THE EFFECTS OF BIAS ON POLARITY-COINCIDENCE DETECTION

Thesis by

William Dean Squire

In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1964

(Submitted April 22, 1964)

ACKNOWLEDGMENTS

My special gratitude is expressed to my adviser, Professor H. C. Martel, for his interest and advice and for the many stimulating discussions during all phases of the work reported in this thesis. In addition, I should like to thank Professors D. J. Braverman, J. N. Franklin, and T. L. Grettenberg for helpful discussions. I am also indebted to Mr. Benedict Freedman who, in a private communication, suggested a very useful approximating function for the error function.

The U. S. Naval Ordnance Test Station furnished generous financial support under the terms of a research contract with the California Institute of Technology. For this support I am most grateful. Thanks are also due to Miss Doris Barnhart, who typed the text, and to Mr. Arthur Kuhn, who prepared the figures.

Most of all, I wish to express my unbounded gratitude to my wife, Shirley, for her understanding, patience, and encouragement, without which this work would never have been completed.

ABSTRACT

A polarity-coincidence detector with biased polarity indicators (biased hard limiters) is studied to determine the effects of the bias on the detector output mean value, output variance, and output signal-to-noise power ratio.

The mean value of the detector output is derived for the case of Gaussian input signal and Gaussian input noise, with arbitrary spectra for the signal and noise. The mean value is expressed as a function of the input signal-to-noise power ratio, and as a function of the biases in the input channel and the reference channel polarity indicators. It is shown that the effect of bias is to introduce a spurious component in the output mean value.

The variance of the detector output is derived for the case of Gaussian input signal and Gaussian input noise, with RC low-pass spectra for signal and noise, for small input signal-to-noise power ratios only. The variance is expressed as a function of the biases in the input channel and reference channel polarity indicators, and as a function of the input signal-to-noise band-width ratio. It is shown that the effect of bias is to introduce a spurious component in the output variance.

The output signal-to-noise power ratio (the square of the output mean divided by the output variance) is derived for both an ideal polarity-coincidence detector (no bias) and for a biased polarity-coincidence detector. The output signal-to-noise power ratio is expressed as a function of the input signal-to-noise power ratio, the biases in the input channel and the reference channel polarity indicators, and the input signal-to-noise band-width ratio, for small input signal-to-noise power ratios only.

It is shown that the output signal-to-noise power ratio of an ideal polarity-coincidence detector is degraded from that of a correlation detector. When the input signal-to-noise band-width ratio is unity, the degradation is about 1.4 db. The degradation increases to about 2.2 db. when the input signal-to-noise band-width ratio becomes either very small or very large. It is also shown that the output signal-to-noise power ratio of a biased polarity-coincidence detector is degraded from that of an ideal polarity-coincidence detector. A simple expression for the degradation is presented. Limits on the biases are given, such that when the biases are smaller than these limits, the degradation of the output signal-to-noise power ratio is negligible.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

A cross-correlator is a device for implementing the mathematical operation of multiplying two quantities, $x_1(t)$ and $x_2(t)$, and integrating the product for a finite length of time, T. If $x_1(t)$ consists of a noise component and possibly a known signal component and if $x_2(t)$ is proportional to the known signal component, then the device is called a correlation detector.

It is well known that a correlation detector is optimum, in a certain sense, for detecting signals in the presence of noise. Specifically, if the signal contains finite energy and if the noise is white Gaussian, then a correlation detector is equivalent to a matched-filter and also to a likelihood-ratio detector (1).

Because of its optimum properties, it is desirable to provide various methods of implementing a correlation detector. Methods of implementation can be divided into two categories - analog and digital. A correlation detector is equivalent to a linear filter and an analog implementation preserves this linearity. Since general and powerful methods exist for the analysis of linear systems, it is natural that analog implementations of the correlation detector have been investigated extensively. On the other hand, a digital implementation is inherently non-linear and consequently only limited investigations of digital implementations have been made. Nevertheless, for many applications, digital implementations have advantages relative to analog methods, and in fact, their use is becoming widespread.*

It is the purpose of this thesis to examine certain properties of an extremely simple form of digital correlation detection, utilizing only two levels of digitization, called polarity-coincidence detection. A non-ideal polarity-coincidence detector, in which the polarity-indicating devices have bias, is investigated.

An inherent characteristic common to all digital systems is that continuous functions are quantized or "digitized". This quantization is accomplished by dividing the domain of the function into a number of intervals or "levels", each with an assigned number to characterize it. All values of the function in an interval are replaced by the characterizing number assigned to that interval. The number of divisions is called the level of digitization. Generally, a digital system contains some sort of data storage apparatus or "memory". The amount of memory required, the complexity of computation and, consequently, the size of the system increase with the level of digitization. Thus it is desirable to restrict the level of digitization to as small a value as is consistent with the required precision.

---

* If a general purpose digital computer programmed to calculate the cross-correlation between two functions is included as an implementation, then indeed a large class of correlation calculations has been implemented digitally.

At least two levels of digitization must be used if any of the characteristics of the original function are to be preserved. Usually, in two level digitizing, the function domain is divided into its positive and negative parts and either the numbers +1 and -1 or +1 and 0 are assigned. It would seem at first glance that two level digitization would destroy so much of the character of the original function as to be useless in a digital implementation of a correlation detector. In fact, however, the performance of a digital correlation detector using two levels of digitization is degraded hardly at all relative to the performance of a perfect correlation detector.* Moreover, the implementation of a two-level digital cross-correlator is extremely simple, due to the fact that the multiplication operation, by a simple artifice, can be replaced by an addition operation. From the following table it is clear that $a \cdot b = 1 - 2(c + d)$, where $+$ denotes addition modulo 2. Thus, in this kind of digital system, multiplication can be replaced by addition modulo 2 after a simple transformation is made.

---

* It should be noted that although the two-level digital correlation detector has nearly optimum performance in the detection of a single signal in the presence of noise, it nevertheless has some limitations. Specifically, it is similar in its behavior in some respects to that of an FM receiver with strong limiting followed by a phase detector. E.G. the desired signal is suppressed by a strong (undesired) signal component near in frequency to the desired signal, due to the power normalizing properties of the digitizing process. This phenomenon has been examined by several investigators (2,3 and 4). See Appendix VII.

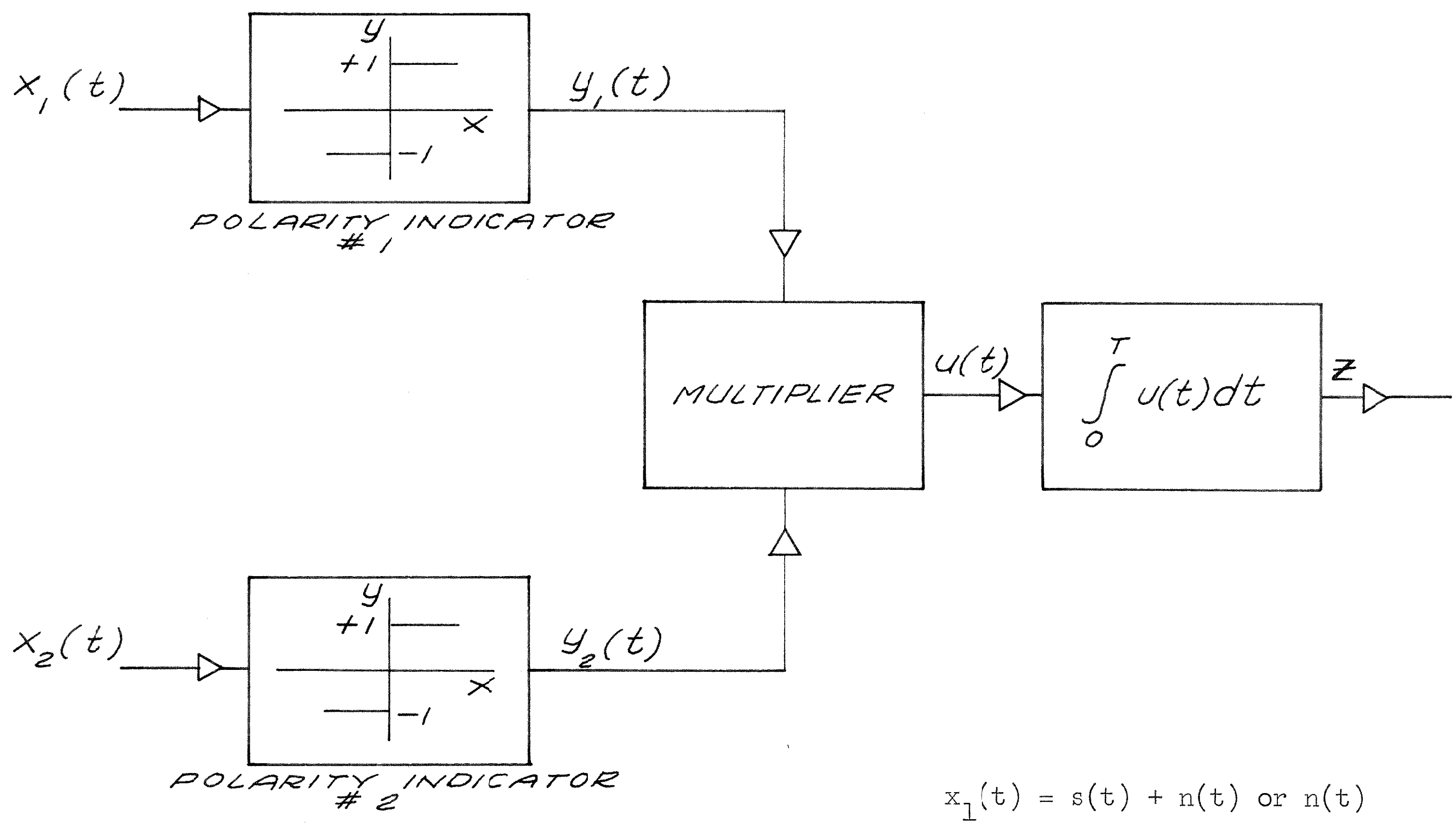
a	+1	+1	-1	-1	c	+1	+1	0	0
b	+1	-1	+1	-1	d	+1	0	+1	0
a·b	+1	-1	-1	+1	c + d	0	+1	+1	0

Because of its simplicity and near optimum performance, there is considerable interest in the two-level digital correlation detector and a number of papers discussing it have appeared in the literature in recent years. Some of these will be discussed below. Two-level digital cross-correlators have been constructed by several investigators (5,6 and 7).

A block diagram of a simple two-level digital correlation detector is shown in Figure 1.1. The two-level digitizers shown in the figure are also variously known as ideal limiters, polarity indicators, sign generators and zero-crossing generators. The corresponding detector is variously called a two-level digital correlation detector, an ideal limiting correlation detector, a polarity-coincidence detector or a zero-crossing correlation detector. In this thesis the term polarity-coincidence detector will be used.

Summary of Literature
Related to Polarity-Coincidence Detection

A review of the literature related to polarity-coincidence detection is given in Appendix VII. A brief summary is given here.



$x_1(t) = s(t) + n(t)$ or $n(t)$

$x_2(t) = s_o(t) = a \cdot s(t)$ with $a = \sigma_o/\sigma_s$

Block Diagram of Polarity-Coincidence Detector

Figure 1.1

There has been interest among statisticians for many years in the evaluation of multi-dimensional Gaussian integrals over regions bounded by hyperplanes, planes, lines, etc. (8, 9, 10 and 11). Such integrals have a direct relation to the problem of determining the effects of two-level digitizing on the detection of signals in Gaussian noise. This relation will be examined in detail in the body of this thesis.

The properties of the auto-correlation function and the signal-to-noise power ratio at the output of an ideal limiter* and at the output of a band-pass limiter** have been discussed in several papers (12 through 19). Most of these have considered narrow-band inputs consisting of sinusoidal signals in narrow-band Gaussian noise. A few have examined wide-band signals and noise. For a single sinusoidal signal in narrow-band Gaussian noise or for a very narrow-band non-sinusoidal signal in narrow-band Gaussian noise, the degradation in signal-to-noise power ratio and signal detectability at the output of a band-pass limiter is approximately 1 db. For multiple sinusoidal signals in Gaussian noise or for wide-band non-sinusoidal

---

* An ideal limiter is a device whose output is +1 for positive inputs and -1 for negative inputs. (The value of the output when the input is zero is usually not important. Often it is arbitrarily set equal to zero.) Thus the ideal limiter is a two-level digitizer of the polarity-indicator type.

** A band-pass limiter is an ideal limiter followed by an ideal band-pass filter. The band-pass limiter is intended to operate only on narrow-band inputs and its band-pass filter has a pass-band corresponding to the narrow-band input.

signals in Gaussian noise, the signal-to-noise power ratio and signal detectability may be seriously degraded (6 db. or more).

The polarity-coincidence detector has been investigated for inputs consisting of a narrow-band Gaussian signal in narrow-band Gaussian noise, the signal and noise having identically shaped spectra (20). The degradation in output signal-to-noise power ratio relative to a correlation detector (no limiting) is less than 1 db. In the same paper (20), the performance of a polarity-coincidence coherency detector* is compared with that of a correlation coherency detector** (no limiting) for the same type of input as before - Gaussian signal and Gaussian noises, the noises having equal power and identically shaped spectra. Again the degradation is less than 1 db. Additional papers discussing the performance of a polarity-coherency detector relative to a correlation coherency detector have appeared recently (21,22).

For a more complete discussion of the above references and also of related literature, see Appendix VII.

---

* The polarity-coincidence coherency detector is a correlation coherency detector** in which the inputs have been ideally limited before multiplication.

** The correlation coherency detector is a cross-correlator whose inputs are: $s_1(t)$ consisting of a noise component $n_1(t)$ and possibly a signal component $s(t)$; $x_2(t)$ consisting of a noise component $n_2(t)$ and possibly the same signal component $s(t)$.

Description of the Area of Investigation Covered in this Thesis

Certain properties of a non-ideal polarity-coincidence detector, in which the polarity indicators have bias, are investigated. In subsequent discussion, this non-ideal polarity-coincidence detector will be called a biased polarity-coincidence detector and will be denoted by BPCD.

The behavior of the BPCD in response to a Gaussian signal in the presence of Gaussian noise is examined. The following properties are investigated:

a. The effects of bias on the post-detection or output mean value. No restrictions are imposed on the form of the signal or noise spectrum and the mean value is derived for arbitrary input signal-to-noise ratios.

b. The effects of bias on the post-detection or output variance. A general expression is obtained for small input signal-to-noise ratios. Numerical values are obtained only for signals and noise with RC low-pass spectra.

c. Using the post-detection or output signal-to-noise power ratio as a criterion of performance, the BPCD is compared with the polarity-coincidence detector (no bias) and with the correlation detector (no limiting).

REFERENCES

1. Davenport, W. B. and W. L. Root: Random Signals and Noise, McGraw-Hill, New York, pp. 343-345; 1958.

2. Jones, J. J.: "Hard-Limiting of Two Signals in Random Noise," IEEE Trans. on Information Theory, vol. IT-9, pp. 34-42; January, 1963.

3. Rubin, W. L. and S. K. Kamen: "S/N Ratios in a Two-Channel Band-Pass Limiter," Proc. IEEE, vol. 51, pp. 389-390; February, 1963.

4. Cahn, C. R.: "A Note on Signal-to-Noise Ratio in Band-Pass Limiters," IRE Trans. on Information Theory, vol. IT-7, pp. 39-43; January, 1961.

5. Kaiser, J. F. and R. K. Angell: "New Techniques and Equipment for Correlation Computation," Technical Memorandum 7668-TM-2, Servomechanisms Lab., Dept. of Electrical Engineering, MIT.; December, 1957.

6. Goldstein, R. M.: Radar Exploration of Venus, Ph.D. Dissertation, Dept. of Electrical Engineering, California Institute of Technology; 1962.

7. Whitehouse, H. J.: "Parallel Digital Delay-Line Correlator," Proc. IEEE, vol. 51, no. 1, p. 237; January, 1963.

8. Kendall, M. G.: "Proof of Relations connected with the Tetrachoric Series and its Generalization," Biometrika, vol. 32, pp. 196-198; 1941.

9. Moran, P. A. P.: "Rank Correlation and Product-Moment Correlation," Biometrika, vol. 35, pp. 203-206; 1948.

10. David, F. N.: "A Note on the Evaluation of the Multivariate Normal Integral," Biomctrika, pp. 458-459; 1953.

11. U. S., National Bureau of Standards: Tables of the Bivariate Normal Distribution Function and Related Functions; 1959.

12. VanVleck, J. H.: The Spectrum of Clipped Noise, Radio Research Lab., Harvard Univ.; July, 1943.

13. Davenport, W. B.: "Signal-to-Noise Ratios in Band-Pass Limiters," Jour. of Applied Physics, vol. 24, no. 6, pp. 720-727; 1953.

14. McFadden, J. A.: "The Correlation Function of a Sine Wave Plus Noise after Extreme Clipping," IRE Trans. on Information Theory, vol. IT-2, pp. 82-83; June 1956.

15. Blachman, N. M.: "The output Signal-to-Noise Ratio of a Power-Law Device," Jour. Applied Physics, vol. 24, pp. 783-785; June, 1953.

16. Jones, J. J.: Op. Cit., (2).

17. Rubin, W. L. and S. K. Kamen: Op. Cit., (3).

18. Cahn, C. R.: Op. Cit., (4).

19. Manasse, R., R. Price, and R. M. Lerner: "Loss of Signal Detectability in Band-Pass Limiters," IRE Trans. on Information Theory, IT-4, pp. 34-38; March, 1958.

20. Faran, J. J. and R. Hills: "Correlators for Signal Reception," Technical Memorandum No. 27, Acoustics Research Lab., Harvard Univ., pp. 58-65; September, 1952.

21. Wolff, S. S., J. B. Thomas, and T. R. Willians: "The Polarity-Coincidence Correlator: A nonparametric Detection Device," IRE Trans. on Information Theory, vol. IT-8, pp. 5-9; January, 1962.

22. Ekre, H.: "Polarity-Coincidence Correlation Detection of a Weak Noise Source," IEEE Trans. On Information Theory, vol. IT-9, pp. 18-23; January, 1963.

CHAPTER II

THE POLARITY-COINCIDENCE DETECTOR WITH BIAS

It is impossible to construct a perfect limiter or polarity indicator. Bias exists in all physical devices, however refined their design and construction may be. The sources of bias in an actual device may be varied. For example, in an electronic polarity indicator employing tubes or transistors, the bias may arise from contact potentials, junction leakage currents, operating point drift, etc.

Since bias is inevitable, it is desirable to determine its effect on the polarity-coincidence detector. To that end, this thesis presents the investigation of the effects of bias for certain types of inputs and with certain assumptions to be described in the subsequent development.

First, however, functional and mathematical descriptions of the biased polarity-coincidence detector (BPCD) will be given.

A biased polarity indicator is a two-level digitizer which divides the input function domain into two regions, $x > b$ and $x < b$, where b is the bias value, and which assigns the output value $y = +1$ to the first region and the output value $y = -1$ to the second region. An ideal limiter or (unbiased) polarity indicator is then a special case of the biased polarity indicator, with zero bias.

A BPCD is a polarity-coincidence detector whose polarity indicators have (not necessarily equal) bias. Thus a BPCD is a device which operates on two inputs, $x_1(t)$ and $x_2(t)$. First it digitizes the two inputs at two levels. This is accomplished by a pair of biased polarity indicators with bias values b_1 and b_2. The outputs of the polarity indicators, y_1 and y_2, are then multiplied. The product, u, can have only the values $+1$ and -1, since y_1 and y_2 have only the values $+1$ and -1.* Finally, the product, $u(t)$, is integrated for a finite length of time, T. The output of the integrator is a real number, z. Since the integrand, $u(t)$, has the property $|u| = 1$, then the output, z, has the property $|z| \leq T$.

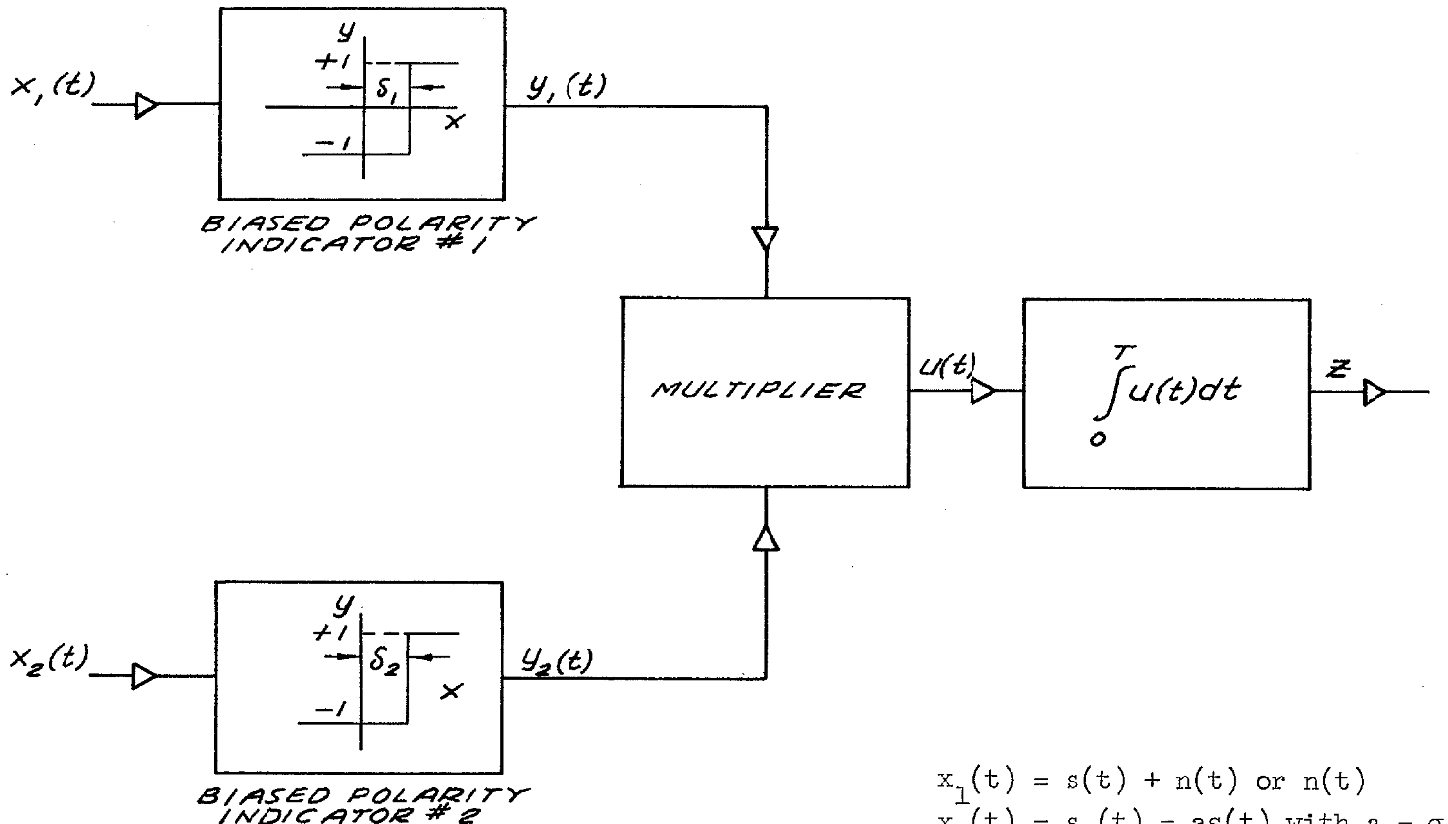
A block diagram of the BPCD is given in Figure 2.1. The transfer functions for the biased polarity indicators are shown inside the blocks representing them. The transfer functions are:

$$y_i = \begin{cases} +1 & \text{for } x_i > b_i \\ 0 & \text{for } x_i = b_i \\ -1 & \text{for } x_i < b_i \end{cases}$$

where $i = 1$ or 2.

---

* The output of the polarity indicators when the input equals b has been ignored, since the input equals b with probability zero for all inputs considered in this thesis.



$x_1(t) = s(t) + n(t)$ or $n(t)$
$x_2(t) = s_o(t) = as(t)$ with $a = \sigma_o/\sigma_s$

Block Diagram of Biased Polarity-Coincidence Detector
Figure 2.1

In the remainder of this thesis, it will be assumed that:

$x_1(t) = s(t) + n(t)$ ("signal present" case), where s and n are the signal and noise with variances $\sigma_s{}^2$ and $\sigma_n{}^2$ respectively, or that

$x_1(t) = n(t)$ ("signal absent" case), and that

$x_2(t) = s_o(t) = a \cdot s(t)$, where $s_o(t)$ is a locally available replica of $s(t)$, $a = \sigma_o/\sigma_s$, $\sigma_o{}^2$ is the variance for s_o and $\sigma_s{}^2$ is the variance for s.

The function $x_1(t)$ is the received stimulus and $x_2(t)$ is the local reference signal.

The output, z, is related to the outputs of the biased polarity indicators by the equation

$$z = \int_0^T u(t)dt = \int_0^T y_1(t)y_2(t)dt \qquad (2.1)$$

and the square of the output, z^2, which will also be required for subsequent developments, is

$$z^2 = \int_0^T\int_0^T u(t)u(\theta)dtd\theta = \int_0^T\int_0^T y_1(t)y_2(t)y_1(\theta)y_2(\theta)dtd\theta \quad . \qquad (2.2)$$

Expressions for the mean of z are derived in Chapter III and expressions for the variance of z are derived in Chapter IV. Expressions for the mean, variance and signal-to-noise power ratio for the output of an ideal correlation detector (no limiting) are derived in Chapter V, for purposes of comparison. Then expressions for the post-detection or output signal-to-noise power ratio are presented for the (unbiased) polarity-coincidence detector and for the biased polarity-coincidence detector. Finally, using the post-detection or output signal-to-noise power ratio as a criterion of performance, the biased polarity-coincidence detector, the (unbiased) polarity-coincidence detector and the correlation detector are compared. A summary of results and conclusions is presented in Chapter VI.

CHAPTER III

THE MEAN VALUE OF THE OUTPUT OF A BIASED POLARITY-COINCIDENCE DETECTOR

In this chapter expressions are derived for the mean value of the BPCD output. First, general expressions will be derived and then they will be specialized for Gaussian signal and noise inputs.

Before proceeding with the development, a number of assumptions will be stipulated which will be used throughout this chapter and, in fact, will be assumed to apply throughout the remainder of this thesis except where specific statements to the contrary are made.

Assumptions

(A1) The inputs are $x_1(t) = s(t) + n(t)$ ("signal present" case) or $x_1(t) = n(t)$ ("signal absent" case), and $x_2(t) = s_o(t)$.

(A2) $n(t)$ and $s(t)$ are sample functions from wide-sense stationary random processes with variances σ_n^2 and σ_s^2 .

(A3) $\Pr\{s(t) + n(t) = b_1\} = 0$ and $\Pr\{s_o(t) = b_2\} = 0$.

(A4) $s_o(t)$ is a locally available replica of $s(t)$. It is identical to $s(t)$ except possibly for amplitude; $s_o(t) = a \cdot s(t)$ where $a = \sigma_o/\sigma_s$ and σ_o^2 is the variance of s_o .

(A5) The $s(t)$ and $n(t)$ (and hence the $s_o(t)$ and $n(t)$) processes have statistically independent first order distributions.

(A6) The density function for $n(t)$ is even in n .

(Thus $E\{n\} = 0$.)

(A7) The density function for $s(t)$ (and hence for $s_o(t)$) is even in s (and s_o). (Thus $E\{s\} = E\{s_o\} = 0$.)

Additional assumptions concerning n, s and s_o will be made at various stages later in the development. The assumptions given above are consistent with all additional assumptions which are to be made.

3.0 General Expressions for the Mean Value of z.

From equation 2.1, the output of the BPCD is

$$z = \int_0^T u(t)dt = \int_0^T y_1(t)y_2(t)dt \quad .$$

The mean or expected value of z is then

$$\mu_z = E\{z\} = E\left\{\int_0^T u(t)dt\right\} = \int_0^T E\{u(t)\}dt$$
$$= \int_0^T E\{y_1(t)y_2(t)\}dt \quad .$$

By assumption A2, $E\{y_1(t)y_2(t)\}$ is time independent* and can be written $E\{y_1y_2\}$. Thus the mean of z is

---

* By assumption A2, s and n have first and second order time independent statistics. Thus x_1 and x_2 and hence y_1 and y_2 also have first and second order time independent statistics.

$$\mu_z = \int_0^T E\{y_1y_2\}dt = T\cdot E\{y_1y_2\} \quad . \tag{3.0-1}$$

$E\{y_1y_2\}$ is evaluated as follows: $y_1y_2 = +1$ when $x_1(t) > b_1$ and $x_2(t) > b_2$ or when $x_1(t) < b_1$ and $x_2(t) < b_2$. Similarly, $y_1y_2 = -1$ when $x_1(t) > b_1$ and $x_2(t) < b_2$ or when $x_1(t) < b_1$ and $x_2(t) > b_2$. Also, $y_1y_2 = 0$ when $x_1(t) = b_1$ or when $x_2(t) = b_2$. Upon introducing assumption A1, these conditions become: $y_1y_2 = +1$ when $s(t) + n(t) > b_1$ and $s_o(t) > b_2$ or when $s(t) + n(t) < b_1$ and $s_o(t) < b_2$. Similarly, $y_1y_2 = -1$ when $s(t) + n(t) > b_1$ and $s_o(t) < b_2$ or when $s(t) + n(t) < b_1$ and $s_o(t) > b_2$. Also, $y_1y_2 = 0$ when $s(t) + n(t) = b_1$ or when $s_o(t) = b_2$.

$E\{y_1y_2\}$ is then written, on introducing assumption A3, as

$$E\{y_1y_2\} = Pr\{s+n > b_1, s_o > b_2\} + Pr\{s+n < b_1, s_o < b_2\}$$

$$- Pr\{s+n > b_1, s_o < b_2\} - Pr\{s+n < b_1, s_o > b_2\} \quad .$$

Next, s is written in terms of s_o by introducing assumption A4. $s = s_o/a$, where $a = \sigma_o/\sigma_s$. Substituting this in the above equation for $E\{y_1y_2\}$ and also substituting the abbreviation $b_3 = b_1 - s_o/a$ yields

$$E\{y_1y_2\} = Pr\{n > b_3, s_o > b_2\} + Pr\{n < b_3, s_o < b_2\}$$

$$- Pr\{n > b_3, s_o < b_2\} - Pr\{n < b_3, s_o > b_2\} \quad .$$

Replacing the right hand side by the corresponding probability integrals yields

$$E\{y_1 y_2\} = \int_{b_2}^{\infty} ds_o \int_{b_3}^{\infty} g(n,s_o)dn - \int_{b_2}^{\infty} ds_o \int_{-\infty}^{b_3} g(n,s_o)dn$$

$$+ \int_{-\infty}^{b_2} ds_o \int_{-\infty}^{b_3} g(n,s_o)dn - \int_{-\infty}^{b_2} ds_o \int_{b_3}^{\infty} g(n,s_o)dn$$

where $g(n,s_o)$ is the joint probability density function for n and s_o. By assumption A5, $n(t)$ and $s(t)$ (and hence $s_o(t)$) are sample functions from random processes with statistically independent first order distributions, so $g(n,s_o)$ can be written $g(n,s_o) = f(n)h(s_o)$ where $f(n)$ is the marginal density for n and $h(s_o)$ is the marginal density for s_o. The mean, written in terms of $f(n)$ and $h(s_o)$, according to equation 3.0-1, is

$$\mu_z = T\cdot E\{y_1 y_2\}$$

$$= T\left[\int_{b_2}^{\infty} \left\{ \int_{b_3}^{\infty} f(n)dn - \int_{-\infty}^{b_3} f(n)dn \right\} h(s_o)ds_o \right.$$

$$\left. + \int_{-\infty}^{b_2} \left\{ \int_{-\infty}^{b_3} f(n)dn - \int_{b_3}^{\infty} f(n)dn \right\} h(s_o)ds_o \right] .$$

The integrals inside the braces can be simplified by the introduction of assumption A6, as follows: Write

$$\int_{-\infty}^{b_3} f(n)dn - \int_{b_3}^{\infty} f(n)dn = \int_{-\infty}^{0} f(n)dn + \int_{0}^{b_3} f(n)dn$$

$$- \int_{0}^{\infty} f(n)dn + \int_{0}^{b_3} f(n)dn \quad .$$

Since by assumption A6 $f(n)$ is an even function, the first and third integrals on the right side cancel and the right side becomes

$$2 \int_{0}^{b_3} f(n)dn \quad .$$

Substituting this result into the preceding equation for μ_z yields

$$\mu_z = T \cdot E\{y_1 y_2\}$$

$$= 2T \left[\int_{-\infty}^{b_2} ds_o \int_{0}^{b_3} f(n)h(s_o)dn - \int_{b_2}^{\infty} ds_o \int_{0}^{b_3} f(n)h(s_o)dn \right] \quad .$$

Note that b_3 is a function of s_o .

Next, the noise and reference signals are normalized. The following notation is introduced in order to simplify the subsequent equations:

$t = n/\sigma_n$; $u(t) = \sigma_n f(\sigma_n t)$ is the density function for t, the normalized noise variate.*

$\lambda = s_o/\sigma_o$; $v(\lambda) = \sigma_o h(\sigma_o \lambda)$ is the density function for λ , the normalized reference signal.

$N = \sigma_s^2/\sigma_n^2 = \sigma_o^2/a^2\sigma_n^2$ is the input signal-to-noise power ratio.

$P = \sigma_s^2 + \sigma_n^2$ is the total input power.

$\delta_1 = b_1/\sqrt{P}$ is the normalized bias for the input polarity indicator.

$\delta_2 = b_2/\sigma_o$ is the normalized bias for the reference signal polarity indicator.

$\eta_1 = -\sqrt{N}\,\lambda$ is a limit of integration.

$\eta_2 = b_3/\sigma_n = (\sqrt{P}/\sigma_n)\delta_1 + \eta_1$ is another limit of integration.

Equivalent expressions are $\eta_2 = \sqrt{1+N}\,\delta_1 + \eta_1$ or $b_1/\sigma_n - \sigma_o\lambda/a\sigma_n$ or $b_1/\sigma_n - \sqrt{N}\lambda$. Note that $\sqrt{P}/\sigma_n = \sqrt{1+N}$.

---

* Clearly, here, t is not the time and $u(t)$ is not the output of the multiplier as they were previously. Because of the limited number of symbols available, a certain amount of duplication will occur in the notation. The context should make the meaning clear in each case.

In terms of this new notation, the preceding equation for μ_z becomes

$$\mu_z = 2T \int_{-\infty}^{\delta_2} d\lambda \int_{0}^{\eta_2} u(t)v(\lambda)dt + 2T \int_{\delta_2}^{\infty} d\lambda \int_{\eta_2}^{0} u(t)v(\lambda)dt \quad . \quad (3.0\text{-}2)$$

Note that η_2 is a function of λ. In the second term of the above equation, the order of the limits on the second integral has been reversed. To compensate for this, the sign preceding the term has been made positive. This is done for future convenience.

The integrals in equation 3.0-2 will later be interpreted in terms of volumes under the surface $u(t)v(\lambda)$ over the regions in the λ, t-plane indicated by the limits of integration.

In order to evaluate these integrals, several sets of conditions on N, δ_1 and δ_2 will be considered. These are listed below.

<u>Case A. N = 0.</u> This is the "signal absent" case. $\sigma_s = 0$. It corresponds to the null hypothesis.

<u>Case B. N $\neq$ 0.</u> This is the "signal present" case. $\sigma_s \neq 0$. It corresponds to the alternative hypothesis.

B_1.) $\delta_1 = \delta_2 = 0$. This is the unbiased or ideal polarity indicator case.

B_2.) $\delta_1 = 0$, $\delta_2 \neq 0$. The input polarity indicator is ideal but the reference signal polarity indicator has bias (or the reference signal has a D.C. component).*

B_3.) $\delta_1 \neq 0$, $\delta_2 = 0$. The input polarity indicator has bias (or the input stimulus has a D.C. component)* but the reference signal polarity indicator is ideal.

B_4.) $\delta_1 \neq 0$, $\delta_2 \neq 0$. The polarity indicators both have bias (or both the input stimulus and the reference signal have D.C. components).*

Case C. $N \to \infty$. This is the noiseless case.

All the above cases, including cases A and C, can be obtained as limiting cases from case B_4. However, it is instructive to examine them individually.

Case A. $N = 0$

This is the case in which no signal is received. Tt will correspond to the null hypothesis in subsequent discussions of detection properties.

---

* It should be noted that a D.C. component in x_1 or x_2 is equivalent to a bias of equal magnitude and opposite sign in the corresponding polarity indicator. Thus the results of this thesis are applicable to systems with inputs and references having non-zero means as well as to systems with biased polarity indicators.

Since $N = 0$, then $\delta_1 = b_1/\sigma_n$, $\eta_1 = 0$ and $\eta_2 = \delta_1$. Denoting the mean for this case by μ_o , equation 3.0-2 becomes

$$\mu_o = 2T \int_{-\infty}^{\delta_2} d\lambda \int_0^{\delta_1} u(t)v(\lambda)dt + 2T \int_{\delta_2}^{\infty} d\lambda \int_{\delta_1}^{0} u(t)v(\lambda)dt \quad .$$

Since the limits of integration are now all constants (in equation 3.0-2 the limits on the inner integrals are functions of the variable of integration for the outer integrals), the order of integration can be interchanged directly. Thus

$$\mu_o = 2T \int_0^{\delta_1} \left\{ \int_{-\infty}^{\delta_2} v(\lambda)d\lambda - \int_{\delta_2}^{\infty} v(\lambda)d\lambda \right\} u(t)dt \quad .$$

The integrals inside the braces can be simplified by the introduction of assumption A7. Proceeding in a manner analogous to that used for simplifying the equation for μ_z on page 20, by the evenness of $v(\lambda)$ the quantity inside the braces becomes

$$2 \int_0^{\delta_2} v(\lambda)d\lambda \quad .$$

Substituting this result into the above equation for μ_o yields

$$\mu_o = 4T \int_0^{\delta_1} u(t)dt \int_0^{\delta_2} v(\lambda)d\lambda \quad . \qquad (3.0\text{-}3)$$

Note that since $u(t)$ and $v(\lambda)$ are even functions, then μ_o is odd in δ_1 and δ_2 individually but even in δ_1 and δ_2 simultaneously.

For an ideal (unbiased) polarity-coincidence detector the mean of the output should be zero when there is no signal present in $x_1(t)$, since in that case there is no component in $y_1(t)$ which is correlated with $y_2(t)$. Equation 3.0-3 gives the mean, μ_o, of the BPCD output when no signal is present. In equation 3.0-3 the values $\delta_1 = 0$ and $\delta_2 = 0$ correspond to an ideal polarity-coincidence detector and, for these values, the equation yields $\mu_o = 0$ as is required.

Equation 3.0-3 shows that the effect of bias is to cause the output to have a spurious non-zero mean when no signal is present. However, the non-zero mean occurs only if both polarity indicators have bias. If either one is ideal, then the mean is zero. This is an important point, because in some systems the reference signal polarity indicator can be regarded as ideal. For example, if a general purpose digital computer were to be programmed as a polarity-coincidence detector, the sign of the reference signal would be generated by the computer directly. To generate an analog reference signal and then pass it through a polarity indicator would be superfluous. In such a case, since there is no polarity indicator to introduce bias, the mean of the output would have the desirable property of being zero in the no signal case regardless of bias in the input channel.

<u>Case B. $N \neq 0$</u>

This is the case in which a signal is received. It will correspond to the alternative hypothesis in subsequent discussions of detection properties.

B_1.) $\delta_1 = \delta_2 = 0$.

This is the "signal present" case for an ideal polarity-coincidence detector. The results of this case will be used for comparison with the results of the non-ideal cases.

Since $\delta_1 = 0$, then $\eta_2 = \eta_1 = -\sqrt{N}\lambda$. Denoting the mean in this case by μ_1 , equation 3.0-2 becomes

$$\mu_1 = 2T \int_{-\infty}^{0} d\lambda \int_{0}^{\eta_1} u(t)v(\lambda)dt + 2T \int_{0}^{\infty} d\lambda \int_{\eta_1}^{0} u(t)v(\lambda)dt$$

$$= 2T(U_{R_1} + U_{R_2}) \quad , \qquad (3.0\text{-}4)$$

where U_A denotes the volume under the surface $u(t)v(\lambda)$ over the region A in the λ,t-plane. The regions R_1 and R_2 are shown in figure 3.0-1. Note that η_1 is a function of λ .

As in Case A, the preceding equation can be further simplified by introducing assumption A7. Since then both u and v are even functions in their respective arguments, it follows that uv is even in t and λ simultaneously and hence is symmetric about the origin in the λ,t-plane. An inspection of figure 3.0-1 reveals that the regions R_1 and R_2 are mirror images in the origin. Therefore, the first and second integrals of the above equation are equal and

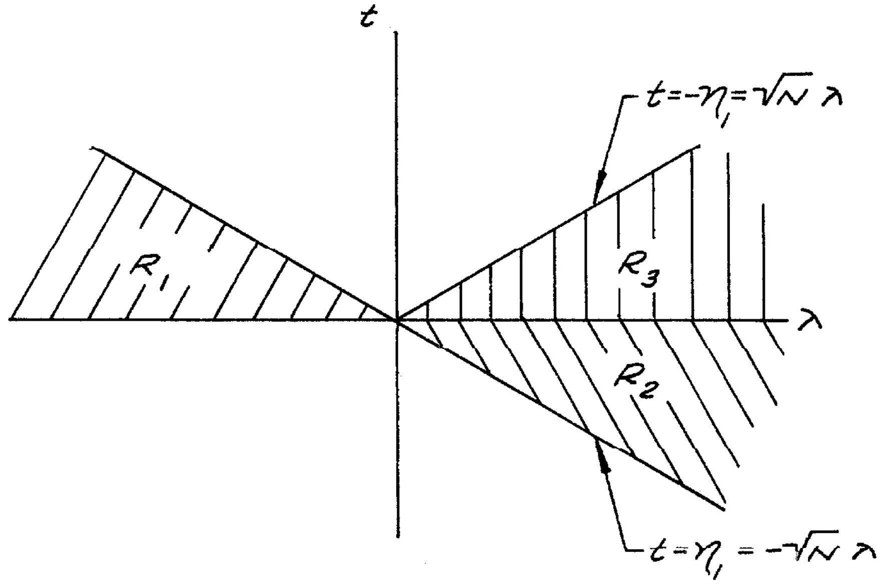


Figure 3.0-1

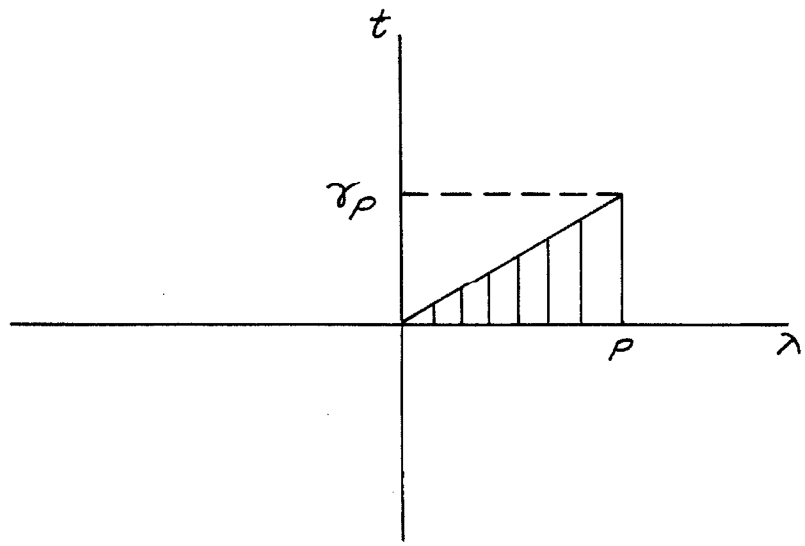


Figure 3.0-2

$$\mu_1 = 4T \cdot U_{R_2} \quad . \tag{3.0-5}$$

An equivalent expression follows from the evenness of $u(t)$.

$$\mu_1 = 4T \cdot U_{R_3} \quad , \tag{3.0-6}$$

where R_3 also is shown in figure 3.0-1.

In the subsequent development, integrals representing the volume under the surface $u(t)v(\lambda)$ over particular right triangular regions of finite extent will occur frequently. Therefore, a notation for such volumes will now be introduced.

Let $U(p,\gamma p)$ denote the volume under the surface $u(t)v(\lambda)$ over the right triangle with base of length p along the λ-axis, with acute vertex at the origin and with altitude of height γp parallel to the t-axis, in the first quadrant of the λ,t-plane, as shown in figure 3.0-2. The integral representation for this volume is

$$U(p,\gamma p) = \int_0^p d\lambda \int_0^{\gamma\lambda} u(t)v(\lambda)dt \quad . \tag{3.0-7}$$

When u and v are even functions, U is even in p and odd in γ.

It is clear that the mean for Case B_1 as given by equation 3.0-6 can be expressed as a limiting value of $U(p,\gamma p)$.

$$\mu_1 = 4T \cdot \lim_{p\to\infty} U(p,\sqrt{N}\, p) \quad . \tag{3.0-8}$$

When $N = 0$, then $\mu_1 = 0$ according to equation 3.0-8. This is consistent with equation 3.0-3. Moreover, $\lim_{N\to\infty} \mu_1 = T$, the maximum possible mean value.

Cases A (with $\delta_1 = \delta_2 = 0$) and B_1 provide the mean value for an ideal polarity-coincidence detector. Case A (with δ_1 and/or $\delta_2 \neq 0$) and the following cases provide the mean value for non-ideal polarity-coincidence detectors.

B_2.) $\delta_1 = 0$, $\delta_2 \neq 0$.

This is the "signal present" case with an ideal input polarity indicator but with a biased reference signal polarity indicator.*
Since $\delta_1 = 0$, then $\eta_2 = \eta_1 = -\sqrt{N}\,\lambda$. Denoting the mean in this case by μ_2 , equation 3.0-2 becomes

$$\mu_2 = 2T \int_{-\infty}^{\delta_2} d\lambda \int_0^{\eta_1} u(t)v(\lambda)dt + 2T \int_{\delta_2}^{\infty} d\lambda \int_{\eta_1}^{0} u(t)v(\lambda)dt$$

which can be written

$$\mu_2 = 2T\left[\int_{-\infty}^{0} d\lambda \int_0^{\eta_1} u(t)v(\lambda)dt + \int_0^{\delta_2} d\lambda \int_0^{\eta_1} u(t)v(\lambda)dt \right.$$

$$\left. + \int_{\delta_2}^{0} d\lambda \int_{\eta_1}^{0} u(t)v(\lambda)dt + \int_0^{\infty} d\lambda \int_{\eta_1}^{0} u(t)v(\lambda)dt \right] \quad .$$

---

* See footnote on page 23.

The first and last integrals are those which appear in equation 3.0-4 for μ_1. Clearly, the second and third integrals are equal. Thus

$$\mu_2 = \mu_1 + 4T \int_0^{\delta_2} d\lambda \int_0^{\eta_1} u(t)v(\lambda)dt \quad . \tag{3.0-9}$$

By assumption A6, this can be written

$$\mu_2 = \mu_1 - 4T \int_0^{\delta_2} d\lambda \int_0^{-\eta_1} u(t)v(\lambda)dt \quad . \tag{3.0-10}$$

Finally, by equation 3.0-7, substituting $-\eta_1 = \sqrt{N}\,\lambda$,

$$\mu_2 = \mu_1 - 4T\cdot U(\delta_2, \sqrt{N}\ \delta_2) = \mu_1 + e_2 \tag{3.0-11}$$

where $e_2 = -4T\cdot U(\delta_2, \sqrt{N}\ \delta_2)$. The mean μ_2 is even in δ_2, due to the evenness of u and v.

Equation 3.0-11 shows that the mean for the non-ideal case with $\delta_1 = 0$ but with $\delta_2 \neq 0$ is the mean for the ideal case plus an error term, e_2. The error term is always negative and is even in δ_2. The error term becomes zero for $\delta_2 = 0$ and for $N = 0$. This is consistent with the results of Cases A and B_1.

The relative error is a useful indication of how serious the error is. The relative error in the mean, μ_z, of a non-ideal system is defined as

$$\rho_z = \frac{\mu_z - \mu_1}{\mu_1} \quad . \tag{3.0-12}$$

where μ_1 is the mean for the ideal system.

For the present case, the relative error, ρ_2, is

$$\rho_2 = \frac{e_2}{\mu_1} = - \frac{U(\delta_2, \sqrt{N}\ \delta_2)}{\lim\limits_{p \to \infty} U(p, \sqrt{N}\ p)} \tag{3.0-13}$$

ρ_2 is even in δ_2, due to the evenness of u and v. Thus the error depends only on the magnitude of the bias in the reference signal polarity indicator and not on the sign of the bias.

B_3.) $\delta_1 \neq 0$, $\delta_2 = 0$.

This is the "signal present" case with an ideal reference signal polarity indicator but with a biased input polarity indicator.*
Since $\delta_2 = 0$, equation 3.0-2 becomes

$$\mu_3 = 2T \int_{-\infty}^{0} d\lambda \int_{0}^{\eta_2} u(t)v(\lambda)dt + 2T \int_{0}^{\infty} d\lambda \int_{\eta_2}^{0} u(t)v(\lambda)dt \quad , \tag{3.0-14}$$

where μ_3 is the mean for Case B_3. Upon making the transformation $\lambda \to -\lambda$ and introducing the evenness assumptions A6 and A7, the first integral of equation 3.0-14 becomes

---

* See footnote on page 23.

$$2T \int_0^\infty d\lambda \int_0^{\eta_3} u(t)v(\lambda)dt$$

where $\eta_3 = (\sqrt{P}/\sigma_n)\delta_1 - \eta_1$. Substituting this into equation 3.0-14 yields

$$\mu_3 = 2T \int_0^\infty \left\{ \int_0^{\eta_3} u(t)dt + \int_{\eta_2}^0 u(t)dt \right\} v(\lambda)d\lambda$$

$$= 2T \int_0^\infty \int_{\eta_2}^{\eta_3} u(t)v(\lambda)dtd\lambda \quad . \qquad (3.0\text{-}15)$$

The limits on the inner integral are $\eta_2 = (\sqrt{P}/\sigma_n)\delta_1 + \eta_1$ and $\eta_3 = (\sqrt{P}/\sigma_n)\delta_1 - \eta_1$. Since $u(t)$ is even, then the inner integral is even in δ_1. Therefore, μ_3 is even in δ_1.

A form which does not reveal the evenness in μ_3 so obviously but which is more convenient for subsequent developments follows: Equation 3.0-14 can be rewritten

$$\mu_3 = 2T\left[\int_{-\infty}^0 d\lambda \int_0^{\eta_1} u(t)v(\lambda)dt + \int_{-\infty}^0 d\lambda \int_{\eta_1}^{\eta_2} u(t)v(\lambda)dt\right.$$

$$\left. + \int_0^\infty d\lambda \int_{\eta_2}^{\eta_1} u(t)v(\lambda)dt + \int_0^\infty d\lambda \int_{\eta_1}^0 u(t)v(\lambda)dt\right] \quad .$$

The first and last integrals are those which appear in equation 3.0-4 for μ_1. Thus

$$\mu_3 = \mu_1 + 2T\left[\int_{-\infty}^{0} d\lambda \int_{\eta_1}^{\eta_2} u(t)v(\lambda)dt - \int_{0}^{\infty} d\lambda \int_{\eta_1}^{\eta_2} u(t)v(\lambda)dt\right] \quad .$$

$$= \mu_1 + e_3 \tag{3.0-16}$$

where

$$e_3 = 2T\left[\int_{-\infty}^{0} d\lambda \int_{\eta_1}^{\eta_2} u(t)v(\lambda)dt - \int_{0}^{\infty} d\lambda \int_{\eta_1}^{\eta_2} u(t)v(\lambda)dt\right] \quad .$$

Note that η_1 and η_2 are functions of λ.

Equation 3.0-16 shows that the mean for the non-ideal case with $\delta_2 = 0$ but with $\delta_1 \neq 0$ is the mean for the ideal case plus an error term, e_3. Since μ_3 is even in δ_1 and since μ_1 is independent of δ_1, then e_3 is even in δ_1. The error in μ_3 depends only on the magnitude of the bias in the input polarity indicator and not on the sign of the bias.

The relative error is, according to equation 3.0-12,

$$\rho_3 = e_3/\mu_1 \quad . \tag{3.0-17}$$

Since e_3 is even in δ_1, then ρ_3 is even in δ_1.

<u>B_4.) $\delta_1 \neq 0$, $\delta_2 \neq 0$.</u>

This is the "signal present" case with both biased input and biased reference signal polarity indicators.* Equation 3.0-2 can be rewritten

$$\mu_4 = 2T\left[\int_{-\infty}^{0} d\lambda \int_{0}^{\eta_2} + \int_{0}^{\delta_2} d\lambda \int_{0}^{\eta_2} + \int_{\delta_2}^{0} d\lambda \int_{\eta_2}^{0} + \int_{0}^{\infty} d\lambda \int_{\eta_2}^{0}\right] u(t)v(\lambda)dt$$

which in turn can be written

$$\mu_4 = 2T\left[\int_{-\infty}^{0} d\lambda \int_{0}^{\eta_1} + \int_{-\infty}^{0} d\lambda \int_{\eta_1}^{\eta_2} + \int_{0}^{\delta_2} d\lambda \int_{0}^{\eta_1} + \int_{0}^{\delta_2} d\lambda \int_{\eta_1}^{\eta_2} + \int_{\delta_2}^{0} d\lambda \int_{\eta_2}^{\eta_1}\right.$$

$$\left. + \int_{\delta_2}^{0} d\lambda \int_{\eta_1}^{0} + \int_{0}^{\infty} d\lambda \int_{\eta_2}^{\eta_1} + \int_{0}^{\infty} d\lambda \int_{\eta_1}^{0}\right] u(t)v(\lambda)dt$$

where μ_4 is the mean for Case B_4. The first and last integrals are those which appear in equation 3.0-4 for μ_1. The third and sixth integrals are those which appear in the error term part of equation 3.0-9 for μ_2. Thus their sum equals $e_2 = \mu_2 - \mu_1$. The second and seventh integrals are those which appear in the error term part of equation 3.0-16 for μ_3. Thus their sum equals $e_3 = \mu_3 - \mu_1$. Therefore,

---

* See footnote on page 23.

$$\mu_4 = \mu_1 + e_2 + e_3 + 4T \cdot I(\delta_1, \delta_2)$$

$$= \mu_2 + \mu_3 - \mu_1 + 4T \cdot I(\delta_1, \delta_2)$$

$$= \mu_1 + e_4 \qquad (3.0\text{-}18)$$

where

$$e_4 = e_2 + e_3 + 4T \cdot I(\delta_1, \delta_2)$$

and

$$I(\delta_1, \delta_2) = \int_0^{\delta_2} d\lambda \int_{\eta_1}^{\eta_2} u(t)v(\lambda)dt \quad .$$

Equation 3.0-18 shows that the mean for the most general non-ideal case with $\delta_1 \neq 0$ and $\delta_2 \neq 0$ is the mean for the ideal case plus an error term, e_4. The error term, e_4, consists of a part which depends on δ_1 only and is identical to the error term when $\delta_1 \neq 0$ but $\delta_2 = 0$, a part which depends on δ_2 only and is identical to the error term when $\delta_2 \neq 0$ but $\delta_1 = 0$, and a part which depends on both δ_1 and δ_2. The mean, μ_4, and the error, e_4, are even in δ_1 and δ_2 simultaneously (i.e. $\mu_4(\delta_1,\delta_2) = \mu_4(-\delta_1,-\delta_2)$) but not in δ_1 and δ_2 individually.

The relative error is, according to equation 3.0-12,

$$\rho_4 = e_4/\mu_1 \quad . \tag{3.0-19}$$

Since e_4 is even in δ_1 and δ_2 simultaneously, then ρ_4 is also.

Other Cases as Limiting Cases of Case B_4

Case A. $N = 0$

In this case, $\eta_2 = \delta_1$ and $\eta_1 = 0$. Therefore, $\mu_1 = 0$, $\mu_2 = 0$ and $\mu_3 = 0$. Thus, from equation 3.0-18,

$$\mu_4 = 4T\int_0^{\delta_2} d\lambda \int_0^{\delta_1} u(t)v(\lambda)dt = \mu_o, \text{ as it should.}$$

Case B. $N \neq 0$.

B$_1$.) $\delta_1 = \delta_2 = 0$. In this case, $\eta_2 = \eta_1$ and $\mu_2 = \mu_3 = \mu_1$. Also, $I(\delta_1, \delta_2) = 0$. Thus, from equation 3.0-18, $\mu_4 = \mu_1$, as it should.

B$_2$.) $\delta_1 = 0$, $\delta_2 \neq 0$. In this case, $\eta_2 = \eta_1$ so that $\mu_3 = \mu_1$ and $I(\delta_1, \delta_2) = 0$. Thus, from equation 3.0-18, $\mu_4 = \mu_2$ as it should.

B$_3$.) $\delta_1 \neq 0$, $\delta_2 = 0$. In this case, $\mu_2 = \mu_1$ and $I(\delta_1, \delta_2) = 0$. Therefore, from equation 3.0-18, $\mu_4 = \mu_3$ as it should.

Case C. $N \to \infty$.

In deriving the expressions for Cases A and B, the noise variate, n, was normalized by dividing it by σ_n. For the present case, it will be necessary to use some other normalization, because this is the noiseless case and $N \to \infty$ may correspond to $\sigma_n \to 0$. A new variate, n_o, will now be introduced which is related to n in the same way that s_o is related to s. In the probability statements on page 18, the expressions $s = (\sigma_s/\sigma_o)s_o$ and $n = (\sigma_n/\sigma_o)n_o$ are substituted for s and n. n_o is a noise variate identical to n except possibly for amplitude. It has variance σ_o^2, the same as s_o. The probability relations become

$$E\{y_1 y_2\} = \Pr\{x_o > b_1,\ s_o > b_2\} + \Pr\{x_o < b_1,\ s_o < b_2\}$$

$$- \Pr\{x_o > b_1,\ s_o < b_2\} - \Pr\{x_o < b_1,\ s_o > b_2\}$$

where $x_o = (\sigma_s s_o + \sigma_n n_o)/\sigma_o$. $E\{y_1 y_2\}$ can be written

$$E\{y_1 y_2\} = \Pr\{s_o > a \cdot b_1 - n_o/\sqrt{N},\ s_o > b_2\}$$

$$+ \Pr\{s_o < a \cdot b_1 - n_o/\sqrt{N},\ s_o < b_2\}$$

$$- \Pr\{s_o > a \cdot b_1 - n_o/\sqrt{N},\ s_o < b_2\}$$

$$- \Pr\{s_o < a \cdot b_1 - n_o/\sqrt{N},\ s_o > b_2\}$$

In the limit as $N \to \infty$,

$$E\{y_1 y_2\} = \Pr\{s_o > a \cdot b_1,\ s_o > b_2\} + \Pr\{s_o < a \cdot b_1,\ s_o < b_2\}$$

$$- \Pr\{s_o > a \cdot b_1,\ s_o < b_2\} - \Pr\{s_o < a \cdot b_1,\ s_o > b_2\}$$

When $a \cdot b_1 \geq b_2 (\delta_1 \geq \delta_2,$ since $\lim_{N \to \infty} \delta_1 = \frac{b_1}{\sigma_s}$), then

$$\mu_\infty = T\left[\int_{a \cdot b_1}^{\infty} h(s_o) ds_o + \int_{-\infty}^{b_2} h(s_o) ds_o - \int_{b_2}^{a \cdot b_1} h(s_o) ds_o\right] \quad .$$

The preceding equation can be written

$$\mu_\infty = T\left[1 - 2 \int_{b_2}^{a \cdot b_1} h(s_o) ds_o\right] \quad .$$

On substituting $\lambda = s_o / \sigma_o$, this becomes

$$\mu_\infty = T\left[1 - 2 \int_{\delta_2}^{b_1/\sigma_s} v(\lambda) d\lambda\right] \quad .$$

But as $N \to \infty$, $\delta_1 = b_1/\sqrt{P} \to b_1/\sigma_s$. Therefore,

$$\mu_\infty = T\left[1 - 2 \int_{\delta_2}^{\delta_1} v(\lambda) d\lambda\right] \quad .$$

When $a \cdot b_1 \leq b_2$, $(\delta_1 \leq \delta_2)$, a similar argument leads to the result

$$\mu_\infty = T\left[1 - 2\int_{\delta_1}^{\delta_2} v(\lambda)d\lambda\right] .$$

Finally, then,

$$\mu_\infty = T\left[1 - 2\int_{\min(\delta_1,\delta_2)}^{\max(\delta_1,\delta_2)} v(\lambda)d\lambda\right] . \qquad (3.0\text{-}20)$$

This result is valid whether $N \to \infty$ due to $\sigma_n \to 0$ or to $\sigma_s \to \infty$. However, if $\sigma_s \to \infty$, then $\delta_1 \to 0$ and the limits of integration become $\min(0,\delta_2)$ and $\max(0,\delta_2)$. This result could have been derived as a limiting case from case B_4 but the result would not have revealed the behavior when $N \to \infty$ due to $\sigma_n \to 0$.

From equation 3.0-20, it is clear that in the ideal case where $\delta_1 = \delta_2 = 0$, the mean value is $\mu_{1\infty} = T$. This value is the maximum possible value for the output z and hence is the maximum possible value for the mean of z.

The expressions derived above for the BPCD output mean value show that whenever a signal is present in the input, then bias in either channel alone or in both channels simultaneously causes a spurious component to be present in the output mean value. If no signal is present in the input, then bias in either channel alone will not cause the presence of a spurious component in the mean - in the absence of a signal, a spurious component in the mean occurs only if there is bias in both the input and reference channels simultaneously.

Since the output mean value is the output "signal", the presence of a spurious component increases the chance of making an error in deciding if a signal is present or not. Therefore, the presence of a spurious component in the output mean value has a degrading effect on the BPCD. For a more complete discussion, see Chapter V.

The various forms of the mean of the BPCD output derived above will be used in Section 3.1 to obtain expressions for the output mean value when the noise in the input channel is Gaussian.

3.1 The Mean Value When the Noise is Gaussian.

In addition to the assumptions introduced previously (assumptions A1 through A7), it will now be assumed that the noise, $n(t)$, is a sample function from a stationary Gaussian random process with zero mean and with variance σ_n^2. This assumption will be in force for the remainder of Chapter III.

(A3-1) $n(t)$ is a sample function from a stationary random process with density function

$$f(n) = \frac{1}{\sqrt{2\pi}\sigma_n} \exp(n^2/2\sigma_n^2) \quad .$$

Upon making the transformation $t = n/\sigma_n$, the density function for t becomes in this case

$$u(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} \quad .$$

When this expression for $u(t)$ is substituted in the equations of section 3.0, the following results are obtained: From equation 3.0-2,

$$\mu_z = 2T \int_{-\infty}^{\delta_2} \left\{ \int_0^{\eta_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt \right\} v(\lambda) d\lambda - 2T \int_{\delta_2}^{\infty} \left\{ \int_0^{\eta_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt \right\} v(\lambda) d\lambda .$$

But

$$\int_0^{\eta_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt = \tfrac{1}{2} \, \mathrm{Erf}(\eta_2/\sqrt{2}) \qquad ,$$

where

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy \qquad .$$

Thus

$$\mu_z = T \int_{-\infty}^{\delta_2} v(\lambda) \mathrm{Erf}(\eta_2/\sqrt{2}) d\lambda - T \int_{\delta_2}^{\infty} v(\lambda) \mathrm{Erf}(\eta_2/\sqrt{2}) d\lambda \qquad . \quad (3.1\text{-}1)$$

Note that η_2 is a function of λ.

Case A. $N = 0$

From equation 3.0-3,

$$\mu_o = 4T \int_0^{\delta_2} v(\lambda) d\lambda \int_0^{\delta_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$$

$$= 2T \cdot \mathrm{Erf}(\delta_1/\sqrt{2}) \int_0^{\delta_2} v(\lambda) d\lambda \qquad . \qquad (3.1\text{-}2)$$

Case B. $N \neq 0$

B_1.) $\delta_1 = \delta_2 = 0$. In this case, $\eta_2 = \eta_1$. Thus, from equation 3.1-1,

$$\mu_1 = T \int_{-\infty}^{0} v(\lambda) \mathrm{Erf}(\eta_1/\sqrt{2}) d\lambda - T \int_{0}^{\infty} v(\lambda) \mathrm{Erf}(\eta_1/\sqrt{2}) d\lambda \quad . \qquad (3.1\text{-}3)$$

Note that η_1 is a function of λ.

At this point it will again be convenient to interpret the preceding integrals as volumes under the surface $u(t)v(\lambda)$ where $u(t)$ is now defined by assumption A3-1.

Let $W(p,\gamma p)$ denote the volume under the surface $v(\lambda)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}$ over the right triangle with base of length p along the λ-axis, with acute vertex at the origin and with altitude of height γp parallel to the t-axis, in the first quadrant of the λ,t-plane, as shown in figure 3.0-2. The integral representation for this volume is

$$W(p,\gamma p) = \int_0^p d\lambda \int_0^{\gamma\lambda} v(\lambda) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$$

$$= \frac{1}{2}\int_0^p v(\lambda) \mathrm{Erf}(\gamma\lambda/\sqrt{2}) d\lambda \quad . \qquad (3.1\text{-}4)$$

When v is an even function, W is even in p and odd in γ.

From equation 3.0-8 it is clear that the mean for Case B_1 with Gaussian noise can be expressed as a limiting value of $W(p,\gamma p)$.

$$\mu_1 = 4T\cdot\lim_{p\to\infty} W(p,\sqrt{N}\,p) \quad . \tag{3.1-5}$$

<u>B_2.) $\delta_1 = 0$, $\delta_2 \neq 0$.</u> Equation 3.0-11 continues to be valid:

$$\mu_2 = \mu_1 + e_2 \quad , \tag{3.1-6}$$

where e_2 is now

$$e_2 = -4T\cdot W(\delta_2,\sqrt{N}\,\delta_2) \quad .$$

From equation 3.0-13, the relative error is

$$\rho_2 = e_2/\mu_1 = -\frac{W(\delta_2,\sqrt{N}\,\delta_2)}{\lim_{p\to\infty} W(p,\sqrt{N}\,p)} \tag{3.1-7}$$

<u>B_3.) $\delta_1 \neq 0$, $\delta_2 = 0$.</u> Equation 3.0-16 continues to be valid:

$$\mu_3 = \mu_1 + e_3 \quad , \tag{3.1-8}$$

where e_3 is now

$$e_3 = T \int_{-\infty}^{0} v(\lambda) \left[\mathrm{Erf}(\eta_2/\sqrt{2}) - \mathrm{Erf}(\eta_1/\sqrt{2}) \right] d\lambda$$

$$- T \int_{0}^{\infty} v(\lambda) \left[\mathrm{Erf}(\eta_2/\sqrt{2}) - \mathrm{Erf}(\eta_1/\sqrt{2}) \right] d\lambda \quad .$$

From equation 3.0-17, the relative error is

$$\rho_3 = e_3/\mu_1 \tag{3.1-9}$$

where e_3 and μ_1 are the appropriate expressions for the Gaussian noise case, as just given.

<u>B_4.) $\delta_1 \neq 0$, $\delta_2 \neq 0$.</u> Equation 3.0-18 continues to be valid:

$$\mu_4 = \mu_1 + e_4 \quad , \tag{3.1-10}$$

where $e_4 = e_2 + e_3 + 4T \cdot I(\delta_1, \delta_2)$ and $I(\delta_1, \delta_2)$ is now

$$I(\delta_1, \delta_2) = \tfrac{1}{2} \int_{0}^{\delta_2} v(\lambda) \left[\mathrm{Erf}(\eta_2/\sqrt{2}) - \mathrm{Erf}(\eta_1/\sqrt{2}) \right] d\lambda \quad .$$

As before, the relative error is $\rho_4 = e_4/\mu_1$ but with the expressions for e_4 and μ_1 appropriate to the Gaussian noise case.

Case C. $N \to \infty$

Since this is the zero noise case (or the infinite signal in finite noise case), the form of the noise distribution has no effect on the mean value of the system output. Thus equation 3.0-20 is unchanged.

$$\mu_\infty = T\left[1 - 2 \int_{\min(\delta_1,\delta_2)}^{\max(\delta_1,\delta_2)} v(\lambda)d\lambda\right] \quad . \qquad (3.1\text{-}11)$$

This result can be derived directly from equation 3.0-20, since the distribution of the noise variate does not appear in that equation either explicitly or implicitly.

The properties attributed to the mean and relative error in the various cases of section 3.0 (evenness in δ_1 or δ_2, etc.) have not been affected by the assumption of Gaussian noise. Therefore a discussion of these properties has not been repeated here.

The expressions derived above for the mean value of the BPCD output will be used in the next section to obtain expressions for the output mean value when both the signal and noise are Gaussian.

3.2 The Mean Value When the Signal and Noise Both are Gaussian.

In this section the effects of bias on the mean value of the output of a polarity-coincidence detector are considered for inputs consisting of Gaussian signal and Gaussian noise. Therefore, in addition to the assumptions introduced previously (assumptions A1 through A7 and assumption A3-1), it will now be assumed that the signal, $s(t)$, is a sample function from a stationary Gaussian random process with zero mean and with variance σ_s^2. This assumption will be in force only in the present section.

(A3-2) $s_o(t)$ is a sample function from a stationary random process with density function

$$h(s_o) = \frac{1}{\sqrt{2\pi}\,\sigma_o} \exp(s_o^2/2\sigma_o^2) \quad .$$

Assumption A3-2 is specified in terms of s_o rather than s, since it is s_o which is used in the subsequent equations. s and s_o have identical distributions except for their variances which are related by the expression $a = \sigma_o/\sigma_s$.

Upon making the transformation $\lambda = s_o/\sigma_o$, the density function for λ becomes in this case

$$v(\lambda) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\lambda^2} \quad .$$

When this expression for $v(\lambda)$ is substituted in the equations of section 3.1, the following results are obtained:

From equation 3.1-1,

$$\mu_z = T \int_{-\infty}^{\delta_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\lambda^2} \operatorname{Erf}(\eta_2/\sqrt{2})d\lambda - T\int_{\delta_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\lambda^2} \operatorname{Erf}(\eta_2/\sqrt{2})d\lambda \quad . \tag{3.2-1}$$

<u>Case A. N = 0</u>

From equation 3.1-2,

$$\mu_o = 2T \cdot \mathrm{Erf}(\delta_1/\sqrt{2}) \int_0^{\delta_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\lambda^2} d\lambda$$

or

$$\mu_o = T \cdot \mathrm{Erf}(\delta_1/\sqrt{2}) \cdot \mathrm{Erf}(\delta_2/\sqrt{2}) \quad . \qquad (3.2\text{-}2)$$

The remarks following equation 3.0-3 apply also to equation 3.2-2.

Before proceeding with the case $N \neq 0$, it will be convenient to introduce a new volume integral. Let $V(p,\gamma p)$ denote the volume under the standard bivariate normal surface with zero correlation, $\frac{1}{2\pi} e^{-\frac{1}{2}t^2} e^{-\frac{1}{2}\lambda^2}$, over the right triangular region of integration with base of length p along the λ-axis, with acute vertex at the origin and with altitude of height γp parallel to the t-axis, in the first quadrant. This is the same region of integration as was discussed in sections 3.0 and 3.1 and as was shown in figure 3.0-2. Since this volume integral will be used extensively in subsequent developments, the region of integration is shown again for convenience in figure 3.2-1.

The integral representation for $V(p,\gamma p)$ is

$$V(p,\gamma p) = \frac{1}{2\pi} \int_0^p d\lambda \int_0^{\gamma\lambda} e^{-\frac{1}{2}t^2} e^{-\frac{1}{2}\lambda^2} dt$$

$$= \frac{1}{2\sqrt{2\pi}} \int_0^p e^{-\frac{1}{2}\lambda^2} \mathrm{Erf}(\gamma\lambda/\sqrt{2}) d\lambda \quad . \qquad (3.2\text{-}3)$$

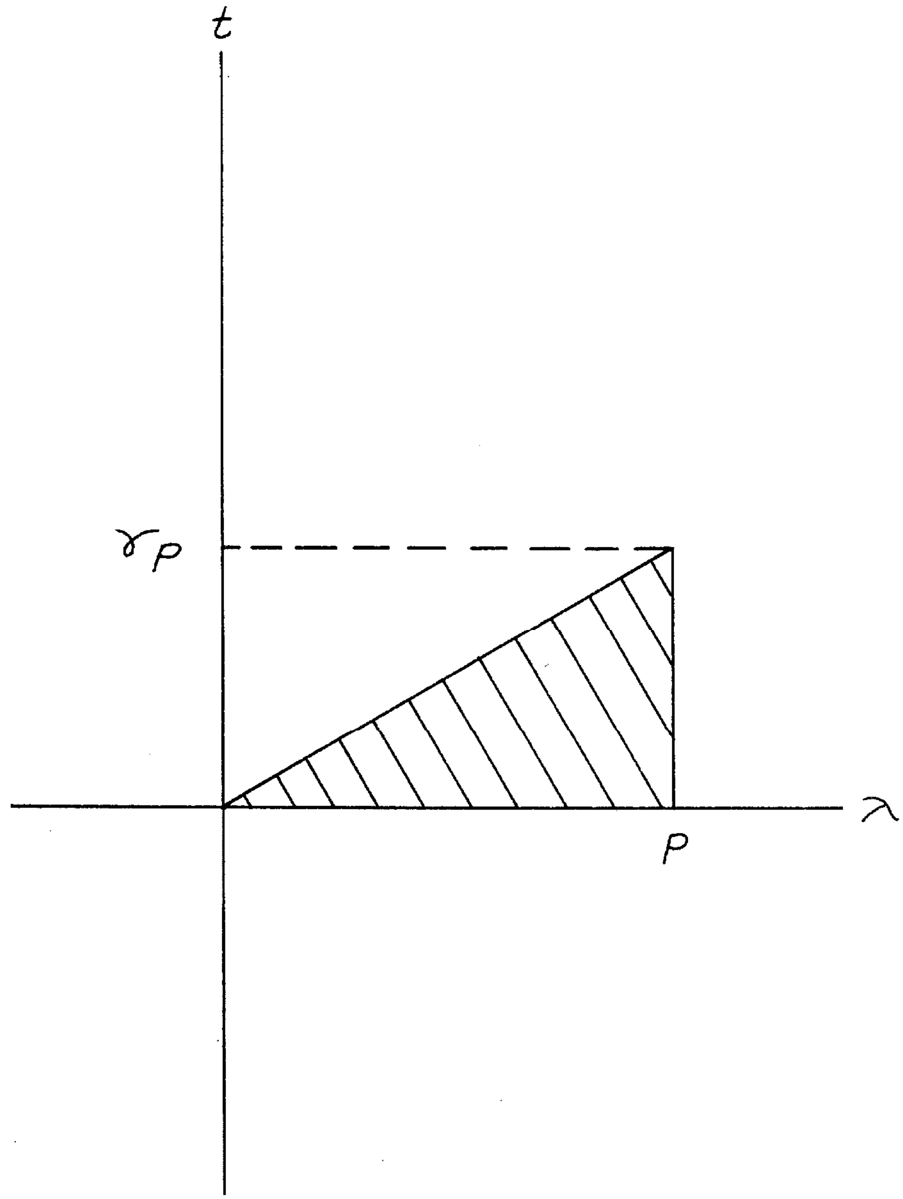


Figure 3.2-1

This integral is one of the multivariate Gaussian integrals with linear boundaries of integration whose importance to the analysis of the BPCD was mentioned in the summary of related literature in Chapter I. This integral is important in statistics. Consequently, a thorough investigation of its properties along with tabulations of its values can be found in the literature (1).

$V(p,\gamma p)$ and $V(\gamma p,p)$ are tabulated in tables III and IV of the reference cited above (1). Both functions are tabulated only for values of γ between zero and one. Values of $V(p,\gamma p)$ with $\gamma > 1$ can be obtained from the table for $V(\gamma p,p)$ with $0 < \gamma \leq 1$ as follows: Let $q = \gamma p$, $\nu = 1/\gamma$. Then $V(\nu q,q) = V(p,\gamma p)$. When $\gamma > 1$, then $0 < \nu < 1$ and values of $V(\nu q,q)$ for this range of ν are tabulated. Similarly, values of $V(\gamma p,p)$ for $\gamma > 1$ can be found from values of $V(p,\gamma p)$ with $0 < \gamma \leq 1$.

A detailed discussion of the properties of $V(p,\gamma p)$, based essentially on that found in the reference cited above (1), is given in Appendix I. For later convenience, the simplest properties are described below without proof.

$$V(p,\gamma p) = V(-p,-\gamma p) \tag{v1}$$

$$V(p,\gamma p) = -V(-p,\gamma p) = -V(p,-\gamma p) \tag{v2}$$

$$V(0,\gamma 0) = V(p,0p) = 0 \tag{v3}$$

$$\lim_{p\to\infty} V(p,\gamma p) = \frac{1}{2\pi}\tan^{-1}(\gamma) \tag{v4}$$

These properties are trivial consequences of the defining equation.

The integral, $V(p,\gamma p)$, has not been evaluated in closed form in terms of the elementary functions or even in terms of a finite number of terms of higher functions. The integral can be evaluated approximately in closed form when p and γ are restricted to certain regions in the p,γ-plane. Moreover, by replacing the error function in the integrand by an approximating function, the integral can be evaluated in closed form for all values of p and γ with a result which differs from the exact value by a small and calculable amount. (I am indebted to Benedict Freedman for suggesting the approximating function method. In a private communication, he described a specific approximating function which makes it possible to evaluate the above integral in closed form with extremely small error. See Appendix II.)

Case B. $N \neq 0$

B_1.) $\delta_1 = \delta_2 = 0$. In equation 3.1-5, $W(p,\sqrt{N}p)$ becomes $V(p,\sqrt{N}p)$. Thus,

$$\mu_1 = 4T \cdot \lim_{p\to\infty} V(p,\sqrt{N}\,p) \quad .$$

By property v4 of $V(p,\gamma p)$ stated on page 49, this is

$$\mu_1 = \frac{2T}{\pi}\tan^{-1}(\sqrt{N}) \quad . \tag{3.2-4}$$

B_2.) $\delta_1 = 0, \quad \delta_2 \neq 0.$ From equation 3.1-6,

$$\mu_2 = \mu_1 + e_2 \qquad , \qquad (3.2\text{-}5)$$

where now

$$e_2 = -4T\cdot V(\delta_2, \sqrt{N}\delta_2) \qquad .$$

From equation 3.1-7 and the result just stated,

$$\rho_2 = -2\pi V(\delta_2, \sqrt{N}\delta_2)/\tan^{-1}(\sqrt{N}) \qquad . \qquad (3.2\text{-}6)$$

B_3.) $\delta_1 \neq 0, \quad \delta_2 = 0.$ From equation 3.0-16,

$$\mu_3 = \mu_1 + e_3 \qquad ,$$

where now

$$e_3 = \frac{2T}{2\pi}\left[\int_{-\infty}^{0} d\lambda \int_{\eta_1}^{\eta_2} e^{-\frac{1}{2}(\lambda^2+t^2)}dt - \int_{0}^{\infty} d\lambda \int_{\eta_1}^{\eta_2} e^{-\frac{1}{2}(\lambda^2+t^2)}dt\right] \qquad .$$

Upon transforming to polar coordinates with $\lambda = r\cdot\cos\theta$ and $t = r\cdot\sin\theta$, this equation becomes

$$e_3 = \frac{T}{\pi}\left[\iint_A re^{-\frac{1}{2}r^2}drd\theta - \iint_B re^{-\frac{1}{2}r^2}drd\theta\right]$$

where the regions of integration A and B are shown in figure 3.2-2. Clearly, the integrand is a function only of the radial distance from the origin and is invariant under a rotation. Therefore, if the regions A and B of figure 3.2-2 are replaced by the regions A' and B' of figure 3.2-3, the equation is still valid.

$$e_3 = \frac{T}{\pi}\left[\iint_{A'} re^{-\frac{1}{2}r^2}drd\theta - \iint_{B'} re^{-\frac{1}{2}r^2}drd\theta\right]$$

Moreover, the integrand is invariant under a reflection in the ζ-axis of figure 3.2-3. Thus, that part of the second integral corresponding to the region in B' below the dotted line $\xi = -\sqrt{N}\,\zeta$ cancels the first integral; that part of the second integral corresponding to the region between the ζ-axis and the dotted line $\xi = -\sqrt{N}\,\zeta$ is equal to that part of the second integral corresponding to the region between the solid line $\xi = \sqrt{N}\,\zeta$ and the ζ-axis. Therefore, after returning to rectangular coordinates,

$$e_3 = -\frac{2T}{\pi}\int_0^{\delta_1} d\zeta \int_0^{\sqrt{N}\zeta} e^{-\frac{1}{2}(\zeta^2+\xi^2)}d\xi \quad .$$

But this is the defining integral for $V(\delta_1, \sqrt{N}\,\delta_1)$. Therefore,

$$\mu_3 = \mu_1 + e_3 \tag{3.2-7}$$

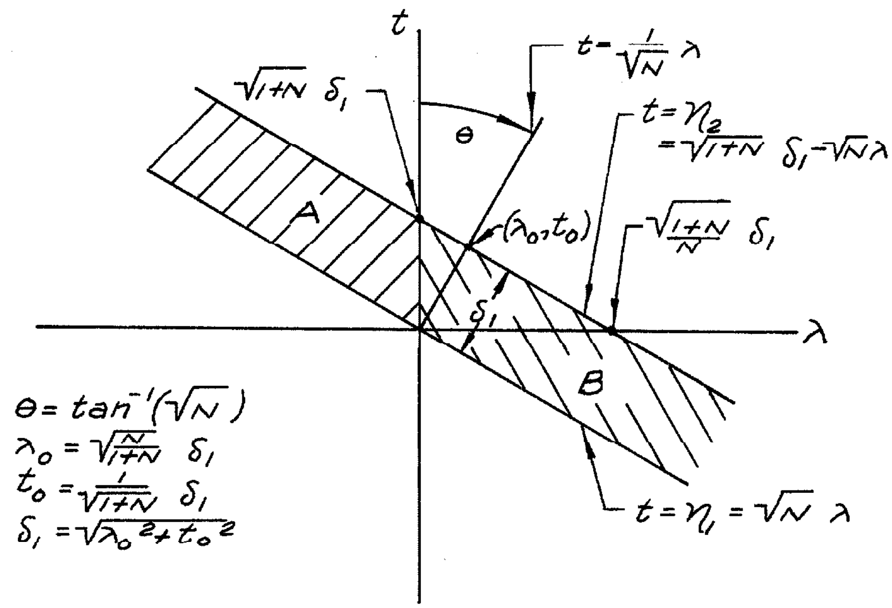


Figure 3.2-2

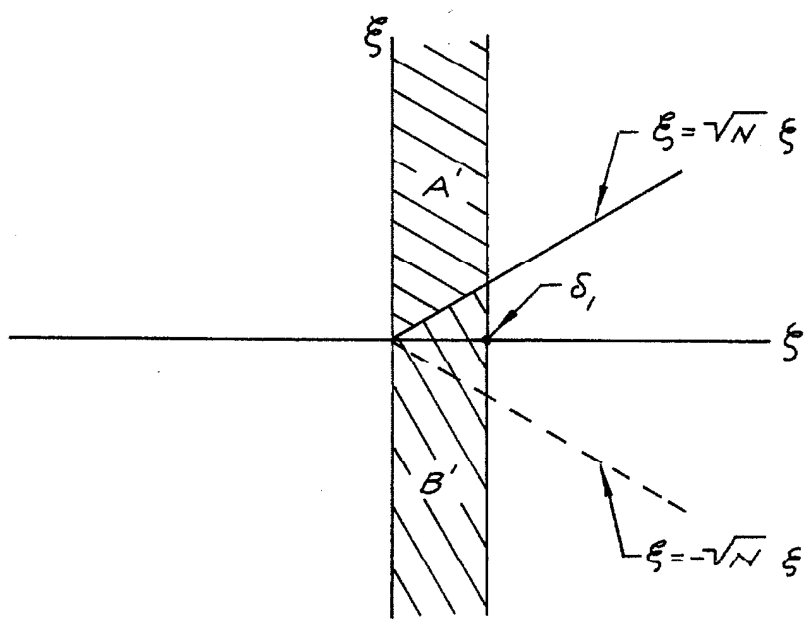


Figure 3.2-3

where

$$e_3 = -4T \cdot V(\delta_1, \sqrt{N}\ \delta_1) \qquad . \tag{3.2-8}$$

From equation 3.1-9 and the result just stated,

$$\rho_3 = -2\pi V(\delta_1, \sqrt{N}\ \delta_1)/\tan^{-1}(\sqrt{N}) \qquad . \tag{3.2-9}$$

It should be noted that equation 3.2-7 for e_3 is identical to equation 3.2-5 for e_2 with δ_1 substituted for δ_2.

<u>B_4.) $\delta_1 \neq 0$, $\delta_2 \neq 0$.</u> The method used to evaluate the integrals for case B_3 will be used repeatedly for case B_4. This method makes use of the fact that, when expressed in polar coordinates, the integrand is invariant under rotation. Thus the integral over any right triangle, T, of the form shown in figure 3.2-4, with acute vertex at the origin, can be rotated into a triangle of the form shown in figure 3.2-1, the integrand being invariant under such a rotation. Thus

$$\frac{1}{2\pi} \iint_T e^{-\frac{1}{2}(\lambda^2+t^2)} dt d\lambda = V(p, \gamma p) \qquad .$$

Similarly, the triangle T in figure 3.2-5 can be rotated into the dotted triangle, the integrand being invariant under the rotation. By the evenness of $e^{-\frac{1}{2}t^2}$, the integral over the dotted triangle is the same as the integral over the mirror image in the λ-axis of the

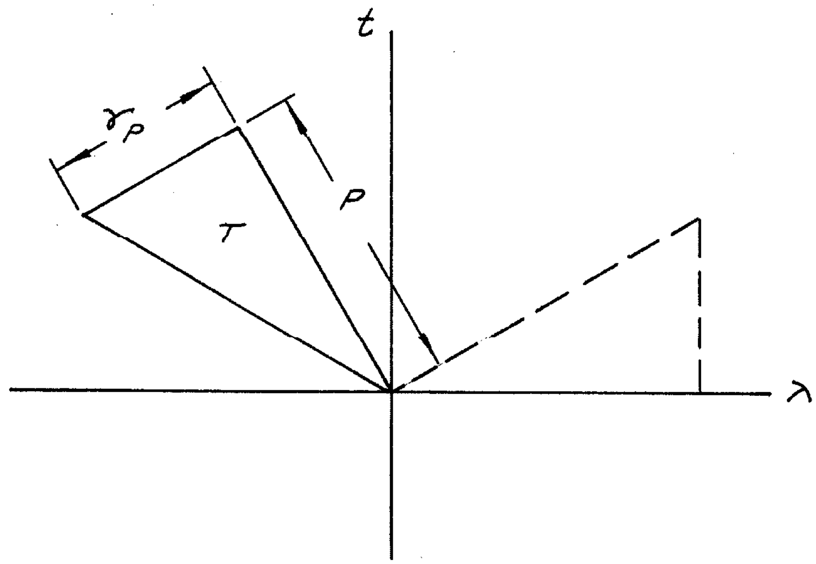


Figure 3.2-4

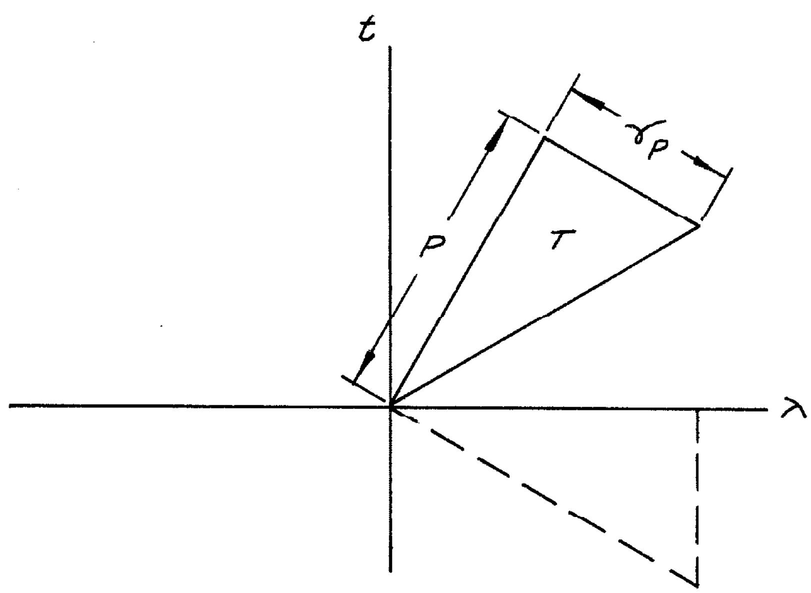


Figure 3.2-5

dotted triangle. Thus, this integral is also equal to $V(p,\gamma p)$. The preceding argument has shown that the integral of

$\frac{1}{2\pi} e^{-\frac{1}{2}(\lambda^2+t^2)}$ over any right triangle with acute angle at the origin has the value $V(p,\gamma p)$ where p is the length of the side adjacent to the origin and γp is the length of the side opposite the origin.

From equation 3.0-18,

$$\mu_4 = \mu_2 + \mu_3 - \mu_1 + 4T\cdot I(\delta_1,\delta_2) \tag{3.2-10}$$

where now

$$I(\delta_1,\delta_2) = \frac{1}{2\pi} \int_0^{\delta_2} d\lambda \int_{\eta_1}^{\eta_2} e^{-\frac{1}{2}(\lambda^2+t^2)} dt \quad . \tag{3.2-11}$$

In order to facilitate the evaluation of this integral, case B_4 will be subdivided into several cases. First, some notation will be introduced. For the significance of the notation, refer to figure 3.2-6. This is the figure corresponding to case i) below, but the notation and its significance will be the same for all the cases below. The region of integration is a parallelogram, with vertical sides consisting of the t-axis and the verticle line $\lambda = \delta_2$, and with slant sides consisting of the line $t = \eta_1 = -\sqrt{N}\,\lambda$ and the line $t = \eta_2 = \sqrt{1+N}\,\delta_1 + \eta_1$, as shown in figure 3.2-6. The vertex at the intersection of $\lambda = \delta_2$ with $t = \eta_2$ is denoted by the coordinates (δ_2,t_2). The intersection of the line $t = \lambda/\sqrt{N}$ with $t = \eta_2$ is denoted by the coordinates (λ_o,t_o).

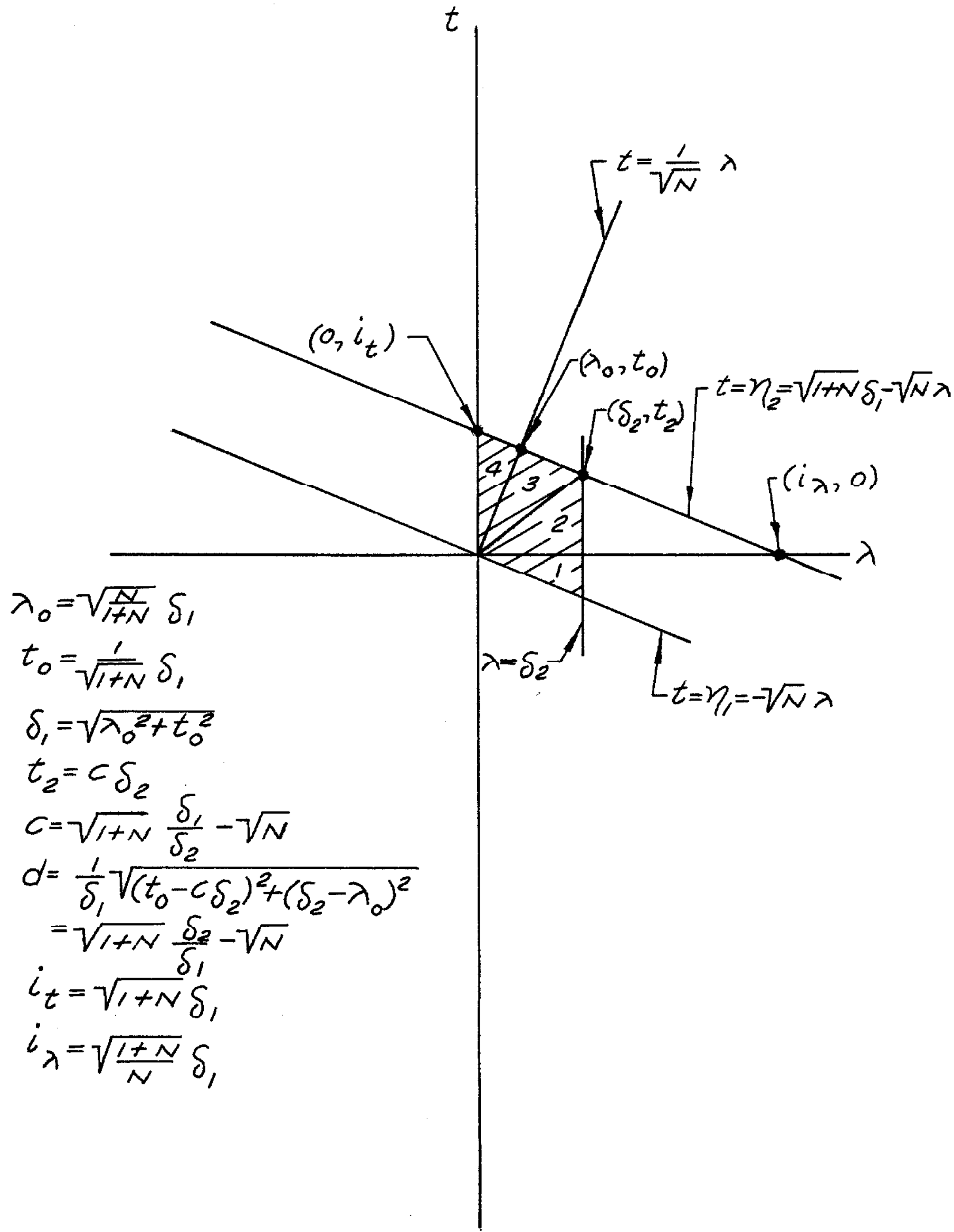

t
$t = \frac{1}{\sqrt{N}} \lambda$
$(0, i_t)$
(λ_0, t_0)
$t = \eta_2 = \sqrt{1+N}\,\delta_1 - \sqrt{N}\,\lambda$
(δ_2, t_2)
$(i_\lambda, 0)$
4
3
2
1
λ
$\lambda = \delta_2$
$t = \eta_1 = -\sqrt{N}\,\lambda$
$\lambda_0 = \sqrt{\frac{N}{1+N}}\,\delta_1$
$t_0 = \frac{1}{\sqrt{1+N}}\,\delta_1$
$\delta_1 = \sqrt{\lambda_0^2 + t_0^2}$
$t_2 = c\,\delta_2$
$c = \sqrt{1+N}\,\frac{\delta_1}{\delta_2} - \sqrt{N}$
$d = \frac{1}{\delta_1}\sqrt{(t_0 - c\,\delta_2)^2 + (\delta_2 - \lambda_0)^2}$
$= \sqrt{1+N}\,\frac{\delta_2}{\delta_1} - \sqrt{N}$
$i_t = \sqrt{1+N}\,\delta_1$
$i_\lambda = \sqrt{\frac{1+N}{N}}\,\delta_1$

Figure 3.2-6

Simple algebraic manipulations establish the following relations:

$$\lambda_o = \sqrt{N/(1+N)}\ \delta_1 \quad .$$

$$t_o = \sqrt{1/(1+N)}\ \delta_1 \quad .$$

$$t_2 = c\delta_2, \quad \text{where} \quad c = \sqrt{1+N}\ (\delta_1/\delta_2) - \sqrt{N} \quad .$$

The distance between the lines $t = \eta_1$ and $t = \eta_2$ is $\sqrt{\lambda_o^2 + t_o^2} = \delta_1$.

The distance between the points (λ_o, t_o) and (δ_2, t_2) is $D = d\delta_1$, where $d = (1/\delta_1)\sqrt{(t_o - c\delta_2)^2 + (\delta_2 - \lambda_o)^2} = \sqrt{1+N}\ (\delta_2/\delta_1) - \sqrt{N}$.

The t-axis intercept, i_t, of the line $t = \eta_2$ is $\sqrt{1+N}\ \delta_1$.

The λ-axis intercept, i_λ, of the line $t = \eta_2$ is $\sqrt{(1+N)/N}\ \delta_1$.

Case B_4 is subdivided as follows:

<u>$\delta_1 \geq 0$</u>

i.) $0 \le \lambda_o \le \delta_2 \le \sqrt{(1+N)/N}\ \delta_1$. For the significance of this inequality, refer to figure 3.2-6. It means that the order of the points on the line $t = \eta_2$ is i_t, (λ_o, t_o), (δ_2, t_2), i_λ from left to right. The region of integration is subdivided into four triangles of type T, as shown in figure 3.2-6. The following table presents the quantities related to these triangles.

Triangle No	Base p	Altitude/Base γ	$\iint_T$
1	δ_2	$\sqrt{N}$	$V(\delta_2, \sqrt{N}\ \delta_2) = \mu_1 - \mu_2$
2	δ_2	c	$V(\delta_2, c\delta_2)$
3	δ_1	d	$V(\delta_1, d\delta_1)$
4	δ_1	$\sqrt{N}$	$V(\delta_1, \sqrt{N}\ \delta_1) = \mu_1 - \mu_3$

Note that for the conditions imposed on δ_1 and δ_2 by the inequality at the beginning of this subdivision, c and d are positive.

From equations 3.2-10 and 3.2-11, using the results tabulated above,

$$\mu_4 = \mu_1 + 4T \cdot V(\delta_1, d\delta_1) + 4T \cdot V(\delta_2, c\delta_2) \quad .$$

The subdivisions of case B_4 remaining to be considered are:

$\underline{\delta_1 \geq 0}$

ii.) $0 \leq \lambda_o \leq \sqrt{(1+N)/N}\ \delta_1 \leq \delta_2$,

iii.) $0 \leq \delta_2 \leq \lambda_o \leq \sqrt{(1+N)/N}\ \delta_1$,

iv.) $\delta_2 \leq 0 \leq \lambda_o \leq \sqrt{(1+N)/N}\ \delta_1$,

$\underline{\delta_1 \leq 0}$

v.) $\sqrt{(1+N)/N}\ \delta_1 \leq \lambda_o \leq 0 \leq \delta_2$,

vi.) $\sqrt{(1+N)/N}\ \delta_1 \leq \lambda_o \leq \delta_2 \leq 0$,

vii.) $\delta_2 \leq \sqrt{(1+N)/N}\ \delta_1 \leq \lambda_o \leq 0$,

viii.) $\sqrt{(1+N)/N}\ \delta_1 \leq \delta_2 \leq \lambda_o \leq 0$.

These subdivisions of case B_4 represent various permutations of the order in which the points i_t, (λ_o, t_o), i_λ and (δ_2, t_2) occur on the line $t = \eta_2$. By methods analogous to that used for subdivision i.) above, it can be shown that the expression derived above for μ_4 holds for all eight subdivisions. Thus, for case B_4

$$\mu_4 = \mu_1 + 4T \cdot V(\delta_1, d\delta_1) + 4T \cdot V(\delta_2, c\delta_2)$$

or

$$\mu_4 = \mu_1 + e_4 \qquad , \tag{3.2-12}$$

where

$$e_4 = 4T\cdot V(\delta_1, d\delta_1) + 4T\cdot V(\delta_2, c\delta_2) \qquad . \tag{3.2-13}$$

with

$$c = \sqrt{1+N}\ (\delta_1/\delta_2) - \sqrt{N}$$

and

$$d = \sqrt{1+N}\ (\delta_2/\delta_1) - \sqrt{N} \qquad .$$

The relative error, $\rho_4 = e_4/\mu_1$ is

$$\rho_4 = 2\pi \frac{4T\cdot V(\delta_1, d\delta_1) + 4T\cdot V(\delta_2, c\delta_2)}{\tan^{-1}(\sqrt{N})} \qquad . \tag{3.2-14}$$

The remarks following equation 3.0-18 apply also to equation 3.2-12. In addition, note that e_4 and ρ_4 are symmetric in δ_1 and δ_2. I.E., $e_4(\delta_1, \delta_2) = e_4(\delta_2, \delta_1)$ and $\rho_4(\delta_1, \delta_2) = \rho_4(\delta_2, \delta_1)$.

For purposes of tabulating and plotting, the form

$$e_3 = 4T\cdot V(\delta_1,[\sqrt{1+N}\ k-\sqrt{N}]\delta_1) + 4T\cdot V\left(k\delta_1,\left[\frac{\sqrt{1+N}}{k} - \sqrt{N}\right]k\delta_1\right) \tag{3.2-15}$$

will be used, where $k = \delta_2/\delta_1$. The parameter k is introduced because tabulating and plotting is considerably simplified as a consequence. The simplifications are obtained as follows:

Since $e_4(\delta_1,\delta_2) = e_4(\delta_2,\delta_1)$, the plane $\delta_1 = \delta_2 (k = 1)$ in the δ_1,δ_2, e_4-space is a plane of symmetry. Moreover, since $V(p,\gamma p) = V(-p,-\gamma p)$, it follows that $e_4(\delta_1,\delta_2) = e_4(-\delta_1,-\delta_2)$, so that for fixed k, e_4 is even in δ_1 (since when k is fixed, a change in the sign of δ_1 implies a change in the sign of δ_2). Thus, the e_4-axis (the line $\delta_1 = \delta_2 = 0$ in the δ_1,δ_2,e_4-space) is an axis of symmetry. The point δ_1,δ_2,e_4 has a mirror image $-\delta_1,-\delta_2,e_4$ in this axis which is the value for $e_4(-\delta_1,-\delta_2)$. Since $e_4(\delta_1,\delta_2) = e_4(\delta_2,\delta_1)$ by the first symmetry and $e_4(\delta_2,\delta_1) = = e_4(-\delta_2,-\delta_1)$ by the second symmetry, it follows that $e_4(\delta_1,\delta_2) = e_4(-\delta_2,-\delta_1)$, so that the plane $\delta_2 = -\delta_1 (k = -1)$ is also a plane of symmetry. The symmetry relations are shown in figure 3.2-7.

It is clear from the symmetry relations that e_4 for all values of δ_1 and k can be obtained from the values for $0 \le \delta_1$, $-\delta_1 \le \delta_2 \le \delta_1$ or equivalently, from the values for $-1 \le k \le 1$, $\delta_1 \ge 0$. This is the shaded region in figure 3.2-7.

Of course, all the preceding symmetry statements about e_4 are also valid for μ_4 and ρ_4, since μ_1 is independent of δ_1 and δ_2.

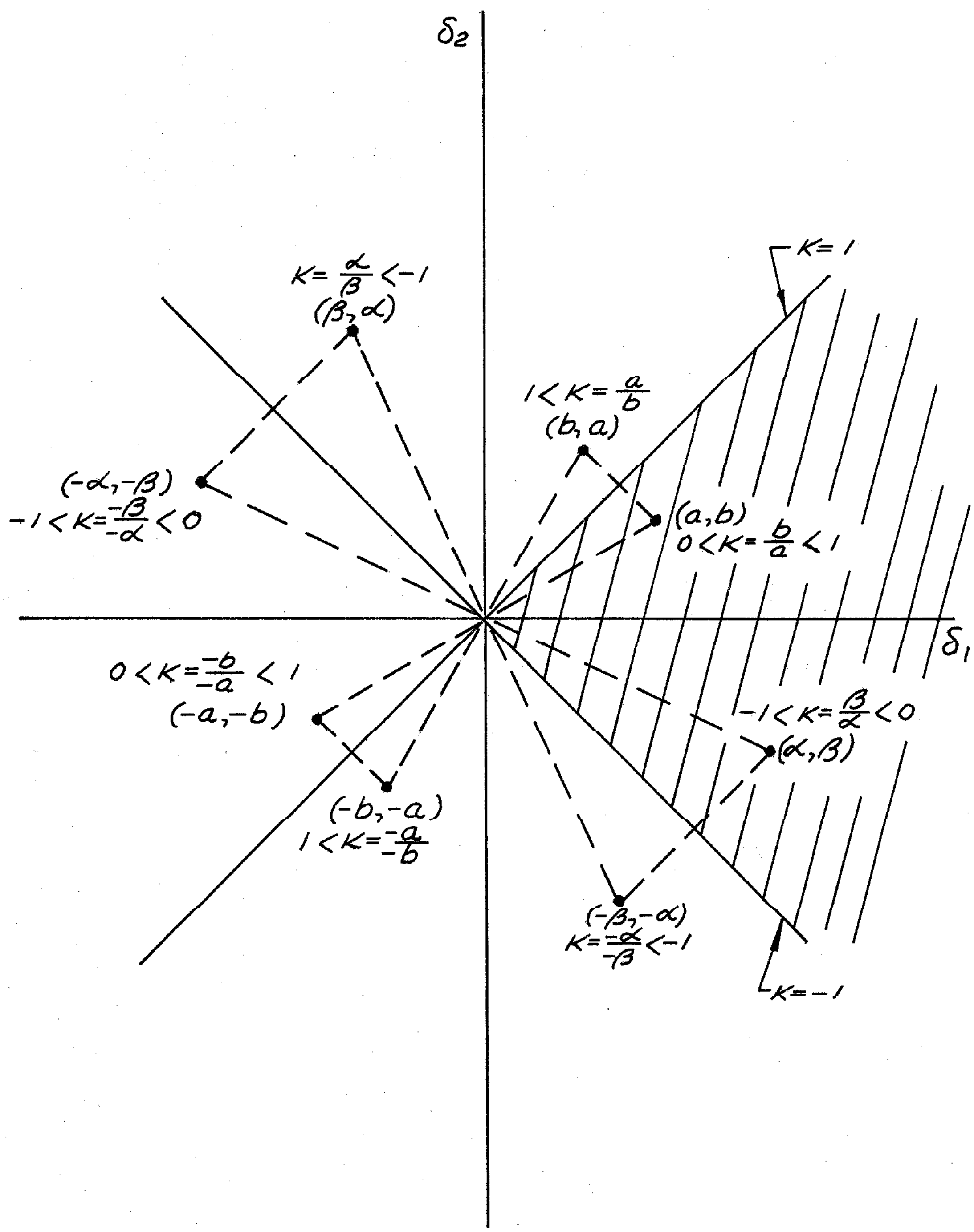

δ2
δ1
K= 1
K= -1
K= α/β <-1
(β, α)
1< K= a/b
(b, a)
(-α, -β)
-1< K= -β/-α <0
(a, b)
0< K= b/a <1
0< K= -b/-a <1
(-a, -b)
-1< K= β/α <0
(α, β)
(-b, -a)
1< K= -a/-b
(-β, -α)
K= -α/-β <-1

Figure 3.2-7

Case C. $N \to \infty$

In equation 3.1-11, $v(\lambda)$ becomes $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\lambda^2}$. Thus,

$$\mu_\infty = T - T \cdot \mathrm{Erf}[\max(\delta_1, \delta_2)/\sqrt{2}] + T \cdot \mathrm{Erf}[\min(\delta_1, \delta_2)/\sqrt{2}] \quad . \qquad (3.2\text{-}16)$$

The properties attributed to the mean and relative error in the various cases of section 3.0 (evenness in δ_1 and δ_2, etc.) have not been affected by the assumption of Gaussian signal and Gaussian noise. In particular, the remarks at the end of section 3.0 apply also to Section 3.2 (and 3.1). Therefore, a discussion of these properties has not been repeated here. However, the assumption of Gaussian signal and Gaussian noise results in some additional properties (e.g. $\mu_4(\delta_1, \delta_2) = \mu_4(\delta_2, \delta_1)$) which have been mentioned as they occur in the preceding discussion.

The expressions derived above for the mean value of the BPCD output will be used in Chapter V to examine the performance of the BPCD as a detector.

The mean value for the BPCD output was examined also for the case when the input signal is sinusoidal and the input noise is Gaussian. Again, the integrals could not be evaluated in closed form in terms of the well known functions. The integrals were evaluated approximately by substituting an approximating function for the error function factor in the integrand of the integral expressions for the output mean.

Since the other factors in the integrand are not the same in this case as they are in the Gaussian signal and Gaussian noise case, it is natural that the approximating function for the error function factor in the integrand must be different also in order to make the approximate integral evaluable. Although the analysis for this case is straightforward, it is extremely involved due to the complexity of the approximating function which is required. Since the variance for the sinusoidal signal case is not treated in this thesis, the complicated analysis for the mean for the sinusoidal signal case will not be presented here. The variance and mean for sinusoidal signal and Gaussian noise will be treated in a future report.

SUMMARY

For convenience, the results of Chapter III are summarized below.

Section 3.0. The General Case.

Case A. $N = 0$

This is the "signal absent" case, corresponding to the null hypothesis.

$$\mu_o = 4T \int_0^{\delta_1} u(t)dt \int_0^{\delta_2} v(\lambda)d\lambda \quad . \qquad (3.0\text{-}3)$$

The mean, μ_o, is even in δ_1 and δ_2 simultaneously but odd in δ_1 individually and in δ_2 individually. Ideally, the mean of the output should be zero when $N = 0$. The effect of bias is to cause a spurious non-zero component in the mean of the output. However, for the spurious component to be present, there must be bias in both the input channel and the reference channel. If the bias is zero in either channel, then there is no spurious component present.

Case B. $N \neq 0$

This is the "signal present" case, corresponding to the alternative hypothesis.

B_1. $\delta_1 = \delta_2 = 0$.

This is the ideal polarity-coincidence detector for $N \neq 0$.

$$\mu_1 = 4T \cdot \lim_{p \to \infty} U(p, \sqrt{N}\, p) \quad , \tag{3.0-8}$$

where $U(p, \gamma p)$ is defined in section 3.0. The mean increases monotonically with N, and $\lim_{N \to \infty} \mu_1 = T$.

<u>B_2. $\delta_1 = 0$, $\delta_2 \neq 0$.</u>

This is the "signal present" case when the reference polarity indicator has bias.

$$\mu_2 = \mu_1 + e_2 \quad , \tag{3.0-11}$$

where

$$e_2 = -4T \cdot U(p, \sqrt{N}\, p) \quad .$$

Bias in the reference channel causes a spurious term to appear in the output mean value. This spurious or error term, e_2, is even in δ_2 and is always negative, thus degrading the mean from the ideal value.

<u>B_3. $\delta_1 \neq 0$, $\delta_2 = 0$.</u>

This is the "signal present" case when the input polarity indicator has bias.

$$\mu_3 = \mu_1 + e_3 \quad , \tag{3.0-16}$$

where

$$e_3 = 2T\left[\int_{-\infty}^{0} d\lambda \int_{\eta_1}^{\eta_2} u(t)v(\lambda)dt - \int_{0}^{\infty} d\lambda \int_{\eta_1}^{\eta_2} u(t)v(\lambda)dt\right] \quad .$$

Bias in the input channel causes a spurious term to appear in the output mean value. This spurious or error term is even in δ_1.

B_4. $\delta_1 \neq 0, \quad \delta_2 \neq 0$.

This is the "signal present" case when the input and reference polarity indicators both have bias.

$$\mu_4 = \mu_1 + e_4 \quad , \tag{3.0-18}$$

where

$$e_4 = e_2 + e_3 + 4T \cdot I(\delta_1, \delta_2)$$

and

$$I(\delta_1, \delta_2) = \int_{0}^{\delta_2} d\lambda \int_{\eta_1}^{\eta_2} u(t)v(\lambda)dt \quad .$$

Bias in both the input and reference channels causes a spurious term to appear in the output mean value. This spurious or error term is even in δ_1 and δ_2 simultaneously but not individually.

Case C. $N \to \infty$

This is the infinite signal-to-noise ratio case.

$$\mu_{\infty} = T\left[1 - 2\int_{\min(\delta_1,\delta_2)}^{\max(\delta_1,\delta_2)} v(\lambda)d\lambda\right] \quad . \qquad (3.0\text{-}20)$$

In the ideal case when $\delta_1 = \delta_2 = 0$, then $\mu_{1\infty} = T$, which is the maximum value that the output mean value can have.

Section 3.1. The Gaussian Noise Case.

There are no essential differences between the Gaussian noise case of section 3.1 and the more general case of section 3.0. Since the noise distribution has now been specified, the expressions for μ_z for the various cases become

Case A. $N = 0$

$$\mu_o = 2T\cdot\mathrm{Erf}(\delta_1/\sqrt{2})\int_0^{\delta_2} v(\lambda)d\lambda \quad . \qquad (3.1\text{-}2)$$

Case B. $N \neq 0$

B_1. $\delta_1 = \delta_2 = 0$.

$$\mu_1 = 4T\cdot\lim_{p\to\infty} W(p,\sqrt{N}\,p) \quad , \qquad (3.1\text{-}5)$$

where $W(p,\gamma p)$ is defined in section 3.1.

$\underline{B_2.\quad \delta_1 = 0, \quad \delta_2 \neq 0.}$

$$\mu_2 = \mu_1 + e_2 \quad , \tag{3.1-6}$$

where

$$e_2 = -4T \cdot W(\delta_2, \sqrt{N}\ \delta_2) \quad .$$

$\underline{B_3.\quad \delta_1 \neq 0, \quad \delta_2 = 0.}$

$$\mu_3 = \mu_1 + e_3 \quad , \tag{3.1-8}$$

where

$$e_3 = T \int_{-\infty}^{0} v(\lambda) \left[\mathrm{Erf}(\delta_2/\sqrt{2}) - \mathrm{Erf}(\delta_1/\sqrt{2})\right] d\lambda$$

$$- T \int_{0}^{\infty} v(\lambda) \left[\mathrm{Erf}(\delta_2/\sqrt{2}) - \mathrm{Erf}(\delta_1/\sqrt{2})\right] d\lambda \quad .$$

$\underline{B_4.\quad \delta_1 \neq 0, \quad \delta_2 \neq 0.}$

$$\mu_4 = \mu_1 + e_4 \quad , \tag{3.1-10}$$

where

$$e_4 = e_2 + e_3 + 4T \cdot I(\delta_1, \delta_2)$$

and

$$I(\delta_1, \delta_2) = \frac{1}{2} \int_0^{\delta_2} v(\lambda)\left[\mathrm{Erf}(\eta_2/\sqrt{2}) - \mathrm{Erf}(\eta_1/\sqrt{2})\right] d\lambda \quad .$$

Case C. $N \to \infty$

For this case, equation 3.0-20, given above, is unchanged.

Section 3.2. The Gaussian Signal and Gaussian Noise Case.

In this case, all of the properties stated above for the general case remain valid and, due to the specification of the signal and noise distributions, additional properties arise. These are described below as they occur.

Case A. $N = 0$

$$\mu_0 = T \cdot \mathrm{Erf}(\delta_1/\sqrt{2}) \cdot \mathrm{Erf}(\delta_2/\sqrt{2}) \quad . \tag{3.2-2}$$

Case B. $N \neq 0$

B_1. $\delta_1 = \delta_2 = 0$.

$$\mu_1 = \frac{2T}{\pi} \tan^{-1}(\sqrt{N}) \quad . \tag{3.2-4}$$

B_2. $\delta_1 = 0, \quad \delta_2 \neq 0$.

$$\mu_2 = \mu_1 + e_2 \quad , \tag{3.2-5}$$

with

$$e_2 = -4T\cdot V(\delta_2, \sqrt{N}\ \delta_2) \quad ,$$

where $V(p, \gamma p)$ is defined in section 3.2.

B_3. $\delta_1 \neq 0$, $\delta_2 = 0$.

$$\mu_3 = \mu_1 + e_3 \quad , \qquad (3.2\text{-}7)$$

with

$$e_3 = -4T\cdot V(\delta_1, \sqrt{N}\ \delta_1) \quad .$$

The expressions for the mean in cases B_2 and B_3 are identical except that in one case the variable is δ_2 and in the other case it is δ_1.

B_4. $\delta_1 \neq 0$, $\delta_2 \neq 0$.

$$\mu_4 = \mu_1 + e_4 \quad , \qquad (3.2\text{-}12)$$

with

$$e_4 = 4T\cdot V(\delta_1, d\delta_1) + 4T\cdot V(\delta_2, c\delta_2)$$

where

$$c = \sqrt{1+N}\ (\delta_1/\delta_2) - \sqrt{N} \quad .$$

and

$$d = \sqrt{1+N}\ (\delta_2/\delta_1) - \sqrt{N} \quad .$$

The error, e_4, is symmetric in δ_1 and δ_2. I.e., $e_4(\delta_1,\delta_2) = e_4(\delta_2,\delta_1)$. The e_4 axis in the δ_1,δ_2,e_4-space is an axis of symmetry. The $\delta_1 = \delta_2$ plane $(k = 1)$ in the δ_1,δ_2,e_4-space is a plane of symmetry. The $\delta_1 = -\delta_2$ plane in the δ_1,δ_2,e_4-space is a plane of symmetry.

Case C. $N \to \infty$

$$\mu_\infty = T - T\cdot \mathrm{Erf}[\max(\delta_1,\delta_2)/\sqrt{2}] + T\cdot \mathrm{Erf}[\min(\delta_1,\delta_2)/\sqrt{2}] \quad . \quad (3.2\text{-}16)$$

In general, Gaussian signal and Gaussian noise or not, bias in the input or reference polarity indicators causes a spurious component in the BPCD output mean value. When a signal is present, bias in either channel is sufficient to cause a spurious component. When no signal is present, there is a spurious component only if bias is present in both channels.

REFERENCE

1. U.S., National Bureau of Standards: Tables of the Bivariate Normal Distribution Function and Related Functions; 1959.

CHAPTER IV

THE VARIANCE OF THE OUTPUT OF A BIASED POLARITY-COINCIDENCE DETECTOR

In this chapter expressions are derived for the variance of the BPCD output for various conditions of the input. First, the expected value of a quantity which is required in the derivation of the variance will be investigated. This quantity is the product of the multiplier output, $u(t)$ at time t (see figure 2.1), with itself at a different time, θ. The expected value of this quantity will be denoted for the present simply as E.

In section 4.0, general expressions for E are derived, subject only to the restrictions imposed by the assumptions of the previous chapter (assumptions A1 through A7) and by six new assumptions to be introduced in this chapter. In this general form, E is expressed in terms of integrals which cannot be further reduced. In section 4.1 the Gaussian signal and Gaussian noise assumptions are introduced. The expressions for E are not simplified significantly as a consequence (except for the case $N = 0$). The resulting integrals cannot be evaluated in closed form, and approximate methods of evaluating them are shown to be extremely complicated. Consequently, only the case for $N = 0$, which is fairly tractable, is examined. It is argued that this case suffices for ascertaining the performance of the BPCD for small input signal-to-noise ratios. In section 4.2, the expressions for E obtained in section 4.1 are used to derive expressions for the variance of the BPCD output, when $N = 0$. In section 4.3 an analysis is presented of the errors which result from the introduction of various approximations in evaluating σ_z^2, when $N = 0$.

Assumptions

(A8) $\Pr\{s(t) + n(t) = b_1,\ s(\theta) + n(\theta) = b_1\} = 0$ and $\Pr\{s_o(t) = b_2,\ s_o(\theta) = b_2\} = 0$, where t and θ are two different times.

(A9) The $s(t)$ and $n(t)$ (and hence the $s_o(t)$ and $n(t)$) processes have statistically independent second order distributions.

(A10) The joint density function for $n(t)$ and $n(\theta)$ is symmetric in $n(t) = n_i$ and $n(\theta) = n_j$.

(A11) The joint density function for $s(t)$ and $s(\theta)$ is symmetric in $s(t) = s_i$ and $s(\theta) = s_j$. (Hence, the joint density function for $s_o(t)$ and $s_o(\theta)$ is symmetric in $s_o(t) = s_{oi}$ and $s_o(\theta) = s_{oj}$.)

(A12) The joint density function for $n(t)$ and $n(\theta)$ is even in $n(t) = n_i$ and $n(\theta) = n_j$ simultaneously but not individually. I.e., $f(n_i,n_j) = f(-n_i,-n_j)$ but $f(n_i,n_j) \neq f(n_i,-n_j)$ and $f(n_i,n_j) \neq f(-n_i,n_j)$, where $f(n_i,n_j)$ is the joint density function for n_i and n_j.

(A13) The joint density function for $s(t)$ and $s(\theta)$ is even in $s(t) = s_i$ and $s(\theta) = s_j$ simultaneously but not individually. (Hence, the joint density function for $s_o(t)$ and $s_o(\theta)$ is even in $s_o(t) = s_{oi}$ and

$s_o(\theta) = s_{oj}$ simultaneously but not individually.)

4.0 General Expressions for the Expected Value of $u(t)u(\theta)$*

From equation 2.2, the output squared of the BPCD is

$$z^2 = \int_0^T\int_0^T u(t)u(\theta)dtd\theta = \int_0^T\int_0^T y_1(t)y_2(t)y_1(\theta)y_2(\theta)dtd\theta \quad .$$

The expected value of z^2 is then

$$E\{z^2\} = E\left\{\int_0^T\int_0^T u(t)u(\theta)dtd\theta\right\} = \int_0^T\int_0^T E\{u(t)u(\theta)\}dtd\theta$$

or

$$E\{z^2\} = \int_0^T\int_0^T E\{y_1(t)y_2(t)y_1(\theta)y_2(\theta)\}dtd\theta \quad .$$

For convenience, the notation $y_1(t) = y_{1i}$, $y_2(t) = y_{2i}$, $y_1(\theta) = y_{1j}$, $y_2(\theta) = y_{2j}$ will be used. Thus, the subscript i indicates the time t and the subscript j indicates the time θ. When a quantity appears with the subscript i,j it indicates either i or j may be used. E.g., $y_{1i,j}$ may be either y_{1i} or y_{1j}. When the subscript i,j appears in an expression more than once, it

---

* $u(t)$ is here the output of the multiplier at time t, as distinguished from the distribution of the random variate, t, to be introduced later.

means that either i or j may be used, but the same choice must be used throughout the expression. E.g., $y_{1i,j}y_{2i,j}$ may be either $y_{1i}y_{2i}$ or $y_{1j}y_{2j}$.

In terms of this notation,

$$E\{z^2\} = \int_0^T\int_0^T E\{y_{1i}y_{2i}y_{1j}y_{2j}\}dtd\theta \quad . \tag{4.0-1}$$

Since the variance of the output, σ_z^2, is related to $E\{z^2\}$ by $\sigma_z^2 = E\{z^2\} - \{\mu_z\}^2$, the central problem of deriving an expression for the variance is to evaluate the integral representing $E\{y_{1i}y_{2i}y_{1j}y_{2j}\}$. The evaluation is accomplished as follows:

$y_{1i} = +1$ when $x_1(t) > b_1$, $y_{1i} = -1$ when $x_1(t) < b_1$; $y_{1j} = +1$ when $x_1(\theta) > b_1$, $y_{1j} = -1$ when $x_1(\theta) < b_1$. And $y_{2i} = +1$ when $x_2(t) > b_2$, $y_{2i} = -1$ when $x_2(t) < b_2$; $y_{2j} = +1$ when $x_2(\theta) > b_2$, $y_{2j} = -1$ when $x_2(\theta) < b_2$. Also, $y_{1i} = 0$ when $x_1(t) = b_1$, $y_{1j} = 0$ when $x_1(\theta) = b_1$; $y_{2i} = 0$ when $x_2(t) = b_2$, $y_{2j} = 0$ when $x_2(\theta) = b_2$. (For the meaning of x_1 and x_2, refer to figure 2.1.)

Upon introducing assumption A1 (see Chapter III), these conditions become: $y_{1i,j} = +1$ when $s_{i,j} + n_{i,j} > b_1$, $y_{1i,j} = -1$ when $s_{i,j} + n_{i,j} < b_1$. And $y_{2i,j} = +1$ when $s_{oi,j} > b_2$, $y_{2i,j} = -1$ when $s_{oi,j} < b_2$. Also, $y_{1i,j} = 0$ when $s_{i,j} + n_{i,j} = b_1$ and $y_{2i,j} = 0$ when $s_{oi,j} = b_2$.

Next, the normalized biases defined in Chapter III are introduced. $\delta_1 = b_1/\sqrt{P} = b_1/\sigma_n\sqrt{1+N}$ and $\delta_2 = b_2/\sigma_o = b_2/a\sigma_s$, where $a = \sigma_o/\sigma_s$. (The introduction of assumption A2 is implied by the introduction of the normalized biases.) The conditions given above describe the boundaries of the regions where y_1 and y_2 are +1 and -1. On the boundaries, y_1 and y_2 are zero. With the normalized biases, the boundary expressions become:

$$s_{i,j} + n_{i,j} \gtreqless b_1 = \sigma_n\sqrt{1+N}\ \delta_1 \qquad \text{i)}$$

or, after introducing assumption A4 relating s and s_o,

$$n_{i,j} \gtreqless \sigma_n\sqrt{1+N}\ \delta_1 - s_{i,j} = \sigma_n\sqrt{1+N}\ \delta_1 - (\sigma_s/\sigma_o)s_{oi,j}$$

from which

$$n_{i,j}/\sigma_n \gtreqless \sqrt{1+N}\ \delta_1 - \sqrt{N}\ s_{oi,j}/\sigma_o \quad .$$

$$s_{oi,j} \gtreqless b_2 \quad \text{or} \quad s_{oi,j}/\sigma_o \gtreqless b_2/\sigma_o - \delta_2 \quad . \qquad \text{ii)}$$

Next, the normalized noise and signal variates defined in Chapter III are introduced. $t = n/\sigma_n$, $\lambda = s_o/\sigma_o$. The boundary expressions become:

$$t_{i,j} \gtreqless \sqrt{1+N}\ \delta_1 - \sqrt{N}\ \lambda_{i,j} \quad ; \quad \lambda_{i,j} \gtreqless \delta_2$$

or

$$t_{i,j} \gtreqless \eta_{2i,j} \quad ; \quad \lambda_{i,j} \gtreqless \delta_2 \quad ,$$

where

$$\eta_{2i,j} = \sqrt{1+N}\ \delta_1 - \sqrt{N}\ \lambda_{i,j} \quad .$$

The region corresponding to $+1$ is obtained by choosing the $>$ sign, -1 by choosing the $<$ sign and 0 by choosing the $=$ sign.

On introducing assumption A3 and the new assumption A8, E can be written

$$E\{y_{1i}y_{1j}y_{2i}y_{2j}\} = p_1 - p_2 - p_3 + p_4 - p_5 + p_6$$

$$+ p_7 - p_8 - p_9 + p_{10} + p_{11} - p_{12} + p_{13} - p_{14}$$

$$- p_{15} + p_{16} \quad , \tag{4.0-2}$$

where

$$p_1 = \Pr\{\lambda_i > \delta_2, \quad \lambda_j > \delta_2, \quad t_i > \eta_{2i}, \quad t_j > \eta_{2j}\}$$

$$p_2 = \Pr\{\lambda_i > \delta_2, \quad \lambda_j < \delta_2, \quad t_i > \eta_{2i}, \quad t_j > \eta_{2j}\}$$

$$p_3 = \Pr\{\lambda_i < \delta_2, \quad \lambda_j > \delta_2, \quad t_i > \eta_{2i}, \quad t_j > \eta_{2j}\}$$

$$p_4 = \Pr\{\lambda_i < \delta_2, \quad \lambda_j < \delta_2, \quad t_i > \eta_{2i}, \quad t_j > \eta_{2j}\}$$

$$p_5 = \Pr\{\lambda_i > \delta_2, \quad \lambda_j > \delta_2, \quad t_i > \eta_{2i}, \quad t_j < \eta_{2j}\}$$

$$p_6 = \Pr\{\lambda_i > \delta_2, \quad \lambda_j < \delta_2, \quad t_i > \eta_{2i}, \quad t_j < \eta_{2j}\}$$

$$p_7 = \Pr\{\lambda_i < \delta_2, \quad \lambda_j > \delta_2, \quad t_i > \eta_{2i}, \quad t_j < \eta_{2j}\}$$

$$p_8 = \Pr\{\lambda_i < \delta_2, \quad \lambda_j < \delta_2, \quad t_i > \eta_{2i}, \quad t_j < \eta_{2j}\}$$

$$p_9 = \Pr\{\lambda_i > \delta_2, \quad \lambda_j > \delta_2, \quad t_i < \eta_{2i}, \quad t_j > \eta_{2j}\}$$

$$p_{10} = \Pr\{\lambda_i > \delta_2, \quad \lambda_j < \delta_2, \quad t_i < \eta_{2i}, \quad t_j > \eta_{2j}\}$$

$$p_{11} = \Pr\{\lambda_i < \delta_2, \quad \lambda_j > \delta_2, \quad t_i < \eta_{2i}, \quad t_j > \eta_{2j}\}$$

$$p_{12} = \Pr\{\lambda_i < \delta_2, \quad \lambda_j < \delta_2, \quad t_i < \eta_{2i}, \quad t_j > \eta_{2j}\}$$

$$p_{13} = \Pr\{\lambda_i > \delta_2, \quad \lambda_j > \delta_2, \quad t_i < \eta_{2i}, \quad t_j < \eta_{2j}\}$$

$$p_{14} = \Pr\{\lambda_i > \delta_2, \quad \lambda_j < \delta_2, \quad t_i < \eta_{2i}, \quad t_j < \eta_{2j}\}$$

$$p_{15} = \Pr\{\lambda_i < \delta_2, \quad \lambda_j > \delta_2, \quad t_i < \eta_{2i}, \quad t_j < \eta_{2j}\}$$

$$p_{16} = \Pr\{\lambda_i < \delta_2, \quad \lambda_j < \delta_2, \quad t_i < \eta_{2i}, \quad t_j < \eta_{2j}\}$$

A typical integral representing the above probability expression is

$$p = \int d\lambda_i \int d\lambda_j \int dt_i \int dt_j \; w(\lambda_i, \lambda_j \; t_i, t_j)$$

where $w(\lambda_i, \lambda_j, t_i, t_j)$ is the joint density function for λ_i, λ_j, t_i and t_j and the limits are chosen according to the appropriate set of inequalities in p_1 through p_{16}. By assumption A9, n and s (and hence t and λ) are sample functions from random processes with statistically independent second order distributions, so $w(\lambda_i, \lambda_j, t_i, t_j)$ can be written $w(\lambda_i, \lambda_j, t_i, t_j) = u(t_i, t_j)v(\lambda_i, \lambda_j)$ where $u(t_i, t_j)$ is the marginal density for t_i and t_j and $v(\lambda_i, \lambda_j)$ is the marginal density for λ_i and λ_j. The above integral, written in terms of $u(t_i, t_j)$ and $v(\lambda_i, \lambda_j)$, is

$$p = \int d\lambda_i \int v(\lambda_i, \lambda_j) \left[\int dt_i \int u(t_i, t_j) dt_j \right] d\lambda_j \qquad .$$

By introducing assumption A10 that $u(t_i, t_j) = u(t_j, t_i)$ and assumption A11 that $v(\lambda_i, \lambda_j) = v(\lambda_j, \lambda_i)$, a number of relations can be established between the preceding sixteen probability expressions. These relations are obtained by applying the transformations $t_i \rightarrow t_j$, $t_j \rightarrow t_i$ and $\lambda_i \rightarrow \lambda_j$, $\lambda_j \rightarrow \lambda_i$ to the above integral with the various limits of integration as indicated by the probability expressions p_1 through p_{16}, as follows:

$$p_2) \quad p_2 = \int_{\delta_2}^{\infty} d\lambda_i \int_{-\infty}^{\delta_2} v(\lambda_i,\lambda_j)\left[\int_{\eta_{2i}}^{\infty} dt_i \int_{\eta_{2j}}^{\infty} u(t_i,t_j)dt_j\right]d\lambda_j \quad .$$

On substituting $t_i \to t_j$, $t_j \to t_i$ and $\lambda_i \to \lambda_j$, $\lambda_j \to \lambda_i$ and using the symmetric properties of $v(\lambda_i,\lambda_j)$ and $u(t_i,t_j)$, noting that due to the transformation, $\eta_{2i} \to \eta_{2j}$ and $\eta_{2j} \to \eta_{2i}$, the result is

$$p_2 = \int_{-\infty}^{\delta_2} d\lambda_i \int_{\delta_2}^{\infty} v(\lambda_i,\lambda_j)\left[\int_{\eta_{2i}}^{\infty} dt_i \int_{\eta_{2j}}^{\infty} u(t_i,t_j)dt_j\right]d\lambda_j \quad .$$

But the limits on this integral correspond to p_3. Therefore,

$$\underline{p_2 = p_3}$$

By the preceding method, relations can be found in exactly the same fashion for:

$p_5)$ $\quad\quad$ $\underline{p_5 = p_9}$

$p_6)$ $\quad\quad$ $\underline{p_6 = p_{11}}$

$p_7)$ $\quad\quad$ $\underline{p_7 = p_{10}}$

$p_8)$ $\quad\quad$ $\underline{p_8 = p_{12}}$

$p_{14})$ $\underline{p_{14} = p_{15}}$

Substituting these relations into equation 4.0-2 yields

$$E\{y_{1i}y_{1j}y_{2i}y_{2j}\} = p_1 - 2p_2 + p_4 - 2p_5 + 2p_6 + 2p_7 - 2p_8 + p_{13} - 2p_{14} + p_{16} \quad . \tag{4.0-3}$$

By introducing assumption A12 that $u(t_i,t_j) = u(-t_i,-t_j)$ and assumption A13 that $v(\lambda_i,\lambda_j) = v(-\lambda_i,-\lambda_j)$, additional relations can be established between the p's. These relations are obtained by applying the transformations $t_i \to -t_i$, $t_j \to -t_j$ and $\lambda_i \to -\lambda_i$, $\lambda_j \to -\lambda_j$ to the typical integral, with appropriate limits, as follows:

$$p_1) \quad p_1(\delta_1,\delta_2) = \int_{\delta_2}^{\infty} d\lambda_i \int_{\delta_2}^{\infty} v(\lambda_i,\lambda_j)\left[\int_{\eta_{2i}}^{\infty} dt_i \int_{\eta_{2j}}^{\infty} u(t_i,t_j)dt_j\right]d\lambda_j \ .$$

On substituting $t_i \to -t_i$, $t_j \to -t_j$ and $\lambda_i \to -\lambda_i$, $\lambda_j \to -\lambda_j$ and using the evenness properties of u and v, with the notation $\eta_{3i,j} = \sqrt{1+N}\ \delta_1 + \sqrt{N}\ \lambda_{i,j}$, the result is

$$p_1(\delta_1,\delta_2) = \int_{-\infty}^{-\delta_2} d\lambda_i \int_{-\infty}^{-\delta_2} v(\lambda_i,\lambda_j)\left[\int_{-\infty}^{-\eta_{3i}} dt_i \int_{-\infty}^{-\eta_{3j}} u(t_i,t_j)dt_j\right]d\lambda_j .$$

But $-\eta_{3i,j}(\delta_1) = \eta_{2i,j}(-\delta_1)$. Therefore,

$$p_1(\delta_1,\delta_2) = \int_{-\infty}^{-\delta_2} d\lambda_i \int_{-\infty}^{-\delta_2} v(\lambda_i,\lambda_j)\left[\int_{-\infty}^{\eta_{2i}(-\delta_1)} dt_i \int_{-\infty}^{\eta_{2j}(-\delta_1)} u(t_i,t_j)dt_j\right]d\lambda_j \ .$$

The limits on this integral correspond to $p_{16}(-\delta_1,-\delta_2)$. Therefore,

$$\underline{p_{16}(\delta_1,\delta_2) = p_1(-\delta_1,-\delta_2)}$$

By the preceding method, relations can be found in exactly the same fashion for:

p_2) $\quad p_{15}(\delta_1,\delta_2) = p_2(-\delta_1,-\delta_2)$

and from the previously derived relation between p_{14} and p_{15},

$$\underline{p_{14}(\delta_1,\delta_2) = p_2(-\delta_1,-\delta_2)} \quad .$$

p_4) $\quad \underline{p_{13}(\delta_1,\delta_2) = p_4(-\delta_1,-\delta_2)}$

p_5) $\quad p_{12}(\delta_1,\delta_2) = p_5(-\delta_1,-\delta_2) \quad ,$

and from the previously derived relation between p_8 and p_{12},

$$\underline{p_8(\delta_1,\delta_2) = p_5(-\delta_1,-\delta_2)} \quad .$$

p_6) $\quad p_{11}(\delta_1,\delta_2) = p_6(-\delta_1,-\delta_2) \quad ,$

and from the previously derived relation between p_6 and p_{11},

$$\underline{p_6(\delta_1,\delta_2) = p_6(-\delta_1,-\delta_2)} \quad .$$

$p_7)$ $$p_{10}(\delta_1,\delta_2) = p_7(-\delta_1,-\delta_2)$$

and from the previously derived relation between p_7 and p_{10},

$$\underline{p_7(\delta_1,\delta_2) = p_7(-\delta_1,-\delta_2)} \quad .$$

Substituting these relations in equation 4.0-3 yields

$$\begin{aligned} E\{y_{1i}y_{1j}y_{2i}y_{2j}\} &= [p_1(\delta_1,\delta_2) + p_1(-\delta_1,-\delta_2)] \\ &- 2[p_2(\delta_1,\delta_2) + p_2(-\delta_1,-\delta_2)] + [p_4(\delta_1,\delta_2) + p_4(-\delta_1,-\delta_2)] \\ &- 2[p_5(\delta_1,\delta_2) + p_5(-\delta_1,-\delta_2)] + [p_6(\delta_1,\delta_2) + p_6(-\delta_1,-\delta_2)] \\ &+ [p_7(\delta_1,\delta_2) + p_7(-\delta_1,-\delta_2)] \quad , \end{aligned} \tag{4.0-4}$$

where the terms in p_6 and p_7 can be replaced by

$$[p_6(\delta_1,\delta_2) + p_6(-\delta_1,-\delta_2)] = 2p_6(\delta_1,\delta_2)$$

and

$$[p_7(\delta_1,\delta_2) + p_7(-\delta_1,-\delta_2)] = 2p_7(\delta_1,\delta_2)$$

if it is convenient to do so.

From the result just derived, it is clear that $E\{y_{1i}y_{1j}y_{2i}y_{2j}\}$ is even in δ_1 and δ_2 simultaneously but not individually. I.e., $E(\delta_1,\delta_2) = E(-\delta_1,-\delta_2)$, but $E(\delta_1,\delta_2) \neq E(-\delta_1,\delta_2)$ and $E(\delta_1,\delta_2) \neq E(\delta_1,-\delta_2)$.

Equation 4.0-4 provides the general expression for $E\{y_{1i}y_{1j}y_{2i}y_{2j}\}$, subject only to the assumptions A1 through A13. Each term, p, is an integral of the form

$$\int d\lambda_i \int v(\lambda_i,\lambda_j)\left[\int dt_i \int u(t_i,t_j)dt_j\right]d\lambda_j$$

with limits as indicated by the inequalities in the table of probabilities following equation 4.0-2. These integrals cannot be further reduced without making some assumptions about the specific nature of u and v. Even when u and v are assumed to be Gaussian density functions, the integrals cannot be evaluated in closed form in terms of elementary functions or even in terms of the well known higher functions. Approximate methods of evaluating them will be discussed later. For the present, a special case will be examined for which the expression for E and also the integrals in the expression are significantly simplified.

$\underline{N = 0}$

This is the "signal absent" case, which corresponds to the null hypothesis in subsequent discussions of detection properties.

For this case, additional relations exist between p_1 and p_4 and between p_6 and p_7. To obtain the relation between p_1 and p_4, the transformation $\lambda_i \rightarrow -\lambda_i$, $\lambda_j \rightarrow -\lambda_j$ is applied to p_1. Since in this case, $\eta_{2i} = \eta_{2j} = \delta_1$, then

$$p_1(\delta_1,\delta_2) = \int_{\delta_2}^{\infty} d\lambda_i \int_{\delta_2}^{\infty} v(\lambda_i,\lambda_j)\left[\int_{\delta_1}^{\infty} dt_i \int_{\delta_1}^{\infty} u(t_i,t_j)dt_j\right]d\lambda_j \quad .$$

Upon introducing the transformation and the evenness of v in λ_i and λ_j implied by assumption A13, this integral becomes

$$p_1(\delta_1,\delta_2) = \int_{-\infty}^{-\delta_2} d\lambda_i \int_{-\infty}^{-\delta_2} v(\lambda_i,\lambda_j)\left[\int_{\delta_1}^{\infty} dt_i \int_{\delta_1}^{\infty} u(t_i,t_j)dt_j\right]d\lambda_j \quad .$$

But the limits on this integral correspond to $p_4(\delta_1,-\delta_2)$ when $N = 0$. Therefore,

$$\underline{p_4(\delta_1,\delta_2) = p_1(\delta_1,-\delta_2)} \qquad \text{when} \quad N = 0$$

To obtain the relation between p_6 and p_7, the transformation $\lambda_i \rightarrow \lambda_j$, $\lambda_j \rightarrow \lambda_i$ is applied to p_6.

$$p_6 = \int_{\delta_2}^{\infty} d\lambda_i \int_{-\infty}^{\delta_2} v(\lambda_i,\lambda_j)\left[\int_{\delta_1}^{\infty} dt_i \int_{-\infty}^{\delta_1} u(t_i,t_j)dt_j\right] d\lambda_j \quad .$$

Upon introducing the transformation and the symmetry of v in λ_i and λ_j implied by assumption A11, this integral becomes

$$p_1 = \int_{-\infty}^{\delta_2} d\lambda_i \int_{\delta_2}^{\infty} v(\lambda_i,\lambda_j)\left[\int_{\delta_1}^{\infty} dt_i \int_{-\infty}^{\delta_1} u(t_i,t_j)dt_j\right]d\lambda_j \quad .$$

But the limits on this integral correspond to p_7 when $N = 0$. Therefore,

$$\underline{p_7 = p_6} \qquad \text{when} \quad N = 0.$$

When these relations are substituted into equation 4.0-4, the result is

$$E\{y_{1i}y_{1j}y_{2i}y_{2j}\} = [p_1(\delta_1,\delta_2) + p_1(-\delta_1,-\delta_2)$$

$$+ p_1(\delta_1,-\delta_2) + p_1(-\delta_1,\delta_2)] - 2[p_2(\delta_1,\delta_2) + p_2(-\delta_1,-\delta_2)]$$

$$- 2[p_5(\delta_1,\delta_2) + p_5(-\delta_1,-\delta_2)] + 2[p_6(\delta_1,\delta_2) + p_6(-\delta_1,-\delta_2)] \ , \qquad (4.0\text{-}5)$$

when $N = 0$.

The integrals representing the individual p's will now be reduced to simpler forms. First, some notation will be introduced. Two integrals will appear repeatedly. They are:

<u>$H_g(h,k;r)$</u>

$$H_g(h,k;r) = \int_h^\infty dx \int_k^\infty g(x,y;r)dy \qquad (4.0\text{-}6)$$

where r is a parameter of the integrand, $g(x,y;r)$.

<u>$\psi_q(h/\sqrt{2})$</u>

$$\psi_q(h/\sqrt{2}) = 2\int_0^h q(x)dx \quad . \qquad (4.0\text{-}7)$$

The apparently awkward argument, $h/\sqrt{2}$, will seem less awkward when the assumption is made that $q(x)$ is the density function for a standard normal distribution. Then $\psi_q(x)$ becomes the error function, $\mathrm{Erf}(x)$.

The function $H_g(h,k;r)$ will be interpreted later as the volume under the surface $g(x,y;r)$ over the region in the x,y-plane to the right of the vertical line $x = h$ and above the horizontal line $y = k$, as shown in figure 4.0-1. The properties of H are discussed in Appendix III for the special case when $g(x,y;r)$ is the density function for a standard bivariate normal distribution with correlation coefficient r. However, the properties of H which are required for the remainder of the present section are identical to the corresponding properties discussed in Appendix III, since a sufficient condition for their validity is that $g(x,y;r)$ be a density function symmetric in x and y and even simultaneously in x and

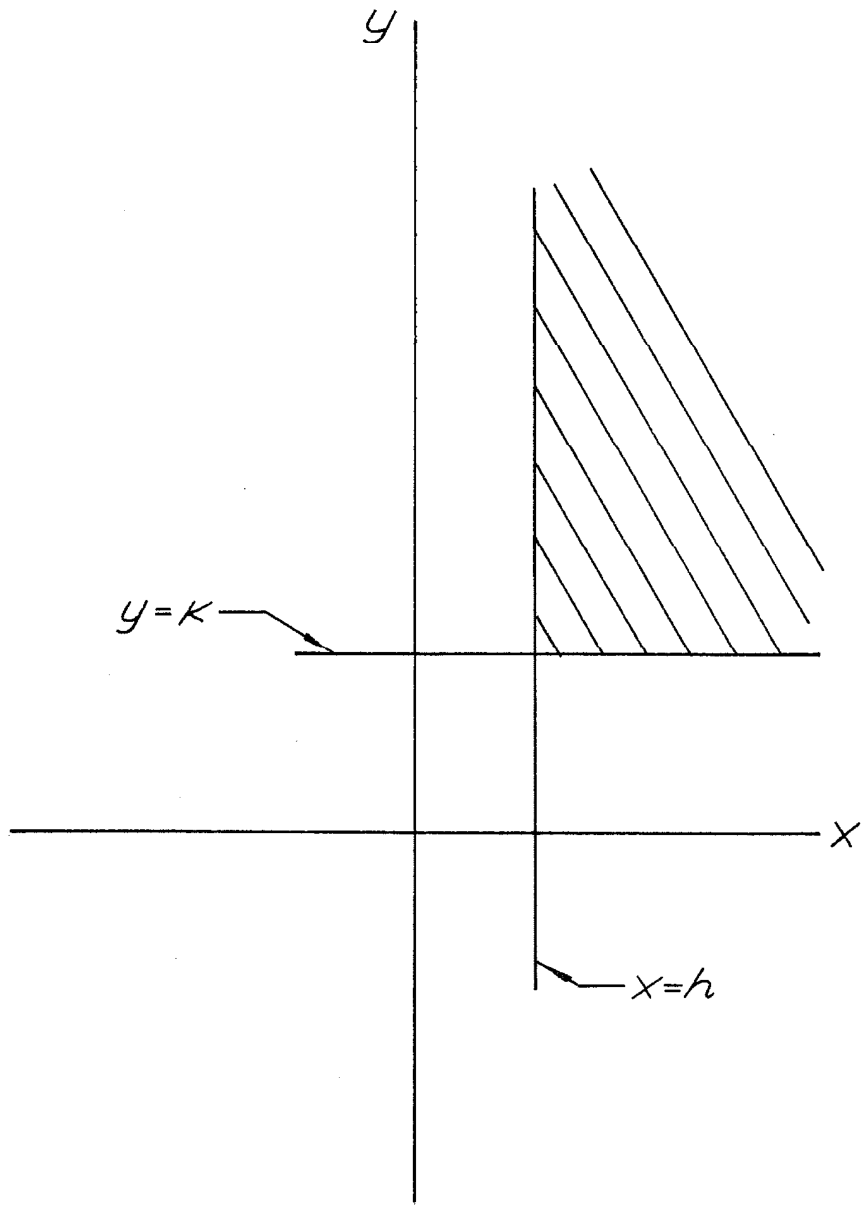


Figure 4.0-1

y - a condition satisfied by the standard bivariate normal density function of Appendix III. Therefore, in the remainder of this section, statements about H will be referred to the corresponding derivation in Appendix III.

The individual p's are reduced as follows:

p_1)
$$p_1 = \int_{\delta_2}^{\infty} d\lambda_i \int_{\delta_2}^{\infty} v(\lambda_i,\lambda_j;r)d\lambda_j \int_{\delta_1}^{\infty} dt_i \int_{\delta_1}^{\infty} u(t_i,t_j;\rho)dt_j$$

$$= H_u(\delta_1,\delta_1;\rho)\cdot H_v(\delta_2,\delta_2;r)$$

where the parameter ρ introduced into the density function u and the parameter r introduced into the density function v will be interpreted later as the correlation coefficients for these two densities.

p_2)
$$p_2 = \int_{\delta_2}^{\infty} d\lambda_i \int_{-\infty}^{\delta_2} v(\lambda_i,\lambda_j;r)d\lambda_j \int_{\delta_1}^{\infty} dt_i \int_{\delta_1}^{\infty} u(t_i,t_j;\rho)dt_j$$

$$= H_u(\delta_1,\delta_1;\rho)\left[\int_{\delta_2}^{\infty} d\lambda_i\left\{\int_{-\infty}^{\infty} v(\lambda_i,\lambda_j;r)d\lambda_j - \int_{\delta_2}^{\infty} v(\lambda_i,\lambda_j;r)d\lambda_j\right\}\right]$$

$$= H_u(\delta_1,\delta_1;\rho)\left[\int_{\delta_2}^{\infty} v(\lambda_i)d\lambda_i - H_v(\delta_2,\delta_2;r)\right]$$

$$= H_u(\delta_1,\delta_1;\rho)\left[\tfrac{1}{2}\{1-\psi_v(\delta_2/\sqrt{2})\} - H_v(\delta_2,\delta_2;r)\right] \quad ,$$

where $v(\lambda_i)$ is the marginal density for λ_i.

$$v(\lambda_i) = \int_{-\infty}^{\infty} v(\lambda_i,\lambda_j;r)d\lambda_j \quad .$$

$p_5)$

$$p_5 = \int_{\delta_2}^{\infty} d\lambda_i \int_{\delta_2}^{\infty} v(\lambda_i,\lambda_j;r)d\lambda_j \int_{\delta_1}^{\infty} dt_i \int_{-\infty}^{\delta_1} u(t_i,t_j;\rho)dt_j$$

$$= \left[\int_{\delta_1}^{\infty} dt_i \left\{\int_{-\infty}^{\infty} u(t_i,t_j;\rho)dt_j - \int_{\delta_1}^{\infty} u(t_i,t_j;\rho)dt_j\right\}\right] H_v(\delta_2,\delta_2;r)$$

$$= \left[\int_{\delta_1}^{\infty} u(t_i)dt_i - H_u(\delta_1,\delta_1;\rho)\right] H_v(\delta_2,\delta_2;r)$$

$$= \left[\tfrac{1}{2}\{1 - \psi_u(\delta_1/\sqrt{2})\} - H_u(\delta_1,\delta_1;\rho)\right] H_v(\delta_2,\delta_2;r)$$

where $u(t_i)$ is the marginal density for t_i.

$$u(t_i) = \int_{-\infty}^{\infty} u(t_i,t_j;\rho)dt_i \quad .$$

p_6)

$$p_6 = \int_{\delta_2}^{\infty} d\lambda_i \int_{-\infty}^{\delta_2} v(\lambda_i,\lambda_j;r)d\lambda_j \int_{\delta_1}^{\infty} dt_i \int_{-\infty}^{\delta_1} u(t_i,t_j;\rho)dt_j$$

$$= \int_{\delta_1}^{\infty} dt_i \left[\int_{-\infty}^{\infty} u(t_i,t_j;\rho)dt_j - \int_{\delta_1}^{\infty} u(t_i,t_j;\rho)dt_j \right]$$

$$\times \int_{\delta_2}^{\infty} d\lambda_i \left[\int_{-\infty}^{\infty} v(\lambda_i,\lambda_j;r)d\lambda_j - \int_{\delta_2}^{\infty} v(\lambda_i,\lambda_j;r)d\lambda_j \right]$$

$$= \left[\int_{\delta_1}^{\infty} u(t_i)dt_i - H_u(\delta_1,\delta_1;\rho) \right] \left[\int_{\delta_2}^{\infty} v(\lambda_i)d\lambda_i - H_v(\delta_2,\delta_2;r) \right]$$

$$= \left[\tfrac{1}{2}\{1 - \psi_u(\delta_1/\sqrt{2})\} - H_u(\delta_1,\delta_1;\rho) \right] \times \left[\tfrac{1}{2}\{1 - \psi_v(\delta_2/\sqrt{2})\} - H_v(\delta_2,\delta_2;r) \right]$$

The preceding expressions for p_1, p_2, p_5 and p_6 can be further simplified by a result presented in Appendix III. According to property p7 of Appendix III,

$$H_g(-h,-k;r) = H_g(h,k;r) + \tfrac{1}{2}\,\psi_q(h/\sqrt{2}) + \tfrac{1}{2}\,\psi_q(k/\sqrt{2}) \quad ,$$

where in this case, the subscript q on ψ corresponds to the marginal density of g in either variate. (Due to the assumption of wide-sense stationarity, A2, the marginal densities of g in both variates are identical.)

$$q(x) = \int_{-\infty}^{\infty} g(x,y)dy \quad \text{or} \quad q(y) = \int_{-\infty}^{\infty} g(x,y)dx \quad .$$

Substituting $h = k = \delta_1$ or δ_2, whichever is appropriate, into the expressions for p_1, p_2, p_5 and p_6 yields

$$p_1(-\delta_1,-\delta_2) = [H_u(\delta_1,\delta_1;\rho) + \psi_u(\delta_1/\sqrt{2})]$$

$$\times[H_v(\delta_2,\delta_2;r) + \psi_v(\delta_2/\sqrt{2})]$$

$$= [H_u + \psi_u]\,[H_v + \psi_v] \quad ,$$

where $H_u = H_u(\delta_1,\delta_1;\rho)$, $H_v = H_v(\delta_2,\delta_2;r)$, $\psi_u = \psi_u(\delta_1/\sqrt{2})$ and $\psi_v = \psi_v(\delta_2/\sqrt{2})$.

$$p_1(\delta_1,-\delta_2) = H_u[H_v + \psi_v] \quad ,$$

$$p_1(-\delta_1,\delta_2) = [H_u + \psi_u]H_v \quad ,$$

$$p_2(-\delta_1,-\delta_2) = [H_u + \psi_u]\,[\tfrac{1}{2} - \tfrac{1}{2}\,\psi_v - H_v] \quad ,$$

$$p_5(-\delta_1,-\delta_2) = [\tfrac{1}{2} - \tfrac{1}{2}\,\psi_u - H_u]\,[H_v + \psi_v] \quad ,$$

and

$$p_6(-\delta_1,-\delta_2) = [\tfrac{1}{2} - \tfrac{1}{2}\,\psi_u - H_u]\,[\tfrac{1}{2} - \tfrac{1}{2}\,\psi_v - H_v] \quad .$$

Substituting the above expressions for the p's in equation 4.0-5 yields

$$E\{y_{1i}y_{1j}y_{2i}y_{2j}\} = 1 - 2(\psi_u + \psi_v) + 4\psi_u\psi_v + 16H_uH_v$$

$$- 4(H_u + H_v) + 8(H_u\psi_v + H_v\psi_u)$$

or

$$E\{y_{1i}y_{1j}y_{2i}y_{2j}\} = [4H_u + 2\psi_u - 1]\ [4H_v + 2\psi_v - 1] \quad . \tag{4.0-8}$$

Equation 4.0-8 provides the general expression for $E\{y_{1i}y_{1j}y_{2i}y_{2j}\}$ when $N = 0$ - the "signal absent" case - subject only to the assumptions A1 through A13. This result will be used in the next section to derive E and hence σ_z^2 when the signal and noise are Gaussian and $N = 0$.

4.1 $E\{y_{1i}y_{1j}y_{2i}y_{2j}\}$ when the Signal and Noise Both are Gaussian.

In this section the effects of bias on $E\{y_{1i}y_{1j}y_{2i}y_{2j}\}$ are considered for inputs consisting of Gaussian signal and Gaussian noise. Therefore, in addition to the assumptions introduced previously (assumptions A1 through A13), it will now be assumed that the noise, $n(t)$, is a sample function from a stationary Gaussian random process with zero mean, with variance σ_n^2, and with correlation coefficient $\rho(\tau)$, where τ is the time difference,

$\theta - t$ and that the signal, $s(t)$, is a sample function from a stationary Gaussian random process with zero mean, with variance σ_s^2, and with correlation coefficient $r(\tau)$.

(A4-1) $n(t)$ is a sample function from a stationary random process with second order density function

$$f(n_i, n_j) = \frac{1}{2\pi\sigma_n^2\sqrt{1-\rho^2}} \exp\left[- \frac{n_i^2 - 2\rho n_i n_j + n_j^2}{2\sigma_n^2(1-\rho^2)} \right]$$

where $n_i = n(t)$, $n_j = n(\theta)$, and $\rho = \rho(\tau)$ with $\tau = \theta - t$; and $s_o(t)$ is a sample function from a stationary random process with second order density function

$$h(s_{oi}, s_{oj}) = \frac{1}{2\pi\sigma_o^2\sqrt{1-r^2}} \exp\left[- \frac{s_{oi}^2 - 2r s_{oi} s_{oj} + s_{oj}^2}{2\sigma_o^2(1-r^2)} \right]$$

where $s_{oi} = s_o(t)$, $s_{oj} = s_o(\theta)$, and $r = r(\tau)$ with $\tau = \theta - t$.

Assumption A4-1 is specified in terms of s_o rather than s, since it is s_o which is used in the subsequent equations. The random variables s and s_o have identical distributions except for their variances which are related by the expression $a = \sigma_o/\sigma_s$.

Upon making the transformation $t_i = n_i/\sigma_n$, $t_j = n_j/\sigma_n$ and $\lambda_i = s_{oi}/\sigma_o$, $\lambda_j = s_{oj}/\sigma_o$, the density functions for t_i, t_j and λ_i, λ_j become in this case

$$u(t_i, t_j; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[- \frac{t_i^2 - 2\rho t_i t_j + t_j^2}{2(1-\rho^2)} \right] \qquad (4.1\text{-}1)$$

and

$$v(\lambda_i, \lambda_j; r) = \frac{1}{2\pi\sqrt{1-r^2}} \exp\left[- \frac{\lambda_i^2 - 2r\lambda_i \lambda_j + \lambda_j^2}{2(1-r^2)} \right] \quad .$$

Substitution of these functions into the integral expressions for the terms of equation 4.0-4, the general equation for E, yields no significant simplifications. The integrals representing the various terms in E (twelve in all) can be written in terms of integrals of the product of a quadratic exponential function with the H function defined in the previous section. A typical integral of the type which occurs is

$$p_1 = \int_{\delta_2}^{\infty} d\lambda_i \int_{\delta_2}^{\infty} v(\lambda_i, \lambda_j; r) H_u(\sqrt{1+N}\ \delta_1 - \sqrt{N}\ \lambda_i,\ \sqrt{1+N}\ \delta_1 - \sqrt{N}\ \lambda_j; \rho) d\lambda_j$$

where according to assumption A4-1 of this section, $v(\lambda_i, \lambda_j; r)$ is an exponential function with a quadratic exponent.

The Freedman approximation function, by which approximate values of $V(p, \gamma p)$ were obtained for use in Chapter III (see Appendixes II and V), can be used to evaluate these integrals approximately in terms of the H and ψ functions defined in the previous section, when $u(t_i, t_j; \rho)$ and $v(\lambda_i, \lambda_j; r)$ are normal density functions. Unfortunately, however, this method has a number of serious limitations. First, the number of integrals which must be evaluated is impractically large. This is due to the fact that the Freedman function has two distinct functional forms depending on the sign of the argument of the error function which it is approximating. Thus, when the Freedman function is substituted for the error function in the integrand of the integral for H, four distinct cases arise, depending on the relation between the variables and the parameter in the argument of H. Four distinct representations for H result, again depending on the relation between the variables and the parameter in the argument of H. Each of these representations is in terms of error functions and quadratic exponential functions, both of which have complicated arguments. In order to proceed, the error functions must be replaced by their Freedman function equivalents. Again, new distinct cases must be considered, depending on the signs of the arguments of the error functions. The total number of distinct forms is now fifteen. These forms are expressed in terms of quadratic exponential functions with very complicated arguments. Seven of the forms have three terms and eight of the forms have six terms. This is a total of sixty-nine terms. By the steps just described, the integrands of each of the twelve p integrals can be reduced approximately to sixty-nine terms of quadratic exponential functions. These can be integrated over limits of the type which occur in the p integrals in terms of the H and ψ functions. However, since there are twelve p integrals to be evaluated, each with sixty-nine terms, the total number of integral terms to be evaluated is eight-hundred twenty-eight - indeed an impractical number!

A second limitation now appears. A simple analysis of the error in E due to the errors of approximation, such as the analysis in Appendix II for the approximation to $V(p,\gamma p)$, is not possible. Although the error in the approximation to the H function is quite small, and the error in the individual terms of the p integrals due to replacing the H function by its approximating function is not very large, nevertheless the error in the sum of the series of eight-hundred twenty-eight terms may be large compared with the sum itself, since in the sum the number of subtractions and additions are about equal. It would require an exhaustive error analysis to estimate the relative error of the sum.

The third limitation is that the final expressions for the p's in terms of the H and ψ functions have arguments in which r and ρ enter in a very complicated manner. Ultimately, it will be necessary to integrate twice in time, in order to obtain expressions for $E\{z^2\}$ and hence σ_z^2. Since r and ρ are functions of the time difference, it seems extremely unlikely that the time integrals could be evaluated.

The most serious of the three limitations in using the Freedman approximation function method is the multiplicity of forms which result from the two distinct forms which the Freedman function has - one for positive arguments of the error function to be approximated, and one for negative arguments. A considerable effort was expended in attempting to find satisfactory approximating functions which do not have this limitation. Surprisingly simple functions were found which can be substituted for the error function in integrals of the types for $V(p,\gamma p)$ and $H_g(h,k;r)$ with only a small resulting error. For example, the function $a \cdot x$ where a is an appropriately chosen constant, is a suitable approximation to $Erf(x)$ in the integral for $V(p,\gamma p)$, with a maximum resulting error of about 0.125. Unfortunately, this error represents a relative error of about 0.5, which is much too large for the applications in this thesis. The approximation $a \cdot x$ is a little too coarse. The function $ae^{-\frac{1}{2}bx^2}$, where a and b are appropriately chosen constants, produces a much smaller error than $a \cdot x$ when substituted for $Erf(x)$ in the integrals for V or H. However, it, like the Freedman function, must assume a different form (namely $-ae^{-\frac{1}{2}bx^2}$) for negative x. As a consequence, it suffers from the same limitations as the Freedman function as regards the number of distinct cases and the resulting number of terms in the integrals. No satisfactory approximating functions were found for the evaluation of the p integrals.

In view of the limitations associated with the approximating function method of evaluating the p integrals, other methods were investigated. The most promising appears to be the following: The p integrals are integrals with a product of two bivariate density functions for the integrand, over regions with linear boundaries. When the distributions corresponding to these two density functions are both standard bivariate normal distributions, these integrals have the form whose importance was mentioned in the summary of related literature in Chapter I. A typical such integral is

$$p_1 = \int_{\delta_2}^{\infty} d\lambda_i \int_{\delta_2}^{\infty} d\lambda_j \int_{\eta_{2i}}^{\infty} dt_i \int_{\eta_{2j}}^{\infty} dt_j v(\lambda_i,\lambda_j;r)u(t_i,t_j;\rho) \quad ,$$

where u and v are standard normal bivariate density functions and η_{2i} and η_{2j} are linear in λ_i and λ_j respectively. Integrals of this type have been studied extensively by statisticians (1,2,3 and 4). For these integrals there always exists a linear transformation such that in the transformed space the limits of integration are all constant (here the transformation is $\lambda_i \to x_1$, $\lambda_j \to x_2$, $t_i \to \sqrt{1+N}\, x_3 - \sqrt{N}\, x_1$, $t_j \to \sqrt{1+N}\, x_4 - \sqrt{N}\, x_2$; then the limits become $\infty \to \infty$, $-\infty \to -\infty$, $\delta_2 \to \delta_2$, $\eta_{2i} \to \delta_1$, $\eta_{2j} \to \delta_1$). Under such transformations, the integrand becomes a quadrivariate normal density function.

Kendall (1, above) has shown that the resulting integral can be expanded in a two dimensional power series in r and ρ, with coefficients which are generalized tetrachoric functions. These tetrachoric functions are expressible as finite series of Hermite polynomials.

In order to obtain $E\{z^2\}$, this two-dimensional power series would be integrated term by term twice in time until the remainder were sufficiently small to make the error in neglecting it acceptable. Thus, it would be necessary to integrate polynomials in r and ρ twice with respect to time, where r and ρ are functions of the time difference, τ. If the forms of r and ρ are suitably chosen (e.g. if r and ρ correspond to RC low-pass spectra for signal and noise), these integrals can be evaluated. Since the series converges rather slowly except for small r and ρ, it would probably be desirable to evaluate the general integral term in closed form, provide this functional form as part of the information stored in a computer memory, and have the computer calculate the sum of as many terms of the integral series as are necessary for the precision required.

Because the tetrachoric series method involves extensive numerical calculations, it will not be developed further in this thesis. The method will be developed and presented in the future as a separate report. Instead, a simplified analysis will be given here which is valid only in a restricted sense. In order to proceed with the development of the simplified analysis, several new assumptions and restrictions will be introduced. However, arguments will be presented which indicate that even with these restrictions, a substantial knowledge of the behavior of the BPCD is acquired.

The first restriction to be introduced is that the signal-to-noise power ratio, N, is small with respect to unity. When N is small, then $E\{z^2\}$, the expected value of the BPCD output, will be due almost entirely to the noise. Therefore for small N, the $N = 0$ value of $E\{z^2\}$ can be used with negligible error. It will be argued later that as the signal-to-noise power ratio increases, the detection quality of a BPCD improves or at least does not deteriorate. Thus, the $N = 0$ case constitutes a lower bound on the detection properties of a BPCD.

$\underline{N = 0}$

Equation 4.0-8, which is the general expression for $E\{y_{1i}y_{1j}y_{2i}y_{2j}\}$ when $N = 0$, is given in terms of the H and ψ functions defined in section 4.0. The first step in analyzing the $N = 0$ case for Gaussian signal and Gaussian noise, then, is to find expressions for H and ψ when the integrands in the defining integrals are normal density functions.

When $g(x,y;r)$ is a standard bivariate normal density function with correlation coefficient r, then $H_g(h,k;r)$ as defined by equation 4.0-6 is denoted by $L(h,k;r)$ and has the equation

$$L(h,k;r) = \int_h^\infty dx \int_k^\infty \frac{1}{2\pi\sqrt{1-r^2}} \exp\left[-\frac{x^2-2rxy+y^2}{2(1-r^2)}\right] dy \quad . \qquad (4.1\text{-}2)$$

This integral is one of the multivariate Gaussian integrals with linear boundaries of integration whose importance to the analysis of the BPCD was mentioned in the summary of related literature in Chapter I. This integral is important in statistics. Consequently, a thorough investigation of its properties along with tabulations of its values can be found in the literature (4).

$L(h,k;r)$ and $L(h,k;-r)$ are tabulated in tables I and II of reference 4. Both functions are tabulated only for positive values of h and k, since functional values of L for negative values of h and/or k can be obtained from functional values of L for positive h and k, as shown in Appendix III.

A detailed discussion of the properties of $L(h,k;r)$, based essentially on that found in reference 4, is given in Appendix III.

For later convenience, the simplest properties are listed below without proof.

$$L(h,k;r) = L(k,h;r) \tag{p1}$$

$$L(h,k;0) = \frac{1}{4}\,[1 - \mathrm{Erf}(h/\sqrt{2})]\,[1 - \mathrm{Erf}(k/\sqrt{2})] \tag{p2}$$

$$L(h,k;-1) = \begin{cases} 0 & \text{if } h+k \geq 0 \\ -\frac{1}{2}[\mathrm{Erf}(h/\sqrt{2}) + \mathrm{Erf}(k/\sqrt{2})] & \text{if } h+k \leq 0 \end{cases} \tag{p3}$$

$$L(h,k;1) = \begin{cases} \frac{1}{2}[1 - \mathrm{Erf}(h/\sqrt{2})] & \text{if } k \leq h, \\ \frac{1}{2}[1 - \mathrm{Erf}(k/\sqrt{2})] & \text{if } k \geq h \end{cases} \tag{p4}$$

$$L(-h,k;r) = -L(h,k;-r) + \tfrac{1}{2}[1 - \mathrm{Erf}(k/\sqrt{2})] \tag{p5}$$

$$L(h,-k;r) = -L(h,k;-r) + \tfrac{1}{2}[1 - \mathrm{Erf}(h/\sqrt{2})] \tag{p6}$$

$$L(-h,-k;r) = L(h,k;r) = \tfrac{1}{2}[\mathrm{Erf}(h/\sqrt{2}) + \mathrm{Erf}(k/\sqrt{2})] \tag{p7}$$

$$L(0,0;r) = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}(r) \tag{p8}$$

$$L(\infty,k;r) = L(h,\infty;r) = 0 \tag{p9}$$

$$L(-\infty,k;r) = \tfrac{1}{2}[1 - \mathrm{Erf}(k/\sqrt{2})] \tag{p10}$$

$$L(h,-\infty;r) = \tfrac{1}{2}[1 - \mathrm{Erf}(h/\sqrt{2})] \tag{p11}$$

An approximate evaluation of the integral for $L(h,k;r)$ when h and k are equal and are small is examined in Appendix IV.

When $q(x)$ is a standard normal density function, then $\psi_q(h/\sqrt{2})$ as defined by equation 4.0-7 becomes

$$\psi_q = 2\int_0^h \frac{1}{2\pi} e^{-\frac{1}{2}x^2} dx \quad .$$

On substituting $u = x/\sqrt{2}$, this becomes

$$\psi_q = \int_0^{h/\sqrt{2}} \frac{2}{\pi} e^{-u^2} du = \mathrm{Erf}(h/\sqrt{2}) \quad . \qquad (4.1\text{-}3)$$

Therefore, when $q(x)$ is the standard normal density function, $\psi_q(h/\sqrt{2})$ equals the error function, $\mathrm{Erf}(h/\sqrt{2})$, whose properties are well known and will not be reviewed here.

Using the preceding results, if equations 4.1-1 are substituted in equation 4.0-8, then

$$E\{y_{1i}y_{1j}y_{2i}y_{2j}\} = [4L(\delta_1,\delta_1;\rho) + 2\mathrm{Erf}(\delta_1/\sqrt{2}) - 1]$$

$$\times [4L(\delta_2,\delta_2;r) + 2\mathrm{Erf}(\delta_2/\sqrt{2}) - 1] \quad . \qquad (4.1\text{-}4)$$

In Appendix III a relation between $L(h,h;r)$ and $V(p,\gamma p)$ is derived and presented in property p13, where $V(p,\gamma p)$ is the function defined in Appendix I and used extensively in Chapter III. If this relation, p13 of Appendix III, is substituted in the preceding equation, the result is

$$E\{y_{1i}y_{1j}y_{2i}y_{2j}\} = \left[8V(\delta_1,\sqrt{\frac{1-\rho}{1+\rho}}\,\delta_1) + \frac{2}{\pi}\sin^{-1}(\rho)\right]$$

$$\times\left[8V(\delta_2,\sqrt{\frac{1-r}{1+r}}\,\delta_2) + \frac{2}{\pi}\sin^{-1}(r)\right] \quad . \tag{4.1-5}$$

The value of the factor

$$F(p;x) = \left[8V(p,\sqrt{\frac{1-x}{1+x}}\,p) + \frac{2}{\pi}\sin^{-1}(x)\right]$$

is of interest for the values $x = 1$, 0 and -1.

$$F(p,1) = 1 \tag{f1}$$

$$F(p,0) = 8V(p,p) \quad (\text{Note that } 0 \le 8V(p,p) \le 1 \text{ for all } p.) \tag{f2}$$

$$F(p,-1) = 2\cdot\mathrm{Erf}(|p|/\sqrt{2}) - 1 \tag{f3}$$

These values give some idea of the way in which the factor F varies with the correlation x. Of course, $0 \le |E| \le 1$ always, so $0 \le |F(p,x)| \le 1$ for all p and x.

δ_1 and δ_2 Small

Next, it will be assumed that δ_1 and δ_2 are small compared with unity. The restriction to small δ_1 and δ_2 does not constitute a practical limitation, since a system is indeed poorly designed if the normalized biases are not small (i.e. if the actual biases are not small compared with the input amplitudes). Further, it will be assumed that $\sqrt{\frac{1-r}{1+r}}$ and $\sqrt{\frac{1-\rho}{1+\rho}}$ are not much larger than unity for any value of τ in the interval $0 \le \tau \le T$. The implications of this last assumption with respect to the types of systems which satisfy it will be discussed later. These two assumptions suffice for approximating $V(p,\gamma p)$ by $V_o(p,\gamma p)$ as defined by equation A2.0-1 of Appendix II. If γ is not very much greater than unity and if p is small compared with unity, then V_o is a good approximation to V.

If the V_o approximation is substituted for V in equation 4.1-5, from equation A2.0-1, the result is

$$E\{y_{1i}y_{1j}y_{2i}y_{2j}\} = \frac{4}{\pi^2}\left[\delta_1^2\sqrt{\frac{1-\rho}{1+\rho}} + \sin^{-1}(\rho)\right] \times\left[\delta_2^2\sqrt{\frac{1-r}{1+r}} + \sin^{-1}(r)\right] \quad . \qquad (4.1\text{-}6)$$

This result will be used in the next section to obtain $E\{z^2\}$ and σ_z^2.

4.2 The Variance of the BPCD Output

The expected value of z^2 is, from equation 4.0-1,

$$E\{z^2\} = \int_o^T \int_o^T \{E\ y_{1i}y_{1j}y_{2i}y_{2j}\}\ dt\ d\theta \quad .$$

In the following development, the expression for E given by equation 4.1-6 will be used, and E will be denoted by $E\{\delta_1,\delta_2;r(\tau),\rho(\tau)\}$ to emphasize the dependence on the parameters δ_1, δ_2, $r(\tau)$ and $\rho(\tau)$. Since r and ρ are both functions of

$\tau = \theta - t$, it will be convenient to make the transformation $\tau = \theta - t$, $\lambda = \theta + t$. The preceding integral then becomes

$$E\{z^2\} = \frac{1}{2} \iint_R E\{\delta_1, \delta_2; r(\tau), \rho(\tau)\} d\theta \, d\tau$$

where the new region of integration, R, is shown in figure 4.2-1. Since both $r(\tau)$ and $\rho(\tau)$ are even functions for real processes, the integral can be taken over the right half of R and multiplied by 2.

$$E\{z^2\} = \int_0^T d\tau \int_\tau^{2T-\tau} E\{\delta_1, \delta_2; r(\tau), \rho(\tau)\} d\lambda \quad .$$

Since λ does not appear in the integrand, the integral with respect to λ can be taken immediately. Thus,

$$E\{z^2\} = 2 \int_0^T (T - \tau) \cdot E\{\delta_1, \delta_2; r(\tau), \rho(\tau)\} d\tau \quad .$$

On substituting from equation 4.1-6, this becomes

$$\begin{aligned} E\{z^2\} = {} & \frac{8}{\pi^2} \delta_1^2 \delta_2^2 \int_0^T (T - \tau) \sqrt{\frac{1-\rho}{1+\rho}} \sqrt{\frac{1-r}{1+r}} \, d\tau \\ & + \frac{8}{\pi^2} \delta_1^2 \int_0^T (T - \tau) \sqrt{\frac{1-\rho}{1+\rho}} \sin^{-1}(r) d\tau \\ & + \frac{8}{\pi^2} \delta_2^2 \int_0^T (T-\tau) \sqrt{\frac{1-r}{1+r}} \sin^{-1}(\rho) d\tau + \frac{8}{\pi^2} \int_0^T (T-\tau) \sin^{-1}(\rho) \sin^{-1}(r) d\tau . \end{aligned} \quad (4.2\text{-}1)$$

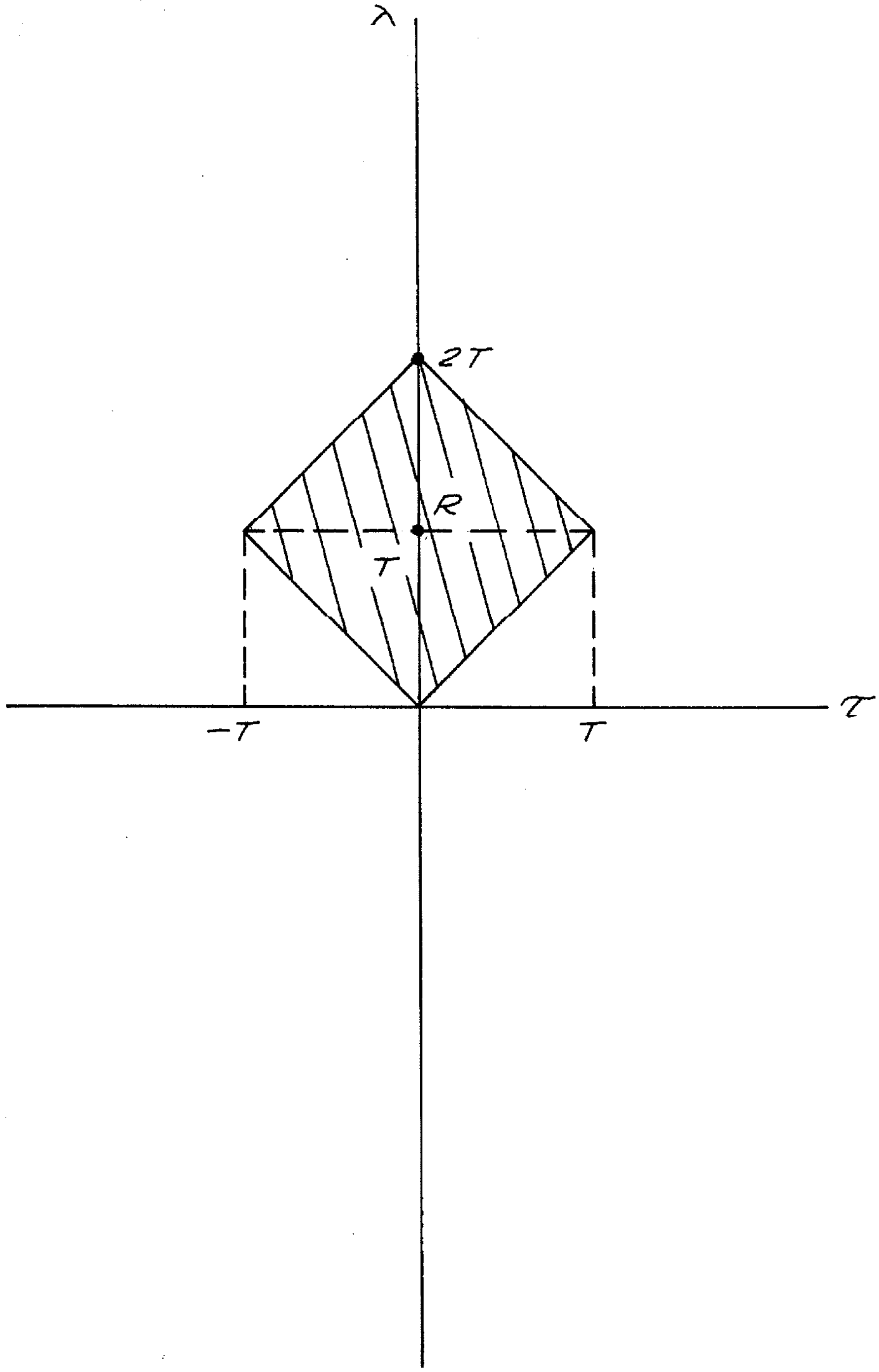


Figure 4.2-1

The last term in equation 4.2-1 represents $E\{z^2\}$ when $\delta_1 = \delta_2 = 0$, i.e. for a perfect polarity-coincidence detector. The first term will be shown later to be μ_o^2, the square of the output mean value for $N = 0$. Since all four terms are positive, it follows that $E\{z^2\}$ and σ_z^2 for a BPCD both are larger than for a perfect polarity-coincidence detector (subject to the assumptions and restrictions introduced above).

In order to proceed further in the development of an expression for the variance, it will be necessary to make additional assumptions. These assumptions will be consistent with the assumption most recently introduced that $\sqrt{\frac{1-r}{1+r}}$ and $\sqrt{\frac{1-\rho}{1+\rho}}$ are not much larger than unity for any value of τ in the interval $0 \le \tau \le T$, and in fact the new assumptions to be introduced will supercede this assumption. Therefore, before further specializing the problem, a brief discussion will be given of some of the types of systems which satisfy the conditions of this previous assumption.

In the first place, the quantity $\sqrt{\frac{1-x}{1+x}}$, where $-1 \le x \le 1$, can be greater than unity only if x is negative; it becomes much larger than unity only as x approaches -1. Therefore, the condition on $\sqrt{\frac{1-r}{1+r}}$ and $\sqrt{\frac{1-\rho}{1+\rho}}$ is equivalent to the assumption that $r(\tau)$ and $\rho(\tau)$ both stay well away from -1 for all values of τ in the interval $0 \le \tau \le T$. Three examples follow - two for which the assumption is valid and one for which it is not valid.

Example 1) A signal (or noise) which has an RC low-pass spectrum has the correlation coefficient $r(\tau) = e^{-|\tau|/\tau_s}$, where τ_s is the correlation duration constant. Clearly, in this case $r(\tau)$ is never negative, so $\sqrt{\frac{1-r}{1+r}}$ is less than unity for all τ.

Example 2) A signal with a rectangular spectrum has the correlation coefficient $r(\tau) = [\sin(2\pi F\tau]/(2\pi F\tau)$, where F is the (one sided) spectral width in cps. The first minimum in $r(\tau)$ occurs for $2\pi F\tau \cong 4.49$, for which $r(\tau) = -0.217$. All subsequent minima have even smaller magnitudes, so that $\sqrt{\frac{1-r}{1+r}}$ is never larger than 1.25 for any τ.

Example 3) A signal with a rectangular band-pass spectrum has the correlation coefficient $r(\tau) = [\cos(2\pi f_c\tau)][\sin(2\pi\frac{B}{2}\tau]/(2\pi\frac{B}{2}\tau)$, where f_c is the center frequency and B is the band-width. The first minimum occurs for $2\pi f_c\tau \cong \pi$, for which $r(\tau) = -\sin(\frac{\pi}{2}\frac{B}{f_c})/(\frac{\pi}{2}\frac{B}{f_c})$. If B/f_c is $0.01\frac{2}{\pi}$, say, then $r(\tau) = -0.999983$, and $\sqrt{\frac{1-r}{1+r}} = 343$. Thus, in order that both p and γp be small, it would be necessary that p be small compared to $1/343 = 0.00292$. If p does not satisfy this condition, then V_o of Appendix II is not a good approximation to V and the equations developed above are not valid. This condition limits p to undesirably small values.

Probably, the argument could be justified that in the region where $[\sin(2\pi\frac{B}{2}\tau)]/(2\pi\frac{B}{2}\tau)$ is not small (i.e. in the region where $r(\tau)$ is not almost zero), the oscillation of the factor $\cos(2\pi f_c\tau)$ prevents any significant contribution to the integral. If $1/B$ is small compared with the integration interval, T, then $r(\tau)$ will be small in magnitude over most of the region of integration. In such a case, the substitution $r(\tau) = 0$ over the whole region of integration probably would not introduce a significant error in the integral, and the approximation of V by V_o would still be valid. Nevertheless, the preceding discussion indicates the care which must be employed in applying the assumptions if the results are to be valid.

The problem remains to evaluate the integrals of equation 4.2-1. In order to accomplish the integration, it will be assumed that both the signal and the noise have RC low-pass spectra but not necessarily with the same correlation duration constant. Notice that if the signal and noise both have RC low-pass spectra, then the restrictions imposed earlier on $\sqrt{\frac{1-r}{1+r}}$ and $\sqrt{\frac{1-\rho}{1+\rho}}$ are met (see example 1, above) so that the approximation of V by V_o is still valid. The correlation coefficients are $r(\tau) = e^{-|\tau|/\tau_s}$ and $\rho(\tau) = e^{-|\tau|/\tau_n}$, where τ_s is the signal correlation duration constant and τ_n is the noise correlation duration constant. Furthermore, it will be assumed that both τ_s and τ_n are very small compared with the integration duration, T. This is equivalent to the assumption that the band-width of the integrator, regarded as a low-pass filter, is very small compared with both the signal band-width and the noise band-width. Such an assumption is satisfied by a large class of communication systems. In fact, when the signal-to-noise power ratio, N, is very small, then T must be made very large in order to detect the signal. Therefore, as $N \to 0$, the assumption is satisfied by most systems which have a value of T large enough to assure satisfactory detection.

The new assumptions introduced will now be summarized for emphasis.

(A4.2-1) The input signal-to-noise power ratio is much less than unity ($N \ll 1$). In this case, $E\{z^2\}$ is approximately equal to the value of $E\{z^2\}$ when $N = 0$, and the $N = 0$ value will be used in place of $E\{z^2\}$.

(A4.2-2) The normalized biases are small with respect to unity ($\delta_1 \ll 1$, $\delta_2 \ll 1$).

(A4.2-3) Both the signal and the noise have RC low-pass spectra, but not necessarily with equal correlation duration constants ($r(\tau) = e^{-|\tau|/\tau_s}$, $\rho(\tau) = e^{-|\tau|/\tau_n}$, where τ_s is the correlation duration constant for the signal and τ_n is the correlation duration constant for the noise). Also, both duration constants are very small with respect to the integration interval $T(\tau_s \ll T$, $\tau_n \ll T)$.

Assumptions A4.2-1 and A4.2-2 were introduced in deriving equation 4.2-1. Assumption A4.2-3 will now be applied in turn to each of the four integrals in equation 4.2-1.

First Integral

The first integral is

$$I_1 = \frac{8}{\pi^2} \delta_1^2 \delta_2^2 \int_0^T (T-\tau) \sqrt{\frac{1-\rho}{1+\rho}} \sqrt{\frac{1-r}{1+r}} \, d\tau \quad . \tag{4.2-2}$$

By assumption A4.2-3, τ_s and τ_n both are small with respect to T. Therefore, both r and ρ are almost zero except for values of τ near the origin relative to T. Thus, $\sqrt{\frac{1-\rho}{1+\rho}}$ and $\sqrt{\frac{1-r}{1+r}}$ are nearly equal to 1 except for values of τ near the origin relative to T (see figure 4.2-2). The contribution to the integral of the small region near the origin where these two functions are not essentially unity is negligible. Therefore,

$$I_1 \cong \frac{8}{\pi^2} \delta_1^2 \delta_2^2 \int_0^T (T-\tau)d\tau$$

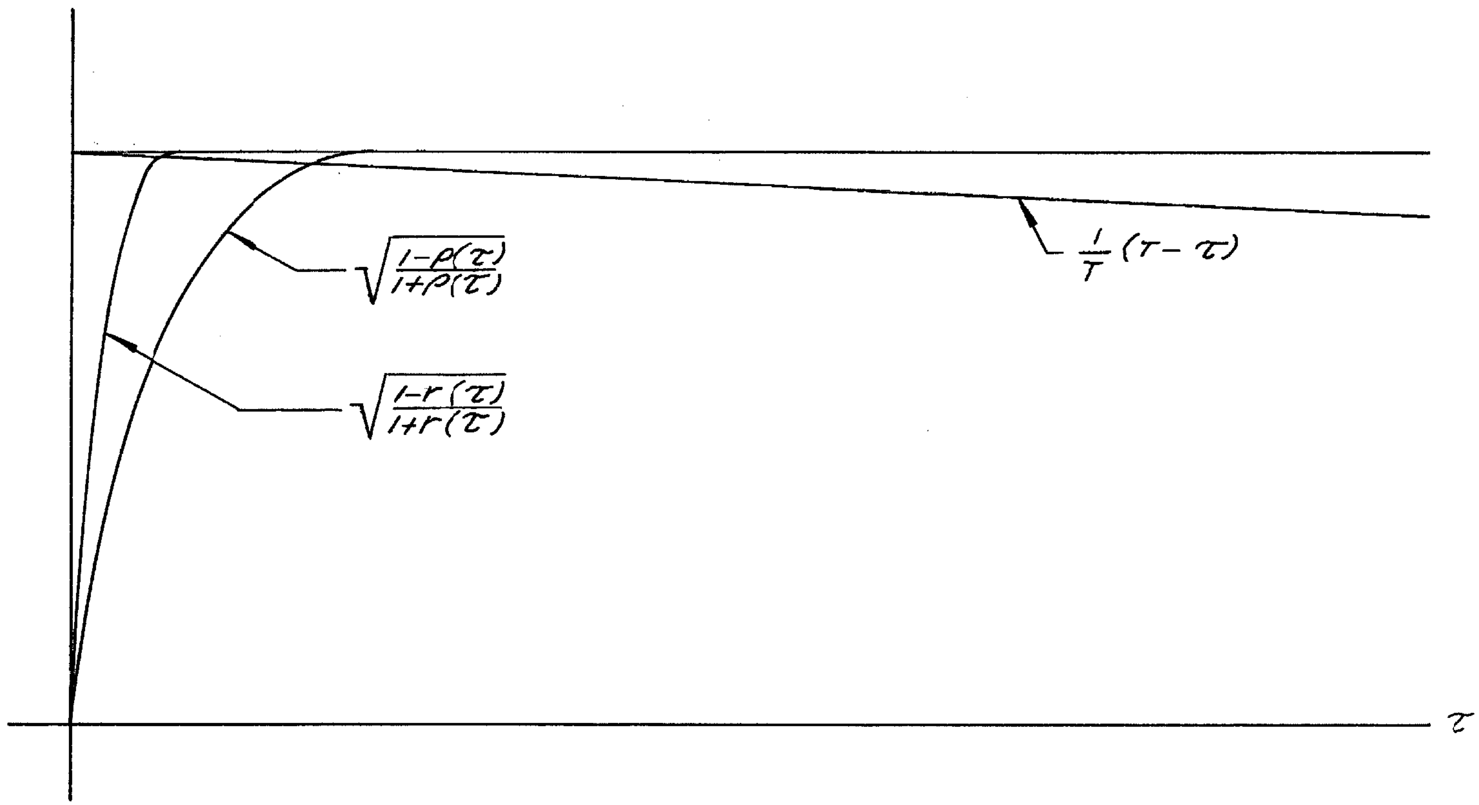
$$= \frac{4}{\pi^2} \delta_1^2 \delta_2^2 T^2 \quad .$$

From equation 3.2-2 of Chapter III, when N = 0 and the signal and the noise are both Gaussian, then the BPCD output mean value is

$$\mu_o = T\cdot \mathrm{Erf}(\delta_1/\sqrt{2})\cdot \mathrm{Erf}(\delta_2/\sqrt{2}) \quad .$$

The error function can be expanded in a power series whose coefficients go to zero with alternating signs. Therefore, the error in using only a finite number of terms of this series is less in magnitude than the first term neglected. From equation 590 of Dwight (5), the first two terms of the series are: $\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} x(1 - \frac{x^2}{3} + \cdots)$. Consequently,

$$\mathrm{Erf}(h/\sqrt{2}) = h\sqrt{\frac{2}{\pi}}$$



Rough Sketch of $\sqrt{\frac{1-r}{1+r}}$ and $\sqrt{\frac{1-\rho}{1+\rho}}$

Figure 4.2-2

with an error smaller in magnitude than $h^3/3\sqrt{2\pi}$ and with the sign of the error the same as the sign of h. Substituting this approximation in the above equation for μ_o yields

$$\mu_o = \frac{2}{\pi} T \delta_1 \delta_2$$

with a relative error less than $\delta_1^2 \delta_2^2/36$, which is indeed small when δ_1 and δ_2 are small.

From this result it follows that

$$I_1 = \frac{4}{\pi^2} \delta_1^2 \delta_2^2 T^2 = \mu_o^2 \qquad (4.2\text{-}3)$$

Second Integral

The second integral is

$$I_2 = \frac{8}{\pi^2} \delta_1^2 \int_0^T (T-\tau) \sqrt{\frac{1-\rho}{1+\rho}} \sin^{-1}(r) d\tau \quad .$$

$\underline{\tau_n/\tau_s \ll 1}$

In this case, the same arguments as were used in obtaining an expression for I_1 show that $\sqrt{\frac{1-\rho}{1+\rho}}$ is essentially unity for values of τ so small that $\sin^{-1}(r)$ and $(T-\tau)$ have changed hardly at all from their $\tau = 0$ values. The contribution to the integral of the small region near the origin where $\sqrt{\frac{1-\rho}{1+\rho}}$ is not essentially unity is negligible. Therefore,

$$I_2 \cong \frac{8}{\pi^2}\, \delta_1^2 \int_0^T (T-\tau)\, \sin^{-1}(r)\, d\tau \quad .$$

But since $\tau_s \ll T$, then $r(\tau)$ is essentially zero for small values of τ for which $(T-\tau)$ has not changed significantly from its $\tau = 0$ value. The contribution to the integral of the region where r and hence $\sin^{-1}(r)$ are essentially zero is negligible (see figure 4.2-3). Therefore, $(T-\tau)$ can be replaced by T with negligible error and

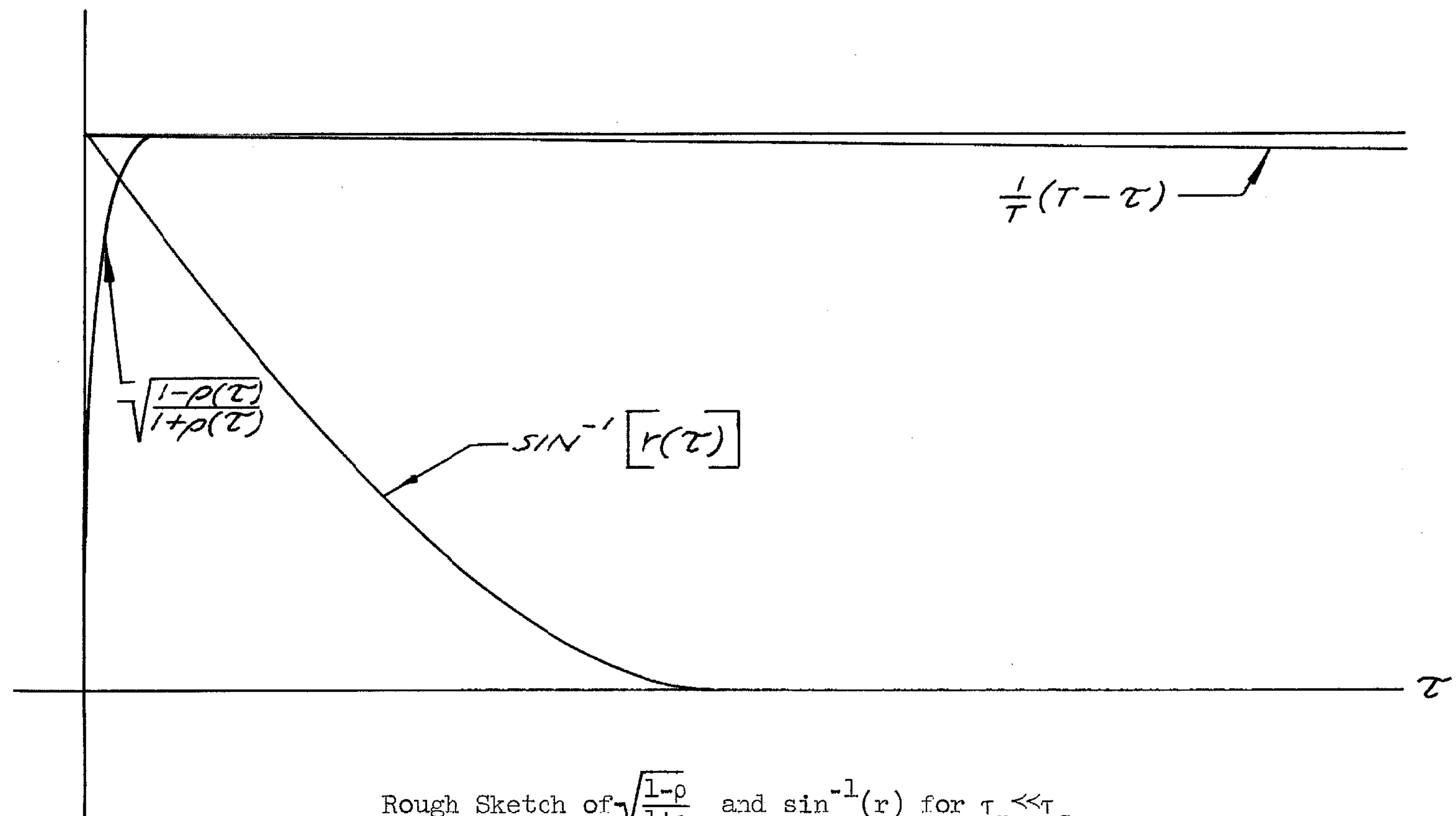
$$I_2 \cong \frac{8}{\pi^2}\, \delta_1^2\, T \int_0^T \sin^{-1}(r)\, d\tau$$

$$= \frac{8}{\pi^2}\, \delta_1^2\, T \int_0^T \sin^{-1}(e^{-\tau/\tau_s})\, d\tau \quad .$$

It is shown in Appendix VI that $\sin^{-1}(e^{-x})$ is approximated very well by the function $e^{-x} + \frac{1}{2}e^{-5x}$, with a relative error in the approximation which nowhere exceeds 0.07. Substituting this approximating function in the above equation yields

$$I_2 \cong \frac{8}{\pi^2}\, \delta_1^2\, T \int_0^T \left(e^{-\tau/\tau_s} + \tfrac{1}{2}e^{-5\tau/\tau_s}\right) d\tau$$

$$\cong \frac{8.8}{\pi^2}\, \delta_1^2\, T\, \tau_s \quad , \qquad \text{since} \quad T/\tau_s \gg 1 \quad .$$



Rough Sketch of $\sqrt{\frac{1-\rho}{1+\rho}}$ and $\sin^{-1}(r)$ for $\tau_n \ll \tau_s$

Figure 4.2-3

$\underline{\tau_n/\tau_s \gg 1}$

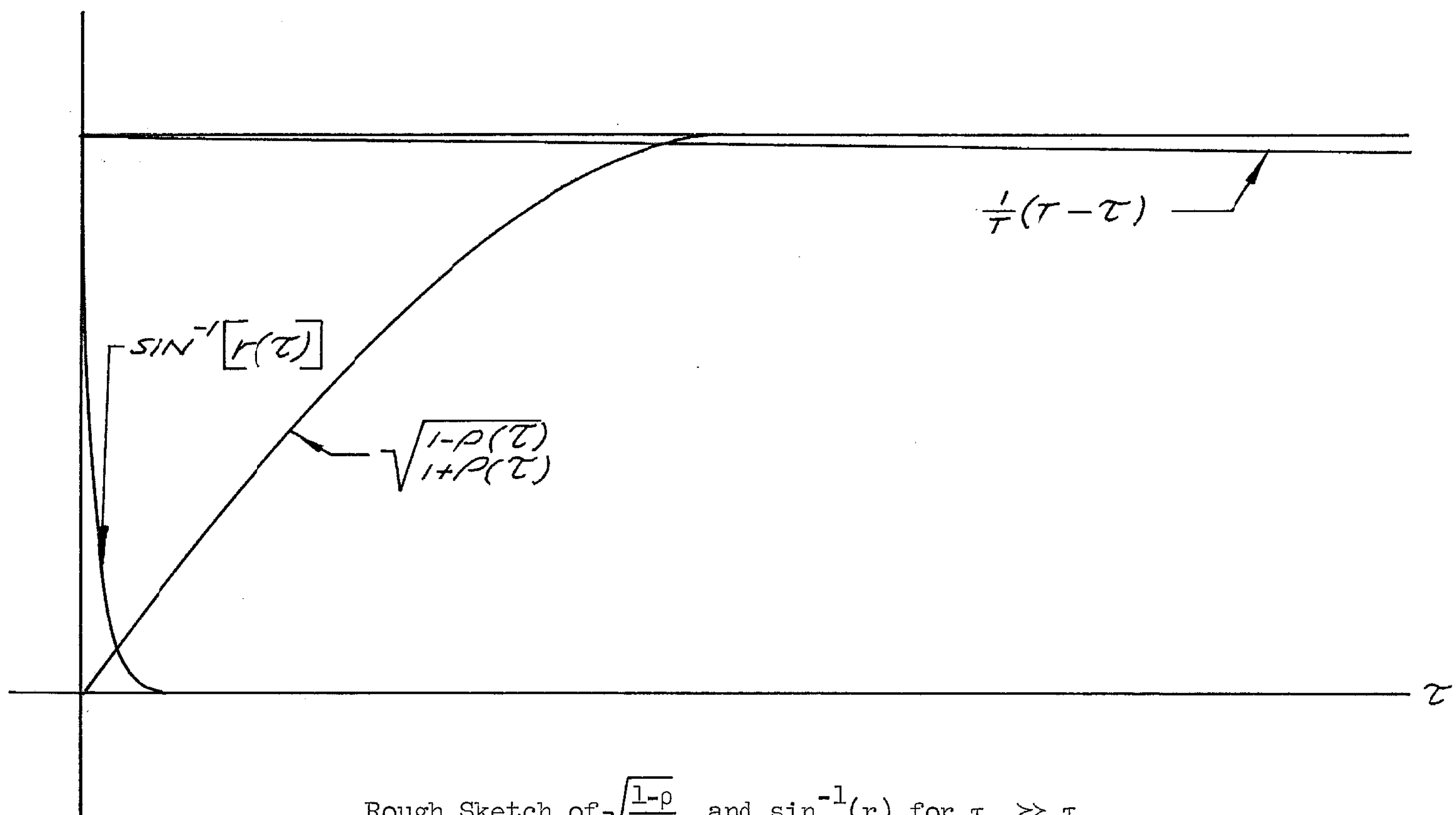
This case is less interesting from a practical point of view than the $\tau_n/\tau_s \ll 1$ case because it implies that the noise band-width is much less than the signal band-width. Such a situation is encountered only infrequently in practice. However, the integral which results would have to be evaluated anyway, since the same integral occurs when I_3, the next integral, is evaluated for the $\tau_n/\tau_s \ll 1$ case.

Since in all cases it is assumed that τ_n and τ_s are both much less than T, then the function $(T-\tau)$ has not changed significantly by the time $\sin^{-1}(r)$ has gone essentially to zero, so it can be replaced by T with negligible error in the integral. Since now $\tau_s \ll \tau_n$, $\sin^{-1}(r)$ will be essentially zero when τ/τ_n is still quite small relative to unity (see figure 4.2-4). In Appendix VI it is shown that for $0 \le x \le 1$, the function $\sqrt{\frac{x}{2}}$ is an excellent approximation to $\sqrt{\frac{1-e^{-x}}{1+e^{-x}}}$, with a relative error nowhere exceeding 0.04 in magnitude. Therefore, on substituting this approximation, as well as the one given above for $\sin^{-1}(e^{-x})$,

$$I_2 \cong \frac{8}{\pi^2}\,\delta_1^2\,T \int_0^T \sqrt{\frac{\tau/\tau_n}{2}}\,\left(e^{-\tau/\tau_s} + \tfrac{1}{6}e^{-5\tau/\tau_s}\right) d\tau \quad .$$

The integral to be evaluated here is of the form

$$I = \frac{1}{a\sqrt{a}} \int_0^{aT} \sqrt{u}\; e^{-u} du \quad .$$



Rough Sketch of $\sqrt{\frac{1-\rho}{1+\rho}}$ and $\sin^{-1}(r)$ for $\tau_n >> \tau_s$

Figure 4.2-4

This integral can be evaluated by a repeated integration by parts and is

$$\frac{1}{a\sqrt{a}}\left[\frac{\sqrt{\pi}}{2}\,\mathrm{Erf}(\sqrt{aT}) - \sqrt{aT}\,e^{-aT}\right] \quad .$$

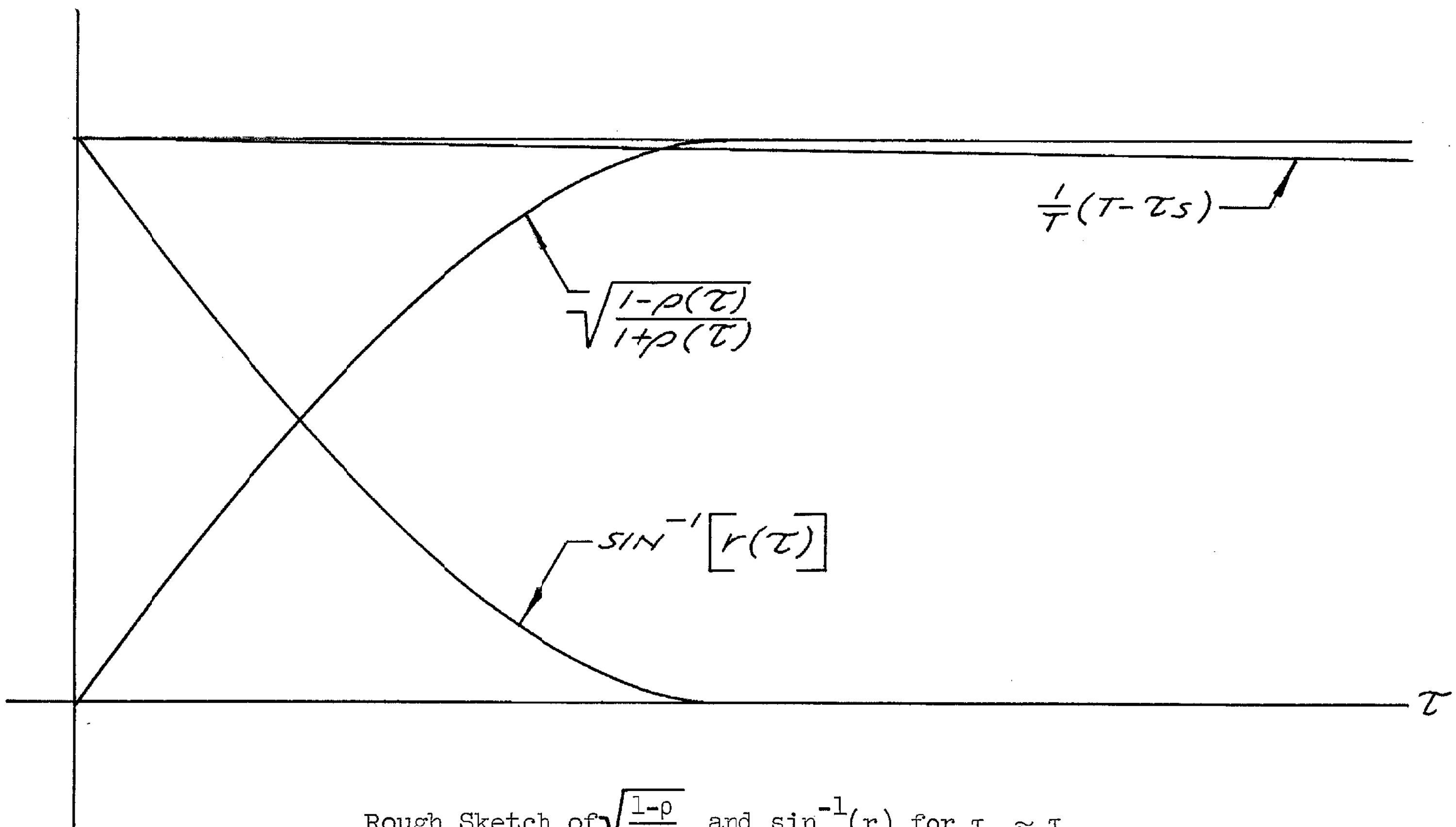
In the above expression, aT is either T/τ_s or $5T/\tau_s$, both of which are very large. Thus, $\mathrm{Erf}(\sqrt{aT}) \cong 1$ and $\sqrt{aT}\,e^{-aT} \cong 0$, so $I \cong \frac{\sqrt{\pi}}{2a\sqrt{a}}$. Substituting this result in the equation above for I_2, with appropriate choices for the constant a, yields

$$I_2 = \frac{3.0}{\pi\sqrt{\pi}}\,\delta_1^2\,T\tau_s\sqrt{\frac{\tau_s}{\tau_n}} \quad .$$

I_2 has the extra factor $\sqrt{\tau_s/\tau_n}$ when $\tau_n/\tau_s \gg 1$, which it does not have when $\tau_n/\tau_s \ll 1$. Otherwise, it is unchanged (except for a ratio of about 2 between the numerical coefficients; this ratio has negligible effect when compared with $\sqrt{\tau_s/\tau_n}$).

<u>General Case. τ_n/τ_s Unrestricted.</u>

In this case, as before, $(T-\tau)$ can be replaced by T without significant error in the integral. However, since τ_n and τ_s may be of the same order of magnitude, neither of the trancendental functions in the integrand dominates the other in its contribution to the integral (see figure 4.2-5). The function $1 - 0.85\,e^{-x}$ is shown in Appendix VI to be an excellent approximation to $\sqrt{\frac{1-e^{-x}}{1+e^{-x}}}$ for $x \geq 1$, with a relative error which nowhere exceeds 0.02 in magnitude.



Rough Sketch of $\sqrt{\frac{1-\rho}{1+\rho}}$ and $\sin^{-1}(r)$ for $\tau_n \approx \tau_s$

Figure 4.2-5

Substituting this approximation, as well as the ones for $\sin^{-1}(e^{-x})$ and $\sqrt{\frac{1-e^{-x}}{1+e^{-x}}}$ when $x \le 1$, in the equation for I_2 yields

$$I_2 = \frac{8}{\pi^2}\,\delta_1^2\,T\left[\int_0^{\tau_n}\sqrt{\frac{\tau/\tau_n}{2}}\,\left(e^{-\tau/\tau_s} + \tfrac{1}{2}e^{-5\tau/\tau_s}\right)d\tau\right.$$

$$\left.+\int_{\tau_n}^{T}\left(1 - 0.85\,e^{-\tau/\tau_n}\right)\left(e^{-\tau/\tau_s} + \tfrac{1}{2}e^{-5\tau/\tau_s}\right)d\tau\right] \quad . \qquad (4.2\text{-}4)$$

These integrals are of types which have been integrated in deriving the preceding expressions for I_2. Using those results with appropriate modifications to accomodate the new constants, and setting $\tau_n/\tau_s = \gamma$, yields

$$I_2 = \begin{cases} \dfrac{8}{\pi^2}\,\delta_1^2\,T\tau_s \left[\begin{array}{l} \tfrac{1}{2}\sqrt{\tfrac{\pi}{2}}\left\{\dfrac{\mathrm{Erf}(\sqrt{\gamma})}{\sqrt{\gamma}} + \dfrac{1}{10}\,\dfrac{\mathrm{Erf}(\sqrt{5\gamma})}{\sqrt{5\gamma}}\right\} \\ + \left(1 - \sqrt{\tfrac{1}{2}}\right)\left(e^{-\gamma} + \dfrac{1}{10}\,e^{-5\gamma}\right) \\ - \dfrac{0.85}{e}\left(\dfrac{\gamma}{1+\gamma}\,e^{-\gamma} + \dfrac{1}{10}\left[\dfrac{5\gamma}{1+5\gamma}\right]e^{-5\gamma}\right) \end{array}\right] & \text{for all values of } \tau_n \text{ and } \tau_s \text{ (for all values of } \gamma), \\ \dfrac{8}{\pi^2}\,\delta_1^2\,T\,\tau_s\left(1 + \dfrac{1}{10}\right) & \text{for } \tau_n \ll \tau_s \ (\gamma \ll 1), \\ \dfrac{8}{\pi^2}\,\delta_1^2\,T\,\tau_s\,\dfrac{1}{\sqrt{\gamma}}\left\{\tfrac{1}{2}\sqrt{\tfrac{\pi}{2}}\left(1 + \dfrac{1}{10\sqrt{5}}\right)\right\} & \text{for } \tau_n \gg \tau_s \ (\gamma \gg 1) \ . \end{cases} \qquad (4.2\text{-}5)$$

Clearly, the cases derived previously for $\tau_n/\tau_s \ll 1$ and for $\tau_n/\tau_s \gg 1$ are limiting cases as $\gamma \to 0$ and $\gamma \to \infty$ of the general equation just derived.

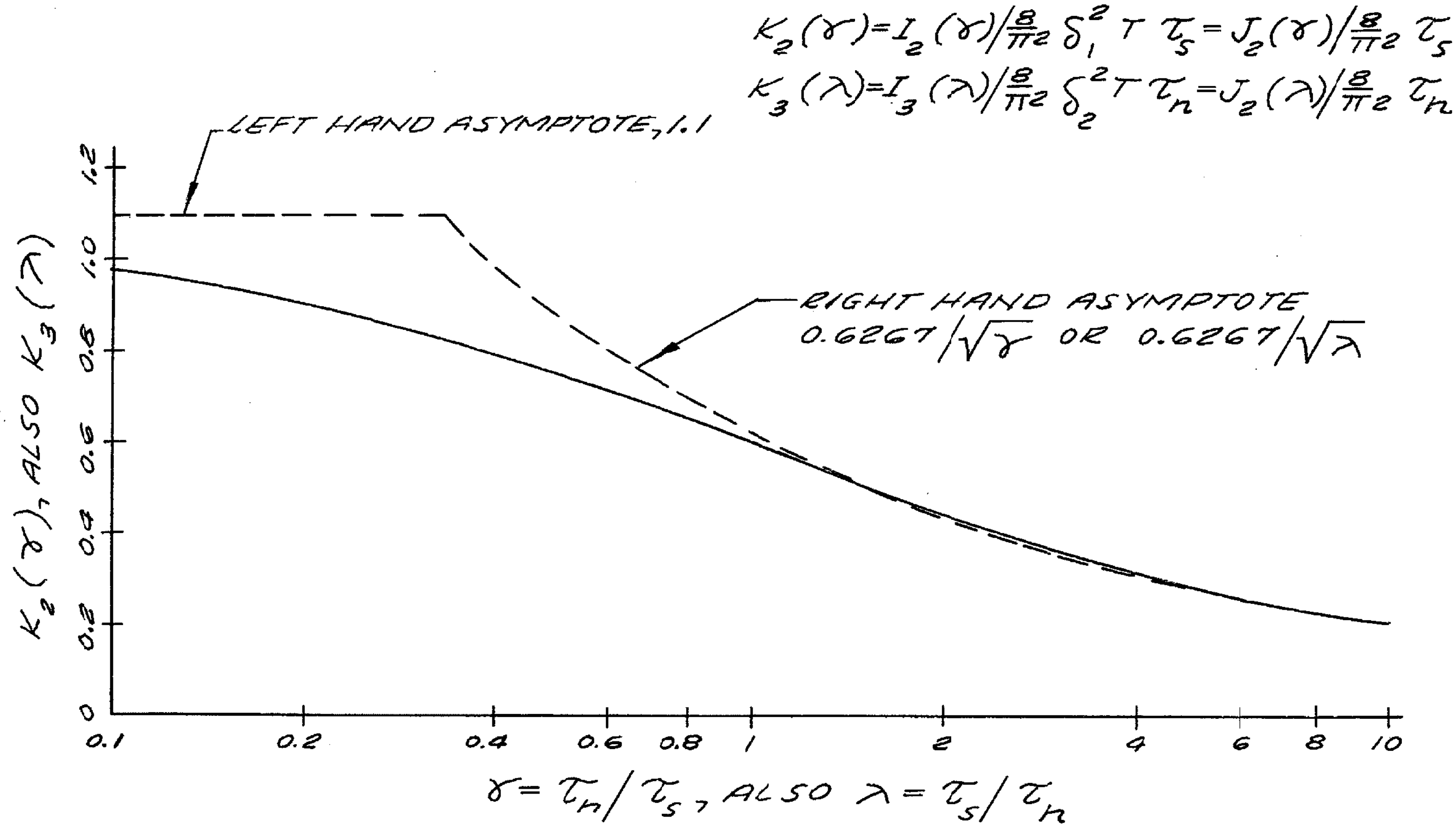
A plot of $K_2 = I_2 / \frac{8}{\pi^2} \delta_1^2 T \tau_s$ as a function of γ is presented in figure 4.2-6. From this plot, it appears that a rough estimate of I_2 can be obtained by using the $\gamma \ll 1$ form for values of $\gamma = \tau_n/\tau_s \le 0.4$ and the $\gamma \gg 1$ form for values of $\gamma \ge 0.4$. The maximum relative error resulting from this crude approximation is 0.25 at $\gamma = 0.4$.

Third Integral

The third integral is

$$I_3 = \frac{8}{\pi^2} \delta_2^2 \int_0^T (T-\tau) \sqrt{\frac{1-r}{1+r}} \sin^{-1}(\rho) d\tau \quad .$$

This integral is identical to I_2 with τ_n and τ_s everywhere interchanged, and with δ_1 replaced by δ_2. Therefore, the equations and properties derived above for I_2 and the plot in figure 4.2-6 for K_2 are valid for $K_3 = I_3 / \frac{8}{\pi^2} \delta_2^2 T \tau_n$, if in them: δ_1 is replaced by δ_2, τ_s is replaced by τ_n, τ_n is replaced by τ_s, and γ is replaced by λ where $\lambda = \tau_s/\tau_n$.



Second and Third Variance Integrals

Figure 4.2-6

$$I_3 = \left[\frac{8}{\pi^2}\,\delta_2^2\,T\,\tau_n\right] \quad X \quad \left[\begin{array}{l} \frac{1}{2}\sqrt{\frac{\pi}{2}}\left\{\frac{\mathrm{Erf}(\sqrt{\lambda})}{\sqrt{\lambda}} + \frac{1}{10}\,\frac{\mathrm{Erf}(\sqrt{5\lambda})}{\sqrt{5\lambda}}\right\} \\ + \left(1 - \sqrt{\frac{1}{2}}\right)\left(e^{-\lambda} + \frac{1}{10}\,e^{-5\lambda}\right) \\ - \frac{0.85}{e}\left(\frac{\lambda}{1+\lambda}\,e^{-\lambda} + \frac{1}{10}\left[\frac{5\lambda}{1+5\lambda}\right]e^{-5\lambda}\right) \end{array}\right]$$

for all values of τ_n and τ_s

(for all values of λ);

$$I_3 = \frac{8}{\pi^2}\,\delta_2^2\,T\,\tau_n\left(1 + \frac{1}{10}\right) \quad \text{for} \quad \tau_s \ll \tau_n \quad (\lambda \ll 1),$$

$$I_3 = \frac{8}{\pi^2}\,\delta_2^2\,T\,\tau_n\,\frac{1}{\sqrt{\lambda}}\,\left\{\frac{1}{2}\sqrt{\frac{\pi}{2}}\left(1 + \frac{1}{10\sqrt{5}}\right)\right\} \qquad (4.2\text{-}6)$$

for $\tau_s \gg \tau_n$ $(\lambda \gg 1)$.

All remarks about I_2 apply to I_3 also, after an appropriate interchange of symbols, as described above.

Fourth Integral

The fourth integral is

$$I_4 = \frac{8}{\pi^2}\int_0^T (T-\tau)\,\sin^{-1}(\rho)\,\sin^{-1}(r)\,d\tau \quad .$$

Since both τ_s and τ_n are small with respect to T, then r and ρ, and hence $\sin^{-1}(r)$ and $\sin^{-1}(\rho)$, are small for all values of τ except for values near the origin relative to T. Therefore, the contribution to the integral for values of τ for which $(T-\tau)$ has changed appreciably from its $\tau = 0$ value is negligible, and $(T-\tau)$ can be replaced by T without a significant error in the integral.

$$I_4 = \frac{8}{\pi^2} T \int_0^T \sin^{-1}(\rho)\ \sin^{-1}(r) d\tau \quad .$$

If the approximation function $e^{-x} + \frac{1}{2}e^{-5x}$ is introduced for $\sin^{-1}(x)$, this integral becomes

$$I_4 = \frac{8}{\pi^2} T \int_0^T (e^{-\tau/\tau_n} + \tfrac{1}{2}e^{-5\tau/\tau_n})(e^{-\tau/\tau_s} + \tfrac{1}{2}e^{-5\tau/\tau_s}) d\tau \quad .$$

Since $T \gg \tau_n$ and $T \gg \tau_s$, the contribution of the upper limit, T, to this integral is negligible and

$$I_4 = \frac{8}{\pi^2} T\ \tau_n \ \{(1 + \frac{1}{20}) \frac{1}{1+\gamma} + \tfrac{1}{2}\ [\frac{1}{1+5\gamma} + \frac{1}{\gamma+5}]\}$$

or equivalently

$$I_4 = \frac{8}{\pi^2} T\ \tau_s \ \{(1 + \frac{1}{20}) \frac{1}{1+\lambda} + \tfrac{1}{2}\ [\frac{1}{1+5\lambda} + \frac{1}{\lambda+5}]\} \quad .$$

The limiting value as $\gamma \to 0$ in the first expression is $1.65 \frac{8}{\pi^2} T\ \tau_n$, or as $\lambda \to 0$ in the second expression is $1.65 \frac{8}{\pi^2} T\ \tau_s$.

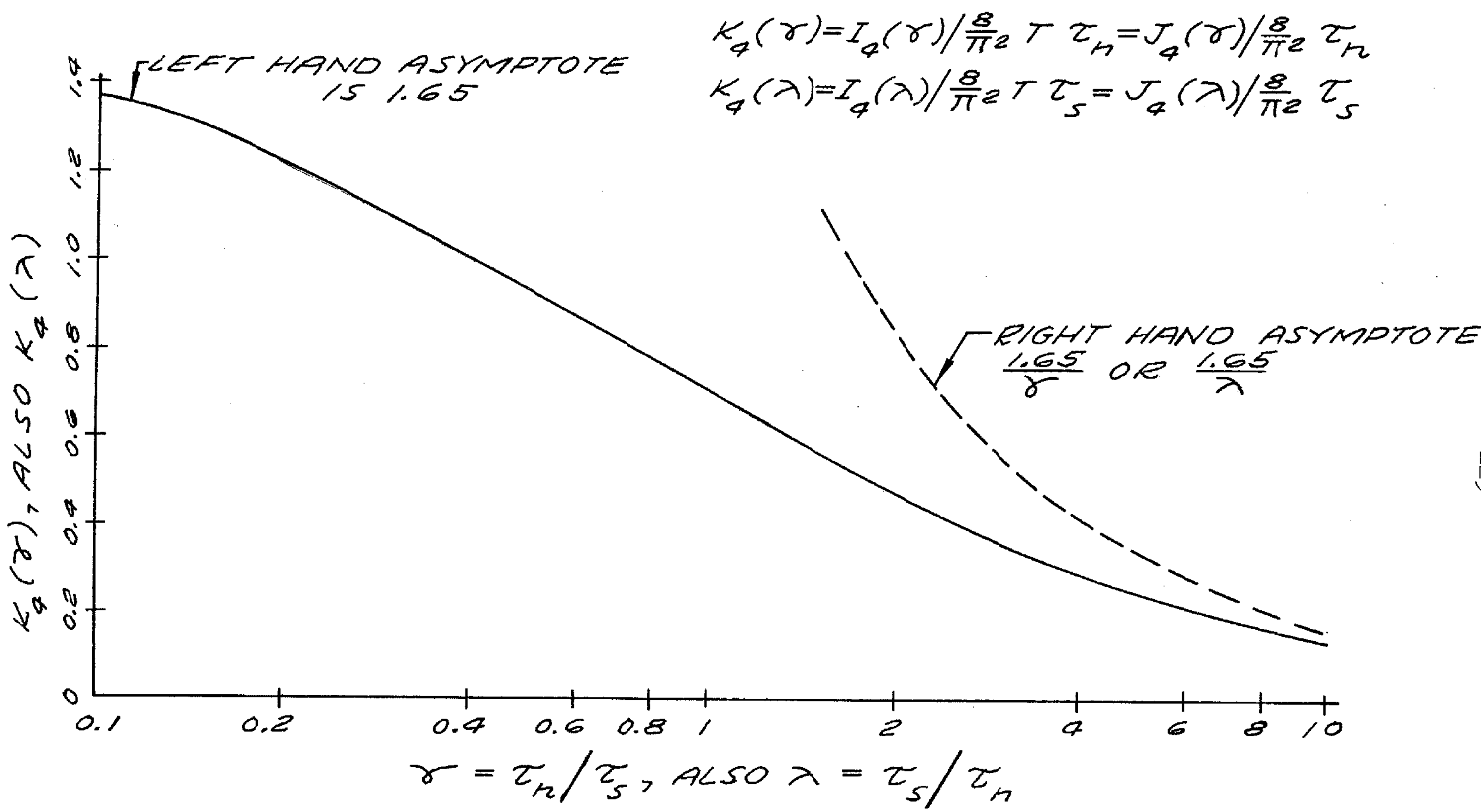
The limiting value as $\gamma \to \infty$ in the first expression, or as $\lambda \to \infty$ in the second expression, is zero. A plot of $K_4 = I_4 / \frac{8}{\pi^2} T \tau_n$ as a function of γ (and of $K_4 = I_4 / \frac{8}{\pi^2} T \tau_s$ as a function of λ) is presented in figure 4.2-7.

A word of caution is in order at this point. γ or λ cannot be varied independently of τ_s or τ_n. In all cases, both τ_s and τ_n are small relative to T. Thus if $\gamma \to 0$, then τ_n must go to zero, since τ_s must remain bounded as required by $\tau_s \ll T$. Conversely, if $\gamma \to \infty$, then τ_s must go to zero, since τ_n must remain bounded as required by $\tau_n \ll T$. Likewise, if $\lambda \to 0$, then τ_s must go to zero and if $\lambda \to \infty$, then τ_n must go to zero. Therefore, the limiting value of I_4 as $\gamma \to 0$ or as $\lambda \to 0$ is actually zero, since τ_n and τ_s appear as factors in the two limiting expressions for I_4 above. That such is the case is obvious from the defining integral. For as τ_n or $\tau_s \to 0$, then $\sin^{-1}(e^{-\tau/\tau_n})$ or $\sin^{-1}(e^{-\tau/\tau_s})$ goes to zero except at $\tau = 0$. Thus the integrand is everywhere zero in the limit except at $\tau = 0$, and the integral is zero.

All the integrals appearing in the expression for $E\{z^2\}$ have now been evaluated and $E\{z^2\}$ is written simply as

$$E\{z^2\} = I_1 + I_2 + I_3 + I_4 \quad . \tag{4.2-8}$$

In each of these integrals, it was necessary to introduce approximations in order to make the integration possible. A discussion of the consequent errors will be postponed until the end of this chapter.



Fourth Variance Integral

Figure 4.2-7

The equations for the variance will now be stated. The variance of the BPCD output, z, is $\sigma_z^2 = E\{z^2\} - \mu_o^2$. But it was shown in the discussion of the integral I_1 that $\mu_o^2 = I_1$. Therefore, $\sigma_z^2 = I_2 + I_3 + I_4$. To emphasize the dependence on the biases and the integration interval, this will be written

$$\sigma_z^2 = T\cdot\delta_1^2\cdot J_2 + T\cdot\delta_2^2\cdot J_3 + T\cdot J_4 \quad , \qquad (4.2\text{-}9)$$

where J_2, J_3 and J_4 are constants for a particular class of signal and noise, given by

$$J_2 = \left[\frac{8}{\pi^2}\tau_s\right] \times \left[\begin{array}{l} \frac{1}{2}\sqrt{\frac{\pi}{2}}\left\{\frac{\mathrm{Erf}(\sqrt{\gamma})}{\sqrt{\gamma}} + \frac{1}{10}\,\frac{\mathrm{Erf}(\sqrt{5\gamma})}{\sqrt{5\gamma}}\right\} \\ + \left(1 - \sqrt{\frac{1}{2}}\right)\left(e^{-\gamma} + \frac{1}{10}\,e^{-5\gamma}\right) \\ - \frac{0.85}{e}\left(\left(\frac{\gamma}{1+\gamma}\right)e^{-\gamma} + \frac{1}{10}\left[\frac{5\gamma}{1+5\gamma}\right]e^{-5\gamma}\right) \end{array}\right]$$

with $\gamma = \tau_n/\tau_s$,

$$J_3 = \left[\frac{8}{\pi^2}\tau_n\right] \times \left[\begin{array}{l} \frac{1}{2}\sqrt{\frac{\pi}{2}}\left\{\frac{\mathrm{Erf}(\sqrt{\lambda})}{\sqrt{\lambda}} + \frac{1}{10}\,\frac{\mathrm{Erf}(\sqrt{5\lambda})}{\sqrt{5\lambda}}\right\} \\ + \left(1 - \sqrt{\frac{1}{2}}\right)\left(e^{-\lambda} + \frac{1}{10}\,e^{-5\lambda}\right) \\ - \frac{0.85}{e}\left(\frac{\lambda}{1+\lambda}\,e^{-\lambda} + \frac{1}{10}\left|\frac{5\lambda}{1+5\lambda}\right|e^{-5\lambda}\right) \end{array}\right]$$

with $\lambda = \tau_s/\tau_n$, (4.2-10)

and

$$J_4 = \frac{8}{\pi^2} \tau_n \left\{ (1 + \frac{1}{20}) \frac{1}{1+\gamma} + \frac{1}{2} \left[\frac{1}{1+5\gamma} + \frac{1}{\gamma+5} \right] \right\}$$

or equivalently

$$J_4 = \frac{8}{\pi^2} \tau_s \left\{ (1 + \frac{1}{20}) \frac{1}{1+\lambda} + \frac{1}{2} \left[\frac{1}{1+5\lambda} + \frac{1}{\lambda+5} \right] \right\} \quad .$$

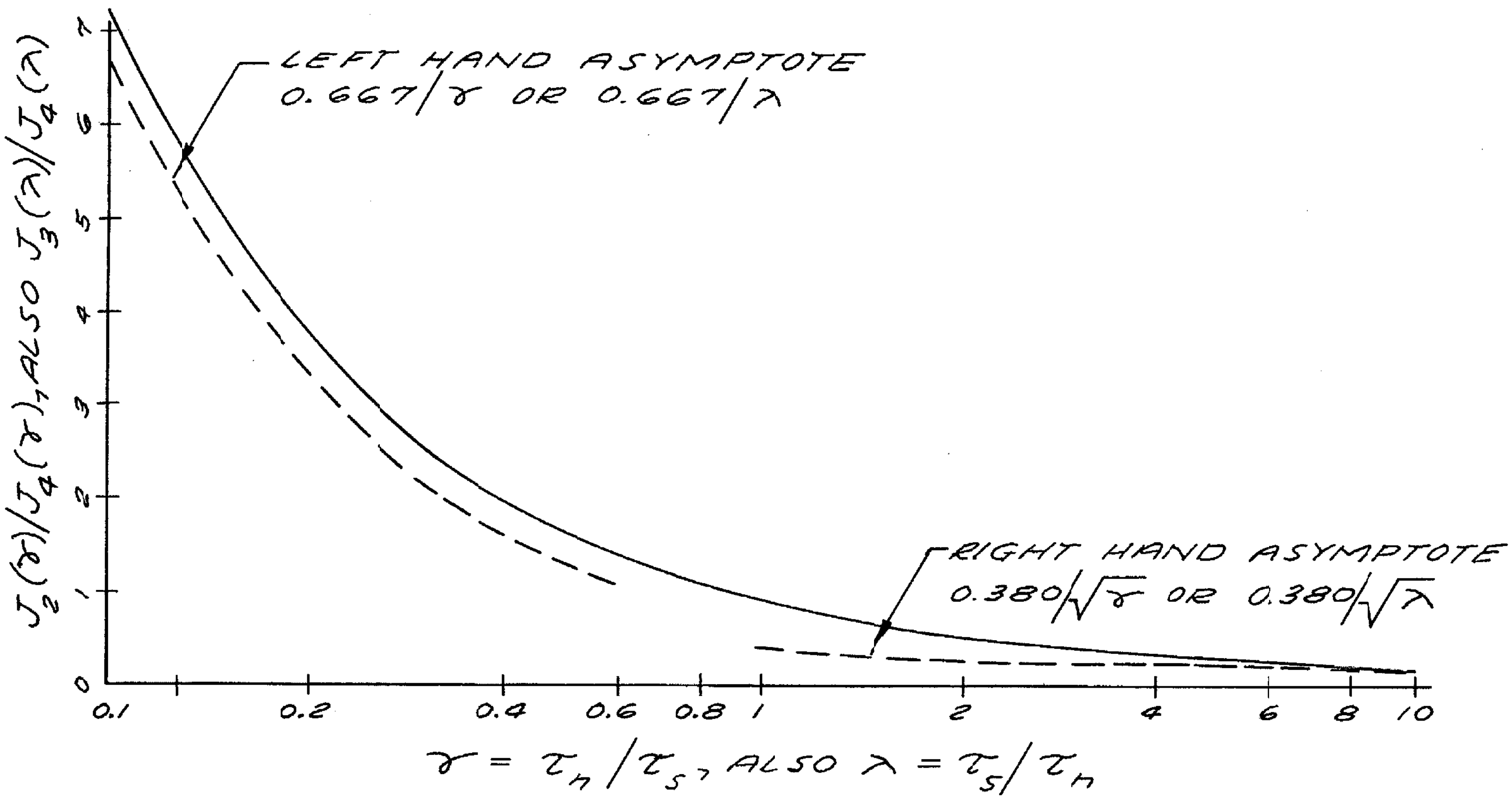
From equation 4.2-9 it appears that the biases affect the variance independently of each other. Since the biases are squared in equation 4.2-9, the sign of the bias has no effect on the variance.

The last term of equation 4.2-9, $T \cdot J_4$, is the variance for a perfect polarity-coincidence detector - i.e. a BPCD with zero bias in both channels.

The ratio J_2/J_4 as a function of γ (or J_3/J_4 as a function of λ) is shown in figure 4.2-8. Because of the $1/\gamma$ (or $1/\lambda$) dependence for small γ (or λ), the J_2 (or J_3) term may not be negligible with respect to J_4 when the noise band-width is small with respect to the signal band-width (or signal band-width is small with respect to the noise band-width) even though δ_1 (or δ_2) is small.

To avoid any possible confusion as to the range of validity of the preceding expression for the variance, the assumptions and restrictions will be re-stated here:

1) The signal is absent; $N = 0$.



Ratio of $J_2(\gamma)$ to $J_4(\gamma)$ or $J_3(\lambda)$ to $J_4(\lambda)$

Figure 4.2-8

2) The (normalized) biases, δ_1 and δ_2, are small with respect to unity.

3) The signal and noise have RC low-pass spectra, with correlation duration constants τ_s and τ_n which are small with respect to the integration interval, T. Equivalently, the integrator band-width is small with respect to the signal and noise band-widths.

It seems reasonable to suppose that a departure from the last assumption - RC low-pass spectra - would not result in a gross change in the constants J_2, J_3 and J_4 if, for this new class of signals and noise, the time domain equivalent of band-width is used for τ_s and τ_n in the expressions for J_2, J_3 and J_4 above.

The properties of the variance just derived, in combination with the properties of the mean value derived in Chapter III, will be used in the next chapter to examine the behavior of the BPCD as a detector.

4.3 Error Analysis

In deriving the preceding equations for the variance, approximations were repeatedly introduced in order to make the analysis tractable. Each of these approximations adds an error to the final result. It is the purpose of this section to determine the magnitude of the accumulative error in the approximations. Each of the integrals - I_1, I_2, I_3 and I_4 - will be considered separately. First, however, some properties of relative errors must be examined.

a) If an integral has an integrand which does not change sign in the region of integration, and if a factor of the integrand is replaced by an approximating function, then the magnitude of the relative error for the resulting approximate integral is less than or equal to the maximum value of the magnitude of the relative error for the approximating function substituted in the integrand. This relation is derived as follows: By definition,

$$\rho = \frac{f_o - f}{f}$$

or

$$f_o - f = \rho f \quad ,$$

where f is the factor in the integrand to be approximated, f_o is the approximating function and ρ is the relative error in the approximation.* Then,

$$I_o - I = \int \rho f g \quad ,$$

where $I = \int fg$, $I_o = \int f_o g$ and g represents the factors in the integrand which are not replaced by an approximating function. The relative error of approximation for the integral is

---

* Note that in this context ρ denotes the relative error, rather than the noise correlation coefficient.

$$\rho_I = \frac{I_o - I}{I}$$

$$= \frac{\int \rho f g}{\int f g}$$

and

$$|\rho_I| = \frac{\left|\int \rho f g\right|}{\left|\int f g\right|} \quad .$$

But

$$\left|\int \rho f g\right| = \left|\int_A \rho f g + \int_B \rho f g\right| = \left|\int_A |\rho| f g - \int_B |\rho| f g\right| \quad ,$$

where A is the region of integration in which ρ is positive and B is the region of integration in which ρ is negative. Then

$$\left|\int \rho f g\right| \leq \left|\int_A |\rho| f g\right| + \left|\int_B |\rho| f g\right| \quad .$$

Let $|\rho|_m$ be the largest value of the magnitude of ρ in the region of integration. Then

$$\left|\int \rho f g\right| \leq |\rho|_m \left|\int_A f g\right| + |\rho|_m \left|\int_B f g\right| \quad .$$

If fg does not change sign in the region of integration, then

$$\left|\int \rho f g\right| \le |\rho|_m \left| \int_A fg + \int_B fg \right| = |\rho|_m \left| \int fg \right| \quad .$$

Therefore,

$$|\rho_I| \le |\rho|_m \quad . \tag{4.3-1}$$

b) If two factors in a function are replaced by approximating functions, and if the relative error for each of these approximations individually is small with respect to unity, then the relative error due to the combined approximation is very nearly equal to the sum of the individual relative errors. This relation is derived as follows: The relative error for the combined approximation is

$$\rho = \frac{f_o g_o - fg}{fg}$$

$$= \frac{f_o}{f} \frac{g_o}{g} - 1 \quad ,$$

where f and g are the two factors to be approximated and f_o and g_o are the corresponding approximating functions. But

$$\rho_f = \frac{f_o - f}{f} \quad \text{or} \quad \frac{f_o}{f} = 1 + \rho_f$$

and

$$\rho_g = \frac{g_o - g}{g} \quad \text{or} \quad \frac{g_o}{g} = 1 + \rho_g \quad ,$$

where ρ_f and ρ_g are the individual relative errors. Thus,

$$\rho = (1 + \rho_f)(1 + \rho_g) - 1 = \rho_f + \rho_g + \rho_f\rho_g \quad .$$

If ρ_f and ρ_g are small with respect to unity, then

$$\rho \cong \rho_f + \rho_g \quad . \tag{4.3-2}$$

c) If g is the reciprocal of f, if f is replaced by an approximating function f_o and if the relative error, ρ_f, in this approximation is small with respect to unity, then the relative error ρ_g for the resulting approximation g_o is very nearly equal to $-\rho_f$. This relation is derived as follows: Since $g = 1/f$, then

$$\rho_g = \frac{g_o - g}{g} = \frac{\frac{1}{f_o} - \frac{1}{f}}{\frac{1}{f}} = -\frac{\frac{f_o - f}{ff_o}}{\frac{1}{f}} = -\frac{f_o - f}{f_o} \cong -\rho_f \quad , \tag{4.3-3}$$

since when ρ_f is small with respect to unity, then $f_o \cong f$ and f_o can be replaced by f in the denominator of the last expression on the right above with negligible error.

d) If f_o is an approximating function for f with relative error ρ_o, if f_1 is an approximating function for f_o with relative error ρ_1, and if ρ_o is small with respect to unity, then f_1 is an approximating function for f with relative error $\rho \cong \rho_o + \rho_1$. This relation is derived as follows: The relative error of f_1 as an approximating function for f is

$$\rho = \frac{f_1 - f}{f} \quad .$$

But the relative error of f_1 as an approximating function for f_o is

$$\rho_1 = \frac{f_1 - f_o}{f_o}$$

from which

$$f_1 = (1 + \rho_1)f_o \quad .$$

Thus,

$$\rho = \frac{(1 + \rho_1)f_o - f}{f} = \frac{f_o - f}{f} + \rho_1 \frac{f_o}{f} \qquad (4.3\text{-}4)$$

$$\cong \rho_o + \rho_1 \quad ,$$

since when ρ_o is small with respect to unity, then $f_o \cong f$ and f_o can be substituted for f in the denominator of the right hand term of the last expression above with negligible error.

The relations derived above for relative errors will now be used to determine the effects of the combined approximations which were introduced into the equations for I_1, I_2, I_3 and I_4.

I_1)

The first approximation made in order to evaluate I_1, as well as the first approximation made in order to evaluate I_2 and I_3, was the substitution of V_o for V in equation 4.1-5 to obtain equation 4.1-6. According to equation A2.0-3 of Appendix II, the relative error due to the approximation of V by V_o is less than $\frac{1}{2}\epsilon$, where in the present case, $\epsilon = (1 + \frac{1-\rho}{1+\rho})\delta_1^2$ or $(1 + \frac{1-r}{1+r})\delta_2^2$, whichever is appropriate. Since neither r nor ρ is negative for any value of τ, then $\epsilon \leq 2\ \delta_1^2$ or $2\ \delta_2^2$, whichever is appropriate. In either case, ϵ is small. It will be assumed henceforth that both δ_1 and δ_2 are less than 0.1 in magnitude. This is a reasonable assumption, since it is quite easy to construct practical polarity indicators whose bias does not exceed 10% of the input level and in fact, a polarity indicator with larger bias would be considered to be poorly designed. If the biases are less than 0.1 in magnitude, then $\epsilon \leq 0.02$ in both cases, and the relative error due to the approximation is less than 0.01.

By property b) for relative errors, derived above, the relative error in the integrand of I_1 due to the substitution of $V_o(\delta_1, \sqrt{\frac{1-\rho}{1+\rho}}\ \delta_1)$ for $V(\delta_1, \sqrt{\frac{1-\rho}{1+\rho}}\ \delta_1)$ and of $V_o(\delta_2, \sqrt{\frac{1-r}{1+r}}\ \delta_2)$ for $V(\delta_2, \sqrt{\frac{1-r}{1+r}}\ \delta_2)$ is equal to the sum of the individual relative errors, each of which is less than 0.01 in this case. Therefore, the relative error in the integrand due to this approximation is less than 0.02.

By property a) for relative errors, derived above, the relative error in the integral due to this approximation, then, is less than 0.02 since the integrand is always positive in the region of integration.

The next and final approximation used in evaluating I_1 was to approximate the transcendental functions in the integrand by a constant - the number 1. If $x = \tau/\max(\tau_s, \tau_n)$ and $a = \min(\tau_s, \tau_n)/\max(\tau_s, \tau_n)$, then $\sqrt{\frac{1-e^{-x}}{1+e^{-x}}}$ and $\sqrt{\frac{1-e^{-x/a}}{1+e^{-x/a}}}$ are the two transcendental factors in the integrand. These two functions are sketched roughly in figure 4.3-1. Since these two functions are factors of the integrand both with values everywhere between zero and one, and since they were both replaced by the number 1, it is clear that the largest possible value of the parameter a (namely, $a = 1$ or $\tau_s = \tau_n$), which results in the smallest possible values for the function $\sqrt{\frac{1-e^{-x/a}}{1+e^{-x/a}}}$, leads to the largest possible error for fixed $T/\max(\tau_s, \tau_n)$. Thus, for the largest possible error, the product of the two factors is $\frac{1-e^{-x}}{1+e^{-x}}$. But this function is easily shown to be

$$\frac{1-e^{-x}}{1+e^{-x}} = \tanh(x/2) \qquad .$$

From equation 4.2-2, on making the transformation $x = \tau/b$, where $b = \max(\tau_s, \tau_n)$ and in this case equals $\min(\tau_s, \tau_n) = \tau_s = \tau_n$, the equation for I_1 becomes

$$I_1 = \frac{8}{\pi^2}\,\delta_1^2\delta_2^2 \cdot b \int_0^{T/b} (T - bx)\,\tanh(x/2)dx \qquad ,$$

and the error due to replacing $\tanh(x/2)$ by the number 1 is

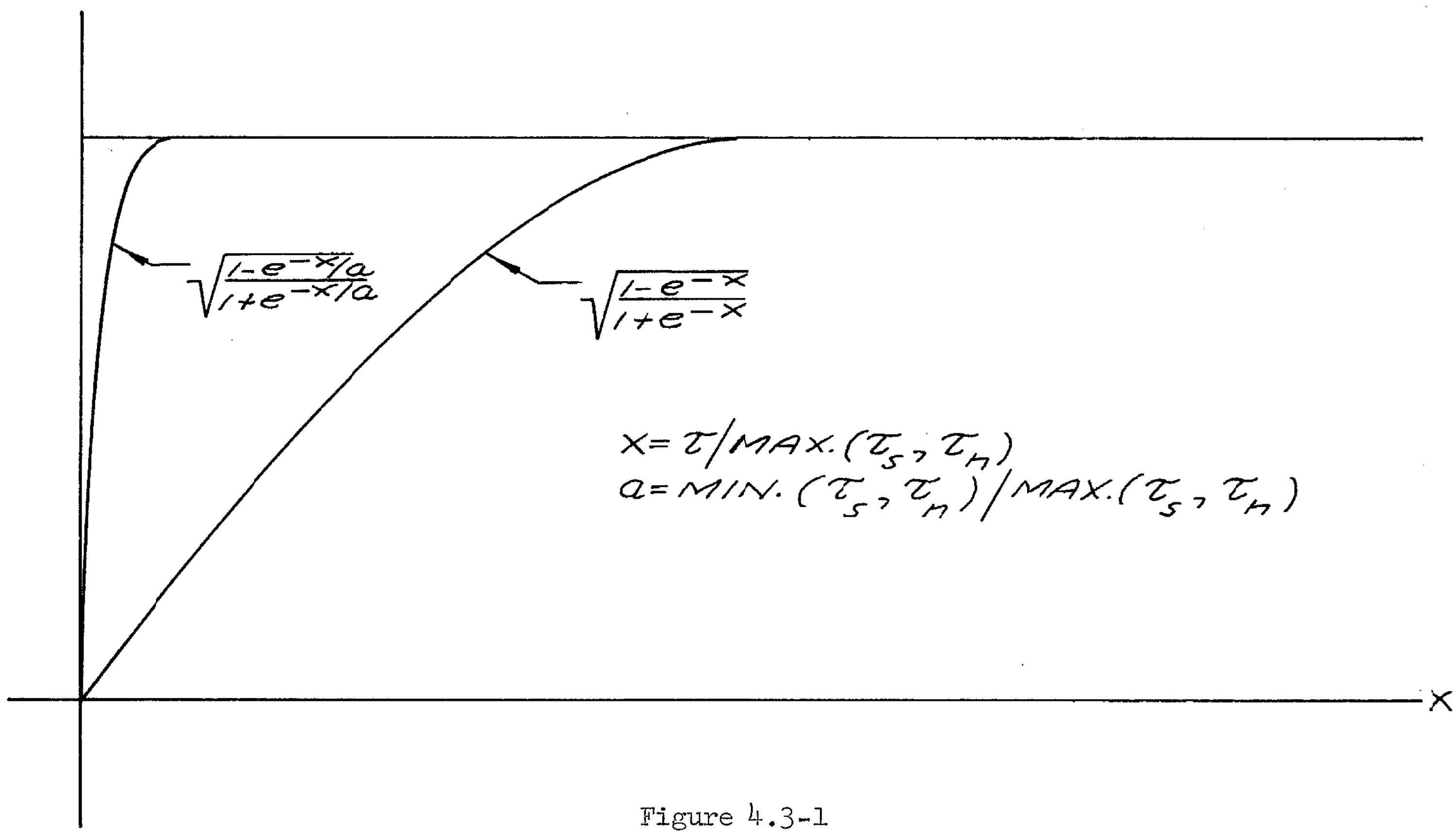


Figure 4.3-1

Rough Sketch of $\sqrt{\frac{1-e^{-x}}{1+e^{-x}}}$ and $\sqrt{\frac{1-e^{-x/a}}{1+e^{-x/a}}}$ for Fixed T/max(τ_s,τ_n)

$$\zeta = \frac{8}{\pi^2} \delta_1^2 \delta_2^2 \left[b \int_0^{T/b} (T-bx)dx - b \int_0^{T/b} (T-bx)\tanh(x/2)dx \right]$$

$$= \frac{8}{\pi^2} \delta_1^2 \delta_2^2 \cdot b \int_0^{T/b} (T-bx) \; [1 - \tanh(x/2)] \; dx \quad .$$

This integral cannot be integrated easily (if at all) in terms of well known functions. However, the integral

$$E = \frac{8}{\pi^2} \delta_1^2 \delta_2^2 \cdot b \int_0^{T/b} T[1 - \tanh(x/2)]dx \quad ,$$

which can be integrated directly, is a bound for ζ. $\zeta \le E$, since $T - bx \le T$ for all x in the region of integration. Performing the integration yields

$$E = \frac{8}{\pi^2} \delta_1^2 \delta_2^2 \; \{T^2 - 2Tb \; \log[\cosh(T/2b)]\}$$

$$= \frac{8}{\pi^2} \delta_1^2 \delta_2^2 \left\{ T^2 - 2Tb \; \log \left[\frac{e^{T/2b} + e^{-T/2b}}{2} \right] \right\}$$

or, since T is much greater than b,

$$E \cong \frac{8}{\pi^2} \delta_1^2 \delta_2^2 \left[T^2 - 2Tb \log \frac{e^{T/2b}}{2} \right]$$

$$\cong \frac{8}{\pi^2} \delta_1^2 \delta_2^2 \cdot 2Tb \log(2)$$

$$\cong 1.13 \ \delta_1^2 \delta_2^2 \ Tb \quad .$$

Thus, $\zeta \leq 1.13 \ \delta_1^2 \delta_2^2$ Tb. But ζ is the largest error which can occur. If $\tau_s \neq \tau_n$, then the error is smaller. Therefore, the error, e, for the approximation is bounded by $e \leq 1.13 \ \delta_1^2 \delta_2^2$ Tb.

The relative error in the approximation is

$$\rho = \frac{e}{I_1} \quad .$$

Substituting the upper bound just derived for e, along with the expression for I_1 from equation 4.2-3,* yields

$$\rho \leq 2.78 \ b/T = 2.78 \max(\tau_s, \tau_n)/T \quad .$$

It will be assumed henceforth that $T/\max(\tau_s, \tau_n) \geq 100$, i.e. that the signal and noise band-widths are at least 100 times the

---

* The expression for I_1 in equation 4.2-3 is, of course, the approximate expression for I_1, whose relative error is to be found. The equation for the relative error, in order to be precise, must have the exact expression for I_1 in its denominator. However, if the approximate expression is very nearly equal to the exact expression as it is in this case, then the substitution of the approximate expression for the exact expression does not introduce a significant error in ρ.

integrator band-width. This is an assumption satisfied by many practical systems. Then, $\rho \leq 0.0278$.

By property d) for relative errors, derived above, the relative error for the approximation just discussed adds to the relative error for the previous approximation. Therefore, the total relative error for I_1 is bounded by

$$\rho_1 \leq 0.048 \quad . \tag{4.3-5}$$

The actual error in I_1 due to the approximations is then bounded by $e_1 \leq 0.048 \frac{4}{\pi^2} \delta_1^2 \delta_2^2 T^2 \leq 2 \times 10^{-6} T^2$, since $\delta_1 \leq 0.1$ and $\delta_2 \leq 0.1$. It was shown in the section in which the approximate expression for I_1 was derived that an approximate expression for μ_o^2 when δ_1 and δ_2 are small is the same as that for I_1, and that the relative error for the approximation to μ_o is less than $\delta_1^2 \delta_2^2/36$. By property b) of relative errors, derived above, the relative error for the approximation to μ_o^2 is then less than $\frac{1}{18} \delta_1^2 \delta_2^2$, and the actual error is then less than

$$\frac{1}{18} \frac{4}{\pi^2} \delta_1^4 \delta_2^4 T^2 \leq 3 \times 10^{-10} T^2 \quad ,$$

since $\delta_1 \leq 0.1$ and $\delta_2 \leq 0.1$. Thus the error in the approximation for μ_o^2 is completely negligible with respect to the error in the approximations for I_1, and the actual error in the equation $I_1 - \mu_o^2 = 0$, which is used in obtaining the variance, is due entirely to the error in the approximation for I_1. It is then,

$$e_o \le 0.02\ \delta_1^2 \delta_2^2\ T^2 \le 2 \times 10^{-6}\ T^2 \qquad (4.3\text{-}6)$$

when δ_1 and δ_2 are less than 0.1.

Since the function $I_1 - \mu_o^2$ is approximately zero, the concept of relative error is meaningless for it.

I_2)

The first approximation made in order to evaluate I_2 was the substitution of V_o for V in equation 4.1-5 to obtain equation 4.1-6. It has been shown in the discussion of the error for I_1 that the relative error due to this approximation is less than 0.01 when δ_1 is less than 0.1.

The next approximation made was the substitution of the function

$$f(x) = \begin{cases} \sqrt{x/2} & \text{for } 0 \le x \le 1 \\ (1 - 0.85\, e^{-x}) & \text{for } 1 \le x \end{cases}$$

in place of the function $\sqrt{(1-e^{-x})/(1+e^{-x})}$. It is shown in Appendix VI that the relative error for this approximation nowhere exceeds 0.04.

Next the approximation $(e^{-x} + \frac{1}{2}e^{-5x})$ was made for $\sin^{-1}(e^{-x})$. It is shown in Appendix VI that the relative error due to this approximation is everywhere less than 0.07.

Then the constant T was substituted for the function $(T-\tau)$. The error in the integral due to this substitution is

$$e = \int_0^T T\cdot g(\tau)d\tau - \int_0^T (T-\tau)g(\tau)d\tau$$

$$= \int_0^T \tau\, g(\tau)d\tau \quad ,$$

where $g(\tau)$ represents all the factors in the integrand except for $(T-\tau)$.

This integral can be integrated, but the result is complicated and is not easily interpreted. However, a simple bound can be found for this integral. The integrand is

$$\tau g(\tau) = f(\tau/\tau_n)(e^{-\tau/\tau_s} + \tfrac{1}{2}e^{-5\tau/\tau_s})\cdot\tau \quad ,$$

where the function $f(x)$ was defined above. The function $\frac{\pi}{2}\, e^{-\tau/\tau_s}$ is greater than $(e^{-\tau/\tau_s} + \frac{1}{2}e^{-5\tau/\tau_s})$ for all positive τ. The function $(1 - 0.8\, e^{-\tau/\tau_n})$ is greater than $f(\tau/\tau_n)$ for all positive τ, and will be used when $\gamma \leq 0.4$. The function $\sqrt{\tau/2\tau_n}$ is greater than $f(\tau/\tau_n)$ for all positive τ, and will be used when $\gamma \geq 0.4$. Thus, for

<u>$\gamma \leq 0.4$</u>

$$e \leq \int_0^T \tau(1 - 0.8\, e^{-\tau/\tau_n})(\tfrac{\pi}{2})\, e^{-\tau/\tau_s}d\tau \quad .$$

On integrating and neglecting terms in $\frac{T}{\tau_s} e^{-T/\tau_s}$, since $T \gg \tau_s$, this becomes

$$e \leq \frac{\pi}{2} \tau_s^2 \left[1 - 0.8\left(\frac{\gamma}{1+\gamma}\right)^2\right] \leq \frac{\pi}{2} \tau_s^2 \quad .$$

$\underline{\gamma \geq 0.4}$

$$e \leq \int_0^T \tau \sqrt{\frac{\tau}{2\tau_n}} \left(\frac{\pi}{2}\right) e^{-\tau/\tau_s} d\tau \quad .$$

On integrating and neglecting terms in $\left(\frac{T}{\tau_s}\right)^a e^{-T/\tau_s}$ where $a = \frac{1}{2}$ or $3/2$, since $T \gg \tau_s$, this becomes

$$e \leq \left(\frac{\pi}{2}\right)^{3/2} \cdot \frac{3}{4} \frac{\tau_s^2}{\sqrt{\gamma}} \leq 1.5 \frac{\tau_s^2}{\sqrt{\gamma}} \quad .$$

The relative error is

$$\rho = \frac{8}{\pi^2} \delta_1^2 e/I_2 \quad .$$

From figure 4.2-6, for $\gamma \leq 0.4$ a good approximation to I_2 is $1.1 \frac{8}{\pi^2} \delta_1^2 T\tau_s$, and for $\gamma \geq 0.4$ a good approximation to I_2 is $\frac{8}{\pi^2} \delta_1^2 T\tau_s \times 0.626/\sqrt{\gamma}$. Therefore,

For $\gamma \leq 0.4$

$$\rho \leq 1.5 \, \tau_s/T \leq 0.015 \quad ,$$

since $\tau_s/T \leq 0.01$, and

For $\gamma \geq 0.4$

$$\rho \leq 2.5 \, \tau_s/T \leq 0.025 \quad ,$$

since $\tau_s/T \leq 0.01$. Thus, for all γ

$\rho \leq 0.025$.

The final approximation made in obtaining an expression for I_2 was to neglect the terms due to the upper limit of integration, T, in the second integral of equation 4.2-4. In the integrated form, the neglected terms are

$$e = 0.85 \left[\frac{\gamma}{1+\gamma} e^{-(1+\gamma)T/\tau_n} + \frac{1}{10} \frac{5\gamma}{1+5\gamma} e^{-(1+5\gamma)T/\tau_n} \right] - \left[e^{-\gamma T/\tau_n} + \frac{1}{10} e^{-5\gamma T/\tau_n} \right] .$$

<u>For $\gamma > 1$</u>

The magnitude of the positive term is less than $1.1\, e^{-\gamma T/\tau_n} \leq 1.1\, e^{-100\gamma}$, since $T \geq 100\tau_n$. The magnitude of the negative term is also less than $1.1\, e^{-100\gamma}$. Thus, the magnitude of the sum of the positive and negative terms is less than $1.1\, e^{-100\gamma}$. $|e| \leq 1.1\, e^{-100\gamma}$.

<u>For $\gamma \leq 1$</u>

The neglected terms can be rewritten

$$e = 0.85 \left[\frac{\gamma}{1+\gamma} e^{-\frac{T}{\tau_s} \frac{1+\gamma}{\gamma}} + \frac{1}{10} \frac{5\gamma}{1+5\gamma} e^{-\frac{T}{\tau_s} \frac{1+5\gamma}{\gamma}} \right] - \left[e^{-T/\tau_s} + \frac{1}{10} e^{-5T/\tau_s} \right] .$$

The magnitude of the positive term is less than $1.1\ e^{-T/\tau_s}$ $\leq 1.1\ e^{-100} \leq 4 \times 10^{-44}$, since $T/\tau_s \geq 100$. The magnitude of the negative terms is also less than 4×10^{-44}. Thus, the magnitude of the sum of the positive and negative terms is less than $4 \times 10^{-44} \cdot |e| \leq 4 \times 10^{-44}$.

The relative error is

$$\rho = \frac{8}{\pi^2} \delta_1^2\ T\tau_s\ e/I_2 \quad .$$

From figure 4.2-6, for $\gamma \geq 1$, a good approximation to I_2 is $0.626 \frac{8}{\pi^2} \delta_1^2\ T\tau_s/\sqrt{\gamma}$, and for $\gamma \leq 1$ a good approximation to I_2 is $1.1 \frac{8}{\pi^2} \delta_1^2\ T\tau_s$. Therefore,

<u>For $\gamma \geq 1$</u>

$$\rho \leq \frac{1.1\ e^{-100\gamma}}{0.626/\sqrt{\gamma}} \leq 7 \times 10^{-44}$$

which is entirely negligible.

<u>For $\gamma \leq 1$</u>

$$\rho \leq 4 \times 10^{-44}$$

which is also entirely negligible.

According to the properties of relative errors developed earlier, the total relative error for all the approximations made in obtaining an expression for I_2 is the sum of the individual errors. Thus,

$$\rho_2 \leq 0.01 + 0.04 + 0.07 + 0.025$$

or

$$\rho_2 \leq 0.145 \tag{4.3-7}$$

The first and last error terms in the preceding sum of errors decrease to zero as δ_1 decreases in magnitude. Thus, by restricting consideration to systems with sufficiently small $|\delta_1|$, the relative error in I_2 can be reduced to 0.11.

I_3)

I_3 is identical to I_2 if τ_s and τ_n are interchanged, δ_1 is replaced by δ_2, and γ is replaced by λ in the expression for I_2. Therefore, the relative error for I_3 is also no larger than 0.145 and can also be reduced to 0.11 by restricting consideration to systems with sufficiently small $|\delta_2|$.

$$\rho_3 \leq 0.145 \tag{4.3-8}$$

I_4)

The first approximation made in order to evaluate I_4 was the substitution of the constant T for the function $(T-\tau)$. The error due to this substitution is

$$e = \int_0^T T \sin^{-1}(e^{-\tau/\tau_n}) \sin^{-1}(e^{-\tau/\tau_s}) d\tau$$

$$- \int_0^T (T-\tau) \sin^{-1}(e^{-\tau/\tau_n}) \sin^{-1}(e^{-\tau/\tau_s}) d\tau \quad .$$

Since $\frac{\pi}{2} e^{-x}$ is greater than $\sin^{-1}(e^{-x})$ for all positive x, then

$$e \leq (\tfrac{\pi}{2})^2 \int_0^T \tau\, e^{-\tau/\tau_n} e^{-\tau/\tau_s} d\tau$$

$$\leq (\tfrac{\pi}{2})^2 \tau_s^2 (\tfrac{\gamma}{1+\gamma})^2 \quad ,$$

where terms in $e^{-T(\frac{1}{\tau_s} + \frac{1}{\tau_n})}$ have been neglected, since $T \gg \tau_s$ and τ_n. The relative error due to this substitution is

$$\rho \leq \frac{8}{\pi^2} e/I_4$$

$$\leq (\tfrac{\pi}{2})^2 \frac{\tau_s}{T} \frac{\gamma}{1+\gamma} \Big/ \left\{ 1.05 + \tfrac{1}{2} \left[\frac{1+\gamma}{1+5\gamma} + \frac{1+\gamma}{\gamma+5} \right] \right\} \quad .$$

This function is less than $1.5\ \tau_s/T$ for all γ. Therefore, $\rho \le 0.015$, since $\tau_s/T \le 0.01$.

The next approximation was to replace the function $\sin^{-1}(e^{-x})$ by the function $e^{-x} + \frac{1}{2} e^{-5x}$. It is shown in Appendix VI that the relative error in this approximation is less than 0.07 in magnitude for all positive x. Since the function $\sin^{-1}(e^{-x})$ appears twice as a factor in the integrand of I_4, the relative error due to this approximation is less than 0.14.

The final approximation made in order to evaluate I_4 was to neglect the contribution of the terms due to the upper limit of integration, T. In the integrated form, the neglected terms are

$$- \tau_n \left\{ \frac{1}{1+\gamma} e^{-\frac{T}{\tau_n}(1+\gamma)} + \frac{1}{20} \frac{1}{1+\gamma} e^{-5\frac{T}{\tau_n}(1+\gamma)} + \frac{1}{2}\left[\frac{1}{1+5\gamma} e^{-\frac{T}{\tau_n}(1+5\gamma)} + \frac{1}{\gamma+5} e^{-\frac{T}{\tau_n}(\gamma+5)} \right] \right\} .$$

Each of these terms corresponds to a term in I_4. The coefficients of the exponentials are identical to the corresponding terms in I_4. Each of the exponentials is less than e^{-T/τ_n} which in turn is less than $e^{-100} \le 4 \times 10^{-44}$, since $T/\tau_n \ge 100$. Therefore, $e \le 4 \times 10^{-44} \cdot I_4$ and

$$\rho \le 4 \times 10^{-44} \quad ,$$

which is entirely negligible.

According to the properties of relative errors presented earlier, the relative error for I_4 due to all the approximations is the sum of the individual relative errors. Thus,

$$\rho_4 \leq 0.015 + 0.14$$

or

$$\rho_4 \leq 0.155 \quad . \tag{4.3-9}$$

SUMMARY

The variance of the output of a BPCD is

$$\sigma_z^2 = T\cdot\delta_1^2\cdot J_2 + T\cdot\delta_2^2\cdot J_3 + T\cdot J_4 \qquad ,$$

where δ_1 is the input channel bias (normalized), δ_2 is the reference channel bias (normalized), T is the integration interval, and J_2, J_3 and J_4 are constants for particular signal and noise correlation durations (or band-widths), given by equations 4.2-10.

The above expression for the variance is valid only if the signal is absent from the input channel, the biases are less than one-tenth of the input and reference levels, and the signal and noise both have RC low-pass spectra with band-widths more than one-hundred times the integrator band-width.

The relative errors in each of the variance terms of the above equation are less than 0.15. In addition, a constant (non-relative) error exists in the above equation, due to the approximate cancellation* of two terms, I_1 and μ_o^2. This constant error is less than $0.02\ \delta_1^2\delta_2^2\ T^2$.

---

* Originally, the expression for $E\{z^2\}$ contained four integrals. In the discussion of the first integral, I_1, it was shown within the accuracy established by the approximations that the square of the mean, μ_o^2, is equal to this integral. Thus, in obtaining the variance from $E\{z^2\}$ by subtracting μ_o^2, these two terms cancel. However, since they are only approximately equal, the error in their difference remains to be taken into account.

It should be stressed that the errors given are upper bounds on the actual errors, and in many situations the actual errors will be significantly less. In any case, these errors are small enough that the detection properties derived for a BPCD in the next chapter will be meaningful.

Moreover, the assumption of RC low-pass spectra for the signal and noise is probably not valid in a real system, since an RC low-pass spectrum implies that the signal or noise has not passed through any multi-pole networks. Even if all the networks in both the transmission and reception ends of a system were single-pole RC networks - a situation which does not occur in nature - their cascaded effect would not be equivalent to a single-pole RC network. Consequently, the assumption that the signal and noise have RC low-pass spectra is at best an approximation. Since the numerical values of J_2, J_3 and J_4 depend through their defining integrals on the functional form of the signal correlation and noise correlation, which in turn depend on the functional form of the signal and noise spectra, the results presented above would be only approximate for real systems even if the integrals were evaluated exactly. The errors due to the approximate evaluation of the integrals then assume less significance.

REFERENCES

1. Kendall, M. G.: "Proof of Relations Connected with the Tetrachoric Series and its Generalization," *Biometrika*, vol. 32, pp. 196-198; 1941.

2. Moran, P. A. P.: "Rank Correlation and Product-Moment Correlation," *Biometrika*, vol. 35, pp. 203-206; 1948.

3. David, F. N.: "A Note on the Evaluation of the Multi-variate Normal Integral," *Biometrika*, pp. 458-459; 1953.

4. U. S., National Bureau of Standards: *Tables of the Bivariate Normal Distribution Function and Related Functions*; 1959.

5. Dwight, H. B.: *Tables of Integrals and Other Mathematical Data*, Macmillan, New York, p. 129; 1947.

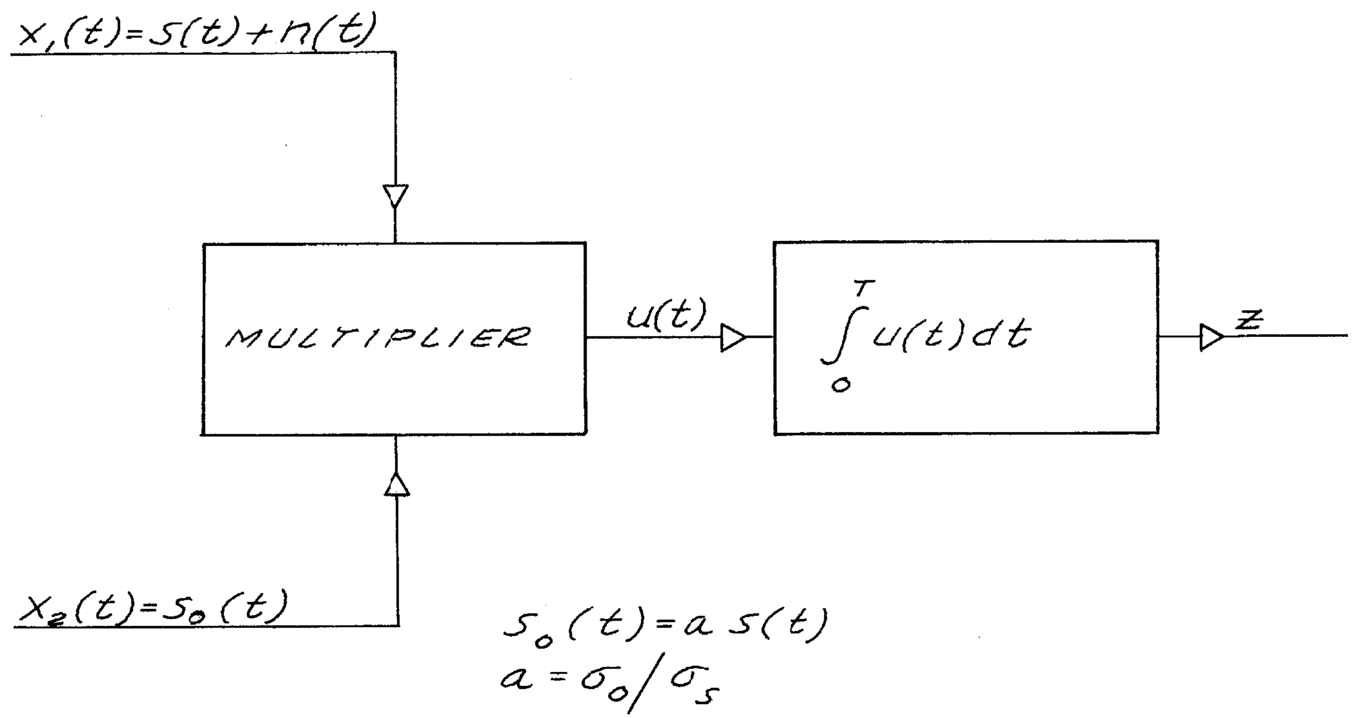
CHAPTER V

THE DETECTION PROPERTIES of a BIASED POLARITY-COINCIDENCE DETECTOR

In this chapter the behavior of a BPCD as a detector is examined. First, the mean and variance of the output for an ideal correlation detector (no limiters in the input and reference channels) will be derived for purposes of comparison. Then, using the output signal-to-noise power ratio as a criterion of performance, the BPCD, the (unbiased) polarity-coincidence detector and the ideal correlation detector will be discussed and compared.

5.0 The Mean and Variance for an Ideal Correlation Detector.

The properties of an ideal correlation detector are well known. However, since the analysis of an ideal correlation detector is straightforward and not very involved, it will be presented here for completeness. In the following analysis it will be assumed that the signal and noise are sample functions from wide-sense stationary independent Gaussian random processes with RC low-pass spectra. (See assumptions A1 through A13, A3-1, A3-2, A4-1, A4.2-1, A4.2-2 and A4.2-3 made previously.) A block diagram of a correlation detector is given in figure 5.0-1.



Block Diagram of Ideal Correlation Detector

Figure 5.0-1

The Mean Value of the Output.

The mean value of z is

$$\mu_z = E\{z\} = E\left\{\int_0^T u(t)dt\right\} = \int_0^T E\{u(t)\}dt$$

$$= \int_0^T E\{x_1(t)x_2(t)\}dt \quad . \tag{5.0-1}$$

By assumption A2 of Chapter III, s, s_o and n have wide-sense stationary distributions. Therefore, x_1 and x_2 have wide-sense stationary distributions and $E\{x_1(t)x_2(t)\}$ is time independent and can be written $E\{x_1x_2\}$. Thus,

$$\mu_z = T\cdot E\{x_1x_2\} \quad . \tag{5.0-2}$$

By assumption A1 of Chapter III, $E\{x_1x_2\}$ is

$$E\{x_1x_2\} = E\{(s+n)s_o\} = E\{ss_o\} + E\{s_on\} \quad .$$

By assumption A4 of Chapter III, $s_o = a\cdot s$, where $a = \sigma_o/\sigma_s$, and by assumption A5 of Chapter III, s_o and n are statistically independent. Moreover, by assumption A6 of Chapter III, $E\{n\} = 0$ and by assumption A7 of Chapter III, $E\{s\} = 0$. Therefore,

$$E\{x_1x_2\} = E\{ss_o\} = a\cdot E\{s^2\} = a\sigma_s^2 = \sigma_o\sigma_s \quad . \tag{5.0-3}$$

Finally, then,

$$\mu_z = T \cdot \sigma_o \sigma_s \quad . \tag{5.0-4}$$

<u>The Variance of the Output.</u>

The expected value of z^2 is

$$E\{z^2\} = E\left\{\iint_0^T u(t)u(\theta)dtd\theta\right\} = \iint_0^T E\{u(t)u(\theta)\}dtd\theta \quad . \tag{5.0-5}$$

Upon introducing the subscript i to indicate the time variable t and the subscript j to indicate the time variable θ, $E\{u(t)u(\theta)\}$ can be written

$$E\{u(t)u(\theta)\} = E\{u_i u_j\} = E\{x_{1i}x_{2i}x_{1j}x_{2j}\} \quad .$$

By assumption A1 of Chapter III, this is

$$E\{u_i u_j\} = E\{(s_i+n_i)s_{oi}(s_j+n_j)s_{oj}\}$$

$$= E\{s_{oi}s_{oj}s_i s_j\} + E\{s_{oi}s_{oj}s_i n_j\}$$

$$+ E\{s_{oi}s_{oj}s_j n_i\} + E\{s_{oi}s_{oj}n_i n_j\} \quad .$$

Since $s_o = a \cdot s$ and $E\{n\} = 0$ and since by assumption A9 of Chapter IV, s and n have statistically independent second order distributions, then

$$E\{u_i u_j\} = a^2\, E\{s_i^2 s_j^2\} + a^2\, E\{s_i s_j\}\, E\{n_i n_j\} \quad . \qquad (5.0\text{-}6)$$

The second term on the right hand side of the above equation can be written immediately in terms of the signal auto-correlation function and the noise auto-correlation function. The first term can be expanded as follows: It is well known that if ζ_1, ζ_2, ζ_3 and ζ_4 are real random variables with a joint Gaussian distribution and with zero means, then

$$E\{\zeta_1\zeta_2\zeta_3\zeta_4\} = E\{\zeta_1\zeta_2\}E\{\zeta_3\zeta_4\} + E\{\zeta_1\zeta_3\}E\{\zeta_2\zeta_4\} + E\{\zeta_1\zeta_4\}E\{\zeta_2\zeta_3\} \quad .$$

By assumption A4-1 of Chapter IV, both the signal and noise have second order Gaussian distributions with zero means. Then s_i, s_i, s_j and s_j, regarded as four random variables, have a fourth order Gaussian distribution with zero means. Therefore,

$$E\{s_i^2 s_j^2\} = E\{s_i^2\}E\{s_j^2\} + 2\, E^2\{s_i s_j\} \quad .$$

Thus,

$$E\{u_i u_j\} = a^2\, E\{s_i^2\}E\{s_j^2\} + 2a^2\, E^2\{s_i s_j\} + a^2\, E\{s_i s_j\}E\{n_i n_j\}$$

$$= a^2\, \sigma_s^4 + 2a^2\, R_s^2(\theta-t) + a^2\, R_s(\theta-t)R_n(\theta-t) \quad , \qquad (5.0\text{-}7)$$

where $R_s(\theta-t) = \sigma_s^2 r(\theta-t)$ and $R_n(\theta-t) = \sigma_n^2 \rho(\theta-t)$ are the autocorrelation functions for the signal and for the noise. Upon substituting the equivalent expressions for R_s and R_n in terms of r and ρ, this equation becomes

$$E\{u_i u_j\} = \sigma_o^2 \sigma_s^2 + 2\,\sigma_o^2 \sigma_s^2 r^2(\theta-t) + \sigma_o^2 \sigma_n^2 r(\theta-t)\rho(\theta-t) \quad .$$

Finally, then,

$$E\{z^2\} = T^2 \sigma_o^2 \sigma_s^2 + 2\,\sigma_o^2 \sigma_s^2 \iint_0^T r^2(\theta-t)\,dt\,d\theta + \sigma_o^2 \sigma_n^2 \iint_0^T r(\theta-t)\rho(\theta-t)\,dt\,d\theta \quad . \tag{5.0-8}$$

Physical Interpretation

Before proceeding with the derivation of the variance for the ideal correlation detector, the physical meaning of the components in the output of the detector as expressed in the above equations will be discussed, and its implications for the BPCD will be examined.

First, it is clear that the amplitude, σ_o, of the reference signal has no effect on the properties of the detector, since it appears as a factor to the first power in the expression for the mean and as a factor to the second power in each of the terms in the expression for the variance.

The mean, given by equation 5.0-4, represents the D.C. component of the detector output. If the output, z, exceeds a pre-established threshold, it is assumed that a signal is present and that the threshold has been exceeded due to the D.C. component caused by the signal's presence. Thus, μ_z is called the "output signal". The quotation marks are employed because μ_z is a real number and not a signal in time in the usual sense.

$E\{z^2\}$, given by equation 5.0-8, has three terms or components. The first term on the right hand side of equation 5.0-8 is the square of μ_z and represents the "output signal" power.

The third term on the right hand side of equation 5.0-8 is the fluctuation term due to the input noise, and is proportional to the input noise power, σ_n^2. It should be stressed that the output of the detector for a single detection attempt, consisting of an integration of the function $x_1 x_2$ for a period of length T, is a single fixed real number - not a function of time - and thus is a fixed constant for any particular detection attempt. The fluctuations in this "constant" output are due to uncertainties in the input. I.e., the value of z is not known before the detection attempt is made, because $n(t)$ is an unknown sample function from a random process. Thus, z is a random variable and if the detection attempt is made repeatedly, assuming a new noise sample function for each repetition, then the a-posteriori value of z presented by the detector at the end of the integration interval T will fluctuate from one detection attempt to the next. These fluctuations in the a-posteriori values of z are usually referred to as the "output noise".

In applications, the detection attempt may be made repeatedly in the presence of a single noise sample function varying randomly through all time, as in a pulsed radar system. However, the frequency with which the detection attempt is repeated is usually small compared with the noise band-width, and consequently the correlation between the noise samples from one detection attempt to the next is small, so that the a-posteriori knowledge of earlier detector outputs gives no a-priori correlation knowledge of the next detector output.

Discussion of the second term on the right hand side of equation 5.0-8 is presented last because in a certain sense, to be made clear below, this term is spurious. This term is the fluctuation term due to the random nature of the signal, and in many applications the signal is not really random.

In those cases where the signal actually is random, the discussion given above for the fluctuations due to the input noise applies also to the fluctuations due to the input signal. Since $s(t)$ is an unknown sample function from a random process, $s(t)$ causes another component of randomness to appear in z in addition to the randomness due to $n(t)$. Thus, as the detection attempt is made repeatedly, assuming a new signal sample function for each repetition, the a-posteriori value of z after the detection attempt will contain additional fluctuations due to the randomness of $s(t)$. These fluctuations are represented by the second term on the right hand side of equation 5.0-8 and are proportional to the input signal power, σ_s^2. Therefore, when $N = \sigma_s^2/\sigma_n^2$ is sufficiently small, the "output noise" is due almost entirely to the input noise and the fluctuations due to the signal can be neglected.

A system in which the signal is a sample function from a random process could be obtained, for example, by selecting a section of duration T from the output of a random noise generator, to be used both for the reference signal and for the signal to be transmitted through a noisy channel to the detector. After waiting a sufficiently long time to assure independence of the next sample (for some processes, the next sample will not be independent no matter how long the waiting period), another section of duration T would be selected from the output of the same random noise generator, to be used again for both the reference and transmitted signals. An alternative method would be to use a new identical but statistically independent noise generator for each new sample of duration T, to be used both as transmitted and reference signal. With this method, the consecutive signals are independent regardless of the waiting period between transmissions.

In many systems, the signal is not a random function. Although it may have been selected originally from a set of sample functions of a random process, once selected, the identical sample function is used for all future signals and reference functions. In the case that the signal has been selected by some means other than a random choice from a set of sample functions (e.g. if the signal is specified arbitrarily by a functional form such as $s(t) = \sin\omega t$), it can still be regarded as a sample function selected at random from a set of sample functions of a random process. (E.g. the function $s(t) = \sin\omega t$ can be regarded as having been selected from the set of sample functions $\sin(\omega t + \phi)$, where ϕ is a random variable uniformly distributed between 0 and 2π).

In this case, where the signal and reference are invariant from one detection attempt to the next, some modifications of the preceding remarks about μ_z and $E\{z^2\}$ are required. In order to make these modifications, it will be necessary to review the steps taken in deriving the expressions for μ_z and $E\{z^2\}$.

Of the equations derived earlier in this section for the mean value of z, equation 5.0-1 is still valid when s and s_o are not random variables but equation 5.0-2 is no longer valid. It becomes instead

$$\mu_z = \int_0^T E\{x_1 x_2\} dt \quad . \tag{5.0-9}$$

Since s and s_o are now non-random, equation 5.0-3 becomes

$$E\{x_1 x_2\} = a \cdot s^2 \quad , \tag{5.0-10}$$

and equation 5.0-4 becomes

$$\mu_z = aT \left[\frac{1}{T} \int_0^T s^2(t) dt \right] \quad . \tag{5.0-11}$$

The quantity inside the brackets of the above equation is the finite time average of $s^2(t)$ over the interval T. Let

$$A_f(T) = \frac{1}{T} \int_0^T f(t) dt$$

denote the finite time average of $f(t)$ over the interval T. The ergodic theorem asserts that if $f(t)$ is a sample function from a strict sense stationary random process,* then

$$\lim_{T\to\infty} A_f(T) = E\{f|I\} \qquad \text{with probability } 1, \qquad (5.0\text{-}12)$$

where I is the element from the Borel field of invariant sets in the sample space** which contains $f(t)$, and $E\{f|I\}$ is the conditional expectation of f given I. If the only invariant sets of the sample space are those with probability 0 and 1, then the process is called ergodic and $\lim_{T\to\infty} A_f(T) = E\{f\}$ with probability 1.

If $f(t)$ is a sample function from a stationary process, and if $g(t) = G\{f(t)\}$ is a function of $f(t)$, then $g(t)$ is a sample function from a stationary process also. Moreover, if the f process is ergodic, then the g process is ergodic also.***

---

* It is assumed that f is a measurable function of t.

** The invariant sets in the sample space are those which under a translation in t differ from their image by at most a set which may depend on t but which has probability 0 for each t.

*** In order that the ergodic theorem hold for g, g as well as f must be a measurable function of t.

Most stationary processes encountered in applications are ergodic. Those stationary processes which are not ergodic can be decomposed into component processes which are ergodic. Thus, there is no loss of generality for practical purposes in assuming that the signal process is ergodic. In fact, it will be assumed later that the signal process is stationary Gaussian with auto-correlation function $R_s(\tau) = e^{-|\tau|/\tau_s}$. It can be shown that this process is indeed ergodic.

It will be assumed for the present case, where $s(t)$ is not a random function, that nevertheless $s(t)$ was obtained originally by sampling from a strict-sense stationary ergodic random process. The present case differs from the case analyzed previously, then, only in that a new sample function for the signal and reference is not obtained for each new transmission and detection attempt. The same sample function, once obtained, is used for all transmissions and detection attempts.

Since $s(t)$ is a sample function from an ergodic process, then $s^2(t)$ is a sample function from an ergodic process also, and by equation 5.0-12.

$$\lim_{T\to\infty}\left[\frac{1}{T}\int_0^T s^2(t)dt\right] = E\{s^2\} = \sigma_s^2 \quad \text{(with probability 1)},$$

where σ_s^2 is the variance of the process from which $s(t)$ was originally selected. Therefore, from equation 5.0-11,

$$\mu_z \cong T\cdot a\sigma_s^2 = T\cdot\sigma_o\sigma_s \qquad \text{(with probability 1)}, \qquad (5.0\text{-}13)$$

which is equal to the expression for μ_z obtained for the previous case with $s(t)$ random. No attempt will be made in this thesis to analyze the error in this approximation. However, the error should be very small if it is assumed that T is large with respect to the duration time for dependence, τ_s, for the signal process.

Within the precision of the approximation, then, the mean of the detector output is independent of whether the signal is a random function or not, so long as the signal has been selected from the sample functions of an ergodic process.

Of the equations derived earlier in this section for $E\{z^2\}$, equations 5.0-5 and 5.0-6 are still valid but equations 5.0-7 and 5.0-8 are not. Since s_i and s_j are now non-random, equation 5.0-6 becomes

$$E\{u_i u_j\} = a^2 s_i^2 s_j^2 + a^2 s_i s_j E\{n_i n_j\}$$

$$= a^2 s_i^2 s_j^2 + a^2 s_i s_j R_n(\theta - t) \quad . \qquad (5.0\text{-}14)$$

Note that the term corresponding to the second term of equation 5.0-7 or the second term of equation 5.0-8 is missing in equation 5.0-14. This term represents the output fluctuation due to the randomness of the signal, and it is natural that it should vanish when the signal is non-random.

Equation 5.0-8 now becomes,

$$E\{z^2\} = a^2T^2\left[\frac{1}{T}\int_0^T s^2(t)dt\right]\left[\frac{1}{T}\int_0^T s^2(\theta)d\theta\right]$$

$$+ a^2 \iint_0^T s(t)s(\theta)R_n(\theta-t)dtd\theta \quad . \qquad (5.0\text{-}15)$$

The first term of this equation is identical to μ_z^2, as given by equation 5.0-11. By an argument similar to but considerably more complicated than the one used for obtaining the approximate equality between μ_z for the non-random signal case and μ_z for the random signal case, it can be shown that the second term in the above equation is approximately equal to the third term of equation 5.0-8, the equation for $E\{z^2\}$ in the random signal case. Again the approximation should be good if it is assumed that T is large with respect to τ_s.

It is clear now that the D.C. component of $E\{z^2\}$ is exactly equal to μ_z^2 in both cases. This was to be expected, of course. Moreover, within the precision of the approximation, the fluctuation term in $E\{z^2\}$ due to the input noise is independent of whether the signal is random or not, so long as the signal has been selected from the sample functions of an ergodic process. The only essential difference between $E\{z^2\}$ when $s(t)$ is random and when $s(t)$ is non-random is the presence of a fluctuation term due to the signal randomness in the former case which is absent in the latter case. When the signal-to-noise power ratio is small, even this difference vanishes.

The corresponding analysis for the BPCD is much more complicated than and not nearly so straightforward as that for the ideal correlation detector. Nevertheless, it can be carried through and the ergodic property of the input signal process can be used to show that the same sort of approximate equality exists between the BPCD means for random and non-random signals and between the BPCD output fluctuations due to the input noise for random and non-random signals. The fluctuation term due to signal randomness did not appear in the equations for the BPCD output variance derived in Chapter IV because it was assumed there that the input signal-to-noise power ratio is zero.

The preceding lengthy digression was presented in order to establish the approximate equality between certain output parameters of a detector when the input signal is a random function and the same parameters when the input signal is a non-random function. All of the analysis preceding the digression and all of the subsequent analysis assumes that the signal is random. This is a convenience in carrying out the analysis. However, many systems utilize non-random signals. Therefore, it was necessary to establish these equalities.

The derivation of the variance for the ideal correlation detector is now continued. Since $\sigma_z^2 = E\{z^2\} - \mu_z^2$, then by equation 5.0-4 and equation 5.0-8,

$$\sigma_z^2 = 2\sigma_o^2\sigma_s^2 \iint_0^T r^2(\theta-t)dtd\theta + \sigma_o^2\sigma_n^2 \iint_0^T r(\theta-t)\rho(\theta-t)dtd\theta \quad .$$

By transforming coordinates to $\tau = \theta - t$, $\lambda = \theta + t$, this integral becomes

$$\sigma_z^2 = \sigma_o^2 \sigma_s^2 \iint_R r^2(\tau) d\lambda d\tau + \tfrac{1}{2} \sigma_o^2 \sigma_n^2 \iint_R r(\tau)\rho(\tau) d\lambda d\tau \quad ,$$

where the region of integration, R, is shown in figure 5.0-2.

Since $r(\tau)$ and $\rho(\tau)$ are both even in τ, the whole integral is equal to twice its value over the right hand half of R. Thus,

$$\sigma_z^2 = 2\sigma_o^2 \sigma_s^2 \int_0^T d\tau \int_\tau^{2T-\tau} r^2(\tau) d\tau + \sigma_o^2 \sigma_n^2 \int_0^T d\tau \int_\tau^{2T-\tau} r(\tau)\rho(\tau) d\tau \ .$$

Since the integrand is independent of λ, the integration with respect to λ can be taken immediately.

$$\sigma_z^2 = 4\sigma_o^2 \sigma_s^2 \int_0^T (T-\tau) r^2(\tau) d\tau + 2\sigma_o^2 \sigma_n^2 \int_0^T (T-\tau) r(\tau)\rho(\tau) d\tau \quad .$$

By assumption A4.2-3 of Chapter IV, $r = e^{-|\tau|/\tau_s}$ and $\rho = e^{-|\tau|/\tau_n}$, where the correlation duration constants, τ_s and τ_n, are both small with respect to T. Then

$$\sigma_z^2 = 4\sigma_o^2 \sigma_s^2 \int_0^T (T-\tau) e^{-2\tau/\tau_s} d\tau + 2\sigma_o^2 \sigma_n^2 \int_0^T (T-\tau) e^{-\tau/\tau_o} d\tau \quad ,$$

where $\tau_o = 1/(\frac{1}{\tau_s} + \frac{1}{\tau_n}) = \tau_s \tau_n/(\tau_s + \tau_n)$.

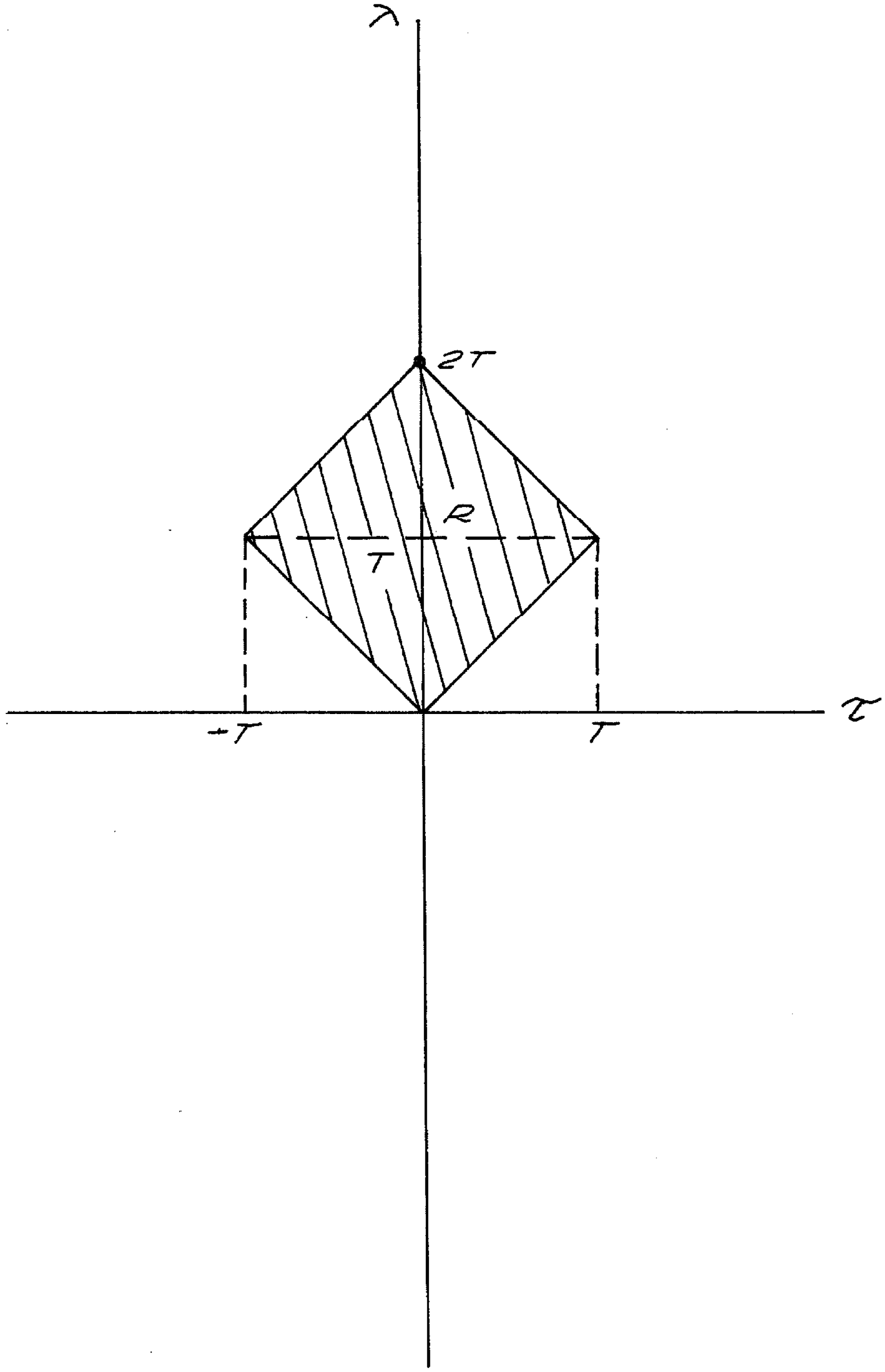


Figure 5.0-2

The preceding integrals can be integrated directly. The result is

$$\sigma_z^2 = 4\sigma_o^2\sigma_s^2\left\{\frac{T\tau_s}{2} - \frac{\tau_s^2}{4}\,(1-e^{-2T/\tau_s})\right\}$$
$$+\ 2\sigma_o^2\sigma_n^2\left\{T\tau_o - \tau_o^2\,(1-e^{-T/\tau_o})\right\} \quad .$$

Since both τ_s and τ_n are small with respect to T, then τ_o is also small with respect to T, because τ_o is less than either τ_s or τ_n. The quantity inside the parantheses in each term above is slightly less than unity. Thus,

$$\sigma_z^2 = 2\sigma_o^2\sigma_s^2\tau_s\left[T-\zeta\tau_s/2\right] + 2\sigma_o^2\sigma_n^2\tau_o\left[T-\xi\tau_o\right] \quad ,$$

where ζ and ξ are slightly less than unity. If it is assumed here, as it was in Chapter IV, that τ_s and τ_n are both less than 0.01T, then

$$\sigma_z^2 \simeq 2\sigma_o^2\sigma_s^2T\tau_s + 2\sigma_o^2\sigma_n^2T\tau_o \quad ,$$

with a relative error in the approximation less than 0.01.

$$|\rho| \leq 0.01 \quad . \qquad (5.0\text{-}16)$$

Upon introducing the notation $N = \sigma_s^2/\sigma_n^2$ and $\gamma = \tau_n/\tau_s$, this equation becomes

$$\sigma_z^2 \cong 2\sigma_o^2\sigma_n^2 T\tau_s \left[N + \frac{\gamma}{1+\gamma}\right] \quad , \qquad (5.0\text{-}17)$$

with a relative error less than 0.01 if τ_s and τ_n both are less than 0.01T.

The term in this equation with N as a factor is the fluctuation term due to the random nature of the signal. According to the discussion presented earlier in this section, if the system is one in which the signal is not random, then the fluctuation term due to signal randomness vanishes. If the signal is random but the input signal-to-noise power ratio, N, is small relative to $\gamma/(1+\gamma)$ then the signal fluctuation term is negligible. Since the variance expressions derived in Chapter IV for the BPCD are for the case that $N = 0$, it will be assumed here that N is small relative to $\gamma/(1+\gamma)$ if the signal is random. This is a reasonable restriction to make, since the purpose in presenting the output parameters of the ideal correlation detector is to compare them with those derived earlier for the BPCD. Thus, whether or not the signal is random, the variance contains only the second term, which is the fluctuation term due to the input noise randomness.

$$\sigma_z^2 = 2\sigma_o^2\sigma_n^2 T\tau_s \left(\frac{\gamma}{1+\gamma}\right) \quad , \qquad (5.0\text{-}18)$$

if the signal is non-random or if N is small relative to $\gamma/(1+\gamma)$.

The "output signal-to-noise power ratio" is defined by

$$N_o = \mu_z^2/\sigma_z^2 \quad . \tag{5.0-19}$$

where N_o denotes the output signal-to-noise power ratio. From equations 5.0-4 and 5.0-18 for the ideal correlation detector, for $s(t)$ non-random or $N \ll \gamma/(1+\gamma)$, this is

$$N_o = \tfrac{1}{2}N \frac{T}{\tau_s} (1+\gamma)/\gamma \tag{5.0-20}$$

$$= \tfrac{1}{2}N \frac{T}{\tau_s} (1+\lambda)$$

where $\lambda = \tau_s/\tau_n$. When $\gamma = \lambda = 1$, i.e. when the noise and signal have identical correlation functions except for amplitude, then

$$N_o = N \frac{T}{\tau_s} \quad .$$

5.1 The "Output Signal-to-Noise Power Ratio" for an Ideal Polarity-Coincidence Detector.

The mean for the output of an ideal (no bias) polarity-coincidence detector is given by equation 3.2-4 of Chapter III. It is

$$\mu_1 = \frac{2T}{\pi} \tan^{-1}(\sqrt{N}) \quad .$$

The variance for the ideal polarity-coincidence detector when $N = 0$ is obtained from equation 4.2-9 of Chapter IV by setting δ_1 and δ_2 equal to zero. It is

$$\sigma_z^2 = T \cdot J_4 = \frac{8}{\pi^2} T\tau_s \left\{ \left(1 + \frac{1}{2 0}\right) \frac{1}{1+\lambda} + \frac{1}{2} \left[\frac{1}{1+5\lambda} + \frac{1}{\lambda+5} \right] \right\} \quad . \tag{5.1-1}$$

Although this expression is valid only when $N = 0$, the value of σ_z^2 for small N will not differ significantly from it. Therefore, equation 5.1-1 will be used also for N small. When N is small, the expression above for the output mean is approximately

$$\mu_1 = \frac{2T}{\pi} \sqrt{N} \quad . \tag{5.1-2}$$

Substituting equations 5.1-1 and 5.1-2 in equation 5.0-19 yields for the output signal-to-noise power ratio

$$N_o = \frac{1}{2} N \frac{T}{\tau_s} \Big/ \left\{ \left(1 + \frac{1}{2 0}\right) \frac{1}{1+\lambda} + \frac{1}{2} \left[\frac{1}{1+5\lambda} + \frac{1}{\lambda+5} \right] \right\} \quad . \tag{5.1-3}$$

Thus, the output signal-to-noise power ratio for the ideal polarity-coincidence detector is proportional to the input signal-to-noise power ratio, just as it is for the ideal correlation detector. In fact, the ideal polarity coincidence detector has an output signal-to-noise power ratio identical to that of the ideal correlation detector except for a degradation factor.

$$N_{o(PCD)} = \Delta \cdot N_{o(CD)} \qquad (5.1\text{-}4)$$

where Δ is the degradation factor, PCD denotes a polarity-coincidence detector and CD denotes a correlation detector. The degradation factor is

$$\Delta = \frac{1}{(1+\lambda)\left\{(1+\frac{1}{20})\frac{1}{1+\lambda} + \frac{1}{2}\left[\frac{1}{1+5\lambda} + \frac{1}{\lambda+5}\right]\right\}} , \qquad (5.1\text{-}5)$$

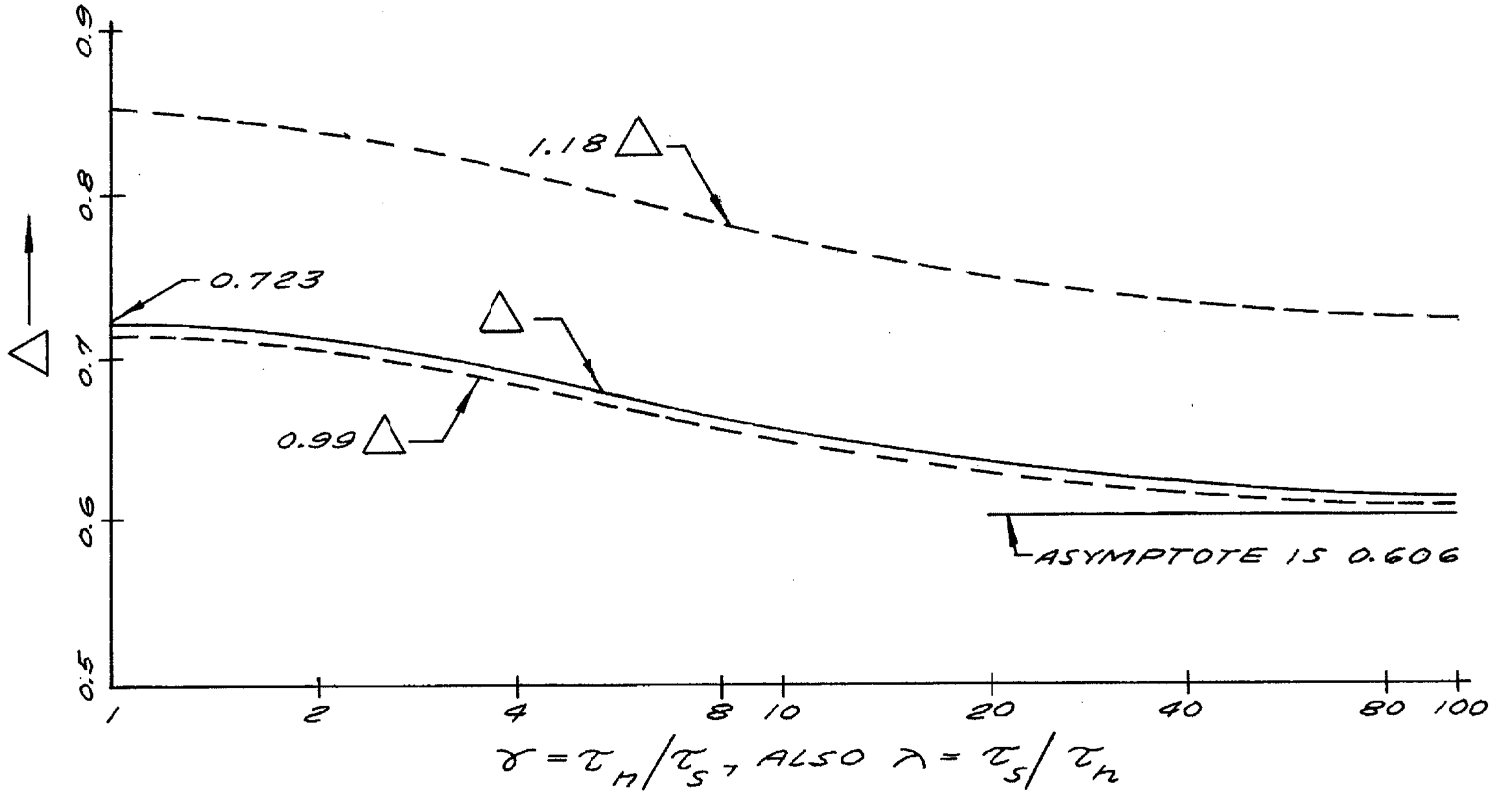
or, since $\gamma = 1/\lambda$,

$$\Delta = \frac{1}{(1+\gamma)\left\{(1+\frac{1}{20})\frac{1}{1+\gamma} + \frac{1}{2}\left[\frac{1}{1+5\gamma} + \frac{1}{\gamma+5}\right]\right\}} . \qquad (5.1\text{-}6)$$

Note that Δ is symmetric in γ and λ. Therefore, only values of γ (or λ) greater than 1 need be considered, since Δ for values of γ (or λ) smaller than 1 can be obtained from Δ for values of γ (or λ) greater than 1 by using the relation $\gamma = 1/\lambda$.

When $\gamma = \lambda = 1$, i.e. when the noise and signal have identical correlation functions except for amplitude, then $\Delta = 1/1.383 = 0.723$- a degradation of 1.4 db. As γ (or λ) goes to zero or infinity, Δ goes to $1/1.65 = 0.606$ - a degradation of 2.2 db.

A plot of the degradation factor is presented in figure 5.1-1. Note that according to equation 5.0-17, as $\gamma \to 0$ (or as $\lambda \to \infty$) the term with N is no longer negligible even for N small with respect to 1. In order that this term be negligible, N must be small with



Degradation Factor, Δ, for a PCD

Figure 5.1-1

respect to $\gamma/(1+\gamma)$. This fact must be born in mind in using the curve in figure 5.1-1. For a fixed N, small with respect to 1, only that part of the curve where $\gamma/(1+\gamma)$ is large with respect to N (or equivalently, where $\frac{1}{1+\lambda}$ is small with respect to $1/N$) is valid, unless the signal is non-random in which case the whole curve is valid. Thus, parts of the curve for $\frac{\gamma}{1+\gamma} \leq 1$ (or $1+\lambda \geq 1$) may not be valid, depending on the value of N, and on whether or not the signal is random. Of course, Δ can be obtained directly from equations 5.0-4, 5.0-17, 5.1-1 and 5.1-2 for any γ and any $N \ll 1$, even when the value indicated by the curve in figure 5.1-1 is not valid.

It should be noted that equations 5.1-5 and 5.1-6 are only approximate. It will be assumed that N is sufficiently small (say $N < 0.01 \frac{\gamma}{1+\gamma}$ if the signal is random) that the error in equation 5.0-18 due to neglecting the effects of signal fluctuations is negligible when compared with the error made in evaluating σ_z^2 in Chapter IV. It will also be assumed that N is sufficiently small (say $N < 0.1$) that the error in equation 5.1-2 is negligible when compared with the error made in evaluating σ_z^2 in Chapter IV. Then the only significant errors are those expressed by equation 4.3-9 of Chapter IV and equation 5.0-16 of the present chapter. According to equation 4.3-9 of Chapter IV, the relative error in equation 4.2-9 of Chapter IV expressing σ_z^2 for the polarity-coincidence detector is less than 0.155 and according to equation 5.0-16, the relative error in equation 5.0-18 expressing σ_z^2 for the correlation detector is less than 0.01. Thus, by the properties of relative errors derived in Chapter IV, the relative error for the approximate degradation given in equation 5.1-5 or 5.1-6 is less than 0.165.

An examination of the approximating function used in Chapter IV which leads to the error in equation 4.2-9 reveals that the error is in the direction of yielding a larger value for σ_z^2 than the actual value. This results in an expression for $N_{o(PCD)}$ which is smaller than the actual value. Thus the error in equation 4.2-9 results in a value of Δ which is smaller than the actual value; the value of Δ given by equation 5.1-5 or 5.1-6 is depressed from the actual value by an amount between 0 and 15.5% due to this error. Similarly, the error in equation 5.0-18 results in a value of σ_z^2 which is larger than the actual value, with the result that the expression for $N_{o(CD)}$ is smaller than the actual value. Thus the error in equation 5.0-18 results in a value of Δ which is larger than the actual value; the value of Δ given by equation 5.1-5 or 5.1-6 is elevated from the actual value by an amount between 0 and 1% due to this error. Therefore, $0.99\Delta \le \Delta_t = 1.18\Delta$, where Δ_t is the true degradation and Δ is the approximate degradation given by equation 5.1-5 or 5.1-6. The upper and lower bounds on Δ_t established by this inequality are indicated in figure 5.1-1 by the dotted curves.

Similar results for particular cases of polarity-coincidence detectors have appeared in the literature. For example, Faran and Hills (1) examined the behavior of a polarity-coincidence detector for Gaussian signal and Gaussian noise, with identical narrow-band RC band-pass spectra (correlation function $r(\tau) = e^{-\omega_f|\tau|} \cdot \cos\omega_o\tau$, where $\omega_f/2\pi$ is the half band-width and $\omega_o/2\pi$ is the center frequency), for the case where N is small. Thus, they examined a system corresponding to the one above for $\gamma = \lambda = 1$, but with an RC band-pass spectrum instead of an RC low-pass spectrum. They found the degradation in performance of the polarity-coincidence detector relative to the correlation detector to be 0.859.

In evaluating the variance both for the polarity-coincidence detector and for the correlation detector, Faran and Hills made simplifying approximations of a nature similar to some of the ones made in this thesis - namely, replacement of the function $T-\tau$ by the constant T. However, they did not analyze the error resulting from this approximation.

Since the difference between the degradation found by Faran and Hills and the degradation found above is of the same order of magnitude as the possible error, and since the Faran and Hills value is for a band-pass spectrum whereas the value derived above is for a low-pass spectrum, the two values are in good agreement.

5.2 The "Output Signal-to-Noise Ratio" for a Biased Polarity-Coincidence Detector.

The mean value of the output of a biased polarity-coincidence detector with (normalized) biases δ_1 in the input channel and δ_2 in the reference channel is given by equation 3.2-12. It is

$$\mu_4 = \mu_1 + e_4 \qquad ,$$

where

$$\mu_1 = \frac{2T}{\pi} \tan^{-1}(\sqrt{N}) \quad ,$$

$$e_4 = 4T \cdot V(\delta_1, d\delta_1) + 4T \cdot V(\delta_2, c\delta_2)$$

with

$$c = \sqrt{1+N}\ (\delta_1/\delta_2) - \sqrt{N} \quad ,$$

$$d = \sqrt{1+N}\ (\delta_2/\delta_1) - \sqrt{N} \quad .$$

When N is small, μ_1 is approximately

$$\mu_1 = \frac{2T}{\pi} \sqrt{N} \quad .$$

The power series for $\tan^{-1}x$ has terms of decreasing magnitude with alternating signs. Therefore, the error in neglecting all terms past the $n\text{-}1^{st}$ is less than the n^{th} term in magnitude. Thus, since the first two terms of the power series for $\tan^{-1}x$ are x and $-x^3/3$ when $|x| < 1$, then the error in the preceding approximation is less than $(\sqrt{N})^3/3$ and the relative error is less than $N/3$. It will be assumed that $N \le 0.1$. Then the relative error is less than 0.033.

The function V which appears in e_4 is discussed in Chapter III and Appendixes I and II. Since it was assumed in Chapter IV that δ_1 and δ_2 both are less than 0.1 in magnitude, and since it is now assumed that $N \le 0.1$, then $d\delta_1 \le 0.1$ and $c\delta_2 \le 0.1$.

Therefore, $(1+d^2)\delta_1^2$ and $(1+c^2)\delta_2^2$ are both less than 0.02 and according to equations A2.0-1 and A2.0-3 of Appendix II, the functions

$$\frac{1}{4\pi}\, d\delta_1^2 \quad \text{and} \quad \frac{1}{4\pi}\, c\delta_2^2$$

are good approximations to $V(\delta_1, d\delta_1)$ and $V(\delta_2, c\delta_2)$ with relative error less than 0.01. Thus the output mean value is approximately

$$\mu_4 = \frac{2T}{\pi}\sqrt{N} + \frac{T}{\pi}\, d\delta_1^2 + \frac{T}{\pi}\, c\delta_2^2$$

$$= \frac{T}{\pi}\left\{[2 - (\delta_1^2 + \delta_2^2)]\sqrt{N} + 2\sqrt{1+N}\;\delta_1\delta_2\right\} \quad .$$

Since δ_1 and δ_2 are less than 0.1 and $N \leq 0.1$, this is approximately

$$\mu_4 = \mu_1 + \frac{2T}{\pi}\,\delta_1\delta_2 = \frac{2T}{\pi}\,[\sqrt{N} + \delta_1\delta_2] \quad . \tag{5.2-1}$$

According to equation 5.2-1, the effect of bias in the input and reference polarity indicators when N is small is to add a spurious D.C. component, $\frac{2T}{\pi}\,\delta_1\delta_2$, to the detector output. At first thought, it might be supposed that this spurious component could be determined by a measurement and then subtracted from the output for subsequent detection attempts. However, this may not be possible. First of all, the output of the detector is a random variable. Therefore, in order to determine the magnitude of the bias effect it would be necessary to average the results of repeated measurements of the detector output, the number of repetitions being large enough to reduce the variance of the averaged output to a value small compared with the averaged output. But the physical sources of the biases (transistor junction leakage current, contact potentials, supply voltage variations, etc.) may not be time stationary. They may change with temperature, operating age of the device or other factors. Thus the biases in the polarity indicators may drift in a random fashion, and as a consequence the D.C. component of the detector output may drift in a random fashion. If the value of the D.C. component changes significantly during the averaging period, the average will not be a good estimate and subtracting it from the detector output will not reduce the spurious D.C. component to zero.

Of course, if the biases are stationary in time, then it is possible to measure the spurious D.C. component in the output and then remove it from subsequent observations of the output. However, in such a case it is more effective to remove the biases at the polarity indicators* because removing them at the indicators will eliminate the term in the variance due to bias as well as the spurious D.C. component.

---

* If the bias in a polarity indicator is time invariant, it is always possible to measure it and then remove it by the simple technique of adding an equal D.C. component to the indicator input.

In order to treat the effects of the spurious D.C. component on the detector when the biases are not time stationary requires a knowledge of the time varying characteristics of the biases - usually statistical in nature. Instead of entering into such a complicated investigation here, it will be assumed that the maximum values of the biases are sufficiently small that the spurious D.C. component is negligible with respect to the D.C. component in the output due to the signal. Such a condition is a reasonable one to impose, because even if the statistics of the spurious term are known, in order to use the system effectively as a detector the D.C. component in the output due to the signal must be large compared with any fluctuating terms.

According to this assumption, then, $\delta_1 \delta_2 \ll \sqrt{N}$. It should be noted that this condition rather than the noise level may set the lower bound on the detectable signal level.

From equation 5.2-1 it is clear that if one of the biases is zero,* then there is no effect on the output mean from the other bias.

Whether one of the biases is zero or not, because of the assumption made above, the effect of the biases on the mean is negligible and

$$\mu_4 = \mu_1 \cong \frac{2T}{\pi} \sqrt{N} \qquad , \qquad (5.2\text{-}2)$$

---

* For a discussion of the type of system in which δ_2 is zero, see page 22.

with a relative error in the approximation less than 0.033, if δ_1 or δ_2 are zero or if $\delta_1\delta_2 \ll \sqrt{N} < 0.3$ $(N < 0.1)$.

The variance for the biased polarity-coincidence detector when $N = 0$ is given by equation 4.2-9. It is

$$\sigma_z^2 = T\cdot\delta_1^2\cdot J_2 + T\cdot\delta_2^2\cdot J_3 + T\cdot J_4 \quad , \tag{5.2-3}$$

where

$$J_2 = \frac{8}{\pi^2}\tau_s \left\{ \begin{array}{l} \frac{1}{2}\sqrt{\frac{\pi}{2}}\left[\frac{\mathrm{Erf}(\sqrt{\gamma})}{\sqrt{\gamma}} + \frac{1}{10}\ \frac{\mathrm{Erf}(\sqrt{5\gamma})}{\sqrt{5\gamma}}\right] \\ + (1 - \sqrt{\frac{\pi}{2}})(e^{-\gamma} + \frac{1}{10}\ e^{-5\gamma}) \\ - \frac{0.85}{e}\ (\frac{\gamma}{1+\gamma}\ e^{-\gamma} + \frac{1}{10}\left[\frac{5\gamma}{1+5\gamma}\right]e^{-5\gamma}) \end{array} \right\}$$

with $\gamma = \tau_n/\tau_s$,

$J_3 = J_2$ with τ_n substituted for τ_s and λ substituted for γ, where $\lambda = \tau_s/\tau_n = 1/\gamma$,

and

$$J_4 = \frac{8}{\pi^2}\tau_n \left\{(1 + \frac{1}{20})\ \frac{1}{1+\gamma} + \frac{1}{2}\left[\frac{1}{1+5\gamma} + \frac{1}{\gamma+5}\right]\right\}$$

or equivalently, the same expression with τ_s substituted for τ_n and λ substituted for γ.

Although equation 5.2-3 is valid only when $N = 0$, the value of σ_z^2 for small N will not differ significantly from it. Therefore, equation 5.2-3 will be used also for N small.

The ratio J_2/J_4 (and also J_3/J_4) is plotted in figure 4.2-8. Since δ_1 and δ_2 are less than 0.1, it can be seen from this curve that for values of $\gamma \geq 1$ the term in σ_z^2 with J_2 as a factor is negligible compared with the term with J_4 as a factor. Likewise, for values of $\lambda \geq 1$ the term in σ_z^2 with J_3 as a factor is negligible compared with the term with J_4 as a factor. Therefore:

For $\gamma \geq 1$

$$\sigma_z^2 = T \cdot \delta_2^2 \cdot J_3 + T \cdot J_4$$

and for $\lambda \geq 1$

$$\sigma_z^2 = T \cdot \delta_1^2 \cdot J_2 + T \cdot J_4 \quad .$$

Since $\gamma \geq 1$ corresponds to $\lambda \leq 1$ and since $\lambda \geq 1$ corresponds to $\gamma \leq 1$, these two expressions suffice for the whole range from $-\infty$ to $+\infty$ for both γ and λ. Moreover, the second expression can be obtained from the first by the substitution of δ_1 for δ_2, γ for λ, τ_s for τ_n and τ_n for τ_s. Therefore, an examination of

$$\sigma_z^2 = T \cdot \delta_2^2 \cdot J_3 + T \cdot J_4 \qquad , \tag{5.2-4}$$

valid for $\gamma \geq 1$, reveals the behavior of σ_z^2 for all values of γ and λ.

The relative error in J_3 and J_4 both is less than 0.155. Therefore, the relative error in σ_z^2 as expressed by equation 5.2-4 is less than 0.155.

According to equation 5.2-4 and the remarks preceding it, the effect of bias in the input and reference polarity indicators is to add a spurious term to the output variance.

Substituting equations 5.2-2 and 5.2-4 in equation 5.0-19 yields for the output signal-to-noise power ratio

$$N_o = \frac{\frac{1}{2}N \cdot (\frac{T}{\tau_s})}{\delta_2^2 K_3(\lambda)/\lambda + K_4(\lambda)} \tag{5.2-5}$$

for $\gamma \geq 1$ (or $\lambda \leq 1$), where $K_3(\lambda) = J_3(\lambda)/\frac{8}{\pi^2}\tau_n$ is plotted in figure 4.2-6 and $K_4(\lambda) = J_4(\lambda)/\frac{8}{\pi^2}\tau_s$ is plotted in figure 4.2-7. When $\gamma \leq 1$, the same expression applies, but with δ_1 substituted for δ_2, γ substituted for λ, τ_n substituted for τ_s and τ_s substituted for τ_n. Thus,

$$N_o = \frac{\frac{1}{2}N \cdot (\frac{T}{\tau_s})/\gamma}{\delta_1^2 K_2(\gamma)/\gamma + K_4(\gamma)} \tag{5.2-6}$$

for $\gamma \leq 1$ (or $\lambda \geq 1$), where $K_2(\gamma) = J_2(\gamma)/\frac{8}{\pi^2}\tau_s$ is also plotted in figure 4.2-6 and $K_4(\gamma) = J_4(\gamma)/\frac{8}{\pi^2}\,\tau_n$ is also plotted in figure 4.2-7.

Since the relative error in the mean as expressed by equation 5.2-2 is less than 0.033 and since the relative error in the variance as expressed by equation 5.2-4 is less than 0.155, then the relative error in N_o as expressed by equation 5.2-5 or 5.2-6 is less than 0.221. (Note that the mean appears squared in equations 5.2-5 and 5.2-6. Thus the contribution to the relative error due to approximating the mean is doubled.)

Equations 5.2-5 and 5.2-6 show that the output signal-to-noise power ratio for the biased polarity-coincidence detector is proportional to the input signal-to-noise power ratio, just as it is for the ideal correlation detector. In fact, the biased polarity-coincidence detector has an output signal-to-noise power ratio identical to that of the ideal correlation detector except for a degradation factor.

$$N_{o(BPCD)} = \Delta N_{o(CD)} \qquad , \qquad (5.2\text{-}7)$$

where Δ is the degradation factor. The discussion of the effects of bias on the output signal-to-noise power ratio can be more conveniently presented in terms of the reciprocal degradation factor, $\Gamma = 1/\Delta$. Then,

$$N_{o(BPCD)} = \frac{1}{\Gamma_{(BPCD)}} N_{o(CD)} \qquad , \qquad (5.2\text{-}8)$$

where $\Gamma_{(BPCD)}$ is given by

$$\Gamma_{(BPCD)} = \begin{cases} (1+\lambda)\left[K_4(\lambda) + \delta_2^2 K_3(\lambda)/\lambda\right] & \text{for } \gamma \geq 1 \\ (1+\gamma)\left[K_4(\gamma) + \delta_1^2 K_2(\gamma)/\gamma\right] & \text{for } \gamma \leq 1 \end{cases} \quad . \tag{5.2-9}$$

The first term on the right hand side of the above equation, $(1+\lambda)K_4(\lambda)$ or $(1+\gamma)K_4(\gamma)$, is the reciprocal degradation factor for the ideal polarity-coincidence detector discussed in section 5.1.

The maximum value of the ratio $K_4(\lambda)/K_3(\lambda)$ for all $\gamma \geq 1$ (i.e. for all $\lambda \leq 1$) is smaller than 1.50 and the minimum value is larger than 1.15. Therefore an approximate idea of the behavior of Γ can be obtained by setting $K_3(\lambda) = K_4(\lambda)/1.5$ (or $K_2(\gamma) = K_4(\gamma)/1.5$ in the second form). Thus, approximately

$$\Gamma_{(BPCD)} = \begin{cases} (1+\lambda)K_4(\lambda)\left[1 + \delta_2^2/1.5\lambda\right] & \text{for } \gamma \geq 1 \\ (1+\gamma)K_4(\gamma)\left[1 + \delta_1^2/1.5\gamma\right] & \text{for } \gamma \leq 1 \end{cases}$$

$$= \Gamma_{(PCD)} \text{ X } \begin{cases} 1 + \delta_2^2/1.5\lambda & \text{for } \gamma \geq 1 \\ 1 + \delta_1^2/1.5\gamma & \text{for } \gamma \leq 1 \end{cases} \quad , \tag{5.2-10}$$

where $\Gamma_{(PCD)}$ is the reciprocal degradation factor for the PCD, where PCD denotes polarity-coincidence detector and BPCD denotes biased polarity-coincidence detector. The factor 1.5 which appears in the denominator of the second term of the various forms of equation 5.2-10 is the asymptotic ratio $K_4(\gamma)/K_2(\gamma)$ as $\gamma \to 0$. This particular number was chosen because it gives to equation 5.2-10 the asymptotically correct form when $\gamma \to 0$ and when $\gamma \to \infty$. The error of approximation for equation 5.2-10 is

$$e = \Gamma_{(PCD)} \frac{\delta^2}{\nu} \left[\frac{1}{1.5} - \nu\, J/J_4\right]$$

where $\nu = \lambda$, $\delta = \delta_2$, $J = J_3(\lambda)$ and J_4 and K_4 are functions of λ if $\gamma \geq 1$, and where $\nu = \gamma$, $\delta = \delta_1$, $J = J_2(\gamma)$ and J_4 and K_4 are functions of γ if $\gamma \leq 1$. The relative error is

$$\rho = \delta^2 \left[\frac{1-1.5\nu J/J_4}{1.5\nu+\delta^2}\right] .$$

Since δ_1 and δ_2 both are less than 0.1,

$$\rho \leq 0.01 \left[\frac{1-1.5\nu J/J_4}{1.5\nu}\right] .$$

An examination of figure 4.2-8 which presents a plot of J/J_4 as a function of λ or γ indicates that the relative error is less than 0.003. Thus equation 5.2-10 is an excellent approximation to equation 5.2-9.

Equation 5.2-10 indicates that when $1 \leq \gamma \leq 0.15/\delta_2^2$ or when $\delta_1^2/0.15 \leq \gamma \leq 1$, the relative error from neglecting the spurious term is less than 0.1. Thus,

$$\Gamma_{(BPCD)} = \Gamma_{(PCD)} \quad \text{for} \quad \delta_1^2/0.15 \le \gamma \le 0.15/\delta_2^2 \quad , \tag{5.2-11}$$

with a relative error less than 0.1.

The conditions on δ_1 and δ_2 such that the degradation in performance of the BPCD is not significantly worse than for a PCD can now be specified. They are:

For $\lambda \ge 1$ $(\gamma \le 1)$

δ_1^2 must be less than $0.15/\lambda$ if the degradation in performance of the BPCD is to be no more than 10% relative to the PCD (i.e. in order that $0.9N_{o(PCD)} \le N_{o(BPCD)} \le N_{o(PCD)}$). This case corresponds to the case where the noise band-width is greater than the signal band-width and is the case commonly encountered in practice. In this case, bias in the reference channel is not significant but bias in the input channel, if sufficiently large, can cause a degradation in the performance of the detector.

For $\gamma \ge 1 (\lambda \le 1)$

δ_2^2 must be less than $0.15/\gamma$ if the degradation in performance of the BPCD is to be no more than 10% relative to the PCD (i.e. in order that $0.9N_{o(PCD)} \le N_{o(BPCD)} \le N_{o(PCD)}$). This case corresponds to the case where the noise band-width is smaller than the signal band-width and is not commonly encountered in practice. In this case, bias in the input channel, if sufficiently large, can cause a degradation in the performance of the detector.

Thus, the degradation in performance of a biased polarity-coincidence detector relative to an ideal polarity-coincidence detector is negligible when $\delta_1^2 \le 0.15/\lambda$ for the $\lambda \ge 1$ case and when $\delta_2^2 \le 0.15/\gamma$ for the $\gamma \ge 1$ case, and has risen to approximately 3 db. when $\delta_1^2 = 1.5/\lambda$ for the $\lambda \ge 1$ case and when $\delta_2^2 = 1.5/\gamma$ for the $\gamma \ge 1$ case.

Equations 5.2-5 and 5.2-6 are subject to the restriction $|\delta_1\delta_2| \ll \sqrt{N}$, and equations 5.2-7 through 5.2-11 are subject to the restrictions $|\delta_1\delta_2| \ll \sqrt{N}$ and $N \ll \gamma/(1+\gamma)$. If these restrictions are not met, the output signal-to-noise ratio for the BPCD can be obtained from the unrestricted* equations for the mean and variance (equations 3.2-12 and 4.2-9) and the result for the BPCD can be compared with the unrestricted result for the CD and the PCD.

A physical explanation for the relation between the BPCD variance and the $\gamma = \tau_n/\tau_s$ ratio follows: Consider the case where $\gamma \gg 1$. Then the noise band-width is much smaller than the signal band-width and a section of the noise sample function and signal sample function might have outputs from the polarity indicators as illustrated in figures 5.2-1 and 5.2-2. In figure 5.2-1, the input channel has no bias but the reference channel has negative bias. Thus, the reference channel polarity indicator output will be positive more often than negative. Therefore, during a positive cycle of the noise, the integral of the product of the polarity indicator outputs will grow from zero as indicated. When the noise goes through a negative cycle, the integral starts to decay back to zero. Such

---

* Note that in the unrestricted equations, it is still assumed that $N \ll 1$.

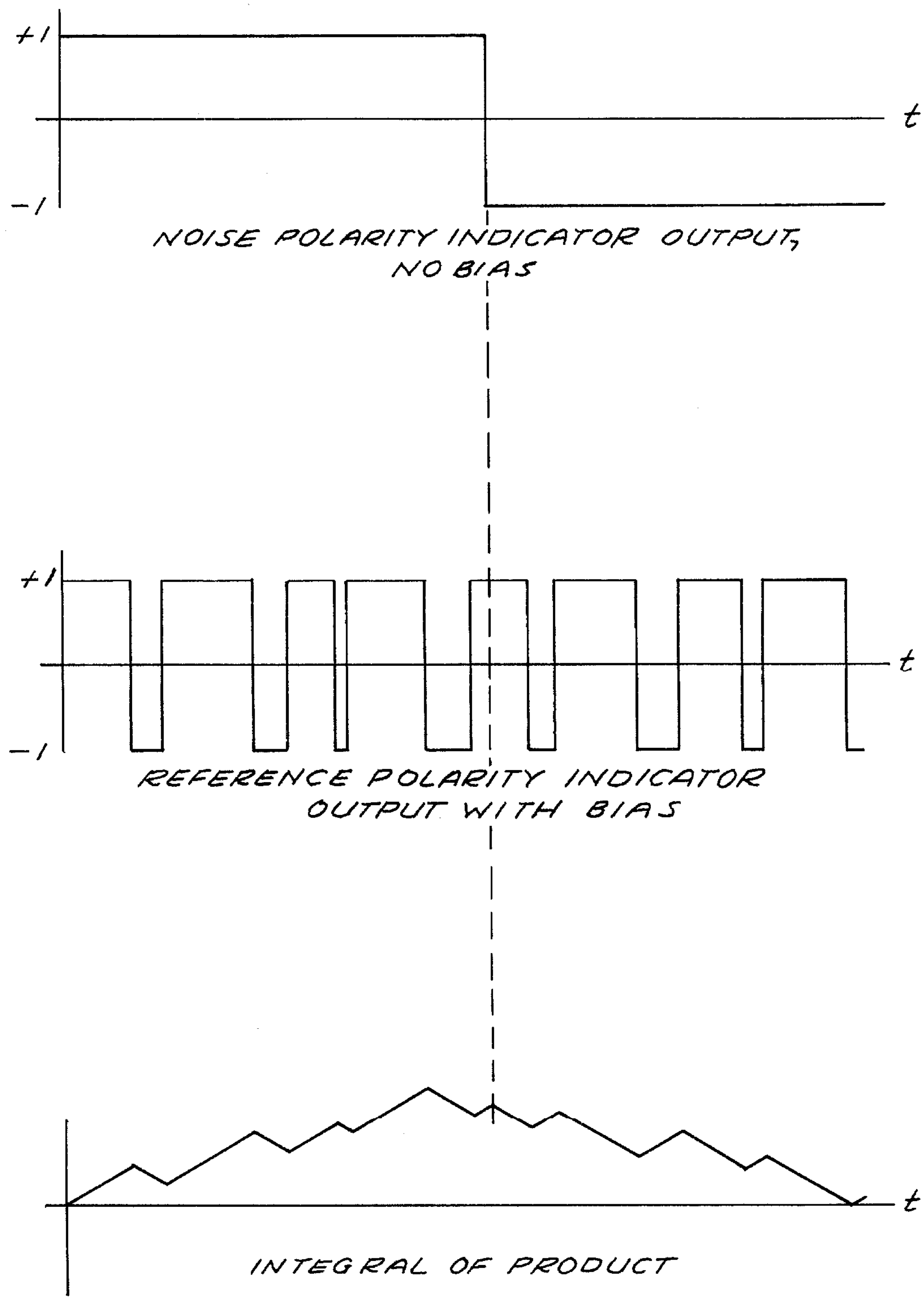

+1
-1
t
NOISE POLARITY INDICATOR OUTPUT,
NO BIAS
+1
-1
t
REFERENCE POLARITY INDICATOR
OUTPUT WITH BIAS
t
INTEGRAL OF PRODUCT

Figure 5.2-1

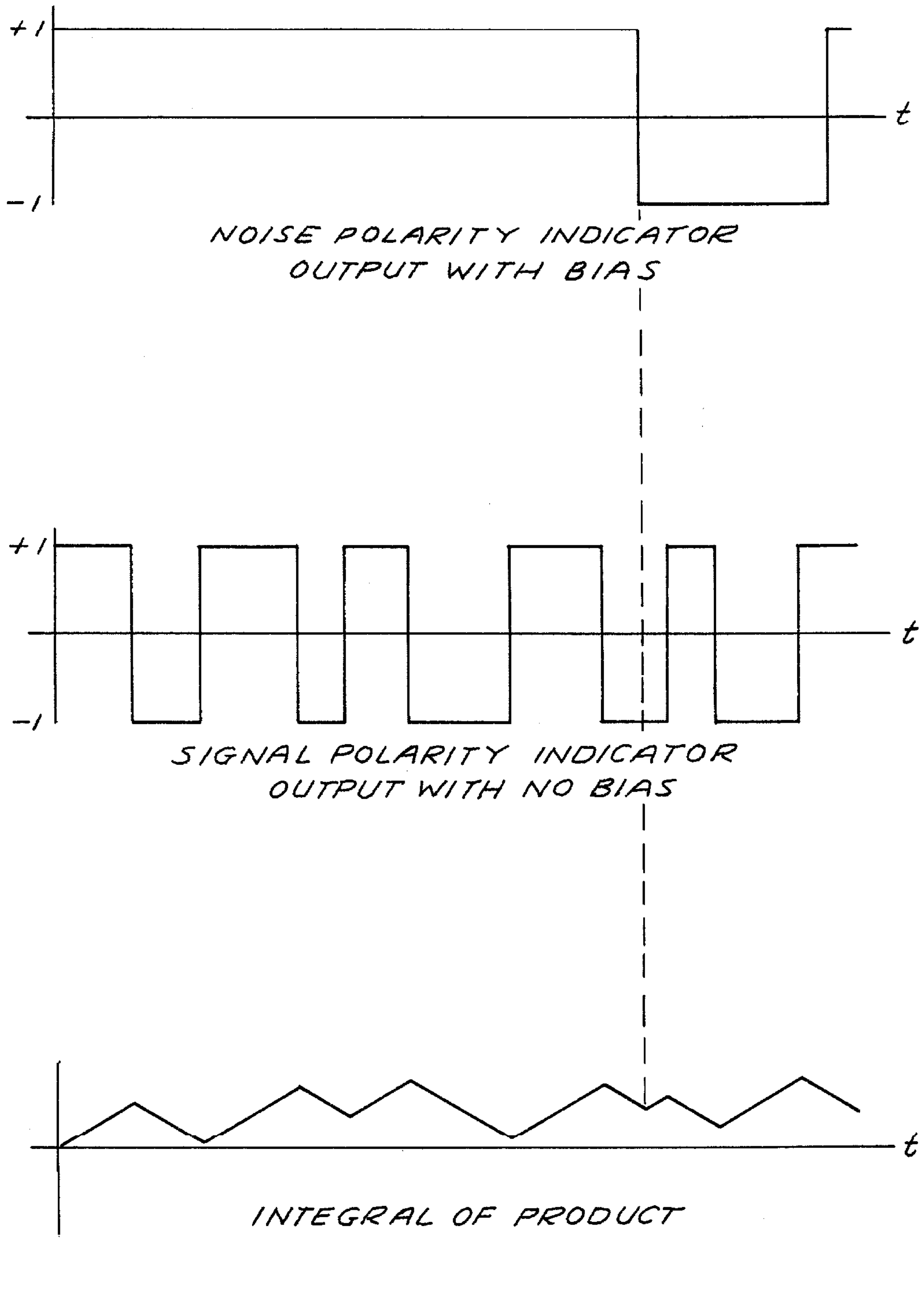

+1
-1
t
NOISE POLARITY INDICATOR
OUTPUT WITH BIAS
+1
-1
t
SIGNAL POLARITY INDICATOR
OUTPUT WITH NO BIAS
t
INTEGRAL OF PRODUCT

Figure 5.2-2

oscillations will continue through each positive and negative cycle of the noise until the end of the integration period. Since the termination of integration at the end of the period T is independent of the times at which the noise changes sign, the integrator output can have any value between the limits shown. The result is a large variance in the output.

On the other hand, when the input channel has negative bias but the reference channel does not have bias, the input channel polarity indicator output will be positive for longer periods of time than negative. However, since the output of the reference channel polarity indicator is now positive and negative for about equal periods of time, the integral of the product of the polarity indicator outputs does not grow and decay as it did for the case discussed above. Therefore, the output has a smaller variance, as shown by figure 5.2-2.

When $\lambda \gg 1$, the signal band-width is much smaller than the noise band-width. Then a section of the noise sample function and signal sample function might have outputs from the polarity indicators as illustrated in figures 5.2-3 and 5.2-4. In figure 5.2-3, the input channel has no bias but the reference channel has negative bias. Thus, the reference channel polarity indicator output will be positive for longer periods of time than negative. However, since the output of the input channel polarity indicator is positive and negative for about equal periods of time, the integral of the product of the polarity indicator outputs does not grow and decay, and the variance is small.

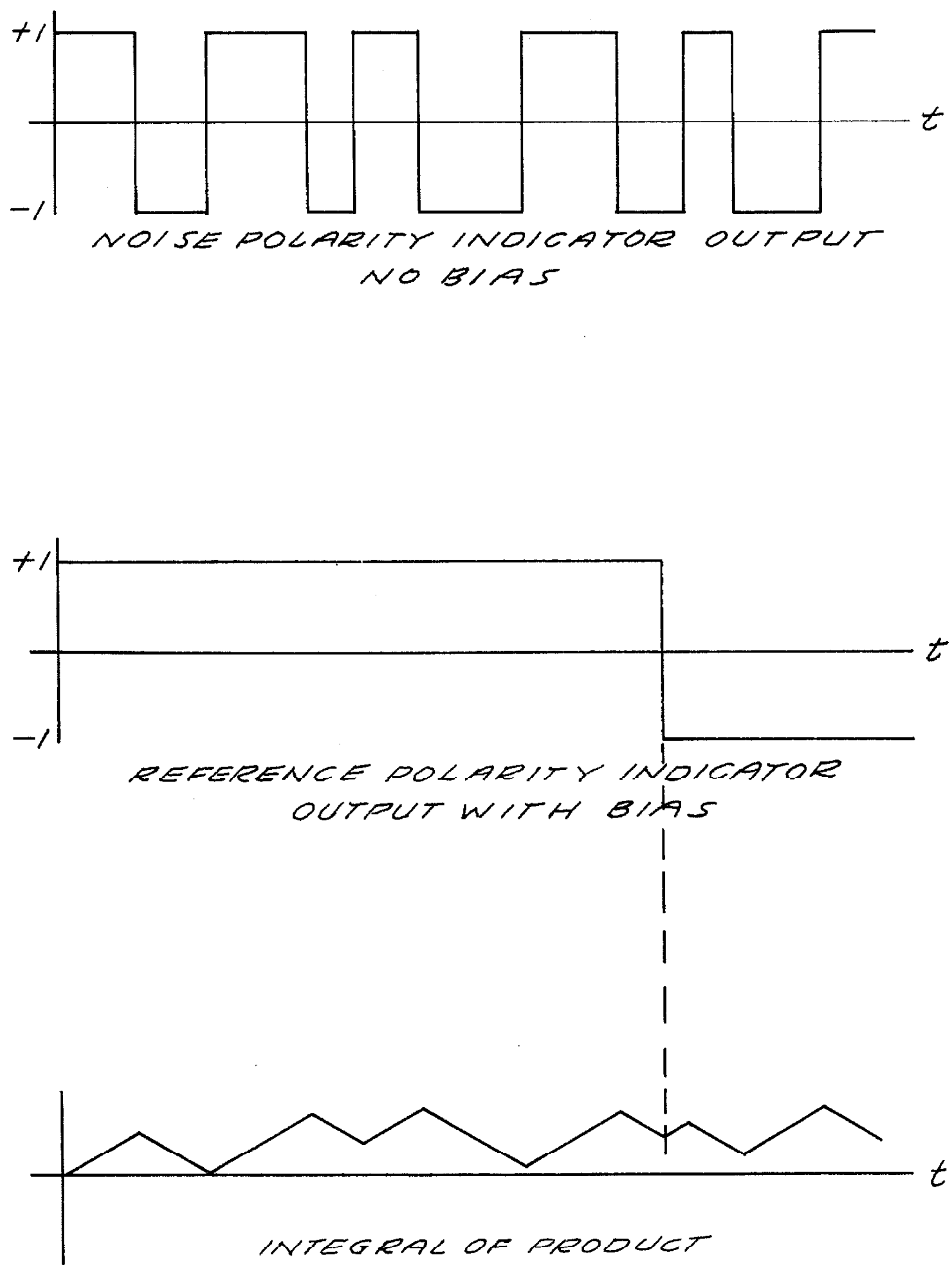

+1
−1
t
NOISE POLARITY INDICATOR OUTPUT
NO BIAS
+1
−1
t
REFERENCE POLARITY INDICATOR
OUTPUT WITH BIAS
t
INTEGRAL OF PRODUCT

Figure 5.2-3

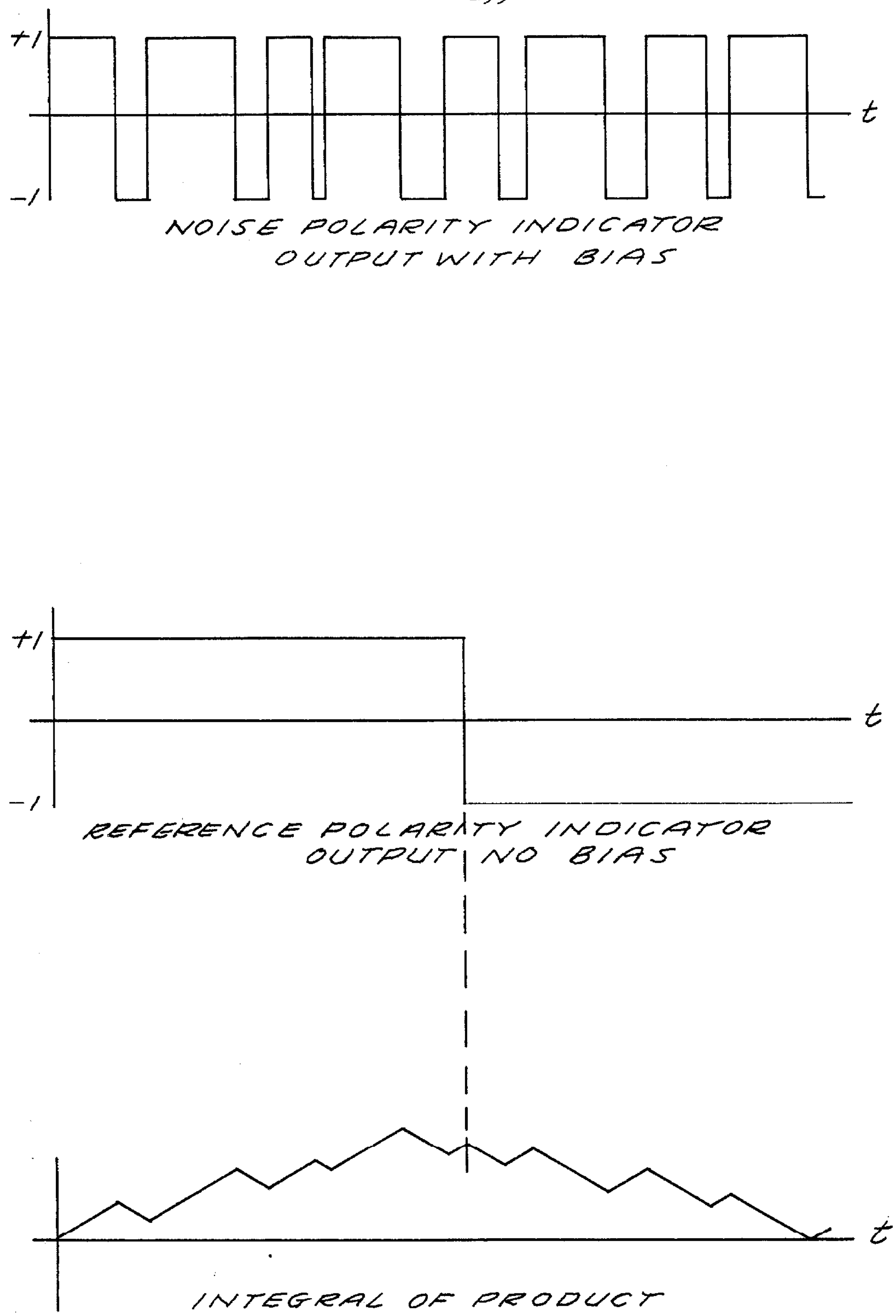

+1
-1
t
NOISE POLARITY INDICATOR
OUTPUT WITH BIAS
+1
-1
t
REFERENCE POLARITY INDICATOR
OUTPUT NO BIAS
t
INTEGRAL OF PRODUCT

Figure 5.2-4

On the other hand, when the input channel has negative bias but the reference channel has no bias, the input channel polarity indicator output will be positive more often than negative. Therefore, during a positive cycle of the reference signal, the integral of the product of the polarity indicator outputs will grow from zero as indicated. When the reference signal goes through a negative cycle, the integral starts to decay back to zero. Such oscillations will continue through each positive and negative cycle of the reference signal until the end of the integration period. Since the termination of integration is independent of the times at which the reference signal changes sign, the integrator output can have any value between the limits shown. The result is again a large variance in the output.

The results of sections 5.0, 5.1 and 5.2, as expressed by equations 5.0-20, 5.1-4 and 5.2-10, indicate that for a fixed input signal-to-noise power ratio, in the case of the three detectors examined and subject to the assumptions and restrictions introduced in deriving the above equations, the output signal-to-noise power ratio is largest for an ideal correlation detector, next largest for an ideal polarity-coincidence detector, and smallest for a biased polarity-coincidence detector. This result is not unexpected. Thus, if output signal-to-noise ratio is used as a criterion for the quality of a detector, then the ordering of the detectors according to decreasing quality is CD, PCD and BPCD.

It should be noted, however, that even though the BPCD is inferior to the PCD as a detector, and the PCD is inferior to the CD as a detector, the degradation in performance is not very serious.

For a PCD, the degradation relative to the CD is between 1.4 db. and 2.2 db., depending on the input signal-to-noise band-width ratio. For a BPCD, the degradation relative to the PCD is negligible, so long as the biases are not too large. (See the conditions on δ_1 and δ_2 for negligible degradation as discussed on page 193.)

SUMMARY

Section 5.0 The Correlation Detector.

The output signal-to-noise power ratio is

$$N_o = \frac{1}{2}N \frac{T}{\tau_s} (1+\lambda) \qquad (5.0\text{-}20)$$

if the signal is non-random or if $N \ll \frac{\gamma}{1+\gamma}$. $\lambda = \tau_s/\tau_n$ and $\gamma = 1/\lambda = \tau_n/\tau_s$.

Section 5.1 The Ideal Polarity-Coincidence Detector.

The output signal-to-noise power ratio is

$$N_o = \frac{1}{2}N \frac{T}{\tau_s} \frac{1}{K_4(\lambda)} \qquad (5.1\text{-}3)$$

if $N \ll 1$. The properties of the function $K_4(\lambda)$ are discussed in Chapter IV.

N_o for a PCD has the same form as for a CD except for a degradation factor.

$$N_{o(PCD)} = \Delta \cdot N_{o(CD)} \qquad (5.1\text{-}4)$$

where Δ is the degradation factor.

$$\Delta = 1/(1+\lambda)K_4(\lambda) = 1/(1+\gamma)K_4(\gamma) \qquad (5.1\text{-}5)$$

The degradation is smallest when $\lambda = \gamma = 1$ (i.e. when the signal-to-noise band-width ratio is unity). For this case, $\Delta = 0.723$ - a degradation of 1.4 db. The degradation is largest when λ (or γ) goes to zero or to infinity (i.e. when the signal-to-noise band-width ratio goes to zero or to infinity). In this case, Δ goes to 0.606 - a degradation of 2.2 db.

Section 5.2 The Biased Polarity-Coincidence Detector.

The output signal-to-noise power ratio is:

For $\gamma \geq 1$

$$N_o = \frac{\frac{1}{2}N(\frac{T}{\tau_s})}{\delta_2^2 K_3(\lambda)/\lambda + K_4(\lambda)} \qquad (5.2\text{-}5)$$

or For $\lambda \geq 1$

$$N_o = \frac{\frac{1}{2}N(\frac{T}{\tau_s})/\gamma}{\delta_1^2 K_2(\gamma)/\gamma + K_4(\gamma)} \quad , \qquad (5.2\text{-}6)$$

if $N \ll 1$ and $\delta_1 \delta_2 \ll \sqrt{N}$.

N_o for a BPCD has the same form as for a CD except for a degradation factor.

$$N_{o(BPCD)} = \frac{1}{\Gamma_{(BPCD)}} N_{o(CD)} \qquad (5.2\text{-}8)$$

where $\Gamma_{(BPCD)}$ is the reciprocal degradation factor.

$$\Gamma_{(BPCD)} \cong \Gamma_{(PCD)} \times \begin{cases} 1 + \delta_2^2/1.5\,\lambda & \text{for} \quad \gamma \geq 1 \\ 1 + \delta_1^2/1.5\,\gamma & \text{for} \quad \lambda \geq 1 \end{cases} \qquad (5.2\text{-}10)$$

where $\Gamma_{(PCD)}$ is the reciprocal degradation factor for a PCD.

REFERENCES

1. Faran, J. J. and R. Hills: "Correlators for Signal Reception," Technical Memorandum No. 27, Acoustics Research Lab., Harvard Univ., pp. 58-65; September, 1952.

2. Davenport, W. B.: "Signal-to-Noise Ratios in Band-Pass Limiters," Jour. of Applied Physics, vol. 24. no. 6, pp. 720-727; 1953.

CHAPTER VI

SUMMARY, CONCLUSIONS AND GENERALIZATIONS

6.0 Summary

Bias in the polarity indicators of a polarity-coincidence detector introduces spurious components into the mean and variance of the detector's output. These spurious components exist both when a signal is present and when a signal is absent.

The Mean Value

Expressions for the mean value of the output of the biased polarity-coincidence detector are presented in equations 3.2-12 and 3.2-13 for the case that the signal and noise are sample functions from stationary Gaussian random processes with zero means, the processes being statistically independent to the first order. These equations are reproduced here.

$$\mu_z = \mu_1 + e \quad , \qquad (3.2\text{-}12)$$

where μ_z is the detector output mean value, μ_1 is the mean value when neither the input nor the reference channel has bias in its polarity indicator, and e is the spurious component. μ_1 is given by equation 3.2-4.

$$\mu_1 = \frac{2T}{\pi} \tan^{-1}(\sqrt{N}) \quad , \qquad (3.2\text{-}4)$$

where T is the integration period for the detector and $N = \sigma_s^2/\sigma_n^2$ is the input signal-to-noise power ratio. The spurious component, e, is given by equation 3.2-13.

$$e = 4T \cdot V(\delta_1, d\delta_1) + 4T \cdot V(\delta_2, c\delta_2) \qquad (3.2\text{-}13)$$

where δ_1 and δ_2 are the biases in the input channel and the reference channel respectively (normalized by the input r.m.s. value and the reference r.m.s. value respectively), $c = \sqrt{1+N}\,(\delta_1/\delta_2) - \sqrt{N}$, $d = \sqrt{1+N}\,(\delta_2/\delta_1) - \sqrt{N}$, and V is an integral of the bivariate standard normal density function with zero correlation over a particular triangular region. The V function is discussed extensively in section 3.2 and in Appendixes I and II.

The Variance

Approximate expressions for the variance of the output of a biased polarity-coincidence detector for the particular case that $N = 0$ (the signal absent case) are presented in equations 4.2-9 and 4.2-10. These equations are reproduced below. They are subject to the following assumptions and restrictions: The signal and noise are sample functions from stationary Gaussian random processes with zero means, the processes being statistically independent to the second order. Both the signal and noise have RC low-pass spectra with normalized correlation functions $r(\tau) = e^{-|\tau|/\tau_s}$ for the signal and $\rho(\tau) = e^{-|\tau|/\tau_n}$ for the noise. The correlation duration constants, τ_s and τ_n, are both much less than T. The normalized biases, δ_1 and δ_2 are both much less than 1.

$$\sigma_z^2 = T\cdot\delta_1^2\cdot J_2 + T\cdot\delta_2^2\cdot J_3 + T\cdot J_4 \quad , \tag{4.2-9}$$

where J_2, J_3 and J_4 are constants depending only on τ_s and τ_n, given by

$$J_2 = \frac{8}{\pi^2}\tau_s \left\{ \begin{array}{l} \frac{1}{2}\sqrt{\frac{\pi}{2}}\left[\frac{\mathrm{Erf}(\sqrt{\gamma})}{\sqrt{\gamma}} + \frac{1}{10}\frac{\mathrm{Erf}(\sqrt{5\gamma})}{\sqrt{5\gamma}}\right] \\ + (1-\sqrt{\frac{\pi}{2}})(e^{-\gamma} + \frac{1}{10}e^{-5\gamma}) \\ - \frac{0.85}{e}(\frac{\gamma}{1+\gamma}e^{-\gamma} + \frac{5\gamma}{1+5\gamma}e^{-5\gamma}) \end{array} \right\} \tag{4.2-10}$$

where $\gamma = \tau_n/\tau_s$,

J_3 is identical to J_2 with γ replaced by $\lambda = 1/\gamma = \tau_s/\tau_n$ and with τ_s replaced by τ_n,

and

$$J_4 = \frac{8}{\pi^2}\tau_n \left\{(1 + \frac{1}{20})\frac{1}{1+\gamma} + \frac{1}{2}\left[\frac{1}{1+5\gamma} + \frac{1}{\gamma+5}\right]\right\}$$

or equivalently,

J_4 is given by the same expression with γ replaced by λ and with τ_n replaced by τ_s.

The first and second terms on the right hand side of equation 4.2-7 above are spurious components due to the bias, and the third term represents the variance for an ideal (no bias) polarity-coincidence detector. The biases affect the variance independently and the effect is independent of the signs of the biases.

Equation 4.2-7 is an approximate expression for the variance. If δ_1 and δ_2 are both less than 0.1 and if τ_s and τ_n are both less than 0.01T, then the relative error in each of the terms of equation 4.2-9 is less than 0.155 and the relative error in the entire expression is less than 0.155.

The quantities $K_2 = J_2/\frac{8}{\pi^2}\tau_s$, $K_3 = J_3/\frac{8}{\pi^2}\tau_n$ and $K_4 = J_4/\frac{8}{\pi^2}\tau_n$ (or in the second form, $K_4 = J_4/\frac{8}{\pi^2}\tau_s$) are plotted in figures 4.2-6 and 4.2-7.

Ideal Correlation Detector

For purposes of comparison, the output mean and variance for an ideal correlation detector (CD) are presented by equations 5.0-4 and 5.0-17 for the same assumptions and restrictions as were imposed on the expressions for the mean and variance of a biased polarity-coincidence detector (BPCD) in the above paragraphs. Thus, the output signal-to-noise power ratio, which is the square of the output mean divided by the output variance, is obtained directly from these two equations, which are reproduced below.

$$\mu_z = T \cdot \sigma_o \sigma_s \qquad (5.0\text{-}4)$$

and

$$\sigma_z^2 = 2\, \sigma_o^2 \sigma_n^2 T\, \tau_s \left[N + \frac{\gamma}{1+\gamma} \right] \quad . \qquad (5.0\text{-}17)$$

Equation 5.0-17 is approximate. The relative error in the approximation is less than 0.01 if τ_s and τ_n both are less than 0.01T.

If an additional restriction is imposed - namely that either N is small with respect to $\gamma/(1+\gamma)$, or that the signal is non-random, then the expression for the output signal-to-noise power ratio is considerably simplified. It becomes

$$N_{o(CD)} = \frac{1}{2}N \frac{T}{\tau_s} (1+\gamma)/\gamma$$

$$= \frac{1}{2}N \frac{T}{\tau_s} (1+\lambda) \quad . \qquad (5.0\text{-}20)$$

Output Signal-to-Noise Ratio for an Ideal Polarity-Coincidence Detector

The output signal-to-noise power ratio for an ideal polarity-coincidence detector (PCD) (no bias) is presented by equation 5.1-3 for the same assumptions and restrictions as were introduced in the paragraphs above dealing with the mean and variance of a BPCD, with one additional assumption. It is assumed that the input signal-to-noise power ratio, N, is small with respect to 1 and that consequently the variance is unaffected by the presence of the signal so that it equals the variance when $N = 0$.

$$N_{o(PCD)} = \frac{1}{2}N \frac{T}{\tau_s} / K_4(\lambda) \quad . \tag{5.1-3}$$

Equation 5.1-3 is approximate. The relative error, which is complicated, is discussed thoroughly in section 5.1.

The output signal-to-noise power ratio for a PCD is proportional to N if $N \ll 1$, just as it is for a CD if either $N \ll \frac{\gamma}{\gamma+1}$ or the signal is non-random. Thus,

$$N_{o(PCD)} = \Delta_{(PCD)} N_{o(CD)} \quad , \tag{5.1-4}$$

where $\Delta_{(PCD)}$ is the performance degradation factor.

$$\Delta_{(PCD)} = 1/(1+\lambda)K_4(\lambda) \tag{5.1-5}$$

or

$$\Delta_{(PCD)} = 1/(1+\gamma)K_4(\gamma) \tag{5.1-6}$$

when $N \ll 1$ and either $N \ll \frac{\gamma}{1+\gamma}$ or the signal is non-random.

A plot of the degradation factor for the PCD in this case is presented in figure 5.1-1. The bounds on the error in the expression for the degradation are also presented in figure 5.1-1.

This result agrees, within the limits of the errors of approximation, with the expression obtained by Faran and Hills (1) for a similar detector in the case when $\gamma = \lambda = 1$.

Output Signal-to-Noise Ratio for a Biased Polarity-Coincidence Detector

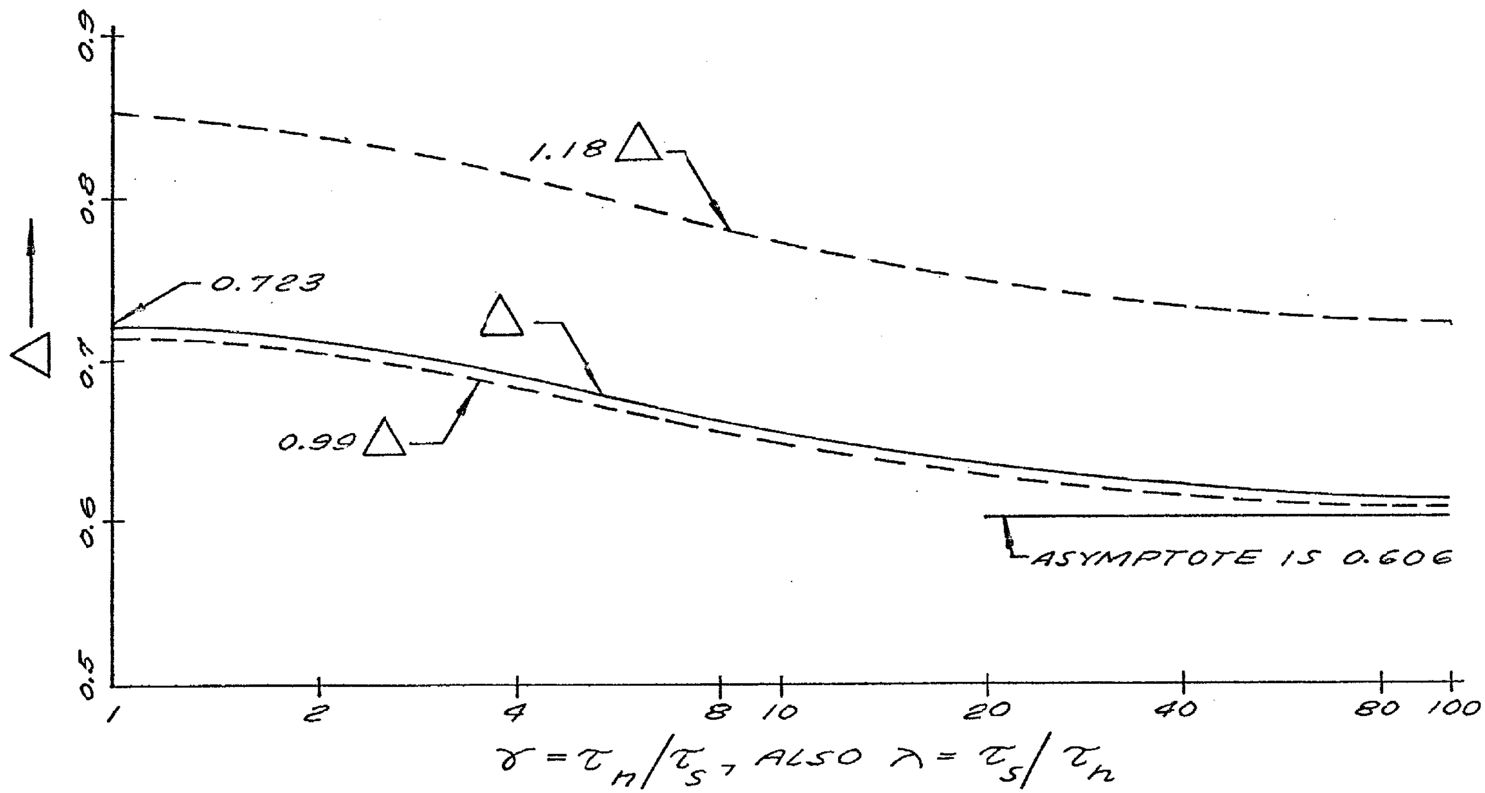
The output signal-to-noise power ratio, which is the square of the output mean divided by the output variance, is obtained for the biased polarity-coincidence detector directly from equations 3.2-12 and 4.2-9 above, and is subject to the assumptions and restrictions introduced in the paragraphs above dealing with the BPCD mean and variance, with the additional assumption that $N \ll 1$ and that consequently the variance is unaffected by the presence of the signal so that it equals the variance when $N = 0$.

If it is assumed in addition that $\delta_1 \delta_2 \ll \sqrt{N}$, then the expression for the output signal-to-noise power ratio is considerably simplified. It becomes:

For $\gamma \geq 1$

$$N_{o(BPCD)} = \frac{\frac{1}{2}N \cdot (\frac{T}{\tau_s})}{\delta_2^2 K_3(\lambda)/\lambda + K_4(\lambda)} \qquad (5.2\text{-}5)$$

and



Degradation Factor, Λ, for a PCD

Figure 5.1-1

For $\lambda \geq 1$

$$N_{o(BPCD)} = \frac{\frac{1}{2}N\cdot(\frac{T}{\tau_s})/\gamma}{\delta_1^2 K_2(\gamma)/\gamma + K_4(\gamma)} \qquad (5.2\text{-}6)$$

These equations are approximate, with a relative error of approximation less than 0.221 when δ_1 and δ_2 are less than 0.1 and τ_s and τ_n are less than 0.01T.

The output signal-to-noise power ratio for a BPCD is proportional to N if $N \ll 1$ and $\delta_1\delta_2 \ll \sqrt{N}$, just as it is for a CD if either $N \ll \gamma/(1+\gamma)$ or the signal is non-random.

Thus,

$$N_{o(BPCD)} = \frac{1}{\Gamma_{(BPCD)}} N_{o(CD)} \quad , \qquad (5.2\text{-}8)$$

where $\Gamma_{(BPCD)} = 1/\Delta_{(BPCD)}$ is the reciprocal degradation factor. It is

$$\Gamma_{(BPCD)} = \Gamma_{(PCD)} \times \begin{cases} 1 + \delta_1^2/1.5\gamma & \text{for} \quad \lambda \geq 1 \\ \\ 1 + \delta_2^2/1.5\lambda & \text{for} \quad \gamma \geq 1 \end{cases} \qquad (5.2\text{-}10)$$

where

$$\Gamma_{(PCD)} = 1/\Delta_{(PCD)}$$

is the reciprocal degradation factor for an ideal polarity-coincidence detector.

Equation 5.2-10 is an approximate equation, with a relative error no larger than 0.1 for any γ or λ, so long as δ_1 and δ_2 are less than 0.003 and $T \geq 100 \max(\tau_s, \tau_n)$.

Bias in the polarity indicators causes a spurious term to be added to the factor $\Gamma_{(BPCD)}$. When $N \ll 1$, $\delta_1 \delta_2 \ll \sqrt{N}$ and $N \ll \frac{\gamma}{1+\gamma}$ or the signal is non-random, then in order that the spurious term in $\Gamma_{(BPCD)}$ be no larger than 10% of $\Gamma_{(PCD)}$, it is necessary and sufficient that:

For $\lambda \geq 1$ $\quad$ δ_1^2 be less than $0.15/\lambda$

For $\gamma \geq 1$ $\quad$ δ_2^2 be less than $0.15/\gamma$.

6.1 Conclusions

Subject to assumptions and restrictions which are stated at the end of this section, the following statements are valid:

Polarity-Coincidence Detector

The output signal-to-noise power ratio of a polarity-coincidence detector is degraded from that of an ideal correlation detector by about 1.4 db. when the input signal-to-noise band-width ratio is unity. The degradation increases to about 2.2 db. when the input signal-to-noise band-width ratio becomes either very small or very large.

Thus, ideal limiting of the input stimulus and reference signal before correlating does not seriously degrade the output signal-to-noise ratio of a correlation detector.

Biased Polarity-Coincidence Detector

The presence of bias in the polarity indicators of a polarity-coincidence detector introduces spurious components in both the mean and the variance of the detector output. These spurious components cause a degradation in detector output signal-to-noise power ratio relative to the ideal (no bias) polarity-coincidence detector.

In order that the degradation in output signal-to-noise power ratio relative to the ideal polarity-coincidence detector be less than 10% (0.05 db.) it is necessary and sufficient that:

<u>For $\lambda \geq 1$</u> $\delta_1^2 \leq 0.15/\lambda$

<u>For $\gamma \geq 1$</u> $\delta_2^2 \leq 0.15/\gamma$,

where $\lambda = \tau_s/\tau_n$, $\gamma = 1/\lambda$, τ_s and τ_n are the signal correlation duration and noise correlation duration constants respectively, δ_1 is the (normalized) bias in the input channel polarity indicator and δ_2 is the (normalized) bias in the reference channel polarity indicator.

The detectors, listed in order of decreasing quality according to the criterion of output signal-to-noise power ratio are - the ideal correlation detector, the ideal polarity-coincidence detector and the biased polarity-coincidence detector. However, the degradations in performance for the PCD relative to the CD and for the BPCD relative to the PCD are not very serious. (1.4 db. to 2.2 db. for the PCD relative to the CD, according to equations 5.1-4 and 5.1-5, and negligible for the BPCD relative to the PCD if the conditions stated on page 193 for δ_1 and δ_2 are satisfied.)

Assumptions and Restrictions

The signal and noise are sample functions from stationary Gaussian random processes with zero means which are statistically independent to the second order. Both the signal and noise have RC low-pass spectra with normalized correlation functions $r(\tau) = e^{-|\tau|/\tau_s}$ for the signal and $\rho(\tau) = e^{-|\tau|/\tau_n}$ for the noise. The correlation duration constants, τ_s and τ_n, are both much less than the integration period, T, of the correlator. The normalized biases, δ_1 and δ_2, are both much less than 1. The input signal-to-noise power ratio is either much less than $\gamma/(1+\gamma)$, where $\gamma = \tau_n/\tau_s$, or the signal is non-random. N is much less than 1. $\delta_1\delta_2$ is much less than $\sqrt{N}$. For $N \ll 1$, the variance of the detector output is independent of N.

6.2 Generalizations

Conceptually, the problem of analyzing the effects of bias in a polarity-coincidence detector is straightforward. The mean of the output of the detector (for the type of signal and noise considered in this thesis) is expressed in an uncomplicated way in terms of a function, $V(p, \gamma p)$, which is tabulated in the literature and whose properties are simple enough that the effects of bias on the mean are quite easily interpreted. The variance is expressed in terms of several probability integrals which have simple meaning. However, these integrals cannot be evaluated in closed form even in the special case of RC low-pass Gaussian signal and noise considered in this thesis. Therefore, simplifying assumptions were introduced in order to make the analysis tractable. These assumptions limit the generality of the results considerably. Specifically, the validity of the results of this thesis has been demonstrated only for the case of small input signal-to-noise ratios. In particular, the validity of the ranking, according to the output signal-to-noise power ratio, of the three detectors examined in this thesis has been demonstrated only for the case of small input signal-to-noise ratios.

It is reasonable, according to one's intuition, to suppose that this ranking of the detectors is preserved when the input signal-to-noise ratio is not small. It would seem to be surprising if the ideal correlation detector were not superior to the ideal polarity-coincidence detector and if the ideal polarity-coincidence detector were not superior to the biased polarity-coincidence detector in the general case. Caution must be exercised, however, in making such conjectures - for it is known that in some cases the ideal polarity-coincidence detector has a larger output signal-to-noise ratio than the ideal correlation detector. For, example, Davenport (2) has shown that the output signal-to-noise ratio of a band-pass limiter is larger than the input signal-to-noise ratio for large input signal-to-noise ratios. (This does not imply, however, that at large signal-to-noise ratios the ideal polarity-coincidence detector is superior to the ideal correlation detector in any other sense than in the signal-to-noise ratio sense. For example, in the statistical hypothesis testing sense, the polarity-coincidence detector is probably inferior to the ideal correlation detector at all signal-to-noise ratios, at least for Gaussian signals and noise.)

A method for analyzing the output variance when the input signal-to-noise ratio is not restricted to small values is outlined in Chapter IV of this thesis. It employs a power series expansion of the integrator input correlation function in terms of the detector input signal and noise correlation functions, with coefficients of the series in terms of tetrachoric series of Hermite polynomials.

An analysis of the output variance according to this method will be the subject of a future report. This analysis will provide the answer to the validity of the results of this thesis for values of the input signal-to-noise ratio which are not small.

There is one area in which generalization seems fairly safe. The output variance of a biased polarity-coincidence detector, for RC low-pass Gaussian signal and noise with small input signal-to-noise ratio, as given by equation 4.2-9 is

$$\sigma_z^2 = T\cdot\delta_1^2\cdot J_2 + T\cdot\delta_2^2\, J_3 + T\cdot J_4$$

where J_2, J_3 and J_4 are constants depending only on the signal and noise correlation functions. The signal and noise correlation functions enter into the expressions for J_2, J_3 and J_4 in the integrands of the integrals which define these constants. If low-pass spectra other than the RC type are considered, the detailed structure of the integrands in the defining expressions for J_2, J_3 and J_4 will of course be altered - but the gross structure will still correspond to the low-pass character of the signal and noise spectra. Thus, the integrals may be expected to suffer only minor changes in value if the signal and noise band-widths are unchanged. Therefore, qualitative properties of the detector as presented in this thesis should hold for Gaussian signal and noise with low-pass spectra more general than the RC type.

REFERENCES

1. Faran, J. J. and R. Hills: "Correlators for Signal Reception," Technical Memorandum No. 27, Acoustics Research Lab., Harvard Univ., pp. 58-65; September, 1952.

2. Davenport, W. B.: "Signal-to-Noise Ratios in Band-Pass Limiters," Jour. of Applied Physics, vol. 24. no. 6, pp. 720-727; 1953.

APPENDIX I

PROPERTIES OF $V(p,\gamma p)$

The function $V(p,\gamma p)$ is defined by

$$V(p,\gamma p) = \int_0^p d\lambda \int_0^{\gamma\lambda} z(\lambda)z(t)dt \qquad \text{(A1-1)}$$

where $z(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ is the density function for the standard normal variate. A related function is

$$V(\gamma p,p) = \int_0^{\gamma p} d\lambda \int_0^{\lambda/\gamma} z(\lambda)z(t)dt \quad . \qquad \text{(A1-2)}$$

The following properties of $V(p,\gamma p)$ and $V(\gamma p,p)$ are easily derived from the definitions:

$V(\gamma p,p) = V(q,\nu q)$ where $q = \gamma p$ and $\nu = 1/\gamma$. This equation results from direct substitution in equation A1-2. Thus, functional values for $\gamma > 1$ can be obtained from the twin function for values of $\gamma < 1$.

By making the transformations $\gamma \rightarrow -\gamma$ and $t \rightarrow -t$ as required and by making use of the evenness of $z(\lambda)$ and $z(t)$, the following relations are derived:

$$V(p,\gamma p) = V(-p,-\gamma p) \qquad \text{(v1)}$$

$$V(p,\gamma p) = -V(-p,\gamma p) = -V(p,-\gamma p) \qquad \text{(v2)}$$

Thus, functional values for negative p and γ can be obtained from functional values for positive p and γ.

It is obvious from the definitions that

$$V(0,\gamma 0) = V(p,0p) = 0 \tag{v3}$$

Relations corresponding to v1, v2 and v3 can be found for $V(\gamma p,p)$ by using the identity $V(\gamma p,p) = V(q,\nu q)$ with $q = \gamma p$ and $\nu = 1/\gamma$.

Additional properties become apparent after equations A1-1 and A1-2 are written in polar coordinates. Let $\lambda = r\cdot\cos\theta$, $t = r\cdot\sin\theta$. Then

$$V(p,\gamma p) = \frac{1}{2\pi}\int_0^{\beta} d\theta \int_0^{p/\cos\theta} r\cdot e^{-\frac{1}{2}r^2}\,dr$$

$$= \frac{1}{2\pi}\int_0^{\beta} (1-e^{-\frac{1}{2}p^2/\cos^2\theta})\,d\theta \quad , \tag{A1-3}$$

where $\beta = \tan^{-1}(\gamma)$, and

$$V(\gamma p,p) = \frac{1}{2\pi}\int_0^{\delta} d\theta \int_0^{\gamma p/\cos\theta} r\cdot e^{-\frac{1}{2}r^2}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{\delta} (1-e^{-\frac{1}{2}\gamma^2 p^2/\cos^2\theta})\,d\theta \tag{A1-4}$$

where $\delta = \tan^{-1}(1/\gamma)$.

From equation A1-3,

$$\lim_{p\to\infty} V(p,\gamma p) = \frac{1}{2\pi}\int_0^{\beta} d\theta = \frac{1}{2\pi}\tan^{-1}(\gamma) \qquad \text{(v4)}$$

From equation A1-4,

$$\lim_{p\to\infty} V(\gamma p,p) = \frac{1}{2\pi}\int_0^{\delta} d\theta \qquad \text{for} \quad \gamma \neq 0$$

$$= 0 \qquad \text{for} \quad \gamma = 0$$

or

$$\lim_{p\to\infty} V(\gamma p,p) = \frac{1}{2\pi}\tan^{-1}(1/\gamma) \qquad \text{for} \quad \gamma \neq 0$$

$$= 0 \qquad \text{for} \quad \gamma = 0 \quad .$$

On introducing the well known identity

$$\tan^{-1}(1/\gamma) = \pi/2 - \tan^{-1}(\gamma) \quad ,$$

the preceding equation can be written

$$\lim_{p\to\infty} V(\gamma p,p) = \frac{1}{2\pi}\tan^{-1}(1/\gamma) \quad \Big\} \text{ for } \gamma \neq 0 \tag{v5}$$

$$= 1/4 - \frac{1}{2\pi}\tan^{-1}(\gamma)$$

$$= 0 \qquad \text{for } \gamma = 0 \quad .$$

It is obvious from the definitions that

$$\lim_{\gamma\to\infty} V(p,\gamma p) = \frac{1}{2}\int_0^{|p|} z(\lambda)d\lambda = \frac{1}{4}\text{Erf}\ (|p|/\sqrt{2}) \quad , \tag{v6}$$

and

$$\lim_{\gamma\to\infty} V(\gamma p,p) = 0 \quad . \tag{v7}$$

$V(p,\gamma p)$ and $V(\gamma p,p)$ can be interpreted as the volumes under the standard bivariate normal surface with zero correlation, over the regions in the γ,t-plane as shown in figures A1-1 and A1-2.

A plot of $V(p,\gamma p)$, with γ as a parameter, is presented in figure A1-3. This plot is based on the approximations for $V(p,\gamma p)$ derived in Appendix II.

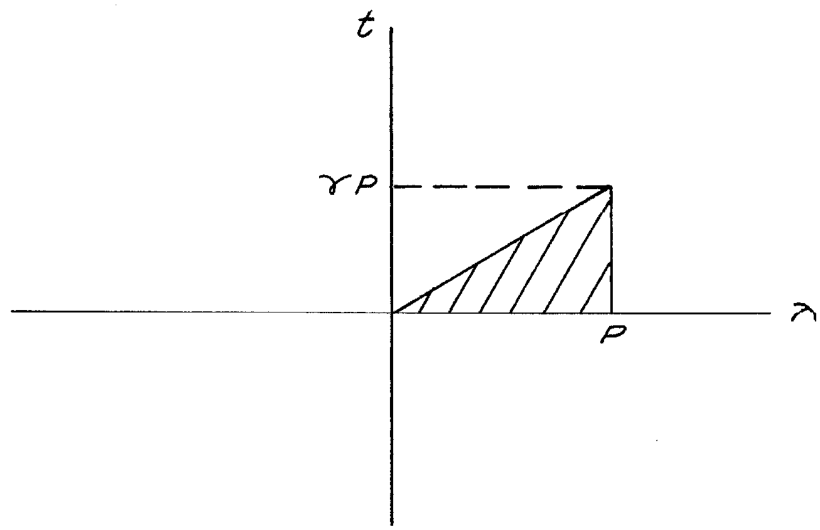


Figure A1-1

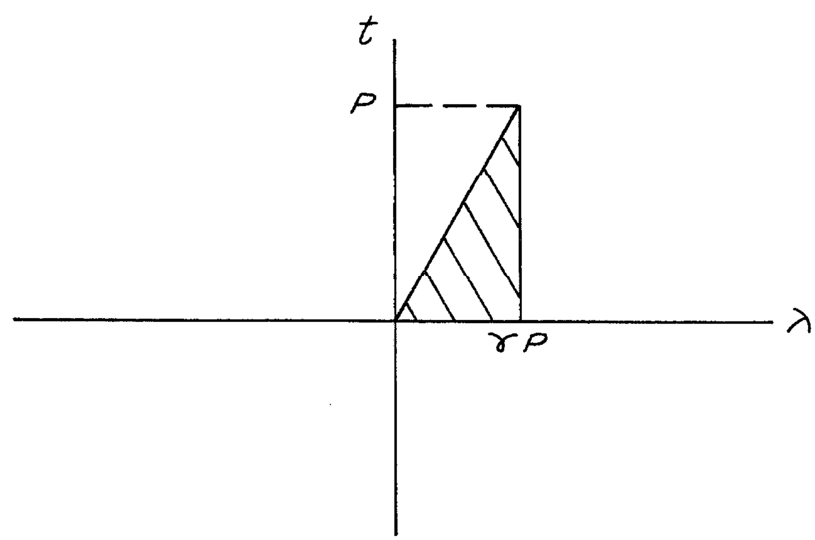
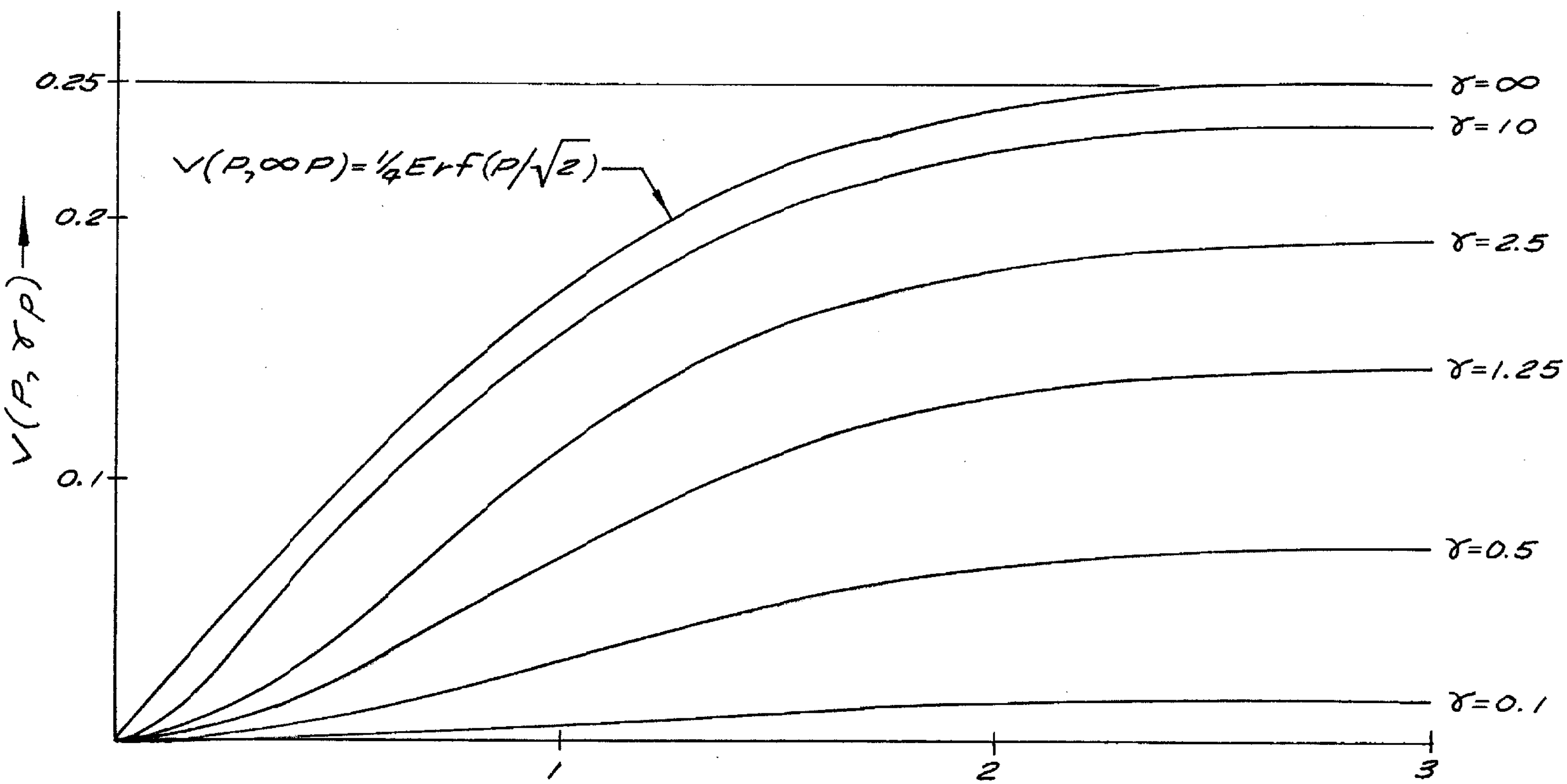


Figure A1-2



The Function V(p,γp)

Figure A1-3

APPENDIX II

APPROXIMATE EVALUATIONS OF THE INTEGRAL $V(p,\gamma p)$

The integral $V(p,\gamma p)$ can be evaluated approximately in closed form in several different ways. The integral to be evaluated is

$$V(p,\gamma p) = \frac{1}{2\pi} \int_0^p d\lambda \int_0^{\gamma\lambda} e^{-\frac{1}{2}\lambda^2} e^{-\frac{1}{2}t^2} dt \quad .$$

The triangle over which the integration takes place is shown in Figure A2-1.

A2.0 p and γp small Compared with Unity.

If the exponent of the integrand when $\lambda = p$ and $t = \gamma p$ is denoted by $\frac{1}{2}\epsilon$ (i.e. if $(1+\gamma^2)p^2 = \epsilon$), then the following inequalities hold for all λ, t in the region of integration, R : $0 \le \frac{1}{2}(t^2 + \lambda^2) \le \frac{1}{2}\epsilon$; thus $0 \le 1-e^{-\frac{1}{2}(t^2+\lambda^2)} \le 1-e^{-\frac{1}{2}\epsilon}$.

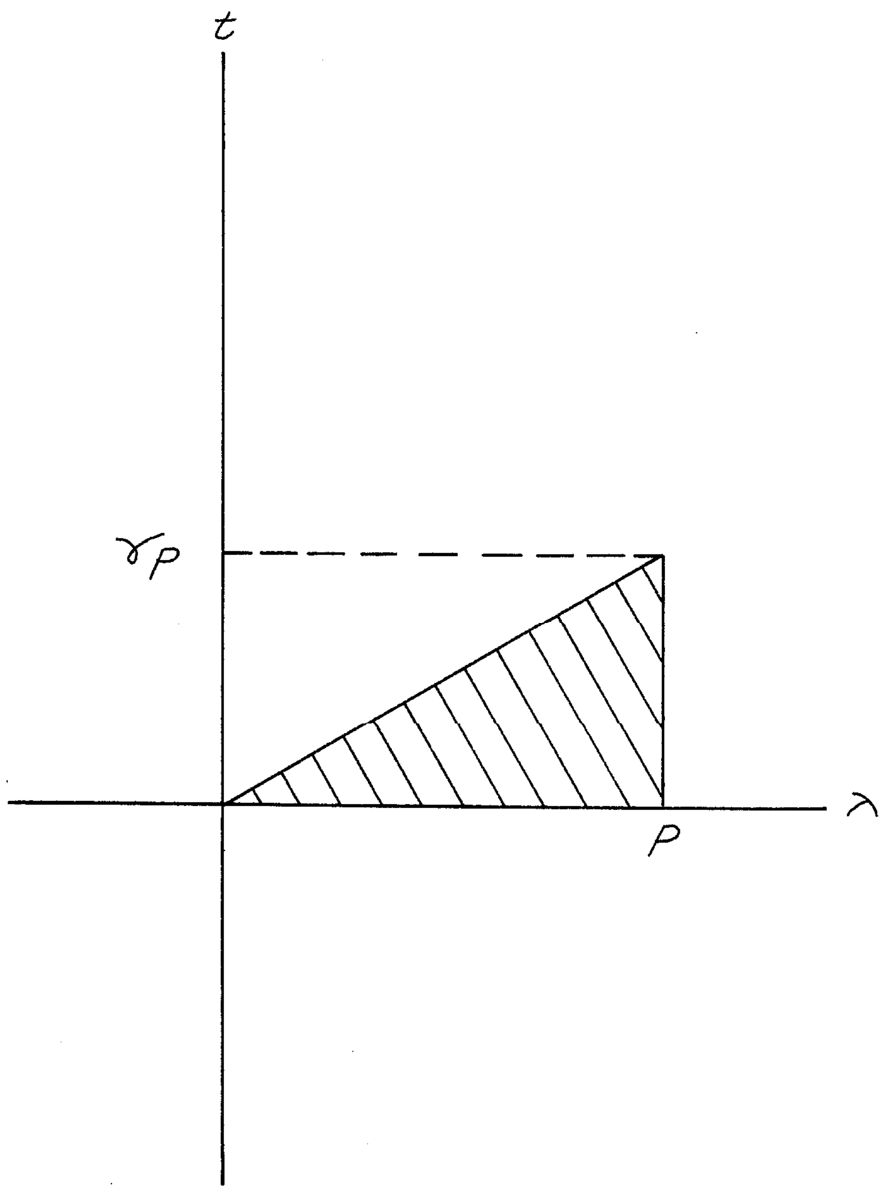


Figure A2-1

But $1-e^{-\frac{1}{2}\epsilon} \le \frac{1}{2}\epsilon$ for all non-negative ϵ. Therefore, $0 \le 1-e^{-\frac{1}{2}(t^2+\lambda^2)} \le \frac{1}{2}\epsilon$. Upon integrating, this inequality becomes

$$0 \le \frac{1}{2\pi}\int_0^p d\lambda \int_0^{\gamma\lambda} dt - \frac{1}{2\pi}\int_0^p d\lambda \int_0^{\gamma\lambda} e^{-\frac{1}{2}\lambda^2} e^{-\frac{1}{2}t^2} dt \le \frac{1}{2\pi}\frac{\epsilon}{2}\int_0^p d\lambda \int_0^{\gamma\lambda} dt$$

or

$$0 \le \frac{1}{2\pi}\frac{\gamma p^2}{2} - V(p,\gamma p) \le \frac{1}{2}\epsilon \frac{1}{2\pi}\frac{\gamma p^2}{2} \quad .$$

From this expression it is clear that $\frac{1}{2\pi}\frac{\gamma p^2}{2}$ is a good approximation to $V(p,\gamma p)$ when ϵ is small compared with unity.

$$V_o(p,\gamma p) = \frac{1}{2\pi}\frac{\gamma p^2}{2} \tag{A2.0-1}$$

is an approximation to $V(p,\gamma p)$ with error

$$e = V_o(p,\gamma p) - V(p,\gamma p) \le \frac{1}{2}\epsilon \cdot V_o(p,\gamma p) \quad . \tag{A2.0-2}$$

For ϵ small compared with unity (i.e. for $(1+\gamma^2)p^2 \ll 1$), the error is small relative to V_o and V. Then the relative error is approximately

$$\rho \cong \frac{V_o - V}{V_o} \quad .$$

V_o is substituted for V in the denominator, since the difference between V_o and V is small compared with V. Thus, from equation A2.0-1,

$$\rho \cong \tfrac{1}{2}\epsilon \qquad . \tag{A2.0-3}$$

A2.1 γ Small Compared with Unity.

By transforming to polar coordinates, additional approximations to $V(p,\gamma p)$ can be found which are precise when γ is small compared with unity. Let $\lambda = r\cdot\cos\theta$, $t = r\cdot\sin\theta$. Then

$$V(p,\gamma p) = \frac{1}{2\pi}\int_0^{\beta} d\theta \int_0^{p/\cos\theta} r\cdot e^{-\frac{1}{2}r^2}\,dr$$

$$= \frac{1}{2\pi}\int_0^{\beta} (1-e^{-\frac{1}{2}p^2/\cos^2\theta})\,d\theta \qquad ,$$

where $\beta = \tan^{-1}(\gamma)$.

The following inequalities hold for all θ in the range of integration: $1 \le 1/\cos^2\theta \le 1/\cos^2\beta = 1+\gamma^2$. Thus

$0 \le e^{-\frac{1}{2}p^2/\cos^2\beta} = e^{-\frac{1}{2}p^2(1+\gamma^2)} \le e^{-\frac{1}{2}p^2/\cos^2\theta} \le e^{-\frac{1}{2}p^2}$ or

$0 \le 1-e^{-\frac{1}{2}p^2} \le 1-e^{-\frac{1}{2}p^2/\cos^2\theta} \le 1-e^{-\frac{1}{2}p^2(1+\gamma^2)}$. Upon integrating, these inequalities become

$$\frac{1}{2\pi}(1-e^{-\frac{1}{2}p^2})|\tan^{-1}(\gamma)| \le |V(p,\gamma p)| \le \frac{1}{2\pi}(1-e^{-\frac{1}{2}p^2(1+\gamma^2)})|\tan^{-1}(\gamma)| \quad .$$

When γ is small compared with unity, the upper and lower bounds are nearly equal and $\frac{1}{2\pi}(1-e^{-\frac{1}{2}p^2})\tan^{-1}(\gamma)$ is a good approximation to $V(p,\gamma p)$.

$$V_1(p,\gamma p) = \frac{1}{2\pi}(1-e^{-\frac{1}{2}p^2})\tan^{-1}(\gamma) \cong \frac{\gamma}{2\pi}(1-e^{-\frac{1}{2}p^2}) \qquad \text{(A2.1-1)}$$

for small γ is an approximation to $V(p,\gamma p)$ with error

$$e = V_1(p,\gamma p) - V(p,\gamma p) \qquad \text{(A2.1-2)}$$

$$|e| \le \frac{1}{2\pi}(e^{-\frac{1}{2}p^2} - e^{-\frac{1}{2}p^2(1+\gamma^2)})\tan^{-1}(|\gamma|) \quad .$$

$$\le \frac{1}{2\pi} e^{-\frac{p^2}{2}} \left[\frac{p^2|\gamma|^3}{2}\right] \qquad \text{for small } \gamma \quad .$$

For γ small compared with unity, the error is small relative to V_1 and V. Then the relative error is approximately

$$\rho \cong (V_1 - V)/V_1 \quad .$$

V_1 is substituted for V in the denominator, since the difference between V_1 and V is small compared with V. Thus, from equations A2.1-1 and A2.1-2,

$$|\rho| \cong e^{-\frac{1}{2}p^2}(1-e^{-\frac{1}{2}p^2\gamma^2})/(1-e^{-\frac{1}{2}p^2}) \quad .$$

$$\cong \tfrac{1}{2}p^2\gamma^2\, e^{-\frac{1}{2}p^2}/(1-e^{-\frac{1}{2}p^2}) \le \gamma^2 \quad \text{for all} \quad p. \qquad \text{(A2.1-3)}$$

The preceding approximate expression for ρ is small for all values of p, so long as γ is small.

A2.2 $\gamma \to \infty$.

The integral can be evaluated exactly in the limit as γ goes to infinity.

$$\lim_{\gamma\to\infty} V(p,\gamma p) = \frac{1}{2\pi}\int_0^{|p|} d\lambda \int_0^{\infty} e^{-\frac{1}{2}(t^2+\lambda^2)}dt = \frac{1}{2\pi}\int_0^{|p|} e^{-\frac{1}{2}\lambda^2}d\lambda \int_0^{\infty} e^{-\frac{1}{2}t^2}dt \qquad \text{(A2.2-1)}$$

or

$$V_2(p,\infty p) = \frac{1}{4}\,\mathrm{Erf}\,(|p|/\sqrt{2}\,) \quad .$$

A2.3 p Small Compared with Unity.

When p is small compared with unity, the exponential factor of the integrand with exponent $-\frac{1}{2}\lambda^2$ is nearly unity. If this factor is replaced by unity, the integral can be evaluated in closed form.

$$V_3(p,\gamma p) = \frac{1}{\sqrt{2\pi}}\int_0^{p} \tfrac{1}{2}\mathrm{Erf}\,(\gamma\lambda/\sqrt{2}\,)d\lambda = \frac{1}{\sqrt{2\pi}}\,\frac{\sqrt{2}}{\gamma}\int_0^{\gamma p/\sqrt{2}} \mathrm{Erf}(x)dx \quad .$$

This integral can be evaluated by a contour integration in the complex plane. However, it appears in a small table of error function integrals assembled by Middleton (1), so the derivation will not be given here. The result is

$$V_3(p,\gamma p) = \frac{p}{2\sqrt{2\pi}} \operatorname{Erf}(\gamma p/\sqrt{2}\,) - \frac{1}{2\pi\gamma}\left(1-e^{-\frac{1}{2}\gamma^2 p^2}\right) \quad . \tag{A2.3-1}$$

A bound on the error is found as follows: For all λ in the range of integration, the inequality

$$0 \le 1-e^{-\frac{1}{2}\lambda^2} \le 1-e^{-\frac{1}{2}p^2}$$

is valid. Thus the error is

$$e = \frac{1}{\sqrt{2\pi}} \int_0^p \left(1-e^{-\frac{1}{2}\lambda^2}\right)\tfrac{1}{2}\operatorname{Erf}(\gamma\lambda/\sqrt{2})\,d\lambda$$

$$\le \frac{1}{\sqrt{2\pi}} \int_0^p \left(1-e^{-\frac{1}{2}p^2}\right)\tfrac{1}{2}\operatorname{Erf}(\gamma\lambda/\sqrt{2})\,d\lambda$$

$$\le \left(1-e^{-\frac{1}{2}p^2}\right)V_4(p,\gamma p)$$

$$\le (p^2/2)V_4(p,\gamma p) \quad \text{for small } p.$$

For p small compared with unity, the error is small relative to V_3 and V. Then the relative error is approximately

$$\rho \cong (V_3 - V)/V_3 \quad .$$

V_3 is substituted for V in the denominator, since the difference between V_3 and V is small compared with V. Thus, from equation A2.3-2,

$$\rho \cong p^2/2 \quad \text{for small } p. \qquad \text{(A2.3-3)}$$

A2.4 $p \to \infty$.

The limit of V as $p \to \infty$ is derived in Appendix I. In the limit, the integral can be evaluated in closed form. It is

$$\lim_{p \to \infty} V(p, \gamma p) = \frac{1}{2\pi} \tan^{-1}(\gamma) \quad . \qquad \text{(A2.4-1)}$$

The preceding results provide good approximations for V when p and γ are restricted to certain regions of the p,γ-plane (p and γp small, γ small, p small, $p \to \infty$ and $\gamma \to \infty$). In the next section, no restrictions will be placed on p and γ. An approximation to V which is good for all values of p and γ is found.

A2.5 The Approximating Function Method.

In Appendix V it is shown that the Freedman approximation function is an excellent approximation to $\phi(t) = \frac{1}{2}\mathrm{Erf}(t/\sqrt{2})$. The Freedman approximation function is

$$\phi_a(t) = \begin{cases} \frac{1}{2}\left[1-e^{-\frac{1}{2}(\alpha t^2+\beta t)}\right] & \text{for} \quad t \geq 0 \\ -\frac{1}{2}\left[1-e^{-\frac{1}{2}(\alpha t^2-\beta t)}\right] & \text{for} \quad t \leq 0 \end{cases} \qquad \text{(A2.5-1)}$$

where $\alpha = 0.72$ and $\beta = 1.58$.

If this expression is substituted for $\mathrm{Erf}(\gamma\lambda/\sqrt{2})$ in the equation

$$V(p,\gamma p) = \frac{1}{\sqrt{2\pi}} \int_0^p e^{-\frac{1}{2}\lambda^2} \tfrac{1}{2}\mathrm{Erf}(\gamma\lambda/\sqrt{2})d\lambda \qquad ,$$

the result is

$$V_5(p,\gamma p) = \frac{1}{\sqrt{2\pi}} \int_0^p e^{-\frac{1}{2}\lambda^2} \tfrac{1}{2}\left[1-e^{-\frac{1}{2}(\alpha\gamma^2\lambda^2+\beta\gamma\lambda)}\right]d\lambda \qquad ,$$

where consideration is restricted to non-negative γp. To consider only non-negative γp is sufficiently general, since V for negative γp can be obtained from V for positive γp using the properties of V given in Appendix I. Performing the indicated integration,

$$V_5(p,\gamma p) = \tfrac{1}{4}\mathrm{Erf}(p/\sqrt{2}) - \frac{\sigma}{2}\, e^{\mu^2/2\sigma^2} \frac{1}{\sqrt{2\pi}} \int_{-\mu/\sigma}^{(p-\mu)/\sigma} e^{-\frac{1}{2}\zeta^2} d\zeta \quad ,$$

where $\zeta = (\lambda-\mu)/\sigma$, $\mu = -\beta\gamma/2(1+\alpha\gamma^2) = -\sigma^2\beta\gamma/2$ and $\sigma = 1/\sqrt{1+a\gamma^2}$.

Finally, then

$$V_5(p,\gamma p) = \frac{1}{4}\left\{\mathrm{Erf}(p/\sqrt{2}) - \sigma e^{\mu^2/2\sigma^2}\left[\mathrm{Erf}\left(\frac{p-\mu}{\sigma\sqrt{2}}\right) + \mathrm{Erf}(\mu/\sigma\sqrt{2})\right]\right\} \quad . \tag{A2.5-2}$$

The error is bounded in magnitude as follows:

$$|e| = \left| \frac{1}{\sqrt{2\pi}} \int_0^p e^{-\frac{1}{2}\lambda^2} \left\{\phi_a(\gamma\lambda) - \tfrac{1}{2}\mathrm{Erf}(\gamma\lambda/\sqrt{2})\right\} d\lambda \right|$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_0^p e^{-\frac{1}{2}\lambda^2} \left|\phi_a(\gamma\lambda) - \tfrac{1}{2}\mathrm{Erf}(\gamma\lambda/\sqrt{2})\right| d\lambda \quad .$$

Let ϵ be any bound on $\left|\phi_a(\gamma\lambda) - \tfrac{1}{2}\mathrm{Erf}(\gamma\lambda/\sqrt{2})\right|$. Then

$$|e| \leq \epsilon \frac{1}{\sqrt{2\pi}} \int_0^p e^{-\frac{1}{2}\lambda^2} d\lambda = \tfrac{1}{2}\epsilon\mathrm{Erf}(p/\sqrt{2}) \quad . \tag{A2.5-3}$$

It is shown in Appendix V that $|\phi_a(t) - \tfrac{1}{2}\mathrm{Erf}(t/\sqrt{2})|$ is less than 0.0016 for all t. Therefore,

$$|e| \leq 0.0008\ \mathrm{Erf}(p/\sqrt{2}) \leq 0.0008 \quad . \tag{A2.5-4}$$

According to equation 4.3-1 of Chapter IV, when the integrand of an integral does not change sign in the region of integration, then the relative error for the integral due to substituting an approximating function for a factor in the integrand has a magnitude which is smaller than - or at most equal to - the maximum magnitude of the relative error for the approximating function. The integrand in the integral for $V(p,\gamma p)$ does not change sign in the region of integration. Moreover, it is shown in Appendix V that the relative error due to approximating $\frac{1}{2}\mathrm{Erf}(t/\sqrt{2})$ by $\phi_a(t)$ has a maximum magnitude of 0.01. Therefore, the relative error for the approximation $V_5(p,\gamma p)$ is no greater than 0.01 in magnitude.

$$|\rho| \leq 0.01 \quad . \tag{A2.5-5}$$

REFERENCES

1. Middleton, D.: Introduction to Statistical Communication Theory, McGraw-Hill, New York, p. 1027; 1960.

APPENDIX III

PROPERTIES OF L(h,k;r)

The function $L(h,k;r)$ is defined by

$$L(h,k;r) = \int_h^\infty dx \int_k^\infty v(x,y;r)dy \quad , \qquad \text{(A3-1)}$$

where $v(x,y;r)$ is the standard bivariate normal density function with correlation coefficient r.

$$v(x,y;r) = \frac{1}{2\pi\sqrt{1-r^2}} \exp\left[- \frac{x^2-2rxy+y^2}{2\left(1-r^2\right)}\right] \quad .$$

Upon setting $w = (y-rx)/\sqrt{1-r^2}$, equation A3-1 becomes

$$L(h,k;r) = \int_h^\infty dx \int_{j(x)}^\infty z(x)z(w)dw \quad , \qquad \text{(A3-2)}$$

where $j(x) = (k-rx)/\sqrt{1-r^2}$ and $z(x)$ is the density function for the standard normal variate.

$$z(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \quad .$$

The inner integral can be written

$$\int_{j(x)}^{\infty} z(w)dw = \int_{0}^{\infty} z(w)dw - \int_{0}^{j(x)} z(w)dw$$

$$= \tfrac{1}{2}\left[1 - \mathrm{Erf}\{j(x)/\sqrt{2}\}\right] \quad .$$

Thus, equation A3-2 can be written

$$L(h,k;r) = \int_{h}^{\infty} z(x)\cdot\tfrac{1}{2}\left[1 - \mathrm{Erf}\{j(x)/\sqrt{2}\}\right]dx$$

$$= \frac{1}{4}\left[1 - \mathrm{Erf}(h/\sqrt{2})\right] - \tfrac{1}{2}\int_{h}^{\infty} z(x)\mathrm{Erf}\left\{\frac{k-rx}{\sqrt{2}\sqrt{1-r^2}}\right\}dx \quad . \tag{A3-3}$$

The density function $v(x,y;r)$ is symmetric in x and y, and is even in x and y simultaneously but not individually. I.e., $v(x,y;r) = v(y,x;r)$ and $v(x,y;r) = v(-x,-y;r)$, but $v(x,y;r) \neq v(-x,y;r)$ and $v(x,y;r) \neq v(x,-y;r)$. Using these properties of v and applying the transformations $x \to -x$, $y \to -y$ and $x \to y$, $y \to x$ as required, it is easily shown that the following properties are valid for $L(h,k;r)$.

$$L(h,k;r) = L(k,h;r) \tag{p1}$$

$$L(h,k;0) = \frac{1}{4}\left[1 - \mathrm{Erf}(h/\sqrt{2})\right]\left[1 - \mathrm{Erf}(k/\sqrt{2})\right] \tag{p2}$$

$$L(h,k;-1) = \begin{cases} 0 & \text{if } h+k \geq 0 \\ -\frac{1}{2}\left[\mathrm{Erf}(h/\sqrt{2}) + \mathrm{Erf}(k/\sqrt{2})\right] & \text{if } h+k \leq 0 \end{cases} \tag{p3}$$

$$L(h,k;1) = \begin{cases} \frac{1}{2}\left[1 - \mathrm{Erf}(h/\sqrt{2})\right] & \text{if } k \leq h, \\ \frac{1}{2}\left[1 - \mathrm{Erf}(k/\sqrt{2})\right] & \text{if } k \geq h \end{cases} \tag{p4}$$

$$L(-h,k;r) = -L(h,k;-r) + \frac{1}{2}\left[1 - \mathrm{Erf}(k/\sqrt{2})\right] \tag{p5}$$

$$L(h,-k;r) = -L(h,k;-r) + \frac{1}{2}\left[1 - \mathrm{Erf}(h/\sqrt{2})\right] \tag{p6}$$

$$L(-h,-k;r) = L(h,k;r) + \frac{1}{2}\left[\mathrm{Erf}(h/\sqrt{2}) + \mathrm{Erf}(k/\sqrt{2})\right] \tag{p7}$$

Thus, functional values for negative h and/or k can be obtained from functional values for positive h and k.

$$L(0,0;r) = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}(r) \tag{p8}$$

$$L(\infty,k;r) - L(h,\infty;r) = 0 \tag{p9}$$

$$L(-\infty,k;r) = \frac{1}{2}\left[1 - \mathrm{Erf}(k/\sqrt{2})\right] \tag{p10}$$

$$L(h,-\infty;r) = \frac{1}{2}\left[1 - \mathrm{Erf}(h/\sqrt{2})\right] \tag{p11}$$

The following property, which relates $L(h,k;r)$ to $V(p,\gamma p)$, the function discussed in Chapter III and Appendix I, is not obvious and will be derived.

$$L(h,k;r) = V\left[h, \frac{k-rh}{\sqrt{1-r^2}}\right] + V\left[k, \frac{h-rk}{\sqrt{1-r^2}}\right] \qquad \text{(p12)}$$

$$+ \frac{1}{4} - \frac{1}{4}\left[\mathrm{Erf}(h/\sqrt{2}) + \mathrm{Erf}(k/\sqrt{2})\right] + \frac{1}{2\pi}\sin^{-1}(r) \quad ,$$

where $V(a,b) = V(a,\left(\frac{b}{a}\right)a)$ is the function defined in Appendix I. When $h = k$,

$$L(h,h;r) = 2V(h, \sqrt{\frac{1-r}{1+r}}\,h) + \frac{1}{4} - \frac{1}{2}\mathrm{Erf}(h/\sqrt{2}) + \frac{1}{2\pi}\sin^{-1}(r) \quad . \qquad \text{(p13)}$$

The derivation of property p12 follows: The region of integration, R, for the integral of equation A3-2 is the shaded region in the x,w-plane shown in Figure A3-1. This region can be synthesized as follows:

R = right half-plane $- R_1 - R_2 - R_3 - R_4 - R_5 - R_6 - R_7$, where R_1 through R_7 are shown in Figure A3-1. This can be written

$$R = rhp - R_1 + R_4 + R_5 - R_8 - R_9 \quad ,$$

where $R_8 = R_2 + R_3 + R_4 + R_5$ and $R_9 = R_4 + R_5 + R_6 + R_7$.

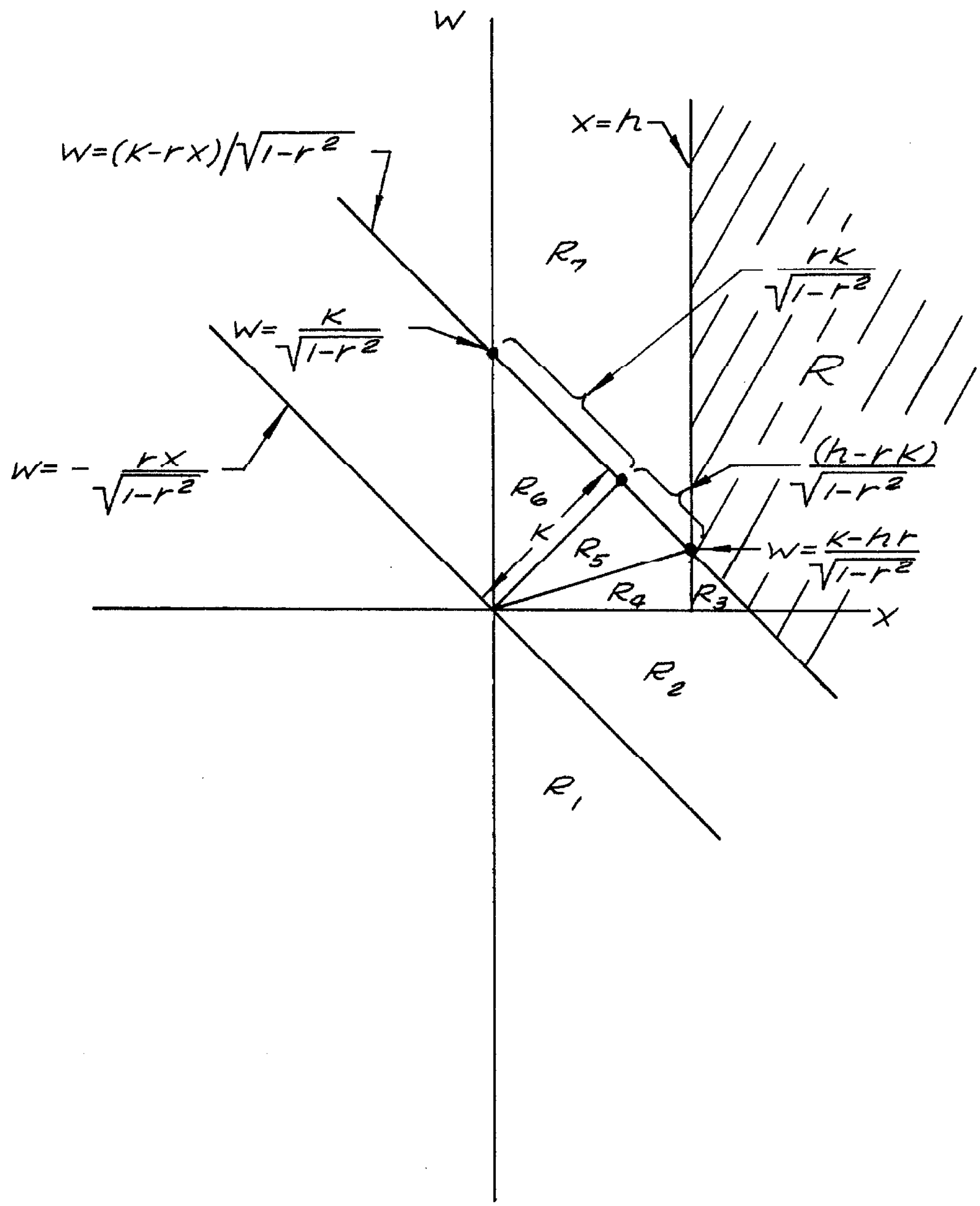

W
X=h
W=(K-rX)/√(1-r²)
R7
rK/√(1-r²)
W=K/√(1-r²)
R
W=-rX/√(1-r²)
(h-rK)/√(1-r²)
R6
K
R5
W=(K-hr)/√(1-r²)
R4
R3
X
R2
R1

Figure A3-1

Triangles R_4 and R_5 are right triangles, each with an acute vertex at the origin. R_1 can be regarded as the limiting case as p goes to infinity of a right triangle with an acute vertex at the origin and with side adjacent to the origin of length $\frac{r}{\sqrt{1-r^2}}\,p$ and side opposite the origin of length p. R_8 can be regarded as the limiting case as γ goes to infinity of a right triangle with acute vertex at the origin and with side adjacent to the origin of length k and side opposite the origin of length γk. R_9 can be regarded as the limiting case as γ goes to infinity of a right triangle with an acute vertex at the origin and with side adjacent to the origin of length h and side opposite the origin of length γh. Thus, R_1, R_4, R_5, R_8 and R_9 all are right triangles of the type T discussed in section 3.2, page 54. It is shown there that the double integral of $z(x)z(w)$ over any triangle of type T is equal to $V(p,\gamma p)$, where p is the length of the side adjacent to the origin and γp is the length of the side opposite the origin. Since the integral over the first quadrant is clearly equal to $L(0,0;0)$, then the integral over the right half-plane is equal to $2L(0,0;0)$. Therefore, on introducing various limiting values for $V(p,\gamma p)$ and $L(h,k;r)$ as needed from Appendix I and III,

$$L(h,k;r) = 2L(0,0;0) - \frac{1}{4} + \frac{1}{2\pi}\tan^{-1}\frac{r}{\sqrt{1-r^2}} - \frac{1}{4}\,\mathrm{Erf}(h/\sqrt{2})$$

$$- \frac{1}{4}\,\mathrm{Erf}(k/\sqrt{2}) + V\left(h, \frac{k-rh}{\sqrt{1-r^2}}\right) + V\left(k, \frac{h-rk}{\sqrt{1-r^2}}\right) \quad .$$

Since $\tan^{-1} \frac{r}{\sqrt{1-r^2}} = \sin^{-1}(r)$ and $L(0,0;0) - \frac{1}{4}$, finally,

$$L(h,k;r) = V\left[h, \frac{k-rh}{1-r^2}\right] + V\left[k, \frac{h-rk}{1-r^2}\right]$$

$$+ \frac{1}{4} - \left[\frac{1}{4} \operatorname{Erf}(h/\sqrt{2}) + \operatorname{Erf}(k/\sqrt{2})\right] + \frac{1}{2\pi} \sin^{-1}(r) \quad ,$$

as was to be proved.

To make the proof complete, it would be necessary to consider other values of h,k and r for which the relative location of the points of intersection of the line $w = \frac{k-rx}{\sqrt{1-r^2}}$ with the line normal to it through the origin, with the w-axis, and with the line $x = h$ occur in a different order than is shown in Figure A3-1. However, the proofs for these other cases are similar to the one just given. Therefore, they will be omitted.

$L(h,k;r)$ can be interpreted as the volume under the standard bivariate normal surface with correlation coefficient r, over the region in the x,y-plane as shown in Figure A3-2, as well as the volume under the standard bivariate normal surface with zero correlation coefficient over the region in the x,w-plane as shown in Figure A3-1.

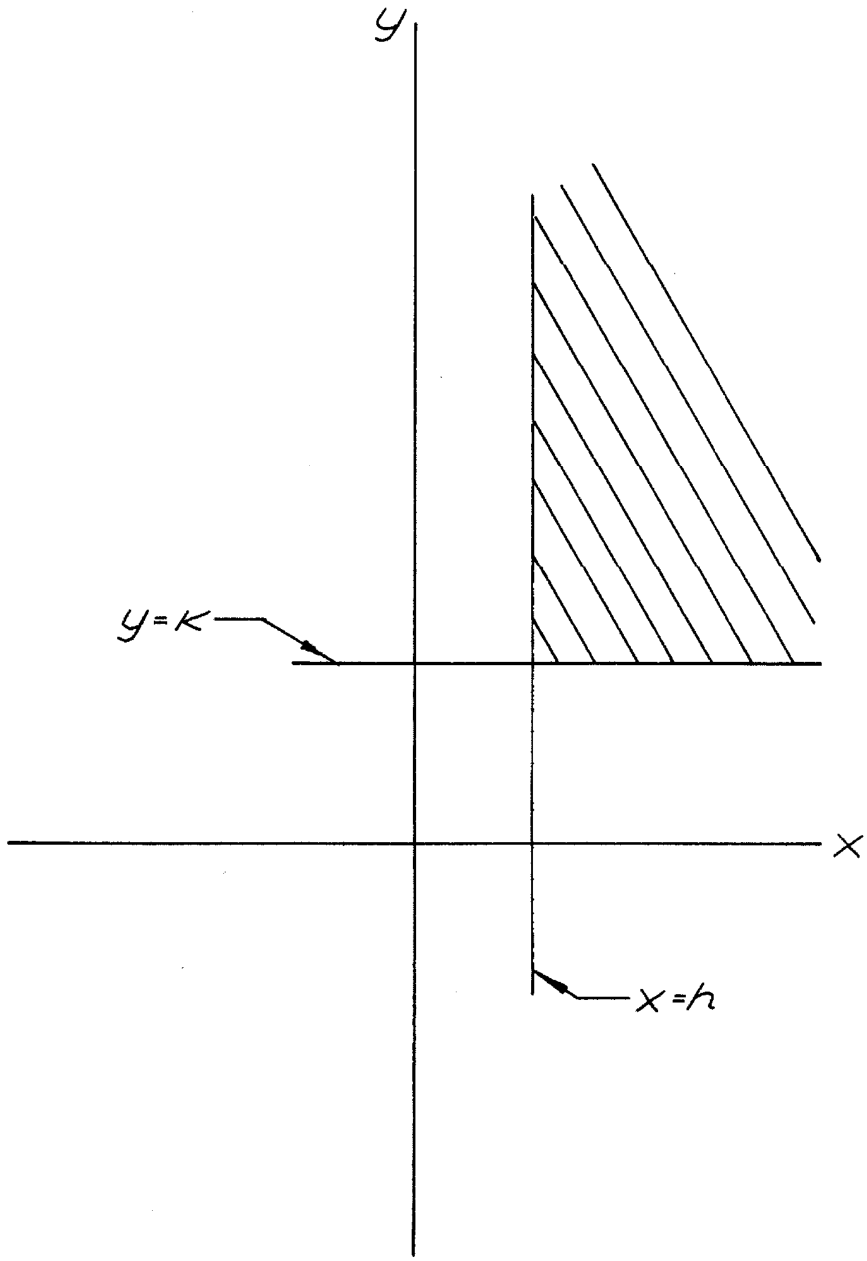


Figure A3-2

APPENDIX IV

APPROXIMATE EVALUATIONS OF THE INTEGRAL L(h,h;r)

The relation derived between $L(h,h;r)$ and $V(p,\gamma p)$ in Appendix III together with the approximation equations derived for $V(p,\gamma p)$ in Appendix II can be used for obtaining approximations for $L(h,h;r)$. From property p13 of Appendix III,

$$L(h,h;r) = 2V\left(h,\sqrt{\frac{1-r}{1+r}}\,h\right) + \frac{1}{4} - \frac{1}{2}\,\mathrm{Erf}(h/\sqrt{2}) + \frac{1}{2\pi}\sin^{-1}(r) \quad .$$

From equation A2.0-1 of Appendix II, when p and γp are both small, then

$$V(p,\gamma p) \cong \frac{1}{2\pi}\,\frac{\gamma p^2}{2}$$

with an error smaller in magnitude than $\frac{1}{2}\epsilon\,\frac{1}{2\pi}\,\frac{\gamma p^2}{2}$ where $\epsilon = (1+\gamma^2)p^2$, and with the sign of the error the same as the sign of γ. Moreover, $\mathrm{Erf}(h/\sqrt{2})$ can be expressed in a power series with alternating signs in the terms. Therefore, the error in using only a finite number of terms of this series is less in magnitude than the first term neglected. Thus, from equation 590 of Dwight (1),

$$\tfrac{1}{2}\mathrm{Erf}(h/\sqrt{2}) \cong h/\sqrt{2\pi}$$

with an error smaller in magnitude than $h^3/6\sqrt{2\pi}$ and with the sign of the error the same as the sign of h.

From these considerations, it follows that when h and $\sqrt{\frac{1-r}{1+r}}\,h$ are both small compared with unity, then

$$L(h,h;r) \simeq \frac{1}{2\pi}\sqrt{\frac{1-r}{1+r}}\,h^2 + \frac{1}{4} - h/\sqrt{2\pi} + \frac{1}{2\pi}\sin^{-1}(r) \qquad \text{(A4-1)}$$

with an error smaller in magnitude than

$$\frac{1}{2}\left(1 + \frac{1-r}{1+r}\right)h^2 + h^3/6\sqrt{\pi} \quad .$$

Clearly, the error is small for small h, so long as r is not near in value to -1, because if r is not near in value to -1 then $\frac{1-r}{1+r}$ will not be much larger than unity and the error term for the V approximation will not be large.

The preceding approximate expression for $L(h,h;r)$ is valid only for small h and for r not too near in value to -1. An approximate expression valid for all h and all r can be obtained, of course, merely by substituting for V from equation A2.5-6 of Appendix II, and for $\mathrm{Erf}(h/\sqrt{2})$ from equation A5-2 from Appendix V. The result is much more complicated than the one given above in equation A4-1.

REFERENCE

1. Dwight, H.: Tables of Integrals and other Mathematical Data, Macmillan, New York, p. 129; 1947.

APPENDIX V

AN APPROXIMATING FUNCTION FOR THE ERROR FUNCTION

In equation A1-1, which defines the function $V(p,\gamma p)$, the inner integral can be written $\frac{1}{2}\text{Erf}(\gamma\lambda/\sqrt{2})$. Thus, the equation can be written

$$V(p,\gamma p) = \frac{1}{2}\int_0^p z(\lambda)\cdot\text{Erf}(\gamma\lambda/\sqrt{2})d\lambda \quad , \qquad \text{(A5-1)}$$

where $z(\lambda) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\lambda^2}$ is the density function for the standard normal variate.

This integral cannot be evaluated in terms of the elementary functions or even in terms of the well known higher functions. Extensive tables of $V(p,\gamma p)$ and $V(\gamma p,p)$ have been published by the National Bureau of Standards (1). These tables are discussed in Section 3.2 of Chapter III of this thesis. Even though tables of V are available, it would be convenient to have a fairly simple functional representation for V in order to apply the results of Chapter III to the detection properties of a BPCD. I am indebted to Benedict Freedman who, in a private communication, suggested an approximating function for the error function which when substituted for $\text{Erf}(\gamma\lambda/\sqrt{2})$ in the above equation makes it possible to evaluate the integral quite simply. The evaluation of $V(p,\gamma p)$ using this approximation function is presented in Appendix II.

The Freedman approximation function is

$$\phi_a(t) = \begin{cases} \frac{1}{2}\left[1-e^{-\frac{1}{2}(\alpha t^2+\beta t)}\right] & \text{for } t \geq 0 \\ -\frac{1}{2}\left[1-e^{-\frac{1}{2}(\alpha t^2-\beta t)}\right] & \text{for } t \leq 0 \end{cases} \qquad \text{(A5-2)}$$

where $\alpha = 0.72$ and $\beta = 1.58$. According to Freedman, these values of α and β are the optimum values with two decimal places for minimizing the maximum magnitude of the error due to the approximation. He further states that new optimum values of α and β obtained by increasing the number of decimal places do not result in a significant reduction in the maximum magnitude of the error.

A tabulation of the error, e, and the relative error, ρ, for the approximation is given in table A5-1. For $t \geq 0.5$, the values of $\phi_a(t)$ and $\frac{1}{2}\mathrm{Erf}(t/\sqrt{2})$ were calculated from available tables of the error function and of the exponential function. However, for $t \leq 0.5$, the values calculated from the available tables of the error function and the exponential function were not precise enough. Thus, for $t \leq 0.5$, the values presented in the table were calculated from the power series expansion of the error function and of the exponential function.

From the table, it is apparent that the relative error has a maximum magnitude of 0.01.

$$|\rho| \leq 0.01 \quad . \qquad \text{(A5-3)}$$

t	$\phi_a(t)$	$\frac{1}{2}\mathrm{Erf}(t/\sqrt{2})$	$e(t)$	$\rho(t)$
10^{-4}	3.9500×10^{-5}	3.9894×10^{-5}	-3.94×10^{-7}	-0.00988
2×10^{-4}	7.8999×10^{-5}	7.9788×10^{-5}	-7.89×10^{-7}	-0.00989
4×10^{-4}	1.5800×10^{-4}	1.5958×10^{-4}	-1.58×10^{-6}	-0.00990
8×10^{-4}	3.1601×10^{-4}	3.1915×10^{-4}	-3.14×10^{-6}	-0.00984
0.001	3.9502×10^{-4}	3.9894×10^{-4}	-3.92×10^{-6}	-0.00983
0.002	7.9007×10^{-4}	7.9788×10^{-4}	-7.81×10^{-6}	-0.00979
0.004	1.5804×10^{-3}	1.5958×10^{-3}	-1.54×10^{-5}	-0.00965
0.008	3.1615×10^{-3}	3.1915×10^{-3}	-3.00×10^{-5}	-0.00940
0.01	3.9523×10^{-3}	3.9893×10^{-3}	-3.70×10^{-5}	-0.00927
0.02	7.9089×10^{-3}	7.9784×10^{-3}	-6.95×10^{-5}	-0.00871
0.04	1.5832×10^{-2}	1.9593×10^{-2}	-1.21×10^{-4}	-0.00758

(continued on next page)

TABLE A5-1

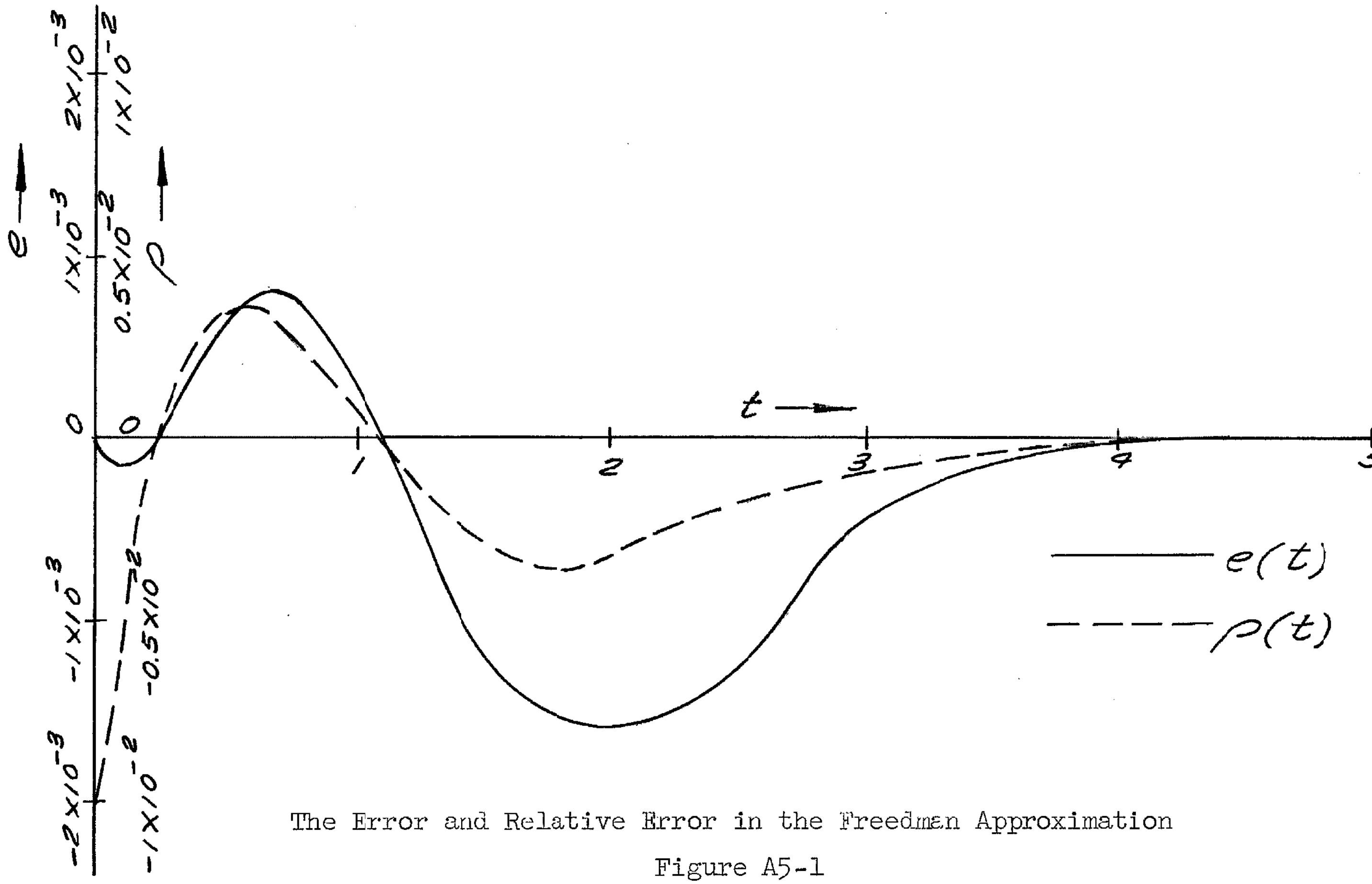
(continued from preceding page)

t	$\phi_a(t)$	$\frac{1}{2}Erf(t/\sqrt{2})$	$e(t)$	$\rho(t)$
0.08	3.1703×10^{-2}	3.1881×10^{-2}	-1.78×10^{-4}	-0.00558
0.1	3.9640×10^{-2}	3.9828×10^{-2}	-1.88×10^{-4}	-0.00472
0.2	7.9180×10^{-2}	7.9257×10^{-2}	-7.70×10^{-5}	-0.00097
0.4	1.5587×10^{-1}	1.5542×10^{-1}	$+4.50 \times 10^{-4}$	+0.00290
0.5	1.9214×10^{-1}	1.9146×10^{-1}	$+6.80 \times 10^{-4}$	+0.00355
0.8	2.8893×10^{-1}	2.882×10^{-1}	$+7.0 \times 10^{-4}$	+0.00243
1.0	3.4168×10^{-1}	3.414×10^{-1}	$+3.0 \times 10^{-4}$	+0.00082
1.5	4.3199×10^{-1}	4.332×10^{-1}	-1.2×10^{-3}	-0.0028
2.0	4.7559×10^{-1}	4.772×10^{-1}	-1.6×10^{-3}	-0.00337
3.0	4.9817×10^{-1}	4.9864×10^{-1}	-4.7×10^{-4}	-0.00094
4.0	4.9994×10^{-1}	4.99967×10^{-1}	-3.00×10^{-5}	-0.00006

TABLE A5-1 (cont.)

The error, e, and the relative error, ρ, are plotted in figure A5-1. The curves for e and ρ show plainly that the points tabulated in table A5-1 are taken at rather coarse intervals. Nevertheless, the curves are reasonably accurate representations of the behavior of e and ρ. In particular, there are not any unnoticed gross departures from the curves in between the computed points. That this is true can be established by examining the first and second derivatives of e, both of which can be written in closed form. Their zeros occur approximately at the places suggested by the curve for e. If there were any marked departures of the actual values of e from those indicated by the curve, there would have to be some zeros not shown by the curve in either the first or second derivative of e. Since there are none, it is reasonable to conclude that the curve is a good replica of e.

In the applications of this thesis, it is the relative error which is important rather than the actual error. Therefore, slightly different values for α and β might be used in order to minimize the maximum magnitude of the relative error. However, to find the optimum values of α and β analytically requires the solution of a fairly complicated transcendental equation involving extensive numerical calculations. Since the maximum magnitude of ρ is sufficiently small for the purposes of this thesis if the above values of α and β are used, these calculations were not carried out.



The Error and Relative Error in the Freedman Approximation

Figure A5-1

REFERENCES

1. Tables of the Bivariate Normal Distribution Function and Related Functions, Applied Mathematics Series # 50, National Bureau of Standards, Washington, D.C., 1959.

APPENDIX VI

SOME APPROXIMATING FUNCTIONS

In this appendix approximating functions are presented for $\sin^{-1}(e^{-x})$ and $\sqrt{\frac{1-e^{-x}}{1+e^{-x}}}$.

A6.0 $\sin^{-1}(e^{-x})$

The function $e^{-x} + \frac{1}{2}e^{-5x}$ is a good approximation to $\sin^{-1}(e^{-x})$ for all positive x. These two functions are tabulated in table A6.0-1. The error and the relative error for the approximation are also tabulated in table A6.0-1. The two functions, $\sin^{-1}(e^{-x})$ and $e^{-x} + \frac{1}{2}e^{-5x}$, are plotted in figure A6.0-1 for values of x between 0 and 2.5. The relative error for the approximation is plotted in figure A6.0-2. From the curve in figure A6.0-2 it is clear that the maximum magnitude of the relative error for this approximation is 0.07.

$$e^{-x} + \tfrac{1}{2}e^{-5x} \cong \sin^{-1}(e^{-x}) \quad \text{for} \quad x \geq 0 \quad , \qquad \text{(A6.0-1)}$$

with a relative error which nowhere exceeds 0.07 in magnitude.

The value $\frac{1}{2}$ for the coefficient and 5 for the exponential multiplier were chosen for computational convenience. The choice of slightly different values for these two constants would reduce the maximum magnitude of the relative error, but the computational complexity would be increased considerably.

x	$\sin^{-1}(e^{-x})$	$e^{-x} + \frac{1}{2}e^{-5x}$	c	ρ
0.025	1.35	1.42	0.070	0.051
0.05	1.26	1.34	0.080	0.063
0.075	1.19	1.27	0.080	0.068
0.10	1.13	1.21	0.080	0.070
0.125	1.080	1.15	0.070	0.056
0.15	1.036	1.097	0.061	0.059
0.2	0.960	1.00	0.040	0.042
0.3	0.835	0.853	0.018	0.022
0.4	0.734	0.738	0.004	0.006
0.5	0.652	0.647	-0.005	-0.007

(continued on next page)

TABLE A6.0-1

(continued from preceding page)

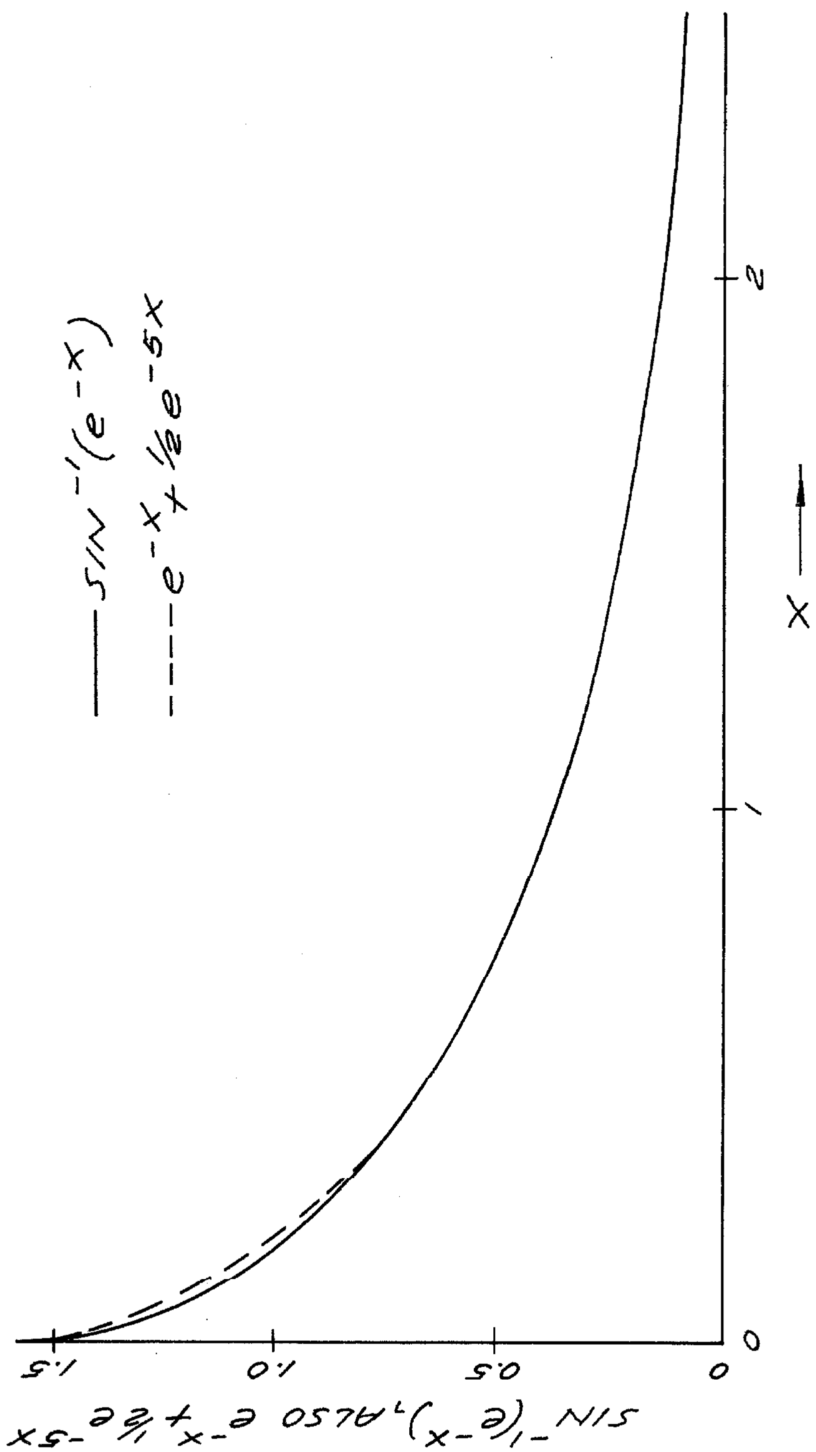
x	$\sin^{-1}(e^{-x})$	$e^{-x} + \frac{1}{2}e^{-5x}$	e	ρ
0.6	0.582	0.574	-0.008	-0.015
0.7	0.521	0.512	-0.009	-0.017
0.8	0.465	0.458	-0.007	-0.015
0.9	0.420	0.413	-0.007	-0.017
1.0	0.377	0.371	-0.006	-0.016
1.1	0.340	0.335	-0.005	-0.015
1.2	0.305	0.302	-0.003	-0.010
1.3	0.275	0.273	-0.002	-0.008
1.4	0.250	0.247	-0.003	-0.012
1.5	0.225	0.223	-0.002	-0.010

TABLE A6.0-1 (cont.)

(continued from preceding page)

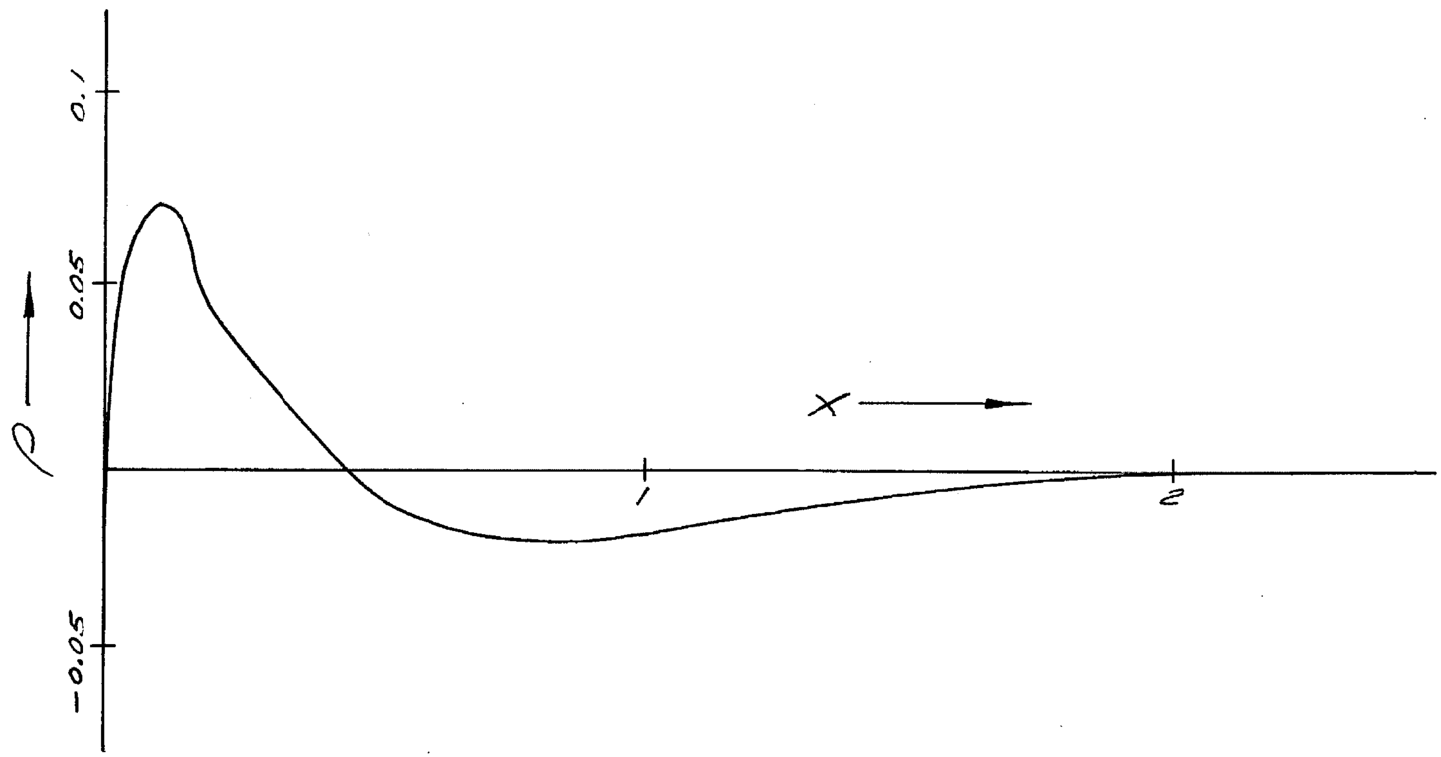
x	$\sin^{-1}(e^{-x})$	$e^{-x} + \frac{1}{2}e^{-5x}$	e	p
1.6	0.203	0.202	-0.001	-0.005
1.7	0.184	0.183	-0.001	-0.005
1.8	0.166	0.165	-0.001	-0.005
2.0	0.136	0.135	-0.001	-0.007
2.1	0.122	0.122	000000	000000
2.2	0.111	0.111	000000	000000
2.3	0.100	0.100	000000	000000
2.4	0.0908	0.0907	-0.001	-0.001
2.5	0.0821	0.0821	000000	000000

TABLE A6.0-1 (cont.)



$\sin^{-1}(e^{-x})$ and $e^{-x} + \frac{1}{2}e^{-5x}$

Figure A6.0-_



Relative Error in $e^{-x} + \frac{1}{2}e^{-5x}$ as an Approximation to $\sin^{-1}(e^{-x})$

Figure A6.0-2

A6.1 $\sqrt{\frac{1-e^{-x}}{1+e^{-x}}}$.

The function $\sqrt{x/2}$ is a good approximation to $\sqrt{\frac{1-e^{-x}}{1+e^{-x}}}$ for $0 \le x \le 1$. These two functions are tabulated in table A6.1-1. The error and relative error for the approximation are also tabulated in table A6.1-1. The two functions, $\sqrt{\frac{1-e^{-x}}{1+e^{-x}}}$ and $\sqrt{x/2}$, are plotted in figure A6.1-1 for values of x between 0 and 1. The relative error for the approximation is plotted in figure A6.1-2. From the curve in figure A6.1-2, it is clear that the maximum magnitude of the relative error for this approximation is 0.04 for $0 \le x \le 1$.

$$\sqrt{x/2} \cong \sqrt{\frac{1-e^{-x}}{1+e^{-x}}} \quad \text{for} \quad 0 \le x \le 1, \tag{A6.1-1}$$

with a relative error which nowhere exceeds 0.04 in magnitude.

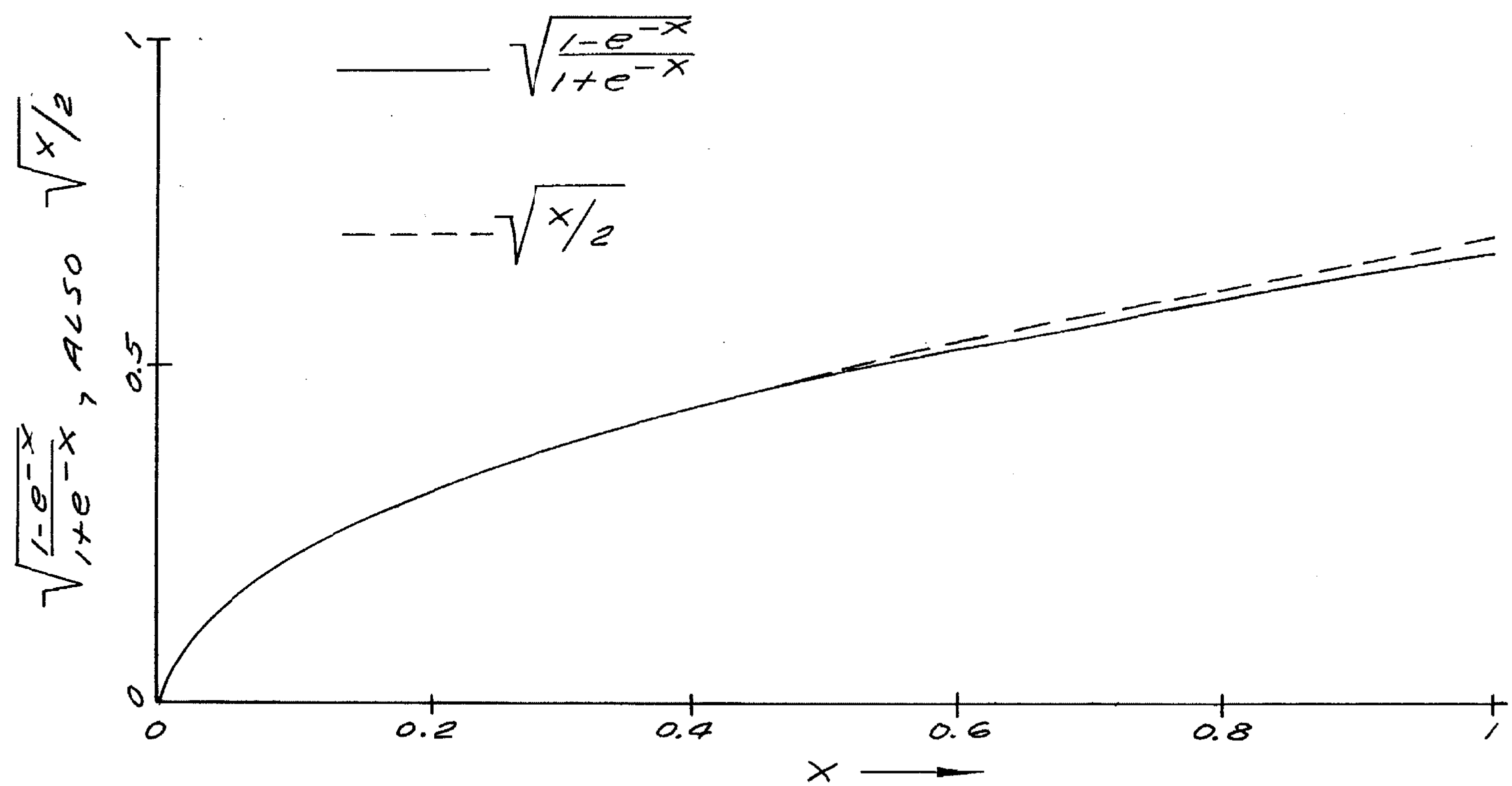
The function $1 - 0.85e^{-x}$ is a good approximation to $\sqrt{\frac{1-e^{-x}}{1+e^{-x}}}$ for $1 \le x$. These two functions are tabulated in table A6.1-2. The error and relative error for the approximation are also tabulated in table A6.1-2. The two functions, $\sqrt{\frac{1-e^{-x}}{1+e^{-x}}}$ and $1 - 0.85e^{-x}$, are plotted in figure A6.1-3 for values of x between 1 and 2.5. The relative error for the approximation is plotted in figure A6.1-4. From the curve in figure A6.1-4 it is clear that the maximum magnitude of the relative error for this approximation is 0.02 for $1 \le x$.

$$1 - 0.85e^{-x} \cong \sqrt{\frac{1-e^{-x}}{1+e^{-x}}} \quad \text{for} \quad 1 \le x, \tag{A6.1-2}$$

with a relative error which nowhere exceeds 0.02 in magnitude.

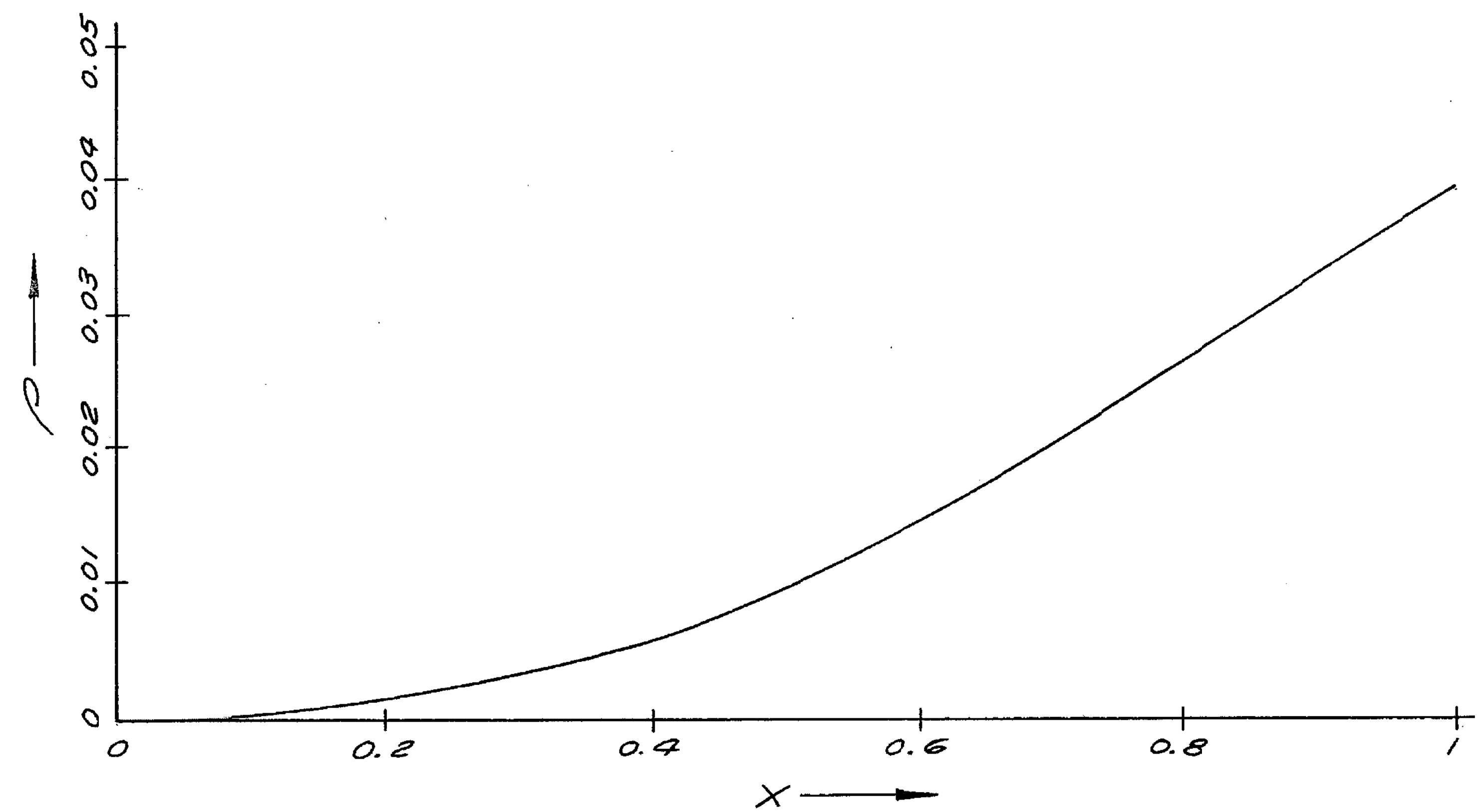
x	$\sqrt{\frac{1-e^{-x}}{1+e^{-x}}}$	$\sqrt{x/2}$	e	ρ
0.1	0.224	0.224	000	000
0.2	0.316	0.316	000	000
0.3	0.396	0.387	0.001	0.003
0.4	0.444	0.447	0.003	0.007
0.5	0.495	0.500	0.005	0.010
0.6	0.539	0.548	0.009	0.017
0.7	0.580	0.591	0.011	0.019
0.8	0.615	0.632	0.017	0.028
0.9	0.650	0.670	0.020	0.031
1.0	0.680	0.707	0.027	0.040

TABLE A6.1-1



$\sqrt{\frac{1-e^{-x}}{1+e^{-x}}}$ and $\sqrt{x/2}$

Figure A6.1-1

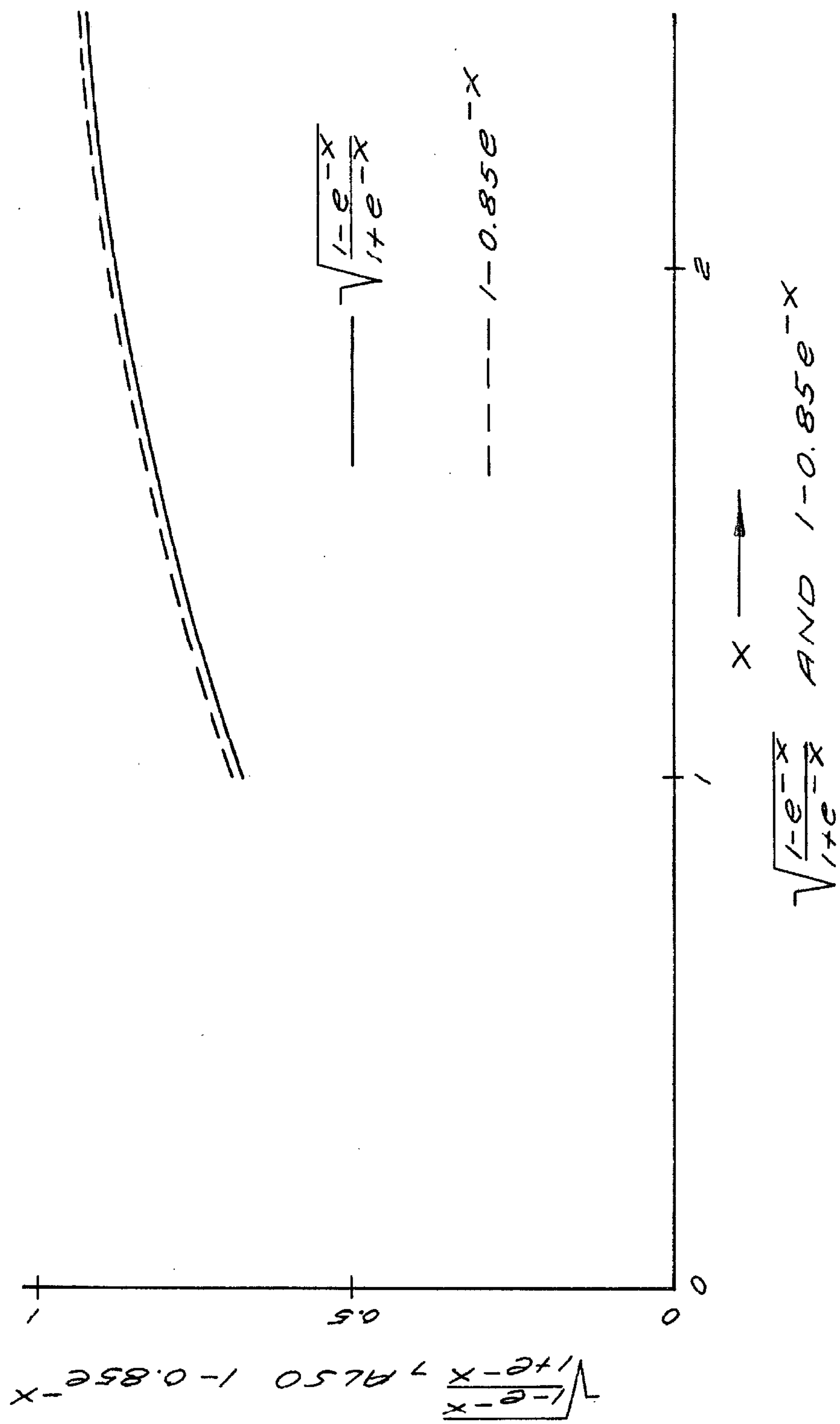


The Relative Error of $\sqrt{x/2}$ as an Approximation to $\sqrt{\frac{1-e^{-x}}{1+e^{-x}}}$

Figure A6.1-2

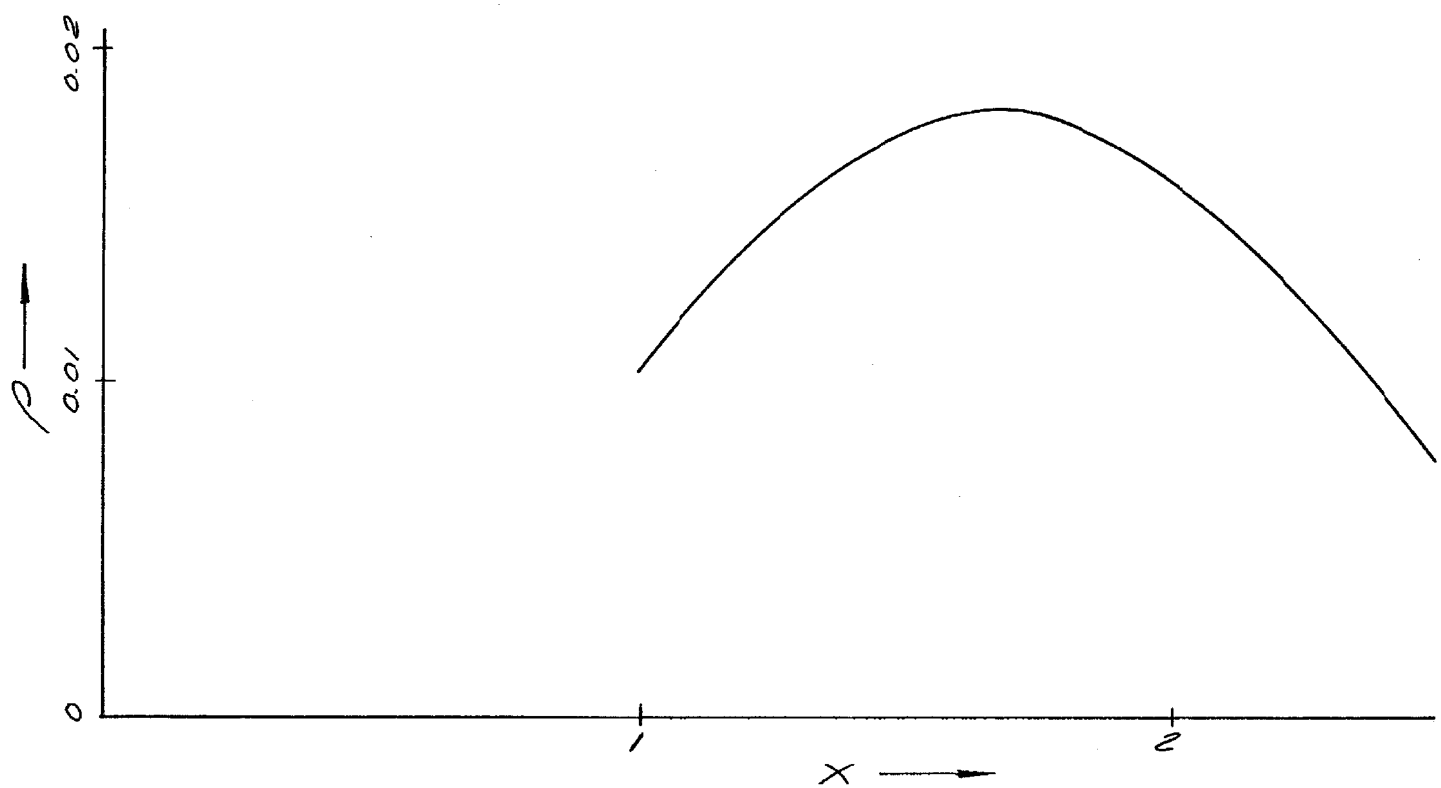
x	$\sqrt{\frac{1-e^{-x}}{1+e^{-x}}}$	$1 - 0.85e^{-x}$	e	ρ
1.0	0.680	0.687	0.007	0.01
1.2	0.732	0.744	0.012	0.016
1.4	0.777	0.790	0.013	0.017
1.6	0.815	0.828	0.013	0.016
1.8	0.845	0.860	0.015	0.018
2.0	0.872	0.885	0.013	0.015
2.2	0.894	0.906	0.012	0.013
2.4	0.914	0.923	0.009	0.010

TABLE A6.1-2



$\sqrt{\frac{1-e^{-x}}{1+e^{-x}}}$
$1-0.85e^{-x}$
0
1
2
0
0.5
1
X
$\sqrt{\frac{1-e^{-x}}{1+e^{-x}}}$ AND $1-0.85e^{-x}$
$\sqrt{\frac{1-e^{-x}}{1+e^{-x}}}$, ALSO $1-0.85e^{-x}$

Figure A6.1-3



Relative Error for $1 - 0.85e^{-x}$ as an Approximation to $\sqrt{\frac{1-e^{-x}}{1+e^{-x}}}$

Figure A6.1-4

From equations A6.1-1 and A6.1-2 it follows that

$$f(x) \cong \sqrt{\frac{1-e^{-x}}{1+e^{-x}}} \quad \text{for all } x, \qquad \text{(A6.1-3)}$$

where

$$f(x) = \begin{cases} \sqrt{x/2} & \text{for } 0 \le x \le 1 \\ 1 - 0.85e^{-x} & \text{for } 1 \le x, \end{cases} \qquad \text{(A6.1-4)}$$

with a relative error which nowhere exceeds 0.04.

APPENDIX VII

SURVEY OF THE LITERATURE RELATED TO POLARITY-COINCIDENCE DETECTION

There is a distinct lack of agreement in the literature on the terminology to be used for various types of correlation devices. Therefore, in order to make the subsequent discussion of the literature self-consistent, the following terminology is introduced.

A cross-correlator is a device which implements the mathematical operation of multiplying two inputs, $x_1(t)$ and $x_2(t)$, and integrating the product for a finite length of time, T. The cross-correlator is called by various names depending on the form of the two inputs.

Correlation Detector

$x_1(t)$ consists of a noise component $n(t)$ and possibly a known signal component $s(t)$. $x_2(t) = k \cdot s(t)$ is proportional to $s(t)$. In this case the device is called a matched-signal cross-correlation detector, shortened to "correlation detector" and denoted by CD.

Correlation Coherency Detector

$x_1(t)$ consists of a noise component $n_1(t)$ and possibly a signal component $s(t)$. $x_2(t)$ consists of a noise component $n_2(t)$ and the same signal component as appears in $x_1(t)$ - the component $s(t)$ - which appears in $x_2(t)$ if and only if it appears in $x_1(t)$. In this case the device is called a cross-correlation coherency detector, shortened to "correlation coherency detector" and denoted by CCD.

Auto-Correlator

$x_1(t)$ and $x_2(t)$ are equal (except possibly for a shift in time). In this case the device is called an "auto-correlator," denoted by AC.

Polarity-Coincidence Detector and Polarity-Coincidence Coherency Detector

If the two inputs are passed through ideal polarity indicators (ideal limiters) before multiplication, new names are applied to the devices. The correlation detector is then called a "polarity-coincidence detector," denoted by PCD. The correlation coherency detector is then called a "polarity-coincidence coherency detector," denoted by PCCD. The auto-correlator is then called a "polarity-coincidence auto-correlator," denoted by PCAC.

In addition, a device called a "band-pass limiter" is often discussed in the literature. A band-pass limiter is an ideal limiter (polarity indicator) followed by an ideal band-pass filter.

In discussing the literature related to polarity-coincidence detection, the terminology introduced above will be used, even though it may not correspond to that found in the literature being discussed.

Ideal Limiting

The earliest work on the effects of two-level digitizing on random processes seems to have been done by J. H. Van Vleck in 1943 (1). He examined the effects of ideal limiting on the auto-correlation function of a Gaussian process. The principal result is the so-called arc-sine law. It relates the output auto-correlation function of the ideal limiter to the input auto-correlation function, if the input is Gaussian, by the extremely simple equation: $R(\tau) = \frac{2}{\pi} \sin^{-1}[r(\tau)]$ where $r(\tau)$ is the auto-correlation function for the input process and $R(\tau)$ is the auto-correlation function for the output process. Using the arc-sine relationship, Van Vleck obtained the output spectra for various inputs with low-pass and band-pass spectra.

Band-Pass Limiters

Van Vleck's work for the case of band-pass limiters was extended by Davenport (2, 1953) and McFadden (3, 1956). Using the characteristic function method of Rice, Davenport derived the output auto-correlation function for an oddsymmetry power-law device with a sinusoidal signal plus narrow-band Gaussian noise input. The resulting expression is a double series in the harmonics of the signal and the signal and noise intermodulation terms. Davenport examined the behavior of the band-pass limiter by considering only the fundamental frequency term in the case of the power-law device with zero exponent. He found for small input signal-to-noise power ratios, $N \ll 1$, that the output signal-to-noise power ratio, N_o, is reduced by the factor $\pi/4$ (1 db.), and for large input signal-to-noise power ratios, $N \gg 1$, that N_o is increased by a factor of 2 (3 db.). McFadden examined the same problem but restricted consideration to $N \ll 1$. By so doing, he was able to obtain the output auto-correlation function as a simple power series, whose first term is the same as Van Vleck's arc-sine term and whose higher order terms serve as correction terms when a sinusoidal signal with amplitude small relative to the noise amplitude is present.

Blachman (4, 1953) examined the same problem by applying Rice's direct method. He found the signal-to-noise power ratio at the output of an arbitrary power-law device in the spectral region of the m^{th} harmonic of the input sinusoid. By imposing the conditions for an odd symmetry zero-order power-law device and setting $m = 1$, he obtained the following expression for N_o for a band-pass limiter:

$$N_o = \frac{\pi N[I_o(\frac{1}{2}N) - I_1(\frac{1}{2}N)]^2}{4e^N - \pi N[I_o(\frac{1}{2}N) - I_1(\frac{1}{2}N)]^2} ,$$

where N is the input signal-to-noise power ratio and N_o is the output signal-to-noise power ratio, valid for all N. I_o is the Bessel function of second kind and zero order and I_1 is the Bessel function of second kind and first order. This expression agrees with Davenport's result when N goes to zero or to infinity.

Jones (5, 1963) extended Davenport's result to the case of two sinusoidal signals in the presence of narrow-band Gaussian noise. Using Davenport's method, he derived the auto-correlation function at the output of an ideal limiter as a triple series with integrals of Bessel functions as coefficients. He then considered the effect of passing the output through an ideal band-pass filter and obtained an expression for the output signal-to-noise power ratio of an ideal band-pass limiter. The principal result is that if one signal is much stronger than both the noise and the other signal, then its output signal-to-noise power ratio is increased by a factor of 2 relative to its input signal-to-noise power ratio (just as it would be if the weak signal were not present), whereas the output signal-to-noise power ratio of the weak signal is decreased by a factor of 2 relative to its input signal-to-noise power ratio. I.e., the strong signal has suppressed the weak signal. However, if both signals are weak relative to the noise, then their output signal-to-noise power ratios are both decreased by a factor of $\pi/4$ relative to their input signal-to-noise power ratios (just as the output signal-to-noise power ratio of a single signal which is weak relative to the noise is decreased by a factor of $\pi/4$). Rubin and Kamen (6, 1963) examined a somewhat different problem. They considered two sinusoidal signals separated in frequency, each centered in a narrow band of Gaussian noise, and derived the signal-to-noise power ratios for both signals at the output of an ideal double band-pass limiter consisting of an ideal limiter followed by an ideal band-pass filter with two pass-bands, each centered at one of the sinusoidal frequencies. The results are similar to those obtained by Jones (5, above) but are modified slightly, because the two sinusoidal signals in this case lie in non-contiguous noise bands.

Cahn (7, 1961) examined the band-pass limiter from a point of view entirely different from that of the investigators cited above. He restricted consideration to the case of small input signal-to-noise ratios and represented the signal by symmetric and anti-symmetric side-bands of a strong carrier (the noise or a strong interfering sinusoidal component). By so doing, he obtained extremely simple expressions for the output signal-to-noise or interference first order statistics. He obtained specific expressions in the case of inputs consisting of two sinusoidal signals, a sinusoidal signal and strong Gaussian noise, a sinusoidal signal and a strong non-Gaussian noise or interference, and a wide-band signal in the presence of a strong noise of arbitrary (wide or narrow) band-width. (The other investigations cited above are restricted to sinusoidal signals in narrow-band Gaussian noise.)

Manasse, Price and Lerner (8, 1958) investigated the effect of a band-pass limiter on signal detectability. They assumed that Davenport's result (2, above) holds for an arbitrary signal (not necessarily sinusoidal) in the presence of narrow-band Gaussian noise so long as the signal band-width is narrow relative to the noise band-width and the input signal-to-noise ratio is small. They then obtained criteria for the band-pass limiter output signal detectability in terms of the output signal energy and output noise power. By comparing this result with the signal detectability when there is no limiting, they obtained an expression for the degradation in signal detectability due to band-pass limiting. They evaluated the degradation numerically for three cases: Rectangular noise spectrum - with a degradation of 1.16 (0.7 db.), Gaussian shaped noise spectrum - with a degradation of 1.118 (0.5 db.) and an optically shaped noise spectrum - with a degradation of 1.059 (0.3 db.). Moreover, they made the interesting discovery that the degradation can be made to go to zero by the local addition of a noise whose spectral density in the original pass-band is small relative to the spectral density of the original noise but whose band-width is so great that its total power is large relative to the original noise power.

Polarity-Coincidence Devices

Faran and Hills (9, 1952) compared the performance of the polarity-coincidence detector (PCD) with that of the ideal correlation detector (CD) (no limiting) and also compared the performance of the polarity-coincidence coherency detector (PCCD) with that of the ideal correlation coherency detector (CCD) (no limiting) for a Gaussian signal in the presence of Gaussian noise, signal and noise having identically shaped RC band-pass spectra, for the case of small input signal-to-noise ratios. For both the PCD and the PCCD the degradation in performance relative to the ideal detector is 0.7 db.

Wolff, Thomas and Williams (10, 1962) compared the performance of the polarity-coincidence coherency detector with that of the correlation coherency detector and also with that of a Neyman-Pearson detector, for a general class of signal and noise inputs, but with the additional assumption that the inputs are sampled at the Nyquist rate and that the inputs are statistically independent. They investigated the degradation in detection probability for a fixed type I error (false detection probability). For a Gaussian signal and noise with small input signal-to-noise ratio, as the sample size goes to infinity, the degradation of the PCCD relative to the CCD is 2.47 (4 db.) and relative to the Neyman-Pearson detector is 5.0 (7 db.). The increase in degradation over that found by Faran and Hills appears to be due primarily to sampling.

Ekre (11, 1963) compared the polarity-coincidence coherency detector with the ideal correlation coherency detector both with sampling and without sampling, for Gaussian signal and noise with identically shaped spectra and small input signal-to-noise ratios. He obtained numerical values for the degradation in output signal-to-noise ratio for three types of input spectra: RC low-pass, RC band-pass and rectangular low-pass. His results show that a considerably higher sampling rate than is required by the Nyquist criterion must be used in order to assure a small degradation. A degradation of 10 db. is common for a sampling rate in the order of the Nyquist rate.

REFERENCES

1. Van Vleck, J. H.: The Spectrum of Clipped Noise, Radio Research Lab., Harvard Univ.; July, 1943.

2. Davenport, W. B.: "Signal-to-Noise Ratios in Band-Pass Limiters," Jour. of Applied Physics, vol. 24, no. 6, pp. 720-727; 1953.

3. McFadden, J. A.: "The Correlation Function of a Sine Wave Plus Noise after Extreme Clipping," IRE Trans. on Information Theory, vol. IT-2, pp. 82-83; June 1956.

4. Blachman, N. M.: "The Output Signal-to-Noise Ratio of a Power-Law Device," Jour. Applied Physics, vol. 24, pp. 783-785; June, 1953.

5. Jones, J. J.: "Hard-Limiting of two Signals in Random Noise," IEEE Trans. on Information Theory, vol. IT-9, pp. 34-42; January, 1963.

6. Rubin, W. L. and S. K. Kamen: "S/N Ratios in a Two-Channel Band-Pass Limiter," Proc. IEEE, vol. 51, pp. 389-390; February, 1963.

7. Cahn, C. R.: "A Note on Signal-to-Noise Ratio in Band-Pass Limiters," IRE Trans. on Information Theory, vol. IT-7, pp. 39-43; January, 1961.

8. Manasse, R., R. Price, and R. M. Lerner: "Loss of Signal Detectability in Band-Pass Limiters," IRE Trans. on Information Theory, IT-4, pp. 34-38; March, 1958.

9. Faran, J. J. and R. Hills: "Correlators for Signal Reception," Technical Memorandum No. 27, Acoustics Research Lab., Harvard Univ., pp. 58-65; September, 1952.

10. Wolff, S. S., J. B. Thomas, and T. R. Williams: "The Polarity-Coincidence Correlator: A Nonparametric Detection Device," IRE Trans. on Information Theory, vol. IT-8, pp. 5-9; January, 1962.

11. Ekre, H.: "Polarity-Coincidence Correlation Detection of a Weak Noise Source," IEEE Trans. on Information Theory, vol. IT-9, pp. 18-23; January, 1963.