BOOTSTRAP THEORY AND CERTAIN PROPERTIES OF THE HADRON AXIAL VECTOR CURRENT

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ABSTRACT

The matrix element $\langle \Omega^- | j_\mu^A | \Xi^0 \rangle$ is calculated. A similar calculation for $\langle \Delta | j_\mu^A | N \rangle$ is compared with experiment. The relationship of the Goldberger-Treiman relation to the bootstrap principle is discussed. Approximate symmetry predictions for the axial vector current are compared with the S-matrix calculations. Implications of the bootstrap principle for equal time commutators of hadron currents are discussed.

Dedicated to
Michel Melkanoff

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I. INTRODUCTION

At present there are two basic approaches to the problem of strong interactions. One approach is based on the properties of scattering amplitudes (1). The other approach is based on the properties of currents (2). For the scattering amplitude approach there is a guiding principle, the bootstrap principle, which states that all strongly interacting particles are composites of other strongly interacting particles. Although no one has succeeded in formulating the bootstrap principle in a general mathematical way it is understood how to apply the principle in various specific situations of interest; for example, to calculate the properties of low-lying hadron states. For the approach based on currents, on the other hand, there is no principle known at present which is sufficiently general to enable one to calculate all properties of hadrons. However, it does appear that hadron currents have simple properties which can be formulated in a precise mathematical way. Moreover, it is expected that the current approach will eventually be extended so as to provide a complete solution of the hadron problem.

The relation between the bootstrap principle and the behavior of hadron currents has already been discussed in general terms by Dashen and Frautschi (3). Their discussion was based on an S-matrix theory of hadron currents which is originally due to Chew, Goldberger, Low, and Nambu (4). By applying the bootstrap principle to the S-matrix theory they obtained predictions for the behavior of hadron currents which were generally in accord with observation. In this thesis we will discuss in more detail certain implications of the bootstrap theory for the hadron axial vector current. From the experimental point of view the axial vector current is interesting because of its connection with the weak interactions. From the theoretical point of view there is added interest due to the fact that there appears to be some kind of deep connection between the axial vector current and the strong interaction properties of hadrons. It is hoped that this study will be of some interest in both of these connections.

Our calculations will pertain mainly to matrix elements of the axial vector current among baryon octet and decuplet states. The principal tool we will employ in our study is the S-matrix theory of currents. From the theoretical point of view the simplest place among baryon octet and decuplet states in which to apply an S-matrix theory of the axial vector current is to a calculation of the matrix element $\langle \Omega^- | J_\mu^A | \Xi^o \rangle$.

In section II we illustrate in detail how S-matrix theory can be used to calculate $\langle \Omega^- | J_\mu^A | \Xi^o \rangle$. This matrix element is of interest in connection with Ω^- leptonic decay and the calculation is carried far enough to enable us to estimate the Ω^- leptonic decay rate. A similar calculation for $\langle \Delta | J_\mu^A | N \rangle$ is also discussed and compared with experimental data on neutrino production of pions.

One of the conjectured properties of the axial vector current is that it is "partially conserved" (6, 7). As a consequence of this hypothesis there is a sort of universality relation between matrix elements of the axial vector current and pseudoscalar meson couplings. The relation between the axial vector coupling for neutron β-decay and the πNN coupling constant was first discovered by Goldberger and Treiman (8) from another point of view. In section III we discuss the connection between the Goldberger-Treiman relation and its generalizations and the bootstrap principle. We take advantage of the reciprocal bootstrap relation between N and Δ (9) to show that the Goldberger-Treiman relations for the N-N and N-Δ transitions are consistent with S-matrix calculations of these transitions. We also show that when the S-matrix calculation is extended to SU(3) then one obtains the SU(3) generalization of the Goldberger-Treiman relation.

In section IV approximate symmetry predictions for the axial vector current are discussed. We show that the S-matrix calculations of matrix elements of the axial vector current for B_8-B_8 and Δ_{10}-B_8 transitions are generally consistent with the predictions of approximate symmetry. The theoretical reasons for the appearance of approximate symmetries in S-matrix calculations are then discussed. It is shown that the bootstrap principle implies that the equal time commutator of two hadron currents is a current. The bootstrap principle is then used to study the form of the algebra generated. It is shown that it is consistent with the observed approximate symmetry.

II. $\langle \Omega^- | j_\mu^A | \Xi \rangle$

The matrix element $\langle \Omega^- | j_\mu^A | \Xi \rangle$ provides an example of a matrix element of the axial vector current which can be treated in a fairly simple way by S-matrix methods. The matrix element is of interest in connection with Ω^- β-decay. Furthermore, it is a convenient place in which to compare a dynamical calculation with predicted properties of the axial vector current. In terms of Lorentz invariants we have

$$\langle \Omega^- | j_\mu^A(0) | \Xi^o \rangle = \frac{G}{\sqrt{2}} \sin\theta \; \psi_\alpha [f_1(\lambda^2)\delta_{\alpha\mu} + f_2(\lambda^2) k_\alpha \gamma_\mu + f_3(\lambda^2) k_\alpha P_\mu + f_4(\lambda^2) k_\alpha k_\mu] \psi \tag{1}$$

where ψ is a Dirac spinor and ψ_α is a Rarita-Schwinger spin 3/2 spinor (The absence of a γ_5 in the matrix element is a reflection of the fact that ψ_α is odd under parity). The meaning of the momenta is given in Table I. We will use the convention $a \cdot b = a_o b_o - \underset{\sim}{a} \cdot \underset{\sim}{b}$.

Table I

Particle	4-Momentum	Energy	Mass
Ω^-	P_μ	M	M
Ξ^o	p_μ	E	m
"θ"	$k_\mu = P_\mu - p_\mu$	k_o	λ

Our conventions for γ-matrices and ψ's are explained in Appendix A. Equation 1 tells us that the matrix element is characterized by the four amplitudes $f_1, \ldots, f_4$. These amplitudes are related to the multipole amplitudes that are familiar in low energy nuclear physics. For example f_1 is a linear combination of the E1, L1, M2 and induced pseudoscalar amplitudes. In the following we will show how to calculate the f's by using S-matrix methods.

Our approach to the problem of calculating $\langle \Xi | j_\mu^A | \Omega^- \rangle$ will be based on a study of the amplitude for $\theta + \Xi \rightarrow \bar{\kappa} + \Xi$ where θ transforms like an axial vector current. In particular we will find the f's by looking at the residue of the Ω pole in those amplitudes that lead to an $I = 0$ $P_{3/2}$ final state. In general, there are four amplitudes for $\theta + \Xi \rightarrow \bar{\kappa}\,\Xi$ in a state of given isotopic spin, angular momentum, and parity. Finding the residue of the Ω pole in the four amplitudes that lead to the $I = 0$ $P_{3/2}$ final state will give us four relations for $f_1, \ldots, f_4$.

The reason that we chose to study transitions to a $\bar{\kappa}\,\Xi$ state rather than some other communicating state (e.g. a $\pi\bar{\kappa}\,\Xi$ state) is that the amplitudes for transition to the $\bar{\kappa}\,\Xi$ state are by far the simplest to calculate. This because one need only take into account the $\bar{\kappa}\,\Xi$ channel in calculating the amplitude for $\theta + \Xi \rightarrow \bar{\kappa} + \Xi$, whereas finding the amplitudes for transition to another state is necessarily a multichannel problem. It is this circumstance that makes the calculation of $\langle \Omega^- | j_\mu^A | \Xi^o \rangle$ a simple problem from the S-matrix theory point of view.

An axial vector current can be split into a piece that transforms like an axial vector and a piece that transforms like a pseudoscalar; viz. $J_\mu^A = J_{1\mu}^A + \frac{k_\mu}{\lambda^2} J_o^A$ where $J_{1\mu}^A = J_\mu^A - \frac{k_\mu k_\nu J_\nu^A}{\lambda^2}$ and $J_o^A = k_\nu J_\nu^A$. It will be convenient to calculate the matrix elements of these pieces separately. Let us first consider the axial vector part. If θ transforms like an axial vector then we can make an angular momentum and parity decomposition of the amplitude for $\theta + \Xi \rightarrow \bar{\kappa} + \Xi$ in terms of the polarization 3-vector. It will be convenient to use the multipole decomposition given in Appendix B. The allowed transitions and corresponding amplitudes are listed in Table II. The notation for the amplitudes is analogous to the notation that is used by CGLN (4) for pion photoproduction amplitudes.

Table II

Multipole Order	Total Angular Momentum	Parity	Meson Angular Momentum	Amplitude
j	j + 1/2	$(-1)^j$	j + 1	M_{j+1^-}
j	j - 1/2	$(-1)^j$	j - 1	M_{j-1^+}
j	j + 1/2	$(-1)^{j+1}$	j	E_{j^+}, L_{j^+}
j	j - 1/2	$(-1)^{j+1}$	j	E_{j^-}, L_{j^-}

From the table we see that the M_{1^+}, E_{1^+}, and L_{1^+} amplitudes lead to a $P_{3/2}$ final state. In the following we will assume that these amplitudes also lead to an $I = 0$ final state. The residue of the Ω pole in these amplitudes can be related to $f_1, \ldots, f_4$ by calculating the Feynman amplitude corresponding to Figure 1 and using equations B. 5. The result for the residues in E_{1^+}/q, L_{1^+}/q, and M_{1^+}/qk^2 is

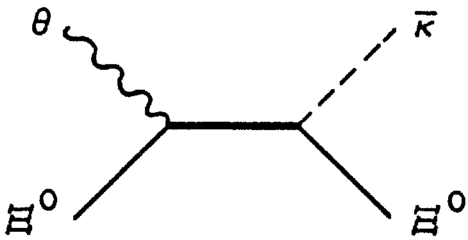
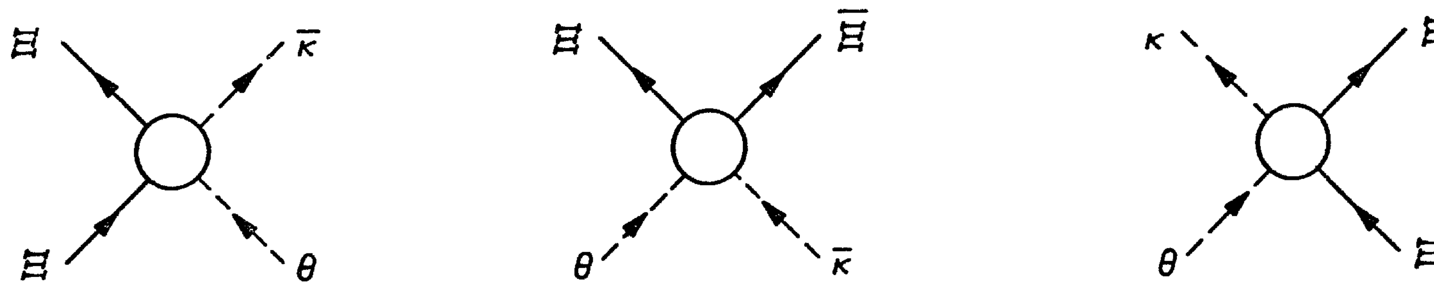


Figure 1



(a) s-channel (b) t-channel (c) u-channel

Figure 2

$$\gamma_E = \frac{1}{\sqrt{6\pi}} \frac{[(M+m)^2 - \lambda^2]^{1/2}}{M^2} \left\{ f_1 + \frac{M}{(M+m)^2 - \lambda^2} k^2 f_2 \right\}$$

$$\gamma_L = \frac{1}{\sqrt{6\pi}} \frac{[(M+m)^2 - \lambda^2]^{1/2}}{M^2} \left\{ f_1 + \left(\frac{k}{k_o}\right) \frac{2M(M+m)}{(M+m)^2 - \lambda^2} kf_2 + \left(\frac{k}{k_o}\right) Mkf_3 \right\} g_\Omega \quad (2)$$

$$\gamma_M = \frac{1}{\sqrt{6\pi}} \frac{[(M+m)^2 - \lambda^2]^{-1/2}}{M^2} \left(\frac{f_2}{M}\right) g_\Omega$$

where g_Ω is the $\bar{\kappa}\Xi\Omega^-$ coupling constant. If we define the $\bar{\kappa}\Xi\Omega^-$ coupling as $i\gamma_\Omega \psi_\alpha q_\alpha \psi$ then (in the following kaon mass = 1):

$$g_\Omega^2 = \frac{(M+m)^2 - 1}{16\pi} \frac{2\gamma_\Omega^2}{3} \; . \quad (3)$$

It is evident from equations 2 that if we can calculate γ_E, γ_L, and γ_M then we can determine f_1, f_2, and f_3.

Suppose we write $h_\Phi(W) = \rho_\Phi \Phi_{1^+}(W)$ where ρ_Φ is a factor which removes any singularities in Φ_{1^+} of kinematic origin. Then the only singularities in $h_\Phi(W)$ will be poles and branch cuts associated with intermediate states for the three processes of Figure 2. This means that the discontinuities in h_Φ will be determined by unitarity and crossing symmetry; therefore we may hope to determine $h_\Phi(W)$ by combining unitarity and crossing symmetry with the use of dispersion relations. If we neglect the effect of competing channels then the phase of $h_\Phi(W)$ above the $\bar{\kappa}\Xi$ threshold will be equal to the $I = 0$ $P_{3/2}$ $\bar{\kappa}\Xi$ phase shift (9). Let us define a function D(W):

$$D(W) = (W - M)\exp\left[- \frac{W-M}{\pi} \int_{m+1}^{\infty} \frac{\delta(W)}{(W'-M)(W'-W+iE)} dW' \right] \quad (4)$$

where δ is the $I = 0$ $P_{3/2}$ $\bar{\kappa}\Xi$ phase shift. $D(W)$ is an analytic function except for a cut running from $W = m+1$ to ∞. Furthermore, on this cut $D(W)$ has a phase of $-\delta$. Therefore in the approximation of neglecting competing channels the function $D(W)h_{\Phi}(W)$ will be analytic except for "left hand" cuts; i.e., cuts associated with intermediate states for the processes of Figures 2b and 2c. Therefore, if the function $D(W)h_{\Phi}(W)$ behaves suitably at infinity we can write:

$$h_{\Phi}(W) = \frac{1}{2\pi i D(W)} \int_{\text{l. h. cuts}} \frac{D(W')\text{disc.}[h_{\Phi}(W)]}{W'-W} dW' \quad (5)$$

where disc.$[h_{\Phi}(W)]$ is the discontinuity in $h_{\Phi}(W)$. We will assume that this equation is correct as written without subtractions (this in fact is the basic assumption of our approach). From equations 4 and 5 we find that the residue of the Ω pole in $h_{\Phi}(W)$ is given by

$$R_{\Phi} = \frac{1}{2\pi i} \int_{\text{l. h. cuts}} \frac{D(W')\text{disc}[h_{\Phi}(W')]}{W'-W} dW' \ . \quad (6)$$

This formula will serve as the basis for our calculation of the residue of the Ω pole in the multipole amplitudes.

In principle one can find the value of $h_{\Phi}(W)$ on the left-hand cuts from the amplitudes for the processes of Figure 2b and 2c. Actually, however, we do not know these amplitudes and therefore cannot in general calculate disc. $[h_{\Phi}(W')]$. Nevertheless comparison with the theory of pion photoproduction (11) suggests that for the purpose of calculating R_{Φ} it is not too bad an approximation to determine disc. $[h_{\Phi}(W')]$ from the Σ and Λ exchange diagrams (Figure 3)(at least for small λ^2). In the Chew-Low effective range type approximation (12) this amounts to taking

$$\text{disc.}\left[\frac{E_{1^+}}{q}\right] = \text{disc.}\left[\frac{L_{1^+}}{q}\right] = 2\pi i \frac{4}{\sqrt{2\pi}} \sum_{i=\Lambda,\Sigma} f_i\left(\frac{-G_A^i}{G\sin\theta}\right)\delta(W-W_i)$$

$$\text{disc.}[M_{1^+}] = 0 \tag{7}$$

where $W_\Lambda = 2m - m_\Lambda$, $W_\Sigma = 2m - m_\Sigma$, f_Λ (f_Σ) is the $\bar{\kappa}\,\Xi\,\Lambda\,(\bar{\kappa}\,\Xi\,\Sigma)$ strong coupling constant, and $G_A^{\Xi\Lambda}$ $(G_A^{\Xi\Sigma})$ is the $\Xi\Lambda\,(\Xi\Sigma)$ zero momentum transfer axial vector coupling. We will assume as a first approximation that disc. $[\Phi_{1^+}]$ is given by the above simple expressions. The success of the Chew-Low theory suggests that this will give results which are not too bad for small λ^2. We see immediately from equation 7 that the residue of the Ω pole in M_{1^+} is zero for small λ^2. In order to find the residues in E_{1^+} and L_{1^+} we must know ρ_E, ρ_L, and D(W) for W mear to M. For small λ^2 we can take ρ_E and ρ_L to be simply q^{-1}. As for D(W) we see from equation 4 that for W very near M D(W) is simply W-M. Further if δ behaves reasonably then the variation of the exponential

Figure 3

factor in equation 2 should be negligible for $|W-M| < 1$. Thus we can set $D(W_\Lambda)/W_\Lambda - M = D(W_\Sigma)/W_\Sigma - M = 1$. Using these approximations and equation 7 we find that the residues of the Ω pole in E_{1^+}/q and L_{1^+}/q are given by:

$$\gamma_E \simeq \gamma_L \simeq \frac{4}{\sqrt{2\pi}} \sum_{i=\Lambda,\Sigma} f_i\left(\frac{-G_A^i}{G\sin\theta}\right) . \quad (8)$$

The Chew-Low approximation is exact in the static limit. Therefore, in comparing equations 2 with equation 8 we will set $k = 0$. In the limit $k = 0$ equations 2 give $\gamma_E = \gamma_L$ in agreement with equation 8. Further, comparison between the two equations gives

$$f_1(0) \simeq 4/3 \, \frac{M^2}{M+m} \left[\frac{1}{m+m_\Lambda} \left(\frac{g_{\overline{\kappa}\Xi\Lambda}}{g_\Omega}\right)\left(\frac{-G_A^{\Xi\Lambda}}{G\sin\theta}\right) + \frac{1}{m+m_\Sigma}\left(\frac{g_{\overline{\kappa}\Xi\Sigma}}{g_\Omega}\right)\left(\frac{-G_A^{\Xi\Lambda}}{G\sin\theta}\right)\right] \quad (9)$$

where $g_{\overline{\kappa}\Xi\Lambda}$ and $g_{\overline{\kappa}\Xi\Sigma}$ are the unrationalized pseudoscalar coupling constants. According to the bootstrap principle one can determine the values of the various quantities appearing in this equation from self-consistency. Here we will not be that ambitious. Since at this point we are merely interested in evaluating the matrix element 1 we will use the best estimates for the quantities appearing on the right hand side of the equation to evaluate the left hand side. The ratio's $(g_{\overline{\kappa}\Xi\Lambda}/g_\Omega)$ and $(g_{\overline{\kappa}\Xi\Sigma}/g_\Omega)$ can be estimated from calculations of the $\pi_8 \overline{B}_8 B_8$ and $\pi_8 \overline{\Delta}_{10} B_8$ coupling patterns (13) to be 0.6 and 0.03 respectively. As for $G_A^{\Xi\Lambda}$ and $G_A^{\Xi\Sigma}$ Cabbibo's theory (14) predicts

-0.85 $G\sin\theta$ and 0.02 $G\sin\theta$. Substituting these values into equation 9 gives

$$f_1(0) \simeq 1.2 \quad . \tag{10}$$

We cannot determine the amplitudes f_2 and f_3 within the framework of the Chew-Low approximation. However, we can obtain expressions for the discontinuities in the multipole amplitudes which permit us to estimate f_2 and f_3 by evaluating the Σ and Λ exchange diagrams with the correct kinematics. The Feynman amplitude corresponding to the diagrams of Figure 3 has the form

$$i g^i G_A^i \alpha(\lambda^2) \bar{u} [\gamma_\mu \gamma_5 (\not{p}_1 - \not{q} + m_i) \gamma_5] u \frac{1}{u - m_i^2} \quad . \tag{11}$$

(The induced pseudoscalar term does not contribute since $k \cdot G = 0$). Expressing this in terms of center of mass variables and using equation B.5 gives the following expressions for the "Born" multipole amplitudes:

$$[M_{1^+}]_B = \frac{1}{16\pi W} \left\{ -AQ_2(a) - BQ_1(a) + \frac{C}{5} [Q_1(a) - Q_3(a)] \right\} g^i \frac{G_A^i}{G\sin\theta}$$

$$[E_{1^+}]_B = \frac{1}{16\pi W} \left\{ AQ_2(a) + BQ_1(a) - \frac{3}{5} C [Q_1(a) - Q_3(a)] \right.$$

$$\left. + \frac{2}{3} D [Q_0(a) - Q_2(a)] \right\} g^i \; G\sin \tag{12}$$

$$[L_{1^+}]_B = \frac{1}{8\pi W}\left\{(A-E)Q_2(a) + \frac{C}{5}[3Q_3(a) + 2Q_1(a)] + (-B+F)Q_1(a)\right.$$

$$\left. - \frac{D}{3}[2Q_2(a) + Q_0(a)]\right\} g^i \frac{G_A^i}{G\sin\theta}$$

where $$A = \frac{(E_1+m)^{1/2}(W-m_i)}{2k(E_2+m)^{1/2}} \qquad B = \frac{(E_2+m)^{1/2}(W-m_i)}{2q(E_1+m)^{1/2}}$$

$$C = \frac{q}{(E_1+m)^{1/2}(E_2+m)^{1/2}} \qquad D = \frac{(E_2+m)^{1/2}(E_1+m)^{1/2}}{k}$$

$$E = \frac{k}{k_o}\frac{(W-m_i-2E_2)}{2(E_2+m)^{1/2}(E_1+m)^{1/2}} \qquad F = \frac{k}{k_o}\frac{(W+m_i-2E_2)(E_2+m)^{1/2}(E_1+m)^{1/2}}{2qk}$$

and $$a = \frac{2m^2+2q_ok_o-W^2-m_i^2}{2qk} \ .$$

The Q_j are the usual Legendre functions of the second kind. The Q's arise because in projecting out the multipole amplitudes we have integrals of the form

$$\int_{-1}^{1} \frac{P_j(x)}{[2m^2-W^2+2q_ok_o-2qkx]-m_i^2}\, dx \ .$$

This integral has two short branch cuts along the real axis which for small λ^2 are located near $W = \pm (2m-m_i)$ as well as a cut along the whole imaginary axis. According to our previous prescription we can approximately calculate the residues of the Ω pole in the multipole amplitudes by substituting the discontinuities in the Born amplitudes into equation 6 and integrating along the above cuts. We cannot actually carry out this program, however, because we do not know D(W). Nevertheless, it is expected that the most important contribution will come from the nearby short cut. On this cut D(W) can be approximated by simply W-M. Let us choose the kinematic factors to be

$$\rho_M = (E_1+m)^{1/2}/qk^2(E_1+m)^{1/2}$$

$$\rho_E = \rho_L = 1/q\,(E_1+m)^{1/2}(E_2+m)^{1/2} \quad .$$

Then by approximate numerical evaluation of the integrals we find that for $\lambda^2 \approx 0$ the residues of the Ω pole in the amplitudes M_{1^+}/qk^2, E_{1^+}/q, and L_{1^+}/q are given by

$$\gamma_M \simeq 0$$

$$\gamma_E \simeq (\gamma_E)_{static} \qquad (13)$$

$$\gamma_L \simeq (1.1)\,(\gamma_E)_{static}$$

where $(\gamma_E)_{static}$ is the value given in equation 8. Although the accuracy of our approximations is probably not great enough to tell whether these equations are significant comparison of these expressions with equations 2 gives

$$f_2(0) \simeq 0 \qquad Mmf_3(0) \simeq 0.5 \ . \tag{14}$$

If the values of f_1, f_2 and f_3 do not vary much for $0 \leq \lambda^2 \lesssim 1$ then we can use the values that we have obtained for $f_1(0)$, $f_2(0)$, and $f_3(0)$ to estimate the leptonic decay rates for the Ω^- (since the $\Omega^- - \Xi^0$ transition has $\Delta J = 1$, no-parity change, the axial vector contribution will dominate the decay rate). If we use the values given by Mathews (15) for the phase space integrals then we obtain the predictions listed in Table III. It is interesting to point out that our

Table III

Decay	Rate (sec^{-1})	Branching Ratio
$\Omega^- \rightarrow \Xi^0 + e^- + \bar{\nu}$	$2 \cdot 10^8$	$\sim 2\%$
$\Omega^- \rightarrow \Xi^0 + \mu^- + \bar{\nu}$	$1 \cdot 10^8$	$\sim 1\%$

calculation of Ω leptonic decay is a sort of S-matrix analogue of the usual calculations of Gamow-Teller transitions in nuclei. The information on the Ω "wave-function" is contained in the D function.

So far we have only shown how to compute the axial vector part of the matrix element 1. We will now briefly show how to determine the pseudoscalar part. Let us consider the amplitude for

$\theta_p + \Xi \rightarrow \overline{K} + \Xi$ where θ_p transforms like a pseudoscalar. The angular momentum and parity decomposition of this amplitude is given in Appendix C. For a final state of given isotopic spin, angular momentum, and parity there is just one amplitude. Let us denote the amplitude leading to the I = 0 $P_{3/2}$ final state by f(W). By calculating the Feynman amplitude corresponding to Figure 1 and using equations C. 3 we find that the residue of the Ω pole in f(W)/qk is related to $f_1, .., f_4$ by

$$\gamma_{\underline{p}} = \frac{1}{\sqrt{24\pi}} \frac{[(M+m)^2-\lambda^2]^{1/2}}{M^2} \left\{f_1 + (M-m)f_2 + \frac{1}{2}(M^2-m^2)f_3 + \lambda^2 f_4\right\} g_\Omega. \quad (15)$$

We see that f_4 can in principle be determined from this equation if f_1, f_2, and f_3 are known as functions of λ^2. Thus equations 2 and 15 constitute a complete set of equations for determining $f_1, \ldots, f_4$.

The residue $\gamma_{\underline{p}}$ can be calculated by using the same method that was used to calculate γ_E, γ_L, and γ_M. Let us write $h(W) = \rho f(W)$ where ρ is a factor which removes the kinematic singularities in f(W). Then by using the same arguments that led to equation 6 we find that the residue of the Ω pole in H(W) is given by

$$R = \frac{1}{2\pi i} \int_{\text{l. h. cuts}} \frac{D(W')\text{disc.}[h(W')]}{W'-M} \, dW' \quad (16)$$

where disc.[h(W)] is the discontinuity in h(W). Just as was the case for equation 5 we may approximately evaluate the integral over the left hand cuts by calculating disc.[h(W)] from the Σ and Λ exchange diagrams. The Feynman amplitude corresponding to the diagrams of Figure 3 has the form

$$ig^i[(m+m_i)\frac{G_A}{G\sin\theta}\alpha(\lambda^2) - \lambda^2\beta(\lambda^2)][\gamma_5(\not{p}_1 - \not{q} + m_i)\gamma_5]\frac{1}{u-m_i^2} \tag{17}$$

where β is induced pseudoscalar form factor. Making use of equations C.3 to project out the discontinuity in f(W) and keeping only the contribution from the nearby short cut in equation 16 gives the following expression for $\gamma_{\underline{p}}$ in the limit $\lambda^2 = 0$

$$\gamma_{\underline{p}} \simeq \frac{2}{\sqrt{2\pi}} \sum_{i=\Lambda,\Sigma} f_i\left(\frac{-G_A^i}{G\sin\theta}\right) . \tag{18}$$

Comparison of this expression with equation 15 gives

$$[f_1(0)+(M-m)f_2(0)+\tfrac{1}{2}(M^2-m^2)f_3(0)] \simeq 4\sqrt{3}\,\frac{M^2}{M+m}\sum_{i=\Lambda,\Sigma}\frac{1}{m+m_i}\left(\frac{g_i}{g_\Omega}\right)\left(\frac{-G_A^i}{G\sin\theta}\right). \tag{19}$$

This is numerically consistent with equation 9 because the terms $(M-m)f_2$ and $\frac{1}{2}(M^2-m^2)f_3$ are small.

The problem of calculating the matrix element $\langle\Delta|\, j_\mu^A \,|N\rangle$ is very similar to the problem of calculating $\langle\Omega^-|\, j_\mu^A \,|\Xi^o\rangle$. It is slightly more complicated conceptually because the Δ occurs as a resonance above πN threshold rather than as a bound state. Furthermore, inelastic effects are expected to be of some importance. If the inelastic effects are neglected and the Δ is treated as a stable particle then the details of calculating $\langle\Delta|\, j_\mu^A \,|N\rangle$ are almost identical to the details of calculating $\langle\Omega^-|\, j_\mu^A \,|\Xi^o\rangle$. The main difference is that the discontinuity in the multipole amplitudes is calculated from the nucleon exchange diagram (Figure 4) rather than from the Σ and Λ exchange diagrams. If the calculation is carried out in the way described in the previous two sentences one obtains

$$f_1(0) \simeq 0.9 \qquad f_2(0) \simeq 0 \qquad Mmf_3(0) \simeq 0.1. \tag{20}$$

These values can be used to provide information about the process $\nu + N \rightarrow \Delta + 2$. For example, the forward scattering amplitude is dominated by the axial-vector contribution. In fact the invariant differential cross section in the forward direction is approximately given by

$$\frac{d\sigma}{dt}(0) \simeq \frac{G^2}{12\pi} \, \frac{S - M_\Delta^2}{S - m_N^2} \left(\frac{M_\Delta + m_N}{M_\Delta}\right)^2 |f_1(0)|^2 . \tag{21}$$

An experimental measurement of $d\sigma/dt$ near $t = 0$ (17) gives $f_1(0) \simeq .87$. In order to determine the differential cross section away from the forward direction and the total cross section it is necessary to make some assumption about the dependence of the f's on λ^2 and to take the vector contribution into account. A plausible way to take the λ^2 dependence of the form factors into account is to assume

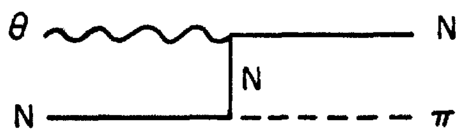


Figure 4

$$f_i(\lambda^2) = \frac{f_i(0)}{\left[1 - \frac{\lambda^2}{M_A^2}\right]^n} \qquad (22)$$

For n = 1 the form factors have a pole structure while for n = 2 they are like the empirical form factors appearing in electron scattering experiments (16). If the values of Equation 20 are used for the $f_i(0)$ and if the contribution of the vector current is estimated by using the CVC hypothesis and photoproduction date (16) then we obtain the total cross-sections shown in Figure 5. The curves we have drawn correspond to n = 2. Some experimental measurements (17) of the total cross-section are also shown. It is seen that the experimental results suggest that M_A is around 900 MeV. This value is consistent with the measurements of the axial vector form factor for $\nu + n \rightarrow p + \mu^-$ (17).

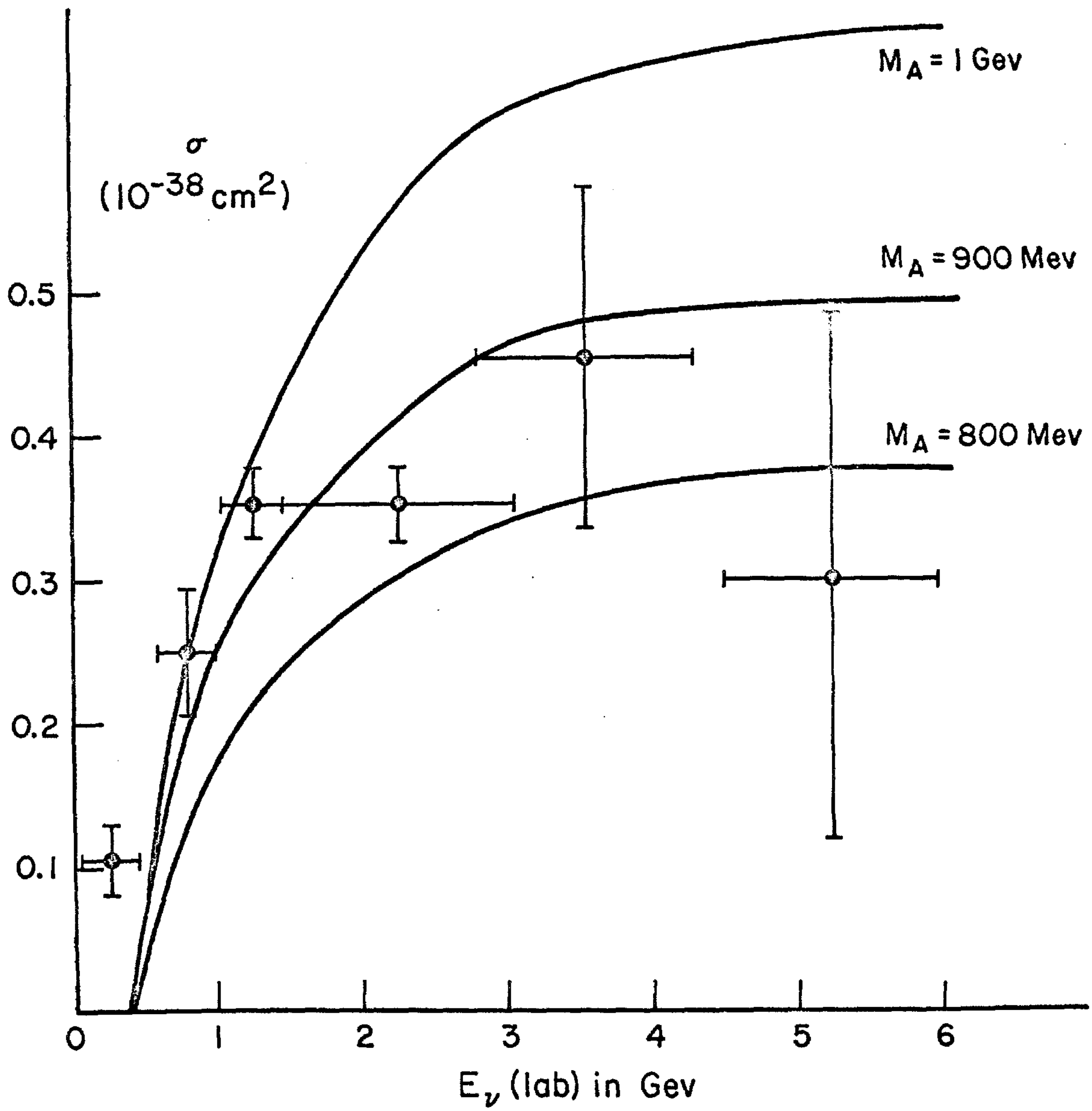


Figure 5. TOTAL CROSS SECTION FOR $\nu + n \rightarrow \Delta^{+} + \mu^{-}$

III. THE GOLDBERGER-TREIMAN RELATION

The Goldberger-Treiman relation for neutron β-decay is

$$2m_N\left(\frac{-G_A}{G\cos\theta}\right) = \frac{g}{f_\pi} \tag{1}$$

where g is the rationalized πNN coupling constant and f_π^{-1} is the pion decay amplitude. f_π^{-1} is normalized so that the rate for $\pi^+ \to \mu^+ + \nu$ is given by (in units pion mass = 1):

$$\Gamma_\pi = \frac{1}{64\pi^2} G^2\cos^2\theta\, m_\mu^2 (1 - m_\mu^2)^2 \left(\frac{f_\pi^2}{4\pi}\right)^{-1} . \tag{2}$$

Experimentally the error in equation 1 is about 8%. If we generalize equation 1 to the baryon octet then in the approximation of the eightfold way (which includes $f_\pi = f_\kappa$) we have that the F/D ratio for the axial vector current should be the same as the F/D ratio for the $\pi_8\bar{B}_8B_8$ couplings. In fact Cabbibo's value of 0.30/0.95 (14) for the F/D ratio for the axial vector current agrees well with the estimates of F/D for the meson couplings.

It has been pointed out (7) that the Goldberger-Treiman relation follows from the assumption that the form factor for the matrix element of the divergence of the axial vector current satisfies an unsubtracted dispersion relation in λ^2 and that this dispersion relation is dominated by the lowest lying pseudoscalar meson state. The actual behavior of the divergence of the axial vector current should of course follow from the bootstrap principle. In this section we will make use of the S-matrix theory of currents to study the zero momentum transfer behavior of the divergence of the hadron axial vector current. In particular we will take advantage of the reciprocal bootstrap relationship between the baryon octet and decuplet (9, 18) to calculate ratios of the zero momentum transfer form factors for octet-octet and octet-decuplet transitions.

Let us consider the matrix element of the divergence of the axial vector current between N and Δ. This matrix element may be written in the form

$$\langle \Delta | \partial_\mu j_\mu^A | N \rangle = i[f_1+(M_\Delta - m_N)f_2+\tfrac{1}{2}(M_\Delta^2 - M_N^2)f_3+\lambda^2 f_4]\, \psi_\alpha k_\alpha \psi \qquad (3)$$

where the f's are defined as in equation II. 1. The form factor here can be calculated by using the S-matrix methods described in the last section. For example, by using the same methods that were used to derive equation II. 13 we find that in the limit $\lambda^2 = 0$

$$[f_1(0)+(M_\Delta - m_N)f_2(0)+\tfrac{1}{2}(M_\Delta^2 - m_N^2)f_3(0)] \simeq \frac{\sqrt{2}}{3}\frac{2M_\Delta^2}{(M_\Delta+m_N)m_N}\left(\frac{g_{\pi NN}}{g_\Delta}\right)\left(\frac{-G_A}{G\cos\theta}\right) \qquad (4)$$

where g_Δ is related to the $\pi\Delta N$ coupling constant γ_Δ by

$$g_\Delta^2 = \frac{(M_\Delta + m_N)^2 - 1}{16\pi}\,\gamma_\Delta^2 \; .$$

By comparison if one assumes that the form factor for $\langle \Delta | \partial_\mu J^A_\mu | N \rangle$ is dominated by the pion pole then one obtains

$$[f_1(0)+(M_\Delta - m_N)f_2(0)+\frac{1}{2}(M_\Delta^2 - m_N^2)f_3(0)] \simeq \frac{\gamma_\Delta}{\sqrt{2}f_\pi} . \quad (5)$$

If we express γ_Δ in terms of g_Δ and use the Goldberger-Treiman relation (equation 1) for f_π^{-1} equation 5 becomes

$$[f_1(0)+(M_\Delta - m_N)f_2(0)+\frac{1}{2}(M_\Delta^2 - m_N^2)f_3(0)] \simeq \sqrt{2}\left\{\frac{2m_N}{M_\Delta + m_N}\left(\frac{g_\Delta}{g_{\pi NN}}\right)\left(\frac{-G_A}{G\cos\theta}\right)\right\}.$$

In order for this to agree with equation 4 we must have

$$\sqrt{2}\left(\frac{g_\Delta}{g_{\pi NN}}\right) = \frac{\sqrt{2}}{3}\left(\frac{g_{\pi NN}}{g_\Delta}\right)$$

or

$$g_\Delta^2 = \frac{1}{3}\left(\frac{M_\Delta}{m_N}\right)^2 g_{\pi NN}^2 \quad . \quad (6)$$

This is exactly the equation for g_Δ^2 that is obtained by Chew and Low (12) and is in agreement with experiment. Thus, we see that equation 5 is consistent with the bootstrap theory.

The considerations of the last paragraph may be summarized in the following way: the two Goldberger-Treiman relations, equations 1 and 5, tell us that at $\lambda^2 = 0$ the ratio of the form factors for the matrix elements of the divergence of the axial vector current is equal to the ratio of the corresponding strong coupling constants; however, this is just what is obtained from the S-matrix theory of currents. The reason for this result is easy to see. The matrix element of the divergence of the axial vector current between N and N or Δ and N has the same form as the pion coupling. Thus, the P-wave amplitudes for $X + N \rightarrow \pi + N$, where X is coupled to the divergence of the axial vector current, have the same crossing relations in the static approximation as the P-wave pion-nucleon scattering amplitudes. Since we expect that the static approximation will be fairly good for $\lambda^2 = 0$ we see immediately that the ratio of the NN and ΔN form factors at $\lambda^2 = 0$ is equal to the ratio of the πNN and $\pi\Delta$N coupling constants.

In deriving equation 4 we calculated the residue of the Δ pole in the amplitude for $X + N \rightarrow \pi + N$ in terms of the residue of the N pole in the amplitude for $X + N \rightarrow \pi + N$. However, we might just as well have calculated the residue of the N pole in terms of the residue of the Δ pole. To see how this would go consider the amplitude for $X + N \rightarrow \pi + N$ where the final πN is in an $I = 1/2$ $P_{1/2}$ final state. The residue of the N pole in this amplitude can be calculated by a formula similar to equation II. 12. The main difference is that the D function will be defined in terms of the $I = 1/2$ $P_{1/2}$ πN phase shift. If the discontinuity in the amplitude on the left hand cut due to the πN intermediate state is approximated by a "pseudopole" due to Δ exchange and if the linear approximation for the D function is made then we find that:

$$\left(\frac{-G_A}{G\cos\theta}\right) \simeq \frac{4\sqrt{2}}{3} \frac{m_N(M_\Delta + m_N)}{2M_\Delta^2} \left(\frac{g_\Delta}{g_{\pi NN}}\right) [f_1(0) + (M_\Delta - m_N) f_2(0) + \frac{1}{2}(M_\Delta^2 - m_N^2) f_3(0)]. \tag{7}$$

This differs from equation 4 by only a factor of 8/9. Thus, we see that there is a sort of reciprocal bootstrap relationship between $(-G_A/G\cos\theta)$ and $[f_1(0)+(M_\Delta - m_N)f_2(0)+\frac{1}{2}(M_\Delta^2 - m_N^2)f_3(0)]$.

Let us now consider what happens when this reciprocal bootstrap relationship is extended to SU(3). In SU(3) the ratio of $\bar{\Delta}_{10}B_8$ axial vector couplings is fixed while for the $\bar{B}_8B_8$ axial vector couplings we can have an F type coupling and a D type coupling. Thus, there will be one independent form factor for $\langle\Delta_{10}|\partial_\mu J_\mu^A|B_8\rangle$ and two independent form factors for $\langle B_8|\partial_\mu J_\mu^A|B_8\rangle$. The generalization of equation 4 may be obtained in a straightforward way by considering the amplitude for $X + B_8 \to \pi_8 + B_8$ where the final $\pi_8 + B_8$ is in a $P_{3/2}$ decuplet state. If one makes use of the bootstrap condition for the strong coupling constants one finds that

$$[f_1(0)+(M-m)f_2(0)+\tfrac{1}{2}(M^2-m^2)f_3(0)] \simeq \frac{2M}{M+m}[\sqrt{2/3}\, g_F+\sqrt{6/5}\, g_D] \quad (8)$$

where g_F and g_D are defined so that the matrix element of the octet axial vector current $g_\mu^{(i)}$ is for zero momentum transfer given by

$$\langle A|\, g_\mu^{(i)}|\, B\rangle = (f_{ABi}\, g_F + d_{ABi}\, g_D)\, \bar{u}_A \gamma_\mu \gamma_5 u_B$$

where f_{ABi} and d_{ABi} are defined in Gell-Mann's paper (32). The left hand side is normalized so that one multiplies by the appropriate SU(3) Clebsh-Gordan coefficient to get the form factor for the transition. Equation 8 will agree with the SU(3) generalization of equation 5 if g_F/g_D is equal to the F/D ratio for the $\pi_8\bar{B}_8B_8$ coupling constants. The SU(3) generalization of equation 7 is complicated by the fact that the octet representation occurs twice in the decomposition of 8×8 and therefore we have a coupled two-channel problem. In the static limit this problem is identical to the problem of magnetic dipole couplings studied by Dashen (18). He showed that if the reciprocal bootstrap relationship is to exist then F/D ratio for the magnetic couplings must equal the F/D ratio for the $\pi_8\bar{B}_8B_8$ couplings. For our problem this means that g_F/g_D equals F/D for the $\pi_8\bar{B}_8B_8$ couplings. However, this is just what the SU(3) generalization of equation 1 predicts!

In summary we can say that S-matrix calculations of matrix elements of the divergence of the axial vector current for octet-octet and octet-decuplet transitions are in essential agreement with the Goldberger-Treiman relations for the SU(3) symmetric case. Further, it appears that the Goldberger-Treiman relation for the N-Δ transition is good in real life. Whether the Goldberger-Treiman relations are good in real life for the strangeness-changing transitions is not clear at present because of uncertainties in the values of the strong coupling constants.

IV. APPROXIMATE SYMMETRY

In the previous two sections we calculated the ratios of various matrix elements of the axial-vector current by making use of the bootstrap principle. One can also obtain predictions for these ratios from approximate symmetry schemes. In this section we will discuss the relation between these two approachs. We will concentrate on the static (19) and collinear (20) SU(6) approximate symmetry schemes since it appears that these are the most successful.

If we assume that the octet and decuplet baryon states belong to the 56 representation of $SU(6)_S$ then we obtain predictions for the matrix elements of the Gamow-Teller operator between the 56 states (21). Because of SU(3) symmetry we can summarize the $[SU(6)]_S$ predictions in the following way: (a) the F/D ratio for Gamow-Teller matrix elements between octet states is 2/3; (b) the amplitude for the Gamow-Teller transition $\Delta^+ S_z = \frac{1}{2} \to n\, S_z = \frac{1}{2}$ is $2/5\ G_A$. Prediction (a) is in accord with the result obtained by applying the bootstrap principle to the SU(3) symmetric static model (3). In our previous notation the Gamow-Teller amplitude for $\Delta^+ S_z = \frac{1}{2} \to n\, S_z = \frac{1}{2}$ is $G/\sqrt{2}\ \cos\theta\ \sqrt{2/3}\ f_1(0)$. Thus, prediction (b) gives

$$f_1(0) = \frac{2\sqrt{3}}{5} \frac{G_A}{G\cos\theta} \quad \text{for } \Delta^+ \to n\, . \tag{1}$$

This differs from the result of our S-matrix calculation by 7%. From SU(3) symmetry we have that $(\Omega^- \to \Xi^0)_{G.T.} = -\sqrt{3}\tan\theta\,(\Delta^+ \to n)_{G.T.}$ so that $SU(6)_S$ predicts that

$$f_1(0) = \frac{6}{5} \left(\frac{-G_A}{G\cos\theta}\right) \quad \text{for} \quad \Omega^- \rightarrow \Xi^o \ . \tag{2}$$

This differs from the S-matrix result (equation II. 10) by 10%. It is interesting that our S-matrix calculations of the amplitudes for $\Delta^+ \rightarrow n$ and $\Omega^- \rightarrow \Xi^o$ follow the SU(3) pattern. This suggests that the generalization of the Cabbibo theory (14) to decuplet-octet transitions will agree with observation. A test of this idea should be possible soon when the relative rates of Y_1^* and Δ production by neutrinos are measured. Overall, we can say that the $SU(6)_S$ predictions are in close numerical agreement with the S-matrix calculations in the SU(3) symmetric case and that there is a correspondence even in the broken symmetry case.

The $SU(6)_W$ symmetry gives the $SU(6)_S$ predictions plus the prediction that the magnetic quadrupole amplitude for $\Delta \rightarrow N$ is zero (22). One can show that the magnetic quadrupole amplitude is proportional to the amplitude f_2 of section II (see for example equation II. 2). Thus, $SU(6)_W$ predicts that $f_2 = 0$. This, however, is in agreement with the results of our S-matrix calculation (equations II. 14 and II. 20).

The close correspondence between the results of our S-matrix calculations and the predictions of approximate symmetry leads us naturally to try to understand why the dynamical calculations should give results which approximately respect a symmetry, in particular $SU(6)_S$ and $SU(6)_W$. At first sight it seems remarkable that our calculations should agree with SU(6) predictions because we have not explicitly taken into account vector mesons. However, it can be argued that inclusion of vector mesons would not greatly change the pattern of our results (23). Thus, it is perhaps not surprising that our calculations would yield approximate SU(6) symmetry. One can, for example, cite some calculations of Cutkosky (24) which show that in some simple bootstrap models self-consistency of degenerate supermultiplets requires that the couplings respect a symmetry group. Since the input to our calculations was an approximately degenerate baryon SU(6) 56 supermultiplet and, effectively, an approximately degenerate meson SU(6) 35 supermultiplet we might expect the results to approximately respect SU(6). The validity of the particular symmetries $SU(6)_S$ and $SU(6)_W$ would require, in addition to approximately degenerate supermultiplets, some special conditions on the S-matrix. For the $SU(6)_W$ symmetry these conditions are probably satisfied in our approximate calculations because we have considered only single particle intermediate states in the three channels (25). As for $SU(6)_S$ it is not clear whether sufficient conditions for its validity are satisfied.

The theoretical foundations for the appearance of approximate symmetries in our calculations can also be approached from the point of view of current commutation relations. It has been shown by Dashen and Gell-Mann (26) and Lee (27) that current algebras can give rise to hadron symmetries provided certain conditions are satisfied; for example, that the sum over intermediate states in the matrix element of the commutator converges rapidly. It has subsequently been pointed out by Dashen and Frautschi (28) that as a consequence of the bootstrap principle the prospects for correlating a current algebra with an approximate symmetry are quite favorable, at least for low-lying hadron states. Below we will show that the bootstrap principle also implies that the currents form an algebra and that the algebra is such that it can explain the $SU(6)_S$ and $SU(6)_W$ symmetries of the baryon 56.

In order to get a hold on the commutator of two currents in an S-matrix theory we must have some way of representing the commutator in terms of a scattering amplitude. A clue as to how this might be done is provided by Heisenberg's original derivation of the commutation relation $[x, p] = i$ (29). He noted that in order to be consistent with the correspondence principle the amplitude for the scattering of light by an oscillator must behave in a certain way. For example, the amplitude for the forward scattering of light by an atom should, according to the correspondence principle, approach the classical Thomson value $Z e^2/m_e$ as the frequency of the light becomes large in comparison with the binding energy. He then showed that the amplitude for the scattering of light will behave in the right way if a certain sum rule for the absorptive part of the forward scattering amplitude (the Thomas-Reiche-Kukn sum rule) is satisfied and that this sum rule is implied by the relation $[x, p] = i$. By analogy we are led to consider the scattering amplitude for $\theta^i + a \to \theta^j + b$, where a and b are hadron states and θ^i and θ^j are coupled to the currents j^i and j^j, in connection with current commutation relations. As a matter of fact Fubini (30) has shown that the content of current commutation relations can be expressed in terms of a sum rule for the absorptive part of these amplitudes. Furthermore, each sum rule is equivalent to a certain statement about the high-energy behavior of the scattering amplitude (31). In discussing the high-energy behavior of these amplitudes we do not have the correspondence principle as a guide; however, we do have the bootstrap principle.

Let us consider the amplitude for $\theta^i + a \to \theta^j + b$ where a and b are single-particle states and θ^i and θ^j are coupled to the currents j^i and j^j. If the invariant amplitudes obey unsubtracted dispersion relations in s for fixed t then the scattering amplitude for the process $\theta^i + a \to \theta^j + b$ will be given by:

$$T^{ij} = (2\pi)^3 \sum_n \frac{1}{2E_n} \left\{ \frac{\langle p_2 | j^j(0) | n\rangle \langle n | j^i(0) | p_1\rangle}{\omega_1 + E_1 - E_n} \delta^3(\underset{\sim}{p}_n - \underset{\sim}{k}_1 - \underset{\sim}{p}_1) \right.$$

$$\left. - \frac{\langle p_2 | j^i(0) | n\rangle \langle n | j^j(0) | p_1\rangle}{\omega_1 - E_2 + E_n} \delta^3(\underset{\sim}{p}_n + \underset{\sim}{k}_2 - \underset{\sim}{p}_1) \right\} . \tag{4}$$

The notation is explained in Table IV. The right-hand side of

Table IV

Particle	4-momentum	3-momentum	Energy
a	p_1	$\underset{\sim}{p}_1$	E_1
b	p_2	$\underset{\sim}{p}_2$	E_2
θ^i	k_1	$\underset{\sim}{k}_1$	ω_1
θ^j	k_2	$\underset{\sim}{k}_2$	ω_2

equation 4 can be related to the commutator of j^i and j^j if the denominator of the two terms in brackets is the same for all n. One way of doing this is to let the energies ω_1 and ω_2 go to infinity while keeping $\underset{\sim}{k}_1$ and $\underset{\sim}{k}_2$ fixed. Another way is to keep all the particles on their mass shell and let the momenta $\underset{\sim}{p}_1$ and $\underset{\sim}{p}_2$ go to infinity. Taking either of these limits T^{ij} becomes $1/\omega_1 \; X^{ij}$ where

$$X^{ij} = \int e^{ik_2 \cdot x} \delta(x_o) \langle p_2 | [j^j(x), j^i(0)] | p_1 \rangle d^4x \quad . \tag{5}$$

We now wish to argue that X^{ij} behaves like a self-consistent current.

Consider the amplitude for $\theta^i + N \rightarrow \theta^j + N + \pi$. If ω_1 is very large then we expect that most of the amplitude for this process will come from emission of the pion from the initial or final nucleon (see Figure 6). For example the amplitude corresponding to emission from an initial nucleon line is proportional to

$$\frac{1}{(p_1 - q)^2 - m^2} \quad .$$

On the other hand the amplitude for emission from an intermediate state is proportional to

$$\frac{1}{(p_n - q)^2 - m_n^2} \times \text{(terms linear in } p_n)$$

where p_n is typically on the order of $p_1 + k_1$. If the sum over intermediate states in equation 4 converges then we can pick some mass M such that the contribution to the sum from intermediate states with $m_n > M$ is negligible. Further, if the convergence is uniform then we can pick one M for all ω_1. Because of the behavior of the form factors it is likely that the convergence is uniform. Therefore, we conclude that as $\omega_1 \rightarrow \infty$ the ratio of the amplitude for emission from an initial or final nucleon to the amplitude for

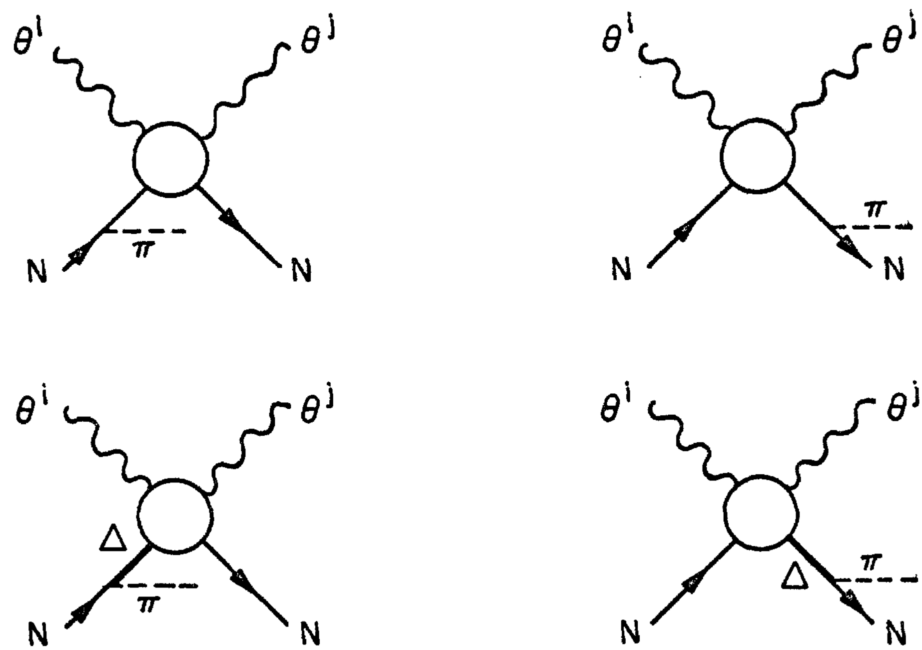


Figure 6. Some typical mechanisms for pion emission from an initial or final nucleon.

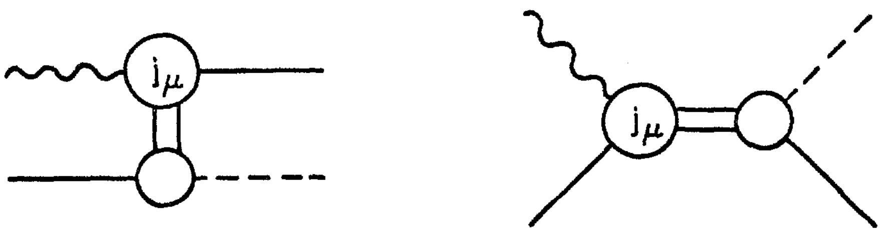


Figure 7. Some contributions to the unitarity condition for a current.

emission from an intermediate state goes like $1/\omega_1$. By comparing the diagrams for emission of a pion from the initial or final nucleon with the unitarity condition for a current (Figure 7) we see that the unitarity condition for $\theta^i + N \to \theta^j + N + n$ will approach the unitarity condition for the matrix element $\langle N|j|N\pi\rangle$ as $\omega_1 \to \infty$. A little thought shows that a similar conclusion will be likely to hold for arbitrary hadron states a and b. Since the self-consistent currents in a bootstrap theory are defined by the unitarity condition the above arguments lead us to the conclusion that X^{ij} defines a self-consistent current.

Having shown that the equal time commutator of two hadron currents is a hadron current let us see if we can guess what algebra is generated. If we consiser the matrix elements of the commutator $[j^i_\nu(\underset{\sim}{x}, 0),\ j^j_0(0)]$ of two vector currents between hadron states then we find that they behave like the matrix elements of a vector current. Now the number of vector currents is probably severely limited by the bootstrap principle. In fact calculations with the static model (3) suggest that there are only nine possible vector currents corresponding to the baryon current plus the octet of vector currents observed in weak and electromagnetic interactions. If these nine currents are the only possible vector currents then the commutator $[j^i_\nu(\underset{\sim}{x}, 0),\ j^j_0(0)]$ will automatically be determined for every case except when it corresponds to $I = 0\ \ Y = 0$. Approximate calculation of the amplitude for $\theta^i + B_{56} \to \theta^j + B_{56}$ suggests that the commutator of two octet currents corresponding to $I = 0\ \ Y = 0$ is the hypercharge current. Therefore for the commutator of two octet vector currents we would have

$$[j^j_\nu(\underset{\sim}{x}, 0),\ j^i_0(0)] = i f_{jik}\, j^k_\nu(0)\delta^3(x) \tag{6}$$

where the f_{ijk} are the SU(3) structure constants (32). Equation 6 is just the algebra of vector currents proposed by Gell-Mann (2). Similar arguments can be put forward to show how algebras involving axial-vector, scalar, and tensor currents can arise. The above discussion is, of course, not a derivation of the current algebra but is only intended to make plausible the fact the bootstrap principle can lead to the algebras which have been correlated with the approximate symmetry.

The above arguments were based on the assumption that the invariant amplitudes for $\theta^i + a \to \theta^j + b$ satisfy unsubtracted dispersion relations in s for t fixed. If this assumption is not true then the amplitude for $\theta^i + a \to \theta^j + b$ cannot be written in form of equation 4. We see no reason to believe, however, that the Low amplitudes $T^{ij}_{\mu\nu}$ will not continue to behave in the same way with respect to, for example, pion emission as the full amplitude. If this is true then one could construct the commutator independently of whether or not there are subtractions. These ideas, in fact, might be used to prove that there are no no-trivial terms in equation 8 involving gradients of δ-functions.

We conclude by pointing out that whatever the algebra is the bootstrap principle implies that it must be consistent with the approximate symmetry of the low-lying hadron states. Consider the commutator of the time-component of a vector and space-component of an axial vector current. This will have a term coupling like $\underset{\sim}{\gamma}\gamma_5$ to the baryon octet. Now by using the methods of section II, one finds that because of the reciprocal bootstrap the ratios of the couplings among octet states is fixed. Furthermore, it is not hard to show that the ratios are such that the algebra projected on the baryon 56 is consistent with the approximate symmetry.

V. CONCLUDING REMARKS

We have shown that S-matrix calculations for matrix elements of the axial vector current between baryon octet and decuplet states are consistent with certain predicted properties of the axial vector current. It should be noted that our demonstration of consistency has been carried out on two levels. On the lowest level we showed that the S-matrix calculations of the matrix elements were numerically close to what was predicted by the properties in question. On a higher level we showed that if the predicted property is expressed in terms of the behavior of an operator then the operator behaves in the S-matrix theory at least approximately as predicted. In the case of the Goldberger-Treiman relation the operator was the divergence of the axial vector current, while in the case of approximate symmetry the operator was the equal time commutator of two currents.

APPENDIX A

The γ_μ are the ordinary Dirac matrices. A familiar representation is

$$\underset{\sim}{\gamma} = \begin{bmatrix} 0 & \underset{\sim}{\sigma} \\ -\underset{\sim}{\sigma} & 0 \end{bmatrix} \qquad \gamma_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .$$

Our γ_5 is defined as $i\gamma_0\gamma_x\gamma_y\gamma_z$. The Dirac spinors ψ satisfy

$$\not{p}\psi = m\psi$$

where $\not{p} = p_\mu \gamma_\mu$. They are normalized so that $\bar{\psi}\psi = 1$ where $\bar{\psi} = \psi^+ \gamma_0$. In terms of Pauli spinors χ

$$\psi = [\frac{E + m}{2m}]^{1/2} (1 + \frac{\underset{\sim}{\alpha} \cdot \underset{\sim}{p}}{E + m} [\begin{smallmatrix} \chi \\ 0 \end{smallmatrix}]$$

where $\underset{\sim}{\alpha} = \gamma_0 \underset{\sim}{\gamma}$. The Rarita-Schwinger spinors ψ_μ satisfy

$$\not{P}_\mu \psi_\mu = M\psi_\mu$$

$$\gamma_\mu \psi_\mu = 0$$

$$P_\mu \psi_\mu = 0 \quad .$$

They are normalized so that $\bar{\psi}_\mu \psi_\mu = -1$. Let us define 4-vector spinors ξ_μ:

$$\xi_\mu = \varepsilon_\mu \, \Theta \, \chi$$

where $\epsilon(1) = 1/\sqrt{2}\,[-1, -i, 0, 0]$, $\epsilon(0) = [0, 0, E/M, P/M]$, $\epsilon(-1) = 1/\sqrt{2}\,[1, -1, 0, 0]$. Then ψ_μ will be given by

$$\psi_\mu = \left[\frac{E+M}{2M}\right]^{1/2} \left(1 + \frac{\underset{\sim}{\alpha} \cdot \underset{\sim}{P}}{E+M}\right) \begin{bmatrix} \xi_\mu \\ 0 \end{bmatrix} .$$

A useful result is

$$\sum_{\text{spins}} \psi_\mu \bar{\psi}_\nu = \left[-\frac{2}{3}\,\delta_{\mu\nu} + \frac{2}{3}\,\frac{P_\mu P_\nu}{M^2} + \frac{1}{3}\,(\gamma_\nu P_\mu - \gamma_\mu P_\nu) + \frac{1}{6}\,(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)\right] (\not{P} + M) .$$

APPENDIX B

In the center-of-mass system the differential cross section for $\theta + B \rightarrow \pi + B$ where θ is an axial vector particle may be written:

$$\frac{d\sigma}{d\Omega} = q/k \left| \chi_f G \chi_i \right|^2 \tag{B. 1}$$

where χ is a Pouli spinor, q and k are the meson and boson 3-momenta, and

$$G = i\sigma \cdot q\sigma \cdot \varepsilon G_1 + i\sigma \cdot \varepsilon\sigma \cdot k G_2 + i\sigma \cdot q\sigma \cdot kq \cdot \varepsilon G_3$$

$$+ iq \cdot \varepsilon G_4 + i\sigma \cdot q\sigma \cdot kk \cdot \varepsilon G_5 + ik \cdot \varepsilon G_6 . \tag{B. 2}$$

We may express G in terms of the multipole amplitudes by using the appropriate projection operators:

$$G = \sum_{j=0}^{\infty} \Big\{ L_j^+ (1 + j + \sigma \cdot L_q) i\varepsilon \cdot k + L_j^- (j - \sigma \cdot L_q) i\varepsilon \cdot k$$

$$- E_j^+ (1 + j + \sigma \cdot L_q) \varepsilon \cdot (k \times L_k) - E_j^- (j - \sigma \cdot L_q) \varepsilon \cdot (k \times L_k) \tag{B. 3}$$

$$- M_{j-1}^+ (\sigma \cdot q)(j - \sigma \cdot L_q) i\varepsilon \cdot L_k - M_{j+1}^- (\sigma \cdot q)(1 + j + \sigma \cdot L_q) i\varepsilon \cdot L_k \Big\} P_j(q \cdot k)$$

where $L_q = -i(q \times \partial_q)$, $L_k = -i(k \times \partial_k)$, and P_j are the Legendre polynomials. Carrying out the operations indicated in A. 3 and comparing with A. 2 gives the following expressions for the G_i:

$$G_1 = \sum_{j=1}^{\infty} \left\{ [(j+2)M_j^+ + E_j^+ + (j-1)M_j^- + E_j^-] P_j'(x) \right\}$$

$$G_2 = \sum_{j=0}^{\infty} \left\{ (j+1)M_j^+ P_{j+1}'(x) + jM_j^- P_{j-1}'(x) \right\}$$

$$G_3 = \sum_{j=1}^{\infty} \left\{ (E_j^+ + M_j^+ + E_j^- - M_j^-) P_j''(x) \right\}$$

$$G_4 = \sum_{j=1}^{\infty} \left\{ (-E_j^+ - M_j^+) P_{j+1}''(x) + (M_j^- - E_j^-) P_{j-1}''(x) \right\} \tag{B. 4}$$

$$G_5 = -G_1 - xG_3 + \sum_{j=0}^{\infty} \left\{ (L_j^- - L_j^+) P_j'(x) \right\}$$

$$G_6 = -G_2 - xG_4 + \sum_{j=0}^{\infty} \left\{ L_j^+ P_{j+1}'(x) - L_j^- P_{j-1}'(x) \right\}$$

where $x = \cos\theta$. The inversion formulae are

$$M_{j^+} = \frac{1}{2(j+1)} \int_{-1}^{1} dx \left\{ -G_1 P_{j+1}(x) + G_2 P_j(x) + G_3 \frac{P_j(x) - P_{j+2}(x)}{2j+3} \right\}$$

$$M_{j^-} = \frac{1}{2j} \int_{-1}^{1} dx \left\{ G_1 P_{j-1}(x) - G_2 P_j(x) - G_3 \frac{P_{j-2}(x) - P_j(x)}{2j-1} \right\}$$

$$E_{j^+} = \frac{1}{2(j+1)} \int_{-1}^{1} dx \left\{ G_1 P_{j+1}(x) - G_2 P_j(x) - G_3 (j+2) \frac{P_j(x) - P_{j+2}(x)}{2j+3} - G_4 (j+1) \frac{P_{j-1}(x) - P_{j+1}(x)}{2j+1} \right\}$$

$$E_{j^-} = \frac{1}{2j} \int_{-1}^{1} dx \left\{ G_1 P_{j-1}(x) - G_2 P_j(x) + G_3 (j-1) \frac{P_{j-2}(x) - P_j(x)}{2j-1} + G_4 j \frac{P_{j-1}(x) - P_{j+1}(x)}{2j+1} \right\} \qquad \text{(B. 5)}$$

$$L_{j^+} = \frac{1}{2} \int_{-1}^{1} dx \left\{ (G_1 + xG_3 + G_5) P_{j+1} + (G_2 + xG_4 + G_6) P_j \right\}$$

$$L_{j^-} = \frac{1}{2} \int_{-1}^{1} dx \left\{ (G_1 + xG_3 + G_5) P_{j-1} + (G_2 + xG_4 + G_6) P_j \right\}.$$

APPENDIX C

Let us write the center-of-mass differential cross-section for $X + B \rightarrow \pi + B$ where X is a pseudoscalar particle as

$$\frac{d\sigma}{d\Omega} = q/k \,|\chi_f \,[f_1 + \sigma\cdot\hat{q}\;\sigma\cdot\hat{k}\, f_2]\;\chi_i\,|^2 \qquad \text{C.1}$$

Because of conservation of angular momentum and parity there is just one amplitude for a given total angular momentum and orbital angular momentum of the π. Let us denote the partial wave amplitudes corresponding to orbital angular momentum ℓ , total angular momentum $\ell \pm 1/2$ by $f_{\ell\pm}$. Then we will have

$$f_1 = \sum_{\ell=0}^{\infty} f_{\ell+}\, P'_{\ell+1}(\chi) - \sum_{\ell=2}^{\infty} f_{\ell-}\, P'_{\ell-1}(\chi)$$
$$f_2 = \sum_{\ell=1}^{\infty} (f_{\ell-} - f_{\ell+})\; P'_{\ell}(\chi) \qquad \text{C.2}$$

These equations can be inverted to give

$$f_{\ell\pm} = \frac{1}{2}\int_{-1}^{1} [\,f_1\, P_{\ell}(\chi) + f_2\, P_{\ell\pm1}(\chi)]\; d\chi \qquad \text{C.3}$$

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