PART I

CYLINDRICAL COUETTE FLOW IN A RAREFIED GAS ACCORDING TO GRAD'S EQUATIONS

PART II

SMALL PERTURBATIONS IN THE UNSTEADY FLOW OF A RAREFIED GAS BASED ON GRAD'S THIRTEEN MOMENT APPROXIMATION

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PART I

CYLINDRICAL COUETTE FLOW IN A RAREFIED GAS ACCORDING TO GRAD'S EQUATIONS

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ABSTRACT

Grad's thirteen moment method is applied to the problem of the shear flow and heat conduction between two concentric, rotating cylinders of infinite length. In order to concentrate on the effects of curvature the problem is linearized by requiring that the Mach number is small compared with unity, and that the temperature difference between the two cylinders is small compared with the mean temperature. The solutions of the linearized Grad equations show a qualitatively correct transition of the cylinder drag from free-molecule flow to the classical Navier-Stokes regime. However the magnitude of the curvature effect on the drag in rarefied flow is not given correctly, because Grad's distribution function ignores the wedge-like domains of influence of the two cylinders.

The solution obtained for the heat transfer rate is physically unrealistic in the free-molecule flow limit, and this result is produced by a cross-coupling between the normal stresses and the radial heat flux imposed by Grad's distribution function. In this simple problem the difficulty can be eliminated by taking the normal stresses to be identically zero and employing a truncated moment method. However, in general this device cannot be utilized in problems involving curved solid boundaries, or when dissipation is considered. One concludes that the choice of the distribution function to be employed in Maxwell's moment equations is dictated by the requirements imposed in the limiting case of highly rarefied gas flows, as well as in the Navier-Stokes regime.

TABLE OF CONTENTS

PART | PAGE

LIST OF FIGURES

LIST OF SYMBOLS

a	radius of inner cylinder (rotating)
b	radius of outer cylinder (stationary)
$\vec{c}$	intrinsic velocity, $\vec{c} = \vec{\xi} - \vec{u}(x, t)$
C_D	drag coefficient
C_H	Stanton number
f	velocity distribution function
f_o	Maxwellian velocity distribution function $= \frac{\rho}{(2\pi RT)^{3/2}} \exp\left\{- c^2/2RT\right\}$
k	coefficient of heat conductivity $= 15/4\ R\mu_1$
p	pressure of gas
p_{ij}	stress (increment over hydrostatic pressure)
Pr	Prandtl number $= \frac{c_p \mu_1}{k}$
q_i	heat flux
r, θ	cylindrical coordinates
R	gas constant
Re	Reynolds number $= \rho_1 U \left(\frac{b-a}{\mu_1}\right)$
T	temperature of gas
u, v	velocity components in x, y direction of Cartesian coordinates
u_r, u_θ	velocity components in r, θ directions of cylindrical coordinates
U	velocity at surface of rotating wall
x, y	rectangular Cartesian coordinates
α	fraction of incident gas molecules specularly reflected from Cylinder walls
λ	mean free path of the gas molecules
μ	coefficient of viscosity
$\vec{\xi}$	molecule velocity

ρ density of gas

Subscripts

1 quantities in the gas at the surface of the inner cylinder

2 quantities in the gas at the surface of the outer cylinder

I. INTRODUCTION

Because of the well-known difficulties encountered in attempting to solve the Maxwell-Boltzmann integro-differential equation, a number of investigators have turned instead to Maxwell's integral equations of transfer.* In this procedure the Maxwell-Boltzmann equation is satisfied in a certain average sense, rather than point-by-point, and the particle velocity distribution function is regarded as a suitable weighting function. The first modern application of Maxwell's technique to fluid mechanics is H. Grad's[2] thirteen-moment method, which utilizes the "local Maxwellian", multiplied by a polynomial of the Chapman-Enskog type. The coefficients of this polynomial contain the corresponding stresses and heat flux quantities, which are now regarded as new dependent variables to be determined by solving thirteen simultaneous first-order differential equations obtained from the Maxwell moment equations. Of course in special problems the number of moments required is much less than thirteen.

Yang[3] and Lees applied Grad's method to the problem of the steady shearing motion and heat conduction between two infinite, parallel flat plates. In order to bring out some of the main features of Grad's method, without becoming involved in undue mathematical complications, the problem is linearized by requiring that the Mach number is small compared with unity, and that the temperature difference between the two plates is small compared with ambient temperature. Reasonable results

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* See Reference 1 for a brief review and discussion of some of this work.

for drag, heat transfer and velocity and temperature profiles were obtained over the whole range of gas densities. In the limit $Re/M \longrightarrow 0$ these results agree with the usual free-molecule flow quantities, while in the opposite limiting case $Re/M >> 1$ they join smoothly to the classical Navier-Stokes and Fourier relations.

Linearized, steady, plane Couette flow is undoubtedly too simple to provide a meaningful test of any integral method. One would like to investigate the influence of dissipation and streamline curvature on molecular effects. Such a study utilizing Grad's equations might be helpful in answering questions about the sensitivity of the results obtained by Maxwell's integral method to the form of the weighting function employed. One of the simplest situations involving curvature is the problem of shear flow and heat conduction between two concentric, rotating cylinders of infinite length (cylindrical Couette flow). In addition, this flow is one of the few that have been studied experimentally over the whole range of gas density by several different investigators[4, 5].

On the theoretical side, Rose[6] was the first to apply Grad's equations to cylindrical Couette flow, but the results were never published. In a private communication Dr. Grad states that no explicit solutions of the non-linear problem were obtained. Chiang[7] also had some difficulties with the non-linear Grad equations for this problem, and he resorted instead to an expansion procedure in powers of M^2/Re. Up to second-order terms this procedure is identical to the Burnett expansion[8], and is not very helpful for rarefied flows. In the present study the problem is linearized by requiring that $M^2 << 1$ and $\Delta T/T << 1$, in order to concentrate on the curvature effect. In Section II the full Grad equations and boundary conditions for steady, cylindrical Couette flow are written down, and the usual conservation integrals are obtained. In Section III the linearized equations and boundary conditions are formulated and solved, and the results compared with experiments and with the expression for the cylinder drag suggested in Reference 3. Section IV contains a critical discussion of the results, some conclusions about the difficulties inherent in utilizing Grad's form of the weighting function, and some observations on the rules that must guide the selection of a suitable weighting function.

II. EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

II. 1. Equations of Motion for Cylindrical Couette Flow

Grad's general equations of motion for a two-dimensional problem in cylindrical coordinates are given in Appendix I. In the case of steady cylindrical Couette flow symmetry requires that all mean quantities are functions of r only; hence $(\partial/\partial t) = (\partial/\partial\theta) = 0$, and we are dealing with ordinary differential equations. The mean quantities involved in the problem are the following:

u_r , u_θ	r and θ components of the velocity vector
q_r , q_θ	r and θ components of the energy flux
p_{rr}, $p_{\theta\theta}$, $p_{r\theta}$	stress components (increment over hydrostatic pressure)
p, ρ, T	thermodynamic variables; pressure, density, and temperature .

Hence, we have ten unknowns to determine.

Grad's thirteen moment approximations furnish a set of nine moment equations for the 2-dimensional case; thus, one more equation is needed. This additional equation is obtained by relating the temperature to a certain second moment of the distribution function f. Since Grad's scheme is set up for monatomic molecules, each of which has three degrees of freedom, the kinetic energy per unit mass is 3/2 RT. An element of kinetic energy is $\frac{1}{2} c^2 f \, d\vec{x} \, d\vec{\xi}$; integration over all values of $\vec{\xi}$ yields the equation of state

$$3/2\, RT = 1/\rho \int \tfrac{1}{2} c^2 f \, d\vec{\xi} = (3/2)(p/\rho)$$

or

$$p = \rho R T \tag{1}$$

The nine moment equations are

Continuity

$$(d/dr)(\rho\, u_r\, r) = 0 \tag{2}$$

Momentum

$$\frac{dp}{dr} + \frac{dp_{rr}}{dr} + \frac{p_{rr} - p_{\theta\theta}}{r} + \rho u_r \frac{du_r}{dr} - \rho \frac{u_\theta^2}{r} = 0 \tag{3}$$

$$\frac{dp_{r\theta}}{dr} + \frac{2p_{r\theta}}{r} + \rho u_r \frac{du_\theta}{dr} + \rho \frac{u_\theta u_r}{r} = 0 \tag{4}$$

Energy

$$\begin{aligned} &\frac{5}{3} p \left(\frac{du_r}{dr} + \frac{u_r}{r} \right) + \frac{2}{3} \left(\frac{dq_r}{dr} + \frac{q_r}{r} \right) + \frac{2}{3} \Big(p_{rr} \frac{du_r}{dr} \\ &+ p_{r\theta} \frac{du_\theta}{dr} + p_{\theta\theta} \frac{u_r}{r} - p_{r\theta} \frac{u_\theta}{r} \Big) + u_r \frac{dp}{dr} = 0 \end{aligned} \tag{5}$$

Stresses

$$\begin{aligned} &\frac{2}{3} p \left(2 \frac{du_r}{dr} - \frac{u_r}{r} \right) + \frac{4}{15} \left(2 \frac{dq_r}{dr} - \frac{q_r}{r} \right) + u_r \frac{dp_{rr}}{dr} \\ &- \frac{2u_\theta}{r} p_{r\theta} + \frac{7}{3} p_{rr} \frac{du_r}{dr} - \frac{2}{3} p_{r\theta} \frac{du_\theta}{dr} - \frac{4}{3} \frac{p_{r\theta}}{r} u_\theta \\ &- \frac{2}{3} \frac{p_{\theta\theta} u_r}{r} + \frac{p_{rr} u_r}{r} = - \frac{p}{\mu} p_{rr} \end{aligned} \tag{6}$$

$$p\left(\frac{du_\theta}{dr}-\frac{u_\theta}{r}\right)+\frac{2}{5}\left(\frac{dq_\theta}{dr}-\frac{q_\theta}{r}\right)+u_r\frac{dp_{r\theta}}{dr}+\frac{u_\theta(p_{rr}-p_{\theta\theta})}{r}$$

$$+p_{rr}\frac{du_\theta}{dr}+2p_{r\theta}\frac{u_r}{r}+2p_{r\theta}\frac{du_r}{dr}-p_{\theta\theta}\frac{u_\theta}{r}=-\frac{p}{\mu}p_{r\theta} \tag{7}$$

$$-\frac{2}{3}p\left(\frac{du_r}{dr}-2\frac{u_r}{r}\right)-\frac{4}{15}\left(\frac{dq_r}{dr}-2\frac{q_r}{r}\right)+\frac{2}{r}u_\theta p_{r\theta}+u_r\frac{dp_{\theta\theta}}{dr}$$

$$+\frac{4}{3}p_{r\theta}\frac{du_\theta}{dr}-\frac{2}{3}\left(p_{rr}\frac{du_r}{dr}-p_{r\theta}\frac{u_\theta}{r}\right)+\frac{7}{3}p_{\theta\theta}\frac{u_r}{r}+p_{\theta\theta}\frac{du_r}{dr}=-\frac{p}{\mu}p_{\theta\theta} \tag{8}$$

<u>Heat Flux</u>

$$\frac{5}{2}p\frac{dRT}{dr}+RT\left(\frac{dp_{rr}}{dr}+\frac{p_{rr}-p_{\theta\theta}}{r}\right)+q_r\left(\frac{du_r}{dr}+\frac{u_r}{r}\right)+u_r\frac{dq_r}{dr}-u_\theta\frac{q_\theta}{r}$$

$$+\frac{11}{5}q_r\frac{du_r}{dr}+\frac{2}{5}\left(q_r\frac{u_r}{r}+q_\theta\frac{du_\theta}{dr}\right)-\frac{7}{5}q_\theta\frac{u_\theta}{r}-\frac{p_{rr}}{\rho}\frac{dp}{dr}+\frac{7}{2}p_{rr}\frac{dRT}{dr}$$

$$-\frac{p_{rr}}{\rho}\left(\frac{dp_{rr}}{dr}+\frac{p_{rr}-p_{\theta\theta}}{r}\right)-\frac{p_{r\theta}}{\rho}\left(\frac{dp_{r\theta}}{dr}+\frac{2p_{r\theta}}{r}\right)=-\frac{2}{3}\frac{p}{\mu}q_r \tag{9}$$

$$RT\left(\frac{dp_{r\theta}}{dr}+\frac{2p_{r\theta}}{r}\right)+q_\theta\left(\frac{du_r}{dr}+\frac{u_r}{r}\right)+u_r\frac{dq_\theta}{dr}+q_r\frac{u_\theta}{r}+\frac{11}{5}q_\theta\frac{u_r}{r}$$

$$+\frac{7}{5}q_r\frac{du_\theta}{dr}+\frac{2}{5}\left(q_\theta\frac{du_r}{dr}-q_r\frac{u_\theta}{r}\right)-\frac{p_{r\theta}}{\rho}\frac{dp}{dr}+\frac{7}{2}p_{r\theta}\frac{dRT}{dr} \tag{10}$$

$$-\frac{p_{r\theta}}{\rho}\left(\frac{dp_{rr}}{dr}+\frac{p_{rr}-p_{\theta\theta}}{r}\right)-\frac{p_{\theta\theta}}{\rho}\left(\frac{dp_{r\theta}}{dr}+\frac{2p_{r\theta}}{r}\right)=-\frac{2}{3}\frac{p}{\mu}q_\theta$$

The right-hand side terms in stresses and heat flux equations are produced by the collision integral. In the heat flux equations, the results of the stress equations are already utilized; hence $-(2/3\ p/\mu)\ q_i$ are the only terms introduced by the collision integral.

II. 2. Boundary Conditions

Since we are dealing with a cylindrical coordinate system which is orthogonal, we have a local Cartesian coordinate system; hence the boundary conditions for plane Couette flow can be applied to the cylindrical case if we simply replace the subscript x by θ and y by r.

At the outer stationary cylinder (r = b) the boundary conditions are as follows[3]:

$$u_r(b) = 0 \tag{11}$$

$$-\frac{p_{r\theta}(b)}{p(b)} + \frac{2(1-\alpha)}{(1+\alpha)} \frac{u_\theta(b)}{[2\pi RT(b)]^{\frac{1}{2}}}\left(1 + \frac{p_{rr}(b)}{2p(b)}\right) + \frac{2}{5}\frac{(1-\alpha)}{(1+\alpha)}\frac{q_\theta(b)}{[2\pi RT(b)]^{\frac{1}{2}}p(b)} = 0 \tag{12}$$

$$-\left[\frac{2\pi}{RT(b)}\right]^{\frac{1}{2}}\frac{q_r(b)}{p(b)} + \frac{4(1-\alpha)}{(1+\alpha)}\left[1 - \frac{T_2}{T(b)} - \frac{p_{rr}(b)}{2p(b)}\left(\frac{3}{2} - \frac{T_2}{T(b)}\right) + \left(1 + \frac{p_{rr}(b)}{2p(b)}\right)\frac{\frac{[u_\theta(b)]^2}{2}}{RT(b)}\right] = 0 \tag{13}$$

At the inner rotating cylinder (r = a)

$$u_r(a) = 0 \tag{14}$$

$$\frac{p_{r\theta}(a)}{p(a)} + \frac{2(1-\alpha)}{(1+\alpha)}\frac{u_\theta(a) - U}{[2\pi RT(a)]^{\frac{1}{2}}}\left(1 + \frac{p_{rr}(a)}{2p(a)}\right) + \frac{2}{5}\frac{(1-\alpha)}{(1+\alpha)}\frac{q_\theta(a)}{[2\pi RT(a)]^{\frac{1}{2}}p(a)} = 0 \tag{15}$$

$$\left[\frac{2\pi}{RT(a)}\right]^{\frac{1}{2}}\frac{q_r(a)}{p(a)} + \frac{4(1-\alpha)}{(1+\alpha)}\left[1 - \frac{T_1}{T(a)} + \frac{p_{rr}(a)}{2p(a)}\left(\frac{3}{2} - \frac{T_1}{T(a)}\right) + \left(1 + \frac{p_{rr}(a)}{2p(a)}\right)\frac{\frac{(u_\theta(a) - U)^2}{2}}{RT(a)}\right] = 0 \tag{16}$$

Equations (11) and (14) represent conservation of mass. Equations (12), (15) and (13), (16) represent conservation of momentum and energy, respectively, at outer and inner walls.

II. 3. General Solutions of Cylindrical Flow

From the integration of Eq. (2) and with the aid of the boundary conditions (11) and (14), we obtain that everywhere

$$u_r = 0 \quad . \tag{17}$$

This result shows clearly that only shear flow exists. By utilizing this result, the rest of the differential equations are simplified enormously and we are able to integrate the conservation equations immediately [Eqs. (4) and (5)]:

$$p_{r\theta} = B/r^2 \tag{18}$$

$$r(q_r + p_{r\theta}\, u_\theta) = c \quad . \tag{19}$$

Equation (18) states that the torque is constant across the annulus, while Eq. (19) states that the flux of heat energy plus the rate at which work is done on the fluid by the shear stress is a constant. So far the integrals obtained are valid for the flow between two concentric cylinders at arbitrary temperature difference and rotating speed. A study of the equations shows that solutions in reasonably simple form are difficult to find. In order to bring out the effect of curvature as simply as possible linearized equations and boundary conditions will be used instead.

III. CYLINDRICAL COUETTE FLOW AT LOW MACH NUMBERS AND SMALL TEMPERATURE DIFFERENCES

III. 1. The Linearization of Equations of Motion and Boundary Conditions

When the inner cylinder is rotating slowly and the temperature difference between the cylinders is kept small, or more precisely if

$$M = U/\sqrt{\gamma R T_1} << 1$$

and

$$\frac{T_1 - T_2}{T_1} >> 1 \quad ,$$

the thermodynamical quantities may be expressed as

$$\begin{aligned} \rho &= \rho_1 + \rho' \\ p &= p_1 + p' \\ T &= T_1 + T' \end{aligned} \tag{20}$$

and the coefficient of viscosity

$$\mu = \mu_1 + \mu' \quad . \tag{21}$$

For the remaining quantities, we have

$$Q = Q' \quad , \tag{22}$$

where Q denotes any velocity component, stress, or heat flux component. Subscript 1 denotes quantities evaluated at the inner cylinder, used here as the reference base, and the prime denotes perturbations.

If the expressions (20) to (22) are introduced into Eqs. (1) to (16) and all quadratic terms of small perturbations are neglected, the equations of motion as well as the boundary conditions are linearized. Furthermore, the tangential quantities u_θ , $p_{r\theta}$, q_θ and the normal quantities ρ, p (or T), p_{rr} , $p_{\theta\theta}$, and q_r are separated. This uncoupling of tangential and normal quantities has been pointed out by Yang and Lees[3, 9] as being typical of the particular linearization procedure. The remaining equations of motion in linearized form are as follows:

Momentum

$$\frac{dp'}{dr} + \frac{dp'_{rr}}{dr} + \frac{p'_{rr} - p'_{\theta\theta}}{r} = 0 \tag{23}$$

Stresses

$$\frac{4}{15}\left(2\frac{dq'_r}{dr} - \frac{q'_r}{r}\right) = -\frac{p_1}{\mu_1}p'_{rr} \tag{24}$$

$$p_1\left(\frac{du'_\theta}{dr} - \frac{u'_\theta}{r}\right) + \frac{2}{5}\left(\frac{dq'_\theta}{dr} - \frac{q'_\theta}{r}\right) = -\frac{p_1}{\mu_1}p'_{r\theta} \tag{25}$$

$$-\frac{4}{15}\left(\frac{dq'_r}{dr} - 2\frac{q'_r}{r}\right) = -\frac{p_1}{\mu_1}p'_{\theta\theta} \tag{26}$$

Heat Flux

$$\frac{5}{2}p_1\frac{dRT'}{dr} - RT_1\frac{dp'}{dr} = -\frac{2}{3}\frac{p_1}{\mu_1}q'_r \tag{27}$$

$$q_\theta' = 0 \tag{28}$$

State

$$p' = \rho' RT_1 + \rho_1 RT' \tag{29}$$

After linearization the shear stress $p_{r\theta}$ is given by a form identical to the Navier-Stokes stress strain rate relation [Eqs. (25) and (28)]; therefore, we obtain the same expression for u_θ as in the incompressible Navier-Stokes solution. However, the normal stresses p_{rr} and $p_{\theta\theta}$ are described by more complicated expressions. Here the heat flux rates are coupled in with the normal stresses. This coupling is inherent in Grad's scheme and as a result introduces difficulties in the heat transfer problem. (See Sections III. 4 and IV.)

As far as Eq. (27) is concerned, we obtain the familiar Fourier conduction law after simply identifying the coefficient of heat conductivity k by $15/4\ R\mu_1$.

The linearized boundary conditions are at r = b,

$$-\frac{p_{r\theta}'(b)}{p_1} + \frac{2(1-\alpha)}{(1+\alpha)} \frac{u_\theta'(b)}{[2\pi R T_1]^{\frac{1}{2}}} = 0 \tag{30}$$

$$-\left(\frac{2\pi}{RT_1}\right)^{\frac{1}{2}} \frac{q_r'(b)}{p_1} + \frac{4(1-\alpha)}{(1+\alpha)}\left[\frac{T_1-T_2}{T_1} + \frac{T_2}{T_1}\frac{T'(b)}{T_1} - \frac{p_{rr}'(b)}{4p_1}\right] = 0 \tag{31}$$

and at r = a,

$$\frac{p_{r\theta}'(a)}{p_1} + \frac{2(1-\alpha)}{(1+\alpha)} \frac{u_\theta'(a) - U}{[2\pi R T_1]^{\frac{1}{2}}} = 0 \tag{32}$$

$$\left(\frac{2\pi}{RT_1}\right)^{\frac{1}{2}} \frac{q_r'(a)}{p_1} + \frac{4(1-\alpha)}{(1+\alpha)}\left[\frac{T'(a)}{T_1} + \frac{p_{rr}'(a)}{4p_1}\right] = 0 \tag{33}$$

III. 2. The Solution of the Linearized Equations of Motion

By simplifying Eqs. (5) and (19) we have

$$q_r' = c/r \quad . \tag{34}$$

The integration of Eq. (25) yields

$$u_\theta' = c_1 r + \frac{B}{2\mu_1 r} \quad , \tag{35}$$

which is the same expression as the incompressible Navier Stokes solution.

Eqs. (24) and (26) give the normal stresses

$$p_{rr}' = \frac{\mu_1}{p_1} \frac{4c}{5r^2} \tag{36}$$

$$p_{\theta\theta}' = - \frac{\mu_1}{p_1} \frac{4c}{5r^2} \tag{37}$$

These normal stresses are identically zero in the Navier Stokes solution because div $\vec{u} = 0$, $u_r = 0$, and $u_\theta = u_\theta(r)$.

Substituting p_{rr}' and $p_{\theta\theta}'$ into Eq. (23) we arrive at

$$dp'/dr = 0 \qquad \text{or} \qquad p' = \text{constant} \tag{38}$$

Since the pressure is defined as $p = p_1 + p'$, we see that p' is just an additive constant; therefore, it may be set equal to zero, i.e., $p = p_1$ throughout the flow field. Knowing q_r' we can integrate Eq. (27) to give

$$T' = -(4/15)\,\frac{c}{\mu_1 R}\,\ln r + c_2 \quad . \tag{39}$$

T' satisfies the Laplace equation and it is natural we obtain the term $\ln r$ in cylindrical coordinates. Finally, with the aid of the equation of state we have

$$\rho' = (4/15)\,\frac{c}{\mu_1 R}\,\frac{\rho_1}{T_1}\,\ln r - (\rho_1/T_1)\,c_2 \quad . \tag{40}$$

III. 3. The Evaluation of Tangential Quantities -- Shear Stress, Tangential Velocity, and Heat Flux

The tangential quantities $p_{r\theta}$, u_θ ($q_\theta = 0$) given by the linearized Grad equations are identical in form with the Navier-Stokes solution, but the boundary conditions are quite different. We shall express these quantities in such a way that the quantity Re/M appears as a parameter. The constants B and c_1 in $p_{r\theta}$ and u_θ may be evaluated by utilizing the boundary conditions (30) and (32). After substituting the results of Eqs. (18), (35), and (38), Eqs. (30) and (32) become

$$-\frac{B}{p_1 b^2} + 2\,\frac{(1-\alpha)}{(1+\alpha)}\,\frac{1}{\sqrt{2\pi R T_1}}\left(c_1 b + \frac{B}{2\mu_1 b}\right) = 0 \tag{41}$$

$$\frac{B}{p_1 a^2} + \frac{2(1-\alpha)}{(1+\alpha)}\,\frac{1}{\sqrt{2\pi R T_1}}\left(c_1 a + \frac{B}{2\mu_1 a}\right) = \frac{2(1-\alpha)}{(1+\alpha)}\,\frac{U}{\sqrt{2\pi R T_1}} \, . \tag{42}$$

By solving these simultaneous equations for B and c_1, we obtain

$$B = \frac{\frac{2b^2}{b-a} U \mu_1 Re/M}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{10\pi}{3}}\left(\frac{a}{b}+\frac{b^2}{a^2}\right)+\frac{a+b}{a}\frac{Re}{M}} \tag{43}$$

$$c_1 = \frac{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{10\pi}{3}} - \frac{Re}{M}\frac{b}{b-a}}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{10\pi}{3}}\left(\frac{a}{b}+\frac{b^2}{a^2}\right)+\frac{a+b}{a}\frac{Re}{M}} \frac{U}{b} \tag{44}$$

where

$$M = \frac{U}{\sqrt{\frac{5}{3} RT_1}} \qquad \text{and} \qquad Re = \rho_1 U \frac{(b-a)}{\mu_1} ,$$

for a monatomic gas.

Hence, we obtain

$$p_{r\theta} = \frac{\frac{2b^2}{b-a} U \mu_1 \frac{Re}{M}}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{10\pi}{3}}\left(\frac{a}{b}+\frac{b^2}{a^2}\right)+\frac{a+b}{a}\frac{Re}{M}} \frac{1}{r^2} \tag{45}$$

$$u_\theta/U = \frac{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{10\pi}{3}}\frac{r}{b} + \frac{Re}{M}\frac{b}{b-a}\left(\frac{b}{r}-\frac{r}{b}\right)}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{10\pi}{3}}\left(\frac{a}{b}+\frac{b^2}{a^2}\right)+\frac{a+b}{a}\frac{Re}{M}} \tag{46}$$

The velocity profile u_θ/U across the annulus is plotted in Figure 2 for different values of Re/M.

The drag coefficient C_D multiplied by the Mach number M is defined as

$$C_D M = \frac{p_{r\theta} M}{\frac{1}{2} \rho_1 U^2}$$

At the stationary wall (r = b), $p_{r\theta}(b) = B/b^2$; hence

$$C_D M = \frac{4}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{10\pi}{3}}\left(\frac{a}{b}+\frac{b^2}{a^2}\right) + \frac{a+b}{a}\frac{Re}{M}} \tag{47}$$

or

$$\frac{1}{C_D M} = \frac{1}{4} \frac{1+\alpha}{1-\alpha} \left(\frac{a}{b} + \frac{b^2}{a^2} \right) \sqrt{\frac{10\pi}{3}} + \frac{1}{2} \left(1 + \frac{b-a}{2a} \right) \frac{Re}{M} \tag{48}$$

For a diffusively reflecting surface ($\alpha = 0$) ,

$$\frac{1}{C_D M} = \frac{1}{4} \left(\frac{a}{b} + \frac{b^2}{a^2} \right) \sqrt{\frac{10\pi}{3}} + \frac{1}{2} \left(1 + \frac{b-a}{2a} \right) \frac{Re}{M} \tag{49}$$

The above expression is in complete agreement with the result obtained by C. Y. Liu for small ratio of annulus width to cylinder radius in his analysis based on L. Lees method.[1] In the limiting case when a approaches b, $1/C_D M$ takes the form of Eq. (50), which is identical to the result for plane Couette flow found by Yang[3] and Lees.

$$1/C_D M = \frac{1}{2} \left(\sqrt{\frac{10\pi}{3}} + \frac{Re}{M} \right) \tag{50}$$

In that paper it was suggested that the drag on the stationary outer cylinder can be written in the form

$$1/C_D M = A\,(b/a) + B\,(b/a)\ Re/M \quad .$$

By analogy with the case of plane Couette flow it was thought that the function B(b/a) should be identical with the expression for drag obtained in the Navier-Stokes regime, and Eq. (49) shows that this supposition is correct. However, the function A(b/a) was taken from the free molecule flow result of Bowyer[4] and Talbot, i.e.,

$$A\,(b/a) = \tfrac{1}{2}\,(b^2/a^2)\ \sqrt{10\pi/3} \quad .$$

Evidently the value of the drag coefficient given by Eq. (49) in the limit $Re/M \longrightarrow 0$ is $2(1 + a^3/b^3)^{-1}$ times larger than the correct value. (See Section IV.)

To determine the slip velocity, we have at r = b

$$\frac{u_\theta(b)}{U} = \frac{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{10\pi}{3}}}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{10\pi}{3}}\left(\frac{a}{b}+\frac{b^2}{a^2}\right)+\frac{a+b}{a}\,\frac{Re}{M}} \tag{51}$$

and at r = a

$$\frac{u_\theta(a)}{U} = \frac{u_\theta(b)}{U} + \frac{\frac{a+b}{a}\,\frac{Re}{M}}{\frac{1+\alpha}{1-\alpha}\left(\frac{a}{b}+\frac{b^2}{a^2}\right)\sqrt{\frac{10\pi}{3}}+\frac{a+b}{a}\,\frac{Re}{M}} \quad . \tag{52}$$

In the limit when $Re/M \longrightarrow \infty$, we obtain

$$\frac{u_\theta(b)}{U} = 0 \tag{53}$$

$$\frac{u_\theta(a)}{U} = 1 \quad . \tag{54}$$

These results represent nothing but the usual no-slip boundary conditions associated with the "Navier-Stokes" limit.

In the other limit when $Re/M \longrightarrow 0$ we obtain

$$\frac{u_\theta(a)}{U} = \frac{u_\theta(b)}{U} = \frac{1}{(a/b) + (b^2/a^2)} \quad . \tag{55}$$

Furthermore, we have

$$\frac{u_\theta(b)}{U} = \frac{u_\theta(a)}{U} = \frac{1}{2} \tag{56}$$

if the gap b-a approaches zero, so that again the results are reduced to that of the plane case in the limit $b/a \longrightarrow 1$. The variations of $\frac{u_\theta(a)}{U}$ and $1/C_D M$ vs. Re/M for a diffusively reflecting surface ($\alpha = 0$) are plotted in Figures 3 and 4, respectively.

III. 4. The Evaluation of Normal Quantities -- Normal Stresses, Normal Heat Flux, and Thermodynamic Variables

As we pointed out earlier, the normal and tangential quantities are uncoupled after the linearization, so that what is left here reduces to the case of steady state heat conduction between two cylinders at rest. The whole problem will be solved after the evaluation of the two remaining constants c and c_2, and this can be done by substituting q_r', p_{rr}', and

T' into the two remaining boundary conditions Eqs. (31) and (33). We have therefore

$$\left(\frac{2\pi}{RT_1}\right)^{\frac{1}{2}}\frac{c}{bp_1} - \frac{4(1-\alpha)}{(1+\alpha)}\left[\frac{T_1-T_2}{T_1} - \frac{T_2}{T_1}\frac{4}{15}\frac{c}{\mu_1 RT_1}\ln b + \frac{T_2}{T_1}\frac{c_2}{T_1} - \frac{c\mu_1}{5p_1^2 b^2}\right] = 0 \tag{57}$$

$$\left(\frac{2\pi}{RT_1}\right)^{\frac{1}{2}}\frac{c}{ap_1} + \frac{4(1-\alpha)}{(1+\alpha)}\left[-\frac{4}{15}\frac{c}{\mu_1 RT_1}\ln a + \frac{c_2}{T_1} + \frac{1}{5}\frac{c\mu_1}{p_1^2 a^2}\right] = 0 \tag{58}$$

By solving these simultaneous equations for c and c_2, we obtain

$$c = \frac{\dfrac{T_1 - T_2}{T_1}}{\left(\dfrac{2\pi}{RT_1}\right)^{\frac{1}{2}}\dfrac{1}{bp_1}\left(1+\dfrac{b}{a}\dfrac{T_2}{T_1}\right)\dfrac{(1+\alpha)}{4(1-\alpha)} + \dfrac{T_2}{T_1}\dfrac{4}{15R\mu_1 T_1}\ln\dfrac{b}{a} + \dfrac{\mu_1}{5p_1^2 b^2}\left(1+\dfrac{b^2}{a^2}\dfrac{T_2}{T_1}\right)} \tag{59}$$

and

$$c_2 = -(T_1 - T_2)\frac{\left(\dfrac{2\pi}{RT_1}\right)^{\frac{1}{2}}\dfrac{1}{ap_1}\dfrac{1+\alpha}{4(1-\alpha)} - \dfrac{4}{15R\mu_1 T_1}\ln a + \dfrac{\mu_1}{5p_1^2 a^2}}{\left(\dfrac{2\pi}{RT_1}\right)^{\frac{1}{2}}\dfrac{1}{bp_1}\left(1+\dfrac{b}{a}\dfrac{T_2}{T_1}\right)\dfrac{(1+\alpha)}{4(1-\alpha)} + \dfrac{T_2}{T_1}\dfrac{4}{15R\mu_1 T_1}\ln\dfrac{b}{a} + \dfrac{\mu_1}{5p_1^2 b^2}\left(1+\dfrac{b^2}{a^2}\dfrac{T_2}{T_1}\right)} \tag{60}$$

Hence the temperature variation across the annulus can be written as ($\alpha = 0$)

$$\frac{T_1 - T}{T_1 - T_2} = \frac{4}{15}\,\frac{\frac{Re}{M}\ln\left(1 + \frac{b-a}{a}\frac{r-a}{b-a}\right)}{\frac{1}{4}\sqrt{\frac{10\pi}{3}}\,\frac{b-a}{b}\left(1 + \frac{b}{a}\frac{T_2}{T_1}\right) + \frac{T_2 Re}{T_1 M}\ln\frac{b}{a} + \frac{5}{4}\frac{M}{Re}\left(\frac{b-a}{b}\right)^2\left(1 + \frac{b^2 T_2}{a^2 T_1}\right)}$$

$$+ \frac{\sqrt{\frac{10\pi}{3}}\,\frac{15}{16}\,\frac{b-a}{a} + \frac{5}{4}\frac{M}{Re}\left(\frac{b-a}{a}\right)^2}{\sqrt{\frac{10\pi}{3}}\,\frac{15}{16}\,\frac{b-a}{b}\left(1 + \frac{b}{a}\frac{T_2}{T_1}\right) + \frac{Re}{M}\frac{T_2}{T_1}\ln\frac{b}{a} + \frac{5}{4}\frac{M}{Re}\left(\frac{b-a}{b}\right)^2\left(1 + \frac{b^2}{a^2}\frac{T_2}{T_1}\right)} \qquad (61)$$

The temperature profile, Eq. (61), across the annulus is plotted in Figure 5 for a diffusively reflected surface ($\alpha = 0$). The temperature of the gas at the surface of the inner cylinder is given by

$$T(a) = T_1 + \frac{b}{a}(T_2 - T_1)\,\frac{\sqrt{\frac{10\pi}{3}} + \frac{4}{3}\frac{1-\alpha}{1+\alpha}\frac{M}{Re}\frac{b-a}{a}}{\left[\sqrt{\frac{10\pi}{3}}\left(1 + \frac{b}{a}\frac{T_2}{T_1}\right) + \frac{T_2}{T_1}\frac{4(1-\alpha)4}{(1+\alpha)15}\frac{Re}{M}\frac{b}{b-a}\ln\frac{b}{a} + \frac{4}{3}\frac{(1-\alpha)}{(1+\alpha)}\frac{M}{Re}\frac{b-a}{b}\left(1 + \frac{b^2}{a^2}\frac{T_2}{T_1}\right)\right]} \qquad (62)$$

In the limiting case when $Re/M \longrightarrow 0$,

$$T_1 - T(a) = (T_1 - T_2)\frac{1}{\frac{a^2}{b^2} + \frac{T_2}{T_1}} \tag{61}$$

Furthermore, when a approaches b and T_2 approaches T_1 we obtain

$$T_1 - T(a) = \tfrac{1}{2}(T_1 - T_2) \tag{62}$$

In the other limiting case when $Re/M \longrightarrow \infty$

$$T_1 - T(a) = 0 \tag{63}$$

As expected there is no temperature jump at normal density. At the surface of the outer cylinder we have in the limiting case when $Re/M \longrightarrow 0$

$$T(b) - T_2 = (T_1 - T_2)\frac{1 - \frac{b^2}{a^2}\frac{T_1 - T_2}{T_1}}{1 + \frac{b^2}{a^2}\frac{T_2}{T_1}} \tag{64}$$

The ratio of the two temperature jumps is

$$\frac{T_1 - T(a)}{T(b) - T_2} = \frac{b^2/a^2}{1 + \frac{b^2}{a^2}\,T_2/T_1} \tag{65}$$

In the limit when a approaches b and T_2 approaches T_1, one recovers the result of plane Couette flow found by Yang[3] and Lees.

$$T_1 - T(a) = T(b) - T_2 = \tfrac{1}{2}(T_1 - T_2) \tag{66}$$

The products of Stanton number C_H and the Mach number M is defined as

$$C_H M = \frac{q_r M}{\rho_1 U c_p (T_1 - T_2)} \quad .$$

At the wall of the inner cylinder we have

$$C_H M = \frac{1}{\frac{5}{2}\left[\frac{1+\alpha}{4(1-\alpha)} \frac{a}{b}\left(1+\frac{b}{a}\frac{T_2}{T_1}\right)\sqrt{\frac{10\pi}{3}} + \frac{T_2}{T_1}\frac{Re}{M} Pr \frac{a}{b-a} \ln\frac{b}{a} + \frac{M}{Re}\left(1+\frac{b^2}{a^2}\frac{T_2}{T_1}\right)\frac{5}{6}\frac{a}{b}\frac{b-a}{b}\right]} \tag{67}$$

or

$$\frac{1}{C_H M} = \frac{(1+\alpha)}{4(1-\alpha)} \frac{a}{b}\left(1+\frac{b^2}{a^2}\frac{T_2}{T_1}\right)\frac{5}{2}\sqrt{\frac{10\pi}{3}} + \frac{T_2}{T_1}\frac{a}{b-a}\ln\frac{b}{a} Pr \frac{Re}{M}$$
$$+ \frac{5}{6}\frac{a^2}{b^2}\left(\frac{b-a}{a}\right)\frac{M}{Re}\left(1+\frac{b^2}{a^2}\frac{T_2}{T_1}\right) \tag{68}$$

One may notice that there are two kinds of limiting processes which lead to quite different results.

(1) Let a approach b at a fixed value of Re/M (also γ is taken to be 5/3 for a monatomic gas), then we again have the result of plane Couette flow[3].

$$1/C_H M = (5/4)\, \frac{(1+\alpha)}{(1-\alpha)} \sqrt{\frac{10\pi}{3}} + Pr\,(Re/M) \tag{69}$$

(2) For arbitrary a and b, we have as $Re/M \longrightarrow \infty$

$$1/C_H M \longrightarrow \frac{T_2}{T_1} \frac{a}{b-a} \ln \frac{b}{a} \, (Re/M) \; Pr \tag{70}$$

or

$$C_H \longrightarrow \frac{T_1}{T_2} \frac{b-a}{a} \frac{1}{\ln b/a} \frac{1}{Pr \; Re/M} \tag{71}$$

which agrees with the classical solution of Fourier heat conduction for steady state.

However, when $Re/M \longrightarrow 0$, we encounter the difficulty that $C_H M$ approaches zero instead of the constant value given by a free molecule calculation.

Thus, in a free molecular limit the temperature distribution and heat transfer are physically unrealistic. (See Section IV.) The plot of $C_H M$ vs. Re/M is given by Figure 6.

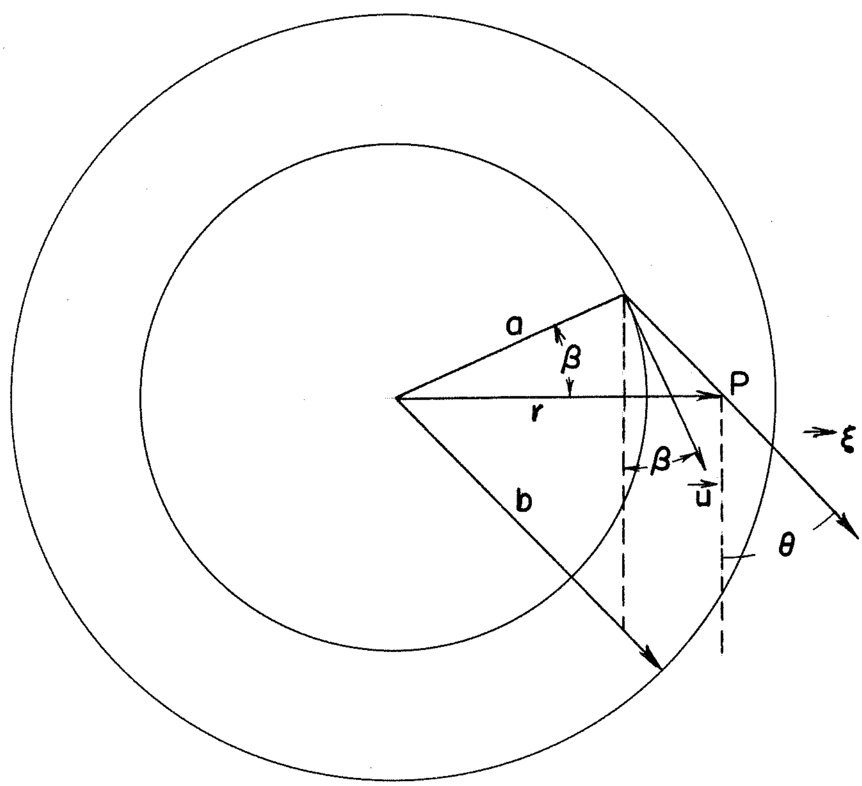
IV. DISCUSSION AND CONCLUSIONS

IV. 1. Cylinder Drag and Shear Flow

A number of writers[10, 1] have pointed out that Grad's distribution function is not expected to be entirely satisfactory for highly rarefied gas flows, because it does not contain the "two-sidedness", or discontinuity in velocity space that is so characteristic of the low density regime. In the present problem the actual distribution function in the limit $Re/M \longrightarrow 0$ for particles emitted diffusely from the inner rotating cylinder is given by the following expression (See sketch.):

$$f_P(\vec{\xi}, r) = n_a \left(\frac{m}{2\pi k T_a} \right)^{3/2} \exp\left\{ - \frac{m}{2kT_a} \left[\vec{\xi} - \vec{u}(\beta) \right]^2 \right\} \qquad (72)$$

$$0 \leq \beta \leq \cos^{-1}(a/r) \quad ,$$



where $\vec{u}(\beta)$ is constant in magnitude, but not in direction, and β is uniquely determined by θ and (r/a)*. The velocity distribution function for particles emitted diffusely from the outer stationary cylinder is similar, except that $\vec{u} = 0$ and n_a, T_a are replaced by n_b, T_b. On the other hand, Grad's distribution function [Eq. (AI. 3)] ignores the wedge-like domains of influence of the two cylinders at the point P, as well as the angular dependence of $\vec{u}$ [Eq. (72)]. In the present problem these omissions lead inevitably to the Navier-Stokes relation for the shear stress $p_{r\theta}$ when $M^2 << 1$ [Eqs. (25) and (28)] **. The boundary conditions [Eqs. (30) and (32)] are reduced to the same form as Maxwell's famous velocity slip relation, thus assuring a <u>qualitatively</u> correct transition of the cylinder drag from free molecule flow to the classical Navier-Stokes regime [Eq. (48)].

When the width of the annulus is small compared to the inner cylinder radius the solutions of the linearized Grad equations for $p_{r\theta}$ and u_θ contain only small errors of order $\left(\frac{b-a}{a}\right)$. But when $\frac{b-a}{a} = 0(1)$ the wedge-like domains of influence of the two cylinders cannot be ignored. For example, in the limit $Re/M \longrightarrow 0$ the effect of the inner cylinder rotation on the gas dies off with radial distance like the solid

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* In fact, $\tan\theta = \dfrac{(r/a) - \cos\beta}{\sin\beta}$, $\cos^{-1}(a/r) \leq \theta \leq \pi/2$.

** Actually this statement is applicable to any function of the form $f = f_o[1 + \phi]$ making the same omissions. We remark that a two-sided function utilizing half-range Maxwellians of the type introduced in Reference 1 yields the Navier-Stokes relation only when $\frac{b-a}{a} << 1$, but not otherwise.

angle subtended by the cylinder at any point in the annulus, and the linear mean velocity distribution given by Eq. (46) no longer represents the true physical situation. In particular, when $b/a >> 1$ the mean velocity given by Eq. (46) approaches zero everywhere, and $u_\theta(a) \longrightarrow 0$, instead of $\frac{1}{2}$. Thus the drag on the inner cylinder given by Eq. (45) is exactly twice the correct free-molecule flow value.

We conclude that the excellent agreement obtained between the solutions of the linearized Grad equations for steady, plane Couette flow[3] in the limit $Re/M \longrightarrow 0$ and the correct free-molecule quantities is somewhat misleading. This linearized plane flow problem is so simple that almost any reasonable distribution function employed in Maxwell's moment equations yields satisfactory results. The present study shows that the magnitude of the effect of streamline curvature on shear drag is not given correctly by Grad's f, but that at least there are no gross physical contradictions, so far as shear drag is concerned. The mean velocity distribution is less satisfactory. Similar conclusions can be drawn from a study of Goldberg's[11] solution of the linearized Grad equations for the "slow" flow over a sphere.

IV. 2. Heat Transfer and Mean Temperature Distribution

Even for small (but finite) values of $(b-a)/a$, the heat transfer rate given by the solution of the linearized Grad equations approaches zero faster than the density in the limit $Re/M \longrightarrow 0$ [Eq. (68)]. On the other hand, if $Re/M \sim (b-a)/\lambda$ is held fixed, while $(b-a)/a \longrightarrow 0$, we recover the results obtained previously[3] for linearized plane Couette flow, and the heat transfer rate approaches the correct free-molecule flow limit as $Re/M \longrightarrow 0$. This non-uniform convergence and physically unrealistic behavior of the heat transfer solutions for linearized cylindrical Couette flow is produced by the coupling between the normal stress p_{rr} and the radial heat flux q_r, which occurs both in the moment equation for p_{rr} [Eq. (24)] and in the energy boundary condition at either cylinder [Eqs. (31) and (33)]. By dimensional analysis one can easily verify that a term of order M/Re is thereby introduced. The coupling between p_{rr} and q_r in the moment equation, in turn, is forced upon us by the term

$$- f_o \frac{q_r c_r}{pRT} \left(1 - \frac{c^2}{5RT} \right)$$

in Grad's distribution function. For example, on the left-hand side of Maxwell's moment equation for p_{rr} one obtains

$$(1/r)\,(d/dr) \left\{ rm \int f \, \xi_r^{\,3} \, d\vec{\xi} \right\},$$

and the term containing q_r in Grad's f evidently gives rise to a term of the form $(1/r)\,(d/dr)\,(rq_r)$ in this equation.

Similarly, the rate of energy transfer to the surface, given by the expression

$$m \int_{c_r < 0} f \; c_r \; (c^2/2) \; d\vec{\xi}$$

contains a term proportional to p_{rr}, because of the term $f_o\,(p_{rr}/2pRT)\,c_r^{\,2}$ appearing in Grad's distribution function. In linearized plane Couette flow $p_{yy} = p_{xx} \equiv 0$ and q_y = constant, so that no cross-coupling occurs. Such cross-couplings do not occur in the moment equation for the shear stress $p_{r\theta}$, because (1) the term containing $p_{r\theta}$ in Grad's f is anti-symmetric; (2) all physical quantities are functions of r alone in this particular problem; (3) $q_\theta \longrightarrow 0$ when $M^2 << 1$.

Because the geometry of the present problem is so simple this dilemma can be resolved by a slight modification of Grad's method. When $M^2 << 1$ the radial momentum equation shows that an acceptable solution is given by $p_{rr} = p_{\theta\theta} = 0$, $p =$ constant [Eqs. (3) and (23)]. But $q_{rr} \sim (1/r)$, so that this solution is clearly incompatible with the stress equations for p_{rr} and $p_{\theta\theta}$ [Eqs. (24) and (26)]. Therefore we must drop these two moment equations entirely, and agree to employ a modified Grad distribution function involving only $p_{r\theta}$, q_r , and q_θ . When this truncated moment method is utilized, the shear flow and cylinder drag are not changed, Eq. (27) for q_r is again reduced to the ordinary Fourier heat-conduction "law", and the energy boundary condition reduces to the well-known temperature-jump condition. Without going into details (Appendix 2), we state that the heat transfer and shear flow problems are now entirely similar in this new framework, and the criticisms of the shear flow results contained in Section IV. 1 are equally applicable to the heat transfer problem.

Of course this simple device is unacceptable in more general flow problems involving streamline curvature, because $p_{rr} \neq p_{\theta\theta} \neq 0$, even when $M^2 << 1$. For example, Goldberg's[11] solution of the linearized Grad equations for "slow" flow over a sphere exhibits the same contradictions in the heat transfer rate in the limit $Re/M \longrightarrow 0$. Here p_{rr} and $p_{\theta\theta}$ do not vanish identically even in the classical Navier-Stokes limit ($Re/M >> 1$), which corresponds to Stokes flow over a sphere. When dissipation is considered (M^2 arbitrary) these normal stresses do not vanish identically even in the simplest geometry of plane Couette flow, and cross-couplings between these stresses and the heat flux are inevitable if the unmodified Grad f is employed. Similar cross-couplings are observed in the problem of the steady, plane shock wave[12], and these cross-couplings are probably responsible for the difficulties that have been encountered in applying Grad's method to this problem.

These remarks are applicable to any f that is a simple extension of the Chapman-Enskog polynomial. Perhaps these difficulties can be avoided by utilizing the two-sided polynomial distribution of the form

$$f = f_o \left[1 + a_o^{\pm} \, c_x + a_1^{\pm} \, c_x c_y + \ldots \right],$$

employed by Gross[10], Jackson, and Ziering for plane, parallel geometry. However, to the author's present knowledge this type of f has been applied only to the case of linearized flows ($M^2 << 1$). It is not clear that such a velocity distribution function can describe the situation for non-linear, highly rarefied gas flows, where two distinctly different Maxwellian streams are usually involved. In this connection we remark that the weighting function (f) introduced in Reference 1, which utilizes two half-range Maxwellians expressed in terms of a certain number of parametric functions, leads to physically consistent results over the whole range of gas density, not only for linearized, cylindrical Couette flow*, but also for non-linear plane Couette flow[13]. In addition, the moment equations derived for the steady plane shock wave do not exhibit any singularities within the shock wave. Of course no integral method is unique, but it appears that the choice of the weighting function f to be employed in Maxwell's moment method is dictated by the requirements imposed in the limiting case of highly rarefied gas flows, as well as in the classical Navier-Stokes regime.

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* Report in preparation.

REFERENCES

1. Lees, L.: A Kinetic Theory Description of Rarefied Gas Flow. GALCIT Hypersonic Research Project, Memorandum No. 51, December 15, 1959.

2. Grad, H.: On the Kinetic Theory of Rarefied Gases. Communications on Pure and Applied Mathematics, Vol. 4, No. 4, pp. 331-407, December, 1949.

3. Yang, H. T. and L. Lees: Plane Couette Flow at Low Mach Number According to the Kinetic Theory of Gases. GALCIT Hypersonic Research Project, Memorandum No. 36, February 1, 1957. Also, Proceedings of the Fifth Midwestern Conference on Fluid Mechanics, 1957, pp. 41-65.

4. Bowyer, J. M. and L. Talbot: Near-Free-Molecule Couette Flow between Concentric Cylinders. University of California, Institute of Engineering Research, Report No. HE-150-139, July 10, 1956, (condensed from Ph. D. thesis, 1956).

5. Kuhlthau, A. R.: The Application of High Rotational Speed Techniques to Low Density Gas Dynamics. Proceedings of the Third Midwestern Conference on Fluid Mechanics, 1953, pp. 495-514, University of Minnesota.

6. Rose, M.: Cylindrical Couette Flow. Physical Review, Vol. 91, No. 2, p. 469, July 15, 1953 (abstract only).

7. Chiang, S. F.: Drag on a Rotating Cylinder at Low Pressures. University of California, Institute of Engineering Research, Report No. HE-150-100, May 19, 1952.

8. Yang, H. T.: Reduction of Grad Thirteen-Moment Equations to Burnett Equations for Slip Flow, (unpublished), GALCIT, 1954.

9. Yang, H. T. and L. Lees: Rayleigh's Problem at Low Mach Number According to the Kinetic Theory of Gases. GALCIT Hypersonic Research Project, Technical Report No. 2, July 15, 1955. Also, Journal of Math. and Physics, M. I. T., Vol. 35, No. 3, pp. 195-235, October, 1956.

10. Gross, E. P.; E. A. Jackson; S. Ziering: Boundary Value Problems in Kinetic Theory of Gases. Annals of Physics, Vol. 1, No. 2, pp. 141-167, May, 1957.

11. Goldberg, R.: The Slow Flow of a Rarefied Gas Past a Spherical Obstacle. Ph. D. Thesis, Department of Mathematics, New York University, April 1, 1954.

12. Grad, H.: The Profile of a Steady Plane Shock Wave. Communications on Pure and Applied Mathematics, Vol. 5, No. 3, pp. 257-300, August, 1952.

13. Lees, L. and C. Y. Liu: Kinetic Theory Description of Plane, Compressible Couette Flow. To be presented at Second International Symposium on Rarefied Gas Dynamics, August 3-6, 1960, Berkeley, California.

14. Bomelburg, H. J.: Heat Loss from Very Thin Heated Wires in Rarefied Gases. The Physics of Fluids, Vol. 2, No. 6, pp. 717-718, November-December, 1959.

APPENDIX I

GRAD'S THIRTEEN MOMENT EQUATIONS IN TWO-DIMENSIONAL CYLINDRICAL COORDINATES

In calculating flow problems one is often mainly interested in certain lower moments of the velocity distribution function rather than the function itself. Therefore, it is natural that one takes the Maxwell integral transport equation as the starting point for applying approximate methods.

The equation is given as

$$\frac{\partial}{\partial t}\int f Q \, d\vec{\xi} + \nabla_R \cdot \left[\int f \vec{\xi} Q \, d\vec{\xi}\right] = \int f \left(\frac{\vec{F}}{m} \cdot \vec{\nabla_\xi} Q\right) d\vec{\xi} + \Delta Q \qquad \text{(I. 1)}$$

where

f is the velocity distribution function

Q is any function of the velocity components of a particle (moment, energy, etc.)

$\vec{\xi}$ and $\vec{R}$ are independent variables

$\vec{F}$ is the external force vector

and

$$\Delta Q = \iiiint (Q - Q') f f_1 V \, d\vec{\xi} \, d\vec{\xi}_1 \, b \, db \, dh \, d\varepsilon \qquad \text{(I. 2)}$$

is the collision integral in which $Q' - Q$ represents the change in Q experienced in a collision.

In Grad's thirteen-moment approximation the distribution function is a linear function of the stresses and heat fluxes, which are now regarded as separate dependent variables not explicitly related to ρ, $\vec{u}$, T, and their derivatives. They are, however, related to the second and third moments of the velocity distribution function f. Thus, in a rectangular Cartesian coordinate system

$$f = f_0 \left[1 + \frac{p_{ij}}{2pRT} c_i c_j - \frac{q_i c_i}{pRT}\left(1 - \frac{c^2}{5RT}\right)\right] \tag{I. 3}$$

where f_o is the local Maxwellian.

By substituting this expression for f (I. 3) into the equation (I. 1), and by taking Q to be equal successively to m , $m\xi_i$, $m(\xi^2/2)$, $m\xi_i\xi_j$ and $m\xi_i(\xi^2/2)$, the thirteen partial differential equations in a rectangular coordinate system (including the conservation relations) are obtained for the thirteen independent moments ρ, $\vec{u}$, T , p_{ij} and q_i . The equations are

Conservation of Mass

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_r}(\rho u_r) = 0 \tag{I. 4}$$

Conservation of Momentum

$$\frac{\partial u_i}{\partial t} + u_r \frac{\partial u_i}{\partial x_r} + \frac{1}{\rho}\frac{\partial P_{i\alpha}}{\partial x_\alpha} = 0 \tag{I. 5}$$

Conservation of Energy

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x_\alpha}(u_\alpha p) + \frac{2}{3} P_{i\alpha}\frac{\partial u_i}{\partial x_\alpha} + \frac{2}{3}\frac{\partial q_r}{\partial x_r} = 0 \tag{I. 6}$$

Stresses

$$\frac{\partial p_{ij}}{\partial t} + \frac{\partial}{\partial X_\alpha}(u_\alpha p_{ij}) + \frac{2}{5}\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial q_\alpha}{\partial x_\alpha}\right)$$
$$+ p_{i\alpha}\frac{\partial u_j}{\partial x_\alpha} + p_{j\alpha}\frac{\partial u_i}{\partial x_\alpha} - \frac{2}{3}\delta_{ij} p_{rs}\frac{\partial u_r}{\partial x_s} \tag{I. 7}$$
$$+ p\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_\alpha}{\partial x_\alpha}\right) = -\frac{p}{\mu} p_{ij}$$

Heat Fluxes

$$\frac{\partial q_i}{\partial t} + \frac{\partial}{\partial x_\alpha}(u_\alpha q_i) + \frac{7}{5} q_r \frac{\partial u_i}{\partial x_r} + \frac{2}{5} q_r \frac{\partial u_r}{\partial x_i} + \frac{2}{5} q_i \frac{\partial u_\alpha}{\partial x_\alpha}$$
$$+ RT\frac{\partial p_{i\alpha}}{\partial x_\alpha} + \frac{7}{2} p_{i\alpha}\frac{\partial RT}{\partial x_\alpha} - \frac{p_{i\alpha}}{\rho}\frac{\partial P_{rs}}{\partial x_s} + \frac{5}{2} p \frac{\partial RT}{\partial x_i} = -\frac{2}{3}\frac{p}{\mu} q_i \tag{I. 8}$$

where the results of stresses are already utilized in the heat flux equations.

Given below also as a reference is the list of all moments involved in Grad's approximation.

$$\rho(\vec{R}, t) = \int m f(\vec{\xi}, \vec{R}, t)\, d\vec{\xi} \tag{I. 9}$$

$$\rho \vec{u}(\vec{R}, t) = \int m \vec{\xi} f\, d\vec{\xi} \tag{I. 10}$$

$$P_{ij} = \int m c_i c_j f\, d\vec{\xi} \tag{I. 11}$$

$$q_{ijk}(\vec{R}, t) = \frac{1}{2}\int m c_i c_j c_k f\, d\vec{\xi} \tag{I. 12}$$

where $\vec{c} = \vec{\xi} - \vec{u}(\vec{R}, t)$ is the intrinsic or relative velocity.

By contraction, the following tensors are produced.

$$P_{ii} = 3p \tag{I. 13}$$

$$q_{ijj} = q_i \tag{I. 14}$$

$$p_{ij} = P_{ij} - p\,\delta_{ij} \tag{I. 15}$$

$$p_{ii} = 0 \tag{I. 16}$$

Grad's Equations in Two-Dimensional Cylindrical Coordinate System

In two-dimensional problems, all quantities are independent of z, hence we may set $w = p_{xz} = p_{yz} = q_z = 0$ a priori, and the number of partial differential equations is reduced to nine.

To write these equations in cylindrical coordinates, one applies the following transformations

$$r = (x^2 + y^2)^{\frac{1}{2}} \qquad \theta = \tan^{-1}\frac{y}{x} \tag{I. 17}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u_r \\ u_\theta \end{pmatrix} \tag{I. 18}$$

$$\begin{pmatrix} q_x \\ q_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} q_r \\ q_\theta \end{pmatrix} \tag{I. 19}$$

and

$$p_{ij} = l_{i\alpha}\, l_{j\beta}\, p_{\alpha\beta} \tag{I. 20}$$

where i , j are related to x , y and α , β are related to r , θ coordinates. $l_{i\alpha}$ and $l_{j\beta}$ are direction cosines between the two coordinate systems

$$\begin{pmatrix} l_{xr} & l_{yr} \\ l_{x\theta} & l_{y\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

The nine moment equations become (without external force)

Conservation of Mass

$$\frac{\partial \rho}{\partial t} + \rho\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right) + u_r \frac{\partial \rho}{\partial r} + \frac{u_\theta}{r}\frac{\partial \rho}{\partial \theta} = 0 \tag{I. 21}$$

Conservation of Momentum

r-component

$$\frac{\partial u_r}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{1}{\rho}\left(\frac{\partial p_{rr}}{\partial r} + \frac{1}{r}\frac{\partial p_{r\theta}}{\partial \theta} + \frac{p_{rr} - p_{\theta\theta}}{r}\right) + \left(u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r}\right) = 0 \tag{I. 22}$$

θ-component

$$\frac{\partial u_\theta}{\partial t} + \frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{\rho}\left(\frac{\partial p_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial p_{\theta\theta}}{\partial \theta} + \frac{2}{r}p_{r\theta}\right) + \left(u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r}\right) = 0 \tag{I. 23}$$

Conservation of Energy

$$\frac{\partial p}{\partial t} + \frac{5}{3}p\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right) + \frac{2}{3}\left(\frac{\partial q_r}{\partial r} + \frac{q_r}{r} + \frac{1}{r}\frac{\partial q_\theta}{\partial \theta}\right) + \left(u_r\frac{\partial p}{\partial r} + \frac{u_\theta}{r}\frac{\partial p}{\partial \theta}\right)$$

$$+ \frac{2}{3}\left(p_{rr}\frac{\partial u_r}{\partial r} + p_{r\theta}\frac{\partial u_\theta}{\partial r} + \frac{p_{r\theta}}{r}\frac{\partial u_r}{\partial \theta} + \frac{p_{\theta\theta}}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{p_{\theta\theta}}{r}u_r - p_{r\theta}\frac{u_\theta}{r}\right) = 0 \qquad \text{(I. 24)}$$

Stresses

"p_{rr}"

$$\frac{\partial p_{rr}}{\partial t} + \frac{2}{3}p\left(2\frac{\partial u_r}{\partial r} - \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r}\right) + \frac{4}{15}\left(2\frac{\partial q_r}{\partial r} - \frac{1}{r}\frac{\partial q_\theta}{\partial \theta} - \frac{q_r}{r}\right) + u_r\frac{\partial p_{rr}}{\partial r} + \frac{u_\theta}{r}\frac{\partial p_{rr}}{\partial \theta}$$

$$-\frac{2}{r}u_\theta p_{r\theta} + \frac{7}{3}p_{rr}\frac{\partial u_r}{\partial r} - \frac{2}{3}p_{r\theta}\frac{\partial u_\theta}{\partial r} + \frac{4}{3}p_{r\theta}\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{4}{3}\frac{p_{r\theta}u_\theta}{r} - \frac{2}{3}\frac{p_{\theta\theta}}{r}\frac{\partial u_\theta}{\partial \theta} \qquad \text{(I. 25)}$$

$$-\frac{2}{3}\frac{p_{\theta\theta}}{r}u_r + \frac{p_{rr}}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{p_{rr}u_r}{r} = -\frac{p}{\mu}p_{rr}$$

"$p_{r\theta}$"

$$\frac{\partial p_{r\theta}}{\partial t} + p\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right) + \frac{2}{5}\left(\frac{\partial q_\theta}{\partial r} + \frac{1}{r}\frac{\partial q_r}{\partial \theta} - \frac{q_\theta}{r}\right) + u_r\frac{\partial p_{r\theta}}{\partial r} + \frac{u_\theta}{r}\frac{\partial p_{r\theta}}{\partial \theta}$$

$$+ \frac{u_\theta p_{rr}}{r} - \frac{u_\theta p_{\theta\theta}}{r} + p_{rr}\frac{\partial u_\theta}{\partial r} + 2\frac{p_{r\theta}}{r}\frac{\partial u_\theta}{\partial \theta} + 2p_{r\theta}\frac{u_r}{r} + 2p_{r\theta}\frac{\partial u_r}{\partial r} + \frac{p_{\theta\theta}}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right) \qquad \text{(I. 26)}$$

$$= -\frac{p}{\mu}p_{r\theta}$$

"$p_{\theta\theta}$"

$$\begin{aligned}
&\frac{\partial p_{\theta\theta}}{\partial t} - \frac{2}{3}p\left(\frac{\partial u_r}{\partial r} - \frac{2}{r}\frac{\partial u_\theta}{\partial \theta} - 2\frac{u_r}{r}\right) - \frac{4}{15}\left(\frac{\partial q_r}{\partial r} - \frac{2}{r}\frac{\partial q_\theta}{\partial \theta} - 2\frac{q_r}{r}\right) + u_r\frac{\partial p_{\theta\theta}}{\partial r} + \frac{u_\theta}{r}\frac{\partial p_{\theta\theta}}{\partial \theta} \\
&+ \frac{2}{r}u_\theta p_{r\theta} + \frac{4}{3}p_{r\theta}\frac{\partial u_\theta}{\partial r} - \frac{2}{3}\left(p_{rr}\frac{\partial u_r}{\partial r} + \frac{p_{r\theta}}{r}\frac{\partial u_r}{\partial \theta} - p_{r\theta}\frac{u_\theta}{r}\right) + \frac{7}{3}p_{\theta\theta}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) \\
&+ p_{\theta\theta}\frac{\partial u_r}{\partial r} = -\frac{p}{\mu}p_{\theta\theta}
\end{aligned} \tag{I.27}$$

Heat Fluxes

"q_r"

$$\begin{aligned}
&\frac{\partial q_r}{\partial t} + \frac{5}{2}p\frac{\partial RT}{\partial r} + RT\left(\frac{\partial p_{rr}}{\partial r} + \frac{1}{r}\frac{\partial p_{r\theta}}{\partial \theta} + \frac{p_{rr} - p_{\theta\theta}}{r}\right) + q_r\left(\frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) \\
&+ u_r\frac{\partial q_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial q_r}{\partial \theta} - \frac{u_\theta q_\theta}{r} + \frac{11}{5}q_r\frac{\partial u_r}{\partial r} + \frac{2}{5}\left(q_r\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + q_r\frac{u_r}{r} + q_\theta\frac{\partial u_\theta}{\partial r}\right) \\
&+ \frac{7}{5}q_\theta\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right) - \frac{1}{\rho}\left(p_{rr}\frac{\partial p}{\partial r} + \frac{p_{r\theta}}{r}\frac{\partial p}{\partial \theta}\right) + \frac{7}{2}\left(p_{rr}\frac{\partial RT}{\partial r} + \frac{p_{r\theta}}{r}\frac{\partial RT}{\partial \theta}\right) \\
&- \frac{p_{rr}}{\rho}\left(\frac{\partial p_{rr}}{\partial r} + \frac{1}{r}\frac{\partial p_{r\theta}}{\partial \theta} + \frac{p_{rr} - p_{\theta\theta}}{r}\right) - \frac{p_{r\theta}}{\rho}\left(\frac{\partial p_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial p_{\theta\theta}}{\partial \theta} + \frac{2p_{r\theta}}{r}\right) = -\frac{2}{3}\frac{p}{\mu}q_r
\end{aligned} \tag{I.28}$$

"q_θ"

$$\begin{aligned}
&\frac{\partial q_\theta}{\partial t} + \frac{5}{2}p\frac{1}{r}\frac{\partial RT}{\partial \theta} + RT\left(\frac{\partial p_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial p_{\theta\theta}}{\partial \theta} + \frac{2p_{r\theta}}{r}\right) + q_\theta\left(\frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) \\
&+ u_r\frac{\partial q_\theta}{\partial r} + \frac{u_\theta}{r}\left(\frac{\partial q_\theta}{\partial \theta} + q_r\right) + \frac{11}{5}\left(q_\theta\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + q_\theta\frac{u_r}{r}\right) + \frac{7}{5}q_r\frac{\partial u_\theta}{\partial r} \\
&+ \frac{2}{5}\left(\frac{q_r}{r}\frac{\partial u_r}{\partial \theta} - q_r\frac{u_\theta}{r} + q_\theta\frac{\partial u_r}{\partial r}\right) - \frac{1}{\rho}\left(p_{r\theta}\frac{\partial p}{\partial r} + \frac{p_{\theta\theta}}{r}\frac{\partial p}{\partial \theta}\right) + \frac{7}{2}\left(p_{r\theta}\frac{\partial RT}{\partial r} + \frac{p_{\theta\theta}}{r}\frac{\partial RT}{\partial \theta}\right) \\
&- \frac{p_{r\theta}}{\rho}\left(\frac{\partial p_{rr}}{\partial r} + \frac{1}{r}\frac{\partial p_{r\theta}}{\partial \theta} + \frac{p_{rr} - p_{\theta\theta}}{r}\right) - \frac{p_{\theta\theta}}{\rho}\left(\frac{\partial p_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial p_{\theta\theta}}{\partial \theta} + \frac{2p_{r\theta}}{r}\right) = -\frac{2}{3}\frac{p}{\mu}q_\theta
\end{aligned} \tag{I.29}$$

APPENDIX II

CALCULATION OF HEAT LOSS FROM VERY THIN HEATED WIRES IN A RAREFIED GAS BASED ON A TRUNCATED FORM OF GRAD'S THIRTEEN MOMENT METHOD

H. J. Bomelburg[14] has performed a series of experiments with fine heated wires of different diameters fastened inside a bell jar to study quantitatively the behavior of heat conductivity in rarefied gases. The temperature of the wire was kept constant and the heat loss at various pressures as measured. The heat loss at normal density ($Kn^* = \infty$) is defined as Q_∞ , and the heat loss at some lower pressure is called Q. The quantity Q/Q_∞ is then plotted against Kn on a logarithmic scale to show that the heat conductivity is dependent on pressure as the mean free path gets to be large compared with the container. The wires used are of aspect ratio well above 1000 and the temperature difference $(T_w - T_b)/T_w$ is approximately 1/10. [Here T_w is the temperature of the wire, and T_b is the temperature of the gas at the wall of the bell jar. In these experiments T_w was about 60°C and T_b was about room temperature 25°C.]

Bomelburg's experiment is very closely related to the linearized cylindrical Couette flow, since the geometry is the same and the temperature difference small. On the other hand, since the bell jar as well as the hot wire are fixed, there is no mean fluid motion, hence the problem

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* Bomelburg defined Kn as d/λ , d being the diameter of the wire and λ the mean free path. His definition is just the reciprocal of the Kn commonly used.

reduces to a pure heat conduction. We hereby propose to treat the case by means of a truncated form of Grad's moment method. [See Section IV, Discussion and Conclusion.] (The problem can be idealized as two-dimensional because of the high aspect ratio and the boundary condition can be linearized because of small temperature difference.) First of all, let us introduce the following symbols:

a	radius of wire
b	radius of bell jar
Re/M	$= \dfrac{\rho_w a (\gamma R T_w)^{\frac{1}{2}}}{\mu_w}$
ρ_w, μ_w, p_w	density, coefficient of viscosity, and pressure of gas at the wire surface .

We say a priori that $p_{rr} = p_{\theta\theta} = 0$ and $p = p_w$ = constant throughout the field. Furthermore, the two stress equations (p_{rr} and $p_{\theta\theta}$) are not used. Symmetry requires that all tangential quantities must vanish. The heat loss Q is given by the energy equation as

$$Q = c/r \quad .$$

The heat flux equation becomes

$$Q = -15/4 \quad R \mu_w (dT/dr)$$

with the boundary conditions

at r = a

$$\left(\frac{2\pi}{RT_w}\right)^{\frac{1}{2}} \frac{Q(a)}{p_w} + \frac{4(1-\alpha)}{(1+\alpha)}\left[\frac{T_b - T_w}{T_w} + \frac{4}{15 R\mu_w} \frac{c}{T_w} \ln \frac{b}{a}\right] = 0$$

at r = b

$$T = T_b \quad .$$

Solving for c , we obtain

$$\frac{Q}{Q_\infty} = \frac{\frac{4(1-\alpha)}{(1+\alpha)} \frac{4}{15} \frac{Re}{M} \ln b/a}{\left[\sqrt{2\pi\gamma} + \frac{4(1-\alpha)}{(1+\alpha)} \frac{4}{15} \frac{Re}{M} \ln \frac{b}{a}\right]}$$

Re/M is proportional to Kn or

$$Re/M = \sqrt{5\pi/6} \quad Kn \quad .$$

For a diffusively reflected surface $\alpha = 0$,

$$Q/Q_\infty = \frac{\frac{16}{15}\sqrt{\frac{5\pi}{6}} \ln \frac{b}{a} Kn}{\sqrt{2\pi\gamma} + \frac{16}{15}\sqrt{\frac{5\pi}{6}} \ln \frac{b}{a} Kn} \quad .$$

Q/Q_∞ vs. Kn is plotted in Figure 7 on log-log scale. For small values of Kn, the two sided solid angle effect becomes more and more important and the curve deviates away from experimental results; however, it shows qualitatively the correct trend. The problem is now being studied at this laboratory by Mrs. Y. L. Wu, utilizing the two-sided Maxwellian introduced in Reference 1.

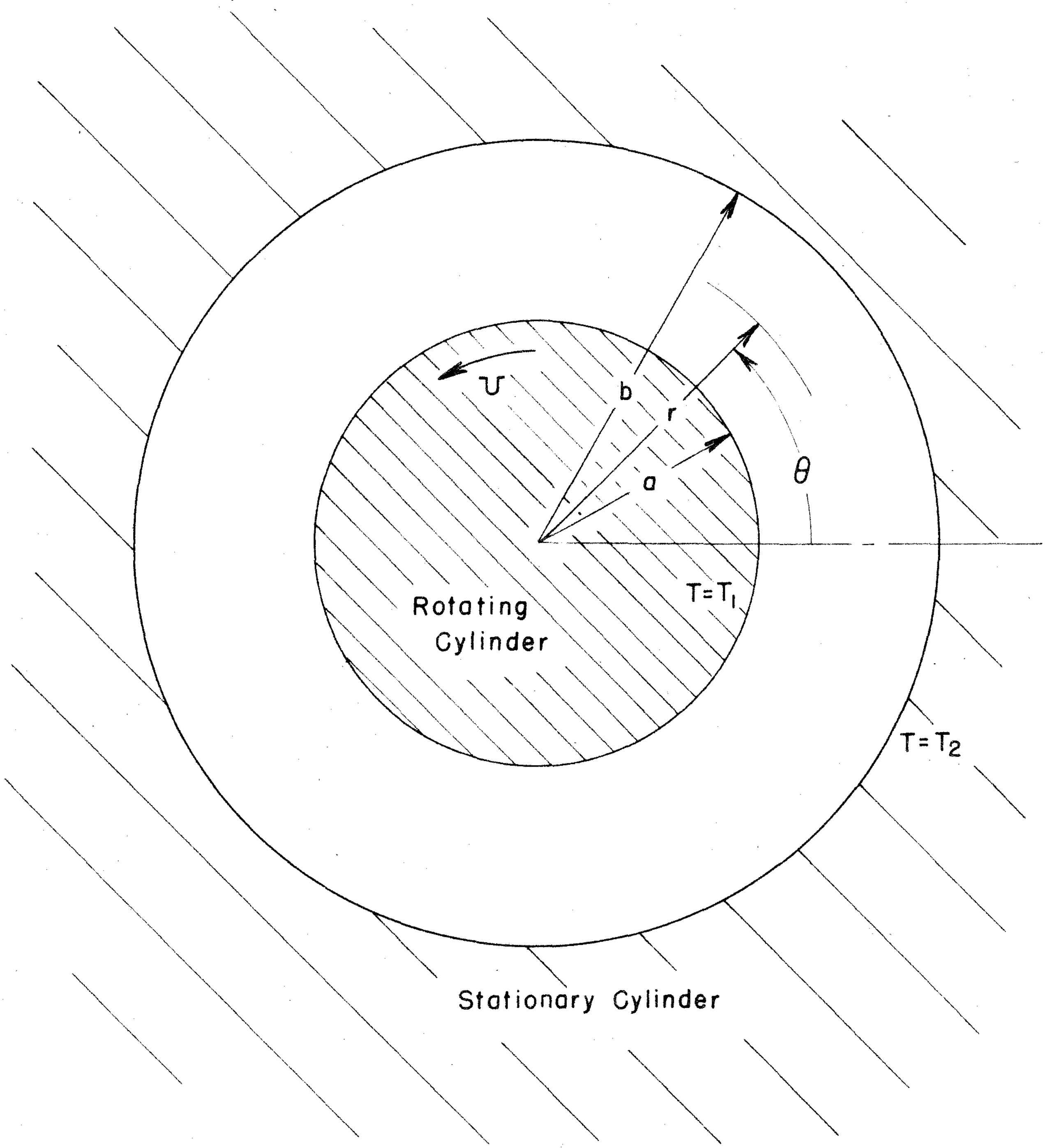

FIG. 1 – SCHEMATIC PICTURE OF THE PROBLEM

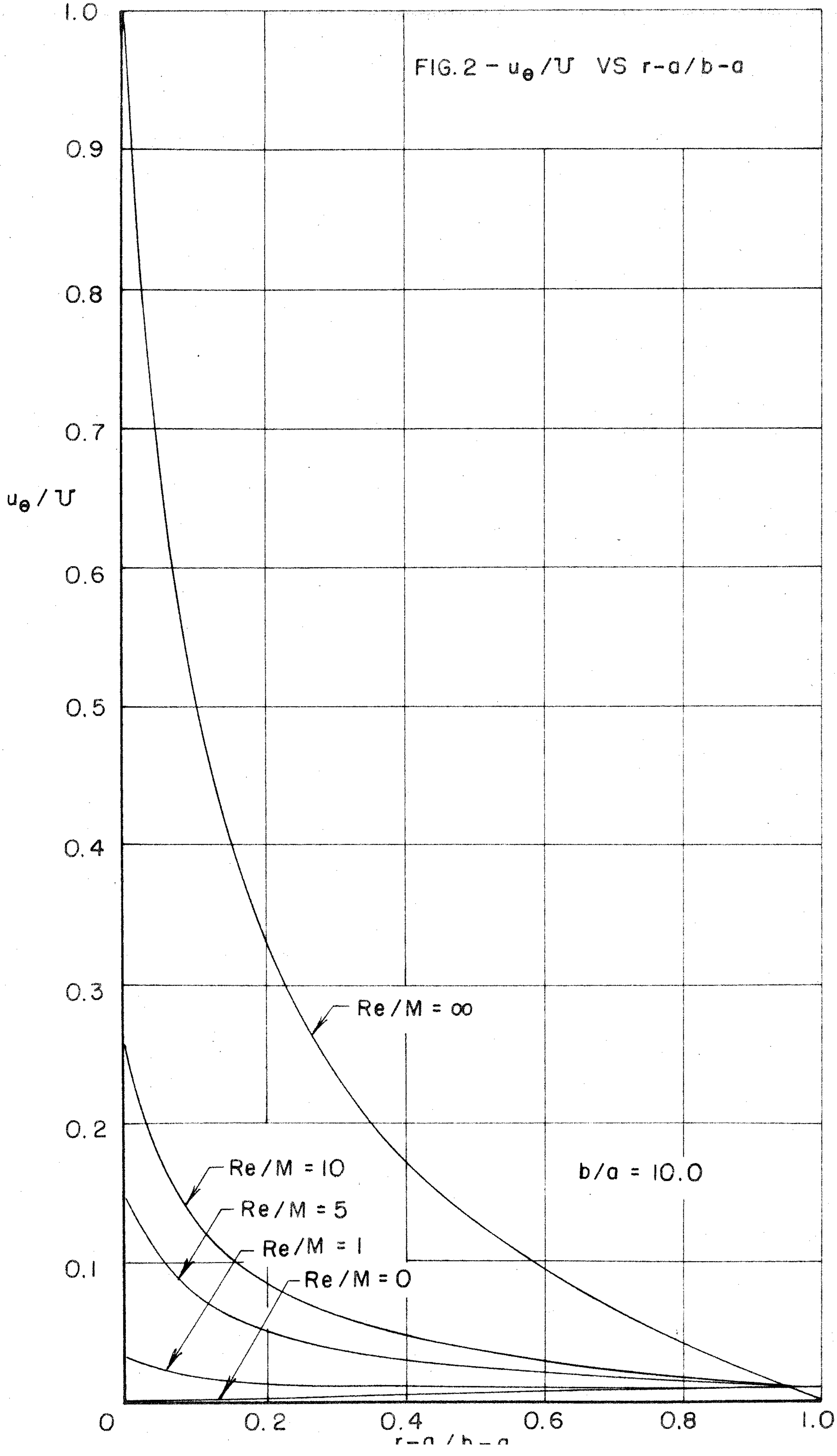
FIG. 2 – u_θ / U VS r-a/b-a
u_θ / U
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.2
0.4
0.6
0.8
1.0
r-a/b-a
Re/M = ∞
Re/M = 10
Re/M = 5
Re/M = 1
Re/M = 0
b/a = 10.0

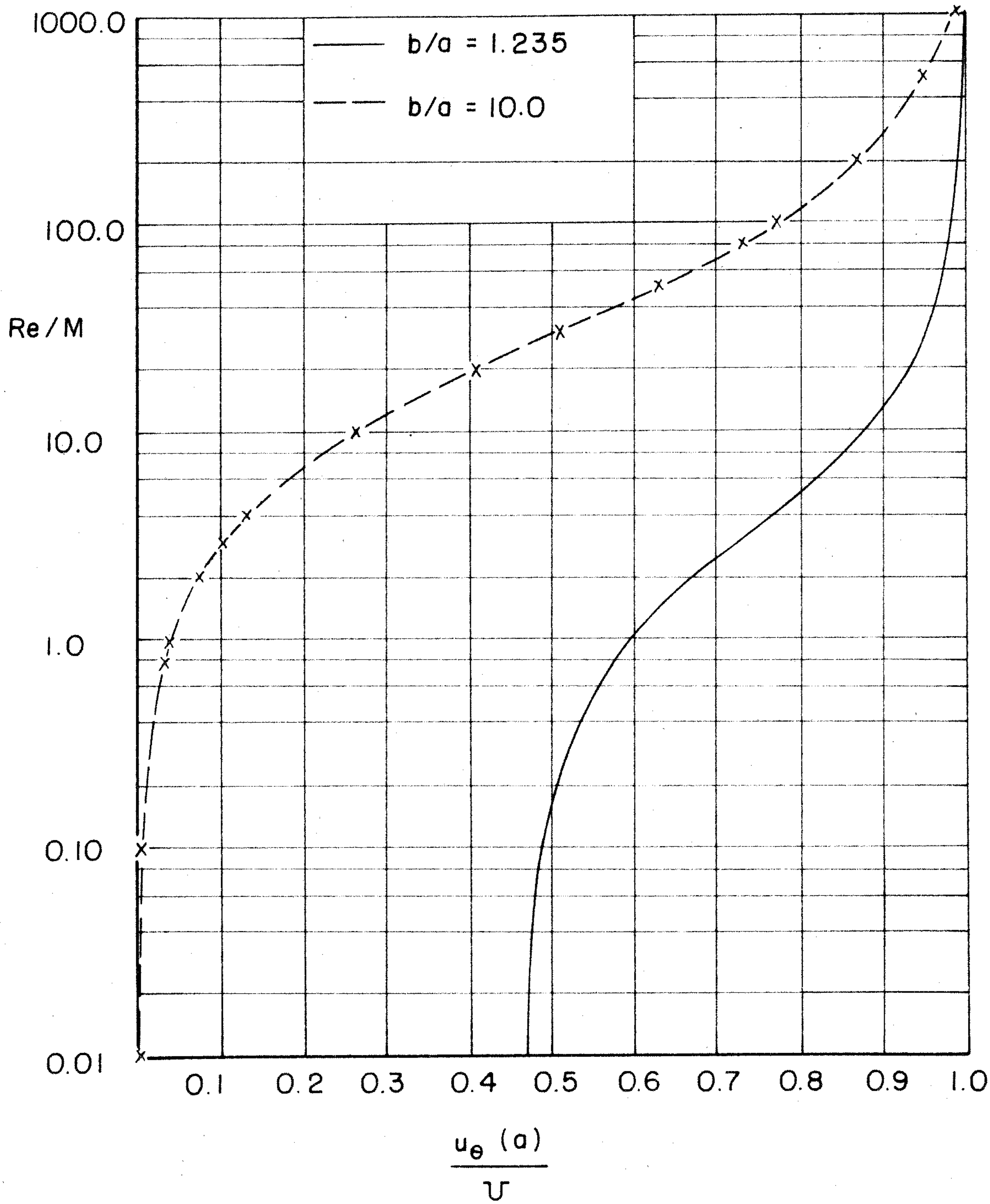

FIG. 3 - $u_\theta (a)/U$ VS Re/M

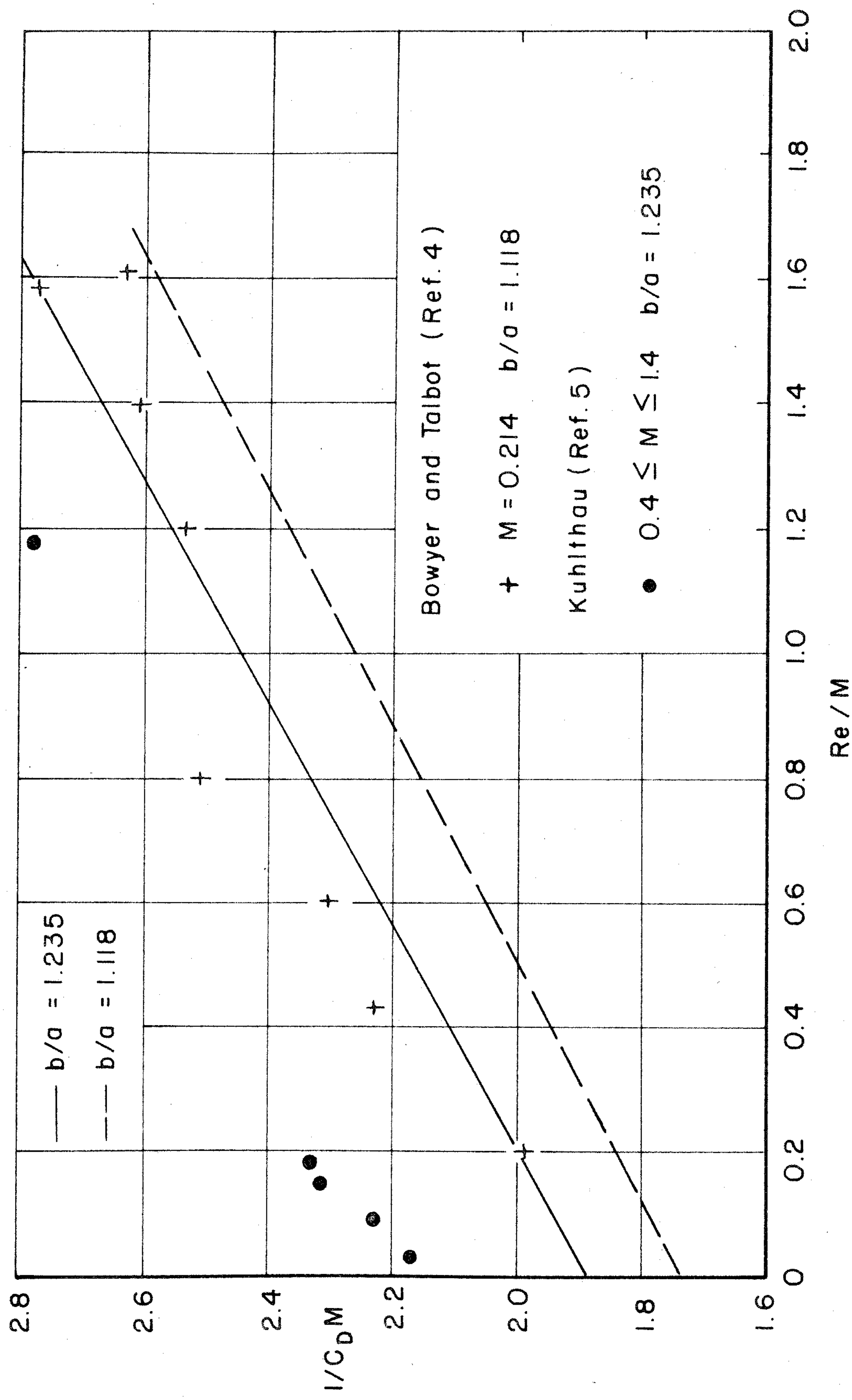

FIG. 4 - $1/C_D M$ VS Re/M

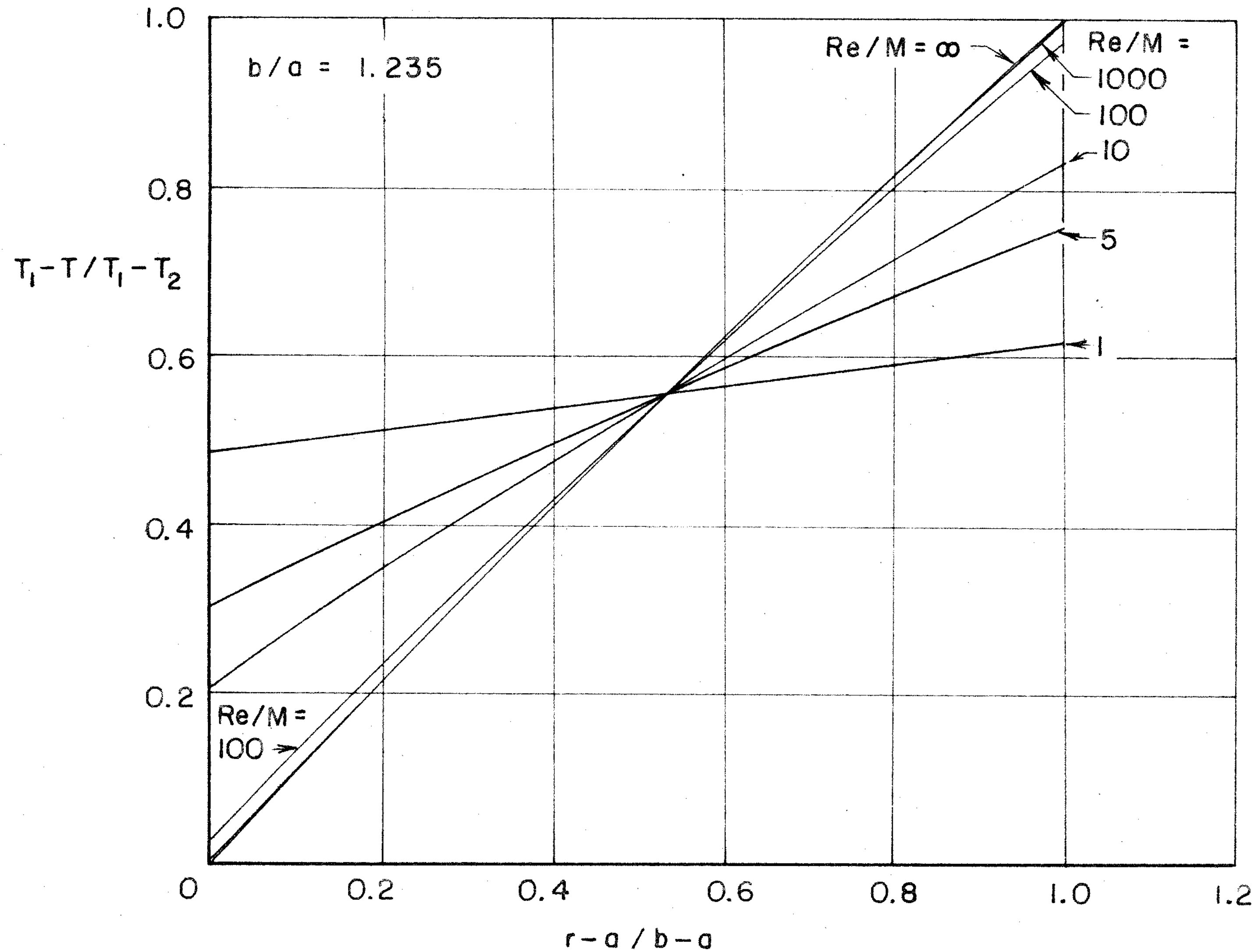

FIG. 5 — $T_1 - T$ / $T_1 - T_2$ VS. $r - a$ / $b - a$

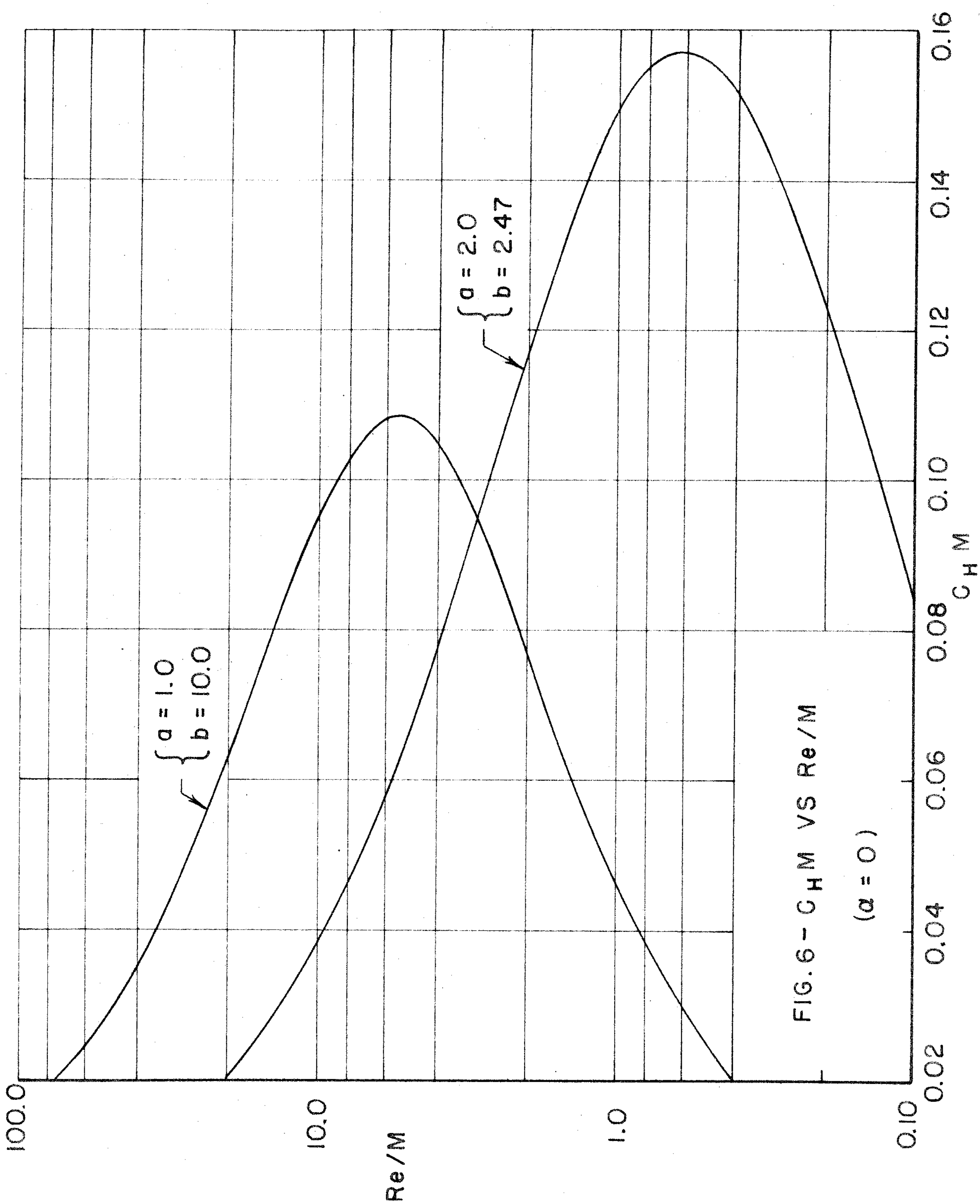

FIG. 6 - $C_H M$ VS Re/M

(α = 0)

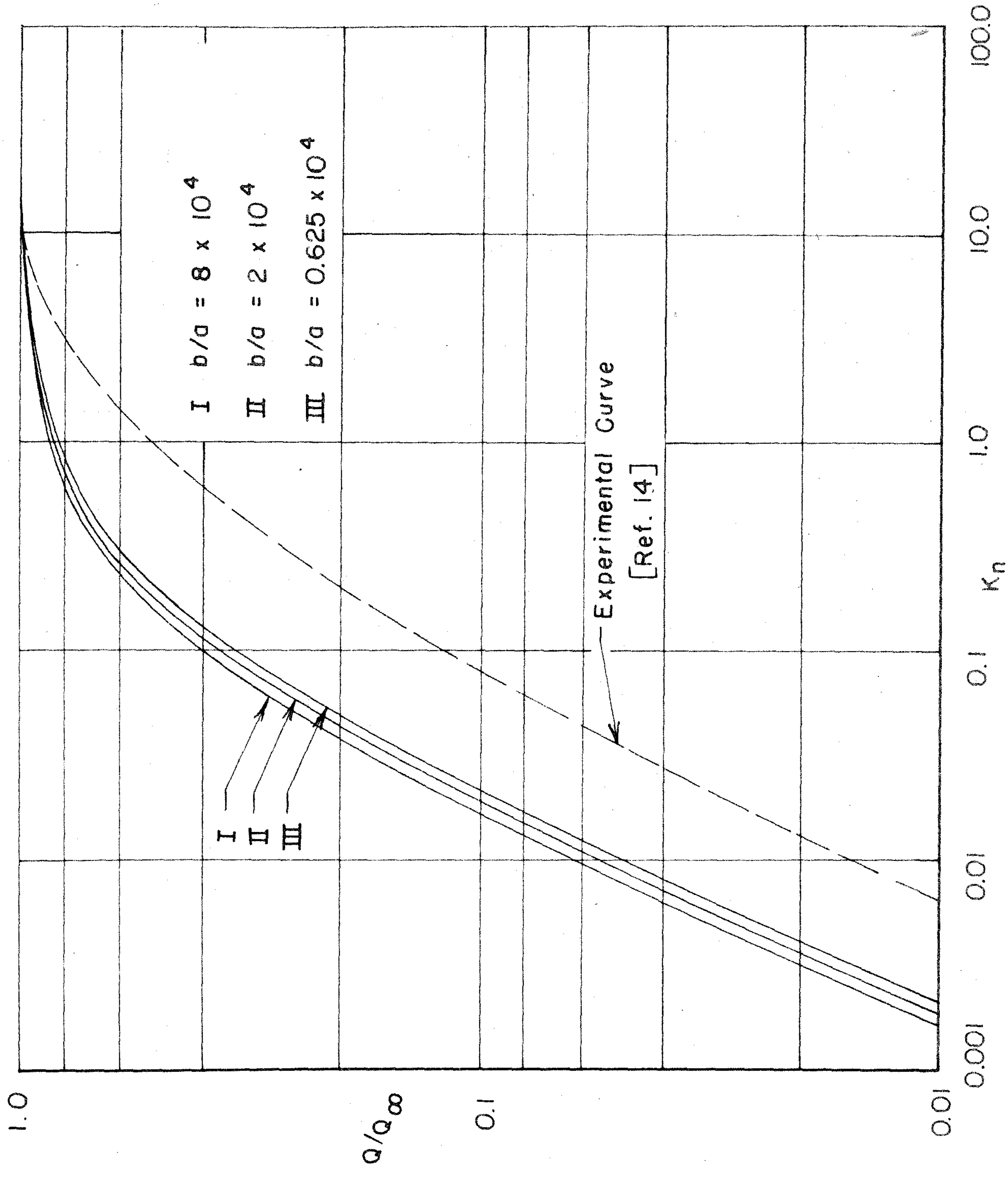

FIG. 7 – Q/Q_∞ VS. K_n

PART II

SMALL PERTURBATIONS
IN THE UNSTEADY FLOW OF A RAREFIED GAS
BASED ON GRAD'S THIRTEEN MOMENT APPROXIMATION

ABSTRACT

In this paper, the unsteady one-dimensional flow of a compressible, viscous and heat conducting fluid is treated, based on linearized Grad's thirteen moment equations. The fluid, initially at rest, is set into motion by some small external disturbances. Our interest is to examine the nature of all the responses. The fluid field extends to infinity in both directions; thus no length is involved, and also there is no solid wall boundary existing in the problem. The nature of the external disturbances is restricted to having a unit impulse in the momentum equation and a unit heat addition in the energy equation. The disturbances are located on an infinite plane normal to the flow direction; and the responses induced correspond to fundamental solutions of the problem. The method of Laplace transforms is applied, and the inverse transforms of all quantities are obtained in integral form. Because of the complicated expressions of the integrands involved, we consider only certain limiting cases which correspond to small and large times from the start of the motion, compared to the average time between molecular collisions. In order to study these limiting cases, it is essential to understand the behavior of the integrand in the complex plane; hence all singularities and branch points are obtained.

When t is small, the integrand is expanded in powers of t to obtain a wave front approximation. All discontinuities are propagated along the characteristics of the linearized system, and a damping term also appears.

At large values of time, the integrand gets its main contribution around the branch points, and these solutions are identical to those obtained from the Navier-Stokes equation.

The fundamental solution of the one-dimensional unsteady flow,

idealized as it seems to be, offers itself as a tool to understand other related problems. The piston problem, as well as the normal quantities in Rayleigh's problem (e.g., normal velocity, normal stress, and thermodynamical quantities), are governed by the same set of equations. Hence, certain parts of the fundamental solutions can be applied directly to these problems. The limiting forms of the normal quantities in Rayleigh's problem are expected to be worked out in another paper in the near future.

TABLE OF CONTENTS

LIST OF SYMBOLS

c	isentropic speed of sound, $c^2 = \gamma p_o/\rho_o$
c_o	isothermal speed of sound, $c_o^2 = p_o/\rho_o$
c_p	specific heat at constant pressure
F	external force
$G^{(i)}$	Green's functions
H	heat addition
K	coefficient of heat conductivity
P	hydrodynamic pressure
p	non-dimensional perturbation pressure
p_o	hydrodynamic pressure of the fluid at rest
q	heat flux
s	non-dimensional perturbation density
$\mathcal{s}$	Laplace transform variable
T	temperature
T_o	temperature of the fluid at rest
x, t	distorted space and time coordinates as defined in the text
x', t'	physical space and time coordinates
γ	ratio of specific heats at constant pressure and constant volume
$\delta(x)\delta(t)$	delta functions, such that $\int\int \delta(x)\delta(t)\, dx\, dt = 1$
θ	non-dimensional perturbation temperature
μ	coefficient of viscosity
μ_o	coefficient of viscosity of the fluid at rest
ρ	density
ρ_o	density of the fluid at rest
p_{ij}, τ	stress increment over the hydrodynamic pressure

I. INTRODUCTION

Grad's thirteen moment equations, derived from kinetic theory considerations, represent a formidable set of non-linear equations far more complicated than any set of the hydrodynamic equations one usually encounters. It would seem reasonable to tackle the simplest possible problems first. If one examines the solutions of Grad's equations in existence, one is not surprised to find that most cases considered so far are linearized[1, 6, 10, 11], and involve very simple geometry[5]. It has been known for a long time that linearized hydrodynamic equations offer solutions of such a nature that one obtains not only the overall picture, but also some typical features of the exact non-linear problems are still retained[7]. The linearization may also be justified by saying that it makes mathematical treatment possible, and thus allows one to carry out a unified discussion of various effects[3, 8, 9]. Furthermore, within the frame of linear theory, superposition can always be applied to construct new solutions. For these reasons, a similar treatment is attempted for Grad's equations.

So far, the solutions obtained for Grad's equations are all for the steady state case, except Rayleigh's problem[11] treated by Yang and Lees. In that particular problem, equations of "acoustic" nature and solutions at least in limiting cases are obtained for a heat insulated plate. The characteristics show an initial linear growth in time, and the solutions show interesting features which are quite different in nature to that of Navier-Stokes[8]. It was also suggested that more non-stationary problems should be taken up for investigation. The present work concerns the one-dimensional unsteady problem, which may be considered as an extension of Rayleigh's problem (normal quantities). On the other hand, it bears a certain resemblance to the piston problem. In both cases the longitudinal waves[7] play an important part.

The fundamental solutions of the problem are the main interest in the present work. Since there is no solid boundary involved, the solutions are relatively simple to obtain. Furthermore, the introduction of impulse functions makes all solutions appear as contour integrals; consequently, studies of limiting cases can be carried out without too much difficulty. Although the problem seems to be quite idealized, solutions obtained yield appreciable amounts of information that are useful, in considering other cases, such as the piston or heat conduction of an infinite plate.

II. LINEARIZED GRAD'S EQUATIONS

The general Grad's thirteen moment equations[4] with external force and heat addition are given below in Cartesian tensor form:

Continuity

$$\frac{\partial \rho}{\partial t'} + \frac{\partial}{\partial x_\alpha'}(\rho u_\alpha) = 0 \tag{1}$$

Momentum

$$\frac{\partial u_i}{\partial t'} + u_\alpha \frac{\partial u_i}{\partial x_\alpha'} + \frac{1}{\rho}\frac{\partial p_{i\alpha}}{\partial x_\alpha'} + \frac{1}{\rho}\frac{\partial P}{\partial x_i'} = F_i' \tag{2}$$

Energy

$$\frac{\partial P}{\partial t'} + \frac{\partial}{\partial x_\alpha'}(u_\alpha P) + \frac{2}{3}(p_{i\alpha} + \delta_{i\alpha} P)\frac{\partial u_i}{\partial x_\alpha'} + \frac{2}{3}\frac{\partial q_\alpha}{\partial x_\alpha'} = H' \tag{3}$$

Stresses

$$\frac{\partial p_{ij}}{\partial t'} + \frac{\partial}{\partial x_\alpha'}(u_\alpha p_{ij}) + \frac{2}{5}\left(\frac{\partial q_i}{\partial x_j'} + \frac{\partial q_j}{\partial x_i'} - \frac{2}{3}\delta_{ij}\frac{\partial q_\alpha}{\partial x_\alpha'}\right) + p_{i\alpha}\frac{\partial u_j}{\partial x_\alpha}$$

$$+ p_{j\alpha}\frac{\partial u_i}{\partial x_\alpha'} - \frac{2}{3}\delta_{ij} p_{\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta'} + P\left(\frac{\partial u_i}{\partial x_j'} + \frac{\partial u_j}{\partial x_i'} - \frac{2}{3}\delta_{ij}\frac{\partial u_\alpha}{\partial x_\alpha'}\right) = -\frac{P}{\mu} p_{ij} \tag{4}$$

Heat Flux

$$\frac{\partial q_i}{\partial t'} + \frac{\partial}{\partial x_\alpha'}(u_\alpha q_i) + \frac{7}{5} q_\alpha \frac{\partial u_i}{\partial x_\alpha'} + \frac{2}{5} q_\alpha \frac{\partial u_\alpha}{\partial x_i'} + \frac{2}{5} q_i \frac{\partial u_\alpha}{\partial x_\alpha} + RT\frac{\partial p_{i\alpha}}{\partial x_\alpha'}$$

$$+ \frac{7}{2} p_{i\alpha}\frac{\partial RT}{\partial x_\alpha'} + \frac{5}{2} P\frac{\partial RT}{\partial x_i'} - \frac{p_{i\alpha}}{\rho}\left(\frac{\partial p_{\alpha\beta}}{\partial x_\beta'} + \delta_{\alpha\beta}\frac{\partial P}{\partial x_\beta'}\right) = -\frac{2}{3}\frac{P}{\mu} q_i \tag{5}$$

Altogether, fifteen unknowns are involved in Eqs. (1) to (5); hence, we need in addition the equation of state, which is also obtained from certain moment relation[4], to complete the set.

$$P = \rho R T \tag{6}$$

Furthermore, from the definition of the moments and also from Eq. (4), there exists the relation

$$p_{ii} = 0 \quad .$$

Therefore, in general, only five stresses are to be solved, and the total number of moment equations reduces to thirteen.

In the following, the analysis will be based on the theory of small perturbations. By small perturbations, we mean that

$$P = p_o (1+p) , \quad \rho = \rho_o (1+s) , \quad T = T_o (1+\theta) \tag{7}$$

where p, θ, s << 1 everywhere, and $|\vec{u}| << c$, c being the isentropic speed of sound. For stresses and heat flux, we have

$$p_{ij}/p_o << 1 \qquad \text{and} \qquad q_i \sim O(p_o u_i) \quad .$$

We also assume

$$\mu = \mu_o (1+\mu') \tag{8}$$

where $\mu_o = \mu_o (T_o)$ and $\mu' << 1$. We can utilize the above relations to linearize Eqs. (1) to (6) by dropping all products and squares of perturbations. A set of linearized equations is obtained in the following form.

Continuity

$$\frac{\partial s}{\partial t'} + \frac{\partial u_\alpha}{\partial x'_\alpha} = 0 \tag{9}$$

Momentum

$$\frac{\partial u_i}{\partial t'} + \frac{1}{\rho_0}\frac{\partial p_{i\alpha}}{\partial x'_\alpha} + \frac{p_0}{\rho_0}\frac{\partial p}{\partial x'_i} = F'_i \tag{10}$$

Energy

$$p_0\frac{\partial p}{\partial t'} + p_0\frac{\partial u_\alpha}{\partial x'_\alpha} + \frac{2}{3}p_0\,\delta_{i\alpha}\frac{\partial u_i}{\partial x'_\alpha} + \frac{2}{3}\frac{\partial q_\alpha}{\partial x'_\alpha} = H' \tag{11}$$

Stresses

$$\frac{\partial p_{ij}}{\partial t'} + \frac{2}{5}\left(\frac{\partial q_i}{\partial x'_j} + \frac{\partial q_j}{\partial x'_i} - \frac{2}{3}\delta_{ij}\frac{\partial q_\alpha}{\partial x'_\alpha}\right)$$
$$+ p_0\left(\frac{\partial u_i}{\partial x'_j} + \frac{\partial u_j}{\partial x'_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_\alpha}{\partial x'_\alpha}\right) = -\frac{p_0}{\mu_0}p_{ij} \tag{12}$$

Heat Flux

$$\frac{\partial q_i}{\partial t'} + RT_0\frac{\partial p_{i\alpha}}{\partial x'_\alpha} + \frac{5}{2}p_0 RT_0\frac{\partial \theta}{\partial x'_i} = -\frac{2}{3}\frac{p_0}{\mu_0}q'_i \tag{13}$$

State

$$p = S + \theta \tag{14}$$

The kind of linearization used above is very common in hydrodynamics. One would get the steady state Oseen's type of equations by applying a Galilean transformation to the above equations[7].

III. ONE-DIMENSIONAL UNSTEADY FLOW

III. 1. Equations of Motion

For the one-dimensional flow problem, the number of moments required is greatly reduced. Here, we have a set of 5 first order partial differential equations instead of the thirteen needed for the general case, and one algebraic (equation of state) equation. The six unknowns to be determined are the following:

p	perturbation pressure
s	perturbation density
θ	perturbation temperature
u	velocity
τ	normal stress
q	heat flux

The quantities p, s, and θ are non-dimensional. If we introduce a new set of coordinates

$$x = \frac{p_0}{\mu_0} x' \qquad , \qquad t = \frac{p_0}{\mu_0} t'$$

the six equations describing the flow are

Continuity

$$\frac{\partial s}{\partial t} + \frac{\partial u}{\partial x} = 0 \tag{15}$$

Momentum

$$\frac{\partial u}{\partial t} + \frac{p_0}{\rho_0}\frac{\partial p}{\partial x} + \frac{1}{\rho_0}\frac{\partial \tau}{\partial x} = F(x,t) \tag{16}$$

Energy

$$\frac{\partial p}{\partial t} + \frac{5}{3}\frac{\partial u}{\partial x} + \frac{2}{3p_0}\frac{\partial q}{\partial x} = H(x,t) \tag{17}$$

Stress

$$\left(\frac{\partial}{\partial t}+1\right)\tau+\frac{8}{15}\frac{\partial q}{\partial x}+\frac{4}{3}p_0\frac{\partial u}{\partial x}=0 \tag{18}$$

Heat Flux

$$\left(\frac{\partial}{\partial t}+\frac{2}{3}\right)q+\frac{p_0}{\rho_0}\frac{\partial \tau}{\partial x}+\frac{5}{2}\frac{p_0^2}{\rho_0}\frac{\partial \theta}{\partial x}=0 \tag{19}$$

State

$$p = s + \theta \tag{20}$$

where

$$F' = \frac{p_0}{\mu_0}F \qquad H' = \frac{p_0^2}{\mu_0}H$$

III. 2. Laplace Transforms with Zero Initial Conditions

Since our purpose is to determine the responses generated by small disturbances in a fluid field originally at rest, we may set t = 0 as the time at which the disturbances are introduced; therefore, only solutions for t > 0 are of interest to us. Hence, the method of Laplace transforms should bring out all solutions, at least in integral form.

The Laplace transform with respect to t of any quantity Q = Q(x, t) is

$$L\{Q\} = \bar{Q} = \int_0^\infty e^{-st} Q(x,t)\,dt \quad . \tag{21}$$

With zero initial condition,

$$L\left\{\frac{\partial Q}{\partial t}\right\} = s\bar{Q} \quad , \tag{22}$$

and the inverse transform is defined as

$$Q = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} e^{st}\bar{Q}\, ds \tag{23}$$

where σ is to the right of all singularities. The transformed equations become ordinary differential equations of the independent variable, x, as follows:

Continuity

$$\frac{d\bar{u}}{dx} = -s\bar{S} \tag{24}$$

Momentum

$$s\bar{u} + \frac{p_o}{\rho_o}\frac{d\bar{p}}{dx} + \frac{1}{\rho_o}\frac{d\bar{\tau}}{dx} = \bar{F}(x;s) \tag{25}$$

Energy

$$\frac{d\bar{q}}{dx} = \frac{3}{2}p_o s\left(\frac{5}{3}\bar{S} - \bar{p}\right) + \frac{3}{2}p_o\bar{H}(x;s) \tag{26}$$

Stress

$$\bar{\tau} = \frac{4}{5}p_o\frac{s}{s+1}\bar{p} \tag{27}$$

Heat

$$\left(s + \frac{2}{3}\right)\bar{q} + \frac{p_o}{\rho_o}\frac{d\bar{\tau}}{dx} + \frac{5}{2}\frac{p_o^2}{\rho_o}\frac{d\bar{\theta}}{dx} = 0 \tag{28}$$

State

$$\bar{p} = \bar{S} + \bar{\theta} \tag{29}$$

III. 3. Solutions of Transformed Equations

In order to solve the six unknowns from Eqs. (24) to (29), we start off by eliminating $\bar{\theta}$ from Eq. (29), $\bar{u}$ from Eq. (24), $\bar{\tau}$ from Eq. (27), and $\bar{q}$ from Eq. (28) successively to arrive at two simultaneous equations for $\bar{s}$ and $\bar{p}$.

$$-\lambda^2(\lambda+1)\bar{s} + \frac{p_o}{\rho_o}\left(\frac{9}{5}\lambda+1\right)\frac{d^2\bar{p}}{dx^2} = (\lambda+1)\frac{d\bar{F}}{dx} \tag{30}$$

$$\left[\frac{p_o}{\rho_o}(\lambda+1)\frac{d^2}{dx^2} - \lambda(\lambda+1)\left(\lambda+\frac{2}{3}\right)\right]\bar{s} - \left[\frac{p_o}{\rho_o}\left(\frac{33}{25}\lambda+1\right)\frac{d^2}{dx^2} - \frac{3}{5}\lambda(\lambda+1)\left(\lambda+\frac{2}{3}\right)\right]\bar{p} = \frac{3}{5}(\lambda+1)\left(\lambda+\frac{2}{3}\right)\bar{H} \tag{31}$$

Cross-differentiation of the above equations yields the governing equations of $\bar{s}$ and $\bar{p}$ respectively.

$$\frac{p_o^2}{\rho_o^2}\left(\frac{9}{5}\lambda+1\right)\frac{d^4\bar{s}}{dx^4} - \frac{p_o}{\rho_o}\lambda\left(\frac{78}{25}\lambda^2+\frac{16}{5}\lambda+\frac{2}{3}\right)\frac{d^2\bar{s}}{dx^2} + \frac{3}{5}\lambda^3(\lambda+1)\left(\lambda+\frac{2}{3}\right)\bar{s} = \frac{p_o}{\rho_o}\left(\frac{33}{25}\lambda+1\right)\frac{d^3\bar{F}}{dx^3} - \frac{3}{5}\lambda(\lambda+1)\left(\lambda+\frac{2}{3}\right)\frac{d\bar{F}}{dx} + \frac{3}{5}\frac{p_o}{\rho_o}\left(\frac{9}{5}\lambda+1\right)\left(\lambda+\frac{2}{3}\right)\frac{d^2\bar{H}}{dx^2} \tag{32}$$

$$\frac{p_o^2}{\rho_o^2}\left(\frac{9}{5}\lambda+1\right)\frac{d^4\bar{p}}{dx^4} - \frac{p_o}{\rho_o}\lambda\left(\frac{78}{25}\lambda^2+\frac{16}{5}\lambda+\frac{2}{3}\right)\frac{d^2\bar{p}}{dx^2} + \frac{3}{5}\lambda^3(\lambda+1)\left(\lambda+\frac{2}{3}\right)\bar{p} = \frac{p_o}{\rho_o}(\lambda+1)\frac{d^3\bar{F}}{dx^3} - \lambda(\lambda+1)\left(\lambda+\frac{2}{3}\right)\frac{d\bar{F}}{dx} + \frac{3}{5}\lambda^2(\lambda+1)\left(\lambda+\frac{2}{3}\right)\bar{H} \tag{33}$$

We notice that both equations have the same homogeneous part, but the inhomogeneous parts differ. In fact, the same linear differential operator governs all unknown quantities to be determined. This behavior is expected, since the linear operator is related to the characteristics of the linearized system. To save writing, we denote

$$a = \frac{p_0^2}{\rho_0^2}\left(\frac{9}{5}s + 1\right)$$

$$b = -\frac{p_0}{\rho_0} s\left(\frac{39}{25}s^2 + \frac{8}{5}s + \frac{1}{3}\right)$$

$$c = \frac{3}{5}s^3(s+1)\left(s + \frac{2}{3}\right)$$

$$f_{\bar{s}} = f_{\bar{s}_F} + f_{\bar{s}_H}$$

$$f_{\bar{p}} = f_{\bar{p}_F} + f_{\bar{p}_H}$$

where

$$f_{\bar{s}_F} = -\frac{\left(\frac{33}{35}s+1\right)}{\frac{p_0}{\rho_0}\left(\frac{9}{5}s+1\right)}\frac{d^3\bar{F}}{dX^3} + \frac{3/5\, s(s+1)(s+2/3)\rho_0/p_0}{\frac{p_0}{\rho_0}\left(\frac{9}{5}s+1\right)}\frac{d\bar{F}}{dX}$$

$$f_{\bar{s}_H} = -\frac{3}{5}\left(s+\frac{2}{3}\right)\frac{d^2\bar{H}}{dX^2}$$

$$f_{\bar{p}_F} = -\frac{(s+1)}{\frac{p_0}{\rho_0}\left(\frac{9}{5}s+1\right)}\frac{d^3\bar{F}}{dX^3} + \frac{s(s+1)(s+2/3)}{\frac{p_0}{\rho_0}\left(\frac{9}{5}s+1\right)}\frac{\rho_0}{p_0}\frac{d\bar{F}}{dX}$$

$$f_{\bar{p}_H} = -\frac{3/5\, s^2(s+1)(s+2/3)}{\frac{p_0}{\rho_0}\left(\frac{9}{5}s+1\right)}\bar{H}$$

The quantities $f_{\bar{S}_F}$ and $f_{\bar{P}_F}$ are introduced by the external forcing term, while $f_{\bar{S}_H}$ and $f_{\bar{P}_H}$ correspond to the heat addition term. Since the equations are linear, the solutions associated with $\bar{\bar{F}}$ and $\bar{H}$ can be treated separately.

Equations (32) and (33) are now written as

$$a\left(\frac{d^2}{dX^2}-\lambda_1\right)\left(\frac{d^2}{dX^2}-\lambda_2\right)\bar{S},\bar{P} = -\left(f_{\bar{S}}, f_{\bar{P}}\right)a \quad . \tag{34}$$

Here we already have factored the fourth order operator

$$a\frac{d^4}{dX^4} + 2b\frac{d^2}{dX^2} + c$$

into the product of two second-order operators, $\left(\frac{d^2}{dX^2}-\lambda_1\right)$ and $\left(\frac{d^2}{dX^2}-\lambda_2\right)$. The λ' are given below.

$$\lambda_{1,2} = -b/a \pm \frac{1}{a}\sqrt{b^2-ac} \quad , \tag{35}$$

or more precisely,

$$\lambda_{1,2} = \frac{s}{\frac{p_0}{\rho_0}\left(\frac{9}{5}s+1\right)}\left[\frac{39}{25}s^2+\frac{8}{5}s+\frac{1}{3} \pm \sqrt{\frac{846}{625}s^4+\frac{324}{125}s^3+\frac{47}{25}s^2+\frac{2}{3}s+\frac{1}{9}}\right] \quad . \tag{35a}$$

The λ's are identical to the quantity $\delta(s)\{f_1(s) \pm f_2(s)\}$ obtained by Yang and Lees in Reference 2, except that here distorted coordinates are used, so that the s differs by a factor of p_0/μ_0 .

So far, the forcing functions F and H are left open. They can be any well-behaved functions. However, our present interest is to find the fundamental solutions; thus, we specify

$$F(x, t) = \delta(x)\delta(t) \tag{36}$$

and

$$H(x, t) = \delta(x)\delta(t) \quad . \tag{37}$$

The term F(x, t) represents a unit impulse in the x, t plane distributed evenly on an infinite plane normal to the x-axis at x = 0 and t = 0. This is equivalent to a uniform impulse of strength μ_o^3/p_o^3 in the x', t' plane located at x' = 0 and t' = 0. Similarly, the term H(x, t) represents a unit heat input introduced at t = 0, and at the plane x = 0. In the physical plane, the addition of heat is of the magnitude of μ_o^3/p_o^4. The integrations of F(x, t) and H(x, t) taken with respect to x and t through any interval including the origin are unity. The reasons that one is interested in the fundamental solutions are the following:

(1) In principle, having found the fundamental solutions of the problem, solutions corresponding to any other given functions can be generated. Furthermore, fundamental solutions themselves yield an appreciable amount of information.

(2) All solutions will appear in the form of contour integrals in the complex s plane. Either these integrations can be performed exactly, or certain limiting forms can be obtained if the integrands become too involved, which turns out to be the case in this problem.

The transforms of F and H are

$$\bar{F}(x; s) = \int_0^\infty e^{-st}\,\delta(x)\,\delta(t)\,dt = \delta(x) \tag{38}$$

and

$$\bar{H}(x; s) = \int_0^\infty e^{-st}\,\delta(x)\,\delta(t)\,dt = \delta(x) \tag{39}$$

and

$$\frac{d^n \bar{F}}{dx^n} = \frac{d^n \bar{H}}{dx^n} = \frac{d^n \delta(x)}{dx^n} \tag{40}$$

is the n^{th} order derivative of $\bar{F}$ or $\bar{H}$. Having specified the forms of $\bar{F}$ and $\bar{H}$, we can write down immediately the solutions of the equations (32) and (33).

$$\bar{S} = \bar{S}_F + \bar{S}_H$$

$$\bar{S}_F = \int_{-\infty}^{+\infty} G^{(2)}(x;\xi) f_{\bar{S}_F}(\xi)\,d\xi\,, \quad \bar{S}_H = \int_{-\infty}^{+\infty} G^{(2)}(x;\xi) f_{\bar{S}_H}(\xi)\,d\xi \tag{41}$$

$$\bar{P} = \bar{P}_F + \bar{P}_H$$

$$\bar{P}_F = \int_{-\infty}^{+\infty} G^{(2)}(x;\xi) f_{\bar{P}_F}(\xi)\,d\xi\,, \quad \bar{P}_H = \int_{-\infty}^{+\infty} G^{(2)}(x;\xi) f_{\bar{P}_H}(\xi)\,d\xi \tag{42}$$

where

$$G^{(2)}(x;\xi) = G^{(2)}(x-\xi) = \frac{1}{\lambda_1 - \lambda_2}\left(G_1^{(1)} - G_2^{(1)}\right)$$

$$G_1^{(1)}(x;\xi) = G_1^{(1)}(x-\xi) = \frac{1}{2\sqrt{\lambda_1}}\, e^{-\sqrt{\lambda_1}\,|x-\xi|}$$

$$G_2^{(1)}(x;\xi) = G_2^{(1)}(x-\xi) = \frac{1}{2\sqrt{\lambda_2}}\, e^{-\sqrt{\lambda_2}\,|x-\xi|}$$

are the Green's functions of the operators $\left(\frac{d^2}{dx^2} - \lambda_1\right)\left(\frac{d^2}{dx^2} - \lambda_2\right)$, $\left(\frac{d^2}{dx^2} - \lambda_1\right)$, and $\left(\frac{d^2}{dx^2} - \lambda_2\right)$, respectively. Substituting the expressions of $f_{\bar{s}}$ and $f_{\bar{p}}$ into their corresponding equations, we obtain

$$\bar{S}_F = -\frac{\left(\frac{33}{25}s+1\right)}{\frac{p_0}{\rho_0}\left(\frac{9}{5}s+1\right)} \int_{-\infty}^{+\infty} G^{(2)}(x-\xi)\,\frac{d^3\delta(\xi)}{d\xi^3}\,d\xi + \frac{\frac{3}{5}s(s+1)(s+\frac{2}{3})\rho_0/p_0}{\frac{p_0}{\rho_0}\left(\frac{9}{5}s+1\right)} \int_{-\infty}^{+\infty} G^{(2)}(x-\xi)\,\frac{d\delta(\xi)}{d\xi}\,d\xi \tag{43}$$

$$\bar{S}_H = -\frac{3}{5}\left(s+\frac{2}{3}\right)\int_{-\infty}^{+\infty} G^{(2)}(x-\xi)\,\frac{d^2\delta(\xi)}{d\xi^2}\,d\xi \tag{44}$$

$$\bar{P}_F = -\frac{(s+1)}{\frac{p_0}{\rho_0}\left(\frac{9}{5}s+1\right)}\int_{-\infty}^{+\infty} G^{(2)}(x-\xi)\,\frac{d^3\delta(\xi)}{d\xi^3}\,d\xi + \frac{s(s+1)(s+\frac{2}{3})\frac{\rho_0}{p_0}}{\frac{p_0}{\rho_0}\left(\frac{9}{5}s+1\right)}\int_{-\infty}^{+\infty} G^{(2)}(x-\xi)\,\frac{d\delta(\xi)}{d\xi}\,d\xi \tag{45}$$

$$\bar{P}_H = -\frac{\frac{3}{5}s^2(s+1)(s+\frac{2}{3})}{\frac{p_0}{\rho_0}\left(\frac{9}{5}s+1\right)}\int_{-\infty}^{+\infty} G^{(2)}(x-\xi)\,\delta(\xi)\,d\xi \tag{46}$$

One would expect that in order to obtain all the transformed quantities, the integral

$$I_n = \int_{-\infty}^{+\infty} G^{(2)}(x-\xi)\,\frac{d^n\delta(\xi)}{d\xi^n}\,d\xi \quad , \tag{47}$$

for n = 0, 1, 2, and 3 must be evaluated. At $x - \xi$, the n^{th} derivative of the Green's function $G^{(2)}(x-\xi)$ with respect to ξ is continuous; therefore we can integrate Eq. (47) by parts. The results are collected in Appendix I.

We now have

$$\bar{S}_F = \frac{(\mathrm{sgn}\,x)\left[\left(\frac{33}{25}s+1\right)\lambda_1 - \frac{3}{5}\frac{\rho_0}{p_0}s(s+1)(s+\frac{2}{3})\right]e^{-\sqrt{\lambda_1}|x|}}{2\frac{p_0}{\rho_0}\left(\frac{9}{5}s+1\right)(\lambda_1-\lambda_2)} + \frac{(\mathrm{sgn}\,x)\left[\left(\frac{33}{25}s+1\right)\lambda_2 - \frac{3}{5}\frac{\rho_0}{p_0}s(s+1)(s+\frac{2}{3})\right]e^{-\sqrt{\lambda_2}|x|}}{2\frac{p_0}{\rho_0}\left(\frac{9}{5}s+1\right)(\lambda_2-\lambda_1)} \tag{48}$$

$$\bar{S}_H = -\frac{\frac{3}{5}(s+\frac{2}{3})\sqrt{\lambda_1}}{2(\lambda_1-\lambda_2)}\,e^{-\sqrt{\lambda_1}|x|} - \frac{\frac{3}{5}(s+\frac{2}{3})\sqrt{\lambda_2}}{2(\lambda_2-\lambda_1)}\,e^{-\sqrt{\lambda_2}|x|} \tag{49}$$

$$\bar{P}_F = \frac{(\mathrm{sgn}\,x)\left[(s+1)\lambda_1 - \frac{\rho_0}{p_0}s(s+1)(s+\frac{2}{3})\right]e^{-\sqrt{\lambda_1}|x|}}{2\frac{p_0}{\rho_0}\left(\frac{9}{5}s+1\right)(\lambda_1-\lambda_2)} + \frac{(\mathrm{sgn}\,x)\left[(s+1)\lambda_2 - \frac{\rho_0}{p_0}s(s+1)(s+\frac{2}{3})\right]e^{-\sqrt{\lambda_2}|x|}}{2\frac{p_0}{\rho_0}\left(\frac{9}{5}s+1\right)(\lambda_2-\lambda_1)} \tag{50}$$

$$\bar{P}_H = -\frac{\frac{3}{5}s^2(s+1)(s+\frac{2}{3})\frac{1}{\sqrt{\lambda_1}}e^{-\sqrt{\lambda_1}|x|}}{2\frac{p_0}{\rho_0}\left(\frac{9}{5}s+1\right)(\lambda_1-\lambda_2)} - \frac{\frac{3}{5}s^2(s+1)(s+\frac{2}{3})\frac{1}{\sqrt{\lambda_2}}e^{-\sqrt{\lambda_2}|x|}}{2\frac{p_0}{\rho_0}\left(\frac{9}{5}s+1\right)(\lambda_2-\lambda_1)} \quad . \tag{51}$$

From Eq. (41), we obtain

$$\bar{\theta}_F = -\frac{(sgn x)\left[\frac{8}{25}s\lambda_1 + \frac{2}{5}\frac{\rho_0}{p_0}s(s+1)(s+\frac{2}{3})\right]}{2\frac{p_0}{\rho_0}(\frac{9}{5}s+1)(\lambda_1-\lambda_2)} e^{-\sqrt{\lambda_1}|x|} - \frac{(sgn x)\left[\frac{8}{25}s\lambda_2 + \frac{2}{5}\frac{\rho_0}{p_0}s(s+1)(s+\frac{2}{3})\right]}{2\frac{p_0}{\rho_0}(\frac{9}{5}s+1)(\lambda_2-\lambda_1)} e^{-\sqrt{\lambda_2}|x|} \tag{52}$$

$$\bar{\theta}_H = -\frac{\frac{3}{5}(s+\frac{2}{3})\left[s^2(s+1) - \frac{p_0}{\rho_0}(\frac{9}{5}s+1)\lambda_1\right]}{2\frac{p_0}{\rho_0}(\frac{9}{5}s+1)(\lambda_1-\lambda_2)\sqrt{\lambda_1}} e^{-\sqrt{\lambda_1}|x|} - \frac{\frac{3}{5}(s+\frac{2}{3})\left[s^2(s+1) - \frac{p_0}{\rho_0}(\frac{9}{5}s+1)\lambda_2\right]}{2\frac{p_0}{\rho_0}(\frac{9}{5}s+1)(\lambda_2-\lambda_1)\sqrt{\lambda_2}} e^{-\sqrt{\lambda_2}|x|} . \tag{53}$$

With the aid of Eqs. (39), (40), and (37), the rest of the transforms are determined as

$$\bar{\tau}_F = \frac{4}{5}p_0 \frac{(sgn x)\left[s\lambda_1 - \frac{\rho_0}{p_0}s^2(s+\frac{2}{3})\right]}{2\frac{p_0}{\rho_0}(\frac{9}{5}s+1)(\lambda_1-\lambda_2)} e^{-\sqrt{\lambda_1}|x|} + \frac{4}{5}p_0 \frac{(sgn x)\left[s\lambda_2 - \frac{\rho_0}{p_0}s^2(s+\frac{2}{3})\right]}{2\frac{p_0}{\rho_0}(\frac{9}{5}s+1)(\lambda_2-\lambda_1)} e^{-\sqrt{\lambda_2}|x|} \tag{54}$$

$$\bar{\tau}_H = -\frac{12}{25}\frac{p_0 s^3(s+\frac{2}{3})}{2\frac{p_0}{\rho_0}(\frac{9}{5}s+1)(\lambda_1-\lambda_2)}\frac{1}{\sqrt{\lambda_1}} e^{-\sqrt{\lambda_1}|x|} - \frac{12}{25}\frac{p_0 s^3(s+\frac{2}{3})}{2\frac{p_0}{\rho_0}(\frac{9}{5}s+1)(\lambda_2-\lambda_1)}\frac{1}{\sqrt{\lambda_2}} e^{-\sqrt{\lambda_2}|x|} \tag{55}$$

$$\bar{q}_F = -\frac{\rho_0 s\sqrt{\lambda_1}\, e^{-\sqrt{\lambda_1}|x|}}{2(\lambda_1-\lambda_2)} - \frac{\rho_0 s\sqrt{\lambda_2}\, e^{-\sqrt{\lambda_2}|x|}}{2(\lambda_2-\lambda_1)} \tag{56}$$

$$\begin{aligned}\bar{q}_H = &-\frac{(sgn x)p_0\left\{\frac{12}{25}s^3 + \frac{3}{2}\left[s^2(s+1) - \frac{p_0}{\rho_0}(\frac{9}{5}s+1)\lambda_1\right]\right\}}{2(\frac{9}{5}s+1)(\lambda_1-\lambda_2)} e^{-\sqrt{\lambda_1}|x|} \\ &-\frac{(sgn x)p_0\left\{\frac{12}{25}s^3 + \frac{3}{2}\left[s^2(s+1) - \frac{p_0}{\rho_0}(\frac{9}{5}s+1)\lambda_2\right]\right\}}{2(\frac{9}{5}s+1)(\lambda_2-\lambda_1)} e^{-\sqrt{\lambda_2}|x|}\end{aligned} \tag{57}$$

$$\bar{u}_F = \frac{\left[\lambda_1 - \frac{\rho_0}{p_0}s(s+\frac{2}{3})\right]\sqrt{\lambda_1}\, e^{-\sqrt{\lambda_1}|x|}}{2s(\lambda_1-\lambda_2)} + \frac{\left[\lambda_2 - \frac{\rho_0}{p_0}s(s+\frac{2}{3})\right]\sqrt{\lambda_2}\, e^{-\sqrt{\lambda_2}|x|}}{2s(\lambda_2-\lambda_1)} \tag{58}$$

and

$$\bar{u}_H = -\frac{(sgn\,x)\frac{p_0}{\rho_0}\frac{3}{5}s(s+\frac{2}{3})e^{-\sqrt{\lambda_1}|x|}}{2\frac{p_0}{\rho_0}(\lambda_1-\lambda_2)} - \frac{(sgn\,x)\frac{3}{5}s(s+\frac{2}{3})e^{-\sqrt{\lambda_2}|x|}}{2(\lambda_2-\lambda_1)} \tag{59}$$

III. 4. Approximations

In the previous Section III. 3, we have determined the transforms of all dependent variables. The exact evaluation of these transforms, however, involves a great deal of difficulty because of the complicated expressions we encounter. Nevertheless, certain approximations can be made without too much trouble. Furthermore, these approximations really represent limiting cases, which interest us. There are two approximations we consider in detail. One is the small time approximation; the other is the large time approximation.

By small times, we mean that the time elapsed from t = 0 is small compared with the average collision time t_f. By large values of time, we mean that the time elapsed from t = 0 is much larger than t_f. The physical significance of these approximations will be taken up again in Section IV.

III. 4. a. Solutions Suitable for Small Values of Time

For small time, we are looking essentially for an expansion in powers of $1/s$. ^{2} by neglecting terms of the order of $1/s$, we have

$$\sqrt{\lambda_{1,2}} \simeq \frac{s}{\sqrt{\frac{(13\mp\sqrt{94})\,p_0}{5\,\rho_0}}} - \frac{5/9}{\sqrt{\frac{(13\mp\sqrt{94})\,p_0}{5\,\rho_0}}}\left[\frac{1}{2} - \frac{1}{\sqrt{94}}\right] \tag{60}$$

The first term contributes in the integral

$$\exp\left[\mp\frac{s|x|}{\sqrt{\frac{(13\mp\sqrt{94})\,p_0}{5\,\rho_0}}}\right]$$

which represents a translation through a distance of

$$|x| = \sqrt{\frac{(13 \mp \sqrt{94})\, p_o}{5 \rho_o}}\; t$$

In order to understand this, let us now examine the characteristics of the equations.

The characteristics ϕ (x, t) = constant of the linearized system are found by the vanishing of the determinant:

$$\begin{vmatrix} \phi_t & 0 & \phi_x & 0 & 0 \\ 0 & \frac{p_o}{\rho_o}\phi_x & \phi_t & \frac{1}{\rho_o}\phi_x & 0 \\ 0 & \phi_t & \frac{5}{3}\phi_x & 0 & \frac{2}{3p_o}\phi_x \\ 0 & 0 & \frac{4}{3}p_o\phi_x & \phi_t & \frac{8}{15}\phi_x \\ -\frac{5}{2}\frac{p_o^2}{\rho_o}\phi_x & \frac{5}{2}\frac{p_o^2}{\rho_o}\phi_x & 0 & \frac{p_o}{\rho_o}\phi_x & \phi_t \end{vmatrix} = 0 \quad (61)$$

Along ϕ (x, t) = constant

$$\frac{dx}{dt} = -\phi_t / \phi_x$$

where dx/dt is the slope of the various characteristic curves. The determinant Eq. (61) thus reduces to an algebraic equation for dx/dt.

$$\frac{dx}{dt}\left[\left(\frac{dx}{dt}\right)^4 - \frac{78}{15}\frac{p_o}{\rho_o}\left(\frac{dx}{dt}\right)^2 + 3\frac{p_o^2}{\rho_o^2}\right] = 0 \qquad (62)$$

With solutions

$$\frac{dx}{dt} = 0 \qquad (63)$$

$$\frac{dx}{dt} = \pm\sqrt{\frac{(13 \mp \sqrt{94})\, p_o}{5 \rho_o}} \qquad (64)$$

the solution (dx/dt) = 0 means that the particle path is one characteristic. The solutions of Eq. (64) represent characteristic directions at "sound speed" (dx/dt), different from the isentropic sound-speed which is the characteristic slope from the Euler equations. All the characteristics obtained here are identical to those given by Yang and Lees[11] corresponding to normal quantities. This behavior is to be expected, since no transverse quantity appears in the one-dimensional problem. Because of the linearization, all "characteristic curves" are straight lines and are known in advance. The characteristics normalized against the isothermal speed of sound are plotted in Figure 1.

From Eq. (64), we see that

$$|x| = \sqrt{\frac{(13 \mp \sqrt{94})\, p_o}{5\, \rho_o}}\; t$$

are characteristic lines; hence the term

$$\exp\left[- \frac{s\,|x|}{\sqrt{\dfrac{(13 \mp \sqrt{94})\, p_o}{5\, \rho_o}}} \right]$$

shifts whatever occurs at x = 0 and t = 0 to

$$|x| = \sqrt{\frac{(13 \mp \sqrt{94})\, p_o}{5\, \rho_o}}\; t$$

at t; i.e., signals travel along wave fronts.

The second term contributes to the integral

$$\exp\left[- \frac{5/9}{\sqrt{\dfrac{(13 \mp \sqrt{94})\, p_o}{5\, \rho_o}}} \left[\frac{1}{2} - \frac{1}{\sqrt{94}} \right] |x| \right]$$

which is a damping term such that all perturbations induced die out exponentially. The transforms after having being expanded into power series of $1/s$ may be represented in the following form:

$$\bar{Q} = \left[a_0 + \frac{a_1}{s} + \frac{a_2}{s^2} + \cdots\right] \exp\left[-\frac{s|x|}{\sqrt{\frac{(13 \mp \sqrt{94})\, p_0}{5\rho_0}}}\right] \cdot \exp\left[-\frac{5/9\,|x|}{\sqrt{\frac{(13 \mp \sqrt{94})\, p_0}{5\rho_0}}}\left(\frac{1}{2} - \frac{1}{\sqrt{94}}\right)\right] \qquad (65)$$

A term by term inversion of this transform gives a delta function as the leading term which is the signal initially introduced. The second term gives a unit step function, and from there on, a power series solution valid across the lines of characteristics. Therefore, we have essentially a wave front approximation.

III. 4. b. Solutions Suitable for Large Values of Time

To evaluate a contour integral, the singularities and branch points of the integrand must be first located in order to understand fully the behavior of the integral. By equating $\lambda_{1,2} = 0$, one finds that the points $s = -1,\ -2/3,\ -5/9$, and 0 are branch points. The point $s = -5/9$ is also an essential singularity. As one can see, they are all located to the left of the imaginary axis in the complex s - plane. In other words, they all have negative real parts. If this were not the case, it would mean all quantities diverge with respect to time, and such behavior is physically impossible.

For large values of time, the integral gets the dominating contribution around the algebraically largest branch point, which in our case is the origin; hence, we can expand all transforms in powers of s. As an example, S_F is worked out in detail. From Eq. (48),

$$\bar{S}_F = \frac{(\operatorname{sgn} x)\left[\left(\frac{33}{25}s+1\right)\lambda_1 - \frac{3}{5}\frac{\rho_0}{p_0}s(s+1)\left(s+\frac{2}{3}\right)\right]e^{-\sqrt{\lambda_1}|x|}}{2\frac{p_0}{\rho_0}\left(\frac{9}{5}s+1\right)(\lambda_1-\lambda_2)} + \frac{(\operatorname{sgn} x)\left[\left(\frac{33}{25}s+1\right)\lambda_2 - \frac{3}{5}\frac{\rho_0}{p_0}s(s+1)\left(s+\frac{2}{3}\right)\right]e^{-\sqrt{\lambda_2}|x|}}{2\frac{p_0}{\rho_0}\left(\frac{9}{5}s+1\right)(\lambda_2-\lambda_1)},$$

By keeping only the leading terms of the expansion,

$$\bar{S}_F \cong \frac{(\operatorname{sgn} x)}{5\frac{p_0}{\rho_0}} e^{-\sqrt{\frac{3}{3}\frac{\rho_0}{p_0}}\sqrt{s}|x|} + \frac{(\operatorname{sgn} x)}{\frac{10}{3}\frac{p_0}{\rho_0}} e^{-\frac{s}{\sqrt{\frac{5}{3}p_0/\rho_0}}(1-\frac{7}{10}s)|x|} \quad . \tag{66}$$

The inverse transform S_F is given by

$$S_F = \frac{(\operatorname{sgn} x)}{5\,p_0/\rho_0}\,\delta_1 + \frac{(\operatorname{sgn} x)}{10/3\;p_0/\rho_0}\,\delta_2 \tag{67}$$

where

$$\delta_1 = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} e^{st-\sqrt{\frac{2}{3}\frac{\rho_0}{p_0}}\sqrt{s}|x|}\,ds \tag{68}$$

$$\delta_2 = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} e^{st-\frac{s}{\sqrt{\frac{5}{3}\frac{p_0}{\rho_0}}}(1-\frac{7}{10}s)|x|}\,ds \quad . \tag{69}$$

The other Q_F physical quantities have leading terms in δ_1 and δ_2, except τ_F, which is of higher order. This behavior is shown in Eq. (27) in which $\bar{\tau} \sim s\,\bar{p}$ for s small. For the part induced by H, we have given some of the results below. These are

$$-S_H = -\frac{3}{5}\sqrt{\frac{2}{3}\frac{\rho_0}{p_0}}\,\delta_3 + \frac{3}{5}\sqrt{\frac{3}{5}\frac{\rho_0}{p_0}}\,\delta_2 \tag{70}$$

$$\theta_H = \frac{3}{5}\sqrt{\frac{2}{3}\frac{\rho_0}{p_0}}\,\delta_3 - \frac{3}{5}\sqrt{\frac{3}{5}\frac{\rho_0}{p_0}}\,\delta_2 \tag{71}$$

and

$$u_H = (\operatorname{sgn} x)\,\frac{3}{5}\,\delta_2 \quad . \tag{72}$$

H and F have different influences on the responses, and especially on the discontinuities (See Section IV.). The δ's are evaluated in Appendix II. We have given below the results.

$$\delta_1 = \frac{|x|}{2\sqrt{\pi}\sqrt{\frac{3}{2}\frac{p_0}{\rho_0}}\; t^{3/2}} \exp\left[- \frac{x^2}{4t\left(\frac{3}{2} p_0/\rho_0\right)} \right] \tag{73}$$

$$\delta_2 = \frac{1}{2}\sqrt{\frac{10}{7}}\,\frac{1}{\sqrt{2\pi t}} \left\{ \exp\left[- \frac{10}{7c^2}\frac{(x-ct)^2}{4t} \right] + \exp\left[- \frac{10(x+ct)^2}{7c^2 \cdot 4t} \right] \right\} \tag{74}$$

$$\delta_3 = \frac{1}{\sqrt{\pi t}} \exp\left\{ - \frac{x^2}{4t\left(\frac{3}{2}\frac{p_0}{\rho_0}\right)} \right\} . \tag{75}$$

IV. DISCUSSION AND CONCLUSIONS

When $t/t_f << 1$ the solutions of the linearized Grad equations show that the original delta function impulse is propagated along two distinct characteristics, representing a "fast" wave and a "slow" wave, compared with the isentropic sound speed. The amplitude of the impulse (or the energy and momentum contained within it) decays exponentially with distance from the plane of origin of the disturbance, and step-function disturbances with "jumps" across both wave fronts are left behind. This behavior is quite different from that predicted by the Navier-Stokes equations. The responses to the force and heat input functions do not differ in any significant way when $t/t_f << 1$.

When $t/t_f >> 1$, the two distinct wave fronts have disappeared, and a disturbance propagating at the isentropic sound speed is observed, accompanied by viscous-conductive diffusion away from this "front". The width of this dissipative zone grows like $\sqrt{\nu t}$, while the perturbation amplitudes decay like $1/\sqrt{\nu t}$. Both force and heat input delta functions produce such waves. The amplitude of the pressure, density, temperature, and flow velocity perturbations are all of the same order, while the viscous stress and heat flux are of higher order, as expected.

In addition to these wave fronts, a "wake" is left behind when $t/t_f >> 1$, but the character of this wake for the heat input and force impulse delta functions is quite different. In the case of the heat function, the density and temperature perturbations in the wake are of the classical form

$$\frac{1}{\sqrt{\nu t}} \exp\left\{-\frac{x^2}{4\nu t}\right\} ,$$

with maximum amplitude at the plane of origin, containing a constant total "area" or heat quantity at all times. The velocity, pressure, etc., are all of higher order. In the case of the force impulse function, however, the density and temperature perturbations act like $\partial/\partial x$ of the classical delta heat function solutions, which means that the maximum amplitude occurs at a distance from the origin $\sim\sqrt{t}$, and the magnitude of this amplitude $\sim t^{-1}$. The area or heat quantity associated with this disturbance decays like $1/\sqrt{t}$. Apparently the application of the delta force function introduces a kind of heat "dipole", or equal and opposite heating and cooling delta function at the origin. Thus, in the case of the heat input function, the wave front and wake perturbations are equally important; but in the case of the force impulse function, the important part of the disturbance is contained in the waves.

For $t/t_f << 1$, Grad's equations furnish a kind of average behavior, as observed in Reference 11 for Rayleigh's problem. It would be desirable to examine the present problem with the aid of a somewhat more sophisticated particle velocity distribution function, in order to account for the fact that particles with velocities faster than the Grad characteristics speeds will carry the disturbance ahead of these "wave fronts". It would also be instructive to study other non-steady flow problems, such as the disturbance produced by the sudden heating of an infinite, stationary flat plate, or the piston problem, which have certain close resemblances to the present problem.

REFERENCES

1. Ai, D. K.: Cylindrical Couette Flow in a Rarefied Gas According to Grad's Equations. GALCIT Hypersonic Research Project, Memorandum No. 56, July 15, 1960.

2. Carlslaw, H. S. and J. C. Jaeger: Operational Methods in Applied Mathematics. Second Edition, Oxford University Press.

3. Cole, J. D. and T. Y. Wu: Heat Conduction in a Compressible Fluid. Journal of Applied Mechanics, Vol. 74, No. 2, pp. 209-213, June, 1952.

4. Grad, H.: On the Kinetic Theory of Rarefied Gases. Communications on Pure and Applied Mathematics, Vol. II, No. 4, pp. 331-407, December, 1949.

5. Grad, H.: The Profile of a Steady Shock Wave. Communications on Pure and Applied Mathematics, Vol. V, No. 3, pp. 257-300, August, 1952.

6. Goldberg, R.: The Slow Flow of a Rarefied Gas Past a Spherical Obstacle. Ph.D. Thesis, Department of Mathematics, New York University, April 1, 1954.

7. Lagerstrom, P. A.; J. D. Cole; L. Trilling: Problems in the Theory of Viscous Compressible Fluids. GALCIT Notes, Prepared under ONR Contract No. N6onr-244, T.O. VIII, March, 1949.

8. Wu, T. Yao-Tsu: Small Perturbations in the Unsteady Flow of a Compressible, Viscous, and Heat-Conducting Fluid. Journal of Mathematics and Physics, Vol. XXXV, No. 1, April, 1956, pp. 13-27.

9. Wu, T. Yao-Tsu: On Problems of Heat Conduction in a Compressible Fluid. Ph.D. Thesis, Guggenheim Aeronautical Library, California Institute of Technology, 1952.

10. Yang, H. T. and L. Lees: Plane Couette Flow at Low Mach Number According to the Kinetic Theory of Gases. GALCIT Hypersonic Research Project, Memorandum No. 36, February 1, 1957. Also, Proceedings of the Fifth Midwestern Conference on Fluid Mechanics, 1957, pp. 41-65.

11. Yang, H. T. and L. Lees: Rayleigh's Problem at Low Mach Number According to the Kinetic Theory of Gases. GALCIT Hypersonic Research Project, Technical Report No. 2, July 15, 1955. Also, Journal of Mathematics and Physics, Vol. 35, No. 3, pp. 195-235, October, 1956.

APPENDIX I

SOME FORMULAS IN CONNECTION WITH THE GREEN'S FUNCTION $G^{(2)}(x;\xi)$

In calculating the fundamental solutions, integrals of the type

$$I_n = \int_{-\infty}^{+\infty} G^{(2)}(x;\xi)\,\frac{d^n\delta(\xi)}{d\xi^n}\,d\xi \qquad n = 0,\ 1,\ 2,\ 3$$

and differentiation of the integrals

$$\frac{dI_n}{dx} = \frac{d}{dx}\int_{-\infty}^{+\infty} G^{(2)}(x;\xi)\,\frac{d^n\delta(\xi)}{d\xi^n}\,d\xi \qquad n = 0,\ 1,\ 2,\ 3$$

appear repeatedly.

In this Appendix, we give the results of these integrals and some useful formulas in connection with the evaluation of these integrals.

$$G^{(2)}(x;\xi) = G^{(2)}(x-\xi) = \frac{1}{2(\lambda_1-\lambda_2)}\left[\frac{1}{\sqrt{\lambda_1}}e^{-\sqrt{\lambda_1}|x-\xi|} - \frac{1}{\sqrt{\lambda_2}}e^{-\sqrt{\lambda_2}|x-\xi|}\right] \qquad (I.\ 1)$$

$$I_0 = \frac{1}{2(\lambda_1-\lambda_2)}\left[\frac{1}{\sqrt{\lambda_1}}e^{-\sqrt{\lambda_1}|x|} - \frac{1}{\sqrt{\lambda_2}}e^{-\sqrt{\lambda_2}|x|}\right] \qquad (I.\ 2)$$

$$I_1 = -\frac{(\mathrm{sgn}\,x)}{2(\lambda_1-\lambda_2)}\left[e^{-\sqrt{\lambda_1}|x|} - e^{-\sqrt{\lambda_2}|x|}\right] \qquad (I.\ 3)$$

$$I_2 = \frac{1}{2(\lambda_1-\lambda_2)}\left[\sqrt{\lambda_1}\,e^{-\sqrt{\lambda_1}|x|} - \sqrt{\lambda_2}\,e^{-\sqrt{\lambda_2}|x|}\right] \qquad (I.\ 4)$$

$$I_3 = -\frac{(sgn\,x)}{2(\lambda_1-\lambda_2)}\left[\lambda_1 e^{-\sqrt{\lambda_1}|x|} - \lambda_2 e^{-\sqrt{\lambda_2}|x|}\right] \quad \text{(I. 5)}$$

$$\frac{dI_{n-1}}{dx} = I_n \qquad n = 1,\ 2,\ 3 \quad \text{(I. 6)}$$

$$\frac{dI_3}{dx} = \frac{1}{2(\lambda_1-\lambda_2)}\left[\lambda_1\sqrt{\lambda_1}\, e^{-\sqrt{\lambda_1}|x|} - \lambda_2\sqrt{\lambda_2}\, e^{-\sqrt{\lambda_2}|x|}\right] - \delta(x) \quad \text{(I. 7)}$$

$$\frac{dI_n}{dx} = \frac{d^{n+1} I_0}{dx^{n+1}} \qquad n = 1,\ 2,\ 3 \quad \text{(I. 8)}$$

$$(sgn\,x) = \begin{matrix} +1 \\ -1 \end{matrix} \qquad \text{for} \quad \begin{matrix} x > 0 \\ x < 0 \end{matrix} \quad \text{(I. 9)}$$

$$\frac{d}{dx}(sgn\,x) = 2\,\delta(x) \quad \text{(I. 10)}$$

$$\frac{d}{dx}|x| = (sgn\,x) \quad \text{(I. 11)}$$

APPENDIX II

COLLECTED RESULTS OF CONTOUR INTEGRALS

Integral 1

$$\delta_1 = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} e^{st - a\sqrt{s}}\, ds \qquad \text{where} \qquad a = \frac{|x|}{\sqrt{\frac{3}{2}\, p_0/\rho_0}}$$

In this integral, $s = 0$ is a branch point; hence, we may consider the contour as the one given in Figure 2.

δ_1 is equivalent to the integral along path I, but we know that I = - III - IV since II and V vanish as R goes to infinity. If we let $s = re^{i\theta}$, thus

$$s = r e^{i\pi} = -r \qquad \text{along III , and}$$

$$s = r e^{-i\pi} = -r \qquad \text{along IV}$$

$$\delta_1 = \frac{1}{\pi}\int_0^{\infty} e^{-rt} \sin a\sqrt{r}\, dr$$

If we introduce the transformation

$$r = \beta^2 \qquad \text{and} \qquad dr = 2\beta\, d\beta \ ,$$

then

$$\delta_1 = \frac{2}{\pi}\int_0^{\infty} \beta\, e^{-\beta^2 t} \sin a\beta\, d\beta$$

$$= -\frac{2}{\pi}\frac{\partial}{\partial a}\int_0^{\infty} e^{-\beta^2 t} \cos a\beta\, d\beta$$

$$= \frac{|x|}{2\sqrt{\pi}\sqrt{\frac{3}{2}\frac{p_0}{\rho_0}}\, t^{3/2}} \exp\left[-\frac{x^2}{4t\sqrt{\frac{3}{2}\frac{p_0}{\rho_0}}}\right]$$

Integral 2

$$\delta_2 = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} e^{st - \frac{s}{\sqrt{\frac{5}{3}\frac{p_0}{\rho_0}}}(1-\frac{7}{10}s)|x|} ds$$

We introduce the new variable, $z = s - 5/7$,

$$\delta_2 = \frac{1}{2\pi i} \exp[\tfrac{5}{7}(t - \tfrac{|x|}{c})] \int_{-i\infty}^{+i\infty} e^{zt + \frac{7}{10}\frac{|x|}{c}z^2} dz$$

Along the path $z = iy$: hence

$$\delta_2 = \frac{1}{\pi} \exp[\tfrac{5}{7}(t - \tfrac{|x|}{c})] \int_{-\infty}^{\infty} e^{iyt - \frac{7}{10}\frac{|x|}{c}y^2} dy$$

$$= \frac{1}{2}\sqrt{\frac{10}{7}} \frac{1}{\sqrt{2\pi t}} \left\{ \exp\left[-\frac{10}{7c^2}\frac{(x-ct)^2}{4t}\right] + \exp\left[-\frac{10}{7c^2}\frac{(x+ct)^2}{4t}\right] \right\}$$

Integral 3

$$\delta_3 = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} e^{st - a\sqrt{s}} \frac{ds}{\sqrt{s}} = \frac{1}{\sqrt{\pi t}} \exp\left[-\frac{x^2}{\frac{3}{2}\frac{p_0}{\rho_0}4t}\right]$$

δ_3 is integrated in the same way as δ_1 .

APPENDIX III

SOME THEOREMS ABOUT FUNDAMENTAL SOLUTIONS

In this Appendix we shall state some simple theorems with proofs about fundamental solutions of some special linear differential equations.

Theorem 1

If $G_i^{(1)}$ is the fundamental solution of

$$(M - \lambda_i)u = -f(x) \qquad i = 1,\ 2 \tag{III. 1}$$

defined by

$$u(x) = \iint^{\infty} G_i^{(1)}(x;\xi)f(\xi)\,d\xi \tag{III. 2}$$

where M stands for any linear differential operator in one, two, or three dimensional space, λ_i are constants with the condition $\lambda_1 \neq \lambda_2$ and $\iint^{\infty}$ means to integrate over all components of the position vector .
Then the fundamental solution of

$$(M - \lambda_1)(M - \lambda_2)u = -f(x) \tag{III. 3}$$

over the same region of the space is given by

$$G^{(2)}(x;\xi) = \frac{1}{\lambda_1 - \lambda_2}\left(G_1^{(1)} - G_2^{(1)}\right) \tag{III. 4}$$

such that

$$u(x) = \iint^{\infty} G^{(2)}(x;\xi)f(\xi)\,d\xi \quad . \tag{III. 5}$$

Proof

Since M is a linear operator and λ_1, λ_2 are constants, $(\lambda_1 \neq \lambda_2)$, the operators $(M - \lambda_1)$ and $(M - \lambda_2)$ are commutable. Thus (III. 3) may also be written as

$$(M-\lambda_2)(M-\lambda_1)u = -f(x) \quad . \tag{III. 6}$$

Considering $(M - \lambda_2)$ u as an unknown function in (III. 3) and applying the definition of fundamental solutions given by (III. 2), we have

$$(M-\lambda_2)u = \iint^{\infty} G_1^{(1)}(x;\xi)f(\xi)d\xi \quad . \tag{III. 7}$$

Similarly, from Eq. (III. 6), we obtain

$$(M-\lambda_1)u = \iint^{\infty} G_2^{(1)}(x;\xi)f(\xi)d\xi \quad . \tag{III. 8}$$

Subtracting Eq. (III. 8) from Eq. (III. 7) yields

$$(\lambda_1-\lambda_2)u = \iint^{\infty} (G_1^{(1)} - G_2^{(1)})f(\xi)\,d\xi \quad . \tag{III. 9}$$

Comparing Eq. (III. 9) with Eq. (III. 5) we obtain, for $\lambda_1 \neq \lambda_2$

$$G^{(2)}(x;\xi) = \frac{1}{\lambda_1-\lambda_2}(G_1^{(1)} - G_2^{(1)}) \quad .$$

This proves the theorem.

Theorem 2

Use the same notations and definitions as in theorem 1, for $i = 1, 2, 3$, $\lambda_1 \neq \lambda_2 \neq \lambda_3$. The fundamental solution of

$$(M-\lambda_1)(M-\lambda_2)(M-\lambda_3)u = -f(x) \tag{III. 10}$$

is given by

$$G^{(3)}(x;\xi) = \frac{G_1^{(1)}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{G_2^{(1)}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} + \frac{G_3^{(1)}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} \quad \text{(III. 11)}$$

such that

$$u(x) = \iint^{\infty} G^{(3)}(x;\xi) f(\xi)\, d\xi$$

Proof

Here operators $(M - \lambda_1)$, $(M - \lambda_2)$ and $(M - \lambda_3)$ are again commutable, so by applying the definitions of $G_i^{(1)}$ given by Eq. (III. 2) to Eq. (III. 10) we have

$$(M-\lambda_2)(M-\lambda_3)u = [M^2-(\lambda_2+\lambda_3)M+\lambda_2\lambda_3]u = \iint^{\infty} G_1^{(1)} f(\xi)\, d\xi \quad \text{(III. 12)}$$

$$(M-\lambda_3)(M-\lambda_1)u = [M^2-(\lambda_3+\lambda_1)M+\lambda_3\lambda_1]u = \iint^{\infty} G_2^{(1)} f(\xi)\, d\xi \quad \text{(III. 13)}$$

$$(M-\lambda_1)(M-\lambda_2)u = [M^2-(\lambda_1+\lambda_2)M+\lambda_1\lambda_2]u = \iint^{\infty} G_3^{(1)} f(\xi)\, d\xi \quad . \quad \text{(III. 14)}$$

For the case $\lambda_1 \neq \lambda_2 \neq \lambda_3$, the following identities are easily proven:

$$\frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} + \frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} = 0 \quad \text{(III. 15)}$$

$$\frac{\lambda_2+\lambda_3}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{\lambda_3+\lambda_1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} + \frac{\lambda_1+\lambda_2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} = 0 \quad \text{(III. 16)}$$

and

$$\frac{\lambda_2\lambda_3}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}+\frac{\lambda_3\lambda_1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}+\frac{\lambda_1\lambda_2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}=1 \qquad \text{(III. 17)}$$

Divide Eq. (III. 12) by $(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)$, Eq. (III. 13) by $(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)$ and Eq. (III. 14) by $(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)$ and adding, using the relations Eqs. (III. 15) through (III. 17), we then obtain

$$u(x)=\iint^{\infty}\left[\frac{G_1^{(1)}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}+\frac{G_2^{(1)}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}+\frac{G_3^{(1)}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right]f(\xi)\,d\xi \;. \qquad \text{(III. 18)}$$

Therefore, the result (III. 11) follows.

APPENDIX IV

SOLUTIONS OF NAVIER-STOKES EQUATIONS AND FOURIER CONDUCTION LAW

The equivalent problem has been treated by T. Y. Wu[8] based on a system of linearized Navier Stokes equations and the Fourier conduction law. The fundamental solutions generated by a unit impulse and a unit heat addition are obtained for the Prandtl number, $Pr = \frac{c_p \mu_o}{K_o}$, taken to be 3/4. The reason for choosing this particular value is to lower the order of one of the differential equations (equation for p). Since at large values of time the solutions of Grad's equations are expected to approach that of the Navier-Stokes, Wu's work is therefore of special interest to us. However, the Prandtl number associated with Grad's equation is 2/3. In order to make any direct comparison, modifications must be made. Here, Wu's problem is reworked without specifying the Prandtl number. The distorted coordinates are introduced and a parallel way of solving the equations is taken to make a step by step comparison with Grad's equations possible. The approximation made for the evaluation of integral transforms are also duplicated for the simple reason that the forms of the integrands obtained from Grad's equations are algebraically more involved, and the present evaluations of the integrals are limited to very rough first approximations.

The following are Wu's original equations in physical coordinates x' and t'.

Continuity

$$\frac{\partial s}{\partial t'} + \frac{\partial u}{\partial x'} = 0 \qquad \text{(IV. 1)}$$

Momentum

$$\frac{\partial u}{\partial t'} + \frac{p_0}{\rho_0}\frac{\partial p}{\partial x'} - \frac{4\nu}{3}\frac{\partial^2 u}{\partial x'^2} = F' \qquad \text{(IV. 2)}$$

Energy

$$\frac{\partial \theta}{\partial t'} - K\frac{\partial^2 \theta}{\partial x'^2} = (\gamma - 1)\frac{\partial s}{\partial t'} + H' \qquad \text{(IV. 3)}$$

State

$$p = s + \theta \qquad \text{(IV. 4)}$$

where F' is the external force, H' is heat addition, $K = K_o/(c_v\rho_o)$, $\nu = \mu_o/\rho_o$, and $K_o = 15/4\ R\mu_o$. Later on γ is taken to be 5/3 for a monatomic gas.

The above equations can be replaced by an equivalent set of first order partial differential equations similar to Grad's scheme in distorted coordinates x and t introduced previously.

The equivalent set of equations are

Continuity

$$\frac{\partial s}{\partial t} + \frac{\partial u}{\partial x} = 0 \qquad \text{(IV. 5)}$$

Momentum

$$\frac{\partial u}{\partial t} + \frac{p_0}{\rho_0}\frac{\partial p}{\partial x} + \frac{1}{\rho_0}\frac{\partial \tau}{\partial x} = F(x,t) \qquad \text{(IV. 6)}$$

Energy

$$\frac{\partial p}{\partial t} + \gamma\frac{\partial u}{\partial x} + \frac{(\gamma-1)}{p_0}\frac{\partial q}{\partial x} = H(x,t) \qquad \text{(IV. 7)}$$

Stress

$$\frac{4}{3} p_o \frac{\partial u}{\partial x} + \tau = 0 \tag{IV. 8}$$

Heat

$$\frac{15}{4} \frac{p_o^2}{\rho_o} \frac{\partial \theta}{\partial x} + q = 0 \tag{IV. 9}$$

State

$$p = s + \theta \tag{IV. 10}$$

where

$$F' = \frac{p_o}{\mu_o} F \qquad H' = \frac{p_o^2}{\mu_o} H$$

Equations (IV. 5), (IV. 6), and (IV. 7) are exactly the same as that of Grad's for $\gamma = 5/3$ since they represent nothing but the conservation of mass, momentum, and energy, respectively. The differences show up in the stress and heat conduction relations. The equation of state is of course unchanged. Unlike Grad's system, the above equations are parabolic, and no finite characteristic speeds exist.

If we apply the method of Laplace transform with zero initial conditions to Eqs. (IV. 5) to (IV. 10) we obtain the following equations for the transformed quantities.

Continuity

$$\frac{d\bar{u}}{dx} = -s\,\bar{s} \tag{IV. 11}$$

Momentum

$$s\,\bar{u} + \frac{p_o}{\rho_o} \frac{d\bar{p}}{dx} + \frac{1}{\rho_o} \frac{d\bar{\tau}}{dx} = \bar{F} \tag{IV. 12}$$

Energy

$$\frac{d\bar{q}}{dX} = \frac{3}{2} p_0 \lambda \left(\frac{5}{3}\bar{S} - \bar{P}\right) + \frac{3}{2} p_0 \bar{H} \tag{IV.13}$$

Stress

$$\bar{\tau} = \frac{4}{3} p_0 \lambda \bar{S} \tag{IV.14}$$

Heat

$$\bar{q} = -\frac{15}{4} \frac{p_0^2}{\rho_0} \frac{d\bar{\theta}}{dX} \tag{IV.15}$$

State

$$\bar{P} = \bar{S} + \bar{\theta} \quad . \tag{IV.16}$$

Successive elimination of the transformed variables leads us to a final equation

$$L\,\bar{Q} = \mathcal{J}_{\bar{Q}} \tag{IV.17}$$

where L is the fourth order differential operator,

$$L = \frac{p_0^2}{\rho_0^2}\left(\frac{4}{3}\lambda + 1\right)\frac{d^4}{dX^4} - \frac{p_0}{\rho_0}\frac{\lambda}{3}\left(2 + \frac{23}{5}\lambda\right)\frac{d^2}{dX^2} + \frac{2}{5}\lambda^3$$

$\bar{Q}$ stands for any one of the six dependent varaibles, and $\mathcal{J}_{\bar{Q}}$ denotes the inhomogeneous part of the equation corresponding to each particular $\bar{Q}$.

The operator L can be factored into two second order operators as follows:

$$L = a\frac{d^4}{dX^4} + 2b\frac{d^2}{dX^2} + c = a\left(\frac{d^2}{dX^2} - \lambda_1\right)\left(\frac{d^2}{dX^2} - \lambda_2\right)$$

where

$$a = \frac{p_o^2}{\rho_o^2}\left(\frac{4}{3}\lambda + 1\right)$$

$$b = -\frac{p_o}{\rho_o}\frac{\lambda}{6}\left(2 + \frac{23}{5}\lambda\right)$$

$$c = \frac{2}{5}\lambda^3$$

and

$$\Lambda_{1,2} = \frac{1}{a}\left(-b \pm \sqrt{b^2 - ac}\right)$$

or more precisely,

$$\Lambda_{1,2} = \frac{\rho_o}{6p_o}\frac{\lambda}{\left(\frac{4}{3}\lambda+1\right)}\left[2 + \frac{23}{5}\lambda \pm \sqrt{4 + 4\lambda + \frac{49}{25}\lambda^2}\right] .$$

Knowing $\mathcal{J}_{\bar{Q}}$, one can write down immediately the solution for each $\bar{Q}$.

$$\bar{Q} = -\frac{1}{a}\int_{-\infty}^{+\infty} G^{(2)}(x;\xi)\,\mathcal{J}_{\bar{Q}}(\xi)\,d\xi \qquad \text{. (IV. 18)}$$

Furthermore, since the equations are linear, we can split each solution into two parts, one corresponding to F and the other to H.

$$\bar{Q} = \bar{Q}_F + \bar{Q}_H \quad , \qquad \mathcal{J}_{\bar{Q}} = \mathcal{J}_{\bar{Q}_F} + \mathcal{J}_{\bar{Q}_H}$$

$$\bar{Q}_F = -\frac{1}{a}\int_{-\infty}^{+\infty} G^{(2)}(x;\xi)\,\mathcal{J}_{\bar{Q}_F}\,d\xi$$

$$Q_H = -\frac{1}{a}\int_{-\infty}^{+\infty} G^{(2)}(x;\xi)\,\mathcal{J}_{\bar{Q}_H}\,d\xi$$

where

$$G^{(2)}(x;\xi) = \frac{1}{2(\Lambda_1 - \Lambda_2)}\left[\frac{1}{\sqrt{\Lambda_1}} e^{-\sqrt{\Lambda_1}|x|} - \frac{1}{\sqrt{\Lambda_2}} e^{-\sqrt{\Lambda_2}|x|}\right]$$

is the Green's function of the operator L. The transforms are given below

$$\bar{S}_F = \frac{(\mathrm{sgn}\,x)(\lambda_1 - \frac{2}{5} s \rho_0/p_0) e^{-\sqrt{\lambda_1}|x|}}{2 \frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_1-\lambda_2)} + \frac{(\mathrm{sgn}\,x)(\lambda_2 - \frac{2}{5} s \rho_0/p_0) e^{-\sqrt{\lambda_2}|x|}}{2 \frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_2-\lambda_1)} \tag{IV. 19}$$

$$\bar{P}_F = \frac{(\mathrm{sgn}\,x)(\lambda_1 - \frac{2}{3} s \rho_0/p_0) e^{-\sqrt{\lambda_1}|x|}}{2 \frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_1-\lambda_2)} + \frac{(\mathrm{sgn}\,x)(\lambda_2 - \frac{2}{3} s \rho_0/p_0) e^{-\sqrt{\lambda_2}|x|}}{2 \frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_2-\lambda_1)} \tag{IV. 20}$$

$$\bar{\theta}_F = -\frac{4}{15} \frac{(\mathrm{sgn}\,x)\, s \rho_0/p_0\, e^{-\sqrt{\lambda_1}|x|}}{2 \frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_1-\lambda_2)} - \frac{4}{15} \frac{(\mathrm{sgn}\,x)\, s \rho_0/p_0\, e^{-\sqrt{\lambda_2}|x|}}{2 \frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_2-\lambda_1)} \tag{IV. 21}$$

$$\bar{\tau}_F = \frac{(\mathrm{sgn}\,x)\frac{4}{3}p_0 s(\lambda_1 - \frac{2}{5} s \rho_0/p_0) e^{-\sqrt{\lambda_1}|x|}}{2 \frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_1-\lambda_2)} + \frac{(\mathrm{sgn}\,x)\frac{4}{3}p_0 s(\lambda_2 - \frac{2}{5} s \rho_0/p_0) e^{-\sqrt{\lambda_2}|x|}}{2 \frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_2-\lambda_1)} \tag{IV. 22}$$

$$\bar{q}_F = -\frac{\rho_0 s \sqrt{\lambda_1}\, e^{-\sqrt{\lambda_1}|x|}}{2(\frac{4}{3}s+1)(\lambda_1-\lambda_2)} - \frac{\rho_0 s \sqrt{\lambda_2}\, e^{-\sqrt{\lambda_2}|x|}}{2(\frac{4}{3}s+1)(\lambda_2-\lambda_1)} \tag{IV. 23}$$

$$\bar{u}_F = \frac{\lambda_1\sqrt{\lambda_1}\, e^{-\sqrt{\lambda_1}|x|}}{2s(\lambda_1-\lambda_2)} - \frac{\frac{2}{3}(1+\frac{4}{5}s)\sqrt{\lambda_1}\, e^{-\sqrt{\lambda_1}|x|}}{2 \frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_1-\lambda_2)} + \frac{\lambda_2\sqrt{\lambda_2}\, e^{-\sqrt{\lambda_2}|x|}}{2s(\lambda_2-\lambda_1)} - \frac{\frac{2}{3}(1+\frac{4}{5}s)\sqrt{\lambda_2}\, e^{-\sqrt{\lambda_2}|x|}}{2 \frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_2-\lambda_1)} \tag{IV. 24}$$

$$\bar{S}_H = -\frac{\frac{2}{5}\sqrt{\lambda_1}\, e^{-\sqrt{\lambda_1}|x|}}{2\frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_1-\lambda_2)} - \frac{\frac{2}{5}\sqrt{\lambda_2}\, e^{-\sqrt{\lambda_2}|x|}}{2\frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_2-\lambda_1)} \tag{IV. 25}$$

$$\bar{P}_H = \frac{(\frac{8}{15}s\sqrt{\lambda_1} - \frac{2}{5}s^2\frac{\rho_0}{p_0}\frac{1}{\sqrt{\lambda_1}})\, e^{-\sqrt{\lambda_1}|x|}}{2\frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_1-\lambda_2)} + \frac{(\frac{8}{15}s\sqrt{\lambda_2} - \frac{2}{5}s^2\frac{\rho_0}{p_0}\frac{1}{\sqrt{\lambda_2}})\, e^{-\sqrt{\lambda_2}|x|}}{2\frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_2-\lambda_1)} \tag{IV. 26}$$

$$\bar{\theta}_H = \frac{\frac{2}{5}\sqrt{\lambda_1}}{2\frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_1-\lambda_2)}\left[1+\frac{4}{3}s-\frac{2}{5}s^2\frac{\rho_0}{p_0}\frac{1}{\lambda_1}\right]e^{-\sqrt{\lambda_1}|x|} - \frac{\frac{2}{5}\sqrt{\lambda_2}\, e^{-\sqrt{\lambda_2}|x|}}{2\frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_2-\lambda_1)}\left[1+\frac{4}{3}s-\frac{2}{5}s^2\frac{\rho_0}{p_0}\frac{1}{\lambda_2}\right] \tag{IV. 27}$$

$$\bar{\tau}_H = -\frac{8}{15}\,\frac{p_0\, s\sqrt{\lambda_1}\, e^{-\sqrt{\lambda_1}|x|}}{2\frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_1-\lambda_2)} - \frac{8}{15}\,\frac{p_0\, s\sqrt{\lambda_2}\, e^{-\sqrt{\lambda_2}|x|}}{2\frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_2-\lambda_1)} \tag{IV. 28}$$

$$\bar{q}_H = \frac{\frac{3}{2}(\operatorname{sgn} x)}{2(\lambda_1-\lambda_2)}\left[p_0\lambda_1 - \frac{\rho_0 s^2}{(\frac{4}{3}s+1)}\right]e^{-\sqrt{\lambda_1}|x|} + \frac{\frac{3}{2}(\operatorname{sgn} x)}{2(\lambda_2-\lambda_1)}\left[p_0\lambda_2 - \frac{\rho_0 s^2}{(\frac{4}{3}s+1)}\right]e^{-\sqrt{\lambda_2}|x|} \tag{IV. 29}$$

$$\bar{u}_H = -\frac{\frac{2}{5}(\operatorname{sgn} x)\, s}{2\frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_1-\lambda_2)}\, e^{-\sqrt{\lambda_1}|x|} - \frac{\frac{2}{5}(\operatorname{sgn} x)\, s}{2\frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_2-\lambda_1)}\, e^{-\sqrt{\lambda_2}|x|} \tag{IV. 30}$$

The Inverse Transforms

The inverse transform of a quantity Q is given by

$$Q = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{st}\,\bar{Q}(s)\,ds \qquad \text{(IV. 31)}$$

where σ is to the right of all singularities. Therefore, it is important to understand the behavior of $\bar{Q}$, or more precisely, the behavior of λ s in this case. We have

$$\lambda_{1,2} = \frac{1}{6\frac{p_0}{\rho_0}}\,\frac{s}{(\frac{4}{3}s+1)}\left(2 + \frac{23}{5}s \pm 2\sqrt{1 + s + 49s^2/100}\right) \quad . \qquad \text{(IV. 32)}$$

By setting $\lambda_{1,2} = 0$, one finds that $s = 0$ and $s = -3/4$ are branch points, and the latter is also being an essential singularity. Since we are mainly interested here in the value of the integrals at large time, a branch point approximation would serve the purpose. The point $s = 0$ undoubtedly plays the dominating part; hence we will focus our attention at this point. Given below is the case of S_F worked out in detail. The rest of the integrals are treated in similar fashion.

The Evaluation of S_F for Large t

If we substitute the λ s into S_F and expand it in powers of s, we get, by keeping only the leading terms

$$\bar{S}_F \cong \frac{(\mathrm{sgn}\,x)\ 3/5\ e^{-\sqrt{\lambda_1^*}|x|}}{2\,p_0/\rho_0} + \frac{(\mathrm{sgn}\,x)\ 3/5\ e^{-\sqrt{\lambda_2^*}|x|}}{2\,p_0/\rho_0} \quad , \qquad \text{(IV. 33)}$$

where

$$\lambda_1^* = \frac{2}{3}\,\frac{s\rho_0}{p_0} \quad , \quad \lambda_2^* = \frac{s^2}{5/3\,p_0/\rho_0}\left(1 - \frac{7}{5}s\right)$$

are obtained by taking $\sqrt{1+s+49s^2/100} = 1 + \frac{1}{2}(s + 49s^2/100) + \cdots$

Therefore, we have

$$\bar{S}_F \cong \frac{(sgn\,x)}{5\,p_0/\rho_0} \exp\left[-\sqrt{\frac{2}{3}\frac{\rho_0}{p_0}}\sqrt{s}\,|x|\right] + \frac{(sgn\,x)}{10/3\,p_0/\rho_0} \exp\left[-\sqrt{\frac{3}{5}\frac{\rho_0}{p_0}}\,s\left(1-\frac{7}{10}s\right)|x|\right] \quad . \qquad \text{(IV. 33a)}$$

The inverse transform of S_F consists of two integrals δ_1 and δ_2, where

$$S_F = \frac{(sgn\,x)}{5\,p_0/\rho_0}\,\delta_1 + \frac{(sgn\,x)}{10/3\,\frac{p_0}{\rho_0}}\,\delta_2 \quad , \qquad \text{(IV. 34)}$$

$$\delta_1 = \frac{|x|}{2\sqrt{\pi}\sqrt{\frac{3}{2}\frac{p_0}{\rho_0}}\;t^{3/2}} \exp\left[-\frac{x^2}{4t\sqrt{\frac{3}{2}p_0/\rho_0}}\right]$$

$$\delta_2 = \frac{1}{2}\sqrt{\frac{10}{7}}\frac{1}{\sqrt{2\pi t}}\left\{\exp\left[-\frac{10}{7c^2}\frac{(x-ct)^2}{4t}\right] + \exp\left[-\frac{10}{7c^2}\frac{(x+ct)^2}{4t}\right]\right\} \quad .$$

The details of evaluation of δ_1 and δ_2 are given in Appendix IV. We find also that, by keeping the leading terms

$$p_F = \frac{(sgn\,x)}{2\,p_0/\rho_0}\,\delta_2 \qquad \text{(IV. 35)}$$

$$\theta_F = -\frac{(sgn\,x)}{5\,p_0/\rho_0}\,\delta_1 + \frac{(sgn\,x)}{5\,p_0/\rho_0}\,\delta_2 \qquad \text{(IV. 36)}$$

$$\bar{\tau}_F \sim s\,\bar{p}_F \quad . \qquad \text{(IV. 37)}$$

Therefore, τ_F is of higher order compared with P_F at large t (small s).

$$u_F = \frac{1}{4} \frac{\delta_2}{\sqrt{5/3 \; p_0/\rho_0}} \tag{IV.38}$$

$$q_F = \frac{3}{8} \frac{p_0 \delta_2}{\sqrt{5/3 \; p_0/\rho_0}} \tag{IV.39}$$

Some of the responses induced by H are given below

$$\bar{S}_H \cong -\frac{3}{10} \frac{1}{\sqrt{3/2 \; p_0/\rho_0}} \frac{e^{-\sqrt{\lambda_1^*}|x|}}{\sqrt{s}} + \frac{3}{10c} e^{-\sqrt{\lambda_2^*}|x|} \tag{IV.40}$$

$$\bar{\theta}_H \cong \frac{3}{10} \frac{1}{\sqrt{3/2 \; p_0/\rho_0}} \frac{e^{-\sqrt{\lambda_1^*}|x|}}{\sqrt{s}} - \frac{3}{10c} e^{-\sqrt{\lambda_2^*}|x|} \tag{IV.41}$$

$$\bar{u}_H = -(\mathrm{sgn}\,x)\; 3/10 \; e^{-\sqrt{\lambda_2^*}|x|} \quad . \tag{IV.42}$$

Hence

$$S_H = -\frac{3}{10} \frac{1}{\sqrt{\frac{3}{2}\frac{p_0}{\rho_0}}} \delta_3 + \frac{3}{10c} \delta_2 \tag{IV.43}$$

$$\theta_H = -S_H \tag{IV.44}$$

and

$$u_H = (\mathrm{sgn}\,x) \frac{3}{10} \delta_2 \tag{IV.45}$$

where

$$\delta_3 = \frac{1}{\sqrt{\pi t}} \exp\left[-\frac{x^2}{3/2 \; p_0/\rho_0 \, 4t}\right] \quad . \tag{IV.46}$$

δ_1 and δ_3 represent the wake, while δ_2 represents the two running sound waves.

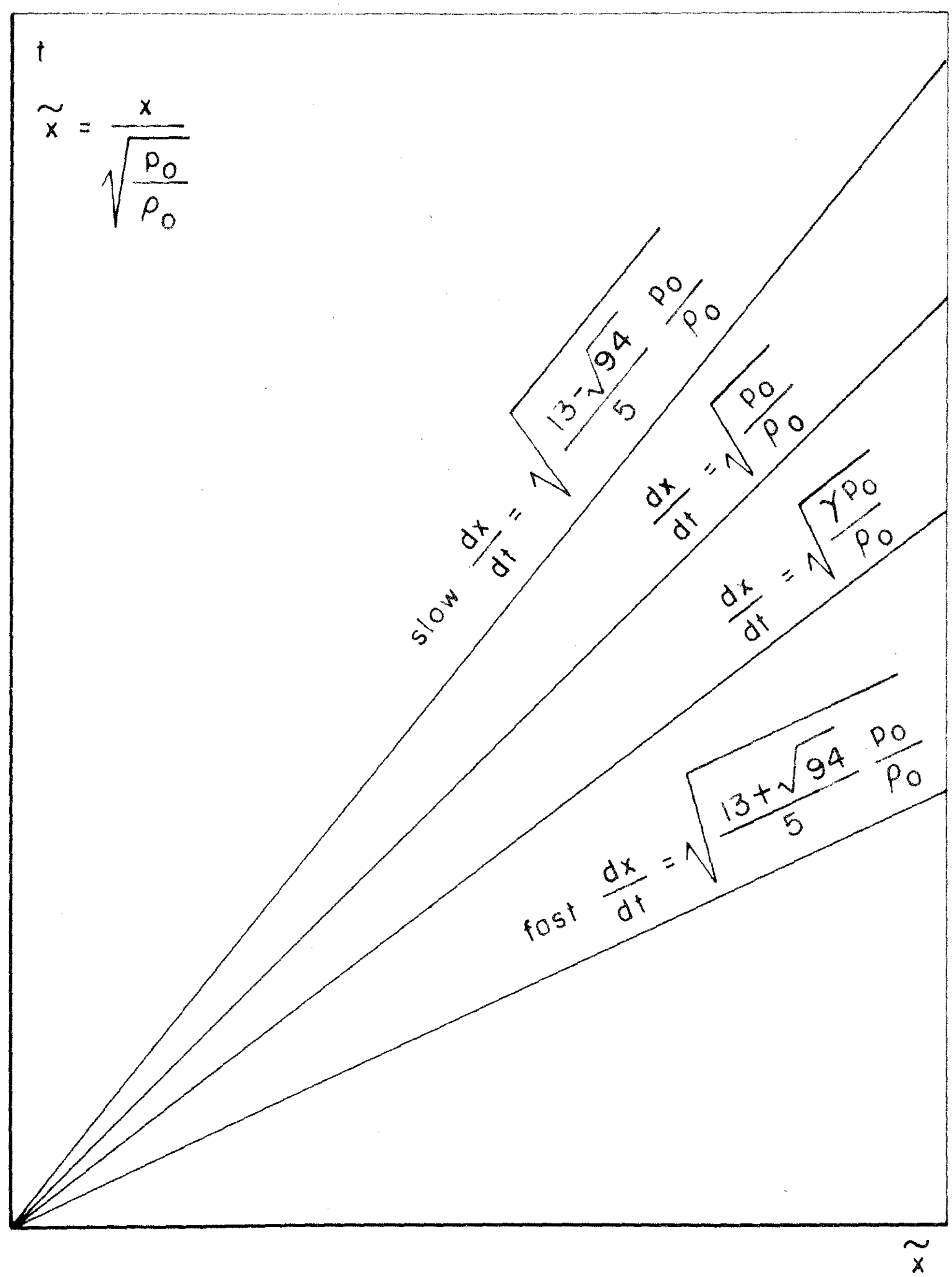

FIG. 1 – CHARACTERISTICS NORMALIZED AGAINST THE ISOTHERMAL SPEED OF SOUND

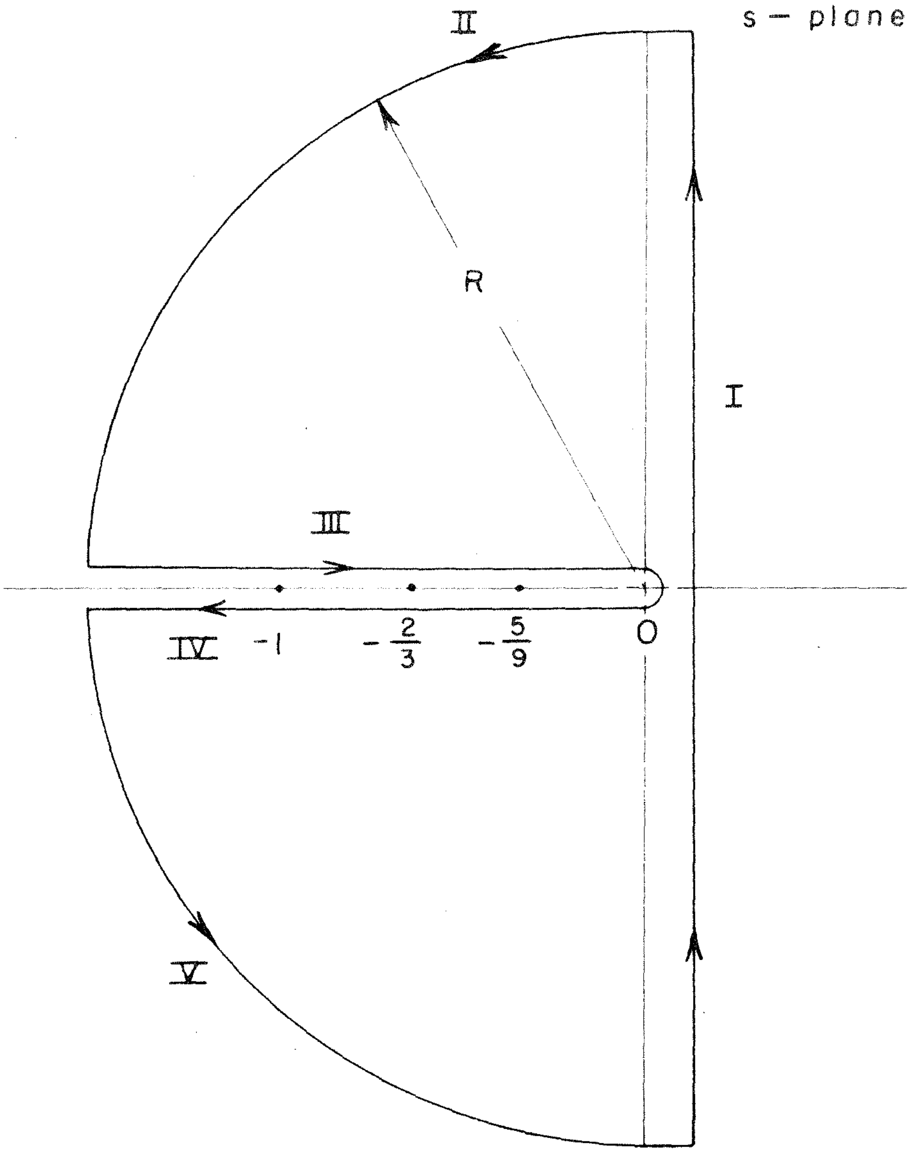

FIG. 2