Wormholes in Euclidean Quantum Gravity

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Abstract

I present a summary of the developments in wormhole physics. I then investigate the (Euclidean time) decay of axion charge that occurs in a 3-sphere of constant volume when there is a small charge violating operator perturbing the Hamiltonian. I demonstrate that in the limit of large Euclidean time T, axion charge decays like CT^{-1}, where C depends only logarithmically on the coefficient of the charge-violating operator. I apply this result to axionic wormholes, and argue that small wormholes will destabilize large wormholes because of this charge decay. In another model, I demonstrate the existence of wormhole solutions with topology $S^1 \times S^2 \times R$. I interpret these wormholes in terms of topological charge violation on flat R^4.

Contents

Chapter 1: Quantum Gravity

Einstein's theory of gravity, General Relativity, is one of the great successes of theoretical physics of the century. As a classical theory, it has withstood all confrontations with experiment, and it has predicted a fabulously rich variety of new phenomena (black holes) and new ways to view nature (relativistic cosmology) that will both further our understanding of astrophysical phenomena and further allow us to confirm its theoretical structure. And yet from a theoretical point of view, it does have a major failing: It has resisted all attempts to be understood in terms of quantum mechanics, another of the great achievements of twentieth-century physics. Quantum mechanics is not just a physical theory; it is a framework into which any fundamental theory of nature must be put. Gravity has so far resisted this procedure.

What are the obstacles that we face when attempting to quantize gravity? Quantum field theory is the currently reigning paradigm for special relativistic quantum theory. The greatest successes of quantum field theory have been in perturbative calculations, where one expands in a small perturbation about an exactly solvable free field theory. Successful perturbation theories must, however, be renormalizable. This means that one can cancel any ultraviolet divergences in the theory by adding some small finite number of counterterms to the Lagrangian. If one attempts to formulate a perturbative theory of gravity, one soon realizes that it is non-renormalizable.

Non-renormalizable theories are perfectly acceptable as descriptions of low-energy physics — the canonical example being the 4-Fermi theory that describes the weak interactions at low energy. Such theories, gravity included, have an intrinsic mass scale at which the effective low-energy theory must break down. For the weak interactions, this scale is ~100 GeV; in fact, we now know that weak interactions are well described by the Weinberg-Salaam model, which is characterized by spontaneous symmetry breaking at about that scale. For gravity, this scale is known as the Planck mass $M_P = 10^{19}$ GeV. The non-renormalizability of gravity tells us that there must be new physics at the Planck scale.

One approach to quantum gravity is to attempt to understand what the nature of the Planck scale physics must be. The most successful attempts at this have been in string theory. String theory is a quantum mechanical theory in which elementary particles (gravitons included) are considered to be one-dimensional, "stringlike" objects rather than zero-dimensional pointlike objects. Because of the stringy nature of gravitons, it appears as though scattering amplitudes become very weak at high (Planck scale) energies, thus controlling the ultraviolet divergences that made quantum gravity non-renormalizable. It may very well be that string theory, or some improved version thereof, is the correct description of physics at the Planck scale.

Because the Planck energy scale is so high, however (almost 10^{15} times the energy that will be available at the SSC), we may never be able to test the Planck scale predictions of string theory. Physics at the Planck scale may still have consequences for measurable aspects of nature. String theory may be able to predict low-energy physics such as particle spectra or coupling strengths. It may also provide the only possible window into the *very* early universe, when such energies may have been present in the big bang. If "naked singularities" exist in nature, Planck scale physics will almost certainly be necessary to understand their properties. Nonetheless, since string theory or any other theory that purports to explain what happens at the Planck scale will be by definition new physics, it is unlikely that we will be able to determine the nature of the new physics from our current theoretical understanding.

It may prove useful then, to approach quantum gravity from the other end: to attempt to understand the consequences of the fact that quantum mechanics appears to be the correct fundamental description of nature, and the fact that general relativity appears to be the correct low-energy theory of gravity. Our most useful beacon in this attempt will simply be the correspondence principle: The correct theory of nature must include general relativity in the classical, low-energy limit. We can then hope to learn something about the correct theory by asking what features this correspondence requires it to have. While this approach is unlikely to lead to the fundamental "theory of everything," we can at least hope that the conclusions we draw are grounded in known physics. Unfortunately, even at this simple level, we are confronted with difficulties that make our analysis uncertain and our path fraught with peril.

There are several different approaches to studying low-energy quantum gravity. One which I will mention here is the canonical approach. In this approach, we construct a Hamiltonian for the gravitational field and attempt to construct wave functions as we do for the Schrödinger equation. We must immediately acknowledge a serious conceptual problem: Because time itself is a dynamical variable in the theory, the Hamiltonian is actually a constraint. For any state $|\Psi\rangle$, $H|\Psi\rangle = 0$. Time seems to have disappeared from the theory, at least time as we usually understand it in quantum mechanics. In the classical theory, this is not a problem; we can always talk about variables that are "clocks," in that they have definite values that are precisely correlated with the time parameter. In the quantum theory, these variables become probabilistic, and thus the interpretation becomes unclear. This is the "problem of time in quantum gravity."

Another angle of attack is to study quantum mechanics in curved spacetime backgrounds. We do not attempt to quantize the metric itself, except perhaps as low-energy gravitons. We may learn a lot about quantum gravity, however, just from the exercise of quantizing fields on arbitrary backgrounds. Already, qualitatively new effects arise: Most important is the Hawking evaporation of black holes,[1] which may lead to violations of quantum-mechanical coherence.

The Euclidean Path Integral

My own research relates to the study of the path integral for quantum gravity. The path integral is a way of calculating quantum-mechanical amplitudes that reproduces all the results of canonical quantization and has proved useful for a variety of calculations in quantum mechanics and quantum field theory. If $S[\phi]$ is the action functional for a theory, one can calculate the quantum-mechanical partition function as follows:

$$Z = \int \mathcal{D}\phi e^{iS[\phi]},$$

which represents a sum over field configurations weighted by e^{iS}. We expect that in the low-energy limit, one ought to be able to calculate amplitudes in quantum gravity by performing a path integral over the metric as well as over any matter fields in the theory. Even if it proves impossible to perform the full path integral, the path integral formalism lends itself particularly well to making semiclassical approximations. In the classical ($\hbar \to 0$) limit, one finds that the path integral is dominated by small fluctuations about classical solutions. For tunnelling processes, which are classically forbidden, one can use the path integral to make small $\hbar$ approximations. In this case, one makes an analytic continuation $t \to -i\tau$, so that

$$Z_E = \int \mathcal{D}\phi e^{-S_E[\phi]},$$

where S_E is the "Euclidean action" for the theory, calculated using the analytic continuation. Tunnelling processes are then represented by Euclidean classical solutions, and their amplitudes can be calculated by integrating over small fluctuations about these solutions. Even when calculating amplitudes for classically allowed processes, the Euclidean path integral often proves useful because of its superior convergence properties.

We now apply the path integral to gravity simply by extending the integration to include the metric degrees of freedom. To do this fully would require a much more thorough understanding of the space of geometries and the integration measure than we have at present.[1] We will work, instead, in the semiclassical approximation, considering only small fluctuations around classical solutions; indeed, we will always leave the integration over small fluctuations as an unevaluated constant. In this approximation, we can estimate the order of magnitude of quantum-mechanical amplitudes as e^{-S}, where S is the classical Euclidean action of the solution.

In quantum gravity, the analytic continuation from Lorentzian to Euclidean metrics is rather different than in field theory. In many cases, it may be impossible. Nonetheless, it is as much a postulate as an approximation that quantities in Euclidean quantum gravity can be calculated using a Euclidean path integral. Indeed, there is some dispute as to how one should interpret the results of such calculations. The principle I will use is as follows. Our goal is to calculate generalized expectation values for quantum-mechanical operators. Such an expectation value is calculated as shown:

$$\langle M \rangle = \int \mathcal{D}g\mathcal{D}\phi \, M e^{-S[g,\phi]},$$

where M is the operator in question, and the integration is over the metric and matter field degrees of freedom. S is the full Euclidean gravitational action.

The Gravitational Action

Before we proceed to discuss what we can learn from this, I will say a few words about the gravitational action. The action for pure general relativity with a

[1] In two dimensions, much progress is being made on a more complete, non-perturbative evaluation of this path integral, and this may bear some relevance to the physics of wormholes.[2]

cosmological constant is

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{16\pi G}R + \Lambda\right).$$

When we perform our analytic continuation to a Euclidean metric, we obtain

$$S_E = \int \mathrm{d}^4x\sqrt{g}\left(-\frac{1}{16\pi G}R + \Lambda\right).$$

I will henceforth stay in Euclidean space, and will refer to this just as S, the action. Now this action plus any matter contribution is sufficient for most calculations, but Gibbons and Hawking found[3] that one must also include a term evaluated on the boundary,

$$S_{GH} = -\frac{1}{8\pi G}\int \mathrm{d}^3x\sqrt{h}(K - K^0),$$

where h is the determinant of the induced metric on the boundary, and K is the trace of the second fundamental form of the boundary; K^0 is the same trace evaluated with the boundary embedded in flat space. The Gibbons-Hawking boundary term is necessary so that the action is properly additive when sewing together spacetimes.

The Euclidean action has a difficulty in that it is unbounded below. The action can be made arbitrarily negative by multiplying the metric by a sufficiently large, rapidly varying conformal factor. If we integrated over all such metrics, we could in no meaningful sense define a convergent path integral. Hawking has proposed that this divergence be controlled by rotating the conformal modes of the metric to imaginary values; this is, however, a rather ad hoc prescription for obtaining finite values. One should note, though, that the unboundedness of the Euclidean action does *not* indicate the presence of classical solutions with arbitrarily low action.

Following the lead of Coleman[4], I will simply assume that in the semiclassical approximation the path integral is dominated by classical solutions of lowest action, and I will ignore the negative conformal modes.

Chapter 2: An Introduction to Wormhole Physics

We have seen how one can define a Euclidean path integral for quantum gravity, and how one should use this to obtain semiclassical results. We would now like to see if there are classical Euclidean solutions that can lead to interesting consequences for physics. There are such solutions; I will discuss the solutions known as "wormholes" and their implications for physics. Wormholes are solutions to the classical Euclidean equations of motion that connect otherwise disconnected flat spacetimes. One can use the same solutions to connect distant regions of the same flat spacetime. One can also consider "semiwormhole" solutions made by cutting the wormhole solutions in two at the throat, or midpoint, of the wormhole. We can interpret this solution as connecting a flat spacetime to a very small "baby universe," which is the boundary at the wormhole throat. It is this solution which we can immediately interpret as a tunnelling process in which a baby universe either splits off or joins on to a flat space. Both of these processes are classically forbidden — in fact, there are no non-singular Lorentzian metrics that exhibit such topology change.[1]

What is the physical effect of such quantum-mechanical topology change? It was originally thought[6] that this would lead to quantum incoherence: pure states would evolve into mixed states. We might expect this because the quantum state of a disconnected baby universe will be fundamentally unobservable. Any such quantum information would be summed over in a quantum density matrix on the flat space;

[1] One can actually get around this by expanding the class of allowed metrics.[5]

thus a pure initial state would evolve to an impure final density matrix.

The Baby Universe Formalism

It was Coleman's insight that this is not necessarily the case[7]. Coleman starts by considering an effective field theory on distance scales larger than the wormhole scale. The effect of a wormhole of given type "i" will be to add a term to the Lagrangian

$$\mathcal{L}_i(\phi, \ldots)(a_i^\dagger + a_i) \equiv \mathcal{L}_i A_i,$$

where $a_i^\dagger$ and a_i are creation and annihilation operators for baby universes. Only the sum of these operators appears, since creation and annihilation of baby universes are indistinguishable processes; they are both represented by the same Euclidean solution. Because the A_i's all commute with each other, we can simultaneously diagonalize them. In a given basis state $|\alpha\rangle$,

$$A_i|\alpha\rangle = \alpha_i|\alpha\rangle.$$

The alpha parameters are much like the theta angle in QCD: They become effective corrections to the constants of nature. We have replaced the information loss of quantum coherence with an indeterminacy of the fundamental physical constants.

The Bilocal Path Integral

I give here a more detailed presentation of the effect of wormholes[2] that is based directly on the path integral. I will use this argument to derive perhaps the

[2] This discussion closely follows that of Klebanov, Susskind, and Banks[8] with some modifications due to Preskill.[9]

most dramatic result of wormhole physics: the "prediction" that the cosmological constant is zero[4]. We are able to predict this because although the constants of nature are indeterminate, they can be thought of as random variables selected from a probability distribution. This distribution is very strongly (in fact, infinitely) peaked at $\Lambda = 0$.

We start by once again writing the expectation value of an observable

$$\langle M \rangle = \frac{1}{Z} \int \mathcal{D}g \mathcal{D}\phi \, M e^{-S[g,\phi;\lambda]}.$$

Here I have written explicitly the dependence of the action on the fundamental constants, λ, and the integration is over all allowed 4-geometries. Z is the same path integral without the operator M inserted; since it is an unimportant (if infinite) normalization constant, I will henceforth absorb it into $\langle M \rangle$. The geometries we will include will have some number of large, flat, connected 4-geometries ("universes") connected by wormholes. We would like to express this in terms of "ordinary physics"; that is to say in terms of

$$\langle M \rangle_\lambda = \frac{1}{Z_\lambda} \int \mathcal{D}g \mathcal{D}\phi \, M e^{-S[g,\phi;\lambda]},$$

where in this case, the path integral is on only a single large flat spacetime and we do not include any wormhole fluctuations. Z_λ is the partition function on a single universe. In the low-energy limit, the effect of inserting a single wormhole into the path integral will be to insert an operator of the form

$$\int \mathrm{d}x_1 \mathrm{d}x_2 \tfrac{1}{2} C_{ij} \mathcal{O}_i(x_1) \mathcal{O}_j(x_2),$$

where $\mathcal{O}_i(x)$ are the elements of some basis of local gauge-invariant operators at x_i and we sum the repeated indices. Note that we have integrated over the coordinates of the wormhole ends. We assume that the constants C_{ij} are independent of position; they will be when x_1 is distant from x_2. For now we assume that the wormholes are dilute, and therefore we can ignore corrections that we will get when x_1 approaches x_2. When we include an arbitrary number of wormholes, this sum (again in the dilute gas approximation) will exponentiate; we write

$$\langle M \rangle = \int \mathcal{D}g\mathcal{D}\phi \, M \exp\left(\tfrac{1}{2}C_{ij} \int \mathrm{d}x_1 \mathrm{d}x_2 \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\right) e^{-S[g,\phi;\lambda]},$$

where the path integral is now over an arbitrary number of "universes" but not over any wormhole fluctuations. We can now make use of the identity

$$\exp(\tfrac{1}{2}C_{ij}V_iV_j) = \int \prod_k \mathrm{d}\alpha_k \exp\left(-\tfrac{1}{2}(C^{-1})_{ij}\alpha_i\alpha_j\right) \exp(-\alpha_l V_l)$$

to rewrite

$$\langle M \rangle = \int \mathcal{D}g\mathcal{D}\phi \prod_k \mathrm{d}\alpha_k \exp\left(-\tfrac{1}{2}(C^{-1})_{ij}\alpha_i\alpha_j\right) M e^{-S[g,\phi;\lambda]-\alpha_l \int \mathrm{d}^4x \mathcal{O}_l(x)},$$

where again the path integral is over multiple disconnected universes. Since $\int \mathrm{d}^4x \mathcal{O}_l(x)$ is just one of the possible terms in the action, we see that it just shifts the constants λ in the exponent. We can rewrite the previous expression somewhat elliptically as

$$\langle M \rangle = \int \mathrm{d}\alpha \int \mathcal{D}g\mathcal{D}\phi \, \exp\left(-\tfrac{1}{2}\alpha^T C^{-1}\alpha\right) M e^{-S[g,\phi;\lambda+\alpha]}.$$

Finally, we note that operators M observable in one universe depend only on values of the fields in that universe. The integrations over the other universes are therefore independent of each other, and will exponentiate to give an expression in terms of "ordinary" path integrals over a single universe,

$$\begin{aligned}\langle M\rangle &= \int d\alpha \exp\left(-\tfrac{1}{2}\alpha^T C^{-1}\alpha\right) \exp\left[\int \mathcal{D}g'\mathcal{D}\phi' e^{-S[g',\phi';\lambda+\alpha]}\right] \int \mathcal{D}g\mathcal{D}\phi\, M e^{-S[g,\phi;\lambda+\alpha]} \\ &\equiv \int d\alpha P(\alpha) Q(\alpha) Z_{\lambda+\alpha}\langle M\rangle_{\lambda+\alpha}.\end{aligned}$$

$P(\alpha)$ is defined to be the Gaussian $\exp\left(-\frac{1}{2}\alpha^T C^{-1}\alpha\right)$; $Q(\alpha) = e^{Z_{\lambda+\alpha}}$.

This result says that the expectation values of operators in the "Euclidean path integral for everything" are given by an integral over a set of parameters α with a given weight of expectation values of the operators calculated "normally," but with coupling constants shifted by the α-parameters. Put more simply, the constants of nature have become random variables selected from a probability distribution that is calculable, at least in principle. This randomness is not exactly something we can measure: any and all experiments that ever have been or ever will be performed can be collected into a single operator M on our universe. The coupling constants as measured by these experiments will all be the same. The probability distribution for coupling constants should instead be thought of as being a distribution of universes, or of initial conditions for our universe. All we can do is to measure the value of α in our universe.

The Constants of Nature

Since we cannot measure the α probability distribution, even in principle, how can it affect physics? We might hope that the probability distribution will be peaked at a particular value for α. If so, we expect that whatever coupling constants we measure in our universe will very likely lie at the selected value. We can then assess our theory by calculating this probability distribution and comparing it with experimental data. We *can* calculate this distribution, but our results will depend even more strongly than before on the details of Euclidean quantum gravity. Nonetheless, we can at least find out what this theory leads us to conclude.

We will find that the probability distribution is relatively unaffected by the constants C_{ij}, so I will not attempt to calculate them. I *will* calculate what I called $Z_{\lambda+\alpha}$:

$$Z_{\lambda+\alpha} = \int \mathcal{D}g\mathcal{D}\phi\, e^{-S[g,\phi;\lambda+\alpha]},$$

where the path integral is over a single universe.

We assume that this path integral is dominated by large, reasonably smooth 4-geometries; if so, we can integrate out all metric and field fluctuations at distance scales much less than the size of the geometry. We will be left with a low-energy effective action for the theory

$$S = \int \mathrm{d}^4x\sqrt{g}\left(\Lambda - \frac{G^{-1}}{16\pi}R + \dots\right),$$

where Λ and G^{-1} are the values of these couplings as shifted by the α's and also as renormalized to the longest distance scales, scales of order the size of the geometry. Since the partition function will be dominated by classical solutions with the lowest action, we need not include light matter fields in the low-energy effective action; they will be at their classical ground state values. Terms depending on higher derivatives of the metric will also be suppressed as we have integrated out all the long wavelength fluctuations.

It remains to solve the Euclidean field equations for this action. They are

$$R_{\mu\nu} = 8\pi G \Lambda g_{\mu\nu}.$$

For a solution of these equations,

$$S = -\Lambda \int \mathrm{d}^4 x \sqrt{g} = -\Lambda V_4,$$

where V_4 is the 4-volume of spacetime. The path integral will therefore be dominated by the classical solution of maximum volume. This solution is a 4-sphere with

$$V_4 = \frac{3}{8G^2\Lambda^2},$$

giving the minimum action

$$S = -\frac{3}{8G^2\Lambda}.$$

So in our approximation,

$$Z_{\lambda+\alpha} = \kappa \exp\left[\frac{3}{8G^2(\alpha)\Lambda(\alpha)}\right],$$

where κ is an unevaluated functional determinant, and I have made explicit the α dependence of Newton's constant and the cosmological constant. The determinant is uninteresting[3] and will be dropped. Our (unnormalized) probability distribution,

$$\exp(-\tfrac{1}{2}C^{-1}\alpha^2) Z_{\lambda+\alpha} e^{Z_{\lambda+\alpha}},$$

clearly has an infinitely strong peak as $G^2\Lambda \rightarrow 0$; our hopes have been fulfilled and we expect that in the observable universe it is overwhelmingly (infinitely!) likely

[3]As long as it is real and positive. There is some dispute[10,11] on this point.

that $G^2\Lambda = 0$. The theory has made a successful "prediction": The cosmological constant is zero.

Consider now the situation as $\Lambda \to 0$. The some probability distribution implies that the constant G^{-1} will typically be at its maximum possible value as a function of the α's. If it could, in fact, become infinite, then the theory would have made an *unsuccessful* prediction: that Newton's constant is zero. In the dilute gas approximation for the wormhole insertions, the shifts in constants are linear in the α's. Since all values of α are integrated over, the distribution will indeed be peaked at $G^{-1} = \infty$. We will see that it is when the α's get large, however, that is exactly when the dilute gas approximation breaks down. It is quite plausible that short-range interactions between the wormhole ends will cause G^{-1} to be bounded above. We hypothesize, then, that this is the case: $G^{-1}(\alpha)$ has a well-defined maximum as the α's run over all possible values. Our probability distribution, then, will be infinitely strongly peaked at the *maximum* value of $G^{-1}(\alpha)$ where $\Lambda(\alpha) = 0^+$.

The Large Wormhole Problem, and its Solution

At this point we should examine one of the underlying assumptions of this analysis: that wormholes exist at a fixed scale and therefore their effects can be incorporated into an effective field theory on longer scales. It is true that the action of a wormhole of size L_w is of order $L_w^2 M_P^2$ — so larger wormholes will be strongly suppressed by e^{-S}. Recall, however, that such effects go into the constants C_{ij} and therefore into the function $P(\alpha)$, which is unimportant in determining the peak of the probability distribution. The important effects come into the function $Q(\alpha)$, that is, from the requirement that G^{-1} is maximized at $\Lambda = 0$. It is this requirement that will determine the density of large wormholes.

In fact, under the assumptions we have made so far, wormholes of any given size will be dense on the background spacetime[12]. After effects of "small wormholes" have been taken into account, the α parameters for large wormholes will still contribute linearly to G^{-1} in the dilute gas approximation. If G^{-1} is to be bounded above, then the dilute gas approximation must break down, and at the maximum of G^{-1}, large wormholes must necessarily be dense. This argument holds for wormholes of any size, and so we have predicted that wormholes will be dense even on macroscopic scales. This is in accord neither with our observations of low-energy physics, nor indeed with the whole framework we have developed for discussing the effects of wormholes.

If we are not to discard the wormhole formalism entirely, we must confront this difficulty head-on. I will examine the "large wormhole problem" more closely in the context of a resolution to this problem proposed by Coleman and Lee[13]. In their proposal there is a further assumption added to the ones we have already made. They assume that the only wormhole configurations one should consider are classical solutions supported by a globally conserved charge. There is some support for this belief: Most or all of the wormhole solutions that have been constructed do have some kind of charge that supports the wormhole throat.[4] Indeed, the

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[4]Some authors have claimed to exhibit wormhole solutions that are not supported by any global charge.[14,15] They usually do this by allowing imaginary values for fields in the theory.

way wormhole solutions are typically constructed is to require specifically that they dominate amplitudes between states of definite global charge.

If wormholes carry some global charge, then the operators they induce in the effective field theory will transform non-trivially under the associated global symmetry. An individual wormhole, therefore, can not induce any terms proportional to the operator **1** (or the operator R, for that matter). This means that there will be no corrections to Λ or G^{-1} in the dilute gas approximation.[5] All corrections will be at least quadratic in α. We can think of such corrections as coming from the effects of pairs of wormhole insertions of opposite charge. Let us write the constant Λ as renormalized by wormholes of sizes L_i as

$$\Lambda = \Lambda_0 - \sum_i b_i |\alpha_i|^2 L_i^{-4}. \tag{2.1}$$

The α's have been adjusted to absorb factors of e^{-S} that would otherwise appear in the C_{ij}'s in $P(\alpha)$. We have written only the contributions that are lowest order in α. We can write corrections to G^{-1} similarly:

$$G^{-1} = G_0^{-1} + \sum_i c_i |\alpha_i|^2 L_i^{-2}.$$

Now the b_i's will, in fact, be positive, as the leading second-order contribution to the ground state energy is negative.[13] We can also write these corrections in terms of the density of wormholes on the background spacetime. Since this density is important for a number of reasons, I will now calculate it somewhat carefully.

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[5]There will, of course, be contributions from ordinary renormalization after we integrate out wormholes. These will still have the same form that I will discuss.

When considering physics at fixed α, we can count the number of wormhole insertions by counting the number of factors of α that appear in amplitudes. In particular, if the index i represents one particular type of wormhole, we count factors of α_i to count insertions of type i wormhole. Let us expand $Z_{\lambda+\alpha}$ in powers of α_i:

$$Z_{\lambda+\alpha} = \sum_N C_N \alpha_i^N .$$

Note that the constants C_N will, of course, depend on the *other* α's, but not on α_i. Then the "mean number of α_i's", that is, the mean number of type i wormholes, is

$$\langle N_i \rangle = \frac{1}{Z_{\lambda+\alpha}} \sum_N N C_N \alpha_i^N = \frac{\alpha_i}{Z_{\lambda+\alpha}} \frac{\partial Z_{\lambda+\alpha}}{\partial \alpha_i}.$$

In our case we estimate

$$Z_{\lambda+\alpha} = \exp\left(\frac{3}{8G^2\Lambda}\right),$$

so

$$\langle N_i \rangle = \frac{\alpha_i}{Z_{\lambda+\alpha}} \frac{\partial Z_{\lambda+\alpha}}{\partial \alpha_i} = -\alpha_i \left[\frac{3(G^{-1})^2}{8\Lambda^2} + \frac{3G^{-1}}{4\Lambda} \frac{\partial G^{-1}}{\partial \alpha_i}\right].$$

Since the 4-volume $V_4 = \frac{3}{8G^2\Lambda^2}$, the density is given by

$$\frac{\langle N_i \rangle}{V_4} = -\alpha_i \frac{\partial \Lambda}{\partial \alpha_i} + 2G\Lambda\alpha_i \frac{\partial G^{-1}}{\partial \alpha_i},$$

and since we will be considering $\Lambda \to 0^+$, we obtain

$$\frac{\langle N_i \rangle}{V_4} = -\alpha_i \frac{\partial \Lambda}{\partial \alpha_i}$$

which, if we use our expansion of Λ, Equation (2.1), is

$$\frac{\langle N_i \rangle}{V_4} = b_i |\alpha_i|^2 L_i^{-4}.$$

We can define a fractional density ν_i to be the fraction of the total volume taken up by wormhole ends of type i. The "end" of a wormhole of scale L_i takes up a volume $\sim L_i^4$, so

$$\nu_i = \frac{\langle N_i \rangle}{V_4} L_i^4 = b_i |\alpha_i|^4.$$

We can now write

$$\Lambda = \Lambda_0 - \sum_i \nu_i L_i^{-4}$$

and

$$G^{-1} = G_0^{-1} + \sum_i \beta_i \nu_i L_i^{-2},$$

where $\beta_i \equiv c_i / b_i$.

In order that G^{-1} not be driven to infinity, there must be some sort of interaction between wormhole ends. We can attempt to account for such an interaction by putting a constraint on the wormhole densities. Clearly, the dilute gas approximation for wormholes of type i will break down when $\nu_i \sim 1$. Let us propose that the interaction has the effect of constraining ν_i to be less than or equal to 1. Then the peak of our distribution for the α's will be where

$$\sum \nu_i L_i^{-4} = \Lambda_0$$

and

$$\sum_i \beta_i \nu_i L_i^{-2} = \text{(a maximum)}.$$

Under the constraint, we can increase G^{-1} the most by making the ν_i's as large as possible for the largest wormholes, since contributions to Λ go as L_i^{-4}, and contributions to G^{-1} go as L_i^{-2}. This, in somewhat more explicit form than given earlier, is the large wormhole problem.

The assumption that wormholes are supported by a conserved global charge, however, will lead us to a resolution of this problem. In the forthcoming discussion we will, on occasion, make further assumptions that are not necessarily justified. This argument, then, should really be thought of as showing that the large wormhole problem is avoidable, not that it is always avoided. The basic concept is very simple: the smallest wormholes remove the charge that supports larger wormholes. In other words, the small wormholes induce charge-violating operators that will destroy the large wormhole solutions. Let us investigate the conditions for small wormholes to destabilize large ones.

We assume that the small wormhole insertions on the large wormhole are uncorrelated, and that the large wormhole is destabilized when the mean square charge fluctuation induced by the small wormholes is equal to the charge on the large one. If L_1 is the size of the large wormhole under consideration, then this wormhole is stable if

$$\sum_{L_i<L_1} \nu_i \left(\frac{L_1}{L_i}\right)^4 q_i^2 \leq q_1^2. \tag{2.2}$$

We now need the relation between the charge of a wormhole and its size. We will use results for wormhole solutions found in a theory of a complex scalar field with a global $U(1)$ invariance. For this case,

$$q_i = \begin{cases} C_1 M_P^2 L_i^2 & \text{for } L_i \ll m^{-1} \\ \\ C_2 M_P^2 L_i/m & \text{for } L_i \gg m^{-1}, \end{cases} \tag{2.3}$$

where m is the mass of the scalar field and $C_{1,2}$ are constants that we will ignore. We make the approximation that one of these two formulas always holds true, depending on whether L_i is less than or greater than m^{-1}. We can then insert (2.3) in (2.2) to find

$$\sum_{L_i<m^{-1}} \nu_i m^2 L_1^2 + \sum_{m^{-1}<L_i<L_1} \nu_i \frac{L_1^2}{L_i^2} \leq 1 \quad \text{for } L_1 > m^{-1} \tag{2.4a}$$

$$\sum_{L_i<L_1} \nu_i \leq 1 \quad \text{for } L_1 < m^{-1}. \tag{2.4b}$$

In Equation (2.4a), all the coefficients of the ν_i's are greater than 1, so using (2.4b) in its place will only weaken the constraint on the ν_i's — any conclusions drawn from Equation (b) will also hold for Equation (a). In addition, whether or not there is a maximum stable wormhole size, we can extend the sum in Equation (b) to be a sum over all wormhole sizes, which leads only to an insignificant error in a constant factor.

We thus derive the constraint

$$\sum_i \nu_i \leq 1. \tag{2.5}$$

We now wish to learn what this constraint implies. Let us first assume that the constants β_i are unity, or at least independent of i.[6] The maximum of G^{-1} will lie on the boundary of the constraint, so

$$\sum_i \nu_i = 1.$$

[6][13] analyzes the extent to which this assumption can be relaxed.

The ν_i's can now be treated as giving a probability distribution for wormhole sizes; we want to maximize $\langle L^{-2}\rangle$, given $\langle L^{-4}\rangle = \Lambda_0$. If we define $x \equiv L^{-2}$, then

$$\langle x\rangle^2 = \langle x^2\rangle - \langle (x - \langle x\rangle)^2\rangle = \Lambda_0 - \mathrm{Var}(x),$$

so we maximize $\langle x\rangle$ by minimizing the variance of x. The variance is minimized when the distribution is concentrated at a particular value. Since $\langle L^{-4}\rangle = \Lambda_0$, then we expect that wormholes will be dense on one scale only: $L_0 \equiv \Lambda_0^{-1/4}$. This is the Coleman-Lee large wormhole suppression mechanism.

One might ask at this point why we did not propose the constraint (2.5) in the first place. After all, it seems perfectly reasonable that interactions between large and small wormholes would result in a requirement that the "total density" of all sizes of wormholes is less than 1. One reason that this is perhaps not as reasonable as it first appears is that it seems difficult to understand in terms of an effective field theory in which the small wormholes are already integrated out. Coleman goes further and states that it makes no sense at all, since putting large wormholes on the background makes *more*, rather than less, space for small wormholes. Even if one does not accept this viewpoint, however, Polchinski has shown[16] that directly applying the excluded volume constraint (2.5) does *not* evade the large wormhole problem. The reason is that the densities ν_i that would appear in such a constraint are "unrenormalized" densities defined on a short distance scale. If one applies such a constraint, he finds that effects at long distance scales are determined by densities which, when properly renormalized both by the effects of wormholes and ordinary field theory, obey the constraint that $\nu_{i,\mathrm{ren}} \le 1$ — from which we obtained the large wormhole problem. It seems, however, that the Coleman and Lee derivation concerns the long-distance variables, and should not fall prey to this pitfall.

We have now derived most of the main results of the wormhole formalism. We would still like to put the Coleman and Lee wormhole suppression on a somewhat firmer effective field theory footing; I will discuss this in the next chapter. After that, I will describe my investigation of the possibility that wormholes of a somewhat different topology contribute to low-energy physics. The reader should now have a background in the ideas that underlie the investigations described in the next two chapters.

Chapter 3: Axion Charge Decay and Wormhole Destabilization

Introduction

I have introduced the physics of wormholes and their effects in the low energy limit of quantum gravity. We have discovered that there exists a potential problem with very large wormholes becoming important, in contradiction with experiment.[12] The proposal of Coleman and Lee that small wormholes destabilize large wormholes[13] seems to provide an acceptable solution to this problem. In particular, it seems to be consistent with an effective field theory understanding of wormhole destabilization. In Reference [17], they explicitly demonstrated this consistency for the case of a massive scalar field. After reviewing their results, I will perform a similar calculation[1] for the case of the axion (a massless particle), showing that the mass gap is not essential for sufficiently rapid charge decay.

The calculation in Reference [17] improved upon the estimates made in [13] by calculating, for a $U(1)$ global charge, the rate of charge decay that is due to the introduction of charge-violating operators. They did this calculation in the theory of a massive complex scalar field. They reached the perhaps surprising conclusion that the charge decay in Euclidean space goes like e^{-2mT}, after a time delay that depends logarithmically on the charge-violating coupling. This means that wormholes much larger than the inverse particle mass will be destabilized because the charge that

[1] This work was published in Reference [18].

would support them decays too quickly. I will not rederive this result, but I will say something about how it was obtained. Coleman and Lee work in a theory with $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$, where

$$\mathcal{L}_0 = \partial^\mu \psi^* \partial_\mu \psi - m^2 \psi^* \psi \tag{3.1}$$

and

$$\mathcal{L}_I = m^2 M^{2-n} 2^{n/2} (\psi^n + \psi^{*n}). \tag{3.2}$$

ψ is a complex scalar field, and M is a mass that parameterizes the strength of the interaction. First, they do an operator derivation of their result, in which they calculate the leading behavior for large time of the quantity

$$e^{-HT} |N\rangle_0,$$

where $|N\rangle_0$ is a state of definite charge, N, and H is the perturbed Hamiltonian. They find that the charge density of the state for large time is given by

$$\rho(T) = \rho(0) e^{-2m(T-T_0)}, \tag{3.3}$$

where

$$T_0 = \frac{2-n}{2mn} \ln \frac{\rho(0)}{M^2 m}. \tag{3.4}$$

They then do an instanton calculation, which leads to the same result for large T, but also predicts that for $T \lesssim T_0$, the charge density is approximately constant before beginning its exponential decay. Finally, they add gravity into the instanton calculation; the resulting solution does not have any wormhole throat, as the charge decays away too quickly. I will now investigate the situation for the case of the axion.

After performing an operator-based calculation similar to the one of Coleman and Lee, I discuss the conditions for wormhole destabilization.

Axionic Wormholes

Does rapid decay of charge in Euclidean space also occur for the axion? In the massive case, the rapid decay is due to the fact that under Euclidean time evolution with e^{-HT}, the states of lowest energy rapidly dominate. Such rapid decay for the axion seems unlikely at first glance: The axion is massless, and thus there seems to be no penalty for large axion charges. However, when discussing wormholes, we must consider what happens in a finite volume, and then there is an increasing minimum energy for increasing axion charge.

Let us start with the theory of an axion field, θ, with a decay constant f, and with $\mathcal{L} = \frac{f^2}{2}\partial_\mu\theta\partial^\mu\theta$. We wish to find the minimum energy configuration for a given charge in a finite volume V (for simplicity, just a 3-sphere). The charge is given by $Q = \int d^3x\ f^2\dot{\theta}$, and the energy is given by $E = \int d^3x\ \frac{f^2}{2}\left(\dot{\theta}^2 + (\vec{\nabla}\theta)^2\right)$. Since any spatial variation of θ increases E without affecting Q, the minimum energy configuration for a given charge will be spatially constant. Replacing integrals with factors of the volume, we find that $Q = Vf^2\dot{\theta}$, and $E = V\frac{f^2}{2}\dot{\theta}^2$, so $E = \frac{Q^2}{2Vf^2}$. This is a very different relation than that for the massive case (there $E = mQ$), but it may lead to similar decay. Ignoring details of the interaction, the decay rate can be roughly approximated by assuming that the rate for decay is given by the difference in energy between states of charge Q and states of charge $Q-1$. That is to say, we expect $\frac{1}{L_{\text{decay}}} \sim \frac{\partial E}{\partial Q} = \frac{Q}{Vf^2}$. This decay rate will be sufficient to destabilize axionic wormholes, if in order of magnitude, $L_{\text{decay}} \ll L_w$. For the axionic wormhole, $L_w^2 \sim \frac{Q}{M_{\text{Pl}} f}$, so $L_{\text{decay}}^2 \ll L_w^2$ yields $\frac{f^2}{M_{\text{Pl}}^2} \ll 1$ as the condition for axionic wormholes to be destabilized. This conclusion seems reassuring: Axionic wormholes will be destabilized for reasonable values of the decay constant. I will investigate this further, however, to see what the effect of the interaction is and just how this decay takes place.

Operator Analysis

Let us start with a theory with $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$, where $\mathcal{L}_0 = \frac{f^2}{2}\partial_\mu\theta\partial^\mu\theta$, and $\mathcal{L}_I = \lambda f^n \cos(n\theta)$. $\mathcal{L}_I$ is the effective interaction that would result from integrating out the effects of charge n wormholes in the theory. We want to calculate the Euclidean time evolution

$$e^{-HT}|N\rangle_0 = \sum_s e^{-E_s T}|s\rangle\langle s|N\rangle_0, \tag{3.5}$$

where $|N\rangle_0$ is the eigenstate of the free Hamiltonian with charge N, and the sum is over eigenstates of the exact Hamiltonian. In perturbation theory, calculating the exact eigenstate for the "corresponding" free eigenstate with an adiabatic method yields

$$|s\rangle = \lim_{\epsilon \to 0^+} T \exp\Big[-i\int_{-\infty}^{0} dt e^{\epsilon t} H_I(t)\Big]|s\rangle_0, \tag{3.6}$$

where $H_I(t)$ is the interaction Hamiltonian in the interaction picture.

Working in the large T limit, we need only consider the lowest-energy states of a given charge at a given order in perturbation theory. This means that only the constant mode of the field is relevant, so $H_0 = \frac{1}{2}Vf^2\dot{\theta}^2$, $H_I = -V\lambda f^n \cos(n\theta)$.

Notice that this is just ordinary quantum mechanics on a circle, where H_0 is the Hamiltonian for a free particle on the circle. The "momentum" of the particle is the axion charge, and is quantized (because we are on a circle). To calculate ${}_0\langle N|s\rangle$, we first notice that $\cos(n\theta) = \frac{1}{2}\left(e^{in\theta} + e^{-in\theta}\right) = \frac{1}{2}(b^{\dagger n} + b^n)$, where $b^\dagger = e^{i\theta}$ is a raising operator for Q, the charge operator. Given any function of Q, $b^\dagger f(Q) = f(Q-1)b^\dagger$. Using this commutation relation, we can explicitly evaluate the time-ordered exponential to any given order in perturbation theory. Noting that only the creation operators in H_I will contribute to the matrix element, we find that

$$H_I(t) = e^{iH_0 t} H_I e^{-iH_0 t} = -\frac{V\lambda f^n}{2} e^{i\frac{2Qn-n^2}{2Vf^2}t} b^{\dagger n}. \tag{3.7}$$

Evaluating the order r contribution to ${}_0\langle N|s\rangle$, we find that

$${}_0\langle N|N-nr\rangle = \left(\frac{V^2 f^{n+2}\lambda}{n^2}\right)^r \frac{1}{r!}\frac{\Gamma\left(\frac{2N}{n}+1-r\right)}{\Gamma\left(\frac{2N}{n}+1\right)}. \tag{3.8}$$

Using Stirling's approximation for the factorial and the gamma functions, setting $x = \frac{N-nr}{N}$, and defining $e^{NF} = e^{-E_{N-nr}T}\langle N-nr|N\rangle_0$, we find

$$\begin{aligned} F = {} & (1-x)\ln V^2 f^{2+n}\lambda - \frac{(1+n)(1-x)}{n}\ln N + \frac{(1-x)(1-2n)}{n}\ln n \\ & + \frac{1+x}{n}\ln(1+x) - (1-x)\ln(1-x) + 2(1-x) - \frac{2}{n}\ln 2 - \frac{Nx^2}{2Vf^2}T. \end{aligned} \tag{3.9}$$

To find the "charge" of $e^{-HT}|N\rangle_0$, one finds the value of x that maximizes F; for large T this should dominate the sum. Ignoring contributions down by $1/T$, we find

$$x = \frac{Vf^2}{NT}\left[-\ln V^2 f^{n+2}\lambda + \frac{(1+n)}{n}\ln N - \frac{(1-2n)}{n}\ln n - 1 + \frac{1}{n}\right]. \tag{3.10}$$

Or in other words, the charge for large T is:

$$\frac{Q(T)}{N} = \frac{Vf^2}{NT} \ln\left[\frac{N^{(1+n)/n} n^{(1-2n)/n} e^{1/n-1}}{V^2 \lambda f^{n+2}}\right] \equiv k\frac{Vf^2}{NT}. \qquad (3.11)$$

We now see that there are both similarities and differences between the case of axion charge and that of $U(1)$ charge in the massive case. In the case of axion charge, as in the massive case, the coupling constant appears only logarithmically, and so has relatively little effect. However, in the case of axion charge, the decay is like T^{-1} rather than e^{-T}. Charge decay in the axion case is therefore much slower, and this may significantly affect our conclusions about whether the charge decay is sufficient to destabilize large wormholes. In addition, the coupling constant affects the overall decay rate, not just the "delay time" for an exponential decay to take effect.

Wormhole Destabilization

Does the axion charge decay succeed in destabilizing large wormholes? In the semiclassical domain, this question can presumably be answered with certainty only by finding out whether the Euclidean equations of motion with a charge-violating interaction still support a wormhole solution. Nonetheless, if we adopt a reasonable criterion for destabilization, I can estimate what ranges of the parameters will result in the desired solution to the large wormhole problem.

Assume that a wormhole is destabilized when the charge is reduced to $\frac{1}{10}$ of its initial value in $\frac{1}{10}$ the radius of the wormhole. If this time satisfies the "large T" condition, then we can calculate whether or not large wormholes are destabilized for particular choices of the parameters. (Note that the large T approximation should be good whenever the charge is a small fraction of the initial charge, which is exactly our condition.) The axionic wormhole charge-size relation is[19–21]:

$$L_w^2 = \frac{Q}{4\sqrt{3}\pi^2 M_{Pl} f}. \tag{3.12}$$

This relation simplifies the condition for wormhole destabilization, which becomes:

$$29k < \frac{M_{Pl}}{f} \tag{3.13}$$

(where k is the logarithm in Equation 3.11).

To simplify this relation still further, we can use the axionic wormhole charge-size relation (3.12) for *small* wormholes of charge $Q = n$. (When these wormholes are integrated out, they give rise to the charge-violating interaction.) Since the coupling λ has mass dimension $4 - n$, $\lambda = \tilde{\lambda} M^{4-n}$, where $\tilde{\lambda}$ is a dimensionless constant and M is the inverse width of the small wormhole throat. Note that M is related to n (or, Q) by Equation 3.12. To discover whether the condition 3.13 will be satisfied, let us look at large wormholes ten times the size of the small wormholes, which are in turn ten times the Planck length in radius. For $f = M_{Pl}$, $n = 6800$, and the condition is $-29(\ln(\tilde{\lambda}) + 450,000) < 1$, which will be satisfied for *very* small values of $\tilde{\lambda}$. For the same wormhole sizes, the smallest n can be is 1. This corresponds to $f = \frac{M_{Pl}}{6800}$ (see Equation 3.12), which gives $260 - 29\ln(\tilde{\lambda}) < 6800$; this inequality will also be satisfied for reasonable values of $\tilde{\lambda}$. Note also that when n and f are fixed, $k \sim (\frac{2}{n} - 4)\ln L + C$, and so if wormholes of a given size are destabilized, wormholes of all larger sizes are also destabilized. (I should point out that when n is small, the parameter in which we expand, $\frac{\lambda f^{n+2}V^2}{n^2}$, will not necessarily be small, but then we should not be surprised if the charge decays rapidly.)

Conclusion

I have demonstrated that for a large class of axion theories, small wormholes seem to destabilize all wormholes of a larger size successfully, assuming a sensible criterion for destabilization. Note that there was no real guarantee that this would work (although an order-of-magnitude calculation suggests that it will). Physics in systems with a mass gap often has large, qualitative differences from physics in systems with massless particles. The Coleman and Lee mechanism may have seemed to depend on the mass gap for charged states, and yet a minor modification of their scheme makes it applicable, almost step by step, to the massless axion. In some sense, this is because in a finite volume there is an effective "mass gap" for creation of axion charge. For almost any form of conserved charge, there will be an energy cost for increasing that charge in a finite volume. One would expect that an analysis similar to the one in this paper would show that any form of wormhole supported by such a charge would be destabilized by weak charge-violating operators. It therefore seems reasonable that the Coleman and Lee mechanism for evading the large wormhole problem would apply to any type of wormhole supported by a conserved global charge.

Chapter 4: $S^1 \times S^2$ Wormholes

Introduction

One of the factors driving the development of the wormhole formalism has been the existence of classical wormhole solutions to the Euclidean equations of motion for gravity and matter. A wormhole solution is typically defined as a solution that is asymptotic to two distinct flat spacetimes. A configuration of wormhole topology that is not a solution to the equations of motion is less convincing evidence of topology change than is a classical solution, as the classical solution may give the dominant contribution to some quantum-mechanical amplitude in the semiclassical limit. Since this is our justification for considering classical solutions, it is reasonable to ask to which quantum-mechanical amplitudes a proposed wormhole solution will give the dominant contribution.

One could propose that the dominant contribution is to an amplitude for the creation or annihilation of a baby universe. Since this process is expected to be unobservable for low-energy processes on a background spacetime, one suspects that this is not a measurable amplitude, and therefore not a physically relevant calculation. One can instead require, however, that the wormhole mediate some process that otherwise could not take place in the underlying field theory. In a field theory with some conserved global charge, we expect that charge violation is just such a process.

The first wormhole solutions found by Giddings and Strominger[19] carried flux associated with a three-index antisymmetric tensor field, or axion. These wormholes thus violated a conserved axion charge. Lee showed[20] how to represent this in terms of a massless scalar field dual to the three-index tensor,

$$H_{\mu\nu\lambda} = \epsilon_{\mu\nu\lambda\sigma} \partial^{\sigma} a.$$

The effect of wormholes can then be represented by operators that violate the symmetry $a \rightarrow a + c$. In order to find the solutions for the scalar field, or for the more general case of a complex scalar field with a $U(1)$ global symmetry,[21] one must constrain the initial and final states to be states of definite charge. This can result in certain terms in the equations of motion changing signs, the net result being that the solutions for the three-index tensor theory are identical to the solutions for the dual scalar field theory, which would not be the case if the equations of motion were applied naively.

Wormhole solutions have also been found in 3-dimensional[1] electromagnetism by Hosoya and Ogura;[22] these solutions carry magnetic flux down the wormhole throat. These solutions are really the direct 3-dimensional analog of the 4-dimensional Giddings-Strominger axionic wormholes. In both cases, the charge that supports the wormhole throat is topologically conserved. This means that the current conservation equation is an identity when expressed in terms of the gauge potential. In three dimensions, the magnetic flux current is

$$j^{\lambda} = \epsilon^{\lambda\mu\nu} F_{\mu\nu}.$$

[1] I will write "n-dimensional" to refer to n-dimensional Euclidean spacetime, with $n-1$ space dimensions and one Euclidean time dimension.

The flux conservation equation, $\partial_\mu j^\mu = 0$, is an identity when F is written in terms of the gauge potential: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

The effects of these wormholes[23] will be similar to the effects of any finite-action monopole solutions that may exist in the theory. In a three-dimensional theory, a monopole solution can be thought of as an instanton that mediates processes that violate magnetic flux. We can express the effect of such flux violation in terms of a scalar field dual to F, just as in the four-dimensional case, we express the effects of axion charge violation in terms of a scalar field dual to H.

In this chapter, we[2] attempt to generalize the magnetic wormhole to four-dimensional electromagnetism. In four dimensions, we still have magnetic flux conservation, in that magnetic flux lines cannot end. (On a spatial slice, $\nabla \cdot \mathbf{B} = 0$.) Loops of magnetic flux can, however, shrink to nothing. We will avoid this in our wormhole solutions by giving the wormholes the topology $S^2 \times S^1 \times R$, so that magnetic flux on the two-sphere can wind around the circle. We will also put a topologically conserved charge on the S^1, namely, the winding number for a periodic scalar field.

We will then discuss the effects of such wormholes. We believe that while insertions of the usual $S^3 \times R$ wormholes appear as pointlike operators at low energy, the effects of $S^1 \times S^2 \times R$ wormholes will appear as looplike operators. We will discuss the consequences of this.

Solutions

We construct our wormholes in a theory that includes the electromagnetic field

[2]This chapter is part of a forthcoming collaboration with John Preskill.

and a massless periodic scalar field (which we will call the axion) coupled to gravity. We want to construct a Euclidean solution with topology $S^1 \times S^2 \times R$. The periodic scalar has a topological charge associated with it, the winding number on the S^1. For the electromagnetic field, magnetic flux on the two-sphere is conserved; this conservation law is topological both in the sense that the current conservation law is an identity, and in the sense that the flux is a topological invariant of the two-sphere if charged fields are added to the theory.[3]

The Euclidean action for this theory is given by

$$S = \int \mathrm{d}^4x\sqrt{g}\left[-\frac{1}{16\pi G}R + \frac{v^2}{2}g^{\mu\nu}\partial_\mu\Theta\partial_\nu\Theta + \frac{1}{4e^2}g^{\mu\nu}g^{\lambda\sigma}F_{\mu\lambda}F_{\nu\sigma}\right].$$

Here v is the axion symmetry-breaking scale; Θ is a periodic scalar field with period 2π. Now the simplest possible ansatz for a wormhole solution with the desired features will have a Euclidean Kantowski-Sachs geometry:[24] each spatial slice will be homogeneous and characterized by the radius of the two-sphere, the radius of the circle, and the topological charges associated with each. The metric for this is

$$\mathrm{d}s^2 = N^2(\tau)\mathrm{d}\tau^2 + a^2(\tau)\mathrm{d}l^2 + b^2(\tau)\mathrm{d}\Omega^2,$$

where l is a periodic coordinate with period 1, and $\mathrm{d}\Omega^2$ is the solid angle element on S^2, $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$. By reparameterizing τ, we can set the lapse function $N^2(\tau)$ to any strictly positive function; for the time being we set it to unity. The quantities a and b are the radii of the circle and the two-sphere, respectively.

[3] This results in the Dirac quantization condition for the magnetic monopole.

We want to solve our field equations for this ansatz. The field equation for Θ is

$$\partial_\mu[\sqrt{g}g^{\mu\nu}\partial_\nu\Theta] = 0.$$

We impose the restriction that Θ is a function of l and τ only, and that derivatives of Θ are functions of τ only:

$$\Theta = T_l(\tau)l + T_0(\tau).$$

Since Θ must have an integral winding number on the circle, it must satisfy the boundary conditions

$$\Theta(l,\tau) = \Theta(l+1,\tau) - 2\pi n,$$

and thus

$$\Theta = 2\pi nl + T_0(\tau).$$

We are considering the case $n \neq 0$, and in this case, $T_0(\tau)$ must be constant to avoid off-diagonal terms in the energy-momentum tensor. We eliminate constant T_0 by shifting Θ, so

$$\Theta = 2\pi nl.$$

For the electromagnetic field there is a field equation and a Bianchi identity:

$$\partial_\mu[\sqrt{g}F^{\mu\nu}] = 0 \quad \text{and} \quad F_{\mu\nu,\rho} + F_{\nu\rho,\mu} + F_{\rho\mu,\nu} = 0.$$

Since we are looking for a purely magnetic solution, $F^{0i} = 0$; therefore,

$$\partial_\mu[\sqrt{g}F^{\mu\nu}] = \partial_0[\sqrt{g}F^{0\nu}] = 0.$$

We see that the field equation is automatically satisfied. In fact, we want to have a homogeneous magnetic field on the two-sphere. In the coordinate basis,[4] our ansatz is

$$F_{\theta\phi} = -F_{\phi\theta} = \frac{\Phi(\tau)}{4\pi}\sin\theta.$$

The Bianchi identity gives $\partial_\tau F_{\theta\phi} = 0$, and thus $\Phi(\tau) = \Phi_0$, a constant. Φ_0 is the conserved magnetic flux on the two-sphere.

The conservation laws allowed us to solve the matter field equations quite directly; the only non-trivial equations are Einstein's equations. Einstein's equations are

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}.$$

For our field content, the energy-momentum tensor is

$$T_{\mu\nu} = v^2[\partial_\mu\Theta\partial_\nu\Theta - \tfrac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\Theta\partial_\beta\Theta] + \frac{1}{e^2}[g^{\alpha\beta}F_{\alpha\mu}F_{\beta\nu} - \tfrac{1}{4}g_{\mu\nu}g^{\alpha\beta}g^{\lambda\sigma}F_{\alpha\lambda}F_{\beta\sigma}],$$

where we have included the metric explicitly. When we substitute our ansatz for the metric and the solutions to the matter equations, Einstein's equations reduce to the following three equations:

$$\frac{2\dot{a}\dot{b}}{ab} + \frac{\dot{b}^2}{b^2} - \frac{1}{b^2} = -\frac{Q_1}{a^2} - \frac{Q_2}{b^4}, \tag{4.1a}$$

$$\frac{2\ddot{b}}{b} + \frac{\dot{b}^2}{b^2} - \frac{1}{b^2} = \frac{Q_1}{a^2} - \frac{Q_2}{b^4}, \tag{4.1b}$$

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} = -\frac{Q_1}{a^2} + \frac{Q_2}{b^4}, \tag{4.1c}$$

[4]The coordinate-basis components of F are not simply the electric and magnetic fields, as would be the case in an orthonormal basis.

where we define $(\dot{\ }) \equiv \frac{\mathrm{d}}{\mathrm{d}\tau}(\ )$, and we set $Q_1 = 8\pi G(2\pi^2 n^2 v^2)$ and $Q_2 = 8\pi G\left(\frac{\Phi_0^2}{32\pi^2 e^2}\right)$. Q_1 and Q_2 are always positive. Note that Equation (4.1a) is a constraint equation for the system of second-order differential equations defined by the other two; one can easily verify that it is conserved by Equations (b) and (c). Equations (b) and (c) are redundant; either can be eliminated without loss of generality. We would now like to find a solution to these equations. We cannot actually solve these equations analytically; instead, some numerical results will be given. Before we do that, however, there is still much that can be said about these equations.

We would like find out whether wormhole solutions exist, and indeed, what exactly we mean by a wormhole solution. Unlike the case where the topology is $S^3 \times R$, there will *not* be any asymptotically Euclidean solutions with topology $S^2 \times S^1 \times R$. First, the ansatz itself cannot be asymptotic to R^4, since the topology forbids it. Second, while there are configurations asymptotic to flat $R^3 \times S^1$ (*i.e.,* $a \to$ constant, $b \to \tau +$ constant), a simple argument shows that these cannot be solutions: Since the circle goes to a constant radius, the energy density that is due to the winding of the scalar goes to a non-zero constant, and thus the curvature cannot go to zero as it must for a flat solution. So what *do* we mean by a wormhole? In our case, we will define a wormhole solution to be a solution such that a) there exists a small "throat" where both of the radii attain a minimum value, and b) some distance outside the throat both radii become much larger than they are near the throat, and are "almost" of the form $a =$ constant, $b = \tau +$ constant . We intend to demonstrate the existence of such solutions.

The first thing we must determine is whether a wormhole throat can form. The condition that we are at a wormhole throat (for a particular value of τ) is that $\dot{a} = \dot{b} = 0$ and $\ddot{a}$, $\ddot{b} \geq 0$. Is this consistent with the equations? Setting $\dot{a}$ and $\dot{b}$ to zero, from the constraint Equation (4.1a) we find a relation between a and b at the throat:

$$\frac{Q_1}{a^2} = \frac{1}{b^2} - \frac{Q_2}{b^4}. \tag{4.2}$$

Using this relation and Equation (4.1b) we find that

$$\frac{\ddot{b}}{b} = \frac{Q_1}{a^2}$$

at the throat, so $\ddot{b}$ will always be greater than zero. Finally, using the previous two relations, at the throat,

$$\frac{\ddot{a}}{a} = \frac{3Q_2}{b^4} - \frac{2}{b^2}.$$

We thus require $Q_2/b^2 > 2/3$ so that $\ddot{a}$ is positive at the throat. Note also that Equation (4.2) implicitly requires that $Q_2/b^2 < 1$ at the throat. At the throat, then, Q_2/b^2 is a free parameter, which must satisfy

$$1 > \frac{Q_2}{b^2} > \frac{2}{3},$$

and one can calculate all other quantities from this (and of course, from Q_1 and Q_2).

Before we go on, let us consider the case $Q_1 = 0$. In Euclidean space, the Kantowski-Sachs ansatz we have given is equivalent to static, spherically symmetric spacetime with periodically identified time, where the Kantowski-Sachs Euclidean time becomes the radial coordinate of the spherically symmetric case. For $Q_1 = 0$, the solution to these equations is well known: it is the Euclidean magnetic Reissner-Nordström solution.[25] These solutions, of course, seem nothing like wormholes, but there is, in fact, a solution with a "throat":

$$b = b_0 \equiv \sqrt{Q_2},$$

$$a = a_0 \cosh(\tau / b_0).$$

The constant radius of the two-sphere is equal to the horizon radius of the extreme Reissner-Nordström black hole. This solution is not even close to being asymptotic to a flat background; therefore, we will ignore it and concentrate on $Q_1 > 0$.

In the general case, $Q_1, Q_2 > 0$, we were unable to obtain analytic solutions. We did obtain some results by numerically integrating the system of ordinary differential equations (4.1). We need only use Equations (4.1a) and (b), which give a first-order differential equation for $a(\tau)$ and a second-order differential equation for $b(\tau)$:

$$\frac{\mathrm{d}a}{\mathrm{d}\tau} = \frac{ab}{2\dot{b}}\left(\frac{1}{b^2} - \frac{\dot{b}^2}{b^2} - \frac{Q_1}{a^2} - \frac{Q_2}{b^4}\right) \tag{4.3a}$$

$$\frac{\mathrm{d}^2 b}{\mathrm{d}\tau^2} = \frac{b}{2}\left(\frac{1}{b^2} - \frac{\dot{b}^2}{b^2} + \frac{Q_1}{a^2} - \frac{Q_2}{b^4}\right). \tag{4.3b}$$

We found numerical solutions by performing integrations with initial conditions set to values appropriate for a wormhole throat:

$$b(0) = b_0 \qquad \left(\frac{2}{3} < \frac{Q_2}{b_0^2} < 1\right)$$

$$a(0) = \left(\frac{Q_1 b_0^4}{b_0^2 - Q_2}\right)^{1/2}$$

$$\dot{b}(0) = \epsilon.$$

Note that $\dot{b}(0)$ is set to a small, non-zero value ϵ. This is necessary, because when $\dot{b} = 0$, Equation (4.1a) merely imposes the constraint (4.2) without fixing $\dot{a}$. Setting $\dot{b}$ to ϵ at $\tau = 0$ does set $\dot{a}(0)$ to (nearly) zero with $a(0)$ and $b(0)$ as given. We set ϵ small enough that computer test runs with the opposite sign for ϵ showed no significant difference. In general, Equation (4.3a) will be undefined whenever $\dot{b}$ goes through zero. The integration package seemed to handle this without significant difficulty. This package[5] implements a Runge-Kutta algorithm of order 4(5) with adaptive step sizing. Graphs of some of the results are displayed in Figures 1–3 at the end of this chapter.

What are some of the general features of the solutions that we can see from the numerical results? Note first that we only need calculate solutions for fixed values of Q_1 and Q_2, here set to 1. We can do this because if we have a pair of functions $a(\tau)$ and $b(\tau)$ which is a solution to Equations (4.1) for any particular values of Q_1 and Q_2, there is a corresponding solution with $\tilde{Q}_1 = \lambda_1^2 Q_1$, $\tilde{Q}_2 = \lambda_2^2 Q_2$ given by

$$\tilde{a}(\tau) = \lambda_1 \lambda_2 a(\tau/\lambda_2)$$

$$\tilde{b}(\tau) = \lambda_2 b(\tau/\lambda_2).$$

The qualitative features of the solutions thus depend only on the parameter Q_2/b_0^2.

We performed numerical integrations that started at the wormhole throat with various values for the free parameter Q_2/b_0^2. When Q_2/b_0^2 is very close to 1, we find a wormhole-like solution, where a starts to rise very rapidly for a time, and then levels off, seeming to approach a constant. The S^2 radius b starts out fairly flat, then goes

[5]The `RungeKutta.m` package supplied with Mathematica 1.2 was used.

to a regime in which $\dot{b}$ is nearly 1. If we continue to integrate to much larger values of τ, a will reach a maximum and start decreasing, eventually collapsing to zero, while b increases rapidly to infinity. When Q_2/b_0^2 is not close to 1, this "nearly flat" behavior never begins; instead, a just reaches a maximum and then collapses while b diverges — the only difference is that this happens much sooner, never allowing the solution to reach a nearly flat regime.

Wormhole Insertions and Topological Charge

We would like to understand what relevance these wormhole solutions have for a theory of quantum gravity. In particular, we would like to know what kind of effects this type of wormhole will have on low-energy physics in flat, four-dimensional spacetime. In order to determine this, we need to understand how these wormhole geometries of $S^1 \times S^2 \times R$ topology can attach to flat R^4.

The answer to this question is actually suggested by the solutions themselves. The metric given (in our ansatz) by $a(\tau) =$ constant and $b(\tau) = \tau$ is flat; flat $R^3 \times S^1$ has this metric with $\tau \in [0, \infty)$. If we consider the subset of this space given by $\tau \in [0, \tau_f]$, we have flat $B^3 \times S^1$, where B^3 is the three-dimensional ball. There are $B^3 \times S^1$ subsets of R^4 that consist of a loop in spacetime with a neighborhood around it. The geometry of these subsets very nearly approximates that of the flat $B^3 \times S^1$ described above, at least in the limit where the loop (the S^1) is long and straight on the scale of the ball around it. After excising such a region from the background, there is a boundary left with topology $S^2 \times S^1$, to which we can attach the $S^2 \times S^1$ boundary of the flat $B^3 \times S^1$ we have constructed, or any other geometry that approximates this near the boundary. Our wormhole solutions almost match this geometry in the $\dot{a} \cong 0$, $\dot{b} \cong 1$ regime. It should then be possible the perturb the geometries of the wormhole solution and the background spacetime in such a way that they can be patched together on the $S^1 \times S^2$ boundary. The geometry formed this way is almost a classical solution, and may in fact closely approximate an exact solution to the equations of motion that connects two large, flat 4-geometries with an $S^1 \times S^2$ throat.

The ansatz we have made is inconsistent with such a solution; this is why we needed to resort to the approximate patching procedure. This also means, however, that some features of the solutions we have found may mislead us in determining properties of the actual solution. For example, the solution we have found may have instabilities that merely indicate that there is an exact, stable solution nearby. Presumably, if we patch our wormhole solution to a solution for the fields in a flat background in a region where both solutions are still reasonably accurate, we can avoid such problems.

What happens in the background when a solution of this type is attached? From the point of view of the background spacetime, the wormhole end appears as a small neighborhood around a closed curve $\mathcal{C}$. As we follow the loop around, we find that the scalar field winds n times. If we look at a three-dimensional slice that intersects the loop, we see magnetic flux coming out; or if our three-dimensional slice contains the loop, it changes the magnetic flux, as follows. Consider the magnetic flux lines in the background "time"-slice before and after the slice in which our loop sits. Since the magnetic flux from the loop changes sign between the "before" and "after" slices, we find that the insertion of the wormhole end creates a loop of magnetic flux in the background. Thus the effect of inserting the wormhole end will be similar to the effect of insertions of an "'t Hooft loop" operator.[26] The 't Hooft loop, $B(\mathcal{C})$, is the analog in one extra dimension of the flux creation operator $\phi(\mathbf{x})$ described in [23] and [26]. Its action on a state (*i.e.,* a time slice) is to perform a singular gauge transformation that has a non-zero winding number along a curve that links the loop. In the path integral, an insertion of $B(\mathcal{C})$ means that one integrates over gauge field configurations such that $\mathcal{C}$ is the world line of a Dirac monopole singularity of the gauge field. There is no particular conservation law that forbids the formation of loops of magnetic flux. But recall that we also have axion winding number as we follow the loop around: The wormhole insertion creates a loop of flux with axion winding number. We now show that this "flux winding number" is topologically conserved.

Consider a loop of flux that winds in the manner described above. The axion winding number is given by

$$2\pi n = \oint_{\mathcal{C}} \nabla\Theta \cdot \mathrm{d}\mathbf{l},$$

where the line integral is along the loop. If we multiply this by the flux, we have

$$2\pi n\Phi = \int \mathbf{B}\cdot \mathrm{d}\mathbf{a} \oint_{\mathcal{C}} \nabla\Theta \cdot \mathrm{d}\mathbf{l} = \int \mathrm{d}^3x \mathbf{B}\cdot\nabla\Theta.$$

This is a conserved charge, because $\mathbf{B}\cdot\nabla\Theta = J^0$, where

$$J^\lambda = \epsilon^{\lambda\mu\nu\rho} F_{\mu\nu} \partial_\rho \Theta,$$

and J^λ is an identically conserved current. So "flux winding" as we have defined it is a topologically conserved charge.

Unfortunately, it also happens to be zero! Note that because of flux conservation, $\mathbf{B} \cdot \nabla\Theta = \nabla \cdot (\mathbf{B}\Theta)$, so our charge is equal to the integral of $\mathbf{B}\Theta$ over the two-sphere at infinity. Since there are no monopoles in the theory, this will be zero. It should have been clear from the start that if the field Θ is continuous everywhere, the winding number around any contractible loop, and therefore *any* loop in R^3, is zero. This points the way to making this charge non-trivial: We must allow singularities in Θ. This is perfectly natural if Θ is actually the field of a complex scalar field ϕ; the singularities of Θ are simply zeros of ϕ.

Low Energy Effects

We would now like to understand the nature of the low-energy effects of allowing these wormholes in the theory. As the insertions are qualitatively different from those of traditional S^3 wormholes, we will generalize the bilocal action formalism for looplike wormhole insertions.

We suppose that a wormhole connects a curve $\mathcal{C}_1$ and a curve $\mathcal{C}_2$ in a background 4-geometry. The effects of this wormhole can be represented by

$$\sum_{ab} C_{ab} B_a(\mathcal{C}_1) B_b(\mathcal{C}_2),$$

where $B_a(\mathcal{C})$ is some set of gauge-invariant operators that depend on fields along $\mathcal{C}$. For $\mathcal{C}_1$ distant from $\mathcal{C}_2$, we can absorb any dependence of this insertion on $\mathcal{C}_1$ or $\mathcal{C}_2$ into the B's; then the C_{ab}'s can be treated as constant in the dilute gas approximation. Following identical arguments to the pointlike case, there exists an α parameter for $B_a(\mathcal{C})$, which gives terms of the effective action

$$S = S_0 + \lambda(\alpha) \int \mathcal{D}\mathcal{C}\, B_a(\mathcal{C}),$$

where we have generalized to non-linear α dependence; the contribution to the action is a path integral over loop space.

This contribution looks very different from the contributions we had from traditional wormholes. It almost looks as though we have a contribution to an effective string field theory, rather than to an ordinary local field theory. This does not mean, however, that we cannot interpret this in terms of more prosaic physics.

Consider ordinary quantum electrodynamics, and the low-energy effective field theory that results from integrating out the electron. Instead of using field theory for the electron, however, we use a first-quantized formalism where one integrates over the path of the electron. The Euclidean action for this is $S = S_0 + S_{\text{el}}$, where

$$S_0 = \int \mathrm{d}^3x \, (\frac{1}{4e^2} F^2)$$

$$S_{\text{el}} = m \int \mathrm{d}l - i \int A_\mu \mathrm{d}x^\mu,$$

where the integrals for S_{el} are along the path of the electron. The path integral for the partition function is

$$Z = \int \mathcal{D}\mathcal{C}\, \mathcal{D}A\, e^{-S},$$

where $\mathcal{C}$ is the path of the electron. Let us integrate out electrons, including an arbitrary number of non-interacting electrons on closed paths. This path integral exponentiates, giving

$$Z = \int \mathcal{D}A \, \exp - \left(S_0 - \int \mathcal{D}\mathcal{C} \exp(i \oint_{\mathcal{C}} A_\mu \mathrm{d}x^\mu - m \int_{\mathcal{C}} \mathrm{d}l) \right).$$

This is just the same form as derived for wormhole insertions, because we have just added a term to the action that is a path integral over loops of an operator that depends on values of fields on the loop. Consider just the electromagnetic coupling term. For the electron, this term gives the Wilson loop, $\exp(i \oint A_\mu dx^\mu)$, but for the wormhole, one will get the 't Hooft loop, as we have discussed. This is the analogous operator for a magnetic monopole. While it is not entirely clear how to impose the constraint that the wormhole insertion winds around a zero of the scalar field, it seems that the effects of wormholes will be similar to the effects of integrating out magnetic monopoles.

The effective Lagrangian obtained from integrating out electrons[27] is well known. To lowest order,[6]

$$\delta S = a(F^2)^2 + b(F\tilde{F})^2,$$

where a and b are calculable (but irrelevant for our purpose) constants, and $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. The Maxwell equations derived from this action are

$$\partial_\mu F^{\mu\nu} = 8a\partial_\mu(F^2 F^{\mu\nu}) + 8b\partial_\mu(F\tilde{F}\tilde{F}^{\mu\nu})$$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0.$$

Thus the virtual electrons create an effective source term for $F^{\mu\nu}$; the Bianchi identity, of course, remains unchanged.

One ought to be able to learn the effects of including monopoles by dualizing the above derivation, *i.e.*, letting $F \leftrightarrow \tilde{F}$. This does *not* change the contribution

[6] We do not include interactions with derivatives of F, as we assume that we are at *very* low frequencies but moderate field strengths.

to the action, but it should change the Maxwell equations, giving a source to $\tilde{F}$ instead of F. Deriving the Maxwell equations in the usual way will not give this; it is necessary to vary the action with respect to a dual potential B_μ such that $\tilde{F}_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. This does seem to be the correct thing to do, as we expect the interaction term for the monopole to be $g_m \oint B_\mu \mathrm{d}x^\mu$. Nonetheless, it would be better to avoid the dual potential altogether.

For another viewpoint on the low-energy effects of $S^1 \times S^2$ wormholes, recall that our first interpretation of the wormhole-induced operator was as one that created or destroyed a loop of magnetic flux that encircles the zero of a scalar field. This can be given a more concrete realization in a theory with vortex solutions. Consider the simplest theory with bosonic superconducting cosmic strings.[28] This theory has two independent $U(1)$ gauge fields A_μ and R_μ, and two complex scalar fields σ and ϕ that are minimally coupled to A and R, respectively. The scalar potential is such that ϕ has a vacuum expectation value, but σ does not. Thus the R gauge symmetry is realized in the Higgs phase, and the A symmetry is realized in the Coulomb phase. The scalar potential also has the property that at the core of a Nielsen-Olesen vortex of the R symmetry, σ has a non-zero expectation value. Since σ carries the charge of the unbroken gauge field A_μ, the σ condensate at the core of the string causes it to be a superconductor. Loops of this string carry persistent currents that are characterized by the winding number of the σ field around the loop. There exist static solutions to the field equations of this theory known as "springs"[29] or "vortons,"[30] that consist of a loop of superconducting string prevented from collapsing by a persistent current and the magnetic field that it generates.

If we identify the $F_{\mu\nu}$ of our wormhole solution with the field strength of the R_μ gauge field, and identify the Θ field of our wormhole solution with the phase of σ, then these vortons are carriers of exactly the topological charge we defined. The theories certainly do not match exactly, but in the limit where the radius of the two-sphere boundary of the wormhole is much less than the radius of the core of the string, we might expect that the wormhole solution can successfully patch on to the vorton solution. In this case, the wormhole insertion induces an operator that creates or destroys a vorton. Since the vorton is quantum-mechanically unstable, the charge is not strictly conserved, but if the vorton lifetime is sufficiently long, the wormhole may be the most important contribution to its decay. The wormhole contribution to the theory could then be described in terms of an effective local field that describes the vorton degrees of freedom, even though this hides somewhat the non-local nature of the wormhole insertions.

At this point we can further address the question of which configurations dominate the path integral. We have postulated the existence of solutions which, for a fixed value of the squared charges Q_1 and Q_2, can be patched to the background spacetime along an arbitrarily long curve $\mathcal{C}$ by adjusting the parameter Q_2/b_0^2 at the wormhole throat. In searching for the lowest action classical solution contributing to a given process, however, we cannot fix the parameters at the throat — we can only fix parameters on the background. For example, if we are looking for the leading contribution to a process which carries away a fixed magnetic flux with a fixed scalar winding number, we should include only the lowest action solution for fixed Q_1 and Q_2. We expect that this will have a circular wormhole solution of some fixed length; this is analogous to the stable static vorton solution with similar parameters. We could also imagine actually searching for processes which annihilate a vorton of a given size as well as charge, the dominant contribution to which will be given by a wormhole of the appropriate size. The correct wormhole solution always depends on the amplitude under consideration.

Conclusions

We have constructed wormhole solutions of topology $S^2 \times S^1 \times R$ in a theory with electromagnetic fields and periodic scalar fields. While not exactly realistic, this theory is a simple example of a theory with topologically conserved charges on both the two-sphere and the circle. These wormhole solutions do not fit the paradigm of having $S^3 \times R$ topology and asymptotic flatness in both directions.

Nonetheless, we see that these wormholes *may* be sensibly interpreted in terms of effective 't Hooft loop operators (or monopole loops) on the background spacetime. There are still a number of loose ends, though. For example, it is an unproven hypothesis that the asymptotically flat solutions suggested by this work actually exist. Even if they do, they may not contribute to the Euclidean path integral in the simple form suggested.

Another issue concerns the Coleman-Lee solution to the large wormhole problem.[13] Although these wormholes are indeed supported by a conserved charge, it is not obvious how this charge can be "drained away" by smaller wormholes. This might be easiest to see in terms of a field operator that creates or destroys vortons. But in this case, the global, topological aspects of the conserved charge seem to be lost.

While these wormholes seem to lie on a somewhat looser foundation than the traditional ones, they may have important things to tell us. If they do make an important contribution to the path integral, they further reinforce the supposition that no global symmetries are safe in quantum gravity, not even topological symmetries on subspaces of the background spacetime.

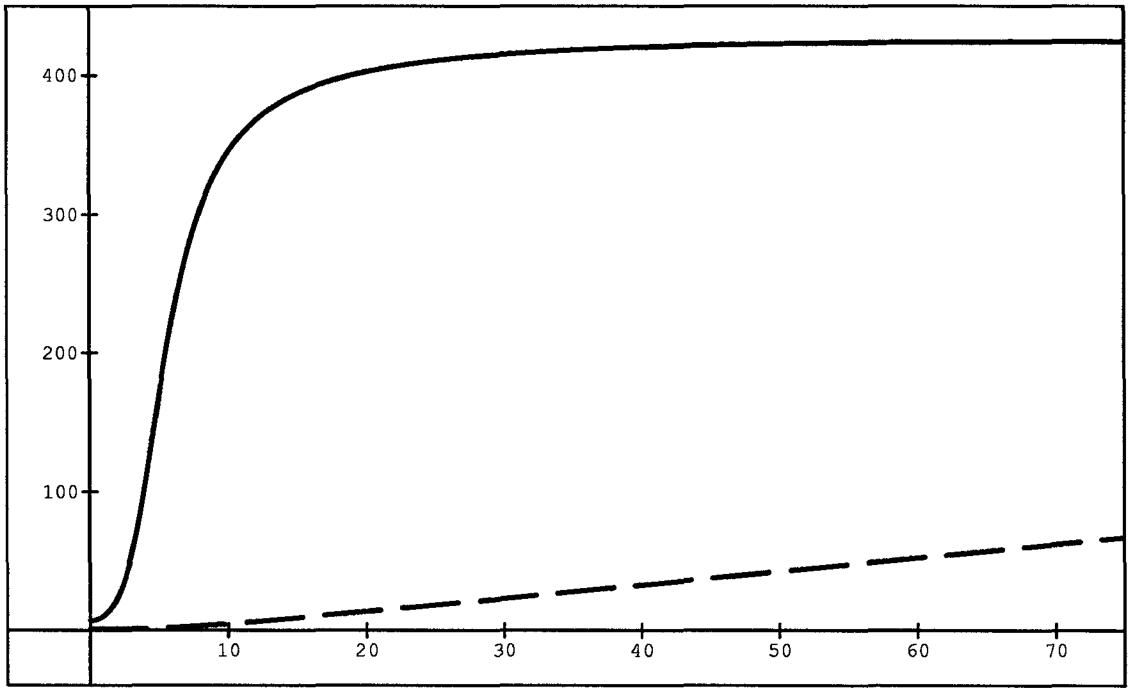

Figure 1: $a(\tau)$ (solid) and $b(\tau)$ (dashed) for $Q_1 = Q_2 = 1$ and $Q_2/b_0^2 = 0.98$.

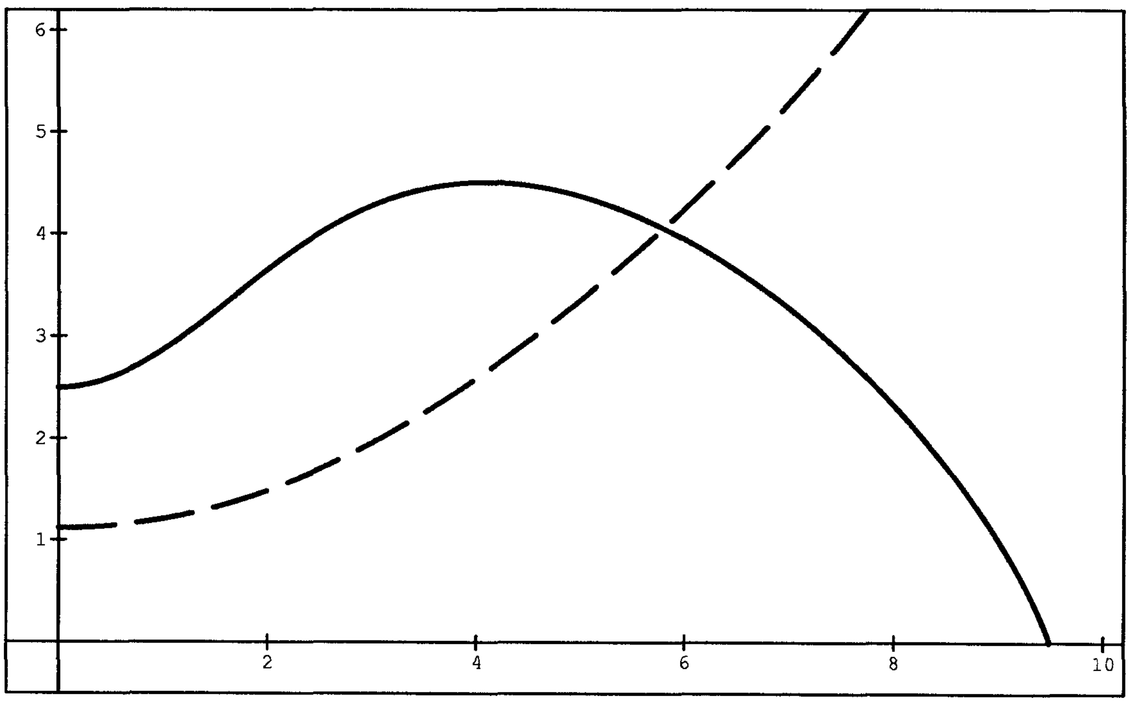

Figure 2: $a(\tau)$ (solid) and $b(\tau)$ (dashed) for $Q_1 = Q_2 = 1$ and $Q_2/b_0^2 = 0.8$. The figure does not show it, but b goes to infinity where a goes to zero.

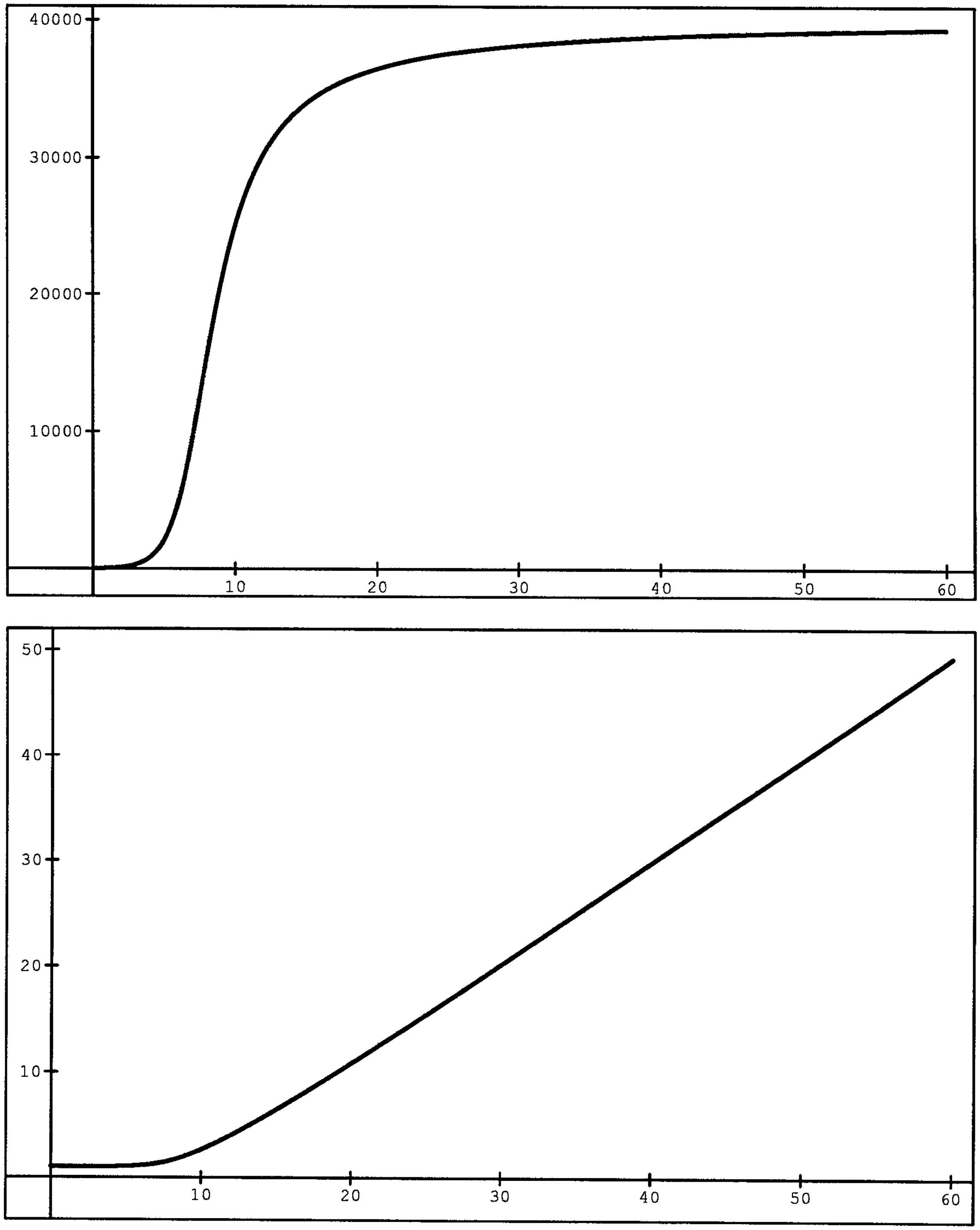

Figure 3: $a(\tau)$ (top) and $b(\tau)$ (bottom) for $Q_1 = Q_2 = 1$ and $Q_2/b_0^2 = 0.999$. The maximum value of a increases dramatically as $Q_2/b_0^2 \to 1$.

Chapter 5: Summary

As physical theories go, the wormhole calculus is a somewhat ambiguous creation. Its fundamental impulses are conservative — one does not attempt to derive new physics, but merely explores the consequences of the physics we have, general relativity and quantum mechanics. To go very far with this input, however, one is frequently forced to make somewhat dubious suppositions or approximations. Nonetheless, the reward for this speculation is rich: We come to the conclusion that the constants of nature are random variables.

Insofar as wormholes represent Planck scale physics, they seem to make no measurable predictions. But, if the results of Reference [4] are correct, there is a strong prediction that the cosmological constant is zero. While it seems unlikely,[9] one may even be able to predict other fundamental constants. If not, though, the randomness inherent in the wormhole formalism may make it impossible to predict anything at all.

There is a some possibility of making direct experimental contact with wormhole physics. The Coleman and Lee large wormhole suppression mechanism predicts that wormholes are dense at a scale set by the bare cosmological constant. If low-energy supersymmetry is a correct theory of nature, then the bare cosmological constant will not be too far above the electroweak symmetry-breaking scale. This means that for interactions at energies at or above this scale, one could no longer consider the wormhole insertions to be pointlike operators. I do not know, however, what their effects would be. If there are weak scale wormholes, one might expect dramatic effects at particle accelerators such as the SSC. In my opinion, however, the current limits on such effects as quantum-mechanical incoherence and baryon-number violation are likely to rule out such accessible wormholes.

Given this, what is the likelihood that experimental physics will help to confirm or refute the theory of wormholes? The most important avenue of approach will be the search for predictions of the constants of nature. Much of the interest in wormhole physics arose with the prediction that the cosmological constant is zero. Indeed, the wormhole theory seems to be the best explanation for this puzzling fact, although alternative attacks on this problem abound.[31] The only other prediction of the wormhole calculus is a failure. Because θ_{QCD} contributes to physics only via instanton effects, which fall off rapidly at short distances, the dependence of Newton's constant on θ_{QCD} is calculable.[32] Unfortunately, this leads to the prediction that $\theta_{QCD} = \pi$, in contradiction with experiment.[1]

Given the mixed record of wormhole physics, why should one spend any time on it at all? I believe that there are a number of reasons. To the extent that wormhole physics is an honest prediction of quantum gravity, it is important to study it as far as it will go. Even if it is not, we learn something about the assumptions we must make and the nature of the Euclidean path integral. It will pay, in the end, to attempt to strengthen the arguments used in deriving the wormhole path integral. This way we can learn about the definite consequences of quantum mechanics and

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[1] This is not the final death of wormhole physics. This incorrect prediction may result from a failure of chiral perturbation theory, rather than from a failure of the wormhole formalism.

gravity.

Another reason is simply that the wormhole solution to the cosmological constant problem is one of the better available solutions to this problem. If we want to keep the baby, it is important to find out what is lurking in the bath water. Finally, the study of certain aspects of the wormhole formalism has taught us some interesting things about ordinary quantum field theory. We have learned, for example, how *very* weak, high-dimension, charge-violating operators can lead to dramatic effects in Euclidean space. The charge decay is much more rapid than one might have expected. We have also learned much about the nature of topologically conserved quantities in quantum field theory and how they can be violated by terms in the effective Lagrangian. In effective field theory, topological symmetries turn out to be much more like ordinary global symmetries than like anything else. The quantum-mechanical relationship between a theory and its classically equivalent dual formulation has also been clarified. Even if the assumptions and predictions of the wormhole formalism are eventually shown to be incorrect, the theory of wormholes is an important addition to theoretical physics.

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