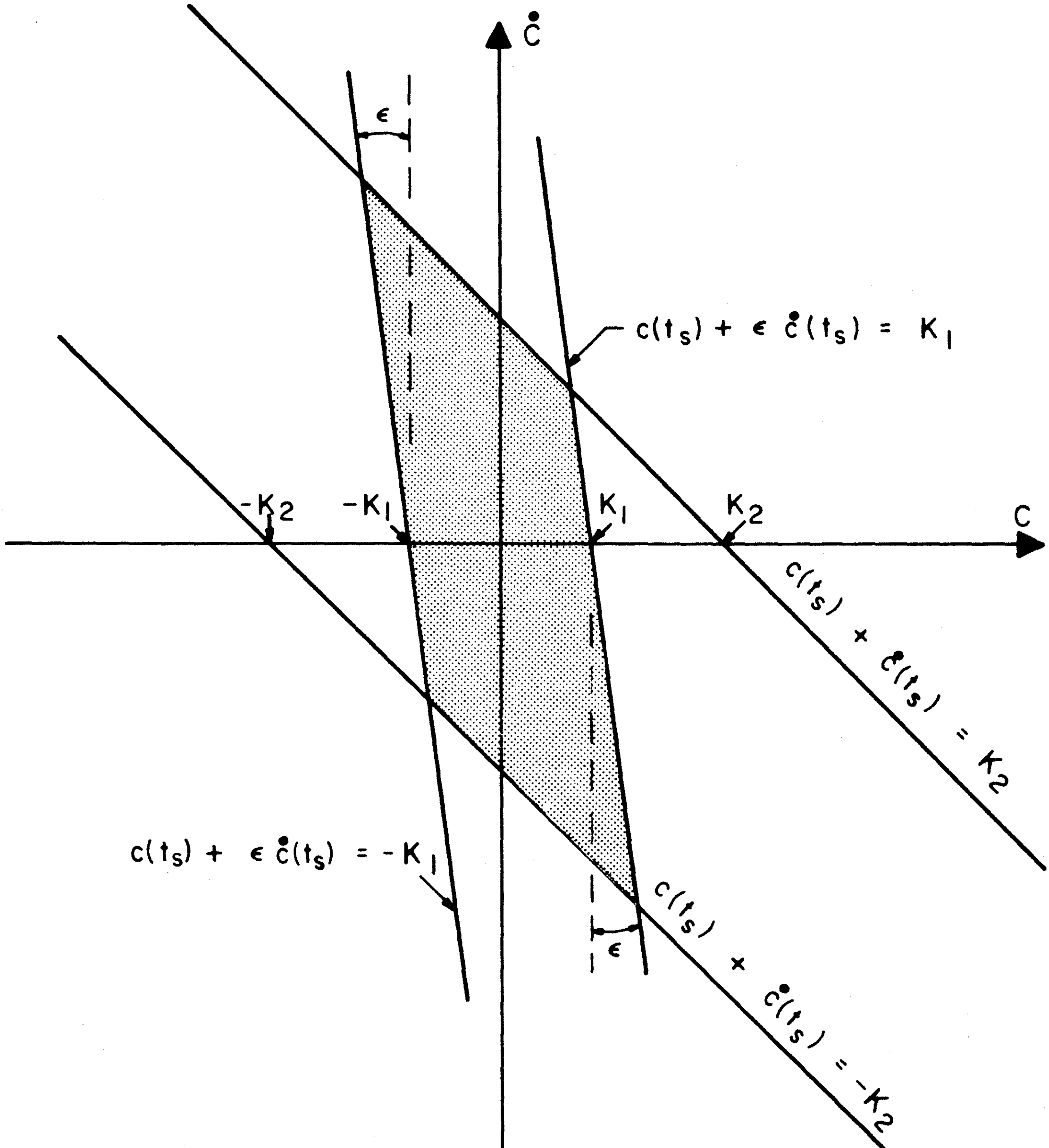


FIGURE 1

PORTRAYAL OF FINAL STATE CONDITIONS IN C, $\dot{C}$ PLANE

The values of the state variables at a time which does not coincide with the beginning or end of a discrete interval are of no interest because of the nature of the forcing function.

Any control sequence that results in the achievement of the above requirements shall be regarded as a satisfactory solution.

IV. PHASE PLANE APPROACH

A. General

In order to evaluate and analyze the effects of various control schemes on the dynamical system, the phase plane method[6,7,8] was employed. This technique consists of portraying the trajectories of the functions $e(t)$ and $\dot{e}(t)$ generated by the forcing function $f(t)$ in a coordinate plane of $\dot{e}$ versus e. It is interesting to note that the trajectories of the state variables are continuous curves even though discontinuities occur in the forcing function.

When a change in the forcing function does occur, a new trajectory is generated with the characteristics of the new forcing function. Since time varies along a given trajectory, every point on that trajectory represents the state of the system at a certain time, t. One can then denote a change of state over a discrete interval of time by specified points on the continuous curve.

The choice of the phase plane method was also governed by the nature of the constraints. The terminal state conditions describe a region centered at the origin in the phase plane. The limits on the initial state of $\dot{e}(0)$ describe the upper and lower boundaries of the phase portrait.

B. Phase Plane Trajectories

Three basic trajectories are required for the forcing functions +1.0, -1.0, and zero. On taking a new origin of time in each interval and setting $f(t) = \Delta$ for simplicity the desired equation is derived by referring to Equation 1:

$$\ddot{c}(t) + \dot{c}(t) = f(t) = \Delta$$

This is equivalent to the system

$$y(t) = \dot{c}(t)$$

$$\dot{y}(t) + y(t) = \Delta$$

noting that

$$\dot{y}/y = \dot{y}/\dot{c} = dy/dc$$

we have

$$\frac{dy}{dc} = \frac{\Delta - y}{y}$$

or

$$dc = \frac{ydy}{\Delta - y}$$

On integrating over one interval T, one obtains

$$\int_{c(0)}^{c(T)} dc = \int_{y(0)}^{y(T)} \frac{ydy}{\Delta - y}$$

or

$$c(T) + \dot{c}(T) = c(0) + \dot{c}(0) + \Delta \log \frac{\Delta - \dot{c}(0)}{\Delta - \dot{c}(T)} \qquad (2)$$

Figure 2 summarizes the three sets of trajectories generated by Equation 2 when $\Delta = \pm 1.0$ or zero.

C. Difference Equations

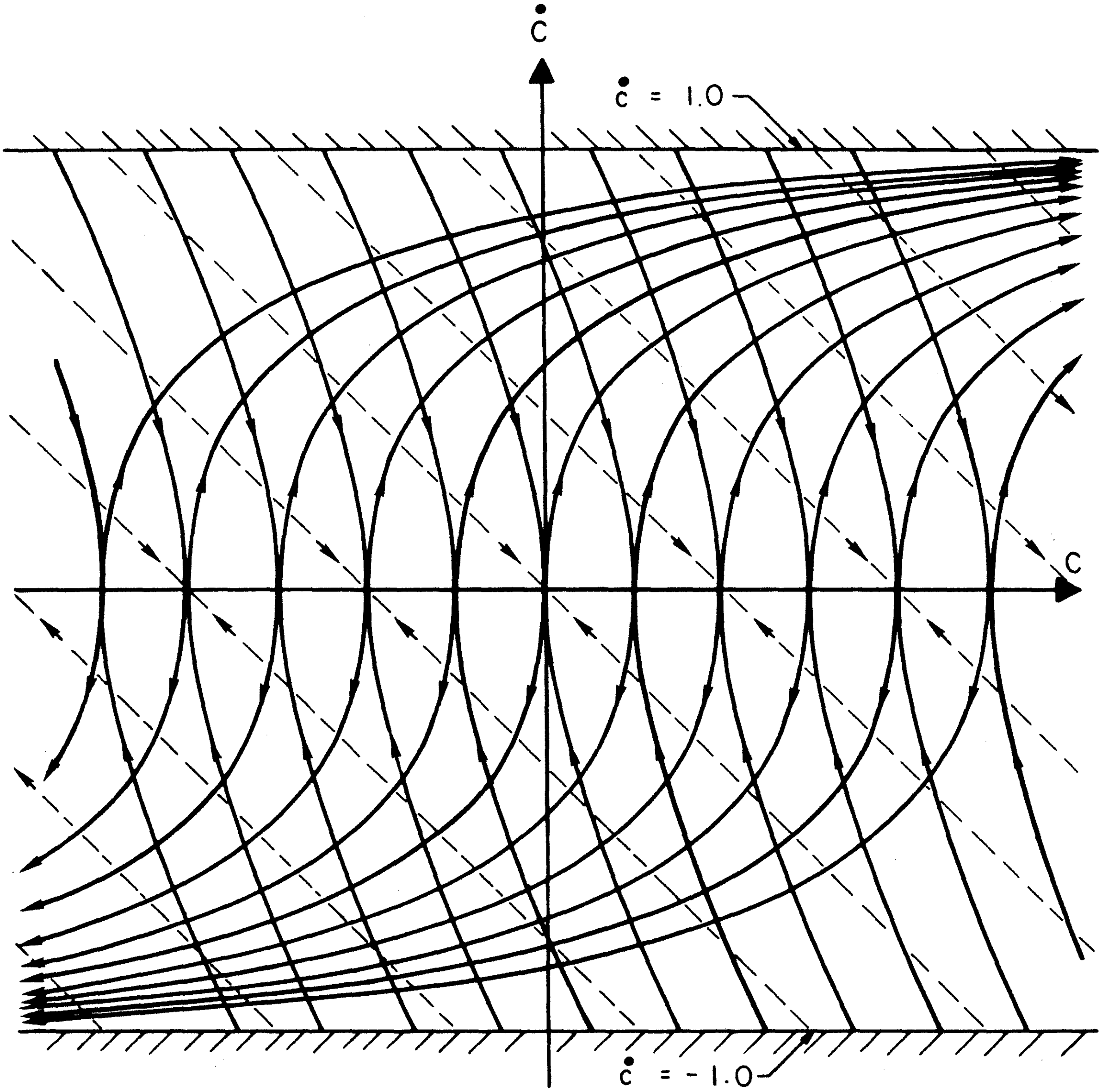
Now, having the paths that the system will follow in the phase plane in each time interval, one may develop additional equations containing time explicity. This step is required in order to be able to determine the state points at discrete intervals on a given trajectory. The necessary equations are obtained by integrating the basic differential equation

$$\ddot{c}(t) + \dot{c}(t) = \Delta$$

and writing the solutions in the form of difference equations; i.e., evaluated at each end of a time interval $0 \leq t \leq T$. This can be done in vector-matrix mathematical language where underlined lower case letters are vectors and capital letters are matrices.

The differential equation of the system is first rewritten in the vector-matrix form:

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + \underline{f}, \text{ where } 0 \leq t \leq T \qquad (3)$$



TRAJECTORIES FOR $\Delta = 1.0$ APPROACH $C = \infty$, $\dot{C} = +1.0$
TRAJECTORIES FOR $\Delta = -1.0$ APPROACH $C = -\infty$, $\dot{C} = -1.0$
TRAJECTORIES FOR $\Delta = 0$ APPROACH $\dot{C} = 0$, $|C|$ ARBITRARY

FIGURE 2

PHASE PLANE TRAJECTORIES

with initial conditions

$$\underline{x}(0) = \underline{c}$$

Through the change of variables

$$x_1 = c \qquad \dot{x}_1 = \dot{c} = x_2$$

$$x_2 = \dot{c} \qquad \dot{x}_2 = \ddot{c} = -x_2$$

we see that the terms A, $\underline{f}$ and $\underline{c}$ in Equation 3 are

$$A = \begin{bmatrix} 0 & 1.0 \\ 0 & -1.0 \end{bmatrix}, \quad \underline{f} = \Delta \begin{Bmatrix} 0 \\ 1.0 \end{Bmatrix}, \quad \text{and} \quad \underline{c} = \begin{Bmatrix} c(0) \\ \dot{c}(0) \end{Bmatrix}$$

The integral solution of Equation 3 over an interval T is of the form

$$\underline{x}(T) = X(T)\underline{c} + X(T)\int_0^T X^{-1}(\tau)\, \underline{f}\, d\tau \qquad (4)$$

where X is a matrix of vectors formed from the solution of the differential equation

$$\ddot{c}(t) + \dot{c}(t) = 0$$

where

$$c(t) = A + Be^{-t}$$

$$\dot{c}(t) = -Be^{-t}$$

and whose constants are evaluated from the relations

$$\underline{x}^1(0) = \begin{Bmatrix} 1.0 \\ 0 \end{Bmatrix}, \text{ and } \underline{x}^2(0) = \begin{Bmatrix} 0 \\ 1.0 \end{Bmatrix}$$

On writing $\mathbb{X}$ in the form

$$\mathbb{X} = \left[\underline{x}^1, \underline{x}^2\right] = \begin{bmatrix} x_1^1 & x_1^2 \\ x_2^1 & x_2^2 \end{bmatrix}$$

where $\underline{x}^1$ and $\underline{x}^2$ are found to be

$$\underline{x}^1 = \begin{Bmatrix} 1.0 \\ 0 \end{Bmatrix}, \text{ and } \underline{x}^2 = \begin{Bmatrix} 1-e^{-t} \\ e^{-t} \end{Bmatrix}$$

we get the necessary matrix

$$\mathbb{X}(t) = \begin{bmatrix} 1 & 1-e^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

which can be formed as $\mathbb{X}(T)\underline{c}$ and $\mathbb{X}(T)\,\mathbb{X}^{-1}(\tau)\underline{f}$ in Equation 4 to yield

$$\underline{x}(T) = \begin{bmatrix} 1 & 1-e^{-T} \\ 0 & e^{-T} \end{bmatrix} \begin{Bmatrix} c(0) \\ \dot{c}(0) \end{Bmatrix} + \Delta \int_0^T \begin{Bmatrix} 1-e^{-(T-\tau)} \\ e^{-(T-\tau)} \end{Bmatrix} d\tau \qquad (5)$$

On performing the final integration we get the basic difference equation in terms of the initial and final states, the sampling time T and the forcing function Δ.

$$\begin{Bmatrix} c(T) \\ \dot{c}(T) \end{Bmatrix} = \begin{bmatrix} 1 & 1-e^{-T} \\ 0 & e^{-T} \end{bmatrix} \begin{Bmatrix} c(0) \\ \dot{c}(0) \end{Bmatrix} + \Delta \begin{Bmatrix} T-1.0+e^{-T} \\ 1-e^{-T} \end{Bmatrix} \qquad (6)$$

on expanding Equation 6 and recombining terms we get

$$c(T) + \dot{c}(T) = c(0) + \dot{c}(0) + T\,\Delta \qquad (7)$$

Now let us apply Equation 6 to a specific situation where the force program is known but the period of application for each force is undetermined. For simplicity we can adopt the following notation for describing a particular forcing sequence

Δ : (+1, 0, -1)

which means that

$\Delta = +1.0$ over the interval $0 \leq t \leq \ell$

$\Delta = 0$ over the interval $\ell \leq t \leq \ell+m$

$\Delta = -1.0$ over the interval $\ell+m \leq t \leq t_s$

$(t_s = \ell + m + n)$

where:

(1) $\ell = \ell_o T$, $m = m_o T$, and $n = n_o T$,

(2) T is the sampling interval and

(3) ℓ_o, m_o, and n_o are positive integers (1, 2, ---, k)

This forcing sequence represents a program for a complete state transference. However, its application must be restricted to those initial states where a minimum time transfer appears likely. Thus the phase plane will ultimately be separated into regions where force programs beginning with $\Delta = +1.0$, -1.0 or zero are preferred.

The difference equations for the forcing sequence Δ : $(+1, 0, -1)$ become

For the interval $0 \leq t \leq \ell$ where $\Delta = +1.0$

$$\begin{Bmatrix} c(\ell) \\ \dot{c}(\ell) \end{Bmatrix} = \begin{bmatrix} 1.0 & 1.0-e^{-\ell} \\ 0 & e^{-\ell} \end{bmatrix} \begin{Bmatrix} c(0) \\ \dot{c}(0) \end{Bmatrix} + \begin{Bmatrix} \ell-1.0+e^{-\ell} \\ 1.0-e^{-\ell} \end{Bmatrix} \qquad (8)$$

and

$$c(\ell) + \dot{c}(\ell) = c(0) + \dot{c}(0) + \ell \qquad (9)$$

For the interval $\ell \leq t \leq \ell+m$ where $\Delta = 0$

$$\begin{Bmatrix} c(\ell+m) \\ \dot{c}(\ell+m) \end{Bmatrix} = \begin{bmatrix} 1.0 & 1.0-e^{-m} \\ 0 & e^{-m} \end{bmatrix} \begin{Bmatrix} c(\ell) \\ \dot{c}(\ell) \end{Bmatrix} \qquad (10)$$

and

$$c(\ell+m) + \dot{c}(\ell+m) = c(\ell) + \dot{c}(\ell) \qquad (11)$$

For the interval $\ell+m \leq t \leq t_s$ where $\Delta = -1.0$

$$\begin{Bmatrix} c(t_s) \\ \dot{c}(t_s) \end{Bmatrix} = \begin{bmatrix} 1.0 & 1.0-e^{-n} \\ 0 & e^{-n} \end{bmatrix} \begin{Bmatrix} c(\ell+m) \\ \dot{c}(\ell+m) \end{Bmatrix} - \begin{Bmatrix} n-1.0+e^{-n} \\ 1.0-e^{-n} \end{Bmatrix} \qquad (12)$$

and

$$c(t_s) + \dot{c}(t_s) = c(\ +m) + \dot{c}(\ +m) - n \qquad (13)$$

For the interval $0 \leq t \leq t_s$ where $\Delta : (+1, 0, -1)$

$$\begin{Bmatrix} c(t_s) \\ \dot{c}(t_s) \end{Bmatrix} = \begin{bmatrix} 1.0 & 1.0-e^{-t_s} \\ 0 & e^{-t_s} \end{bmatrix} \begin{Bmatrix} c(0) \\ \dot{c}(0) \end{Bmatrix} + \begin{Bmatrix} \ell-n+e^{-t_s}-e^{-(m+n)}-e^{-n}+1.0 \\ -e^{-t_s}+e^{-(m+n)}+e^{-n}-1.0 \end{Bmatrix} \qquad (14)$$

and

$$c(t_s) + \dot{c}(t_s) = c(0) + \dot{c}(0) + \ell-n \qquad (15)$$

The necessary equations for the other force programs can be formed in a similar manner. For comparison consider $\Delta : (+1, -1, 0)$

$$\begin{Bmatrix} c(t_s) \\ \dot{c}(t_s) \end{Bmatrix} = \begin{bmatrix} 1.0 & 1.0-e^{-t_s} \\ 0 & e^{-t_s} \end{bmatrix} \begin{Bmatrix} c(0) \\ \dot{c}(0) \end{Bmatrix} + \begin{Bmatrix} \ell-n+e^{-t_s}-2e^{-(m+n)}+e^{-m} \\ -e^{-t_s}+2e^{-(m+n)}-e^{-m} \end{Bmatrix} \qquad (16)$$

and on expanding and recombining Equation 16 we again get Equation 15

$$c(t_s) + \dot{c}(t_s) = c(0) + \dot{c}(0) + \ell-n$$

D. Time in the Phase Plane

For completeness the elapsed time in terms of the initial and final state variables can also be determined from the basic integral expression:

$$t_2 - t_1 = \int_{c(t_1)}^{c(t_2)} \frac{dc(t)}{\dot{c}(t)} = \int_{\dot{c}(t_1)}^{\dot{c}(t_2)} \frac{d\dot{c}(t)}{\ddot{c}(t)}$$

which in this problem becomes

$$T = \int_{\dot{c}(0)}^{\dot{c}(T)} \frac{d\dot{c}(t)}{\Delta - \dot{c}(t)} = \log \frac{\Delta - \dot{c}(0)}{\Delta - \dot{c}(T)} \qquad (17)$$

on substituting this solution in Equation 2 we get:

$$c(T) + \dot{c}(T) = c(0) + \dot{c}(0) + T\,\Delta$$

which is the same as Equation 7.

E. Strategy for Time Optimal Solution

With the foregoing information we are now ready to solve our problem. The following steps were adopted as a strategy for arriving at a time optimal solution.

Step 1. Partitioning of the Phase Plane

By referring to the trajectories in Figure 2 and the terminal state region in Figure 1 (hereinafter called the target) we can begin relating optimum force assignments with initial state conditions in the phase plane. We note that a $\Delta = -1.0$ force will cause initial states in the upper right (first) quadrant of Figure 2 to follow a direct path towards a $\Delta = +1.0$ trajectory. Upon interception, with a change of force to $\Delta = +1.0$ or zero, we see that the trajectory proceeds towards the origin.

Had a force of the opposite sense been chosen initially ($\Delta = +1.0$), the state points in this quadrant would not have proceeded towards the origin as rapidly, regardless of the subsequent forcing sequence. As noted in Figure 3, the first quadrant is then assigned an initial force of $\Delta = -1.0$ except in a region near the origin. Following a similar argument, the initial-force assignment for the third quadrant would be $\Delta = +1.0$.

The second and fourth quadrants contain the bulk of the force changes. By utilizing symmetry we need only investigate the trajectories associated with one quadrant and apply the results to the other with the signs reversed.

Proceeding with the second quadrant we can establish a preliminary upper bounds on the $\Delta = +1.0$ region. This is determined by an extension of a $\Delta = 0$ trajectory from the corner of an arbitrary target. This boundary establishes the earliest point that a change in the initial force should be considered. Once past this boundary we may select a variety of forcing sequences for acquiring the target region in minimum time.

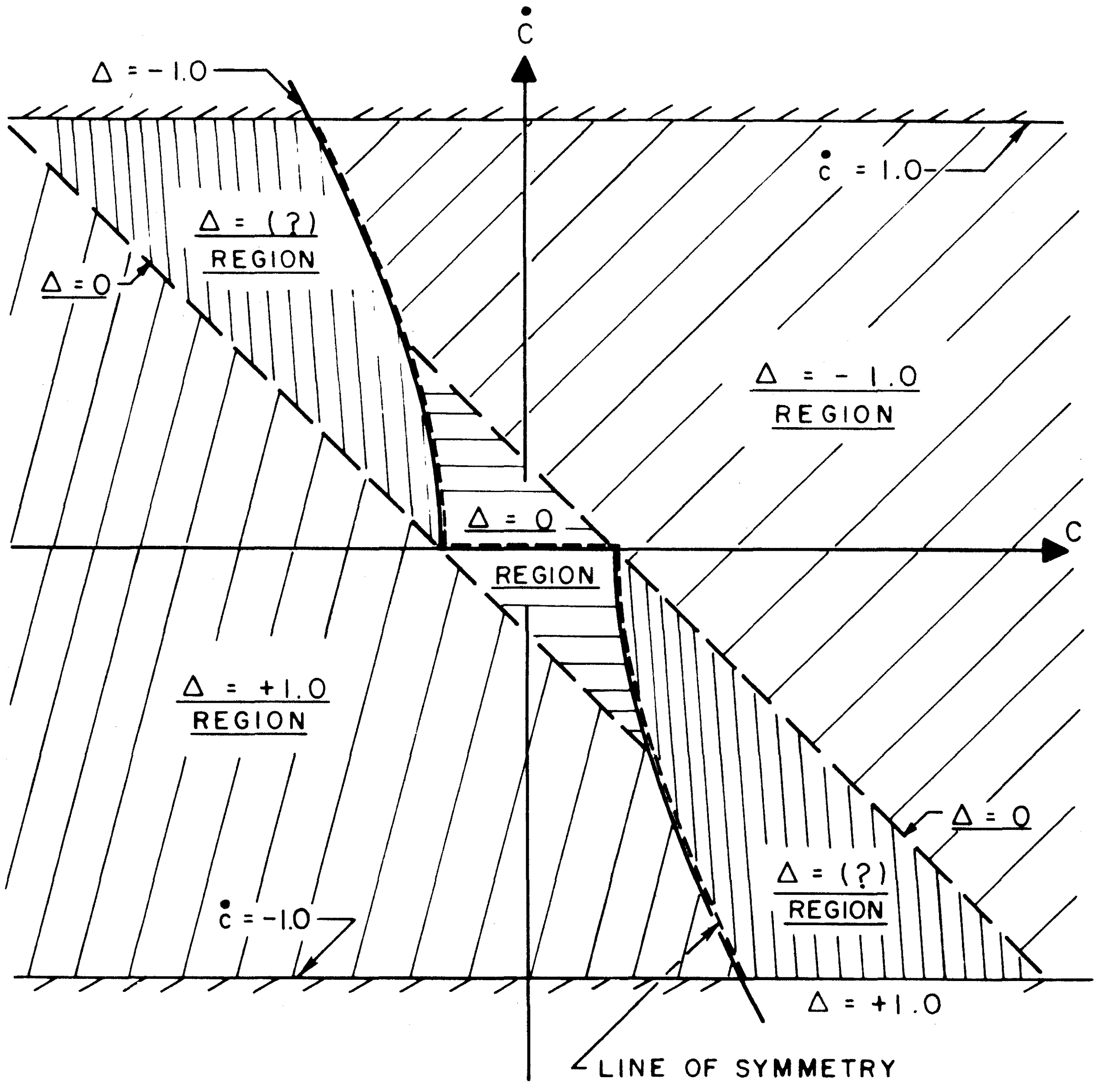


FIGURE 3

INITIAL PARTIONING OF PHASE PLANE BY FORCE ASSIGNMENT

The target can then be represented initially as bounded by segments of $\Delta = 0$ trajectories, and symmetrical segments of $\Delta = \pm 1.0$ trajectories. Further refinements are effected later in the study.

Step 2. Study of the $\Delta = (?)$ Region

By the foregoing arguments we have reduced the problem to a study of the optimal force programs within the pie-shaped regions of the second and fourth quadrants of Figure 3. In attacking the unknown region in the second quadrant we can start by tabulating all the permissible force programs applicable to this quadrant

(1) Δ : (+1, 0, -1)

(2) Δ : (+1, -1, 0)

(3) Δ : (+1, -1)

(4) Δ : (+1)

(5) Δ : (0)

(6) Δ : (0, -1)

These programs may be interpreted as a continuous application of a prescribed force in the order noted. Programs (1) through (4) overlap the $\Delta = +1.0$ region, while programs (5) and (6) apply only in the $\Delta = (?)$ region of Figure 3.

In addition to these six programs we could consider sequences such as Δ : (+1, -1, +1), (+1, 0, +1) and (+1, -1, +1, -1) for this section of the phase plane. These sequences can be eliminated, however, as not being potential time-optimal force programs.

It is interesting to note that the corresponding force program for the related Bellman problem would be Δ : (+1, -1). The program for the related Desoer and Wing study would be Δ : (+1, +δ, -1, -δ), where $|\delta| \leq 1.0$.

The results of this two-step approach for solving the problem are presented in the next section.

V. SOLUTION TO THE PROBLEM

A. General Remarks

On applying the strategems outlined in the previous section one sees that the solution to the problem evolves by trial and error. Once an apparent minimum time forcing sequence is determined, it must be tested against other potential control schemes in order to evaluate its optimality.

No rigorous mathematical criterion could be derived for a direct measure of the quality of an apparent minimum time solution. Instead the IBM 7090 digital computer was utilized for a cross-check on the final solution. This is discussed in the next section.

The following general characteristics were noted as a consequence of evaluating the various force programs in the phase plane.

(1) The choice of a time-optimal force program is dependent on the initial state of the system.

(2) Several force programs may produce a state transference to the target region in equivalent time.

(3) The switching boundaries for changing the applied force may be simple or complex in nature. This depends on the force program chosen for time-optimal control.

B. Proposed Solution to the Problem

The proposed forcing sequence for this problem is basically Δ : (+1, 0, -1) or (-1, 0, +1), depending upon the initial conditions. On comparing this scheme with other possible programs we note that Δ : (+1, -1, 0) provides an equivalent solution to Δ : (+1, 0, -1) for some initial states. This is portrayed in Figure 4. However, in Figure 5 we see that the initial state points (shaded area) subject to Δ : (+1, -1, 0) control lie within the larger region (a-b) subject to Δ : (+1, 0, -1) control.

We can also note in Figure 5 that curves (a) and (b) form simpler switching boundaries than the sawtooth curve required for Δ : (+1, -1, 0).

A simplification of the target region is also proposed. This new region is shown in Figure 6 in conjunction with the proposed switching boundaries (a), (b) and (d). Our target now conforms to the constraints of the problem as set forth in Figure 1. The target boundaries (and switching boundaries) are evaluated in a later section.

We noted previously that the optimum force is dependent on the initial state conditions. Therefore Table I has been prepared so that we may associate these states with the basic force program.

One can see from this Table that all but one sequence represents a portion of the overall program Δ : (+1, 0, -1).

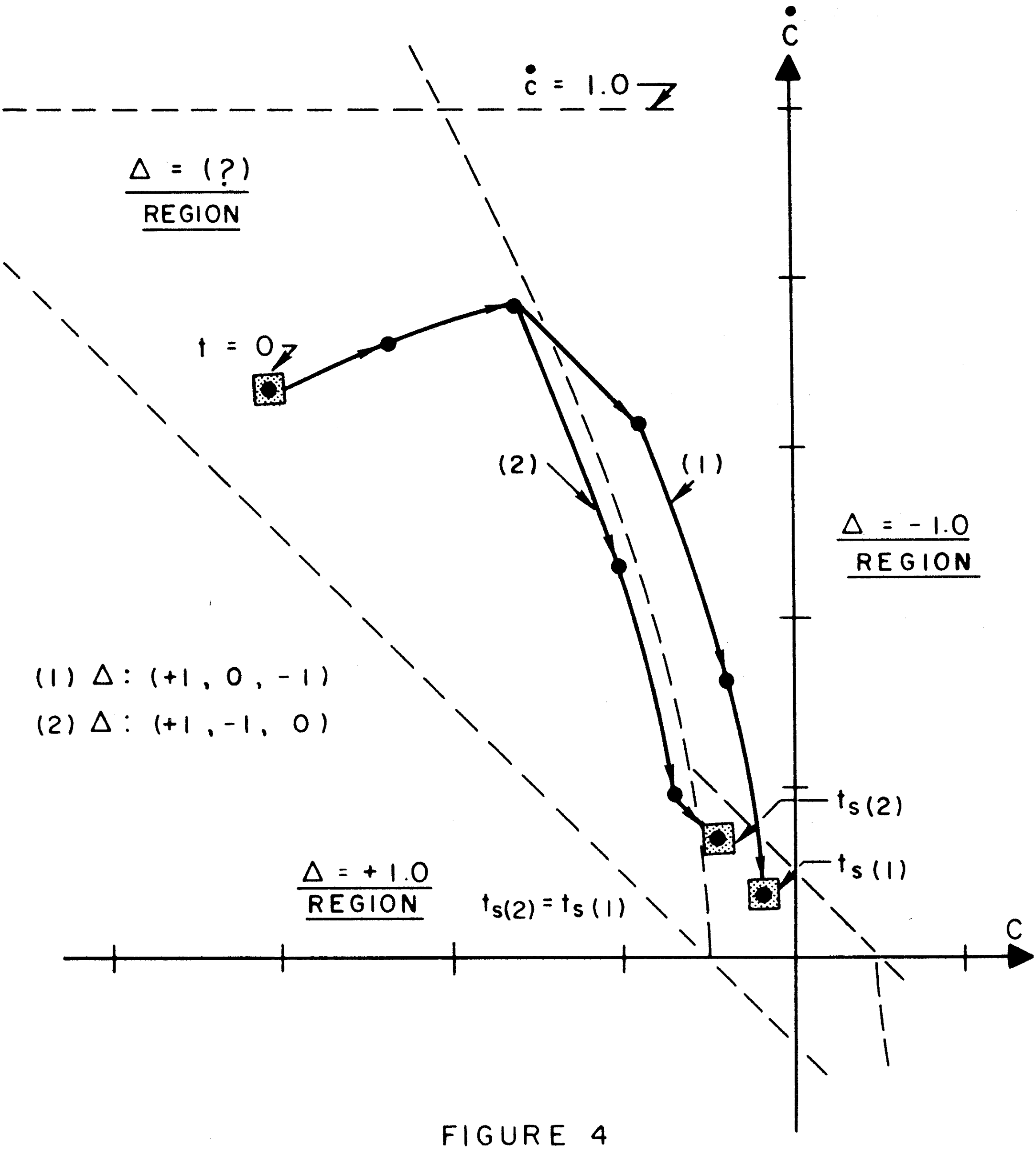


FIGURE 4

PHASE PORTRAIT SHOWING EQUIVALENT TRAJECTORIES FOR Δ: (+1, 0, -1) AND Δ: (+1, -1, 0)

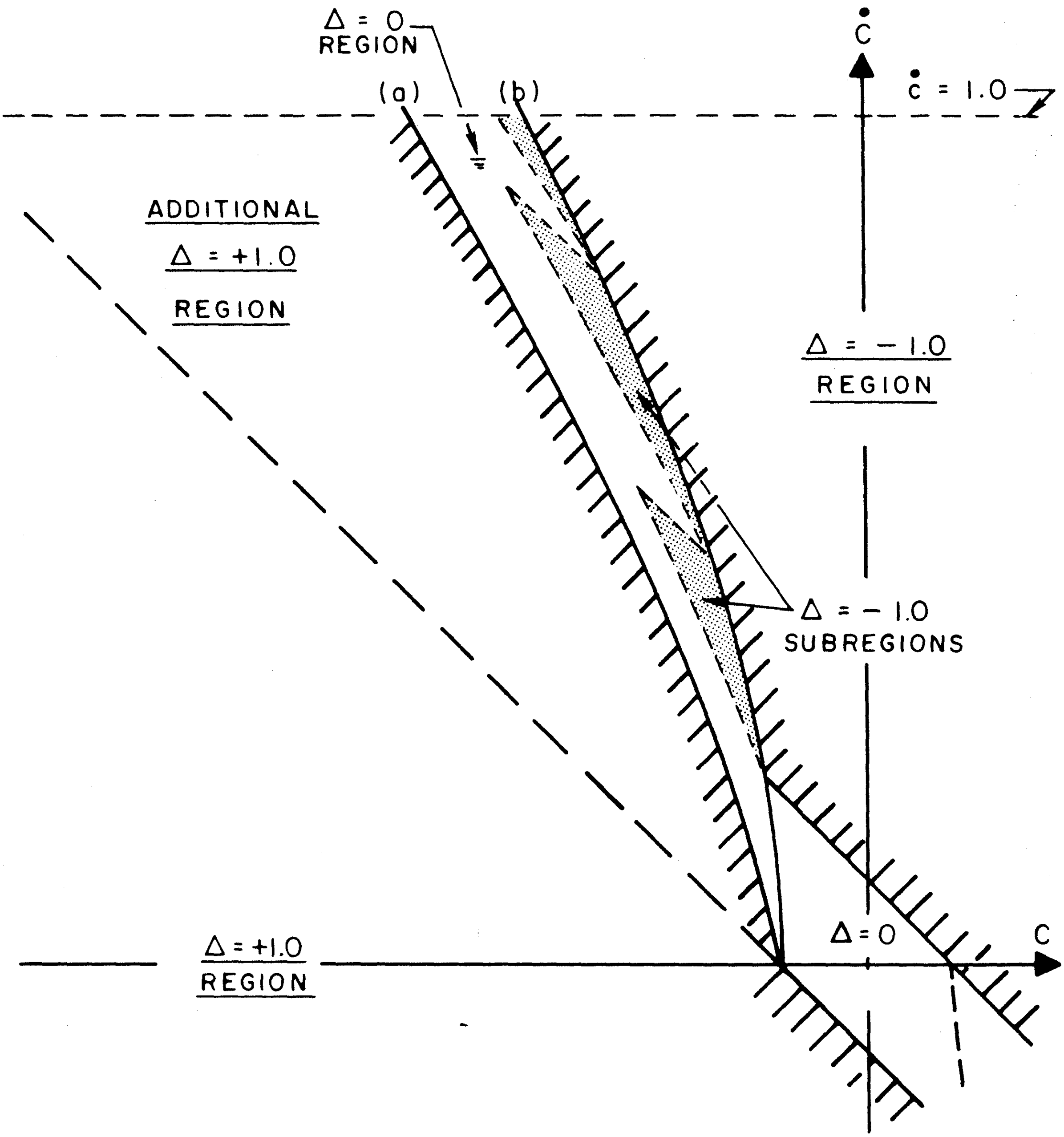


FIGURE 5

REPARTITIONED PHASE PLANE ACCORDING TO OPTIMUM FORCE PROGRAMS

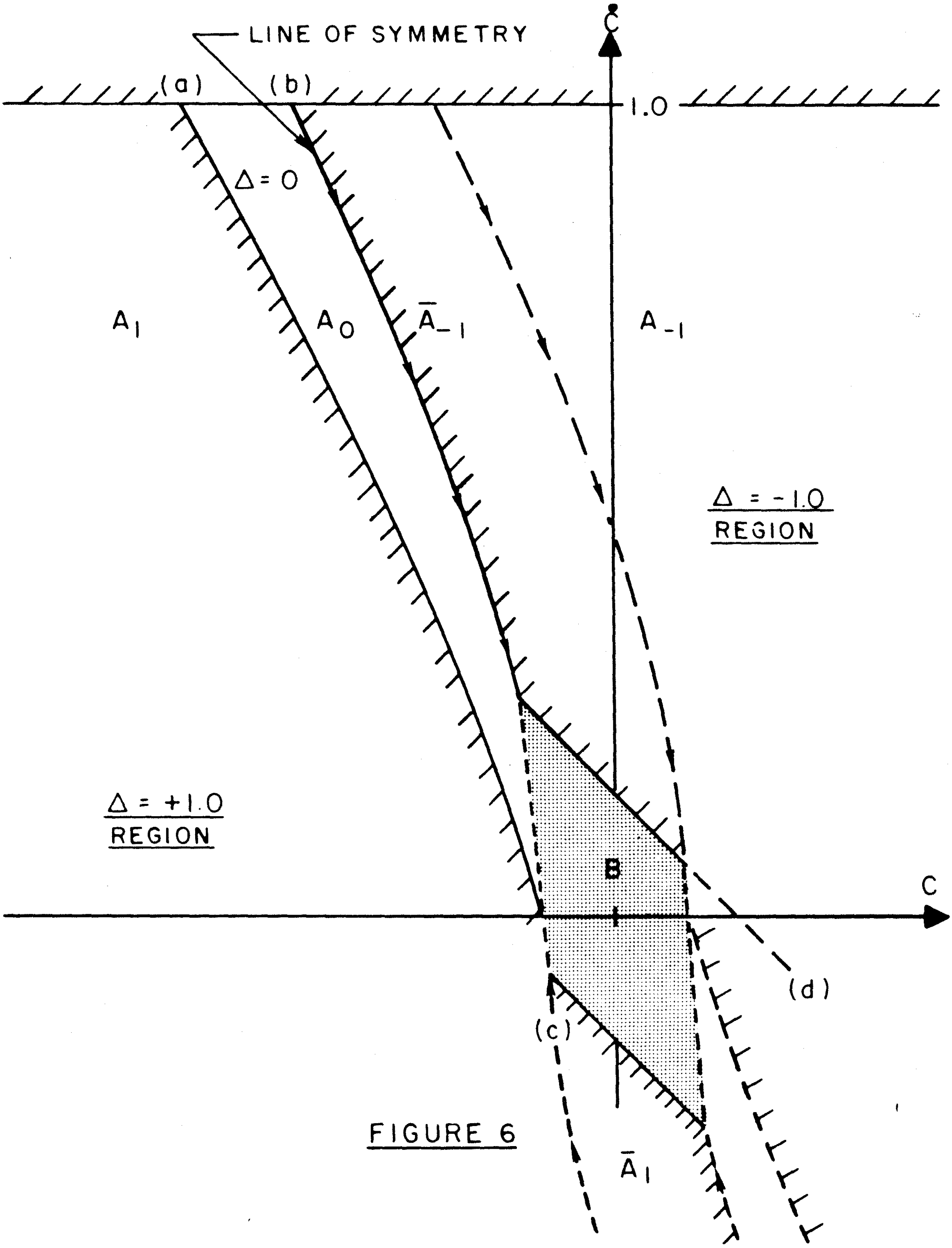


FIGURE 6

TIME - OPTIMAL SWITCHING REGIONS FOR PROPOSED SOLUTION

TABLE I

SUMMARY OF FORCE PROGRAMS

FOR PROPOSED SOLUTION

Initial State Region	Regions Entered at a Sampling Instant	Force Program
A_1	A_o, $\bar{A}_{-1}$, B	Δ : (+1, 0, -1)
A_1	$\bar{A}_{-1}$, B	Δ : (+1, -1)
$\bar{A}_1$	B	Δ : (+1)
A_o	B	Δ : (0)
A_o	$\bar{A}_{-1}$, B	Δ : (0, -1)

Note:

1. $\Delta = 0$ is applied for one sampling interval T
2. System is cut-off when B is entered.

The one exception is Δ : (+1, -1). This additional sequence ensures that any trajectory which oversteps the coast region A_o in Figure 6 will be intercepted at the next sampling period.

For evaluating the optimality of our proposed solution we utilized an independent digital computer solution. This step was required because the problem did not lend itself to a rigorous mathematical treatment. This was due to the constraints of the problem which in turn allowed equivalent solutions to occur under different orders of force application, i.e. Δ : (+1, 0, -1) vs. Δ : (+1, -1, 0).

The digital computer solution consisted of using an IBM 7090 computer to solve Equation 14 for a specified set of initial and final conditions. Equation 14 is restricted to the proposed force program Δ : (+1, 0, -1). The IBM computer then evaluated the variables ℓ, m, and n for the condition where t_s was arbitrarily set equal to 24.0, where

$$t_s = \ell + m + n$$

and

$$1.0 \leq \ell \leq t_s - 2.0$$

$$1.0 \leq m \leq t_s - 2.0$$

$$1.0 \leq n \leq t_s - 2.0$$

T is normalized to unity and ℓ, m, and n are positive integers in Equation 14.

The results of this check, while limited in scope, verified that the proposed force program would effect a state transfer in the same time, and no less, as predicted by the phase plane method.

C. Uniqueness of Proposed Solution

The proposed solution is not unique. There exist a multiplicity of paths for certain initial states lying to the left of region $\bar{A}_{-1}$ of Figure 6.

For example, the proposed optimal force program calls for one coast period in the sequence $\Delta = (+1, 0, -1)$. Whereas in Figure 7 it can be seen that for certain initial states the transference can utilize either one, three, or five coast periods and attain the terminal region in the same total time t_s (and no less).

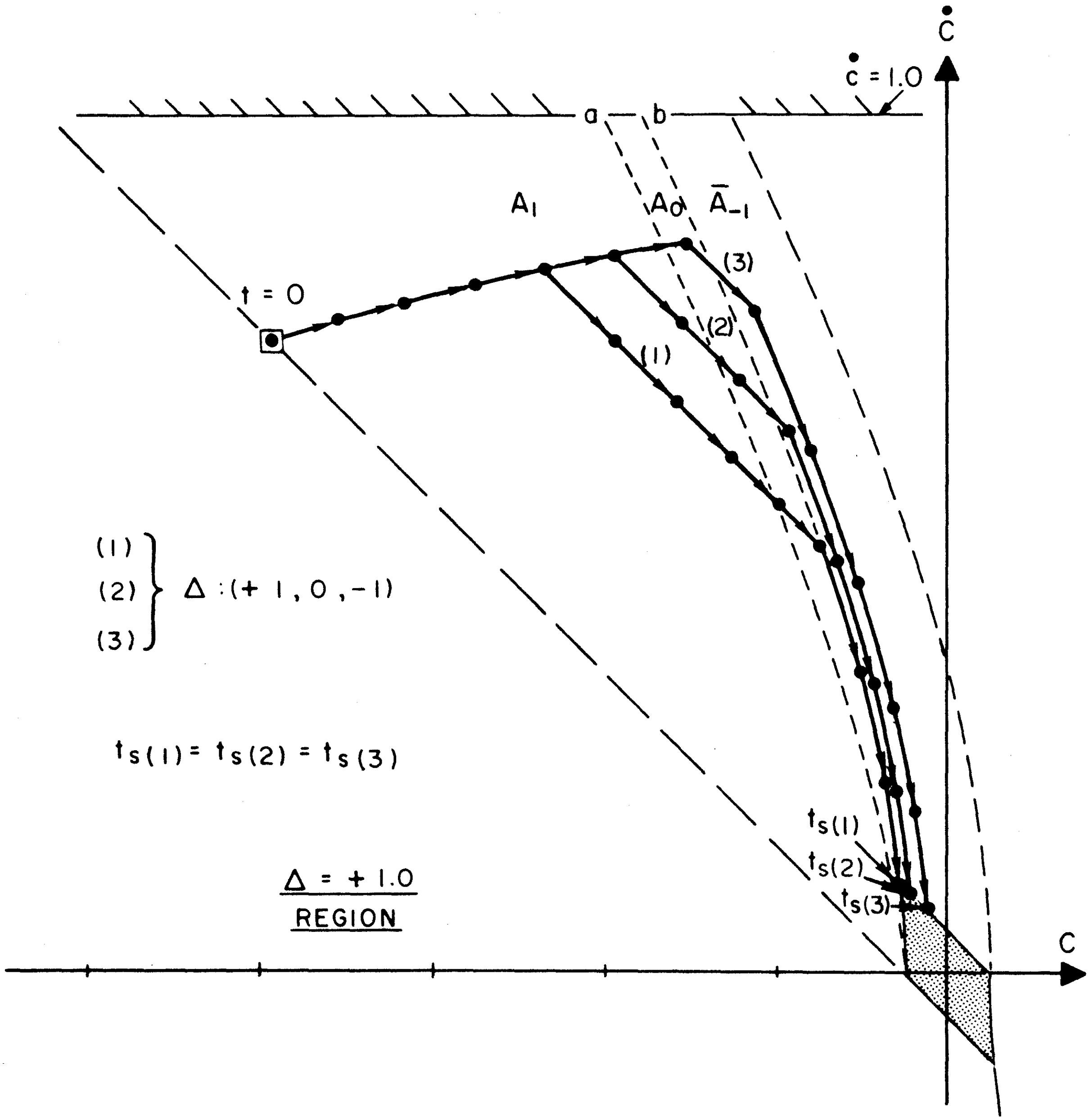


FIGURE 7

PHASE PORTRAIT SHOWING EQUIVALENT TRANSFER PATHS FOR Δ: (+1, 0, −1)

D. Switching and Target Boundaries for Proposed Solution

The switching boundaries which separate the regions of different force application are represented by the lines (a), (b), and (d) in Figure 6. The target or terminal-region boundaries are represented by lines (c) and (d). The following equations describe these contours in the upper half plane

(1) Switching Boundary (a)

$$c(t_k)_a = t_k - K_1 - e^T(e^{t_k} - 1.0) \tag{18}$$

$$\dot{c}(t_k)_a = e^T(e^{t_k} - 1.0) \tag{19}$$

(2) Switching Boundary (b)

$$c(t_k)_b = t_k - K_1 - (e^{t_k} - 1.0) \tag{20}$$

$$\dot{c}(t_k)_b = e^{t_k} - 1.0 \tag{21}$$

where t_k is a dummy variable such that when $t_k = 0$ then $c(0) = -K_1$ and $\dot{c}(0) = 0$

(3) Switching and Target Boundary (d)

$$[c(t) + \dot{c}(t)]_d = K_2 \tag{22}$$

(4) Target Boundary (c)

$$[c(t) + \epsilon\dot{c}(t)]_c = K_1 \tag{23}$$

Equations 18 through 21 are derived in the Appendix. The constants in Equations 22 and 23 are discussed in the following section.

E. Evaluating Constants of the System

Until now, the constants ϵ, K_1, and K_2 have remained arbitrary. These constants are analogous to system accuracy and as such do not bear directly on the determination of the optimal control sequence. The minimum target size for maximum system accuracy is determined by

$$(1) \quad \epsilon = \frac{T/2.0 + K_1 - (e^{T/2 + K_1} - 1.0)}{e^{T/2 + K_1} - 1.0} \qquad (24)$$

$$(2) \quad K_1(\min) = \frac{T}{2.0} + \log\left[\frac{e^{-T} + 1.0}{2.0}\right]^{1/2} \qquad (25)$$

$$(3) \quad K_2(\min) = \frac{T}{2.0} \qquad (26)$$

Thus we see that in order to maintain a high degree of system accuracy the value of T must be kept small. (The above equations are derived in the Appendix.)

VI. SUMMARY AND CONCLUSIONS

A second order dynamical system involving sampled data has been analyzed from the time-optimal control viewpoint. From the results of this analysis we can conclude that

(1) A bang-bang forcing sequence can be used effectively for optimal control, subject to certain constraints.

(2) Bang-bang control with an intermediate coast period produces a state transference between certain initial and final states in minimum time.

(3) The proposed solution does not provide a simpler control scheme than that proposed by Desoer and Wing for a similar problem.

This study has provided some insight into the complexities introduced by the sampling or discrete control process. Discrete control for this class of dynamical systems, e.g. rolling of an airplane or rotation of an electric motor, is considerably more cumbersome than its counterpart in the continuous control field.

VII. APPENDIX

A. Derivation of K_1(min) Equation

The minimum acceptable value of the system constant, K_1, is determined by the choice of control scheme and the maximum value of $\dot{c}(0)$. For an arbitrary force program and $\dot{c}(0) = 1.0$, K_1 should be set equal to $T/2$ for satisfactory switching performance.

A smaller value for K_1 results when one uses the proposed scheme in Figure 6. From the geometry of Figure 8 we see that

$$T = 2K_1 + t_1$$

or

$$K_1 = \frac{T}{2} - \frac{t_1}{2}$$

We can evaluate t_1 using the difference equations derived in the text, where, by applying $\Delta = 0$ to the choice of $\dot{c}(0)$ or $\dot{c}_m$ we get

$$\dot{c}(t_1) = e^{-T} \dot{c}_m$$

on applying $\Delta = +1.0$, for one interval T, to $\dot{c}_m$ we obtain

$$\dot{c}(T) = e^{-T} [\dot{c}_m - 1.0] + 1.0$$

on applying $\Delta = -1.0$, for an interval t_1, to $\dot{c}(T)$ we obtain

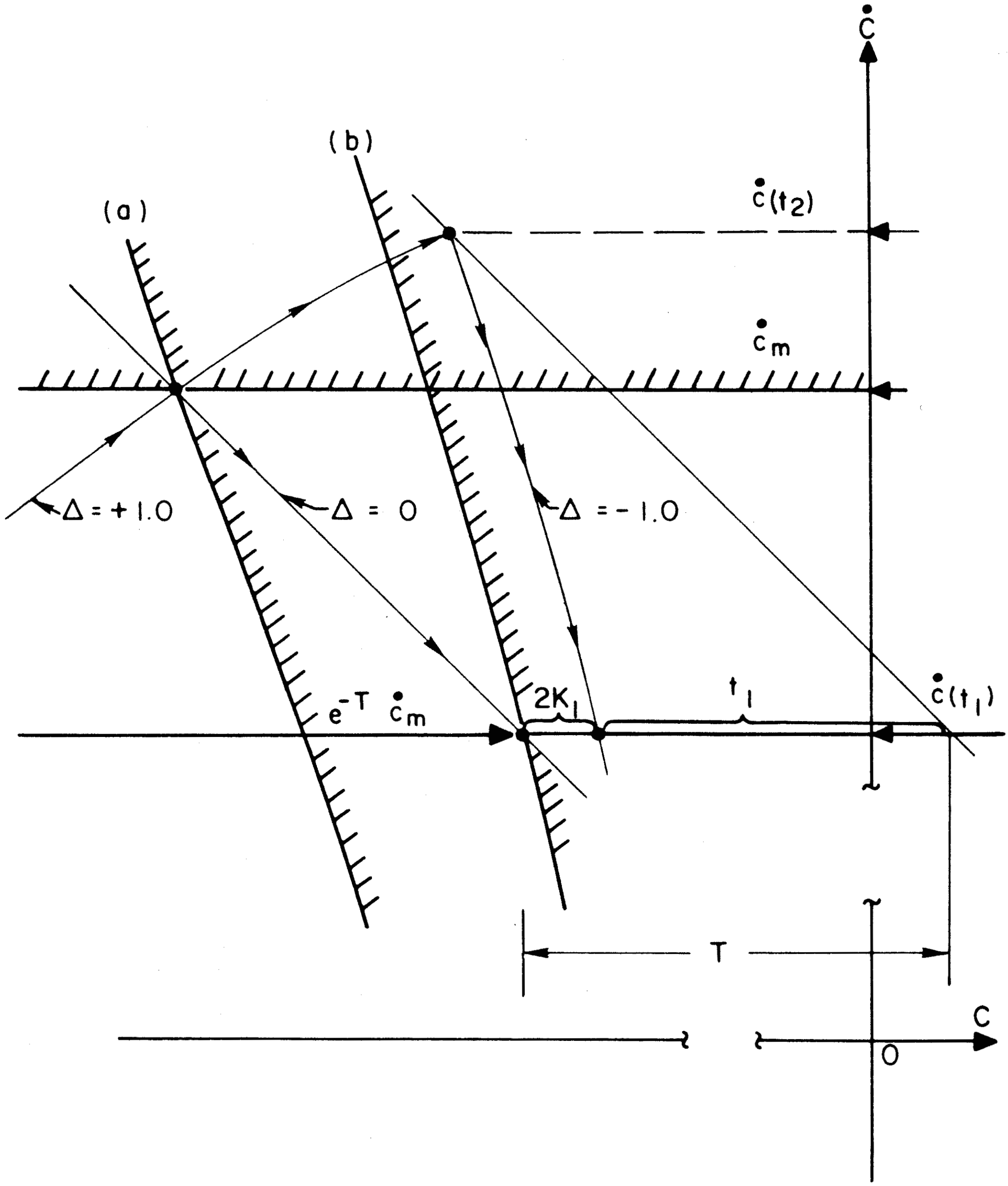


FIGURE 8

PHASE PLANE GEOMETRY FOR EVALUATING K_1(min)

$$\dot{c}(t_1) = e^{-t_1} [\dot{c}(T) + 1.0] - 1.0$$

combining these equations we see that

$$\frac{-t_1}{2} = \log \left[\frac{\dot{c}_m e^{-T} + 1.0}{e^{-T} (\dot{c}_m - 1.0) + 2.0} \right]^{1/2}$$

or when $\dot{c}_m = 1.0$

$$\frac{-t_1}{2} = \log \left[\frac{e^{-T} + 1.0}{2.0} \right]^{1/2}$$

and

$$K_1 = \frac{T}{2.0} + \log \left[\frac{e^{-T} + 1.0}{2.0} \right]^{1/2} = K_1(\min) \Big|_{\dot{c}_m = 1.0} \qquad (25)$$

B. Derivation of ϵ Equation

On deriving an expression for ϵ, we will have the required constants for the target boundary equation

$$c + \epsilon \dot{c} = K_1$$

An expression for ϵ can be obtained in a manner similar to the derivation of K_1. From the geometry of Figure 9 we note that

$$\epsilon = \tan^{-1} \frac{b}{\dot{c}(t_\epsilon)}$$

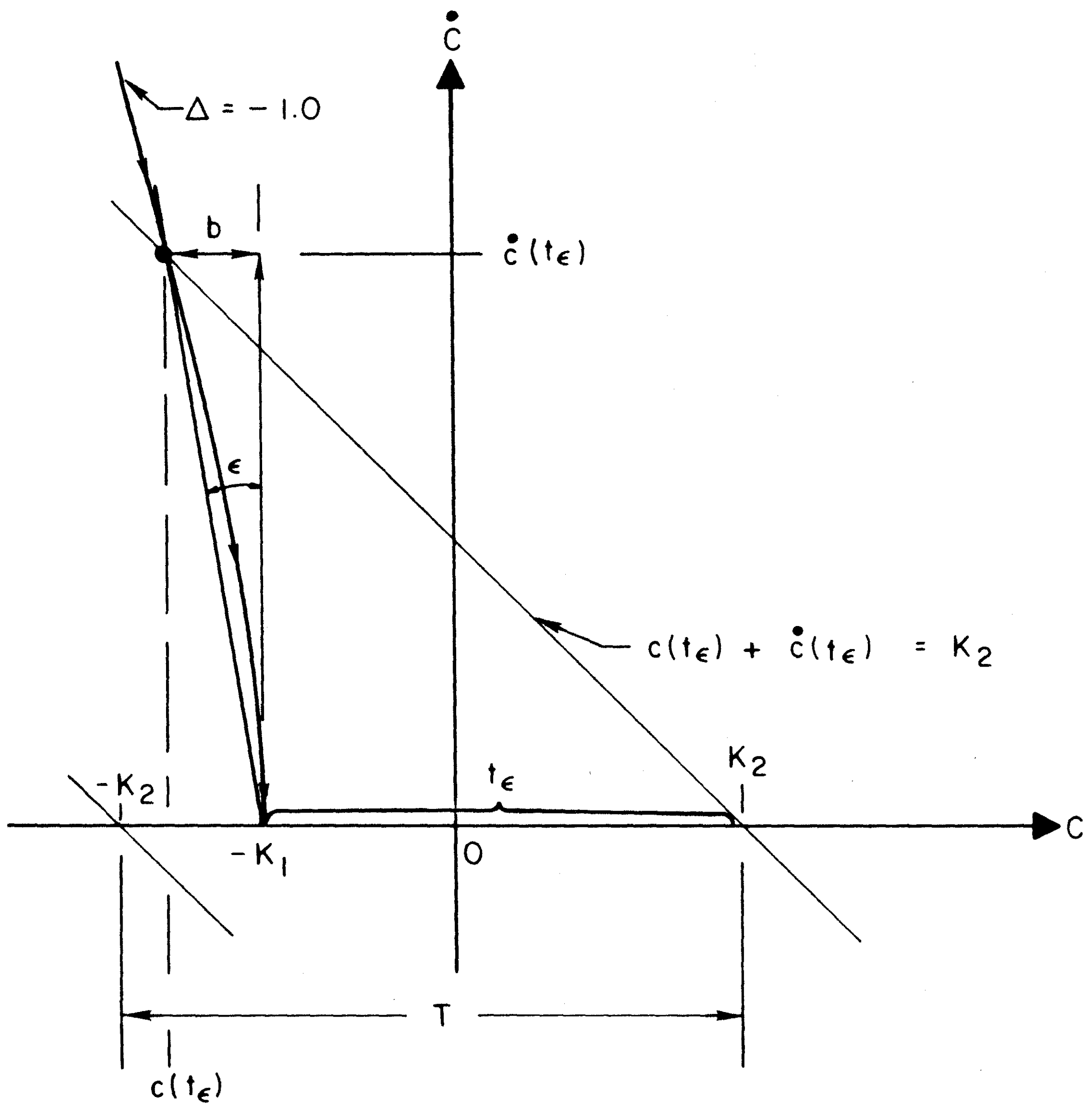


FIGURE 9

PHASE PLANE GEOMETRY FOR EVALUATING ϵ

and

$$b = |c(t_\epsilon) + K_1|$$

$$t_\epsilon = T/2 + K_1$$

and on evaluating $\dot{c}(t_\epsilon)$ and $c(t_\epsilon)$ we see that

$$\dot{c}(t_\epsilon) = e^{t_\epsilon} - 1.0$$

$$c(t_\epsilon) + \dot{c}(t_\epsilon) = T/2$$

or

$$c(t_\epsilon) = \frac{T}{2} - (e^{t_\epsilon} - 1.0)$$

so

$$\epsilon \text{ (radians)} = \tan^{-1} \left| \frac{T/2 + K_1 - (e^{t_\epsilon} - 1.0)}{e^{t_\epsilon} - 1.0} \right|$$

which for small angles $\tan \epsilon \approx \epsilon$ then

$$\epsilon \approx \left| \frac{T/2 + K_1 - (e^{t_\epsilon} - 1.0)}{e^{t_\epsilon} - 1.0} \right|$$

C. Derivation of Switching Boundary (b)

This boundary in the upper half plane represents a section of a $\Delta = -1.0$ trajectory. On applying the boundary conditions $c(T) = -K_1$ and $\dot{c}(T) = 0$ to the basic difference equations we see that

$$\dot{c}(0) = e^{T} - 1.0$$

and

$$c(0) = T - K_1 - (e^{T} - 1.0)$$

for generality we introduce the dummy variable t_k so that

$$c(t_k)_b = t_k - K_1 - (e^{t_k} - 1.0) \qquad (20)$$

$$\dot{c}(t_k)_b = e^{t_k} - 1.0 \qquad (21)$$

D. Derivation of Switching Boundary (a)

This boundary is generated by adding a $\Delta = 0$ sequence to boundary (b) such that

$$\dot{c}(t_k)_a = e^{T}\,\dot{c}(t_k)_b$$

and noting that

$$c(t_k)_a + \dot{c}(t_k)_a = c(t_k)_b + \dot{c}(t_k)_b$$

we then obtain

$$c(t_k)_a = t_k - K_1 - e^T(e^{t_k} - 1.0) \tag{18}$$

$$c(t_k)_a = e^T(e^{t_k} - 1.0) \tag{19}$$

REFERENCES

1. Bellman, R., Glicksburg, I., and Gross, O. "On the 'Bang-Bang' Control Problem," Quarterly of Applied Mathematics, (1956), Vol. 14, pp. 11-18.

2. Desoer, C. A. "The Bang-Bang Servo Problem Treated by Variational Techniques," Information and Control, (1959), Vol. 2, No. 4, pp. 333-348.

3. Tsien, H. S. "Engineering Cybernetics," McGraw-Hill Book Company, Inc., (1954), pp. 136-159.

4. Athanassiades, M. "Bang-Bang Control of Real Pole Systems," Electronics Research Laboratory, University of California, Berkeley, (1961), Series No. 60, Issue No. 377.

5. Desoer, C. A., and Wing, J. "An Optimal Strategy for a Saturating Sampled-Data System," Electronics Research Laboratory, University of California, Berkeley, (1959), Series No. 60, Issue No. 262.

6. Mullin, F. J., and Jury, E. I. "A Phase Plane Approach to Relay Sampled-Data Feedback Systems," Transactions Paper AIEE, (1958), No. 58-1002.

7. Altar, W., and Helstrom, C. W. "Phase-Plane Representation of Sampling Servomechanisms," Westinghouse Research Laboratories, Pittsburgh, Pennsylvania, (1953), Research Report No. 60-94410-14-B.

8. Gille, J-C, Pelegrin, M. J., and Decaulne, P. "Feedback Control Systems," McGraw-Hill Book Company, Inc., (1959), pp. 432-441.