EXOTIC PHENOMENA IN NON-ABELIAN GAUGE THEORIES

Thesis by

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Dedicated to those who know they deserve it

and

in memory of my father.

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Abstract

This thesis deals with some exotic phenomena in non-Abelian gauge theories. More specifically, we study aspects of non-Abelian vortices, non-Abelian Chern-Simons particles, wormhole physics and electroweak strings. Non-Abelian vortices are capable of carrying charges without apparent sources (Cheshire charge). They obey exotic statistics—they generally form irreducible representations of the braid groups of dimensions larger than one. Owing to topological interactions, two vortices scatter non-trivially with each other even in the absence of any classical forces. As a function of the scattering angle, the exclusive cross-section for the vortex-vortex scattering process in the "group eigenstates" is generally multi-valued. Moreover, there can be an exchange contribution even if the two vortices have distinct initial quantum numbers. Thus, two vortices can be indistinguishable without being the same! We also construct exact wave functions for systems of non-Abelian Chern-Simons particles. In wormhole physics, we analyze the measurements of charge and magnetic flux in a wormhole background and show that they are complementary observables. For one thing, this investigation illustrates clearly how charge is conserved in the presence of a wormhole. Finally, we discuss the scattering of fermions from an electroweak string.

"Little steps for tiny feet..."

Richard Feynman

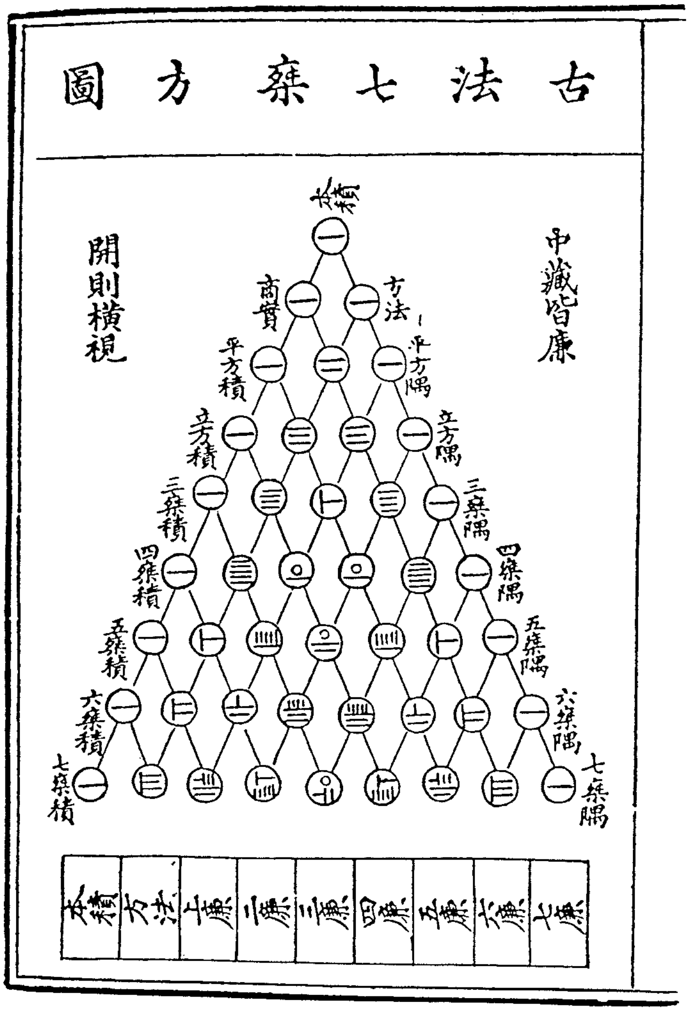

Fig. 80. The 'Pascal' Triangle as depicted in +1303 at the front of Chu Shih-Chieh's *Ssu Yuan Yü Chien*. It is entitled 'The Old Method Chart of the Seven Multiplying Squares' and tabulates the binomial coefficients up to the sixth power.

"This array has been known since the 17th century in Europe as Pascal's Triangle,... Actually it had already appeared in print for the first time,... on the title-page of Apianus' *Arithmetic* (+1527),... But Apianus,... and Pascal would have been rather surprised if they could ever have seen Chu Shih-Chieh*'s *Ssu Yuan Yü Chien*† of +1303,... The fact that Chu speaks of the triangle as old or ancient suggests that the binomial theorem had already been understood at least by the beginning of the +12th century. "

Reference: Joseph Needham, *Science and Civilisation in China* Vol. 3, pages 133-137.

* 朱世傑
† 四元玉鑑, Precious Mirror of the Four Elements

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Chapter 1
Introduction

Quantum field theories and general relativity are the two cornerstones of twentieth century theoretical physics. On the one hand, quantum field theories provide a complete description of all non-gravitational physics. On the other hand, general relativity is a tremendously successful theory of gravitation. Let us consider our first cornerstone. As strong, weak and electromagnetic interactions are all gauge interactions, the importance of gauge theories cannot be overstated. While some of the amazing agreements between the predictions of quantum electrodynamics (QED) and experimental data (e.g., the anomalous magnetic moment of the electron)[1] provide textbook examples of what an exact science should ideally be, some fundamental questions in elementary particle physics such as the confinement of quarks remain unsolved. It would, therefore, be invaluable to explore the subtle aspects of non-Abelian gauge theories, in the hope that insights gained from such scrutiny might shed some light on these fundamental issues. It is in that same spirit that studies of black holes, in particular the information loss paradox in black hole physics, have been made. Incidentally, these two endeavors are not totally unrelated: one of the original motivations for the study of "discrete gauge theories"[2] was the search for "quantum hair" on black holes.[3]

Even though our discussion will be essentially conceptual, we would like to remark that the subject matter of this thesis is not entirely devoid of experimental relevance. For instance, there has been some recent interest in the global analogs of the Aharonov-Bohm effect[4–6] and Cheshire charge.[7] Another related area of potential phenomenological interest would be nonabelions proposed in the fractional quantum Hall effect.[8] It is conceivable that some of the exotic phenomena described here, apart from illuminating some subtle aspects of non-Abelian gauge invariance, will be found to occur in condensed matter systems, particle accelerators and the early universe.

Even though gauge theories are virtually unheard of outside the physics community, they arise naturally in various contexts besides the formulation of fundamental physical laws. The motion of deformable bodies is a good example of how the abstract concepts of gauge theories can provide a useful framework even for discussing everyday realities.

"A cat, held upside-down by its feet and released at rest from a suitable height, will almost always manage to land on its feet, ... , by executing a sequence of deformations beginning and ending at the same shape, a deformable body with nothing to push against and no angular momentum has undergone a net rotation."[9]

A similar effect can be seen in the following example. "Hold your arm straight against your side and point your thumb in the forward direction. Do not rotate your thumb about your arm as you go through the following procedures: Lift your arm sideways until it is level with your shoulder. Then rotate your arm forward so that it sticks straight out in front of you. Finally, drop your arm back to your side. Notice that your thumb no longer juts forward but points in toward your side. While your arm has completed a trajectory and returned to its starting point, your thumb has rotated through an angle of 90° relative to its original direction. It has been globally, but not locally, changed."[10]

In the above example, your thumb is coupled to a slowly changing environment, your arm. By moving your arm, your thumb is "parallel-transported" with a non-rotation prescription. Nevertheless, you find that a cyclic evolution of the environment does lead to a net rotation of your thumb. In analogy, upon travelling adiabatically around a topological defect, the state of a charged particle or another vortex will also change. We call such a change in the state of the particle the holonomy (or the geometric phase) associated with the path. This is interesting because in many cases the charged particle does not experience any force. Yet its dynamics is severely affected by this global effect. The geometric phase has recently shown up in chemical reactions.[11] "If you ignore the geometric phase, you do so at your own risk."[10]

This type of topological interaction will be the subject under study in this thesis, the remainder of which is organized as follows: The concept of vortices (point-like topological defects in two spatial dimensions) is introduced in Chapter 2 which also contains a brief discussion on the topological classification of vortices and the concept of quantum hair. In this thesis, we are interested in a special class of such objects (non-Abelian vortices)[12–19] which are shown to exhibit various exotic properties. For example, a pair of vortices is capable of carrying charges without apparent sources. This unlocalized charge has been called "Cheshire charge"[13,14,20–22] in homage to the Cheshire's cat in "Adventures in the Wonderland". The mechanisms for its existence and its transfer between a pair of vortices and a charged particle will be the main theme of our study in Chapter 3.[21]

Owing to topological interactions, two vortices scatter non-trivially with each other even in the absence of classical forces.[15,19,23] In doing so, they exchange their quantum numbers. In Chapter 4, we compute the exclusive cross section for vortex-vortex scattering.[19] It is found to be a multivalued function of the scattering angle. (More precisely, it exhibits non-trivial monodromy properties.) Moreover, there can be an exchange contribution to the vortex-vortex scattering amplitude that adds coherently with the direct amplitude, even if the two vortices initially have distinct quantum numbers. The existence of an exchange contribution means that it is not possible in principle to keep track of which vortex is which. In this sense, the vortices are indistinguishable, but not the same.[19] We also show how the non-Abelian statistics obeyed by the non-Abelian vortices fits into general discussions of quantum statistics that have appeared in the literature.[24] It is interesting to note that our formula for the differential cross section of the two vortex scattering process coincides with the one derived for the scattering of two scalar particles in 2+1 dimensional gravity.[25] Thus, it may not be too crazy to speculate upon a possible realization of non-Abelian statistics in this context.

Another class of objects obeying non-Abelian statistics arises in theories with non-Abelian Chern-Simons terms. In Chapter 5, we obtain exact wave functions for N non-Abelian Chern-Simons particles using ladder operators.[26] In this example, our insight with multivalued wave functions is used to solve the many-body problem.

In Chapter 6, we apply our concepts to resolve some puzzles in wormhole electrodynamics and chromodynamics.[27] For one thing, our analysis shows us clearly how charge is conserved as a charged particle falls into a wormhole and becomes out of sight to an outside observer. Alternatively, the reader might prefer to envision our space as a thin two-dimensional film, containing objects with Aharonov-Bohm[28] interactions. It might actually be possible to fashion such wormholes in the laboratory. We hope to convey to the reader our belief that the concepts behind the exotic phenomena discussed here are very natural consequences of non-Abelian gauge theories and our results truly lead us to a better understanding of the basic structure of non-Abelian gauge theories.

Up to now, we have only considered the adiabatic transport of particles outside the string core. However, in the case of fermions scattering off a cosmic string, it is well-known that some partial waves of the fermionic wave function will blow up as we approach the core. This may lead to an enhancement of processes which only occur in the string core.[29] We will discuss a similar effect in the context of electroweak strings in Chapter 7.[30]

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Chapter 2
Some Comments on Vortices

2.1 Vortices

A vortex[1] is a stable time-independent solution to a set of classical field equations that has finite energy in two spatial dimensions.[2] Consider the Abelian Higgs model with the Lagrange density:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu \phi|^2 - V(|\phi|), \tag{1}$$

where ϕ is a charged complex scalar field, $D_\mu = \partial_\mu - ieA_\mu$ is the covariant derivative, and

$$V(|\phi|) = \frac{\lambda}{4}(|\phi|^2 - v^2)^2, \tag{2}$$

where v is real and positive. The energy can be written as

$$E = \int d^2r \big[\frac{1}{2}(E_i E^i + B_i B^i) + D_i \phi D^i \phi^\dagger + V(|\phi|)\big]. \tag{3}$$

For the energy to be finite, each of the three non-negative terms must be finite. In particular, for the third term to be finite, $V(|\phi|)$ must approach zero at spatial infinity, and $|\phi|$ must therefore approach v. We may think of two-dimensional space as being bounded by a big circle at $r = \infty$. Thus, we have

$$\phi(r,\theta) \underset{r\to\infty}{\longrightarrow} v e^{i\sigma(\theta)}. \tag{4}$$

Associated with every finite-energy field configuration is a mapping from the circle at spatial infinity to the circle defined by the phase of ϕ, which has a winding number defined by

$$n = \frac{1}{2\pi}[\sigma(\theta = 2\pi) - \sigma(\theta = 0)]. \tag{5}$$

The winding number, being an integer, must be invariant upon smooth deformations of the fields that preserve the finiteness of the energy. That is to say it is a topological invariant. Unlike more familiar conservation laws, a "topological conservation law" is not directly associated with any symmetry of the action.

For the radial part of the second term

$$\int d^2r \left| \left(\frac{1}{r}\frac{\partial}{\partial\theta} - ieA_\theta \right) \phi \right|^2 \tag{6}$$

to be finite, we have

$$A_\theta \underset{r\to\infty}{\longrightarrow} \frac{1}{er}\frac{d\sigma}{d\theta} + \cdots. \tag{7}$$

Hence, the gauge field is a "pure gauge" locally. (i.e., $F_{\mu\nu} \to 0$ as $r \to \infty$.) However, the magnetic flux through the plane can be evaluated using Stokes' Theorem to be

$$\Phi = \oint r d\theta A_\theta = \frac{1}{e}[\sigma(2\pi) - \sigma(0)] = \frac{2\pi}{e}n, \qquad n = 0, \pm 1, \pm 2, \cdots \tag{8}$$

which is quantized. We can, in principle, construct a solition solution (a Nielsen-Olesen vortex) by finding the configuration of lowest energy with, say, unit winding number, and we see that for such a configuration the gauge field cannot be a pure gauge everywhere because the magnetic flux through the plane is non-zero.

A non-singular field configuration with $n \neq 0$ has another important property: the field ϕ must vanish somewhere. For if ϕ had no zeros, its phase σ would be well defined everywhere. Then, by smoothly shrinking the circle at infinity to an infinitesimal circle around the origin, we could smoothly deform the mapping $\sigma(\theta)$, which has winding number $n \neq 0$, to the trivial mapping $\sigma =$ constant. This is clearly impossible. We are forced to conclude that there is at least one point at which σ is ill-defined, because ϕ vanishes.

2.2 General Classification of Topological Defects[3]

Let Ψ be a spatially dependent order parameter subject to a potential $V(\Psi)$ and let G be the symmetry group of the theory so that $V(g\Psi) = V(\Psi)$ for any g in G. Suppose that V attains its minimum at Ψ_0 and that G acts on Ψ_0 non-trivially. Then the set of values of the order parameter $M = \{g\Psi_0 | g \in G\}$ also minimizes V. For $\Psi \in M$, we define the unbroken symmetry group

$$H(\Psi) \equiv \{h \in G | h\Psi = \Psi\} \subset G \tag{9}$$

whose embedding in G depends on the choice of Ψ. The vacuum manifold has the same topology as the coset space. i.e., $M \cong G/H$.

We see in Section 2.1 that a Nielsen-Olesen vortex is classified by its winding number. In the general case, the order parameter of a finite energy configuration must take values in M except possibly in some compact regions C_i which may be regarded as the cores of topological defects (vortices in two spatial dimensions). Such multi-vortex configurations are characterized by *free* homotopy classes of functions from the spatial space with the cores of the vortices excised into the order parameter space (the vacuum manifold). Intuitively, a free homotopy class of functions is an equivalent class of functions under "smooth deformations." One can also define *based* homotopy classes of functions by choosing a base point x_0 (respectively Ψ_0) in the physical space (respectively parameter space) and consider only functions which obey $f(x_0) = \Psi_0$. For a single-vortex in $\mathcal{R}^2$, this based homotopy is given by $\pi_1(M, \Psi_0)$. It has a group structure and is known as the fundamental group of M. One can easily check that in the case of the Nielsen-Olesen vortex $G = U(1)$, H is trivial, $M = S^1$ and $\pi_1(S^1) = Z$.

For any Ψ, it can be shown that one can associate with any path $\mathcal{C}$ connecting Ψ_0 and Ψ, a natural group-theoretic path isomorphism between $\pi_1(M, \Psi_0)$ and $\pi_1(M, \Psi)$. However, the crucial point is that these isomorphisms are path-dependent. In particular, for $\Psi = \Psi_0$, they are inner isomorphisms. In going from free homotopic maps to based homotopic maps, we gain a group structure at the expense of having an ambiguity in the assignment of an element in $\pi_1(M, \Psi_0)$ to an element in the free homotopy class. The reason is that f is freely homotopic to g if and only if $f = hgh^{-1}$ for some $h \in \pi_1(M, \Psi_0)$. i.e., conjugate elements in $\pi_1(M, \Psi_0)$ are freely homotopic. Thus, f and g are equally good for association with a free element. Notice that no matter how many vortices there are, there is still only one conjugation ambiguity. In the case of a vortex pair, we choose two standard paths, beginning and ending at the basepoint x_0, and associate them with the group elements a and b respectively. Under conjugation, we have

$$h : a \rightarrow hah^{-1}, b \rightarrow hbh^{-1}. \tag{10}$$

Modulo an overall conjugation, we describe a n-vortex configuration with a group homomorphism:[4]

$$\pi_1(\Sigma, x_0) \rightarrow \pi_1(M, \Psi_0). \tag{11}$$

This is precisely the moduli space of flat connections modulo gauge transformations:

$\mathrm{Hom}(\pi_1(\Sigma), K)/K$ where $K = \pi_1(M, \Psi_0)$. This object has been under intense study in the context of Chern-Simons theory.[5]

In three spatial dimensions, $\pi_1(M, \Psi_0)$ is useful for the characterization of line defects. Point defects are classified by free homotopy classes of functions from S^2 to M. One can again obtain a group structure by considering based homotopy classes $\pi_2(M, \Psi_0)$. There is once again an ambiguity in going from free homotopy classes to based ones which is given by the natural action of $\pi_1(M, \Psi_0)$ on $\pi_2(M, \Psi_0)$.

2.3 Quantum Hair

It is a widely held belief that the only distinguishing features of a black hole exterior to the horizon are charges carried by massless gauge fields. These are mass, angular momentum and (Abelian or non-Abelian) electric or magnetic charges. This belief is often referred to as the black hole no-hair " theorem".[6] (Challenge to this conjecture has, however, been made.)[7] In this section our concern is with the interesting, but separate issue of "quantum hair"—characteristics of black holes which can only be observed quantum-mechanically. The recent interests in discrete gauge theory[8] are partly motivated by the possibility of quantum hair of discrete gauge charges.[9] Suppose we introduce in the Abelian Higgs model another field η which carries half of the charge as the Higgs field ϕ and does not form condensates. By adiabatically transporting a η particle around a $n = 1$ vortex,

$$D_\mu \eta = (\partial_\mu - ieA_\mu/2)\eta = 0 \tag{12}$$

gives

$$\eta(\phi = 2\pi) = \exp(i\frac{er}{2}\int_0^{2\pi} A_\theta d\theta)\eta(\phi = 0) = -\eta(\phi = 0). \tag{13}$$

The phase acquired by the wave function of an η particle is half of that acquired by a ϕ particle because its charge is half of that of ϕ. Notice that one can detect the Z_2 charge of a particle by Aharonov-Bohm scattering[10] it with a vortex of unit flux, even though there is no classical gauge field. Since this Aharonov-Bohm type interaction is long-ranged, by causality, its result must be insensitive to any local physics. (e.g., the charged particle might have fallen into the event horizon of a black hole.) This forms the basis of the study of quantum hair: black holes may carry quantum-mechanical hair (in this case, Z_2 charges) that is classically undetectable.

2.4 Nematic Liquid Crystals

The order parameter of a nematic liquid crystal describes the local preferred axis in a medium of long molecules with the symmetry of ellipsoids of revolution. One way of specifying the order parameter is to use a unit vector but without an associated direction. (i.e., $\hat{r}$ and $-\hat{r}$ are identified.) Consider the action of the rotation group SO(3) on such a unit vector without an arrow. It is clearly invariant under rotations about the molecular axis which can be chosen, without loss of generality, to be the z-axis. This gives rise to a U(1) symmetry whose generator will be denoted by Q. Moreover, it is also invariant under "flips", i.e., 180° rotations about axes perpendicular to the molecular axis. If one arbitrarily chooses one of these axes and denotes the corresponding 180° rotation by X, one can check easily that a rotation by ϕ about the z-axis, followed by X is equivalent to the action of X followed by a $-\phi$ rotation about the z-axis. i.e.,

$$X e^{i\phi Q} = e^{-i\phi Q} X. \tag{14}$$

Hence, the unbroken subgroup, H, has two connected components $\{e^{i\phi Q} | 0 \leq \phi < 2\pi\}$ and $\{X e^{i\phi Q} | 0 \leq \phi < 2\pi\}$. It is a semi-direct product $H = U(1) \times_{S.D.} Z_2$ rather than a direct product. One may, however, consider the original symmetry group to be $SU(2)$. In that case, the unbroken group which we denote by the same symbol H still has two connected components $\{e^{i\theta Q} | 0 \leq \theta < 4\pi\}$ and $\{X e^{i\theta Q} | 0 \leq \theta < 4\pi\}$.*

What are the vortices in this theory? From the topological classification scheme,

$$\pi_1(G/H) \cong \pi_0(H) = Z_2 = \{a, e\}. \tag{15}$$

(In the above, we have used the exact homotopy sequence and assumed that G is connected and simply-connected.) Hence, we have a Z_2 vortex. Since $\pi_1(M)$ is Abelian, as discussed in section 2.2, there is no distinction between free homotopy

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⋆ One should note that $X^2 = e^{i2\pi Q}$ which is *not* the identity in $SU(2)$. Hence, regarded as a subgroup of $SU(2)$, H is *not* a semi-direct product.

classes and based homotopy classes. Notice that there is no way of continuously defining everywhere a generator Q for the unbroken $U(1)$ in the background of a vortex. As illustrated in Fig. 1, if one defines Q by putting an arrow in the order parameter, one runs into difficulty along a line emanating from the vortex to the spatial infinity. This shows that the generator Q for the $U(1)$ symmetry cannot be globally realized.[11] If one considers a vortex pair instead, the region where Q is undefined can be confined to a finite region enclosing the vortex pair. "Global unrealizability" of unbroken symmetry also occurs when there are non-Abelian *gauge* vortices or monopoles with non-Abelian magnetic charge. As will be discussed in the next chapter, this gives rise to the Cheshire charge carried by a gauge vortex-antivortex pair or a string loop.

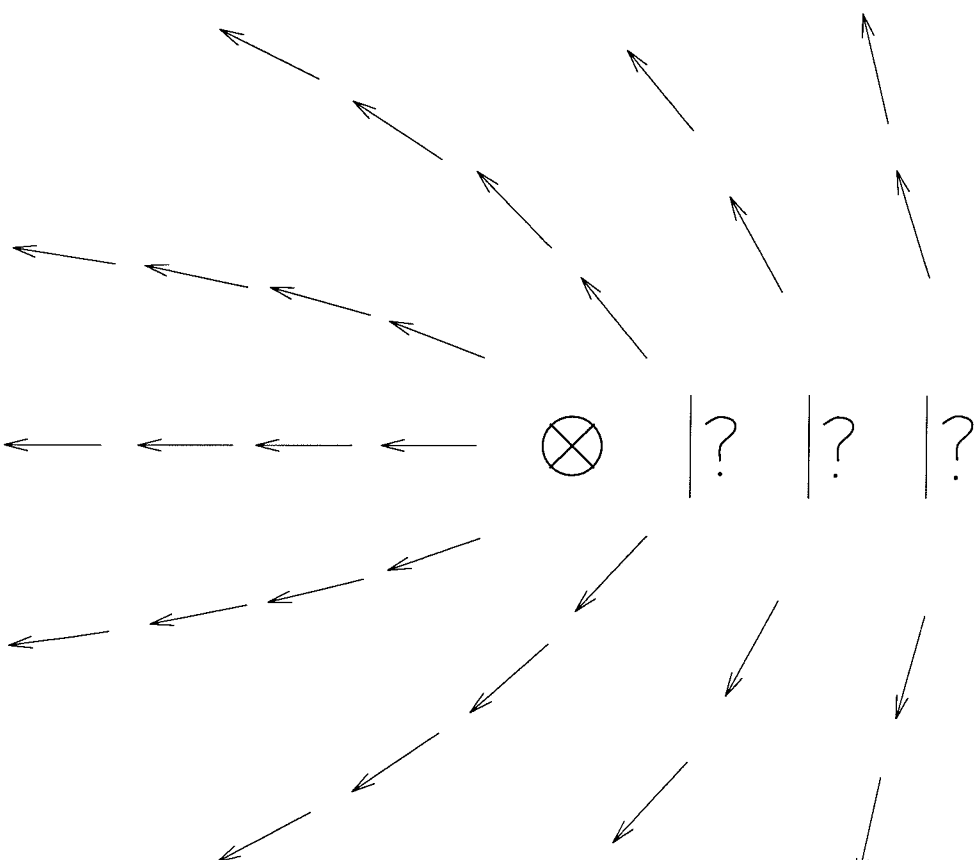

Figure 1

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Chapter 3

Topological Approach to Alice Electrodynamics

3.1 Introduction

In a spontaneously broken non-Abelian gauge theory, charge conjugation can be a *local* symmetry. That is, the unbroken gauge group H may contain both a $U(1)$ factor generated by Q, and an element X of the disconnected component of H such that $XQX^{-1} = -Q$. Such a model contains topologically stable cosmic strings with a remarkable property—when a charged particle is transported around the string, the sign of its charge flips. (The sign of the charge is gauge dependent, but the feature that the sign changes has an unambiguous and gauge–invariant meaning.) This string, which acts as a charge–conjugation looking glass, was first discussed by A. S. Schwarz, who dubbed it the "Alice" string.[1] (The possibility that charge conjugation could be a local symmetry was noted earlier by Kiskis.[2])

A closed loop of Alice string can carry electric charge, and the charge lost by a particle that winds around the string is transferred to the loop. A charged string loop is a peculiar object. It has a long–range electric field, from which its charge can be inferred, yet there is no localized source of charge anywhere on the string or in its vicinity.[3–5] Such charge with no locally identifiable source has been called "Cheshire charge."[4] An Alice string can also carry magnetic Cheshire charge, and can exchange magnetic charge with magnetic monopoles.[3,6,7]

The properties of Alice strings that carry Cheshire charge, and the processes by which charge is exchanged between strings and point particles, have been analyzed previously.[4–7] In this analysis, it is very convenient to employ the unitary gauge. However, in the presence of an Alice string, the gauge transformation that imposes the unitary gauge condition is necessarily singular; it introduces a gauge artifact surface on which fields (the electric and magnetic fields in particular) satisfy nontrivial boundary conditions. At the price of introducing this gauge–artifact singularity, one arrives at an appealing and vivid description of the charge–transfer phenomenon.

In this chapter, we analyze Cheshire charge using a different approach. In the case of magnetic charge, we note that the charge on a string is really a *topological* charge, and that the transfer of charge from magnetic monopole to string has an essentially topological origin. The transfer of topological charge can be described in a manifestly gauge–invariant way. By using global methods, one assuages the concern that the conclusions of previous work were an unfortunate artifact of an illicit gauge choice.

Even in the case of electric charge, global methods provide new insights. We will trace the mechanism of electric charge transfer to a generic topological property of non-Abelian vortices—namely, that when one vortex winds around another, the quantum numbers of both are modified.

The rest of this chapter is organized as follows: In Section 2, we briefly review the simplest model that contains an Alice string, and recall the analysis of Cheshire charge in Ref. 4-7. In Section 3, we describe the long–range interactions between non-Abelian string loops, and use the properties of these interactions to develop a semiclassical theory of Cheshire charge and charge transfer.

In Section 4, we note the subtleties inherent in defining magnetic charge in the presence of loops of Alice string. For the purpose of defining the magnetic charge carried by a particular string loop, it is convenient to introduce an (arbitrary) "basepoint," and a canonical surface (or homotopy class of surfaces) that encloses the loop and is tied to the basepoint. In general, the canonical surface can be chosen in topologically inequivalent ways, and the enclosed magnetic charge depends on this choice. It is just this ambiguity that underlies the transfer of charge from a magnetic monopole to a string loop. We will find that, as a monopole winds around a string loop, the canonical surfaces that are used to define the magnetic charge of both the monopole and the loop are deformed to new (topologically inequivalent) surfaces. Therefore, the charges defined by the original canonical surfaces are modified; charge transfer has taken place.

Section 5 contains some concluding remarks.

3.2 Alice Strings

The simplest model that contains an Alice string has gauge group $SU(2)$ and a Higgs field Φ that transforms as the 5-dimensional irreducible representation of $SU(2)$. We may express Φ as a real symmetric traceless 3×3 matrix that transforms according to

$$\Phi \to M\Phi M^{-1} \ , \quad M \in SO(3) \ . \tag{1}$$

If Φ has an expectation value (in unitary gauge) that can be expressed as

$$\langle\Phi\rangle = v \cdot \mathrm{diag}\ [1,\ 1,\ -2\ ] \ , \tag{2}$$

then the unbroken subgroup of $SU(2)$ is $H = U(1) \times_{S.D.} Z_2$. The unbroken group H has two connected components. The component connected to the identity can be pictured as rotations about a z-axis. Since $SU(2)$ is a double cover of the rotation group, this component, which is isomorphic to $U(1)$, can be expressed as $H_c = \{\exp[i\theta Q] \mid 0 \le \theta < 4\pi\}$, where Q is the $SU(2)$ generator $Q = \frac{1}{2}\sigma_3$. There is also a connected component not connected to the identity of the form $H_d = \{X \exp[i\theta Q] \mid \ 0 \le \theta < 4\pi\}$. This component consists of rotations by 180° about axes that lie in the xy-plane. (X is any such rotation.) Each element Y of H_d anticommutes with Q, $YQY^{-1} = -Q$; it is a "charge–conjugation" operator embedded in the unbroken local symmetry group.

The elements of H_d represent the possible values of the "magnetic flux" of the topologically stable cosmic string excitations of the theory in 3+1 dimensions (or vortex excitations in 2+1 dimensions). In general, the magnetic flux carried by a cosmic string is an element of the unbroken group H that encodes the result of parallel transport along a closed path that encloses the string. To define the magnetic flux we must specify a basepoint x_0 and a closed loop C that starts and ends at x_0 and encircles the string exactly once.(See Fig. 1.) Then the flux is given by the untraced Wilson loop operator

$$h(C, x_0) = P \exp\left(i \int_{(C,x_0)} dx^i \ A_i \right) \ . \tag{3}$$

The flux takes values in $H(x_0)$, the subgroup of the underlying group G that stabilizes the condensate at the point x_0 (since parallel transport around C must return the condensate to its original value).

One can determine what happens to the charge of a particle that travels around an Alice string by considering the behavior of the unbroken symmetry group $H(x_0)$ as it is parallel transported around the string.Consider the situation depicted in Fig. 2, with a single Alice string enclosed by a circle parameterized by ϕ, $0 \leq \phi \leq 2\pi$. At each point on the circle labeled by ϕ, there is a subgroup $H(\phi)$ embedded in G that stabilizes the condensate $\Phi(\phi)$ at that point. The gauge vector potential A_μ relates these subgroups through the equation

$$H(\phi) = U(\phi)H(0)U(\phi)^{-1} , \tag{4}$$

where

$$U(\phi) = P\exp\left(i\int_0^\phi d\phi A_\phi\right) . \tag{5}$$

Note that $U(2\pi) = h(C, x_0)$. It is certainly true that $H(0) = U(2\pi)H(0)U(2\pi)^{-1}$, because $H(2\pi) = H(0)$, but the analogous relation does not hold for the generators of H. Since $U(2\pi) \in H_d$, we have

$$U(2\pi)\ Q\ U(2\pi)^{-1} = -Q . \tag{6}$$

An analogy can be made to the Möbius strip to make it apparent why Q is deformed into $-Q$ upon parallel transport around the circle. The $U(1)$ subgroups $\{H(\phi)\}$ of $SO(3)$ can be represented as undirected lines in $\Re^3$ through the origin that coincide with the axes of the rotation of the $U(1)$ subgroups. Choosing a generator $Q(\phi)$ for $H(\phi)$ at each ϕ is equivalent to choosing a direction for each of these lines. As ϕ varies from 0 to 2π, the lines are twisted into a Möbius strip. There is no continuous way to choose a direction on each of them.

The Möbius twist in the unbroken symmetry group $H(x)$ described above may be discussed more formally in terms of the "global unrealizability" of the unbroken symmetry.[8,4,5] Let $\mathcal{M}$ denote the spatial manifold consisting of $\Re^3$ with the cores of the strings excised. At each point $x \in \mathcal{M}$ is defined the unbroken symmetry group $H(x)$ that stabilizes the Higgs condensate $\Phi(x)$. All these subgroups are isomorphic to the same abstract group H. This structure is a fiber bundle E with model fiber H over the base manifold $\mathcal{M}$. The structure group of the bundle is also H, and H acts on the fibers by conjugation. Locally, in any contractable open subset $U \subset \mathcal{M}$, the fiber bundle has the structure $U \times H$. But generally there does not exist a continuous mapping

$$f : \mathcal{M} \times H \to E . \tag{7}$$

This is because the open sets U_α covering $\mathcal{M}$ can be patched together in a nontrivial way using nontrivial transition functions. In more physical terms, a continuous mapping of the form f is a "global realization" of the unbroken symmetry H considered as an abstract group. (In mathematical language, such a mapping is known as a trivialization of the fiber bundle E.) Clearly, such a realization is not possible in the presence of an Alice string, because such a mapping f would induce a continuous choice of $Q(\phi)$ for $0 \leq \phi < 2\pi$, and we just showed that no such continuous choice exists. ("Global unrealizability" of the unbroken symmetry also occurs when there are monopoles with non-Abelian magnetic charge.[9])

The Möbius twist implies that a charged particle initially with charge q will have charge $-q$ after winding around an Alice string. Of course, the sign of the charge can be changed by a gauge transformation, and therefore has no unambiguous physical meaning. But the statement that the sign changes upon transport around the string is gauge invariant and meaningful. Suppose, for example, that two charges of like sign are initially brought close together; they repel. (See Fig. 3.) Then one charge travels around an Alice string while the other stays behind. When they are brought together again, they attract. Yet the total charge, as measured by an observer far away from the string loop and the point charges, cannot have changed. Where did the missing charge go?

This puzzle is resolved by Cheshire charge.[4,5] In order to understand what happened to the charge, it is convenient to choose a particular gauge—the unitary gauge in which the Higgs field takes the value eq. (2) everywhere. However, the gauge transformation that implements the unitary gauge condition is singular; it has a discontinuity, or cut, on a surface that is bounded by the string loop. (In other words, one can transform to unitary gauge everywhere outside a thin pancake that encloses the string loop. Inside the pancake, the Higgs field twists very rapidly, and the gauge potential is very large. The singularity arises as the width of the pancake shrinks to zero.) As a result, fields on the background of the string loop obey peculiar boundary conditions—the electromagnetic field changes sign on the cut, and charge of a charged matter field flips there.

Because of the peculiar boundary conditions satisfied by the electromagnetic field at the cut, there are solutions to the classical field equations in which the cut appears to be a source of electric (or magnetic) charge, as in Fig. 4. There is not actually any measurable charge density on the cut; the cut is an unphysical gauge artifact. Yet the string loop is charged—it has a long range electric field that can be detected by a distant observer. This electric field has no locally identifiable source; it is "Cheshire charge."

The charge transfer process is sketched in Fig. 5. The initial electric field of a charge-q particle in the vicinity of a string loop is shown in Fig. 5a, and Fig. 5c–e shows how the field changes as the particle travels around the path in Fig. 5b. When the particle crosses the cut, its apparent charge flips from q to $-q$, and the cut seems to acquire the compensating charge $-2q$. It is clear from the final configuration in Fig. 5e that charge $2q$ has been exchanged between the particle and the loop.

Yet there is no gauge–invariant way to pinpoint when the charge transfer took place. The configuration of the electric field lines is gauge invariant, but the direction of the arrows on the field lines is gauge dependent. We can move the cut by performing a singular gauge transformation; this alters the apparent time of the charge transfer without actually changing the physics of the process.

The charge transfer can be characterized in a gauge–invariant manner, as follows: The nontrivial irreducible representations of H are two-dimensional, and can be labeled by the absolute value of the $U(1)$ charge. The tensor product of two irreducible representations decomposes into irreducible representations according to

$$|q_1| \otimes |q_2| = |q_1 + q_2| \oplus |q_1 - q_2| \ . \tag{8}$$

For the charge–loop system described above, the total charge is $|q|$. This charge determines (the absolute value of) the electric flux through a large closed surface that encloses the system, and is of course conserved during the exchange process. Initially, the loop is uncharged and the particle has charge $|q|$. The exchange process leaves (the absolute value of) the charge of the particle unchanged, but produces an excitation of the loop with charge $|2q|$.

So far, we have considered a particular model with Alice strings. Much of the physics discussed in this chapter is independent of the details of that model. We will briefly describe a more general class of models in which Alice-like behavior occurs.[1] Let the unbroken group H to be a subgroup of the simply-connected gauge group G. Topologically stable cosmic strings occur only when $\pi_0(H)$ is nontrivial, so suppose that H has several connected components. Groups of this sort may be constructed as the semi-direct product of a continuous part H_c, which is a connected compact Lie group, and a discrete group D. The semi-direct product $H_c \times_{S.D.} D$ is a generalization of a direct product, defined by a group homomorphism

$$\varphi : D \to \mathrm{Aut}[H_c] \ , \tag{9}$$

where $\mathrm{Aut}[H_c]$ is the group of automorphisms of H_c. Group multiplication is defined using the rule

$$(h_1, d_1) \circ (h_2, d_2) = (h_1 \cdot \varphi_{d_1}(h_2), d_1 \cdot d_2) \ . \tag{10}$$

Strings will have Alice properties if the mapping φ is nontrivial.

In the example described earlier, $D = Z_2$ and the nontrivial automorphism reverses the sign of the generator Q of $H_c = U(1)$. As an example of generalized Alice behavior, consider a model with

$$H = [SU(2)_1 \times SU(2)_2] \times_{S.D.} Z_2 \ , \tag{11}$$

where the nontrivial automorphism is a "parity" operator that interchanges the two $SU(2)$ factors. (With suitable Higgs structure, the gauge group $G = SU(4)$ can be broken to this H.) This model contains an Alice-like string. If an object with representation content (R_1, R_2) under $SU(2)_1 \times SU(2)_2$ is transported around this string, its representation content is changed to (R_2, R_1), and the missing quantum numbers are transferred to the string.

We should also note that a string might exhibit Alice-like behavior, for dynamical reasons, even when such behavior is not topologically required.[4] That is, the flux of a dynamically stable string might assume a value h that is not in the center of H, even though there are elements of the center that lie in the same connected component as h. Then only the subgroup of H that commutes with the flux h can be globally defined in the presence of the string. However, in this case, strictly speaking, the position dependence of the unbroken symmetry group $H(x)$ is not described by a topologically nontrivial bundle. This is because we can trivialize the bundle by smoothly deforming the flux h to an element of the center of H. The bundle is nontrivial only if no element of the center is contained in the same connected component as the flux; that is, only if the Alice behavior is topologically unavoidable.

3.3 Electric Charge

In this section, we describe the electrically charged Alice string, and the charge transfer process, in semiclassical language.[10]

In quantum theory, the electric charge of a state reflects the transformation properties of the state under global gauge transformations. The Alice string classical solution is not a charge eigenstate, but it has a "charge rotor" zero mode. Semiclassical quantization of the zero mode is achieved by constructing linear combinations of the classical string states that *do* have definite charge.[4,5] We need to worry, though, about what is meant by a "global" gauge transformation, since we have seen that gauge transformations are not globally realizable. Fortunately, for the purpose of defining the total charge of a state, it is sufficient to consider a gauge transformation that is constant on and outside a large sphere that encloses all of the charged objects. Inside the sphere, we may deform the gauge transformation so that it vanishes on the core of each string, and on a surface bounded by each string.[5] There is no topological obstruction to constructing this gauge transformation. Strictly speaking, since the flux of a string is defined relative to a basepoint, we should think of the large sphere not as a "free" surface, but rather as a surface tied to the basepoint x_0. That is, the gauge transformation takes the same value at x_0 as on the sphere. (If the total *magnetic* charge enclosed by the sphere is nonzero, then there is a further obstruction, so that the gauge transformations in the disconnected component H_d cannot be defined on the sphere[9]. We defer the discussion of magnetically charged string loops until the next section, and suppose, for now, that the magnetic charge is zero.)

The magnetic flux of the string, defined by eq. (3), takes values in the disconnected component H_d of the unbroken group $H(x_0)$ that stabilizes the condensate at the basepoint x_0. In general, this flux transforms under a transformation $g \in H(x_0)$

according to

$$h(C, x_0) \to g h(C, x_0) g^{-1} \ . \tag{12}$$

In the case of an Alice string, let $|\theta\rangle$ denote the string loop state with flux $h(C, x_0) = Xe^{i\theta Q}$. Under a global H transformation, the transformation property eq. (12) becomes

$$U(e^{i\omega Q})\,|\theta\rangle = |\theta - 2\omega\rangle \ , \tag{13}$$

$$U(Xe^{i\omega Q})\,|\theta\rangle = |2\omega - \theta\rangle \ , \tag{14}$$

where U is the unitary operator acting on Hilbert space that represents the global gauge transformation.

One can construct linear combinations of these "flux eigenstate" string states that transform irreducibly under H. Let

$$|q\rangle = \int_0^{4\pi} \frac{d\theta}{\sqrt{4\pi}}\, e^{i\frac{\theta}{2}q}\,|\theta\rangle \tag{15}$$

(where Q is an integer). It transforms as

$$U(e^{i\omega Q})\,|q\rangle = e^{i\omega q}\,|q\rangle \ ; \tag{16}$$

$$U(Xe^{i\omega Q})\,|q\rangle = e^{i\omega q}\,|{-q}\rangle \ . \tag{17}$$

The two states $|q\rangle$ and $|{-q}\rangle$ thus comprise the basis for an irreducible representation of H.

Only integer-$|q|$ representations of H occur in this decomposition; an Alice string cannot carry half-odd-integer $|q|$. String loops are invariant under the center of $SU(2)$, and so can have no "two-ality."

The semiclassical quantization of the charge rotor of the Alice string is strongly reminiscent of the corresponding treatment of bosonic superconducting strings.[11] But the physical properties of the string are actually remarkably different. Alice strings do not carry persistent currents. Instead, they carry electric charge (or magnetic charge, as we will discuss in the next section).

Now we will discuss the charge transfer process. It will be enlightening to imagine that the charged object that winds through the string loop is itself a loop of Alice string. Then the charge transfer can be regarded as a consequence of a topological interaction between non-Abelian string loops. (We will see in the next section that magnetic charge transfer results from a related topological interaction.)

Consider the system of two string loops $\mathcal{C}_1$ and $\mathcal{C}_2$ shown in Fig. 6a. Suppose that each string is a flux eigenstate, with

$$\begin{aligned} h(C_1, x_0) &= h_1 \ , \\ h(C_2, x_0) &= h_2 \ . \end{aligned} \tag{18}$$

Now suppose that the loop $\mathcal{C}_2$ winds through $\mathcal{C}_1$ as in Fig. 6b. To determine the magnetic flux of the loops after the winding, it is convenient to consider the paths C_1' and C_2' in Fig. 6c. During the winding procedure, these paths are dragged back to the paths C_1 and C_2. Therefore, the flux associated with the paths C_1 and C_2 after the winding is the same as the flux associated with the paths C_1' and C_2' before the winding. One sees that $C_1' = C_1$ and $C_2' = {C_1}^{-1} \circ C_2 \circ C_1$. (Our convention is that $C_2 \circ C_1$ denotes the path that is obtained by traversing first C_1, then C_2.) We therefore find that, after the winding, the flux carried by the string loops is[12−14]

$$\begin{aligned} h'(C_1, x_0) &= h(C_1', x_0) = h_1 \ , \\ h'(C_2, x_0) &= h(C_2', x_0) = {h_1}^{-1} h_2 h_1 \ . \end{aligned} \tag{19}$$

In the case of Alice strings, we denote by $|\theta_1, \theta_2\rangle$ the two–string state with flux $h_1 = X e^{i\theta_1 Q}$ and $h_2 = X e^{i\theta_2 Q}$. Then, if string 2 winds through string 1, eq. (19) becomes

$$|\theta_1, \theta_2\rangle \to |\theta_1, 2\theta_1 - \theta_2\rangle \ . \tag{20}$$

If we construct charge eigenstates as in eq. (15), we find from eq. (20) that the effect of the winding is

$$|\theta_1, q_2\rangle \to e^{i\theta_1 q_2} \, |\theta_1, -q_2\rangle \ , \tag{21}$$

and

$$|q_1, q_2\rangle \to |q_1 + 2q_2, -q_2\rangle \ . \tag{22}$$

Just as in the classical analysis of Section 2, the sign of q_2 flips, and loop 1 acquires a compensating charge.

Of course, we can also analyze (somewhat more straightforwardly) the case in which the charge that winds is a point charge rather than a charged loop. Then eq. (21) follows directly from the gauge transformation property of the charged particle.

3.4 Magnetic Charge

3.4.1 Twisted Flux

In the above discussion of semiclassical quantization, we assumed that the magnetic flux was a constant along the string. But if the unbroken group H is continuous, as in the Alice case, the flux can vary as a function of position along the string loop. Furthermore, if H is not simply connected, then the flux might trace out a noncontractible closed path in H. Then the string loop evidently carries a type of topological charge. This charge is precisely the magnetic charge of the loop.

To define this charge carefully, we should, as usual, select an arbitrary basepoint x_0 and consider the magnetic flux defined by eq. (3). As the path C is smoothly deformed with the basepoint x_0 held fixed, this flux varies smoothly in a given connected component of the group $H(x_0)$.To be specific, consider the family of paths $\{C_\phi \mid 0 \leq \phi < 2\pi\}$, shown in Fig. 7. These paths sweep out a degenerate torus that encloses the string loop. This family $\{C_\phi\}$ is associated with a closed path in $H(x_0)$ that begins and ends at the identity; namely,

$$h(C_\phi, x_0)h^{-1}(C_{\phi=0}, x_0) \ , \quad 0 \leq \phi < 2\pi \ . \tag{23}$$

We have thus found a natural way of mapping a two-sphere that encloses the string loop to a closed loop in H_c, the component of H connected to the identity.

By smoothly deforming the family $\{C_\phi\}$, we may obtain the family of closed paths $\{C'_\phi\}$ shown in Fig. 8. Loosely speaking, $h(C'_\phi, x_0)$ is the flux carried by the string at the point where C'_ϕ wraps around the core of the string. Thus we see that the homotopy class of the path defined by eq. (23) describes how the flux of the string twists as a function of position along the string.

On the other hand, the family $\{C_\phi C^{-1}_{\phi=0}\}$ is equivalent to the family of paths $\{C''_\phi\}$ shown in Fig. 9. But this is just the family of paths used by Lubkin[15,16,6] to define the topological H_c magnetic charge inside a two-sphere. We learn that the element of $\pi_1(H_c)$ that characterizes how the magnetic flux of the string twists is the same as the magnetic charge on the loop.[10,6]

More generally, in the presence of many string loops and pointlike monopoles, we can define the magnetic charge inside any region R whose boundary ∂R is homeomorphic to S^2. The result is a homomorphism

$$h^{(2)} : \ \pi_2[\mathcal{M}, x_0] \to \pi_1[H_c(x_0)] \ , \tag{24}$$

where $\mathcal{M}$ denotes the manifold that is obtained when all string loops and monopoles are removed from $\Re^3$.

3.4.2 Role of the Basepoint

We should now explain why it is important to specify a basepoint x_0 for the purpose of defining the magnetic charge. Naively, it seems that it should be possible to define the magnetic charge enclosed by a "free" surface that is not tied to any basepoint, since the enclosed charge is just the magnetic flux through the surface. But trouble arises if we allow the magnetic charges to move. We can deform the free surface so that it is never crossed by any moving magnetic monopoles or string loops. Nevertheless, the magnetic flux through the surface can change if the surface winds through an Alice string loop.

It will be easier to keep track of charge transfer processes if we define magnetic charge using a surface that is tied to a basepoint. As the charges move, we can again deform the surface so that no monopoles or strings cross it, while keeping the basepoint fixed (as long as no monopoles or strings cross the basepoint). Then the magnetic charge enclosed by the surface remains invariant. However, when a monopole winds around a string loop, the surface enclosing the monopole becomes deformed to a new, topologically inequivalent surface. We can then find how the charge of the monopole has changed by expressing the new surface in terms of the old one. This procedure is closely analogous to our discussion in Section 3 of how the flux of a loop is modified when it winds around another string. There we defined the flux using a standard *path* that became deformed to a new path due to the winding. We can analyze the exchange of magnetic charge using a similar strategy, except that a *surface*, rather than a path, is used to define the charge.

In order to define the magnetic charge enclosed by a free surface $\bar{\Sigma}$ that is homeomorphic to S^2, then, we specify not just the surface, but also a path that attaches the surface to the basepoint x_0. Of course, this path can be chosen in many topologically inequivalent ways; the different choices are classified by $\pi_1[M, x_0]$. Thus, $\pi_1[M, x_0]$ classifies the ambiguity in associating a free surface with an element of $\pi_2[M, x_0]$. There is a corresponding ambiguity in the value of the magnetic charge (given by the homomorphism $h^{(2)}$ defined in eq. (24)) that is associated with a free surface. We resolve this ambiguity by simply choosing a standard convention for the path from the free surface to the basepoint, and sticking with this convention throughout the process under study.

The ambiguity is illustrated by Fig. 10, which shows two inequivalent surfaces Σ and Σ' with basepoint x_0 that are obtained by "threading" the free surface $\bar{\Sigma}$ to the basepoint in two different ways. As shown in Fig. 10d, the surface Σ' can be deformed to a degenerate tube, beginning and ending at x_0, joined to the surface Σ. Since the degenerate tube is equivalent to a closed path β, we may say that the two surfaces differ by an element of $\pi_1[M, x_0]$.

The ambiguity in associating a free surface with an element of $\pi_2[M, x_0]$ can be characterized by a natural homomorphism

$$\tau : \pi_1[\mathcal{M}, x_0] \to \mathrm{Aut}\,(\pi_2[\mathcal{M}, x_0]) \tag{25}$$

that takes (homotopy classes of) closed paths to automorphisms of $\pi_2[M, x_0]$. The mapping τ is defined in the following way: Let $\beta \in \pi_1[M, x_0]$ and $\Sigma \in \pi_2[M, x_0]$. (Below we use the symbols β and Σ to denote both homotopy equivalence classes and particular representatives of the classes.) Then τ_β is an automorphism that takes Σ to a new surface Σ',

$$\tau_\beta : \ \Sigma \to \Sigma' \ , \tag{26}$$

where Σ' is the surface Σ with the degenerate tube β added on. More precisely, let $\beta(t)$, $0 \le t \le 1$ be a parametrized path, with $\beta(0) = \beta(1) = x_0$, and let $\Sigma(\theta, \phi)$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$ be a parametrized surface with $\Sigma(0, \phi) = x_0$. Then the new surface Σ' is

$$\Sigma'(\theta, \phi) = \begin{cases} \beta(2\theta/\pi) & \text{if } 0 \le \theta \le \pi/2 \ , \\ \Sigma(2\theta - \pi, \phi) & \text{if } \pi/2 \le \theta \le \pi \ . \end{cases} \tag{27}$$

Now consider how changing the threading of a free surface to the basepoint modifies the magnetic charge enclosed by the surface. Recall that eq. (3) maps a path that begins and ends at the basepoint to an element of the group $H(x_0)$. If the path is deformed to a homotopically equivalent path, the group element remains in the same connected component of the group. Thus, eq. (3) defines a homomorphism

$$h^{(1)}: \ \pi_1[M, x_0] \rightarrow \pi_0[H(x_0)] \ . \tag{28}$$

If the surface Σ is changed to the surface Σ' by adding the degenerate tube β, then the magnetic charge enclosed by the new surface is related to the magnetic charge enclosed by the original surface according to

$$h^{(2)}(\Sigma') = h^{(1)}(\beta)^{-1} \ h^{(2)}(\Sigma) \ h^{(1)}(\beta) \ . \tag{29}$$

That is, $h^{(2)}(\Sigma')$ is the closed path in H_c (beginning and ending at the identity) that is obtained when $h^1(\beta)$ acts on the closed path $h^{(2)}(\Sigma)$ by conjugation. In the case of the Alice string, eq. (29) simply says that, if β is a path that winds around a string loop, then the magnetic charges enclosed by Σ and Σ' differ by a sign.

3.4.3 Charge Transfer

Eq. (29) is the key to understanding the magnetic charge transfer process, as we will show. First, though, we should recall that $\pi_2[M, x_0]$ has a group structure that allows magnetic charge to be added. The group multiplication law,

$$\circ : \pi_2[\mathcal{M}, x_0] \times \pi_2[\mathcal{M}, x_0] \rightarrow \pi_2[\mathcal{M}, x_0] \ , \tag{30}$$

can be defined as

$$(\Sigma_1 \circ \Sigma_2)(\theta, \phi) = \begin{cases} \Sigma_1(2\theta, \phi) & \text{if } 0 \leq \theta \leq \pi/2 \ , \\ \Sigma_2(2\theta - \pi, \phi) & \text{if } \pi/2 \leq \theta \leq \pi \ , \end{cases} \tag{31}$$

where Σ_1, Σ_2, and $\Sigma_1 \circ \Sigma_2$ are homotopy equivalence class representatives. Group multiplication in π_2 is commutative. Group inversion may be expressed in terms of class representatives as

$$\Sigma^{-1}(\theta, \phi) = \Sigma(\pi - \theta, \phi) \ . \tag{32}$$

We turn to the situation depicted in Fig. 11. Two string loops $\mathcal{C}_1$ and $\mathcal{C}_2$ are shown. We denote by β_1 and β_2 two standard paths, beginning and ending at the basepoint x_0, that wind around the string loops. (These are elements of $\pi_1[M, x_0]$.) We denote by a_1 and a_2 two standard surfaces, based at x_0, that enclose the string loops. (These are elements of $\pi_2[M, x_0]$.) The magnetic charges of the two loops, given by the homomorphism eq. (24), are $h^{(2)}(a_1)$ and $h^{(2)}(a_2)$, respectively.

Now suppose that the loop $\mathcal{C}_2$ winds through the loop $\mathcal{C}_1$ along the path shown in Fig. 11b. We want to determine the magnetic charges of the two loops after this winding. To do so, consider the surfaces a_1' and a_2' shown in Fig. 11c-d. During the winding, these surfaces are dragged back to the surfaces a_1 and a_2, if the surfaces are deformed so that no surface ever touches a string loop. Therefore, the magnetic charge enclosed by a_1, after the winding, is the same as the magnetic charge enclosed by a_1', before the winding. Similarly, the magnetic charge enclosed by a_2, after the winding, is the same as the magnetic charge enclosed by a_2', before the winding.

It remains to find the magnetic charges enclosed by a_1' and a_2' before the winding. Fig. 12a shows a deformation of a_2' that makes it manifest that a_2' can be expressed as

$$a_2' = \tau_{\beta_1}(a_2) \ , \tag{33}$$

where τ_{β_1} is the automorphism of $\pi_2[M, x_0]$ defined by eq.(26)–(27). In Fig. 12b, the surface a_1' is expressed as the sum of two surfaces. The first (outer) surface is just $a_1 \circ a_2$, the surface that encloses both loops. The second (inner) surface is $(a_2')^{-1}$; it is the same as a_2', except with the opposite orientation. We see that

$$a_1' = a_1 \circ a_2 \circ (a_2')^{-1} \ . \tag{34}$$

Finally, we apply eq. (29) to find the magnetic charges after the winding; the result is

$$\begin{aligned} h^{(2)\prime}(a_1) &= h^{(2)}(a_1') = h^{(2)}(a_1 \circ a_2)\left[h^{(2)\prime}(a_2)\right]^{-1} \ , \\ h^{(2)\prime}(a_2) &= h^{(2)}(a_2') = h^{(1)}(\beta_1)^{-1} \ h^{(2)}(a_2) \ h^{(1)}(\beta_1) \ . \end{aligned} \tag{35}$$

Of course, the total magnetic charge is unchanged, because $h^{(2)}(a_1 \circ a_2) = h^{(2)\prime}(a_1 \circ a_2)$.

In the case of the Alice string, the magnetic charge can be labeled by an integer p—the charge in units of the Dirac charge. If the magnetic charges on the string loops are initially p_1 and p_2, and then loop 2 winds through loop 1, eq. (35) says that the charges become modified according to

$$|p_1, p_2\rangle \to |p_1 + 2p_2, -p_2\rangle \ , \tag{36}$$

in accord with the analysis in Section 2.

3.4.4 Dyons

We may also consider dyonic Alice string loops, that carry both magnetic and electric charge. The classical magnetically charged Alice string loop has a charge rotor zero mode, just like the magnetically neutral loop considered in Section 3, and we may proceed with semiclassical quantization in the same manner as before. The only difference from the previous discussion is that, for the magnetically charged loop, there is a topological obstruction to defining global gauge transformations in the disconnected component of the unbroken group H, similar to the obstruction discussed in Ref. 9. (The obstruction occurs because the automorphism that reverses the sign of Q is incompatible the matching condition of a magnetic monopole.) Thus, we obtain states that transform irreducibly under the connected component $H_c = U(1)$, but the states do not transform as representations of the full group.

By decomposing the classical string with magnetic charge p into irreducible representations of $U(1)$, as in Section 3, we find states $|q, p\rangle$ with electric charge q, where q is any integer. Reanalyzing the charge transfer process, we find that, when loop 2 winds through loop 1, the charge assignments change according to

$$|q_1, p_1; q_2, p_2\rangle \to |q_1 + 2q_2, p_1 + 2p_2; -q_2, -p_2\rangle \ . \tag{37}$$

Naturally, both magnetic charge and electric charge are exchanged.

We will comment briefly on how the analysis is modified when the vacuum θ-angle is nonzero. The nonvanishing vacuum angle alters the $U(1)$ transformation properties of states with nonzero magnetic charge, so that eq. (16) is replaced by[17]

$$U(e^{i\omega Q})\,|q, p\rangle = \exp\left[i\omega\left(q + \frac{\theta}{2\pi}p\right)\right]\,|q, p\rangle \ , \tag{38}$$

where Q is the $U(1)$ generator, and q is the charge of the state defined in terms of the electric flux through the surface at spatial infinity. Thus, for magnetically charged string loops, as for all magnetically charged objects, the charge spectrum is displaced away from the integers by $-\theta p/2\pi$. But otherwise, the discussion of electric and magnetic charge transfer is not altered; in particular, eq. (37) still applies.

3.4.5 Linked Loops

The homomorphism defined in eq. (24) assigns a magnetic charge to any region whose boundary is homeomorphic to S^2. But if two string loops link, the magnetic charge on each individual loop is not well defined in general. Only the total magnetic charge of the two loops can be defined. The magnetic field of a pair of linked loops has some interesting properties that we will briefly discuss.

In general, two non-Abelian string loops can link only if the commutator of their fluxes is in the connected component of the unbroken group.⋆ This feature is a consequence of the "entanglement" phenomenon. Suppose that a string loop with flux h_1 and a string loop with flux h_2 cross each other, and become linked. After they cross, a flux $h_1 h_2 {h_1}^{-1} {h_2}^{-1}$ must flow from one loop to the other.[18–21,14] If this commutator is not in the connected component of H, then the commutator flux is itself confined to a stable string. Thus, the two loops must be connected by a segment of string that carries the commutator flux. On the other hand, if the commutator is in the connected component of H, then the commutator flux is unconfined, and the flux will spread out uniformly over the h_1 and h_2 loops. The linked loops will have a long–range magnetic dipole field, though the total magnetic charge of the linked pair is zero.

In the case of the Alice string, consider two linked loops that carry flux $Xe^{i\theta_1 Q}$ and $Xe^{i\theta_2 Q}$, respectively. The commutator flux $e^{i2(\theta_2-\theta_1)Q}$ is in H_c, so that linking is allowed. The strength of the dipole field is proportional to $\theta_2 - \theta_1 \pmod{2\pi}$. If we fix the positions of the loops and specify initial values for θ_1 and θ_2, then, since the dipole field costs magnetostatic energy, the angle $\theta_2 - \theta_1$ will oscillate and the dipole field will become time dependent. These oscillations will cause emission of radiation, and $\theta_2 - \theta_1$ will decay, eventually approaching zero.

3.5 Concluding Remarks

⋆ We thank Tom Imbo for a helpful discussion about this.

3.5.1 Twisted Strings and Pointlike Monopoles

In any model in which a connected gauge group G breaks to a group H that has a disconnected component, there will be topologically stable strings. If, in addition, H contains noncontractible closed paths (and G is simply connected), then a string loop can have a topological twist. We have seen that the topological charge that characterizes the twist on the string is precisely the H magnetic charge. Indeed, magnetically charged string loops exist in any model that contains both strings and magnetic monopoles.

But it is not guaranteed that the twist will be *stable*. Even if the loop is held in place so that it does not contract, it may be energetically favorable for the loop to untwist and spit out a pointlike magnetic monopole. Or conceivably, the monopole may prefer to stick to the string. In other words, the twisting may be localized in the vicinity of a point on the string loop, so that the pointlike magnetic monopole is a "bead" that can slide along the string.†

Unless there is a symmetry to enforce a degeneracy, vortices with different values of the flux are generically expected to have different masses. Correspondingly, twisting the flux on a string loop will generically increase the potential energy stored in the core of the string. Thus, the flux will prefer to stay near the value that minimizes the core energy, and will twist suddenly inside the "bead."

A gradual twist on a string is expected to be stable, then, only if the flux on the string stays in the orbit of the unbroken symmetry group of the theory. This symmetry cannot be a global symmetry. The identity component of the global symmetry group must commute with the gauge group, so the action of a global symmetry cannot rotate the flux continuously. We conclude that, barring accidental degeneracy, stable twists can reside only on Alice-like strings—that is, the unbroken gauge group must act nontrivially on the flux of the string.

Even for an Alice string, there may be values of the magnetic charge that cannot be carried by the string. To illustrate this point, consider a model with a hierarchy

† Here, in contrast to the beads considered in Ref. 22, the magnetic charge carried by the monopole is unconfined.

of symmetry breakdown

$$SU(3) \xrightarrow{v_1} SO(3) \xrightarrow{v_2} O(2) \ , \quad v_1 >> v_2 \ . \tag{39}$$

At the large symmetry breaking mass scale v_1, the gauge group $SU(3)$ is broken to its $SO(3)$ subgroup, embedded so that the 3 of $SU(3)$ transforms as the 3 of $SO(3)$. At the much smaller scale v_2, Alice symmetry breaking occurs, to the unbroken group $O(2)$. Thus, this model contains Alice strings with tension of order v_2^2, and magnetic monopoles with mass of order v_2/e, where e is the gauge coupling. The properties of these strings and monopoles can be accurately described in an effective field theory, with the very-short-distance physics at scale v_1 integrated out. This effective theory is essentially the model discussed in Section 2.

The new feature of this model, though, is that, because $SO(3)$ is embedded in $SU(3)$, there is a heavy monopole with mass of order v_1/e, that carries half the magnetic charge of a light monopole. (The magnetic charge of a light monopole is characterized by a noncontractible loop in $U(1)$ that begins at the identity and ends at a 4π rotation, a loop that is contractible in $SO(3)$. The magnetic charge of the heavy monopole is characterized by a loop that ends at a 2π rotation. This loop is noncontractible in $SO(3)$, but can be contracted in $SU(3)$.) Therefore, the magnetic charge carried by a twisted loop of Alice string is actually an *even* multiple of the magnetic charge quantum in this model. Correspondingly, since all representations of $SU(3)$ are *integer* spin representations of $SO(3)$, the string *can* carry the *electric* charge quantum.

As our general discussion in Section 4.1 indicates, it *is* in a sense possible to construct string configurations with a "half twist" that carry the magnetic charge quantum. Outside this string, the Higgs field takes values in the $SU(3)/O(2)$ vacuum manifold. But unlike the case of a string with a full twist, the Higgs field does not stay within the $SO(3)/O(2)$ submanifold. Heavy Higgs fields (and gauge fields) are necessarily excited in the core of the string. It is energetically favorable for the half twist to shrink to a bead, of to form a heavy monopole that is ejected from the loop. (Which of these scenarios is realized depends on the details of the model.) The *stable* twists on the string can be correctly analyzed using the effective theory, in which the higher scale of symmetry breakdown is disregarded.

Even if v_1 and v_2 are of the same order of magnitude, so that there is no large symmetry breaking hierarchy, the half twist is expected to be unstable in this model. For the Alice string with a stable twist, the "vortices" that are associated with various slices through the string are related by a symmetry, and so are degenerate. There is no symmetry to enforce such a degeneracy for the string with a half twist. So introducing a half twist generically increases the core energy, which destabilizes the twist.

3.5.2 Stable Loops

A loop with Cheshire magnetic charge will have Coulomb energy of order

$$E_{\text{Coulomb}} \sim \frac{p^2}{e^2\ R} \ , \tag{40}$$

where p is the charge in units of the Dirac magnetic charge, e is the gauge coupling, and R is the size of the loop. When the charge p is large, classically stable string loop configurations can be constructed, such that the Coulomb potential energy prevents the loop from collapsing. If the charge is Cheshire charge, then the size R and mass m of a stable loop are, in order of magnitude,

$$R \sim \left(\frac{p}{e}\right) \kappa^{-\frac{1}{2}} \ , \quad m \sim \left(\frac{p}{e}\right) \kappa^{\frac{1}{2}} \ , \tag{41}$$

where κ is the string tension. Though classically stable, these string loops are not expected to be absolutely stable; they will emit elementary monopoles via a quantum tunneling process, assuming that the emission is kinematically allowed.

3.5.3 Cosmological Alice Strings

Because a twisted Alice string carries magnetic charge, we see that a phase transition that produces Alice strings must also produce magnetic monopoles. This observation significantly restricts the role that Alice strings can play in cosmology. If Alice strings ever appeared in the early universe, an unacceptably large abundance of monopoles must have appeared at the same time.[23] Some subsequent mechanism must have reduced the monopole abundance to an acceptable level; we should then ask whether the strings could have survived. For example, if the phase transition that produced the monopoles were followed by an inflationary epoch, the monopole abundance would have been drastically reduced. But, of course, the inflation also would have made Alice strings extremely scarce.

One caveat should be mentioned. The remark in the previous paragraph applies to any model such that the unbroken gauge group H contains a $U(1)$ factor and a charge conjugation operator that acts nontrivially on the $U(1)$ generator Q. But it need not apply to models that exhibit the generalized Alice-like behavior considered at the end of Section 2. In particular, a model with the unbroken group $H = [SU(2)_1 \times SU(2)_2] \times_{S.D.} Z_2$ contains generalized Alice strings. But since H is simply connected, this model contains no magnetic monopoles.

We should note another generic feature of realistic models that contain Alice strings. Charge conjugation is not a manifest exact symmetry in Nature. So if charge conjugation is a *local* symmetry (which must be exact), it must be *spontaneously* broken. In a realistic model, then, Alice strings (at low temperature) are boundaries of charge–conjugation domain walls. If Alice strings were ever present in the early universe, they eventually became confined by walls. Soon after the appearance of the walls, the string network broke up and disappeared.[24,25,6]

3.5.4 Global Strings and Monopoles

Finally, we remark that the discussion of magnetic charge transfer in Section 4 also applies to the line and point defects that arise when a *global* symmetry group G becomes spontaneously broken to a subgroup H. (Such defects can occur in certain condensed matter systems, such as nematic liquid crystals.[19]) By a standard argument,[16,6,19] the magnetic charge, classified by $\pi_1[H]$, is seen to be equivalent to the topological charge of the order parameter Φ, classified by $\pi_2[G/H]$ (assuming that G is simply connected). Thus, our previous analysis applies, without modification, to the transfer of topological charge between a "global monopole" and a "global Alice string."

Recently, Brekke, Fischler, and Imbo[26] have independently investigated the properties of magnetically charged Alice strings.

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FIGURE CAPTIONS

1) The curve C, starting and ending at the point x_0, encloses a loop of cosmic string.

2) A circle, parametrized by ϕ, encloses an Alice string. Corresponding to each point of the circle is an unbroken symmetry group $H(\phi)$ that stabilizes the condensate $\Phi(\phi)$ at that point.

3) Initially two particles carry charge of the same sign. But after one of the particles travels around the string, the particles carry charge of opposite sign.

4) The surface S is a cut at which the electric field changes sign. The loop in (b) carries Cheshire charge.

5) A particle that initially has positive charge travels through a loop of Alice string along the path shown in (b). The electric field during the process is indicated schematically in (c)–(e).

6) The flux on the two string loops $\mathcal{C}_1$ and $\mathcal{C}_2$ is defined with respect to the basepoint x_0 and the paths C_1 and C_2. When $\mathcal{C}_2$ winds through $\mathcal{C}_1$ as in (b), the paths C_1' and C_2' are dragged to C_1 and C_2.

7) The family of closed paths $\{C_\phi \mid 0 \leq \phi < 2\pi\}$ sweeps out a degenerate torus that encloses the Alice string loop.

8) A family of closed paths $\{C_\phi'\}$ obtained by smoothly deforming the family $\{C_\phi\}$.

9) A family of loops C_ϕ'' that sweeps over the surface of a sphere. The loops C_0'' and $C_{2\pi}''$ are degenerate.

10) The free surface $\bar{\Sigma}$ in (a) can be threaded to the basepoint x_0 in inequivalent ways, two of which are illustrated in (b) and (c). The surface (c) can be deformed to (d), which differs from (b) by the degenerate tube β that begins and ends at the basepoint.

11) The magnetic flux of the string loops $\mathcal{C}_1$ and $\mathcal{C}_2$ is defined in terms of the paths β_1 and β_2 shown in (a), and the magnetic charges of the loops are defined in terms of the surfaces a_1 and a_2; the paths and the surfaces are based at the point x_0. When $\mathcal{C}_2$ winds through $\mathcal{C}_1$ as in (b), the surface a_1' shown in (c) is

dragged to a_1, and the surface a_2' shown in (d) is dragged to a_2. The arrows on the surfaces indicate outward-pointing normals.

12) Deformations of the surfaces shown in Fig. 11c-d. In (a), the surface a_2' has been deformed to the degenerate tube β_1 plus the surface a_2. In (b), the surface a_1' has been deformed to the surface $a_1 \circ a_2$ that encloses both loops, plus the inverse of a_2' (that is, a_2' with the orientation reversed); the surface $(a_2')^{-1}$ is the sum of the degenerate tube $(\beta_1)^{-1}$ and the surface $(a_2)^{-1}$.

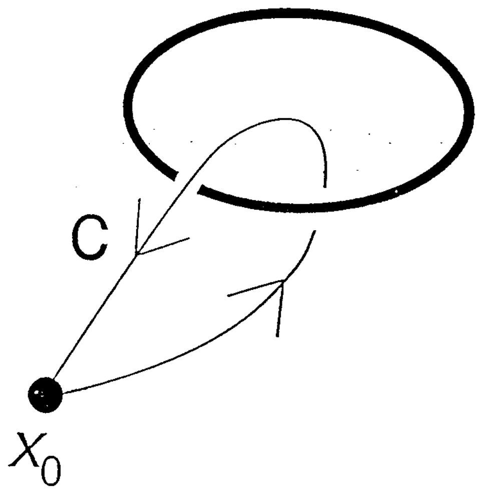

FIGURE 1

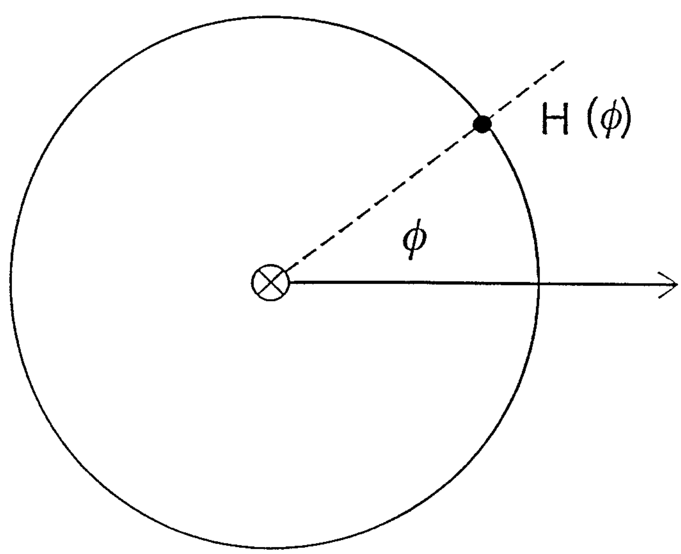

FIGURE 2

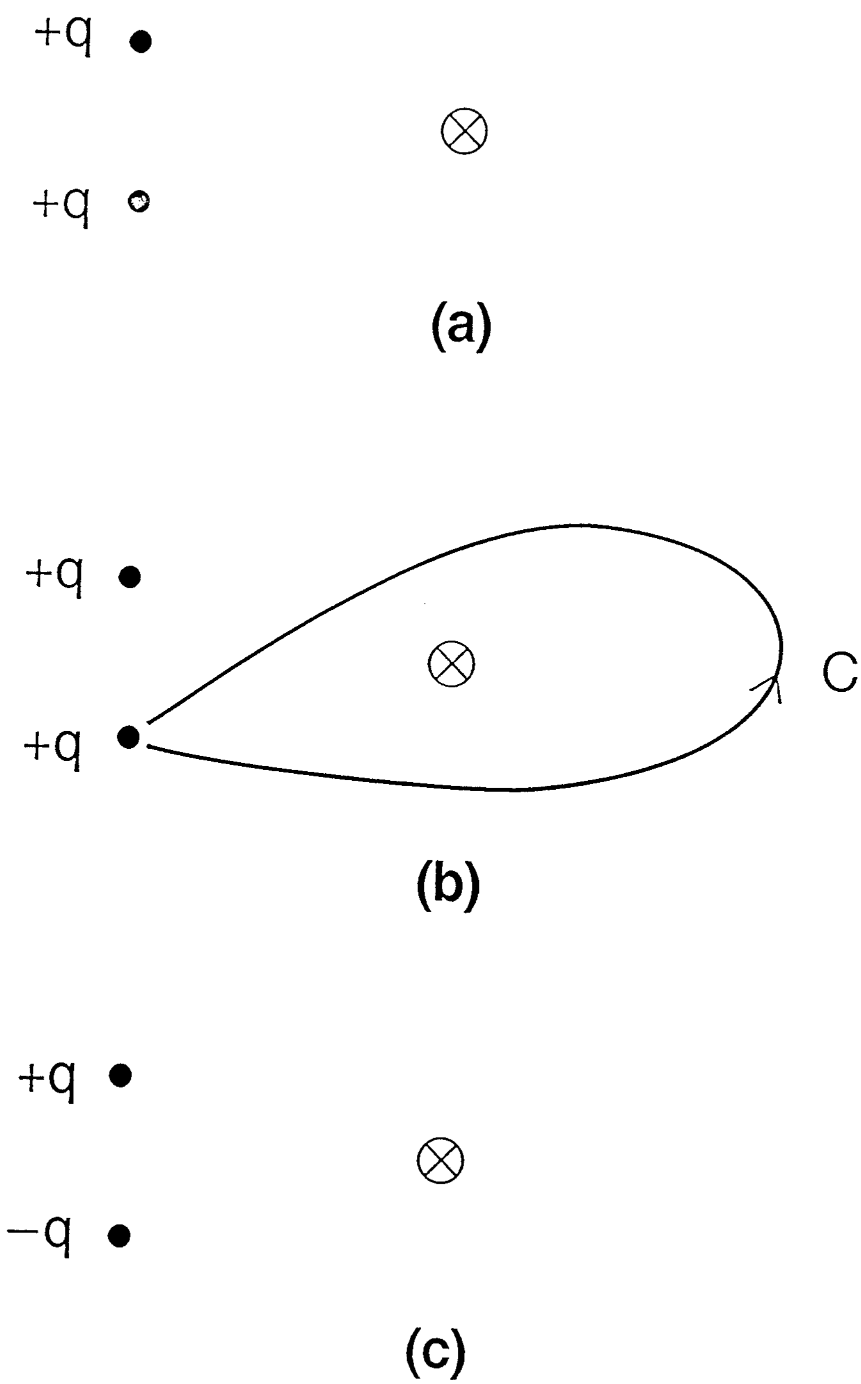

FIGURE 3

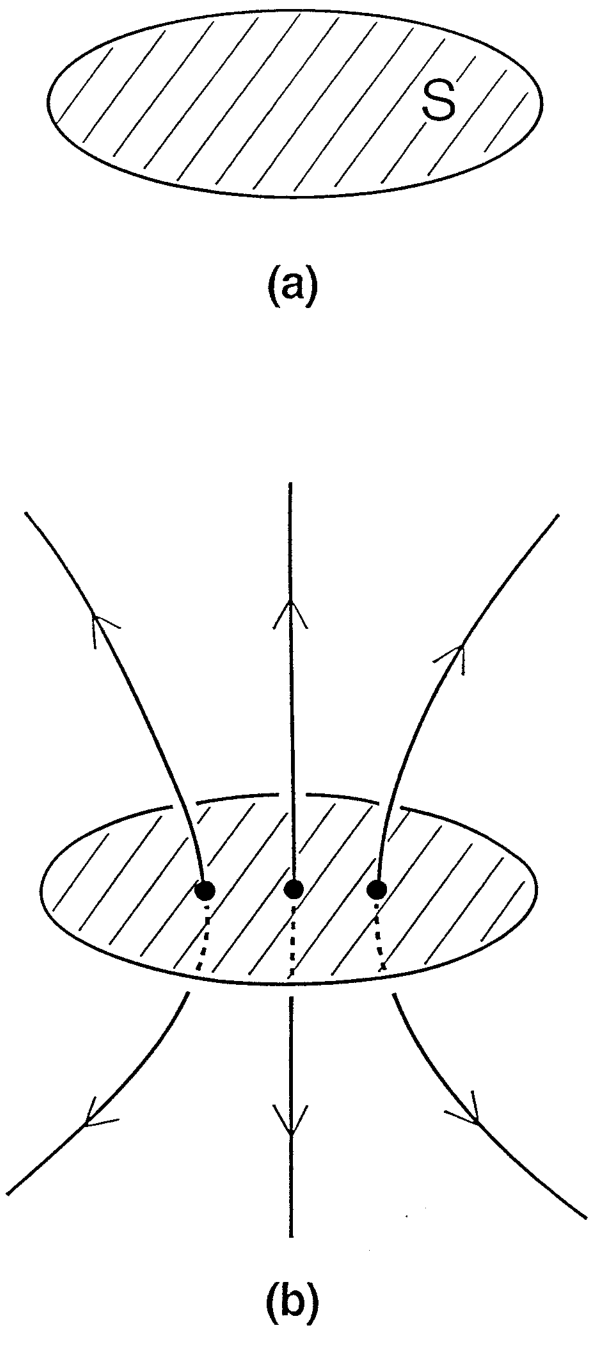

FIGURE 4

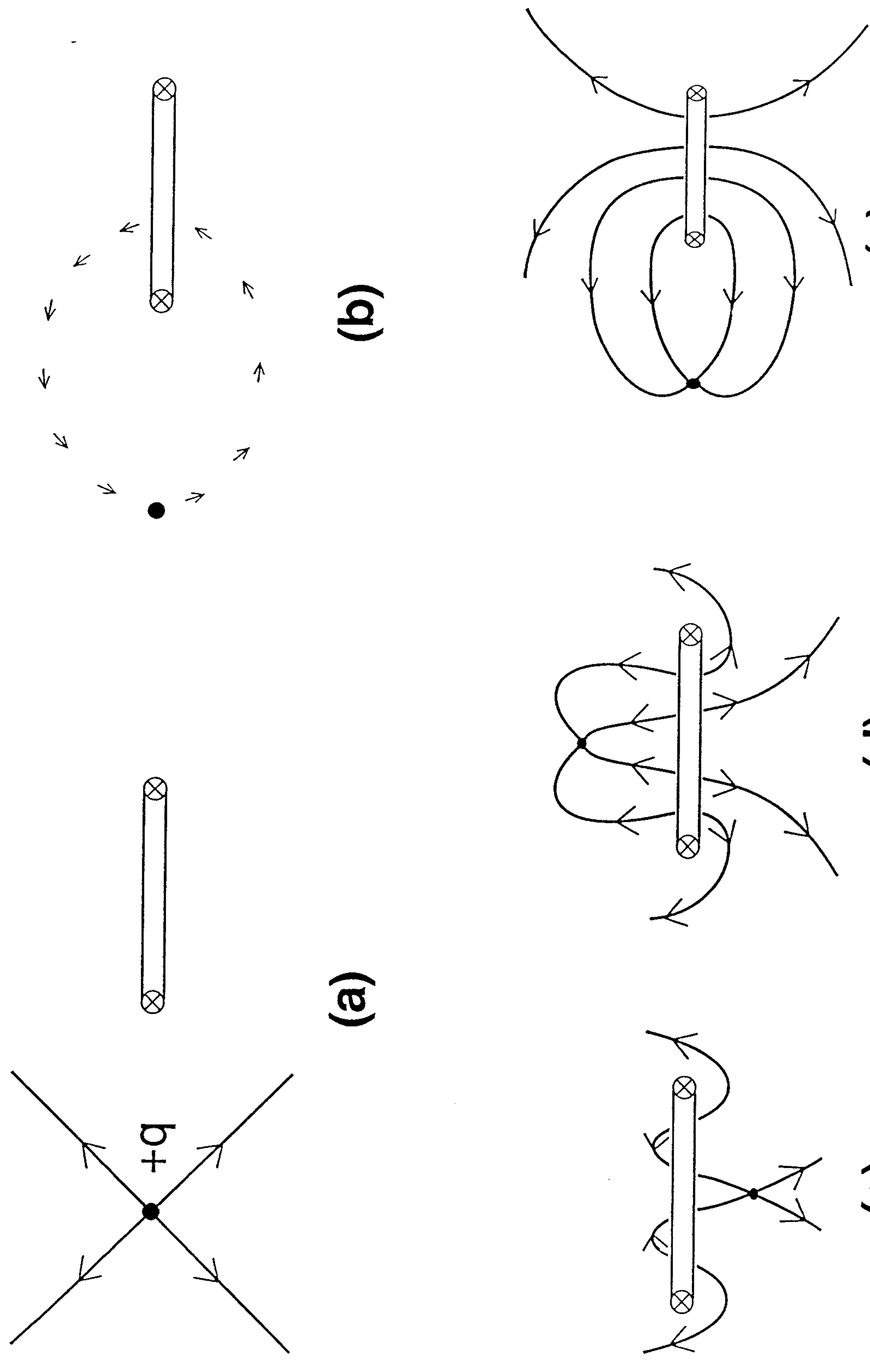

FIGURE 5

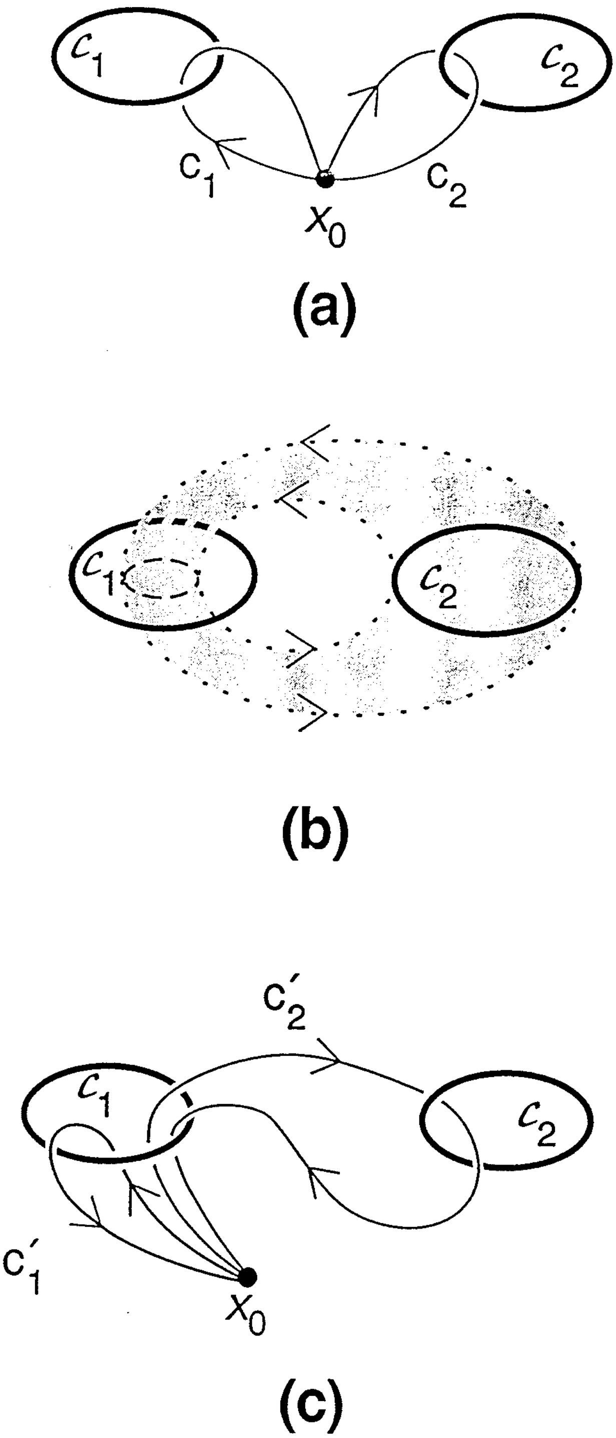

FIGURE 6

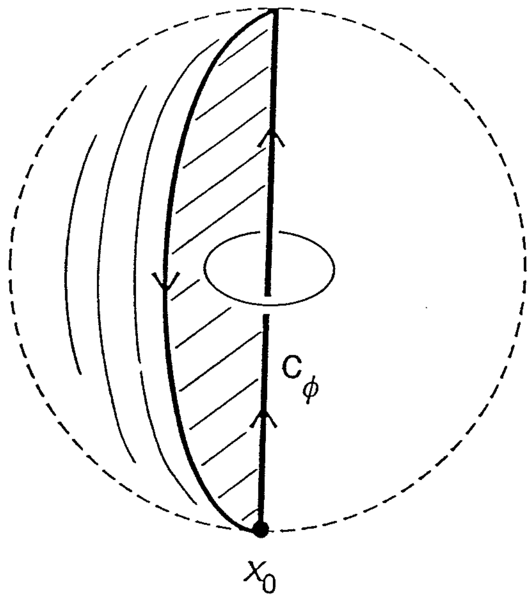

FIGURE 7

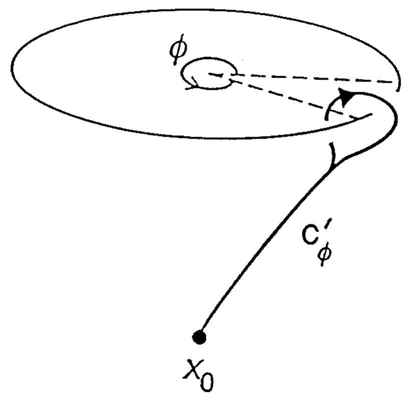

FIGURE 8

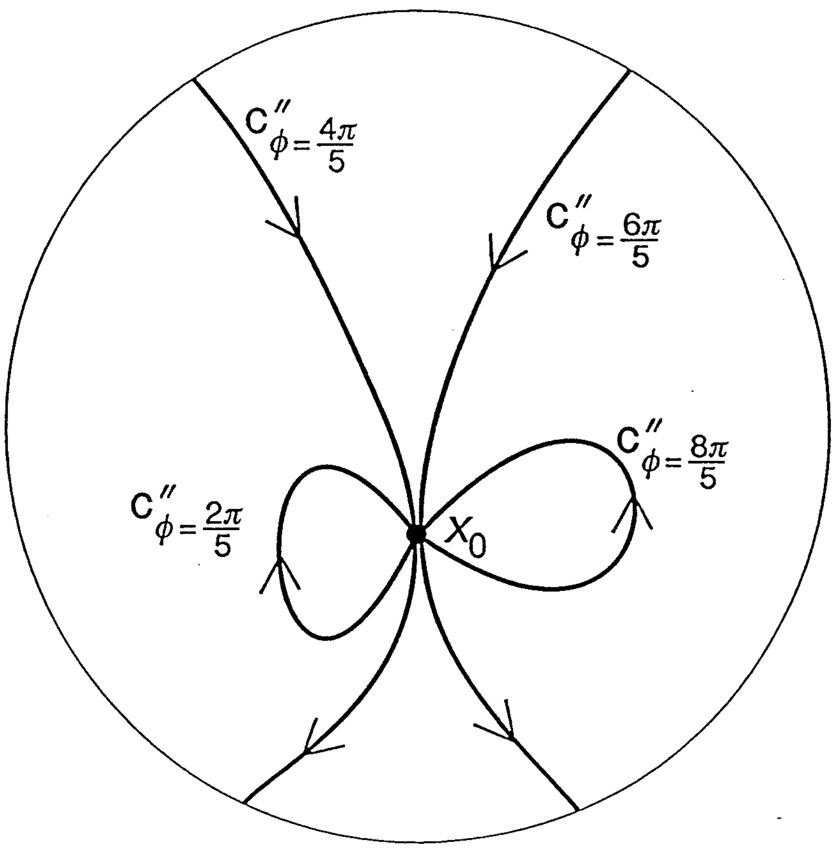

FIGURE 9

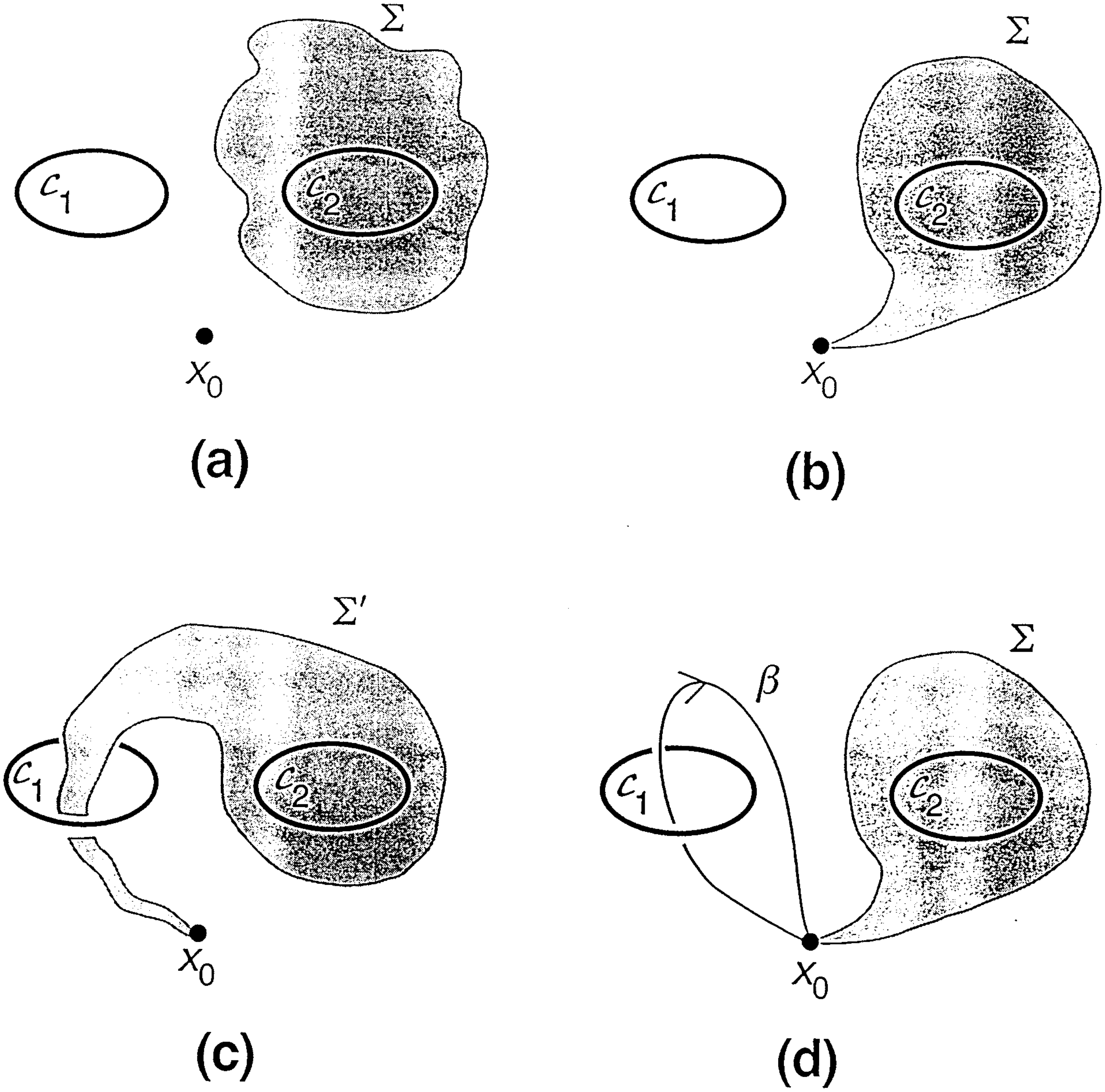
$\overline{\Sigma}$
c_1
c_2
x_0
(a)
Σ
c_1
c_2
x_0
(b)
Σ'
c_1
c_2
x_0
(c)
β
Σ
c_1
c_2
x_0
(d)

FIGURE 10

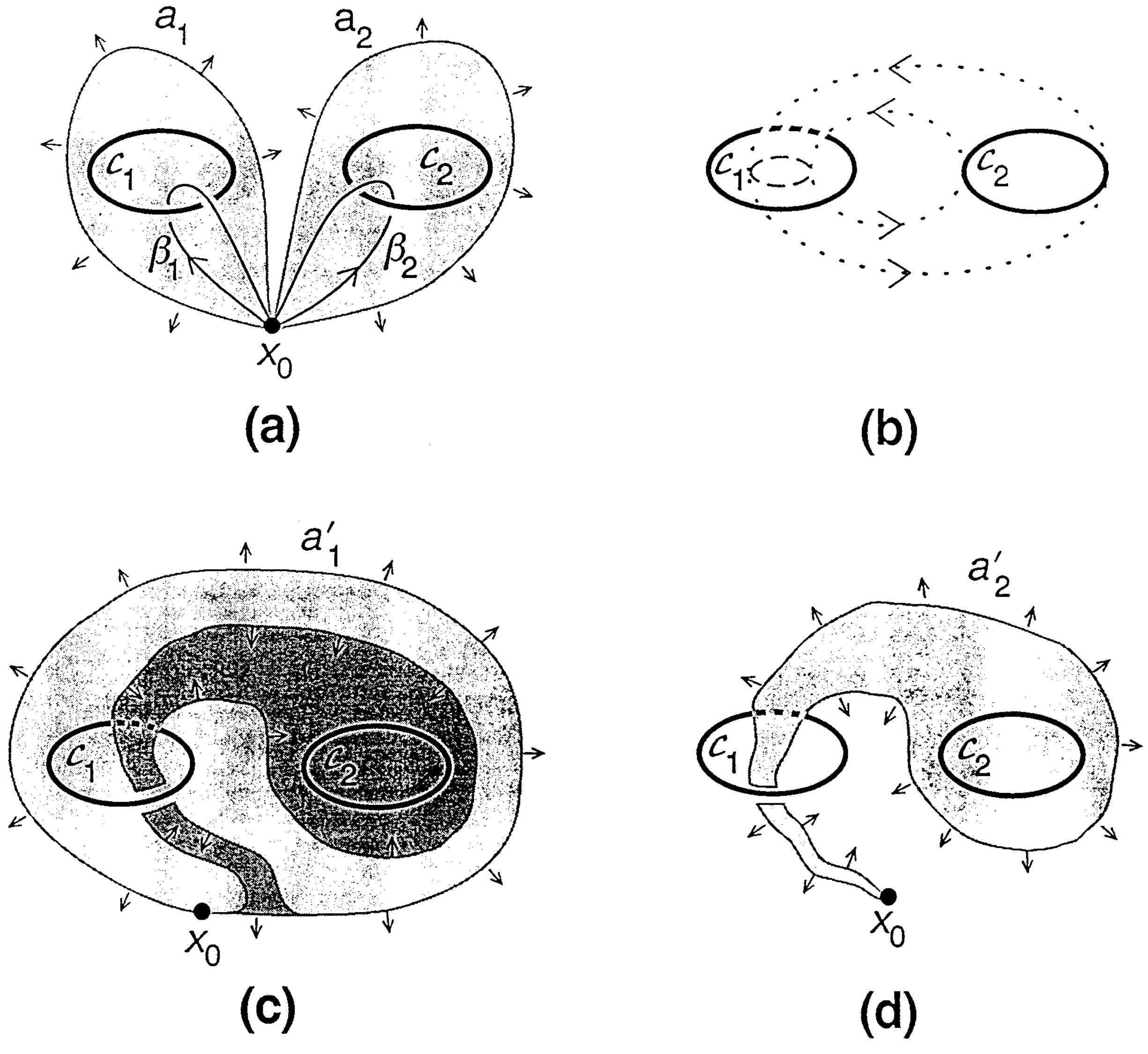

FIGURE 11

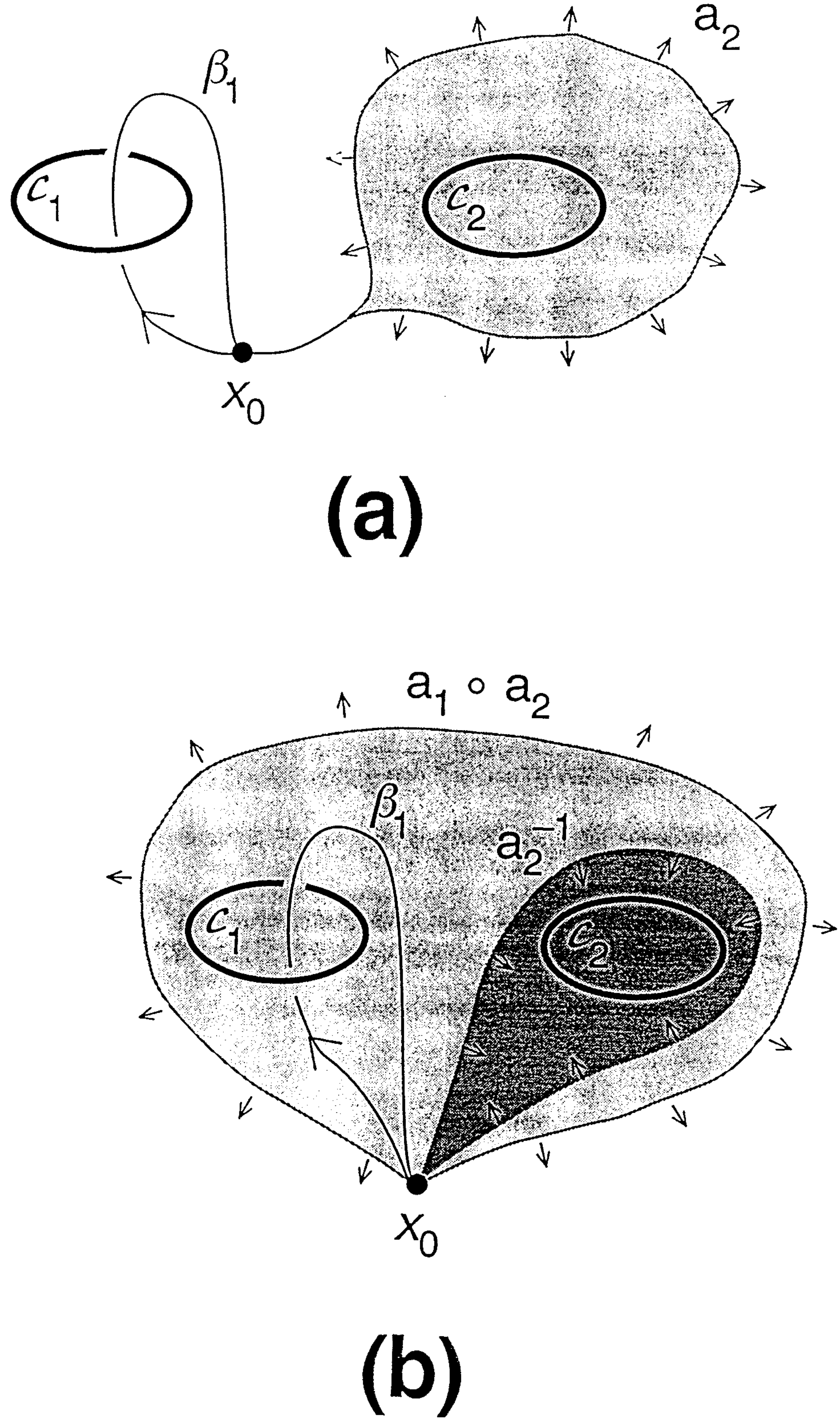

FIGURE 12

Chapter 4
Non-Abelian Vortices and Non-Abelian Statistics

4.1 Introduction

It is well known that exotic generalizations of fermion and boson statistics are possible in two spatial dimensions. The simplest, and most familiar, such generalization is anyon statistics. [1,2] When two indistinguishable anyons are adiabatically interchanged (or one anyon is rotated by 2π), the many-body wave function acquires the phase $e^{i\theta}$, where θ can take any value. An instructive example of an object that obeys anyon statistics is a composite of a magnetic vortex (with magnetic flux Φ) and a charged particle (with charge q). [2] Then the anyon phase arises as a consequence of the Aharonov-Bohm effect, with $e^{i\theta} = e^{iq\Phi}$. Furthermore, anyon statistics is actually known to be realized in nature, in systems that exhibit the fractional quantum hall effect. [3]

It is natural to consider a further generalization: non-Abelian statistics. [4–7] A particular type of non-Abelian statistics is realized by the non-Abelian vortices (and vortex-charge composites) that occur in some spontaneously broken gauge theories. Loosely speaking, the unusual feature of the many-body physics in this case is that the quantum numbers of an object depend on its *history*. In particular, if one vortex is adiabatically carried around another, the quantum numbers of both may change, due to a non-Abelian variant of the Aharonov-Bohm effect. Thus, whether two bodies are identical is not a globally defined notion.

There is no firm observational evidence for the existence of objects that obey this type of quantum statistics. Perhaps such objects will eventually be found in suitable condensed matter systems. (Analogous non-Abelian defects associated with spontaneous breakdown of *global* symmetries are observed in liquid crystals [8] and ^3He. [9]) In any event, the physics of non-Abelian vortices is intrinsically interesting and instructive. For one thing, it forces us to carefully consider some subtle aspects of non-Abelian gauge invariance.

In this chapter, we will focus on the Aharonov-Bohm interactions of a pair of non-Abelian vortices. This is, of course, much simpler and much less interesting than the problem of three or more bodies. Nevertheless, an important conceptual point will be illuminated by our calculation of the vortex-vortex scattering cross section. We will see that this cross section is in general multivalued. While we have learned to be undisturbed, at least in certain contexts, by multivalued wave functions, a cross section is directly observable, and so is ordinarily expected to be a single-valued function of the scattering angle. But the multivaluedness of the cross section for vortex-vortex scattering follows naturally from the ambiguity in assigning quantum numbers to the vortices.

Indeed, multivalued scattering cross sections are a generic consequence of the non-Abelian Aharonov-Bohm effect—they arise in the scattering of a charge off a vortex as well. It is useful to consider the case of the "Alice" vortex, [10–13] which has the property that when a positively charged particle is adiabatically transported around the vortex, it becomes negatively charged. When a positively charged particle scatters from an Alice vortex, the scattered particle may be either positively charged or negatively charged. Thus there are two measurable exclusive cross sections,* $\sigma_+(\theta)$ and $\sigma_-(\theta)$. Though the inclusive cross section $\sigma_{\rm inc} = \sigma_+ + \sigma_-$ is single valued, the exclusive cross sections are not; they are double-valued and obey the conditions

$$\sigma_+(\theta + 2\pi) = \sigma_-(\theta) , \qquad \sigma_-(\theta + 2\pi) = \sigma_+(\theta) . \tag{1}$$

The double-valuedness of the exclusive cross sections is an unavoidable consequence of the feature that a charged particle that voyages around the Alice vortex returns to its starting point with its charge flipped in sign. We might imagine measuring the θ-dependence of the cross section by gradually transporting a particle detector around the scattering center. But then a detector that has been designed to respond to positively charged particles will have become a detector that responds to negatively charged particles when it returns to its starting point. Alternatively, we might catch the scattered particle, and then carry it back along a specified path to a central laboratory for analysis. But then the outcome of the analysis will depend upon the path taken. While we may (arbitrarily) associate a definite path with each value of

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*Strictly speaking, these cross sections do not exist, because there are no asymptotic charged particle states in two-dimensional electrodynamics; see section 7 for further discussion.

the scattering angle, this path cannot vary continuously with θ. A convention for choosing the path artificially restricts the exclusive cross sections to a single branch of the two-valued function, and introduces a discontinuity in the measured cross sections. As we will discuss in more detail below, the cross sections for non-Abelian vortex-vortex scattering have similar multivaluedness properties.

In the case of vortex-vortex scattering (unlike the case of scattering a charged particle off of a vortex), effects of quantum statistics can be exhibited. That is, there may be an exchange contribution to the scattering amplitude that interferes with the direct amplitude. The existence of an exchange contribution means that the two vortices must be regarded as indistinguishable particles—it is not possible in principle to keep track of which vortex is which. The unusual feature of non-Abelian vortex-vortex scattering is that exchange scattering can occur even if the initial vortices are objects with distinct quantum numbers. The vortices are different, yet they are indistinguishable.

Much that we will say in this chapter has been anticipated elsewhere. That the quantum numbers of non-Abelian vortices cannot be globally defined was first emphasized by Bais. [14] (The corresponding observation for defects associated with a spontaneously broken *global* symmetry was made earlier, by Poénaru and Toulouse. [15]) Wilczek and Wu [6] and Bucher [7] discussed the implications for vortex-vortex scattering. E. Verlinde [16] worked out a general formula for the *inclusive* cross section in Aharonov-Bohm scattering, in terms of the matrix elements of the "monodromy" operator, and Bais *et al.* [17] developed a powerful algebraic machinery that can be used to compute these matrix elements (among other things). The main new contributions here are a computation of the *exclusive* cross sections for the various possible quantum numbers of the final state vortices, and an analysis of vortex-vortex scattering that incorporates the exchange of "indistinguishable" vortices. (Wilczek and Wu [6] attempted to calculate the exclusive cross sections, but because they missed the multivaluedness properties of these cross sections, they did not obtain the correct answer.) Once properly formulated, the calculation of these exclusive cross sections is very closely related to the analysis of scattering in (2+1)-dimensional gravity, which was first worked out by 't Hooft [18] and Deser and Jackiw. [19]

The remainder of this chapter is organized as follows: In section 2, we review how the the quantum numbers of non-Abelian vortices are modified by an exchange, and we extend the discussion in section 3 to the case of vortices that also carry charge. We recall the general theory of the quantum mechanics of indistinguishable particles in section 4, and describe how the special case of non-Abelian vortices fits into this general theory. In section 5, we calculate the exclusive cross sections for non-Abelian Aharonov-Bohm scattering of a projectile off of a fixed target. The case of vortex-vortex scattering is analyzed in detail, and we emphasize and explain the multivaluedness properties of these cross sections. The case of two-body scattering in the center-of-mass frame is discussed in section 6. This calculation includes the contribution due to the exchange of "indistinguishable" vortices. In section 7, we extend the previous discussion to the case where the unbroken gauge group is continuous, such as the case of the "Alice" vortex. Section 8 contains our conclusions.

4.2 Non-Abelian Flux and the Braid Operator

We consider, in two spatial dimensions, a gauge theory with underlying gauge group G, which we may take to be connected and simply connected. Suppose that the gauge symmetry is spontaneously broken, and that the surviving manifest gauge symmetry is H. We will assume for now that H is discrete and non-Abelian. The case of continuous H will be briefly discussed in section 7.

This pattern of symmetry breaking will admit stable classical vortex solutions. A vortex carries a "flux" that can be labeled by an element of the unbroken group H. To assign a group element to a vortex, we arbitrarily choose a "basepoint" x_0 and a path C, beginning and ending at x_0 that winds around the vortex. The effect of parallel transport in the gauge potential of the vortex is then encoded in

$$a(C, x_0) = P\exp\left(i\int_{C,x_0} A\right) \in H(x_0)\ . \tag{2}$$

This group element takes a value in the subgroup $H(x_0)$ of G that preserves the Higgs condensate at the point x_0, since transport of the condensate around the vortex must return it to its original value. If H is discrete, then $a(C, x_0)$ will remain unchanged as the path C is smoothly deformed, as long as the path never crosses the cores of any vortices. (The gauge connection is locally flat outside the vortex cores, with curvature singularities at the cores.)

The flux of a vortex can be measured via the Aharonov-Bohm effect. [20,21] We can imagine performing a double slit interference experiment with a beam of particles that transform as some representation R of H. If we then repeat the experiment with the vortex placed between the two slits, the change in the interference pattern reveals

$$\langle u^{(R)}|D^{(R)}(a)|u^{(R)}\rangle \,, \tag{3}$$

where $|u^{(R)}\rangle$ is the internal wave function of the particles in the beam. (The shift in the interference fringes is determined by the phase of this quantity, and the amplitude of the intensity modulation is determined by its modulus.) By measuring this for various $|u^{(R)}\rangle$'s, all matrix elements of $D^{(R)}(a)$ can be determined, and hence, if the representation is faithful, a itself.

However, the flux of the vortex is not a gauge-invariant quantity. A gauge transformation $h \in H(x_0)$ that preserves the Higgs condensate at the basepoint transforms the flux according to

$$h: \quad a \rightarrow hah^{-1} \,. \tag{4}$$

(This gauge transformation is just a relabeling of the particles that are used to perform the measurement of the flux.) Since the gauge transformations act transitively on the conjugacy class in H to which the flux belongs, one might be tempted to say that the flux of a vortex should really be labeled by a conjugacy class rather than a group element. But that is not correct. If there are two vortices, labeled by group elements a and b with respect to the same basepoint x_0, then the effect of a gauge transformation at x_0 is

$$h: \quad a \rightarrow hah^{-1} \,, \quad b \rightarrow hbh^{-1} \,. \tag{5}$$

Thus, if a and b are distinct representatives of the same class, they remain distinct in any gauge.

More generally, we can imagine assembling a "vortex bureau of standards," where standard vortices corresponding to each group element are stored. If a vortex of unknown flux is found, we can carry it back to the bureau of standards and determine which of the standard vortices it matches. (Alternatively, we can find out which antivortex it annihilates.) Thus, though there is arbitrariness in how we assign group elements to the standard vortices, once our standards are chosen there is no ambiguity in assigning a label to the new vortex.

We might have said much the same thing about measuring the color of a quark. Although the color is not a gauge-invariant quantity, we can erect a quark bureau of standards in which standard red, yellow, and blue quarks are kept. When a new quark is found, we can carry it back to the bureau and determine its color relative to our standard basis. However, in the case where there are light gauge fields, curvature of the gauge connection is easily excited. We may find, then, that the outcome of the measurement of the color depends on the path that is chosen when the quark is transported back to the bureau.

In the case where the unbroken gauge group is discrete, there are no light gauge fields. The measurement of the flux of a vortex is unaffected by a deformation of the path that is used to bring the vortex to the bureau of standards, as long as the path does not cross the cores of any other vortices. But when other vortices are present, there is a discrete choice of topologically distinct paths, and the measured flux will in general depend on how we choose to weave the vortex among the other vortices on the way back to the bureau. This ambiguity in measuring the flux is the origin of the "holonomy interaction" among vortices, [14] and of Aharonov-Bohm vortex-vortex scattering. [6,7]

To characterize this interaction, we consider how the fluxes assigned to a pair of vortices are modified when the two vortices are adiabatically interchanged, as depicted in Fig. 1. Here α and β are standard paths, beginning and ending at the basepoint x_0, that are used to define the flux of the two vortices; the corresponding group elements are a and b respectively. When the two vortices are interchanged (in a counterclockwise sense), these paths can be dragged to new paths α' and β', in such a way that no path ever crosses any vortex. Thus, the group elements associated with transport along α' and β' are, after the interchange, still a and b respectively. But the final paths are topologically distinct from the initial paths; from Fig. 1b we see that

$$\alpha' = \beta \rightarrow a \ , \quad \beta' = \beta^{-1}\alpha\beta \rightarrow b \ . \tag{6}$$

(Here, in order to be consistent with the rules for composing path-ordered exponentials, we have chosen an ordering convention in which $\alpha\beta$ denotes the path obtained by first traversing β, then α.) We conclude that, after the interchange, the effect of parallel transport around α is given by the group element aba^{-1}. The effect of the interchange on the two vortex state can be expressed as the action of the *braid operator* $\mathcal{R}$, where

$$\mathcal{R}: \quad |a,b\rangle \rightarrow |aba^{-1},a\rangle \ . \tag{7}$$

Naturally, the braid operator preserves the "total flux" ab that is associated with counterclockwise transport around the vortex pair, for this flux could be measured by a particle that is very far away from the pair, and cannot be affected by the interchange. If the interchange is performed twice (which is equivalent to transporting one vortex in a counterclockwise sense about the other), the state transforms according to

$$\mathcal{R}^2: \quad |a,b\rangle \rightarrow |(ab)a(ab)^{-1},(ab)b(ab)^{-1}\rangle \ ; \tag{8}$$

both fluxes are conjugated by their combined "total flux" ab.

This result has a clear, gauge-invariant meaning. Suppose that two vortices are carried from their initial positions to the vortex bureau of standards along the paths shown in Fig. 2a, and are found to have fluxes a and b. Then if they are carried to the bureau along the alternative paths shown in Fig. 2bc, the outcome of the flux measurement will be different, as expressed in Eq. (8).

4.3 Flux-Charge Composites

The above discussion can be generalized to the case of objects that carry both flux and charge. But there is one noteworthy subtlety. The "charge" of an object is defined by its transformation properties under global gauge transformations. If the object carries flux, however, there is a topological obstruction to implementing the global gauge transformations that do not commute with the flux. [22,11,12] If the flux is a, only a subgroup of H, the centralizer $N(a)$ of the flux, is "globally realizable" acting on the vortex. Thus, the charged states of a vortex with flux a transform as a representation of $N(a)$ rather than the full group H.

We can understand the physical meaning of this obstruction if we think about measuring charge via the Aharonov-Bohm effect. The charge can be measured in a double slit interference experiment, by observing the effect on the interference pattern when various vortices are placed between the slits. But if the particles in the beam carry flux a, and the vortex between the slits carries flux b, then no interference pattern is seen if a and b do not commute. The trouble is that, due to the holonomy interaction, the objects that pass through the respective slits carry different values of the flux when they arrive at the detector, and so do not interfere. (See Fig. 3.) Even more to the point, the slit that the object passed through becomes correlated with the state of the vortex that is placed between the slits, because both fluxes become conjugated as in Eq. (8). Thus, the superposition of particles that passed through the two slits becomes incoherent, and there is no interference. There will be an interference pattern, and a successful charge measurement, only if the flux between the slits commutes with the flux a carried by the particles in the beam. Hence only the transformation properties under $N(a)$ can be measured.

Since the global gauge transformations that can be implemented actually commute with the flux, a non-Abelian vortex that carries charge behaves much like an Abelian flux-charge composite. If the vortex carries flux a and transforms as an irreducible representation ($R^{(a)}$) of $N(a)$, then, since a lies in the center of its centralizer $N(a)$, it is represented by a multiple of the identity in $R^{(a)}$ (by Schur's lemma),

$$D^{(R^{(a)})}(a) = e^{i\theta_{R^{(a)}}}\,\mathbf{1}_{R^{(a)}} \ . \tag{9}$$

Thus, the charged vortices are anyons, and $e^{i\theta_{R^{(a)}}}$ is the anyon phase. A spin-statistics connection holds for these anyons, [2,23] in the sense that an adiabatic interchange of a pair is equivalent to rotating one by 2π—we have $e^{2\pi i J} = e^{i\theta_{R^{(a)}}}$.

The non-Abelian character of the vortices becomes manifest when we consider combining together two flux-charge composites, and decomposing into states of definite charge. The decomposition has the form

$$|a, R^{(a)}\rangle \otimes |b, R^{(b)}\rangle = \oplus_R |ab, R^{(ab)}\rangle \ , \tag{10}$$

where $R^{(a)}$ denotes an irreducible representation of $N(a)$. The nontrivial problem of decomposing a direct product of a representation of $N(a)$ and a representation of $N(b)$ into a direct sum of representations of $N(ab)$ is elegantly solved by the representation theory of quasi-triangular Hopf algebras, as described in the beautiful paper of Bais *et al.* [17] (see also Refs. 25 and 26). This decomposition also diagonalizes the monodromy matrix $\mathcal{M} \equiv \mathcal{R}^2$ that acts on the two vortex state when one vortex winds (counterclockwise) around the other: [27,25]

$$\mathcal{M} \equiv \mathcal{R}^2 = \exp\left[i\left(\theta_{R^{(ab)}} - \theta_{R^{(a)}} - \theta_{R^{(b)}}\right)\right] \ . \tag{11}$$

Eq. (11) follows from Eq. (9) and the spin-statistics connection for anyons, for the action of the monodromy operator is equivalent to a rotation of the vortex pair by 2π, accompanied by a rotation of each member of the pair by 2π in the opposite sense.

A remarkable property of this decomposition is that a pair of uncharged vortices can be combined together to form an object that carries charge. [11,12,17] This is called "Cheshire charge," in homage to the Cheshire cat; the charge can be detected via the Aharonov-Bohm interaction of the pair with another, distant, vortex, but it cannot be localized anywhere on the vortex cores or in their vicinity. Charge can be transferred to or from a pair of vortices due to the Aharonov-Bohm interactions of the pair with another charged object that passes through the two vortices. [28,29,21] Since the pair generically carries a fractional spin given by $e^{2\pi i J} = e^{i\theta_{R(ab)}}$, angular momentum is also transferred in these processes. [30]

4.4 Non-Abelian Quantum Statistics

In this section, we will briefly describe how the non-Abelian statistics obeyed by non-Abelian vortices fits into general discussions of quantum statistics that have appeared in the literature.

In general discussions of the quantum statistics of indistinguishable particles, the following framework is usually adopted: Suppose that the position of each particle takes values in a manifold M (like R^d). For n *distinguishable* particles, we would take the classical configuration space to be $M^n = M \times M \times \cdots \times M$. For *indistinguishable* particles (other than bosons), we must restrict the positions so that no two particles coincide, and we must identify configurations that differ by a permutation of the particles. Thus, the classical configuration space becomes

$$\mathcal{C}_n = [M^n - D_n]/S_n\ , \tag{12}$$

where D_n is the subset of M^n in which two or more points coincide, and S_n is the group of permutations of n objects. In general, this configuration space is not simply connected, $\pi_1(\mathcal{C}_n) \neq 0$.

We may now imagine quantizing the theory by using, say, the path integral method. The histories that contribute to the amplitude for a specified initial configuration to propagate to a specified final configuration divide up into disjoint sectors labeled by the elements of $\pi_1(\mathcal{C}_n)$. We have the freedom to weight the contributions from the different sectors with different factors, as long as the amplitudes respect the principle of conservation of probability. In general, there are distinct choices for these weight factors, which correspond to physically inequivalent ways of quantizing the classical theory. [31]

We can now define an "exchange operator" that smoothly carries the final particle configuration around a closed path in $\mathcal{C}_n$. Although this exchange does not disturb the positions of the particles, it mixes up the different sectors that contribute to the path integral. Since these sectors are weighted differently, in general, the exchange need not preserve the amplitude. This means that the amplitude need not be a single-valued function of the n positions of the final particles. The effect of the exchange can be expressed as the action of a linear operator acting on the amplitude, and because the total probability sums to one, this operator is unitary. By considering the effect of two exchanges performed in succession, we readily see that the exchange operators provide a unitary representation of the the group $\pi_1(\mathcal{C}_n)$. Thus, a unitary representation of $\pi_1(\mathcal{C}_n)$ acting on amplitudes (or wave functions) is a general feature of the quantum mechanics of n indistinguishable particles. (The weight factors appearing in the path integral also transform as a unitary representation of $\pi_1(\mathcal{C}_n)$.)

If the manifold is R^d for $d \geq 3$, then $\pi_1(\mathcal{C}_n) = S_n$, and the exchange operators provide a unitary representation of the permutation group S_n. In addition to the familiar one-dimensional representations associated with Bose and Fermi statistics, non-Abelian representations ("parastatistics") are also possible in principle. But it is known that, in a local quantum field theory, parastatistics can always be reduced to Bose or Fermi statistics by introducing additional degrees of freedom and a suitable global symmetry that acts on these degrees of freedom. [32] For $d = 1$, in this framework, no exchange is possible—the particles cannot pass through each other—and there is no quantum statistics to discuss.

The case $d = 2$ is the most interesting. Then $\pi_1(\mathcal{C}_n)$ is B_n, the braid group on n strands. This is an infinite group with $n-1$ generators $\sigma_1, \sigma_2, \cdots, \sigma_{n-1}$, where σ_j may be interpreted as a (counterclockwise) exchange of the particles in positions j and $j+1$. These generators obey the defining relations

$$\sigma_j \sigma_k = \sigma_k \sigma_j \ , \quad |j-k| \geq 2 \ , \tag{13}$$

and

$$\sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1} \ , \quad j = 1, 2, \cdots, n-2 \tag{14}$$

(The Yang-Baxter relation). It follows from the Yang-Baxter relation that, in a one-dimensional unitary representation of the braid group, all of the σ_j's are represented by a common phase $e^{i\theta}$. This is anyon statistics. But non-Abelian representations of the braid group may also arise in local quantum field theories. Indistinguishable particles in two dimensions that transform under exchange as a non-Abelian unitary representation of the braid group are said to obey non-Abelian statistics.

Our discussion of non-Abelian vortices fits into the general framework outlined above, but with an important caveat. If the vortex flux takes values in an unbroken local symmetry group $H(x_0)$, we treat two vortices with flux a and b as "indistinguishable" if $b = hah^{-1}$ for some $h \in H(x_0)$, and if both vortices have the same charge (transform as the same irreducible representation of the centralizer $N(a) \cong N(b)$). The philosophy is that the particles are regarded as indistinguishable if an exchange of the particles can conceivably occur (in the presence of other particles with suitable quantum numbers) without changing the quantum numbers assigned to the many-particle configuration. The caveat is that these "indistinguishable" particles are not really identical. For example, two vortices with flux a and b are distinct—*e.g.*, the a vortex will not annihilate the antiparticle of the b vortex—if $a \neq b$, even if a and b are in the same conjugacy class.

This classification of the different types of "indistinguishable" vortices can also be described in terms of the representation theory of a quasi-triangular Hopf algebra, or "quantum double". [17,25,26] The quantum double $D(H)$ associated with a finite group H is an algebra that is generated by global gauge transformations and projection operators that pick out a particular value of the flux. A basis for the algebra is†

$$\{P_h a \ , \quad h\, , a \in H\} \ , \tag{15}$$

where P_h projects out the flux value h, and a is a gauge transformation. Since the projection operators satisfy the relations

$$P_h P_g = \delta_{h,g} P_h \ , \quad a P_h a^{-1} = P_{aha^{-1}} \ , \tag{16}$$

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†In Ref. [17,25], the notation ${}^{h}\llcorner_{a}$ is used for $P_h a$.

the multiplication law for the algebra can be expressed as

$$(P_h a) \cdot (P_g b) = \delta_{h,aga^{-1}} (P_h ab) \ . \tag{17}$$

An irreducible representation of the quantum double $D(H)$ can be labeled $([a], R^{(a)})$, where $[a]$ denotes the conjugacy class that contains $a \in H$ and $R^{(a)}$ is an irreducible representation of the centralizer $N(a)$ of a. This is the induced representation of $D(H)$ generated by the representation $R^{(a)}$ of $N(a)$. The space on which this representation acts is a space of charged vortex states that transform irreducibly under the global gauge transformations. In order for an exchange contribution to an amplitude to interfere with the direct amplitude, the two vortices being exchanged must belong to the same irreducible representation of the quantum double.

(If a Chern-Simons term is added to the action of the underlying gauge theory, the situation becomes somewhat more complicated. [17] The Chern-Simons term distorts the charge spectrum of vortices with a specified value of the flux, and unremovable phases can enter the multiplication law of the quantum double. [17,25,33] The vortex states may then transform as a projective (ray) representation under gauge transformations.)

Consider a state of n "indistinguishable" vortices, all with flux conjugate to a, and all transforming as the representation $R^{(a)}$ of the centralizer $N(a)$ (in other words, all of the vortices belong to the irreducible representation $([a], R^{(a)})$ of the quantum double). A basis for these states can be constructed, in which, at each vortex position, we assign a definite flux, and a definite basis state in the vector space on which the representation $R^{(a)}$ acts. Under exchange, these states transform as a representation of B_n that is in general non-Abelian and reducible. This reducible representation can be decomposed into irreducible components. Each irreducible component describes an n-particle state obeying definite "braid statistics."

The point that we wish to emphasize is that the exchange operator will typically modify the quantum numbers that are assigned to the n particle positions. Thus, physical observables, such as transition probabilities or cross sections, need not be invariant under exchange. Instead, the exchange relates the value of the observable for one assignment of quantum numbers to the particle positions to the value of the observable for another choice of quantum numbers. Correspondingly, as we stressed above, the observables are not single-valued functions of the particle positions. Only a subgroup of the braid group returns the quantum numbers to their original values, and so preserves the values of the physical observables. (It is possible to restore the single-valuedness of the many-body wave functions by introducing on the configuration space a suitable connection with nontrivial holonomy. The existence of such a connection does not alter the essential physical point, which is that "indistinguishable" vortices may have distinct quantum numbers that can really be measured by an observer.)

Even distinguishable vortices have non-trivial Aharonov-Bohm interactions, so it is appropriate to broaden this framework slightly. We may consider a many-particle state containing n_1 particles of type 1 (with the type characterized by the class of the flux a, and the charge $R^{(a)}$, or, in other words, by the irreducible representation $([a], R^{(a)})$ of the quantum double), n_2 particles of type 2, and so on. Then an exchange of two particles is permitted only if the particles are of the same type, and the wave function transforms as a unitary representation of the "partially colored braid group" $B_{n_1,n_2,\ldots}$. [30,34]

Within this framework, a general connection between spin and statistics can be derived, assuming the existence of an antiparticle corresponding to each particle. [35,23,24] The essence of the connection is that, if two particles are truly *identical* (carry *exactly* the same quantum numbers), then an exchange of the two particles can be smoothly deformed to a process in which no exchange occurs, but one of the particles rotates by 2π. [35] (The reason that the quantum numbers must be the same is that, for the deformation to be possible, it is necessary for the antiparticle of the first particle to be able to annihilate the second particle.) It follows from the connection between spin and statistics that the effect of an exchange of two objects that are truly identical must be to modify the many-body wave function by the phase $e^{2\pi iJ}$, where J is the spin of the object. We have already remarked in section 3 that this is true for non-Abelian vortices with the same flux and charge. Thus, we find that non-Abelian statistics is perfectly compatible with the connection between spin and statistics.

There are deep connections between the theory of indistinguishable particles in two spatial dimensions and conformally invariant quantum field theory in two-dimensional *spacetime*. These connections have been explored most explicitly in the case of (2+1)-dimensional topological Chern-Simons theories, [36] but appear to be more general. [24] There is a close mathematical analogy between the *particle* statistics in two spatial dimensions that we have outlined here, and the *field* statistics in two-dimensional conformal field theory. In the latter case, all correlation functions can be constructed by assembling "conformal blocks," and the conformal blocks typically transform as a non-trivial unitary representation of the braid group when the arguments of the correlation function are exchanged. (See Ref. [37] for a review.) However, in discussions of conformal field theory, it is usually the case that observables of interest (the correlation functions themselves) are invariant under exchange.

4.5 Vortex-Vortex Scattering

The holonomy interaction between vortices induces Aharonov-Bohm vortex-vortex scattering, as pointed out by Wilczek and Wu [6] and Bucher. [7] Suppose that a vortex that initially carries flux b is incident on a fixed vortex that initially carries flux a. Let us suppose, for now, that the vortices are uncharged.

To understand the behavior of the b vortex propagating on the background of the fixed a vortex, it is convenient to adopt a path integral viewpoint. Consider the two possible paths shown in Fig. 4. If the vortex follows the path that passes below the scattering center, it will arrive at its destination with flux b. But if it follows the path that passes above the scattering center, it arrives carrying the flux aba^{-1}. Thus, if the flux of the scattering center and the flux of the projectile do not commute, the contribution to the path integral from paths that pass below does not interfere with the contribution from paths that pass above. Therefore, a plane wave propagating on the background of the fixed vortex does not remain a plane wave—there is nontrivial scattering.

More generally, the paths can be classified according to how many times they wind around the scattering center (relative to some standard path). The flux of a b vortex that winds around an a vortex k times is modified according to

$$|b\rangle \to |(ab)^k b (ab)^{-k}\rangle \equiv |k\rangle \ . \tag{18}$$

Since the unbroken gauge group H is assumed to be finite, the flux eventually returns to its original value, say after n windings.

The flux of the scattered vortex, then, can take any one of n values. The amplitude for the vortex to arrive at the detector in the flux state $|k\rangle$ defined in Eq. (18) can be found by summing over all paths with winding number congruent to k modulo n. Since only every nth winding sector is included in the amplitude ψ_k for flux channel

k, this amplitude is not a periodic function of the polar angle ϕ with period 2π; rather, the period is $2\pi n$. The n amplitudes are related by the nontrivial monodromy property

$$\psi_k(r, \phi + 2\pi) = \psi_{k+1}(r, \phi) \tag{19}$$

(where $\psi_{k+n}(r,\phi) \equiv \psi_k(r,\phi)$.) Similarly, the exclusive cross section for flux channel k is also multivalued:

$$\sigma_k(\theta - 2\pi) = \sigma_{k+1}(\theta) \ , \tag{20}$$

where $\theta = \pi - \phi$ is the scattering angle. The inclusive cross section

$$\sigma_{\rm inc}(\theta) = \sum_{k=0}^{n-1} \sigma_k(\theta) \tag{21}$$

is single-valued.

As we stressed in the introduction, the multivaluedness of the exclusive cross sections is natural and unavoidable in this context. Whenever we assign a flux to a non-Abelian vortex, we are implicitly adopting a conventional procedure for measuring the flux. For example, the procedure might be to carry the vortex to the "vortex bureau of standards" and analyze it there by performing Aharonov-Bohm interference experiments with various charged particles. Then the multivaluedness arises because, if we carry a vortex in the flux state $|k\rangle$ once around the scattering center (counterclockwise) before returning it to the bureau of standards, the analysis will identify it as the flux state $|k+1\rangle$.

For each value of the scattering angle, we might choose a standard path along which the vortex is to be returned to the bureau for analysis after the scattering event. For example, we might decide to carry it home through the upper half plane for $\theta \in [0, \pi)$ and through the lower half plane for $\theta \in [-\pi, 0)$, as shown in Fig. 5. Then the exclusive cross sections are single-valued, but are discontinuous at $\theta = 0$:

$$\sigma_k(\theta = 0^+) = \sigma_{k+1}(\theta = 0^-) \ . \tag{22}$$

The choice of a standard path amounts to an arbitrary restriction of the n-valued exclusive cross sections to a single branch.

In a sense, the multivaluedness of the wave functions, and of the exclusive cross sections, arises because we have insisted on expressing the flux of the vortices in terms of a multivalued basis—that basis defined by parallel transport of the flux in the background gauge potential of the scattering center. The propagation of the projectile on this background is really non-singular, and the multivaluedness of the amplitudes actually compensates for the multivaluedness of the basis. This is quite analogous to the "singular-gauge" description of ordinary Abelian Aharonov-Bohm scattering. There, expressing the phase of the electron wave function relative to a basis defined by parallel transport is equivalent to performing a singular gauge transformation that gauges away the vector potential and introduces a discontinuity in the wave function. The difference in the non-Abelian case is that the discontinuity corresponds to a jump in observable quantum numbers of the projectile, as explained above. It is natural to use the multivalued basis because it reflects what a team of experimenters would really find if they brought their detectors together to calibrate them alike.

Mathematically, finding the Aharonov-Bohm amplitude for a vortex propagating on the background of a fixed vortex is equivalent to finding the amplitude for a free particle propagating on an n-sheeted surface. (The closely related problem of a free particle propagating on a cone has been discussed in connection with $2+1$ dimensional general relativity. [18,19]) The most convenient way to solve the problem is to transform to a basis of "monodromy eigenstates," since for the elements of this basis the scattering reduces to Abelian Aharonov-Bohm scattering. If the ψ_k's obey the monodromy property Eq. (19), then the monodromy eigenstate basis is

$$\chi_l = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{-2\pi i k l/n} \psi_k \ , \tag{23}$$

with the property

$$\chi_l(r, \phi + 2\pi) = e^{2\pi i l/n} \chi_l(r, \phi) \ . \tag{24}$$

These monodromy eigenstates correspond to states of the two-vortex system that have definite charge, in the sense that they are eigenstates of the gauge transformation $ab \in H$, where ab is the total flux.

We may think of the wave functions χ_l as the coefficients in an expansion of a single-valued wave function in a multivalued basis. That is, we can express a single-valued wave function as

$$|\psi\rangle = \sum_{r,\phi,l} |r,\phi,l\rangle\langle r,\phi,l|\psi\rangle \ , \tag{25}$$

where the basis $|r,\phi,l\rangle$ is "twisted" according to

$$|r,\phi+2\pi,l\rangle = e^{-2\pi i l/n}|r,\phi,l\rangle \ . \tag{26}$$

The coefficients $\chi_l(r,\phi) = \langle r,\phi,l|\psi\rangle$ inherit the property Eq. (24) from the property Eq. (26) of the basis.

By standard methods, [38] we can find the solution to the free-particle nonrelativistic Schrödinger equation that obeys the condition

$$\chi_\alpha(\phi+2\pi) = e^{2\pi i\alpha}\chi_\alpha(\phi) \ , \quad 0 \le \alpha < 1 \ , \tag{27}$$

and matches a plane wave incoming from $\phi = 0$. The asymptotic large-r behavior of this solution is

$$\chi_\alpha \sim e^{-i\vec{p}\cdot\vec{x}} + \frac{e^{ipr}}{\sqrt{r}} f_\alpha(\phi) \ , \quad -\pi < \phi < \pi \ , \tag{28}$$

where

$$f_\alpha(\phi) = \frac{e^{-i\pi/4}}{\sqrt{2\pi p}}\left(\frac{1}{1+e^{i\phi}}\right)e^{i\alpha\phi}\left(e^{-i\alpha\pi} - e^{i\alpha\pi}\right) \ , \quad 0 \le \alpha < 1 \ . \tag{29}$$

Here $e^{-i\alpha\pi}$ is the phase shift for the partial waves with non-negative integer part of the orbital angular momentum and $e^{i\alpha\pi}$ is the phase shift for the partial waves with negative integer part of the orbital angular momentum. The semiclassical interpretation is that wave packets that pass above and below the scattering center acquire a relative phase $e^{2\pi i\alpha}$, the Aharonov-Bohm phase.

There are two subtleties concerning Eq. (28) and (29) that deserve comment. The first subtlety (which is not very important for what follows), is that there is an order of limits ambiguity in the evaluation of the amplitude—the limit $r \to \infty$ does not commute with the limit $\phi \to \pm\pi$. [39] In Eq. (28) and (29), we have taken $r \to \infty$ for fixed ϕ between $-\pi$ and π. Thus, χ_α actually satisfies Eq. (27), although the first term in the asymptotic form Eq. (28) appears not to. (For large r, the phase of the plane wave in Eq. (27) suddenly advances by $e^{2\pi i\alpha}$ as ϕ increases through a narrow wedge near $\phi = \pi$. Of course, if we construct localized wave packets, then the unscattered wave has support at $\phi = 0, \pm\pi$ as $r \to \infty$, and the form of the plane wave away from the forward direction is of no consequence anyway.) The second subtlety, which is very important for what follows, concerns the α dependence of the amplitude. The monodromy condition Eq. 27 depends only on $\alpha - [\alpha]$, where $[\alpha]$ denotes the greatest integer less than or equal to α. Thus, as one can explicitly verify, the amplitude $f_\alpha(\phi)$, when α is not restricted to lie in the range $[0,1)$, takes the same form as Eq. (29), but with α replaced by $\alpha - [\alpha]$. The somewhat surprising feature is that, as a function of α, $f_\alpha(\phi)$ is not differentiable when α is an integer.

The form Eq. (29) for the scattering amplitude in the monodromy eigenstate basis is readily generalized to an arbitrary basis, if we express it in terms of the braid operator $\mathcal{R}$, the square root of the monodromy operator $\mathcal{M}$. The general monodromy condition satisfied by the wave function can be expressed as

$$\psi(\phi + 2\pi) = \mathcal{M}\psi(\phi) , \tag{30}$$

where $\mathcal{M}$ is a unitary matrix acting on internal indices. Then the basis-independent form for the scattering amplitude is

$$\langle \text{out}|f(\phi)|\text{in}\rangle = \frac{e^{-i\pi/4}}{\sqrt{2\pi p}} \left(\frac{1}{1+e^{i\phi}}\right) \langle \text{out}|\mathcal{R}^{\phi/\pi}\left(\mathcal{R}^{-1} - \mathcal{R}\right)|\text{in}\rangle , \tag{31}$$

where $\mathcal{R}$ is defined by $\mathcal{R}^2 = \mathcal{M}$, and $|\text{in}\rangle$, $|\text{out}\rangle$ denote the incoming and outgoing wave functions in internal space. This definition of $\mathcal{R}$ leaves an ambiguity in $\mathcal{R}^{(\phi/\pi \mp 1)}$, and it is important to resolve this ambiguity correctly. Acting on an eigenstate of $\mathcal{M}$ with

$$\mathcal{M} = e^{2\pi i\alpha} , \tag{32}$$

we define

$$\mathcal{R}^{(\phi/\pi \mp 1)} \equiv e^{i(\alpha - [\alpha])(\phi \mp \pi)} . \tag{33}$$

In Eq. (31), the state $|\text{in}\rangle$ is expressed in terms of an arbitrary basis, and we have assumed that the state $|\text{out}\rangle$ is expressed in terms of a basis that is obtained by parallel transport of the in-basis. This out-basis is multivalued, so we have in effect evaluated the amplitude in a "singular gauge."

From Eq. (31), we obtain the cross section

$$\sigma_{\text{in}\to\text{out}}(\phi) = |f(\phi)|^2 = \frac{1}{2\pi p}\left(\frac{1}{4\cos^2\phi/2}\right)\left|\langle \text{out}|\mathcal{R}^{\phi/\pi}\left(\mathcal{R}^{-1} - \mathcal{R}\right)|\text{in}\rangle\right|^2 . \tag{34}$$

By summing $|\text{out}\rangle$ over a complete basis, we obtain the inclusive cross section

$$\sigma_{\text{in}\to\text{all}}(\theta) = \frac{1}{2\pi p}\left(\frac{1}{\sin^2\theta/2}\right)\frac{1}{2}\left(1 - \text{Re}\langle\text{in}|\mathcal{R}^2|\text{in}\rangle\right) \ , \tag{35}$$

where $\theta = \pi - \phi$ is the scattering angle; this is the formula derived by Verlinde. [16]

For monodromy eigenstates with $\mathcal{M} = e^{2\pi i\alpha}$, Eq. (34) reduces to the familiar form of the Aharonov-Bohm cross section,

$$\sigma_\alpha(\theta) = \frac{1}{2\pi p}\left(\frac{\sin^2\pi\alpha}{\sin^2\theta/2}\right) \ , \tag{36}$$

which is a single-valued function of the scattering angle. But the recurring theme of this chapter is that it is often convenient to express the scattering states in terms of a basis other than the monodromy eigenstate basis. Then the exclusive cross sections are in general multivalued, but the inclusive cross section (summed over all possible final state quantum numbers) is always single-valued.

Returning to the special case of (uncharged) vortex-vortex scattering, we obtain the amplitude in the flux eigenstate basis by coherently summing the monodromy eigenstate amplitudes with appropriate phases,

$$\begin{aligned}\langle k|f(\phi)|k=0\rangle &= \frac{1}{n}\sum_{l=0}^{n-1} e^{2\pi ikl/n} f_{l/n}(\phi) \\ &= \frac{e^{-i\pi/4}}{\sqrt{2\pi p}}\left(\frac{i}{2n}\right)\frac{\sin(\pi/n)}{\sin\left[\frac{1}{2n}\left(\phi+(2k+1)\pi\right)\right]\sin\left[\frac{1}{2n}\left(\phi+(2k-1)\pi\right)\right]} \ .\end{aligned} \tag{37}$$

This formula has the expected monodromy property

$$\langle k|f(\phi+2\pi)|k=0\rangle = \langle k+1|f(\phi)|k=0\rangle \ . \tag{38}$$

(Eq. (37) is actually a special case of the the formula derived in (2+1)-dimensional gravity by 't Hooft [18] and Deser and Jackiw. [19])

This amplitude has the infinite forward peak that is characteristic of Aharonov-Bohm scattering. For $\phi = \pi$, the infinite peak occurs in the flux channels $k = 0, -1$ and for $\phi = -\pi$, it occurs in the channels $k = 1, 0$. For ϕ near π, the leading behavior of the amplitude is

$$\langle k=0|f(\phi)|k=0\rangle \sim -\langle k=-1|f(\phi)|k=0\rangle \sim \frac{e^{-i\pi/4}}{\sqrt{2\pi p}}\left(\frac{i}{\phi-\pi}\right) \ . \tag{39}$$

This leading behavior has a simple interpretation. From a path integral viewpoint, the forward peak is generated by paths that pass above or below the scattering center with a large impact parameter, without any winding around the center. If the projectile passes above, it is detected near $\phi = \pi$ as a $k = 0$ vortex (or near $\phi = -\pi$ as a $k = 1$ vortex); if it passes below, it is detected near $\phi = \pi$ as a $k = -1$ vortex (or near $\phi = -\pi$ as a $k = 0$ vortex). Near $\phi = \pi$, the amplitude in the $k = 0, -1$ channels is equivalent to the diffraction pattern generated by a "sharp edge," since paths that wind n times around the scattering center make a negligible contribution. The near-forward amplitude in the $k = 0$ channel comes from summing all of the partial waves with non-negative angular momentum, and the near-forward amplitude in the $k = -1$ channel comes from summing the partial waves with negative angular momentum. Thus, the forward peak in each channel is half as strong as the forward peak for "maximal" ($\alpha = 1/2$) Abelian Aharonov-Bohm scattering.

The inclusive cross section (obtained by summing over all possible final flux channels) can be immediately read off from Eq. (35). If the projectile is a flux eigenstate, and the scattering center is a flux eigenstate whose flux does not commute with that of the projectile, then we have $\langle \mathrm{in}|\mathcal{R}^2|\mathrm{in}\rangle = 0$, and the inclusive cross section takes the universal form

$$\sigma_{\text{flux eigenstate}\to\text{all}}(\theta) = \left(\frac{1}{2}\right)\frac{1}{2\pi p}\left(\frac{1}{\sin^2\theta/2}\right) \; ; \tag{40}$$

that is, half the cross section for maximal Aharonov-Bohm scattering.

So far, we have assumed that the vortex that is being scattered carries no charge. Let us briefly comment on how the analysis is modified when the scattered vortex is charged.

Suppose that the vortex with flux a transforms as some irreducible representation $D^{R^{(a)}}$ of $N(a)$, and that the vortex with flux b transforms as some irreducible representation $D^{R^{(b)}}$ of $N(b)$. And suppose as before that the fluxes return to their original values after the monodromy operator acts n times (that is, after the b vortex winds around the a vortex n times). For charged vortex states, although $\mathcal{M}^n$ preserves the flux values, it acts on the vortex pair as a nontrivial $N(a) \otimes N(b)$ transformation. Specifically, we have

$$\mathcal{M}^n : |a\rangle \otimes |b\rangle \to D^{R(a)}[(ab)^n a^{-n}]|a\rangle \otimes D^{R(b)}[(ab)^n b^{-n}]|b\rangle \, . \tag{41}$$

Note that, since by assumption $(ab)^n a(ab)^{-n} = a$ and $(ab)^n b(ab)^{-n} = b$ (because $\mathcal{M}^n$ preserves the fluxes), $(ab)^n a^{-n} \in N(a)$ and $(ab)^n b^{-n} \in N(b)$.

For the case of scattering a b vortex off of a fixed a vortex, we consider the states $|k\rangle$ defined by

$$|k\rangle \equiv \mathcal{M}^k|b\rangle \ , \quad k = 0, 1, 2, \ldots, n-1 \ , \tag{42}$$

with

$$\mathcal{M}^n|k=0\rangle = D^{R^{(b)}}[(ab)^n b^{-n}]|k=0\rangle \ . \tag{43}$$

To diagonalize the monodromy operator, we first diagonalize the unitary transformation $D^{R^{(b)}}[(ab)^n b^{-n}]$. Corresponding to each eigenstate of this operator with eigenvalue $e^{2\pi i\beta}$ are a set of monodromy eigenstate wave functions

$$\chi_{l,\beta} = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{-2\pi i k(l+\beta)/n} \psi_{k,\beta} \ , \tag{44}$$

with the property

$$\chi_{l,\beta}(r, \phi + 2\pi) = e^{2\pi i(l+\beta)/n} \chi_{l,\beta}(r, \phi) \ . \tag{45}$$

For particular charged states with specified flux, we may evaluate Eq. (31) by coherently superposing the Aharonov-Bohm amplitudes for these monodromy eigenstates.

4.6 Indistinguishable Vortices

The effects of quantum statistics can be seen in the two-body scattering of indistinguishable particles, because exchange scattering can occur; it is possible to lose track of "who's who." In the case of non-Abelian vortices, the exchange effects are more subtle than for Abelian anyons—in general, whether two vortices behave like identical or distinct particles when they are brought together depends on their *history*. Suppose that two identical vortices each carry the flux $a \in H$. If one of the vortices should voyage around another vortex with flux b, and then return to its partner, it would then carry flux bab^{-1}. Hence, if a and b do not commute, it would now be distinct from the other a vortex.

For exchange effects to occur in vortex-vortex scattering, the braid operator must have an orbit of odd order acting on the two vortex state. That is, $\mathcal{R}^n$ must preserve the two vortex state for some odd n. If so, there will be a contribution to the vortex-vortex scattering amplitude in which the two vortices change places, that interferes with the direct amplitude.

As a simple example, consider the permutation group on three objects S_3, where the fluxes are two distinct two-cycles. Then the braid operator defined by Eq. (7) has the orbit

$$\begin{aligned}\mathcal{R}: \quad |(12),(23)\rangle &\to |(13),(12)\rangle \\ &\to |(23),(13)\rangle \\ &\to |(12),(23)\rangle \ ,\end{aligned} \tag{46}$$

of order 3. (See Fig. 6.) Thus, there is an exchange contribution to the scattering of a (12) vortex and a (23) vortex. (In this case, the centralizer of the total flux is Z_3, and the braid eigenstates are the linear combinations of these three states that have definite Z_3 charge.)

Two vortices whose flux belongs to the same conjugacy class of the unbroken group H have the same mass, and we can easily derive a formula for the vortex-vortex scattering amplitude in the center of mass frame, using the same methods as in the previous section. This formula will incorporate the exchange effects whenever the braid operator has an odd orbit acting on the two-vortex state. The two-body wave function in the center of mass frame will now have the property

$$\psi(r, \phi + \pi) = \mathcal{R}\psi(r, \phi) \ , \tag{47}$$

where the braid operator $\mathcal{R}$ is a unitary matrix acting on the internal indices of the wave function. The problem is to solve the free-particle Shrödinger equation subject to this condition.

If the two-body state is a "braid eigenstate,"

$$\chi_\alpha(r, \phi + \pi) = e^{i\pi\alpha}\chi_\alpha(r, \phi) \ , \quad 0 \le \alpha < 2 \ , \tag{48}$$

then the problem is equivalent to anyon-anyon scattering, with statistical phase $e^{i\theta} = e^{i\pi\alpha}$. We can find the solution to the free-particle Schrödinger equation that obeys Eq. (48) and matches plane waves coming from $\phi = 0$ and $\phi = \pi$. The asymptotic large-r behavior of this solution is [40]

$$\chi_\alpha \sim \left(e^{-i\vec{p}\cdot\vec{x}} + e^{i\alpha\pi}e^{i\vec{p}\cdot\vec{x}}\right) + \frac{e^{ipr}}{\sqrt{r}} f_\alpha(\phi)\ , \quad 0 < \phi < \pi\ , \tag{49}$$

where

$$f_\alpha(\phi) = \frac{e^{-i\pi/4}}{\sqrt{2\pi p}}\left(\frac{2}{1-e^{2i\phi}}\right) e^{i\alpha\phi}\left(e^{-i\alpha\pi} - e^{i\alpha\pi}\right)\ , \quad 0 \leq \alpha < 2\ . \tag{50}$$

(As in our discussion of scattering off a fixed target, we remark that the limit $r \to \infty$ does not commute with the limit $\phi \to 0, \pi$. [39] Thus, χ_α actually satisfies Eq. (48), although the first term in the asymptotic form Eq. (49) appears not to.) In an arbitrary basis, in which the braid operator is not necessarily diagonal, we have

$$\langle \text{out}|f(\phi)|\text{in}\rangle = \frac{e^{-i\pi/4}}{\sqrt{2\pi p}}\left(\frac{2}{1-e^{2i\phi}}\right) \langle \text{out}|\mathcal{R}^{\phi/\pi}\left(\mathcal{R}^{-1} - \mathcal{R}\right)|\text{in}\rangle\ , \tag{51}$$

where $|\text{in}\rangle$, $|\text{out}\rangle$ denote the incoming and outgoing two-body wave functions in internal space. As in our discussion of scattering off of a fixed center, there is an ambiguity in the evaluation of $\mathcal{R}^{(\phi/\pi\mp1)}$, and we must now resolve this ambiguity slightly differently than before. If α is not restricted to the range $[0,2)$, then α must be replaced by $\alpha - [[\alpha]]$ in Eq. 50, where $[[\alpha]]$ denotes the greatest *even* integer less than or equal than α. Thus, acting on an eigenstate of $\mathcal{R}$ with eigenvalue

$$\mathcal{R} = e^{i\pi\alpha}\ , \tag{52}$$

we define $\mathcal{R}^{(\phi/\pi\mp1)}$ by

$$\mathcal{R}^{(\phi/\pi\mp1)} = e^{i(\alpha-[[\alpha]])(\phi\mp\pi)}\ . \tag{53}$$

The cross section is

$$\sigma_{\text{in}\to\text{out}}(\phi) = |f(\phi)|^2 = \frac{1}{2\pi p}\left(\frac{1}{\sin^2\phi}\right)\left|\langle \text{out}|\mathcal{R}^{\phi/\pi}\left(\mathcal{R}^{-1} - \mathcal{R}\right)|\text{in}\rangle\right|^2\ . \tag{54}$$

By summing $|\text{out}\rangle$ over a complete basis, we obtain the inclusive cross section

$$\sigma_{\text{in}\to\text{all}}(\theta) = \frac{1}{2\pi p}\left(\frac{1}{\sin^2\theta}\right) 2\left(1 - \text{Re}\langle \text{in}|\mathcal{R}^2|\text{in}\rangle\right)\ , \tag{55}$$

where $\theta = \pi - \phi$ is the scattering angle.

The general problem can be solved by expressing the two-body state as a linear combination of braid eigenstates, and then coherently superposing the anyon-anyon amplitudes. In the case of (uncharged) vortex-vortex scattering, if the initial state is a vortex with flux a coming from $\phi = \pi$ and a vortex with flux b coming from $\phi = 0$, then let us denote by $|k\rangle$ the state obtained when the braid operator $\mathcal{R}$ defined by Eq. (7) acts on the initial state k times

$$|k\rangle \equiv \mathcal{R}^k |a, b\rangle \ . \tag{56}$$

Suppose that the two-vortex state returns to the initial state after $\mathcal{R}$ acts n times. (Note that, in a departure from the notation of the previous section, k and n now denote the number of times the *braid* operator acts on the initial state, rather than the monodromy operator $\mathcal{M} = \mathcal{R}^2$.) Then,

$$\chi_{2l/n} = \sum_{k=0}^{n-1} e^{-2\pi i k l/n} |k\rangle \tag{57}$$

is a braid eigenstate with eigenvalue $e^{i\pi\alpha} = e^{2\pi i l/n}$, and the scattering amplitude in the flux eigenstate basis is

$$\begin{aligned} \langle k|f(\phi)|k=0\rangle &= \frac{1}{n}\sum_{l=0}^{n-1} e^{2\pi i k l/n} f_{2l/n}(\phi) \\ = \frac{e^{-i\pi/4}}{\sqrt{2\pi p}} \left(\frac{i}{n}\right) & \frac{\sin(2\pi/n)}{\sin\left[\frac{1}{n}\left(\phi + (k+1)\pi\right)\right] \sin\left[\frac{1}{n}\left(\phi + (k-1)\pi\right)\right]} \ . \end{aligned} \tag{58}$$

This formula has the desired property

$$\langle k|f(\phi+\pi)|k=0\rangle = \langle k+1|f(\phi)|k=0\rangle \ . \tag{59}$$

Eq. (58) applies for any value of n, but there is an exchange contribution to the amplitude only for odd n. (Note that, if n and k are even, Eq. (59) precisely coincides with Eq. (58), as one would expect.)

The amplitude has the expected infinite peak at $\phi = \pi$ in the channels $k = 0, -2$ and at $\phi = 0$ in the channels $k = \pm 1$. As in our discussion of scattering off of a fixed center, these peaks are generated by paths in which the two vortices pass one another with a large impact parameter, without any winding. If the vortex incident from the right passes above the vortex incident from the left, then, with our conventions, a $k = 0$ state is detected near $\phi = \pi$, and a $k = 1$ state is detected near $\phi = 0$. If the vortex incident from the right passes below, then a $k = -2$ state is detected near $\phi = \pi$, and a $k = -1$ state is detected near $\phi = 0$.

4.7 Continuous Symmetry: The Alice Vortex

So far, we have assumed that the unbroken local symmetry group is a discrete group. In this section, we will briefly consider the properties of non-Abelian vortices when the gauge group is continuous.

If the unbroken gauge group has a non-Abelian Lie algebra, then the gauge interaction is presumably confining. In fact, even if the Lie algebra is Abelian (a product of $U(1)$'s), then charge is logarithmically confined in two spatial dimensions. That is, the Coulomb energy of a charged object is logarithmically infrared divergent. Nevertheless, we might be interested in the Aharonov-Bohm interactions of vortices and charged particles on distance scales that are small compared to the confinement scale, or under circumstances where the Coulomb energy can be safely neglected.

Strictly speaking, there is no Aharonov-Bohm amplitude for the scattering of a charged particle off of a vortex, because there are no asymptotic charged states. Still, the formalism discussed in this chapter finds some application. We can imagine placing a compensating charge far away from the scattering center, and consider the scattering of a wave packet in a bounded region that is small compared to the distance to the compensating charge (or small compared to the confinement distance scale). Furthermore, the charge of a particle behaves like $\hbar e$, where e is a (classical) gauge coupling, so Coulomb effects are of order $(\hbar e)^2$, and are higher order corrections to Aharonov-Bohm scattering in the semiclassical (small $\hbar$) limit. Under suitable conditions, the deflection of the wave packet is described to good accuracy by our general formula for the Aharonov-Bohm amplitude, Eq. (31).

The case of vortex-vortex scattering is more complicated. We can imagine scattering two vortices that are flux eigenstates. (More properly, in the case of continuous gauge symmetry, we should consider narrow "flux wave packets," superpositions of flux eigenstates with small dispersion.) However, a pair of flux eigenstates does not have definite charge; when the state of the pair is decomposed into charge eigenstates, the states of nonzero charge have infrared divergent Coulomb energy. Again, there is a need for a compensating charge. But in this case, the value of the compensating charge must be correlated with the state of the vortex pair. If we trace over the state of the compensating charge, we obtain a density matrix for the vortex pair that is an *incoherent* superposition of charge eigenstates. Thus, the "scattering cross section" is an incoherent sum of the cross sections for the various charge (or braid) eigenstates, and Eq. (31) does not apply.

To make the discussion more definite, let us consider the simplest model that exhibits these features, the "Alice" model. [10–13] The unbroken symmetry group in this case is the semi-direct product of $U(1)$ with Z_2. The group has a component connected to the identity, the $U(1)$ subgroup, that can be parametrized as

$$\{e^{i\omega Q} \ , \quad 0 \le \omega < 2\pi\} \ , \tag{60}$$

where $Q = \sigma_3$ is the $U(1)$ generator. There is also a component that is not connected to the identity,

$$\{i\sigma_2 e^{i\omega Q} \ , \quad 0 \le \omega < 2\pi\} \ . \tag{61}$$

Each element of the disconnected component anticommutes with Q. Thus, the Alice model can be characterized as a generalization of electrodynamics in which charge conjugation is a *local* symmetry.

An "Alice vortex" carries flux that takes a value in the disconnected component of this group. The monodromy operator associated with transport around this vortex, acting on the defining representation of the group, is

$$\mathcal{M}(\omega) = e^{-i\omega Q/2} i\sigma_2 e^{i\omega Q/2} \ . \tag{62}$$

Because $\mathcal{M}$ anticommutes with Q, when a charged particle is transported around the vortex, its charge flips in sign. This monodromy property induces Aharonov-Bohm scattering of the charge eigenstates. Using the prescription Eq. (33), it is straightforward to compute

$$\mathcal{R}^{\phi/\pi}\left(\mathcal{R}^{-1} - \mathcal{R}\right) = e^{-i\omega Q/2}\left(-i\sqrt{2}\right) e^{i\phi/2} \begin{pmatrix} \cos\phi/4 & -\sin\phi/4 \\ \sin\phi/4 & \cos\phi/4 \end{pmatrix} e^{i\omega Q/2} \ . \tag{63}$$

From Eq. (31), we thus obtain the cross section for scattering of charge eigenstates off of a fixed Alice vortex,

$$\sigma_\pm(\theta) = \frac{1}{2\pi p}\frac{1 \pm \sin\theta/2}{4\sin^2\theta/2} \ ; \tag{64}$$

here, σ_+ denotes the cross section when the scattered charge has the same sign as the original projectile, and σ_- is the cross section for charge-flip scattering. Note that these exclusive cross sections respect the relation Eq. (1) anticipated in the introduction.

The case of a charged particle scattering from an Alice vortex is quite similar to the case of vortex-vortex scattering considered in section 5, where the orbit of the monodromy operator has order $n = 2$. There is an important difference, however—the monodromy operator Eq. (62) squares to -1 rather than 1. The property $\mathcal{M}^2 = -1$ holds whenever the charge of the projectile is odd, and hence the cross section Eq. (64) applies for any odd charge. The vanishing of σ_- in the backward direction is easily seen to be a consequence of $\mathcal{M}^2 = -1$; the trajectories with positive and negative *odd* winding number interfere destructively at $\theta = \pi$. If the charge of the projectile is even, then $\mathcal{M}^2 = 1$, and the cross section is given by Eq. (37) for $n = 2$, with $k = 0$ corresponding to σ_+ and $k = 1$ to σ_-.

Now consider the case of vortex-vortex scattering, in the flux eigenstate basis. We denote by $|\omega\rangle$ the vortex state with flux $i\sigma_2 e^{i\omega Q}$. According to Eq. (7), the effect of an exchange on a state of two vortices, each with definite flux, can be expressed as

$$\mathcal{R} : |\omega_1, \omega_2\rangle \rightarrow |2\omega_1 - \omega_2, \omega_1\rangle \ . \tag{65}$$

The exchange preserves the "total flux" $i\sigma_2 e^{i\omega_1 Q} i\sigma_2 e^{i\omega_2 Q} = e^{i(\omega_2 - \omega_1)Q} \equiv e^{i\omega_{\rm tot} Q}$, so an alternative notation is

$$\mathcal{R} : |\omega_1; \omega_{\rm tot}\rangle \rightarrow |\omega_1 - \omega_{\rm tot}; \omega_{\rm tot}\rangle \ , \tag{66}$$

with the flux $\omega_2 = \omega_{\rm tot} + \omega_1$ of the second vortex suppressed.

The two vortex state can be decomposed into states with definite transformation properties under the centralizer of the total flux, which is $U(1)$. These charge eigenstates also diagonalize the braid operator. The action of $U(1)$ on the flux eigenstates is

$$e^{i\epsilon Q} : |\omega_1; \omega_{\rm tot}\rangle \rightarrow |\omega_1 - 2\epsilon; \omega_{\rm tot}\rangle \ , \tag{67}$$

and the charge eigenstates are

$$|q, \omega_{\rm tot}\rangle = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} d\omega' e^{iq\omega'} |2\omega'; \omega_{\rm tot}\rangle \ , \tag{68}$$

where the charge q is an *even* integer. The braid operator acts on the charge eigenstates according to

$$\mathcal{R} : |q, \omega_{\rm tot}\rangle \to e^{iq\omega_{\rm tot}}|q, \omega_{\rm tot}\rangle \ . \tag{69}$$

Formally, we can find the amplitude for a vortex with flux ω_1 to scatter from a fixed center with flux $\omega_2 = \omega_{\rm tot} + \omega_1$ by applying Eq. (31). The result is

$$\langle\ \omega_1'; \omega_{\rm tot}\ {\rm out}|f(\phi)|\omega_1; \omega_{\rm tot}\ {\rm in}\rangle \tag{70}$$

$$= \frac{e^{-i\pi/4}}{\sqrt{2\pi p}}\left(\frac{1}{1+e^{i\phi}}\right)\frac{1}{\pi}\sum_q e^{iq(\omega'-\omega)}\left(e^{i(q\omega_{\rm tot}-[q\omega_{\rm tot}])(\phi/\pi-1)} - e^{i(q\omega_{\rm tot}-[q\omega_{\rm tot}])(\phi/\pi+1)}\right) \tag{71}$$

where q is summed over even integers. We note that it is essential to subtract away the integer part of $q\omega_{\rm tot}$ in order to obtain the correct result. For example, if $\omega_{\rm tot}$ is rational, then the amplitude has support only for discrete values of $\omega' - \omega$. This would not have worked if the integer part had not been subtracted.

However, as noted above, this analysis is moot, because of the need to deal with the infrared divergent Coulomb energy of the states with $q \neq 0$. One way to screen the charge is to place another vortex pair far away, such that the four-vortex system carries total charge zero. But however we arrange to screen the charge, the state of the vortex pair we are studying will be correlated with the state of the compensating charge (unless the vortex pair is in a charge eigenstate). For example, our flux eigenstate becomes

$$|\omega_1; \omega_{\rm tot}\rangle \to \frac{1}{\sqrt{\pi}}\sum_q e^{-iq\omega_1}|q; \omega_{\rm tot}\rangle \otimes |-q; {\rm screen}\rangle \ , \tag{72}$$

where $|-q; {\rm screen}\rangle$ is the state of the screening charge. The vortex pair is actually in the mixed state

$$\rho = \frac{1}{\pi}\sum_q |q; \omega_{\rm tot}\rangle\langle q; \omega_{\rm tot}| \ . \tag{73}$$

The probability distribution for the scattered vortex will be the incoherent sum of the probability distributions for the braid eigenstates.

4.8 Conclusions

This chapter has two recurring themes, relating to the non-Abelian Aharonov-Bohm effect and non-Abelian statistics. The first theme is that the non-Abelian Aharonov-Bohm effect provides a natural setting for multivalued physical observables. A particle that travels around a closed path returns to its starting point as a *different* kind of particle with different quantum numbers. This means that transition probabilities are not single-valued functions of the positions and quantum numbers of the particles in the final state. We have calculated cross sections that exhibit this multivalued character.

The second theme is that two particles that are "indistinguishable" need not be the same. The hallmark of non-Abelian statistics is that there can be an exchange contribution to an amplitude that interferes with the direct amplitude, even if the two particles that are exchanged are distinct objects with different quantum numbers. We have calculated cross sections that include such exchange effects.

These considerations illuminate some subtle aspects of non-Abelian gauge invariance. How do they relate to real phenomenology? There is no firm evidence that objects that obey non-Abelian statistics (called "nonabelions" in Ref. [41]) exist in nature. But it is surely conceivable that nonabelions will eventually be found, in strongly correlated electron systems, [6,41,42] or other frustrated quantum many-body systems. An important question, then, is how would such objects be recognized in laboratory experiments? Much remains to be done to explore the many-body physics of nonabelions. Even the problem of three bodies is not very well understood.

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FIGURES

FIG. 1. Exchange of two vortices. (a) The paths α and β are two standard paths, both beginning and ending at the same basepoint x_0, that are used to define the flux of two vortices. (b) When the vortices are interchanged, these paths are dragged to the new paths $\beta' = \beta^{-1}\alpha\beta$ and $\alpha' = \beta$.

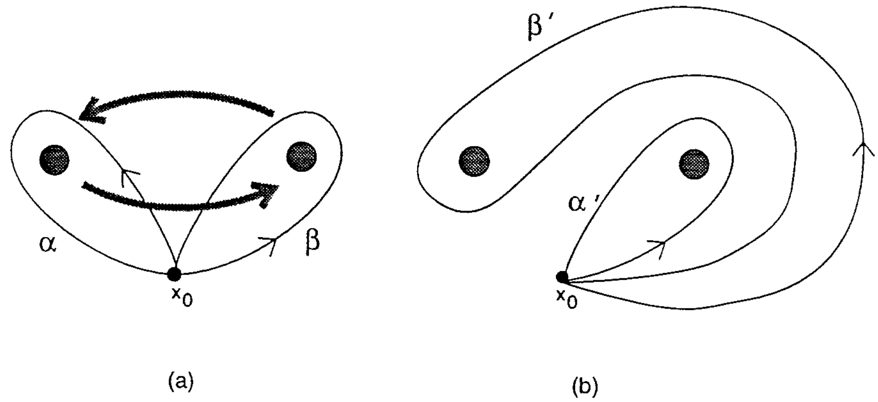
FIG. 2. Vortices can be carried along specified paths to the "Vortex Bureau of Standard," where their flux can be measured. If the two vortices are carried along the paths shown in (a), the fluxes are measured to be a and b, respectively. But if the b vortex goes counterclockwise around the a vortex before voyaging to the Bureau, as in (b), its flux is measured as $(ab)b(ab)^{-1}$. If the a vortex goes counterclockwise around the b vortex before voyaging to the Bureau, as in (c), its flux is measured as $(ab)a(ab)^{-1}$.

FIG. 3. The charge of a particle can be measured via the Aharonov-Bohm effect in a double-slit interference experiment. (a) When a vortex of known flux b is placed between the two slits, the change in the interference pattern measures $\langle u|D^{(R)}(b)|u\rangle$, where $|u\rangle$ denotes the internal state of the charged particle, and (R) is the representation according to which the charged particle transforms. However, if the charged particle is itself a vortex with flux a, there is a restriction on the charges that can be measured. If the a vortex passes through the left slit, as in (b), it arrives at the screen with flux a, and the vortex between the slits remains in the flux state b. If it passes through the right slit, as in (c), it arrives at the screen with flux $(ab)a(ab)^{-1}$, and the flux of the vortex between the slits becomes $(ab)b(ab)^{-1}$. Thus, no interference is seen if a and b do not commute. Because interference occurs only when a and b commute, this experiment can measure only the transformation properties of the charged projectile under the subgroup $N(a)$ that commutes with a.

FIG. 4. Two paths that contribute to the amplitude for a b vortex propagating on the background of a fixed a vortex. If the b vortex passes below the a vortex, it arrives at its destination with flux b; if it passes above the a vortex, it arrives at its destination with flux aba^{-1}. Thus, these two paths do not interfere if a and b do not commute.

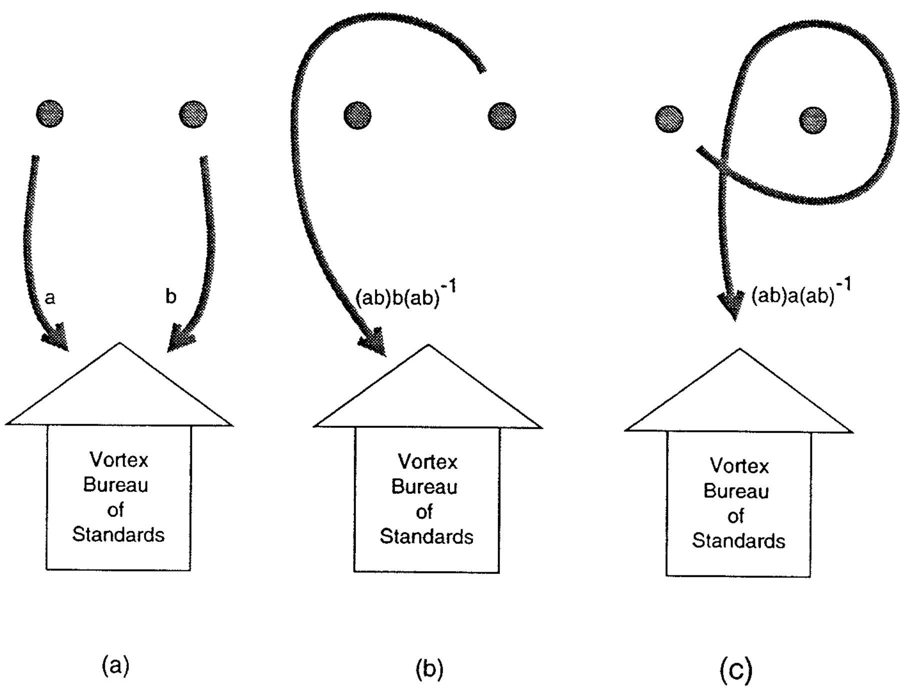
FIG. 5. A convention for measuring the flux of a scattered vortex that is single-valued but discontinuous. If the vortex is scattered into the upper half plane ($0 < \theta < \pi$), it is carried back to the "Vortex Bureau of Standards" above the scattering center; if the vortex is scattered into the lower half plane ($-\pi < \theta < 0$), it is carried back to the Bureau above the scattering center. With this convention, the scattering cross section is discontinuous at $\theta = 0$; the cross section in the "k" channel at $\theta = 0^+$ matches the cross section in the "$k+1$" channel at $\theta = 0^-$.

FIG. 6. Paths contributing to the amplitude for the propagation of a pair of vortices. The initial vortices carry flux taking the values (12) and (23) in S_3. If the vortices braid once as in (b) or twice as in (c), the quantum numbers of the pair are modified. But if the vortices braid three times as in (d), the final quantum numbers match the initial quantum numbers. Thus, paths (a) and (d) add coherently in the amplitude, although the two vortices change places.



β′
α
β
α′
x_0
x_0
(a)
(b)

Fig. 1



a
b
(ab)b(ab)$^{-1}$
(ab)a(ab)$^{-1}$
Vortex Bureau of Standards
Vortex Bureau of Standards
Vortex Bureau of Standards
(a)
(b)
(c)

Fig. 2

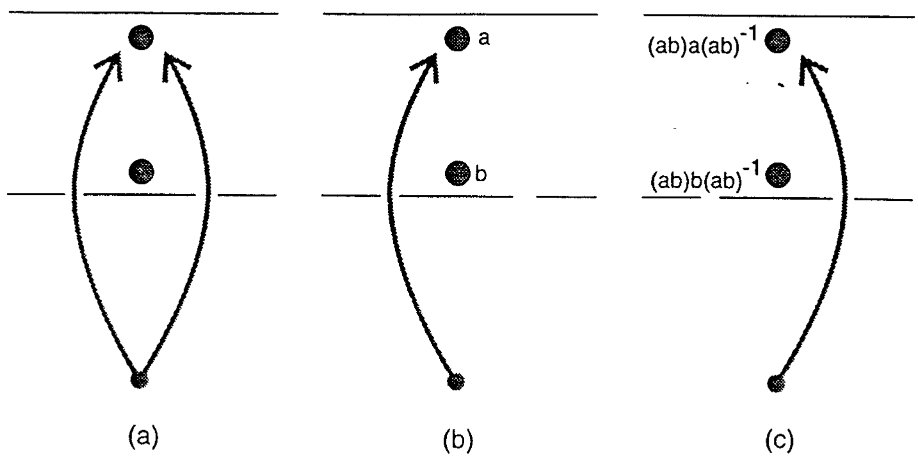


Fig. 3

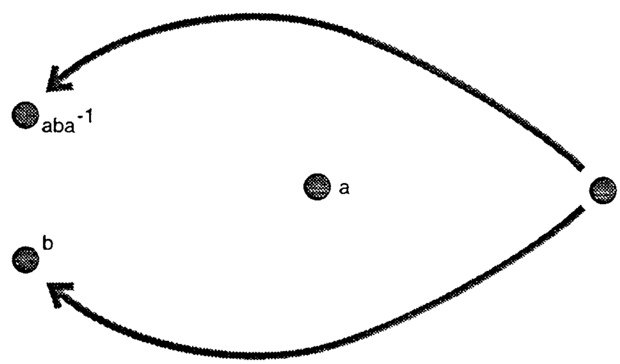


Fig. 4

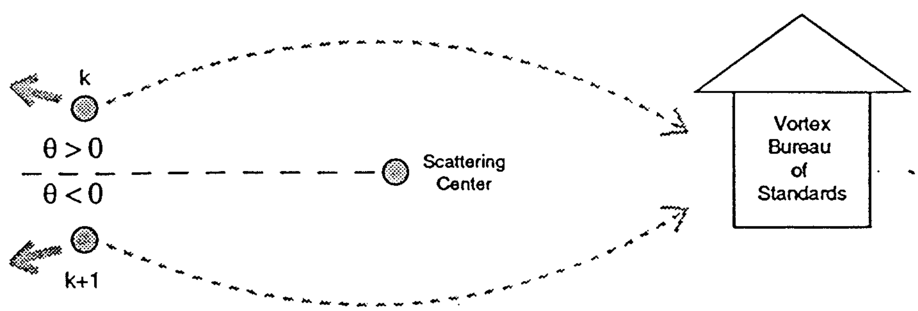


Fig. 5

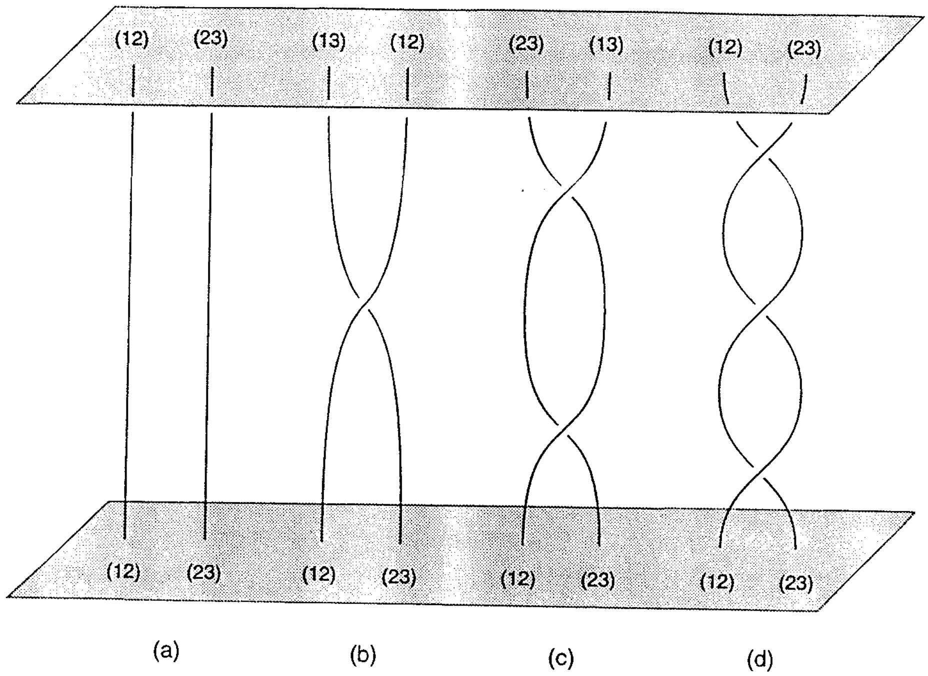


Fig. 6

Chapter 5
Non-Abelian Chern-Simons Particles

5.1 Introduction

It is now well-known that anyons—particles with arbitrary statistics can exist in 2+1 dimensions.[1] Owing to the topological gauge potential, even non-interacting two-anyon states are not (symmetrized) tensor products of single anyon states.[2] Thus the quantum mechanics of systems of many anyons presents a challenge to field theorists.[3] In general, the center of mass motion of the system is not sensitive to the statistics and can be factored out. For two anyons, the relative coordinates present us with a one-body problem which can be solved in many cases. For N anyons, there are $N-1$ relative coordinates, whereas there are $N(N-1)/2$ pairs of particles. These two numbers match only when $N=2$. For $N>2$, the various pair-separation coordinates (in terms of which the statistical gauge potential is easy to write down) are not independent of each other. For this reason, not a single three-anyon system has been completely solved. The many-anyon system has been studied in the mean field approach.[4] Other methods that have been employed include the semi-classical approximation,[5] numerical studies,[6] perturbative analysis from the bosonic or fermionic ends,[7] and the most interesting of all, the ladder operator approach: In the last couple of years, a substantial subset of the exact multi-anyon wavefunctions in a magnetic field has been found with this systematic analysis.[8] This soon generalized to free anyons[9] and anyons of multi-species.[10] While a lot of states are still missing, all the states in the lowest Landau level are obtained by this method.

In this chapter, we apply the ladder operator approach to non-Abelian Chern-Simons (NACS) particles which may be regarded as a generalization of anyons.[11] In the gauge that the Hamiltonian is a free Hamiltonian, the wavefunctions (which have more than one component) are multivalued with non-trivial monodromy properties given by monodromy matrices.[12] By introducing statistical gauge potentials, one has the liberty to work with single-valued wavefunctions. However, since we find multivalued wavefunctions convenient to work with, we will stick to them in the rest of this chapter.

In section 2, we review the ladder operator formalism as applied to anyons. This helps to highlight the differences between the cases of anyons and non-Abelian C-S particles, the object under study in section 3. In particular, of the operators used for anyons, only a subclass of operators, which preserve the monodromy properties of the wavefunctions, are allowed to act on the C-S particles. Nonetheless, our wavefunctions do cover the lowest Landau level. As an application of our formalism, we compute the second virial coefficient of NACS particles. The same set of ladder operators apply to free NACS particles with minor modifications. We also consider systems of multi-species NACS particles. Finally, the relevance of our work to systems of vortices of finite gauge groups is also discussed.

5.2 Anyons

The Hamiltonian for N anyons with charge e and mass m moving on a plane with a constant magnetic field B (perpendicular to the plane) is given by

$$H = \sum_{\alpha=1}^{N} -\frac{1}{2m}(\nabla_\alpha - i\mathbf{a}_\alpha - ie\mathbf{A})^2, \tag{1}$$

where the external gauge field $A^i = -\frac{1}{2}B\epsilon^{ij}x^j$ in the symmetric gauge and the statistical gauge potential

$$a^i_\alpha(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \nu \sum_{\beta\neq\alpha} \epsilon^{ij} \frac{x^j_\alpha - x^j_\beta}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|^2}. \tag{2}$$

By a singular gauge transformation, we can remove $\mathbf{a}_\alpha$ from the Hamiltonian at the expense of using a multivalued wavefunction

$$\psi_{new}(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \exp\left(i\nu \sum_{\alpha<\beta} \theta_{\alpha\beta}\right)\psi_{old}(\mathbf{x}_1, \ldots, \mathbf{x}_N), \tag{3}$$

where $\theta_{\alpha\beta} = \arctan(x^2_\alpha - x^2_\beta)/(x^1_\alpha - x^1_\beta)$. Using the complex notation $z = x^1 + ix^2, \bar{z} = x^1 - ix^2, \partial = \frac{\partial}{\partial z}, \bar{\partial} = \frac{\partial}{\partial \bar{z}}$, the gauge transformed Hamiltonian becomes[8]

$$H = \sum_{\alpha=1}^{N}\left(-\frac{2}{m}\bar{\partial}_\alpha\partial_\alpha + \frac{e^2B^2}{8m}|z_\alpha|^2\right) - \frac{eB}{2m}J, \tag{4}$$

where J is the angular momentum operator in the singular gauge

$$J = \sum_{\alpha=1}^{N} (z_\alpha \partial_\alpha - \bar{z}_\alpha \bar{\partial}_\alpha). \tag{5}$$

Its eigenvalues are shifted from those in the symmetric gauge by a constant $\frac{1}{2}\nu N(N-1)$. (See (10.a).) It is convenient to extract a factor $\exp(-\frac{1}{4}eB\sum_{\alpha=1}^{N}|z_\alpha|^2)$ from the wavefunction. Then the eigenvalue problem becomes

$$\hat{H}\hat{\psi} = (E - \frac{1}{2}N\omega)\hat{\psi}, \tag{6.a}$$

$$J\hat{\psi} = j\hat{\psi}, \tag{6.b}$$

(with $\omega \equiv \frac{eB}{m}$) where the new Hamiltonian $\hat{H}$ and wavefunctions $\hat{\psi}$ are defined by

$$\hat{H} = \sum_{\alpha=1}^{N}\left(-\frac{2}{m}\bar{\partial}_\alpha \partial_\alpha + \frac{eB}{m}\bar{z}_\alpha \bar{\partial}_\alpha\right), \tag{7.a}$$

$$\hat{\psi} = exp\left(\frac{eB}{4}\sum_{\alpha=1}^{N}|z_\alpha|^2\right)\psi. \tag{7.b}$$

Note that the ground state energy is shifted by $\frac{1}{2}N\omega$. We impose two physical requirements for the wavefunctions. First, they must vanish at points of coincidences if $\nu \neq 0$ due to the centrifugal potentials (hard-core requirement). Second, they form Abelian representations of the braid group.

Now we introduce the operators

$$a_\alpha^\dagger = \bar{z}_\alpha - \frac{2}{eB}\partial_\alpha, a_\alpha = \bar{\partial}_\alpha, \tag{8.a}$$

$$b_\alpha^\dagger = z_\alpha - \frac{2}{eB}\bar{\partial}_\alpha, b_\alpha = \partial_\alpha, \tag{8.b}$$

which satisfy $[a_\alpha, a_\beta^\dagger] = [b_\alpha, b_\beta^\dagger] = \delta_{\alpha\beta}$, all other commutators being zero. With respect to these operators, the Hamiltonian $\hat{H}$ in (7.a) and the angular momentum J in (5) can be rewritten as

$$\hat{H} = \omega \sum_{\alpha=1}^{N} a_\alpha^\dagger a_\alpha, \tag{9.a}$$

$$J = \sum_{\alpha=1}^{N} (b_\alpha^\dagger b_\alpha - a_\alpha^\dagger a_\alpha). \tag{9.b}$$

It is trivial to construct two distinct base states (for $0 \le \nu < 2$) with energy and angular momentum eigenvalues:

$$\begin{aligned} \hat{\psi}_I^{(0)} &= \prod_{\alpha<\beta} (z_\alpha - z_\beta)^\nu, \\ E_I^0 &= \frac{1}{2} N\omega, \\ j_I^0 &= \frac{1}{2}\nu N(N-1), \end{aligned} \tag{10.a.}$$

$$\begin{aligned} \hat{\psi}_{II}^{(0)} &= \prod_{\alpha<\beta} (\bar{z}_\alpha - \bar{z}_\beta)^{2-\nu}, \\ E_{II}^0 &= \frac{1}{2} N\omega + \frac{2-\nu}{2} N(N-1)\omega, \\ j_{II}^0 &= -\frac{2-\nu}{2} N(N-1). \end{aligned} \tag{10.b}$$

The general strategy of the ladder operator approach is to construct multi-anyon wavefunctions by acting with step operators on the base states in (10.a) and (10.b). We must, however, respect the statistics and hard-core requirements for the resulting wavefunctions. In order to respect the statistics, we use only symmetric combinations of the step operators. Consider the symmetric operators

$$C_{ln} = \sum_{\alpha=1}^{N} a_\alpha^{\dagger^l} b_\alpha^{\dagger^n}, \tag{11}$$

where l, n are non-negative integers such that $l + n \le N$. (These operators form a basis in the ring of symmetric polynomials in $2N$ variables.) They are step operators in energy and angular momentum which respect the statistics properties of the base states

$$[\hat{H}, C_{ln}] = \omega l C_{ln}, \tag{12.a}$$

$$[J, C_{ln}] = (n - l)C_{ln}. \tag{12.b}$$

They may, however, produce singular states (states with non-vanishing wavefunctions at points of coincidences) which have to be excluded by hand. We must identify which particular C_{ln} produce regular states. These operators can be safely applied to the base states. Consider C_{0n} first. With (8.b), we have

$$C_{0n}\hat{\psi}_I^{(0)} = (\sum_{\alpha=1}^{N} z_\alpha^n)\hat{\psi}_I^{(0)}. \tag{13}$$

Thus they can be safely applied. For C_{1m}, we have

$$\begin{aligned} C_{1m}\hat{\psi}_I^{(0)} &= \sum_{\alpha=1}^{N}(\bar{z}_\alpha - \partial_\alpha)(z_\alpha - \bar{\partial}_\alpha)^m\hat{\psi}_I^{(0)} \\ &= \sum_{\alpha=1}^{N}(\bar{z}_\alpha - \partial_\alpha)z_\alpha^m\hat{\psi}_I^{(0)} \\ &= -\sum_{\alpha=1}^{N} z_\alpha^m\partial_\alpha\hat{\psi}_I^{(0)} + \sum_{\alpha=1}^{N}(\bar{z}_\alpha z_\alpha^m - m z_\alpha^{m-1})\hat{\psi}_I^{(0)}, \end{aligned} \tag{14}$$

where we set $eB = 2$ for simplicity. The seemingly singular first term is in fact regular because

$$\sum_{\alpha=1}^{N} z_\alpha^m\partial_\alpha\hat{\psi}_I^{(0)} = \sum_{\alpha=1}^{N} z_\alpha^m \sum_{\beta\neq\alpha}\frac{\nu}{z_\alpha - z_\beta}\hat{\psi}_I^{(0)} = \nu\sum_{\alpha<\beta}\frac{z_\alpha^m - z_\beta^m}{z_\alpha - z_\beta}\hat{\psi}_I^{(0)}. \tag{15}$$

By a similar proof, one can apply a sequence of operators of the form C_{0n} followed by a sequence of operators of the form C_{1m} to $\hat{\psi}_I^{(0)}$ without generating any singularities. Moreover, states of the form

$$\psi_I^{(l)} = \prod_{\alpha<\beta}(za - zb)^{\nu+2l}, l = 1, 2, \ldots \tag{16}$$

are obtained from the action of C_{0n} on $\hat{\psi}_I^{(0)}$. We can apply a string of operators $C_{n_1m_1}C_{n_2m_2}\ldots C_{n_im_i}$ with $\sum_{j=1}^{i} n_j \leq 2l$ to $\psi_I^{(l)}$ without generating singularities, because such a string contains at most derivatives of order $\sum_{j=1}^{i} n_j$ with respect to z_α.

Thus we see that under suitable conditions, the step operators C_{ln} can be safely applied to the base state $\hat{\psi}_I^{(0)}$ to generate regular new wavefunctions. A similar analysis holds for the other base state $\hat{\psi}_{II}^{(0)}$. Furthermore, closed-form eigenfunctions generated by the action of combinations of operators C_{11}, C_{10} and C_{0m} only have been found, and they can be expressed in terms of the Laguerre functions.[8] In particular, they do not involve the operators C_{1m} with $m > 1$, which however are allowed to act on $\hat{\psi}_I^{(0)}$ to produce regular wavefunctions.

One should also note that the step operator approach only generates a subset of the whole spectrum of wavefunctions.[8] If we naively set ν to be zero or one, we obtain only a subset of the bosonic and fermionic wavefunctions. Unlike the states generated by the step operators, the energies of the missing states show non-linear dependence on the statistical parameter ν in recent numerical studies.[6] However, this is unimportant for what follows.

5.3 Non-Abelian Chern-Simons Particles

Recently, there has been much interest in the non-Abelian generalization of anyons. Non-Abelian Chern-Simons (NACS) particles carry non-Abelian charges and interact with each other through the non-Abelian Chern-Simons term. It has been argued that they may have applications in the fractional quantum Hall effect.[13] Consider a system of N particles each of which carries a statistical charge corresponding to a representation $R_{l_\alpha}, \alpha = 1, \ldots, N$ of a non-Abelian gauge group, which for definite we take to be $G = \mathrm{SU}(2)$. In the holomorphic gauge, the dynamics of N free SU(2) NACS particles is governed by the Hamiltonian[14][12]

$$\begin{aligned} \hat{H} &= -\sum_{\alpha=1}^{N} \frac{1}{m_\alpha} (\nabla_{\bar{z}_\alpha} \nabla_{z_\alpha} + \nabla_{z_\alpha} \nabla_{\bar{z}_\alpha}), \\ \nabla_{z_\alpha} &= \frac{\partial}{\partial z_\alpha} + \frac{2}{k} \sum_{\beta \neq \alpha} \frac{T_\alpha^a T_\beta^a}{z_\alpha - z_\beta}, \\ \nabla_{\bar{z}_\alpha} &= \frac{\partial}{\partial \bar{z}_\alpha}, \end{aligned} \tag{17}$$

where k, a positive integer, is a parameter of the theory and T_α^a are the SU(2)-generators in the representation R_{l_α}.

The wavefunctions take values in the tensor product of these representations.

$$\Psi \in R_{l_1} \otimes \ldots \otimes R_{l_N}. \tag{18}$$

We expand the single-valued wavefunction Ψ in terms of the conformal blocks $F_i \in R_{l_1} \otimes \ldots \otimes R_{l_N}$ (which satisfy $\nabla_\alpha F_i = 0$):

$$\Psi = \sum_i \psi_i F_i. \tag{19}$$

The Hamiltonian acting on the new wavefunction is just the free Hamiltonian. However, the complexity of the problem is hidden in the multivaluedness of the wavefunctions ψ_i. (In fact, it is more "natural" to work with the multivalued wavefunctions ψ_i than the original single-valued wavefunctions Ψ, partly because in the holomorphic gauge the Hamiltonian is not hermitian with respect to the usual inner product. Instead, the inner product is defined in the singular gauge and transformed back to the holomorphic gauge by a non-unitary transformation function which has to be taken into account in the definition of the inner product.[12])

From now on, we stick to the singular gauge. Consider N NACS particles in the same irreducible representation R_l of SU(2) moving in a uniform external magnetic field B. We introduce operators $a_\alpha, a_\alpha^\dagger, b_\alpha, b_\alpha^\dagger$ as in eqn.(8) of section 2 and find that the Hamiltonian is again given by eqn.(9). The only difference lies in the constraints of the monodromy properties of the wavefunctions. In the case of anyons, the wavefunctions have only one component and monodromy leads to acquisition of phases, whereas NACS particles have multi-component wavefunctions whose monodromy properties are given by matrices.

We define

$$\Omega_{\alpha\beta} \equiv \frac{2}{k} \sum_a T_\alpha^a T_\beta^a. \tag{20}$$

Note that $\sum_{\alpha<\beta} \Omega_{\alpha\beta}$, J and $\hat{H}$ commute with each other and are thus good quantum numbers. ($J - \sum_{\alpha<\beta} \Omega_{\alpha\beta}$ is the angular momentum operator in the holomorphic gauge.) We will discuss the diagonalization of $\sum_{\alpha<\beta} \Omega_{\alpha\beta}$ later. For the time being, let us assume this has been done and let $\psi_I \in R_{l_1} \otimes R_{l_2} \otimes \ldots \otimes R_{l_N}$ be a (position-independent) eigenvector of $\sum_{\alpha<\beta} \Omega_{\alpha\beta}$ with eigenvalue Ω. In analogy with the anyon case, we propose applying the same ladder operator approach with the following base states which are expressed as path-ordered line integrals:[15]

$$\hat{\psi}_I^{(0)}(z_1,\ldots,z_N) = P\exp\Big(\int_\Gamma \sum_{\alpha<\beta}(\Omega_{\alpha\beta} - 2m_{\alpha\beta}I)d\ln(z_\alpha - z_\beta)\Big)\psi_I, \tag{21}$$

$$\hat{\psi}_{II}^{(0)}(z_1,\ldots,z_N) = P\exp\Big(\int_\Gamma \sum_{\alpha<\beta}(2n_{\alpha\beta}I - \Omega_{\alpha\beta})d\ln(\bar{z}_\alpha - \bar{z}_\beta)\Big)\psi_I, \tag{22}$$

where Γ is a path in the N-dimensional complex space with one end point fixed and the other being $\zeta = (z_1,\ldots,z_N)$. The $m_{\alpha\beta}$ $(n_{\alpha\beta})$ depend on ψ_I and are the maximal (minimal) integers which make the wavefunctions non-singular at the points of coincidences. This is analogous to the requirement $0 \leq \nu < 2$ in the anyon case. Modulo the terms involving the identity matrix, the first integrand is just the flat Knizhnik-Zamolodchikov connection[16] whereas the second is related to its antiholomorphic analogue. One can easily check that these base states have the desirable monodromy properties. From (5),(6) and (7.a), we have

$$\begin{aligned}
\hat{H}\hat{\psi}_I^{(0)} &= 0,\\
E_I^0 &= \frac{1}{2}N\omega,\\
J\hat{\psi}_I^{(0)} &= \sum_{\alpha=1}^{N} z_\alpha \sum_{\beta\neq\alpha}\frac{(\Omega_{\alpha\beta} - 2m_{\alpha\beta}I)}{z_\alpha - z_\beta}\hat{\psi}_I^{(0)}\\
&= \sum_{\alpha<\beta}(\Omega_{\alpha\beta} - 2m_{\alpha\beta})\hat{\psi}_I^{(0)}\\
&= (\Omega - 2\sum_{\alpha<\beta} m_{\alpha\beta})\hat{\psi}_I^{(0)},\\
\sum_{\alpha<\beta}\Omega_{\alpha\beta}\hat{\psi}_I^{(0)} &= \Omega\hat{\psi}_I^{(0)},
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
\hat{H}\hat{\psi}_{II}^{(0)} &= \sum_{\alpha=1}^{N}\frac{eB}{m}\bar{z}_\alpha\sum_{\beta\neq\alpha}\frac{(2n_{\alpha\beta}I - \Omega_{\alpha\beta})}{\bar{z}_\alpha - \bar{z}_\beta}\hat{\psi}_{II}^{(0)}\\
&= \sum_{\alpha<\beta}\frac{eB}{m}(2n_{\alpha\beta}I - \Omega_{\alpha\beta})\hat{\psi}_{II}^{(0)}\\
&= \frac{eB}{m}[2\sum_{\alpha<\beta} n_{\alpha\beta} - \Omega]\hat{\psi}_{II}^{(0)}
\end{aligned}$$

$$
\begin{aligned}
J\hat{\psi}_{II}^{(0)} &= -\sum_{\alpha=1}^{N} \bar{z}_\alpha \sum_{\beta\neq\alpha} \frac{(2n_{\alpha\beta}I - \Omega_{\alpha\beta})}{\bar{z}_\alpha - \bar{z}_\beta} \hat{\psi}_{II}^{(0)} \\
&= \sum_{\alpha<\beta} (-2n_{\alpha\beta} + \Omega_{\alpha\beta}) \hat{\psi}_{II}^{(0)} \\
&= [-2\sum_{\alpha<\beta} n_{\alpha\beta} + \Omega] \hat{\psi}_{II}^{(0)}, \\
\sum_{\alpha<\beta} \Omega_{\alpha\beta} \hat{\psi}_{II}^{(0)} &= \Omega \hat{\psi}_{II}^{(0)},
\end{aligned}
\tag{24}
$$

where we have used the relation[15]

$$
[\sum_{\alpha<\beta} \Omega_{\alpha\beta}, \Omega_{\gamma\delta}] = 0, \tag{25}
$$

and the fact that ψ_I is an eigenstate of the operator $\sum_{\alpha<\beta} \Omega_{\alpha\beta}$. Mathematically, these commutator relations are just consequences of the integrability condition (infinitesimal pure braid relations) satisfied by the connection. Physically, they follow from the fact that Ω is related to the angular momentum J which is invariant upon monodromy.

Now that we have found the analogous base states, we will apply the ladder operators to them to generate new states. As before, the new states have to respect the statistics. (The NACS particles in the same irreducible representation are regarded as indistinguishable.)[17] Thus, we may only use symmetric combinations of step operators. Also, we have to check that the wavefunctions produced are regular at points of coincidences. There is, however, one crucial difference between the cases of anyons and NACS particles. Even with symmetric step operators, there is no guarantee that the monodromy properties of the wavefunctions are preserved. Any combination of step operators which does not preserve the monodromy properties of the wavefunctions is to be rejected.

First of all, let us consider C_{0n}. As before, we get

$$
C_{0n}\hat{\psi}_I^{(0)} = \left(\sum_{\alpha=1}^{N} z_\alpha^n\right)\hat{\psi}_I^{(0)}. \tag{26}
$$

This shows that C_{0n} can be safely applied to $\hat{\psi}_I^{(0)}$ without changing its monodromy properties or producing singularities. Next we consider C_{1m}.

$$
\begin{aligned}
C_{1m}\hat{\psi}_I^{(0)} &= \sum_{\alpha=1}^{N}(\bar{z}_\alpha - \partial_\alpha) z_\alpha^m \hat{\psi}_I^{(0)} \\
&= -\sum_{\alpha=1}^{N} z_\alpha^m \partial_\alpha \hat{\psi}_I^{(0)} + \sum_{\alpha=1}^{N}(\bar{z}_\alpha z_\alpha^m - m z_\alpha^{m-1})\hat{\psi}_I^{(0)} \\
&= -\sum_{\alpha<\beta}(\Omega_{\alpha\beta} - 2m_{\alpha\beta} I)\frac{z_\alpha^m - z_\beta^m}{z_\alpha - z_\beta}\hat{\psi}_I^{(0)} + \sum_{\alpha=1}^{N}(\bar{z}_\alpha z_\alpha^m - m z_\alpha^{m-1})\hat{\psi}_I^{(0)}.
\end{aligned}
\tag{27}
$$

For $m = 0$,

$$
C_{10}\hat{\psi}_I^{(0)} = \sum_{\alpha=1}^{N} \bar{z}_\alpha \hat{\psi}_I^{(0)}, \tag{28}
$$

which clearly preserves the monodromy property of $\hat{\psi}_I^{(0)}$. When $m = 1$, we have

$$
\begin{aligned}
C_{11}\hat{\psi}_I^{(0)} &= -\sum_{\alpha<\beta}(\Omega_{\alpha\beta} - 2m_{\alpha\beta} I)\hat{\psi}_I^{(0)} + \sum_{\alpha=1}^{N}(\bar{z}_\alpha z_\alpha - 1)\hat{\psi}_I^{(0)} \\
&= [-\Omega + 2\sum_{\alpha<\beta} m_{\alpha\beta} + \sum_{\alpha=1}^{N}(\bar{z}_\alpha z_\alpha - 1)]\hat{\psi}_I^{(0)}.
\end{aligned}
\tag{29}
$$

This shows that C_{11} can be safely applied to the base state. We can say more: strings made up of combinations of the operators C_{0n} (n=1,2,...), C_{10} and C_{11} act on the base state to generate physical states. On the other hand, the operators C_{1m} with $m > 1$ and C_{nm} with $n > 1$ generally change the monodromy property of base state. There is no obvious way of constructing an admissible combination of operators involving them which would preserve the monodromy property of the base state. We therefore reject them as being unphysical and restrict the admissible set of operators to those generated by C_{0n},C_{10} and C_{11}.

The crucial reason why the argument for C_{1m} (with $m > 1$) and C_{nm} (with $n > 1$) as physical operators for anyons do not carry over to the case of non-Abelian C-S particles is that the the various monodromy matrices do not commute. In other words,

$$
[\Omega_{\alpha\beta}, \Omega_{\gamma\delta}] \neq 0. \tag{30}
$$

As in the anyon case, there are again missing states in the spectrum. However, our wavefunctions do cover the entire lowest Landau level as they involve the operators C_{0n} only.

We now consider the construction of closed-form eigenfunctions. In the case of anyons, for $P(z_1, \ldots, z_N, \bar{z}_1, \ldots, \bar{z}_N) \prod_{\alpha<\beta}(z_\alpha - z_\beta)^\nu$ to be an eigenfunction of $\hat{H}$, it follows that the function P has to satisfy a modified differential equation.

$$\sum_{\alpha=1}^{N}\left(-\frac{2}{m}\partial_\alpha + \frac{eB}{m}\bar{z}_\alpha\right)\bar{\partial}_\alpha P - \frac{2}{m}\sum_{\alpha<\beta}\left(\frac{\bar{\partial}_\alpha - \bar{\partial}_\beta}{z_\alpha - z_\beta}\right)\nu P = \left(E - \omega\frac{N}{2}\right)P \qquad (31)$$

An ansatz has been made to construct closed-form eigenfunctions. All the solutions constructed can be expressed in terms of the Laguerre polynomials. They are generated by strings of operators C_{0n},C_{10} andC_{11} only. For NACS particles, we get a similar equation for P, but with ν replaced by $\Omega_{\alpha\beta} - 2m_{\alpha\beta}$. Nevertheless, since these operators are chosen to preserve the monodromy properties of the states. There is every reason to believe that the construction of closed-form solutions will go through with $\frac{1}{2}N(N-1)\nu$ replaced by $\Omega - 2\sum_{\alpha<\beta} m_{\alpha\beta}$.

Finally, we come to the diagonalization of $\sum_{\alpha<\beta}\Omega_{\alpha\beta}$. Consider the identity

$$\sum_a (T_1^a + T_2^a + \ldots + T_N^a)(T_1^a + T_2^a + \ldots + T_N^a) = \sum_{\alpha=1}^{N} T_\alpha^a T_\alpha^a + 2\sum_{\alpha<\beta} T_\alpha^a T_\beta^a. \qquad (32)$$

For SU(2), the left-hand side gives the Casimir operator $J(J+1)$ of the "spin" of the composite made up of the N particles, and the first term on the right-hand side gives the sum of the Casimir operators $\sum_{\alpha=1}^{N} J_\alpha(J_\alpha + 1)$ of the "spins" for the individual particles. (Here we abuse the word "spin" for the internal SU(2) symmetry group. The physical spin (which is a scalar in 2+1 dimensions) of a NACS particle in the j representation is given by $\frac{2}{k}J(J+1)$. Thus, for SU(2), we have

$$\Omega \equiv \sum_{\alpha<\beta}\Omega_{\alpha\beta} = \frac{1}{k}[J(J+1) - \sum_{\alpha=1}^{N} J_\alpha(J_\alpha + 1)]. \qquad (33)$$

We just decompose the composite state into irreducible representations and $\sum_{\alpha<\beta}\Omega_{\alpha\beta}$ would be diagonal in that basis. Actually, we can do better than that. It is easy to check that the operator $T_1^z + T_2^z + \ldots + T_N^z$ commutes with $\hat{H}$, J and $\sum_{\alpha<\beta}\Omega_{\alpha\beta}$. Thus they can be simultaneously diagonalized.

5.4 Second Virial Coefficient and the Large k Limit

In this section, we compute the second virial coefficients for some simple systems of NACS particles. To do so, we need to know all the two-particle states only.

First of all, consider two identical NACS particles in the $j = \frac{1}{2}$ representation of SU(2). From the addition rule for angular momenta, we find that the resulting states consist of a triplet with $\Omega = \frac{1}{2k}$ and a singlet with $\Omega = -\frac{3}{2k}$. For $N = 2$, Ω plays the role of the anyon phase, ν. Let us recall the formula derived by Arovas *et al.*[2] for the second virial coefficient of anyons,

$$B(\nu = 2l + \delta, T) = \lambda_T^2(-\frac{1}{4} + |\delta| - \frac{1}{2}\delta^2), \tag{34}$$

where $|\delta| < 2$. Note that it has a cusp at Bose values $\nu = 2l$. By taking the average over the four two-body states, the second virial coefficient of the NACS particles is given by

$$B(j = \frac{1}{2}, T) = \lambda_T^2[-\frac{1}{4} + \frac{3}{4k} - \frac{3}{8k^2}]. \tag{35}$$

For two particles with $j = 1$, the resulting states have "spins" 2, 1, and 0 (with $\Omega = \frac{2}{k}$, $-\frac{2}{k}$, and $-\frac{4}{k}$ and degeneracies 5, 3 and 1 respectively). We remark that all these states are bosonic if $k = 1$. When $k = 2$, the singlet is a bosonic state whereas others are fermionic. For $k > 4$, all the states are anyonic with $|\nu| < 1$. For $k > 1$, we have

$$B(j = 1, T) = \lambda_T^2[-\frac{1}{4} + \frac{20}{9k} - \frac{8}{3k^2}]. \tag{36}$$

Now we come to the large k limit. For two particles belonging to a representation j with $\lim_{\substack{j\to\infty \\ k\to\infty}} \frac{j^2}{k} = a < 1$, we approximate the sum over all the resulting "spins" $r \le 2j$ by an integral. For example, the $|\delta|$ term is given by

$$\begin{aligned} &\frac{1}{(2j+1)^2}\sum_{r=0}^{2j}(2r+1)\frac{1}{k}\big|[r(r+1) - 2j(j+1)]\big| \\ \sim&\frac{1}{kj^2}\int_{r=0}^{\sqrt{2}j} -r(r^2 - 2j^2) \\ =&\frac{j^2}{k}, \end{aligned} \tag{37}$$

where in the second line we approximate the sum by an integral and divide it into two parts (which happen to be equal) according to the sign of $r^2 - 2j^2$. The δ^2 term can be evaluated in a similar manner. Hence, we get

$$B = \lambda_T^2[-\frac{1}{4} + \frac{j^2}{k} - \frac{j^4}{3k^2}] = \lambda_T^2[-\frac{1}{4} + a - \frac{a^2}{3}]. \tag{38}$$

If $a \to 0$, the last term may be discarded and the second virial coefficient of the NACS particle in the j representation (with physical spin $\frac{2}{k}j(j+1)$) is the same as that of an anyon with *half* of the physical spin as its statistical parameter.

5.5 Concluding Remarks

(1) The N-free-NACS-particle problem can be solved by a similar method. The free Hamiltonian in the "anyon" gauge is given by

$$H = \sum_{\alpha=1}^{N} -\frac{2}{m}\bar{\partial}_\alpha \partial_\alpha. \tag{39}$$

The subtlety is that our base states become unnormalizable.[9] Let us define $r = (\sum_{\alpha=1}^{N} |z_\alpha|^2)^{\frac{1}{2}}$ and consider

$$HM(r)\hat{\psi}_I^{(0)} = \hat{\psi}_I^{(0)} \times -\frac{1}{2m}[d^2/dr^2 + (1/r)(2N - 1 + 2\Omega - 4\sum_{\alpha<\beta} m_{\alpha\beta})]M(r). \tag{40}$$

We have eigenfunctions of the form

$$M_\mu(r) = r^{-\mu} J_\mu(kr), \tag{41}$$

(where $\mu = 2(N - 1 + \Omega - 2\sum_{\alpha<\beta} m_{\alpha\beta})$) with eigenvalues $\hbar^2 k^2/2m$ for H.

(2) Let us now consider the construction of C_{lm} for multi-species non-Abelian C-S particles (particles in various irreducible representations). In this case, when we construct C_{0n}, we do so for each irreducible representation R and symmetrize over particles in this irreducible representation only.[10] Let us call the resulting operator C_{0n}^R. If we construct C_{10}^R and C_{11}^R in a similar manner, we find that these operators have to be rejected: They do not preserve the monodromy properties of the base states because $[\Omega_{\alpha\beta}, \Omega_{\gamma\delta}] \neq 0$, and we are no longer summing over all the particles. Therefore, for C_{10} and C_{11} we do sum over all the particles in the various irreducible representations.

(3) Note that C_{01} and C_{10} represent center of mass excitations. The operator C_{10} was also analyzed by Johnson and Canright,[18] while C_{11} is directly related to the Lie group generator of $SU(1,1)$.[19]

(4) We remark that the operators of C_{1m} and C_{nm} $(m,n > 2)$ do preserve the monodromy property of the base state, if ψ_I in eqns.(20) and (21) is chosen to be a simultaneous eigenstate of all $\Omega_{\alpha\beta}$. The statistics is "Abelianized" in this case. This situation occurs, for example, for some models of non-Abelian vortices of finite gauge groups such as the quarternion group.

(5) The same ladder operator approach may well apply to non-abelian vortices of finite gauge groups.[17][20] Unfortunately, we generally do not know how to construct "smooth" connections which would produce the desirable monodromy in this case.

(6) In a recent paper,[21] Dasnieres de Veigy and Ouvry derived the equation of state of an anyon gas in a strong magnetic field at low temperatures. The idea is that at sufficiently low temperatures, excitations to higher Landau levels can be neglected. Thus one may consider only the lowest Landau states of the anyons, which are covered by the step operators. In fact, apart from the statistical phase factor, the multi-anyon states in the lowest Landau level are tensor-product states of the individual anyon states. By regularizing the grand partition function with a harmonic potential, the equation of state can be obtained. The same decoupling principle should apply to NACS particles. For a fixed base state, modulo the statistical term involving $\Omega_{\alpha\beta}$, the multi-particle wavefunctions in the lowest Landau level are again tensor products of individual particle states. Therefore, in principle, one should be able to derive the equation of state of NACS particles in a strong magnetic field at low temperatures.

Note added. After this manuscript was accepted for publication, it came to our attention that the symmetry group SU$(1,1)$ had also been used for the discussion of two anyons interacting with a Coulomb potential.[22] Besides, an anyon can arise as a composite of magnetic flux with a Dirac particle.[23] In going from the Schrödinger to the Dirac equation, the Zeeman interaction of the spin with the magnetic field has to be included. This may give rise to extra delta function potentials at points of coincidences which can possibly make the hard-core requirement inappropriate. Thus, in some cases, for $0 \leq \nu < 1$, the second base state $\hat{\psi}^{(0)}_{\rm II}$ should be replaced by $\prod_{\alpha<\beta}(\bar{z}_\alpha - \bar{z}_\beta)^{-\nu}$ and is singular at points of coincidences. [See [23] for details.]

Nevertheless, the ladder operator approach goes through with this minor modification. An evaluation of the second virial coefficient has also been performed in a recent preprint by Taejin Lee.[24]

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Chapter 6

Complementarity in Wormhole Chromodynamics

6.1 Introduction

Many years ago, Wheeler[1] and Misner and Wheeler[2] proposed that electric field lines trapped in the topology of a multiply-connected space might explain the origin of electric charge. Consider a three-dimensional space with a handle (or "wormhole") attached to it, where the cross section of the wormhole is a two-sphere. On this space, the source-free Maxwell equations have a solution with electric field lines caught inside the wormhole throat. One mouth of the wormhole, viewed in isolation by an observer who is unable to resolve the small size of the mouth, cannot be distinguished from a pointlike electric charge. Only when the observer inspects the electric field more closely, with higher resolution, does she discover that the electric field is actually source free everywhere.

It is also interesting to consider what happens when a charged particle traverses a wormhole.⋆ (Of course, this "pointlike" charge might actually be one mouth of a smaller wormhole.) Suppose that, initially, the mouths of the wormhole are uncharged (no electric flux is trapped in the wormhole). By following the electric field lines, we see that after an object with electric charge Q traverses the wormhole, the mouth where it entered the wormhole carries charge Q, and the mouth where it exited carries charge $-Q$. Thus, an electric charge that passes through a wormhole transfers charge to the wormhole mouths.

In this chapter, we will address two (closely related) puzzles associated with this type of charge transfer process. Our first puzzle concerns the quantum mechanics of charged particles in the vicinity of a wormhole. We can compute the amplitude for the particle to propagate from an initial position to a final position by performing

⋆ Note that we are assuming that the wormhole is traversable. This assumption would be valid for a non-dynamical wormhole three-geometry, but it is in conflict with the "topological censorship" theorem[3] that can be proved in classical general relativity (with suitable assumptions about the positivity of the energy-momentum tensor). The traversability of the wormhole might be enforced by quantum effects. Alternatively, the reader might prefer to envision our space as a thin two-dimensional film, containing objects with Aharonov-Bohm interactions. Such wormholes might actually be fashioned in the laboratory!

a sum over histories. Naively, one would expect this sum to include histories that traverse the wormhole, and that the contribution to the path integral due to these histories should be combined coherently with the contribution due to histories that do not traverse the wormhole. In fact, the histories can be classified according to their "winding number" around the wormhole, which can take any integer value, and one expects that all of the winding sectors should be combined coherently. Upon further reflection, though, one sees that, for charged particles, this naive expectation must be incorrect. Long after the final position of the particle has been detected, an observer can measure the charge of one of the wormhole mouths. If the mouth was uncharged initially, and carries charge nQ finally, then the observer concludes that the charged particle must have entered that mouth of the wormhole n times. Because the winding sectors are perfectly correlated with the charge transferred to the mouth, the amplitudes associated with different numbers of windings cannot interfere with one another. The puzzle in this case is to understand more clearly the mechanism that destroys the coherence of the different winding sectors.

Our second puzzle arises in a non-Abelian gauge theory, such as quantum chromodynamics. Suppose that a wormhole initially carries no color charge, and consider what happens when a "red" quark traverses the wormhole. (We can give a gauge-invariant meaning to the notion that the quark is red by establishing a "quark bureau of standards" at some preferred location, and carefully preserving a standard red (R) quark, blue (B) quark, and yellow (Y) quark there. When we say that a quark at another location is red, we mean that if it is parallel transported back to the bureau of standards, its color matches that of the standard R quark. This notion is especially simple if we assume that there are no color magnetic fields, so that parallel transport is unaffected by smooth deformations of the path.) An observer who watches the red quark enter one mouth of the wormhole concludes that the mouth becomes a red source of color electric field.[†] But the other mouth of the wormhole is initially in a color-singlet state, and it cannot suddenly acquire a long-range color electric field as the quark emerges from the mouth. Thus, after the traversal, the quark and mouth

† We are assuming that the wormhole is being examined on a sufficiently short distance scale that the effects of color confinement can be neglected.

must be in the color-singlet state

$$\frac{1}{\sqrt{3}}\left(|R\rangle_{\text{quark}}\otimes\left|\bar{R}\right\rangle_{\text{mouth}}+|B\rangle_{\text{quark}}\otimes\left|\bar{B}\right\rangle_{\text{mouth}}+|Y\rangle_{\text{quark}}\otimes\left|\bar{Y}\right\rangle_{\text{mouth}}\right)\ . \qquad (1)$$

The puzzle in this case is to understand why the quark that emerges from the wormhole is not simply in the color state $|R\rangle$, and how the correlation between the color of the quark and the color of the mouth is established.

The resolution of these puzzles involves some peculiar features of the Aharonov-Bohm effect[4] on non-simply connected manifolds. The essential concept is the magnetic flux "linked" by the wormhole. If a particle with charge Q is carried around a closed path that traverses a wormhole (in a $U(1)$ gauge theory), it in general acquires an Aharonov-Bohm phase $e^{iQ\Phi}$, where Φ is the flux associated with the path. (This flux is defined modulo the flux quantum $\Phi_0 = 2\pi/e$, where e is the charge quantum.) If magnetic field strengths vanish everywhere, this flux is a topological invariant, unchanged by smooth deformations of the path. The crucial point is that the flux Φ and the charge of a wormhole mouth are complementary observables—if the mouth has a definite charge (like zero), then the flux does not take a definite value. Summing over the different possible values of the flux generates the decoherence of the winding sectors described above, and also (in the non-Abelian case) causes the red quark that traverses the wormhole to emerge in the state Eq. (1).

6.2 Wormhole complementarity

Let us now analyze these Aharonov-Bohm interactions in greater detail. We will use a notation that is appropriate when the gauge group G is a finite group. This will serve to remind the reader that our analysis applies to the case of a *local discrete symmetry*.[5–7] For the case of a continuous gauge group, one need only replace sums by integrals in some of the expressions below. When the gauge group is discrete (and also when it is continuous), the electric charge of an object, including a wormhole mouth, can be measured in principle by scattering a loop of *cosmic string* (or a closed magnetic solenoid) off of the object. For ease of visualization, we will carry out our explicit analysis for the case of two spatial dimensions, so that charges are measured by scattering magnetic *vortices*. The analysis in three spatial dimensions is similar.

There are actually two types of topological magnetic flux associated with a wormhole in two spatial dimensions, for there are two topologically distinct paths for which

Aharonov-Bohm phases can be measured, as shown in Fig. 1. The path α encloses one mouth of the wormhole, and we will denote the group element associated with parallel transport around this path as $a \in G$. The path β passes through both wormhole mouths, and we denote the associated group element as $b \in G$. We refer to these group elements as the α-flux and β-flux of the wormhole, and denote the corresponding quantum state of the wormhole as $|a,b\rangle_{\text{wormhole}}$. (Of course, in three spatial dimensions, the analog of the path α is contractible, if the cross section of the wormhole is a two-sphere, and there is no topological α-flux.)

Now, we can measure the *electric* charge of a wormhole mouth by winding a vortex around the mouth, and observing the Aharonov-Bohm phase acquired by the vortex. However, winding the vortex around the mouth will also change the state $|a,b\rangle$ of the wormhole. For our purposes, it will be sufficient to consider the special case in which $a = e$, the identity. (A more general analysis of non-Abelian Aharonov-Bohm interactions on topologically nontrivial spaces can found in Ref. 8). As shown in Fig. 2, we may enclose the vortex with a closed path γ; we denote the group element associated with transport around γ as $h \in G$, and refer to it as the flux of the vortex. As the vortex winds counterclockwise around the wormhole mouth, the path $\beta\gamma^{-1}$ is deformed to β. (Here, $\beta\gamma^{-1}$ denotes the path that is obtained by tracing γ^{-1} *first*, followed by β.) Thus, when the vortex winds around the mouth, the flux associated with $\beta\gamma^{-1}$ *before* the winding becomes the flux associated with β after the winding; we conclude that the state of wormhole and vortex is modified according to

$$|e,b\rangle_{\text{wormhole}} \otimes |h\rangle_{\text{vortex}} \rightarrow |e,bh^{-1}\rangle_{\text{wormhole}} \otimes |h\rangle_{\text{vortex}} . \tag{2}$$

Eq. (2) is the centerpiece of our analysis. It says that if the wormhole is in the "flux eigenstate" $|e,b\rangle$, then any attempt to use Aharonov-Bohm interference to measure the electric charge of one mouth is doomed to failure. If we scatter a vortex off of the mouth (with vortex flux $h \neq e$), whether the vortex passed to the left or the right of the mouth is perfectly correlated with the state of the wormhole, and therefore no interference is seen; the probability distribution of the scattered vortex is the incoherent sum of the probability distributions for vortices that pass to the left and pass to the right.

However, by superposing the wormhole states of definite β-flux, we can construct states with definite charge. (We need only decompose the regular representation of

G into irreducible representations.) In particular, in the state

$$|0\rangle_{\text{wormhole}} = \frac{1}{\sqrt{n_G}} \sum_{b\in G} |e,b\rangle_{\text{wormhole}} \tag{3}$$

(where n_G is the order of the group G), each mouth of the wormhole has zero charge. To see this, consider carrying the h-vortex around one mouth of this wormhole. It is easy to see that the state of the wormhole is unmodified, so that the Aharonov-Bohm phase acquired by the vortex is trivial. On the other hand, suppose that we try to measure the β-flux of the wormhole by carrying a charged particle along the path β. Let us denote the initial state of the particle as $|v\rangle_{\text{particle}}$, and let (ν) be the irreducible representation of G according to which the state transforms. Then if we carry this particle around the path β where the wormhole is initially in the state $|0\rangle_{\text{wormhole}}$, the state of particle and wormhole is modified according to

$$\begin{aligned}&|\text{initial}\rangle \equiv |v\rangle_{\text{particle}} \otimes |0\rangle_{\text{wormhole}} \rightarrow \\ &|\text{final}\rangle \equiv \frac{1}{\sqrt{n_G}} \sum_{b\in G} D^{(\nu)}(b)\,|v\rangle_{\text{particle}} \otimes |e,b\rangle_{\text{wormhole}} \ ;\end{aligned} \tag{4}$$

thus the overlap of the final state with the initial state is

$$\langle\text{final}\,|\text{initial}\rangle = \frac{1}{n_G} \sum_{b\in G} \langle v|\, D^{(\nu)}(b)\,|v\rangle = \begin{cases} 1, & \text{if } (\nu) = \text{trivial} \ ; \\ 0, & \text{otherwise} \ . \end{cases} \tag{5}$$

Unless (ν) is trivial, the state of the particle that has been carried through the wormhole is orthogonal to the original state. Hence we recover our earlier conclusion that, for charged particles propagating on the wormhole geometry, paths that traverse the wormhole add incoherently with paths that do not.

We see that the wormhole cannot simultaneously have a definite β-flux and a definite charge. We call this property "wormhole complementarity." It is intimately related to the complementary connection between magnetic and electric flux that was first emphasized by 't Hooft,[9] and was generalized to the non-Abelian case in Ref. 10.

By decomposing the regular representation Eq. (2) into irreducible representations, we obtain states in which the wormhole *mouth* has a definite charge. The charge of a mouth should not be confused with the "Cheshire charge"[6,11] carried by the whole wormhole. To measure the charge of the whole wormhole, we would wind a vortex around *both* mouths of the wormhole. In this process, the state of vortex and wormhole is modified according to[8]

$$
\begin{aligned}
&|a,b\rangle_{\text{wormhole}} \otimes |h\rangle_{\text{vortex}} \rightarrow \\
&\left|hah^{-1}, hbh^{-1}\right\rangle_{\text{wormhole}} \otimes \left|h\left(aba^{-1}b^{-1}\right)h\left(aba^{-1}b^{-1}\right)^{-1}h^{-1}\right\rangle_{\text{vortex}} .
\end{aligned}
\tag{6}
$$

Note that $aba^{-1}b^{-1}$ is the "total flux" of the wormhole, the flux associated with a path that encloses both mouths. Charge measurement is possible only if the initial and final vortex states are not orthogonal, so that interference can occur. Therefore, the flux h of the vortex must commute with the total flux of the wormhole—the charge that can be detected is actually a representation of $N(aba^{-1}b^{-1})$, the centralizer of the total flux.[12,6,11] States of definite Cheshire charge are obtained by decomposing the wormhole states $|a,b\rangle$ into states that transform irreducibly under the action Eq. (6), where $h \in N(aba^{-1}b^{-1})$.

Of course, to an observer with poor resolution, the wormhole mouths look like pointlike particles, and the Cheshire charge of the wormhole coincides with the Cheshire charge of vortex pairs that has been discussed elsewhere.[10,13–14] For example, if $b = e$ then the mouths appear to be a vortex with flux a and an anti-vortex with flux a^{-1}. In the case $a = e$ that we have considered, neither wormhole mouth carries any flux, and the states $|e,b\rangle_{\text{wormhole}}$ are transformed as

$$
|e,b\rangle_{\text{wormhole}} \otimes |h\rangle_{\text{vortex}} \rightarrow \left|e, hbh^{-1}\right\rangle_{\text{wormhole}} \otimes |h\rangle_{\text{vortex}} \tag{7}
$$

when the vortex winds around the wormhole. The states of definite Cheshire charge are obtained by superposing the flux eigenstates $|e,b\rangle_{\text{wormhole}}$, with b taking values in a particular conjugacy class of G. Specifically the states

$$
|0,[b]\rangle_{\text{wormhole}} = \frac{1}{\sqrt{n_{[b]}}} \sum_{b' \in [b]} \left|e, b'\right\rangle_{\text{wormhole}} \tag{8}
$$

(where $[b]$ denotes the class containing b, and $n_{[b]}$ is the order of that class) have trivial total charge, although each wormhole mouth carries charge in these states.

The peculiar behavior we found for Aharonov-Bohm scattering off of a wormhole mouth, when the wormhole is in a flux eigenstate, can be given a more conventional interpretation if we think of the wormhole as a pair of charged particles in a particular (correlated) state. For example, the flux eigenstate $|e,e\rangle_{\text{wormhole}}$ can be decomposed as

$$\begin{aligned}|e,e\rangle_{\text{wormhole}} &\equiv |0,[e]\rangle_{\text{wormhole}}\\ &=\sum_{\nu} C_\nu \sum_i \frac{1}{\sqrt{n_\nu}}\,|e_i,\nu\rangle\otimes|e_i^*,\nu\rangle\ ,\quad \sum_\nu |C_\nu|^2=1\,,\end{aligned}\tag{9}$$

where the $|e_i,\nu\rangle$'s are a basis for the space on which the irreducible representation (ν) acts, and n_ν is the dimension of this representation. This is a superposition of states in which the two particles (the mouths) have nontrivial charges, and are in a combined state of trivial charge. Experiments involving one of the mouths are described by a mixed density matrix of the form

$$\rho=\sum_\nu |C_\nu|^2\frac{1}{n_\nu}\mathbf{1}_\nu\,,\tag{10}$$

and Aharonov-Bohm scattering of the h-vortex off the mouth enables us to measure

$$\mathrm{tr}\ D(h)\rho=\sum_\nu |C_\nu|^2\frac{1}{n_\nu}\chi^{(\nu)}(h)=\begin{cases}1, & h=e\ ;\\ 0, & \text{otherwise}\ ,\end{cases}\tag{11}$$

where $\chi^{(\nu)}$ denotes the character of the representation. (The second equality in Eq. (11) follows from the property Eq. (2).) From the group orthogonality relations, we see that $|C_\nu|^2=n_\nu^2/n_G$. Thus Aharonov-Bohm scattering enables us to determine the probability that the wormhole mouth carries charge (ν), but does not determine the relative phases of the C_ν's.[13] When we think of it as a point particle, the unusual thing about a wormhole mouth is that it is natural to consider a state such that the mouth is in a superposition of particle states with different gauge charges.

6.3 Charge transfer

Now let us suppose that, after the wormhole in the initial state $|0\rangle_{\text{wormhole}}$ is traversed by the charged particle in the initial state $|v\rangle_{\text{particle}}$, we attempt again to measure the charges of the two mouths. If an h-vortex is carried around the mouth that the charged particle *entered*, then the state of wormhole, particle, and vortex is modified according to

$$\begin{aligned}&\frac{1}{\sqrt{n_G}}\sum_{b\in G} D^{(\nu)}(b)\,|v\rangle_{\text{particle}}\otimes|e,b\rangle_{\text{wormhole}}\otimes|h\rangle_{\text{vortex}}\to\\ &\frac{1}{\sqrt{n_G}}\sum_{b\in G} D^{(\nu)}(b)\,|v\rangle_{\text{particle}}\otimes\left|e,bh^{-1}\right\rangle_{\text{wormhole}}\otimes|h\rangle_{\text{vortex}}\ ,\end{aligned}\tag{12}$$

so that the overlap of the initial state with the final state is

$$\text{overlap} = \frac{1}{n_G} \sum_{b,b' \in G} \langle v | \, D^{(\nu)}(b')^{-1} D^{(\nu)}(b) \, | v \rangle \cdot \langle e, b' | \, e, bh^{-1} \rangle = \langle v | \, D^{(\nu)}(h) \, | v \rangle \; . \quad (13)$$

This is exactly the same as the overlap we would have obtained if the vortex had been carried around the initial charged particle. Thus, as we anticipated, the charge of the particle has been transferred to the mouth of the wormhole.

But if we measure instead the charge of the *other* mouth, we obtain a rather different result. It is actually most instructive to consider carrying the h-vortex around *both* the charged particle and the other wormhole mouth. A variant of the argument given earlier shows that carrying the vortex counterclockwise around this mouth changes the wormhole state $|e, b\rangle$ to $|e, hb\rangle$. We thus find that the state of wormhole, particle, and vortex is modified according to

$$\begin{aligned} &\frac{1}{\sqrt{n_G}} \sum_{b \in G} D^{(\nu)}(b) \, |v\rangle_{\text{particle}} \otimes |e, b\rangle_{\text{wormhole}} \otimes |h\rangle_{\text{vortex}} \to \\ &\frac{1}{\sqrt{n_G}} \sum_{b \in G} D^{(\nu)}(h) D^{(\nu)}(b) \, |v\rangle_{\text{particle}} \otimes |e, hb\rangle_{\text{wormhole}} \otimes |h\rangle_{\text{vortex}} \, , \end{aligned} \quad (14)$$

and that the overlap of the initial state with the final state is

$$\text{overlap} = \frac{1}{n_G} \sum_{b,b' \in G} \langle v | \, D^{(\nu)}(b')^{-1} D^{(\nu)}(hb) \, | v \rangle \cdot \langle e, b' | \, e, hb \rangle = 1 \, . \quad (15)$$

Thus the Aharonov-Bohm phase is trivial, and we conclude that the charged particle and mouth are combined together into a singlet state, again as anticipated.

Eq. (1) is a special case of this result. We now understand that if the wormhole mouth initially carries no color charge, that means that the color holonomy associated with traversing the wormhole does not take a definite value. Thus the red quark emerges from the wormhole mouth carrying indefinite color, but with its color perfectly anti-correlated with the color of the mouth. Furthermore, after the (initially) red quark passes through the wormhole, the wormhole state is a superposition of a color octet and color singlet, so that Cheshire charge has been transferred to the wormhole.⋆

⋆ $SU(3)_{\text{color}}$ Cheshire charge has also been discussed recently by Bucher and Goldhaber.[15]

In summary, we have seen that the β-flux "linked" by a wormhole and the charge of a wormhole mouth cannot simultaneously have definite values. We call this property "wormhole complementarity." If the β-flux has a definite value, then each wormhole mouth is in an incoherent superposition of charge eigenstates, and there is no Aharonov-Bohm interference when a vortex (or cosmic string) scatters off the mouth. If the charge of each mouth has a definite value, then the wormhole is in a coherent superposition of β-flux eigenstates. Thus, after a colored particle traverses the wormhole, its color is correlated with that of the wormhole mouth from which it emerged.

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FIGURE CAPTIONS

1) Two non-contractible paths α and β, beginning and ending at an arbitrarily chosen basepoint x_0, on the wormhole geometry. The group elements associated with parallel transport around these paths are the α-flux and β-flux of the wormhole.

2) A vortex winds around one mouth of the wormhole, as shown in (a). If the path $\beta\gamma^{-1}$ shown in (b) is deformed during the winding of the vortex, so that the vortex never crosses the path, $\beta\gamma^{-1}$ evolves to the path β.

FIG. 1

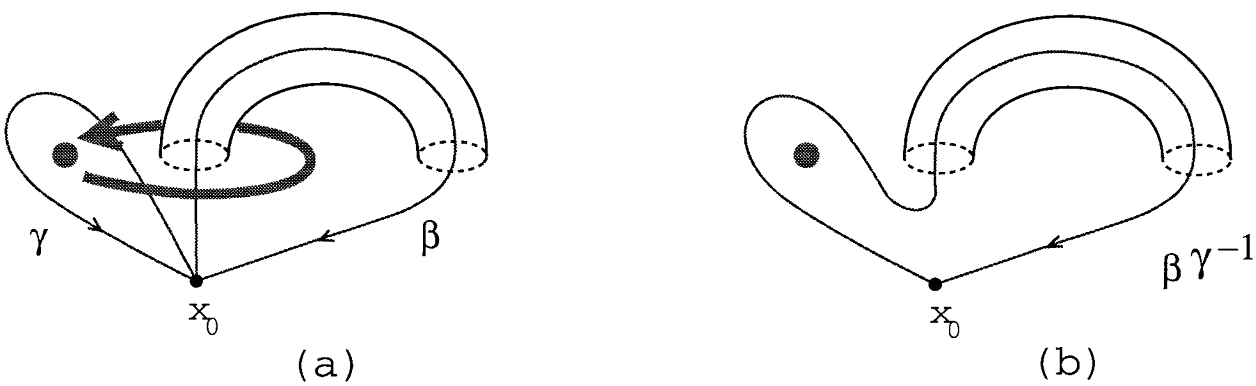

FIG. 2

Chapter 7
Scattering from Electroweak Strings

7.1 Introduction

Some years ago, Callan[1] and Rubakov[2] (see also Wilczek[3]) showed that a grand unified monopole may catalyze baryon number violating processes with strong interaction cross sections rather than the much smaller geometric cross sections. This enhancement effect can be understood as a consequence of a large amplification of the fermionic wave functions near the location of the monopole.[4]

A similar enhancement of cross section also occurs for cosmic strings with fractional fluxes:[5,6] Alford, March-Russell, and Wilczek[5] studied the fermion number violating process in a cosmic string core due to a Yukawa coupling. In their model, there are two fermions with equal $U(1)$ charges and two scalar fields. The first scalar field, η, acts as the Higgs field and thus condenses outside string core. The second scalar field, ϕ, which has a Yukawa coupling to the two fermion fields, condenses within the core. In the limit $kR \ll \nu R \ll 1$, where k is the momentum of the incident fermion, R is the size of the core and $\nu = \lambda < \phi >$ (λ being the Yukawa coupling constant), they found generic enhancement by large factors over the naive, geometric cross section. Maximal enhancement occurs when $\frac{d\sigma}{d\theta} \sim \frac{1}{k}$.

A prominent feature of their results is that while a large enhancement of the fermion number violating process is a general phenomenon, its actual magnitude is extremely sensitive to the $U(1)$ charge of the fermions. For instance, changing the charge from $\alpha = 1/2$ to $\alpha = 1/4$ results in a diminuation of the inelastic cross section by around 15 orders of magnitude. By assigning baryon numbers to the fermions and scalars, their results imply that the exact magnitude of the baryon number violating process is very sensitive to the details of the grand unification model under consideration. Since there are uncertainties in our experimental determination of low energy parameters such as the Weinberg angle, it might be hard for us to say for sure whether a model is phenomenologically feasible. The point is that a slight error made in our determination of the values of such parameters leads to a huge variation in the rate of baryogenesis and may render a feasible model unfeasible and vice versa.

For this reason, we would like to ask the following question: Is this sensitivity a generic feature or is it model-dependent? In other words, can we construct a model where the inelastic cross section is less sensitive to the values of the parameters?

A hint to the answer to this question comes from the investigation made by Perkins *et al.*[6] In their paper, the cross section for a baryon number violating process was derived using first order perturbation theory in quantum field theories. The transition matrix element between an initial state $|\psi\rangle$ and a final state $|\psi'\rangle$ is given by $\mathcal{A} = \langle\psi'| \int d^4x \mathcal{L}_I(x) \, |\psi\rangle$. The computation was divided into two steps. Firstly, they evaluated $\mathcal{A}$ using free fermion spinors, resulting in the "geometric" cross section. In the second step, they solved the Dirac equation with the appropriate boundary conditions to determine the amplitude of the spinor at the core radius R and defined the amplification factor A as the ratio of the amplitude of the actual spinor to that of a free spinor. Since the cross section is proportional to $\mathcal{A}^2$ and since $\mathcal{A}$ involves two spinors, the catalysis cross section is enhanced by a factor A^4 over the geometric cross section. This argument relies on the fact that the amplification factor for the initial state is the same as that for the final state. It might be possible for us to construct models with different amplification factors for the initial and final states. If the two amplifications have opposite dependence on the parameters, the overall amplification, which is the product of the two amplification factors, will then be insensitive to the parameters in the model.

In this chapter, we study the scattering of a charged fermion from an electroweak string. We show that for $0 < \sin^2\theta_w < 1/2$ (θ_w =Weinberg angle), $\omega \sim k \sim m_e$ and $kR \ll 1$, the helicity flip differential cross section for electrons is of the order m_e^{-1}. A delicate cancellation of the dependence of the two amplification factors on the Weinberg angle indeed occurs within this regimé. We would like to remark that the differential cross section in this regimé is dominated by a single partial wave and is thus independent of angle.

Incidentially, our results illustrate that the analysis of the enhancement effects for cosmic strings can be extended to a wider class of string defects: the semi-local strings[7] and the electroweak strings.[8] These recently discovered defects occur in theories where the fundamental group of the vacuum manifold is trivial. Thus, they are at best metastable.[9] While Z-flux carrying electroweak strings are unstable in the Weinberg-Salam model, various mechanisms for stabilizing a Z-flux string have been proposed. First, one can add a linear time dependence of the Goldstone boson to obtain a stable spinning vortex solution.[10] Second, fermions that are massive outside the core become massless inside. It is clear that there are superconducting zero modes in the core. These bound states tend to stabilize the non-topological solitons.[11] Another possibility would be to consider extensions of the electroweak model or topological strings carrying Z-flux which are formed in an earlier phase transition.[12]

Baryogenesis during the weak phase transition is particularly interesting as it may eventually be experimentally verifiable. Consider the following wild speculative scenario: non-topological electroweak strings are formed at the electroweak phase transition. They are stabilized by some mechanism (either one of the above or a combination or some other means). Baryogenesis occurs inside their cores. Baryogenesis due to electroweak strings in the two-Higgs model has been discussed in the literature.[13] It would be interesting to understand the relevance of our results to baryogenesis in future investigations.

After the completion of an earlier version of this chapter, we received a revised manuscript by Davis, Martin and Ganoulis[14] which also discussed electrons scattering off electroweak strings for $0 \leq \sin^2\theta_w \leq 1/2$ and $k \ll m$ or $k \gg m$. In this chapter, we consider the whole parameter space $0 \leq \sin^2\theta_w \leq 1$ in the regimé $m \sim k$. In particular, our analysis applies to semi-local strings, which correspond to $\sin^2\theta_w = 1$.

The plan of this chapter is as follows. In section 2, we review the subject of the electroweak strings and describe a simple model of the field configuration that we will be working with. Using a partial wave analysis, the differential cross section for the helicity flip process of electrons for various values of θ_w will be computed in section 3. In particular, we show that the cross section is proportional to m_e^{-1} for $0 < \sin^2\theta_w < 1/2$, $\omega \sim k \sim m_e$ and $kR \ll 1$. Moreover, the result for the semi-local strings can be obtained from that of the electroweak strings by setting $\sin^2\theta_w$ to 1. In our concluding remarks in section 4, we also note that helicity is violated outside the core of an axion string. Thus, it makes no sense to discuss helicity conserving and helicity flip cross sections in this context.

7.2 Extended Abelian Higgs Model and Electroweak Strings

Consider an extension of the Abelian Higgs model with $N = 2$ complex scalars Φ with their overall phase gauged and an $SU(2)$ global symmetry. The most general renormalizable Lagrangian in four dimensions consistent with these symmetries is

$$\mathcal{L} = |D_\mu \Phi|^2 - \frac{1}{2}\lambda\left(|\Phi|^2 - \eta^2\right)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{1}$$

The field Φ acquires a vacuum expectation value of magnitude η and the symmetry is spontaneously broken into a global $U(1)$. It has also been shown that the Nielsen-Olesen vortex solutions[15] of the Abelian Higgs model (the case with N=1) carry over to the extended Abelian Higgs model. However, the stability of such vortex solutions becomes a dynamical question and depends on the ratio of the masses of the Higgs and vector particles.[9]

Now the extended Abelian Higgs model is precisely the Weinberg-Salam model[16] with the $SU(2)$ charge set equal to zero. By gauging the $SU(2)$ symmetry, one obtains string solutions in the electroweak theory. Such electroweak strings are non-topological and unstable in the minimal electroweak theory. They may, however, be made metastable in some extended models.

Consider an electron moving in the background field of an electroweak string. The relevant part of the Lagrangian is

$$\mathcal{L} = i\bar{L}\gamma^\mu D_\mu L + i\bar{e}_R\gamma^\mu D_\mu e_R - f_e(\bar{L}e_R\Phi + \Phi^\dagger \bar{e}_R L), \tag{2}$$

where $\bar{L} = (\bar{\nu}, \bar{e}_L)$, f_e is the Yukawa coupling constant, Φ is the usual Higgs doublet, and the covariant derivative has the form

$$D_\mu = \partial_\mu + \frac{i\alpha\gamma}{2}Z_\mu, \tag{3}$$

where $\gamma = e/(\sin\theta_w \cos\theta_w)$ (θ_w being the Weinberg angle) and the $Z-$coupling, α, is given by[17]

$$\alpha = -2(T_3 - Q\sin^2\theta_w), \tag{4}$$

where T_3 is weak isospin and Q is electric charge. Note that for electrons and down quarks,

$$\alpha_L = \alpha_R + 1, \tag{5}$$

and there is a marked asymmetry between left and right fields.

For explicit calculations, consider the following simple model[14] of the field configuration.

$$
\begin{aligned}
\Phi &= (\, \Phi^{+} \quad \Phi^{0} \,) = f(r) e^{i\theta} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
Z_{\phi} &= -v(r)/r \\
Z_r &= W = A = 0 \\
f(r) &= \begin{cases} 0 & r < R \\ \frac{\eta}{\sqrt{2}} & r > R \end{cases} \\
v(r) &= \begin{cases} 0 & r < R \\ \frac{2}{\gamma} & r > R \end{cases}
\end{aligned}
\tag{6}
$$

where Z and W are the gauge bosons and A is the photon field. We expect our results to be insensitive to the detail of the core model. A discussion about this issue can be found in Ref. 6.

Writing $e_L = \begin{pmatrix} 0 \\ \psi \end{pmatrix}$ and $e_R = \begin{pmatrix} \chi \\ 0 \end{pmatrix}$, in the representation

$$
\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & -\sigma^j \\ \sigma^j & 0 \end{pmatrix}, \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{7}
$$

the Hamiltonian is

$$
H = \begin{pmatrix} -i\sigma^j D_j^R & f_e f e^{-i\theta} \\ f_e f e^{i\theta} & i\sigma^j D_j^L \end{pmatrix}. \tag{8}
$$

The equations of motion for ψ and χ are

$$
\begin{aligned}
\omega\chi + i\sigma^j D_j^R \chi - f_e f e^{-i\theta}\psi &= 0 \\
\omega\psi - i\sigma^j D_j^L \psi - f_e f e^{i\theta}\chi &= 0.
\end{aligned}
\tag{9}
$$

Note the phase $e^{i\theta}$ and the coupling of ψ to χ via the mass term. Inside the core, there is no coupling and electron is massless. The helicity operator is given by

$$
\Sigma \cdot \Pi = \begin{pmatrix} \sigma \cdot \pi_R & 0 \\ 0 & \sigma \cdot \pi_L \end{pmatrix} = \begin{pmatrix} -i\sigma^j D_j^R & 0 \\ 0 & -i\sigma^j D_j^L \end{pmatrix}. \tag{10}
$$

To see that helicity is not conserved, we compute its commutator with the Hamiltonian and find it to be non-zero inside the core.[18]

$$
[H, \Sigma \cdot \Pi] = i f_e \begin{pmatrix} 0 & \sigma^j (D_j \Phi^0)^* \\ \sigma^j D_j \Phi^0 & 0 \end{pmatrix}. \tag{11}
$$

Note that helicity violating processes can only occur in the string core. They can, however, be enhanced by an amplification of the fermionic wave function at the core radius. In the following section, we perform a detailed calculation of the differential cross section for such scattering processes.

7.3 Scattering Amplitude

We try the usual partial wave decomposition.

$$\chi(r,\theta) = \sum_{l=-\infty}^{\infty} \begin{pmatrix} \chi_1^l(r) \\ i\chi_2^l(r)e^{i\theta} \end{pmatrix} e^{il\theta}$$
$$\psi(r,\theta) = \sum_{l=-\infty}^{\infty} \begin{pmatrix} \psi_1^l(r) \\ i\psi_2^l(r)e^{i\theta} \end{pmatrix} e^{i(l+1)\theta}. \tag{12}$$

Making use of

$$\sigma^j D_j = \begin{pmatrix} 0 & e^{-i\theta}(D_r - iD_\theta) \\ e^{i\theta}(D_r + iD_\theta) & 0 \end{pmatrix}, \tag{13}$$

we substitute (12) into (9) to obtain

$$\begin{array}{llll} \omega\chi_2^l & + \left(\frac{d}{dr} - \frac{l}{r} + \frac{\alpha_R\gamma\nu}{2r}\right)\chi_1^l & -f_e f\psi_2^l & = 0 \\ \omega\chi_1^l & - \left(\frac{d}{dr} + \frac{l+1}{r} - \frac{\alpha_R\gamma\nu}{2r}\right)\chi_2^l & -f_e f\psi_1^l & = 0 \\ \omega\psi_2^l & - \left(\frac{d}{dr} - \frac{l+1}{r} + \frac{\alpha_L\gamma\nu}{2r}\right)\psi_1^l & -f_e f\chi_2^l & = 0 \\ \omega\psi_1^l & + \left(\frac{d}{dr} + \frac{l+2}{r} - \frac{\alpha_L\gamma\nu}{2r}\right)\psi_2^l & -f_e f\chi_1^l & = 0. \end{array} \tag{14}$$

(a) Internal Solution ($r < R$)

In this region, $f = v = 0$, so the equations of motion (14) reduce to

$$\begin{array}{lll} \omega\chi_2^l & + \left(\frac{d}{dr} - \frac{l}{r}\right)\chi_1^l & = 0 \\ \omega\chi_1^l & - \left(\frac{d}{dr} + \frac{l+1}{r}\right)\chi_2^l & = 0 \\ \omega\psi_2^l & - \left(\frac{d}{dr} - \frac{l+1}{r}\right)\psi_1^l & = 0 \\ \omega\psi_1^l & + \left(\frac{d}{dr} + \frac{l+2}{r}\right)\psi_2^l & = 0. \end{array} \tag{15}$$

Thus, ψ and χ are decoupled from each other in the string core. Combining the first two equations and setting $z = \omega r$, we obtain

$$\frac{1}{z}\frac{d}{dz}\left(z\frac{d}{dz}\right)\chi_1^l + \left(\frac{z^2 - l^2}{z^2}\right)\chi_1^l = 0. \tag{16}$$

This is none other than Bessel's equation of order l. By regularity at the origin, the solution is

$$\chi_1^l = c_l J_l(\omega r). \tag{17}$$

This together with the second equation implies

$$\chi_2^l = c_l J_{l+1}(\omega r). \tag{18}$$

By a similar argument, ψ_1^l and ψ_2^l satisfy Bessel's equations of order $l+1$ and $l+2$ respectively and the internal solution is

$$\begin{pmatrix} \chi \\ \psi \end{pmatrix} = \sum_{l=-\infty}^{\infty} \begin{pmatrix} c_l J_l(\omega r) \\ i c_l J_{l+1}(\omega r) e^{i\theta} \\ d_l J_{l+1}(\omega r) e^{i\theta} \\ i d_l J_{l+2}(\omega r) e^{2i\theta} \end{pmatrix} e^{il\theta}. \tag{19}$$

(b) External Solution $(r > R)$

Outside the string core, we decompose our wave functions into eigenfunctions of the helicity operator. i.e.,

$$\begin{aligned} (\sigma \cdot \pi_R)\chi &= -i\sigma^j D_j^R \chi &= \pm k\chi \\ (\sigma \cdot \pi_L)\psi &= -i\sigma^j D_j^L \psi &= \pm k\psi. \end{aligned} \tag{20}$$

From eqn.(14), this gives

$$\begin{aligned} (\omega \mp k)\chi &= f_e f \psi = m\psi \\ (\omega \pm k)\psi &= f_e f \chi = m\chi. \end{aligned} \tag{21}$$

Defining

$$\nu = l - \alpha_R \tag{22}$$

and $z' = kr$, eqn.(20) yields

$$\chi_2^l = \mp \left(\frac{d}{dz'} - \frac{\nu}{z'} \right) \chi_1^l, \tag{23}$$

where $-$ $(+)$ is taken for a positive (negative) helicity state. Thus, the external solution is

$$\begin{pmatrix} \chi \\ \psi \end{pmatrix} = \sum_{l=-\infty}^{\infty} \begin{pmatrix} Z_\nu(kr) \\ \pm i Z_{\nu+1}(kr) e^{i\theta} \\ B^\pm Z_\nu(kr) e^{i\theta} \\ \pm i B^\pm Z_{\nu+1}(kr) e^{2i\theta} \end{pmatrix} e^{il\theta}. \tag{24}$$

In the above, $B^\pm = \frac{m}{\omega \pm k}$, the superscript $\pm$ in B denotes the helicity and $\pm$ in the front of the second and fourth components take the same sign as the helicity for

$Z_\nu = J_\nu$, N_ν and H_ν and opposite sign for $Z_\nu = J_{-\nu}$, $N_{-\nu}$ and $H_{-\nu}$. Here N_ν and H_ν are Neumann and outgoing Hankel functions respectively. Note that it is kr rather than ωr which appears in the arguments of our functions because electrons are massive outside the core. Another point to note is that whereas the second and third components of the internal solutions satisfy Bessel's equation of the same order, the corresponding components of the external solutions satisfy Bessel's equations of orders $\nu+1$ and ν respectively. This relative shift in the order is due to the asymmetry between left and right, i.e., $\alpha_L = \alpha_R + 1$.

(c) Asymptotic Solution

Consider performing a scattering experiment with an incoming plane wave of positive helicity electrons. Since helicity is violated in the core, the scattered wave consists of both positive and negative helicity components. We find that, as $r \to \infty$, the external solution takes the form

$$\sum_{l=-\infty}^{\infty} e^{il\theta}\left[\begin{pmatrix} (-i)^l J_l \\ i(-i)^l J_{l+1} e^{i\theta} \\ B^+(-i)^l J_l e^{i\theta} \\ iB^+(-i)^l J_{l+1} e^{2i\theta} \end{pmatrix} + \frac{f_l e^{ikr}}{\sqrt{r}}\begin{pmatrix} 1 \\ e^{i\theta} \\ B^+ e^{i\theta} \\ B^+ e^{2i\theta} \end{pmatrix} + \frac{g_l e^{ikr}}{\sqrt{r}}\begin{pmatrix} 1 \\ -e^{i\theta} \\ B^- e^{i\theta} \\ -B^- e^{2i\theta} \end{pmatrix}\right]. \tag{25}$$

It is easy to check that the second and third terms are the positive and negative helicity components of the scattered waves respectively.

We divide the problem of matching the asymptotic wave functions into two cases.

(i) For $\nu \geq 0$ or $\nu \leq -1$, we take $Z_\nu^1 = J_\nu$ and $Z_\nu^2 = N_\nu$. The external wave function is therefore

$$\begin{pmatrix} (a_l J_\nu & + & b_l N_\nu & + & A_l J_\nu & + & B_l N_\nu) e^{il\theta} \\ i(a_l J_{\nu+1} & + & b_l N_{\nu+1} & - & A_l J_{\nu+1} & - & B_l N_{\nu+1}) e^{i(l+1)\theta} \\ (a_l B^+ J_\nu & + & b_l B^+ N_\nu & + & A_l B^- J_\nu & + & B_l B^- N_\nu) e^{i(l+1)\theta} \\ i(a_l B^+ J_{\nu+1} & + & b_l B^+ N_{\nu+1} & - & A_l B^- J_{\nu+1} & - & B_l B^- N_{\nu+1}) e^{i(l+2)\theta} \end{pmatrix} \tag{26}$$

Making use of the asymptotic large x forms

$$J_\mu(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\mu\pi}{2} - \frac{\pi}{4}\right)$$

$$N_\mu(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\mu\pi}{2} - \frac{\pi}{4}\right), \tag{27}$$

we match coefficients of $e^{il\theta}\frac{e^{\pm ikr}}{\sqrt{r}}$ in eqns.(25) and (26) to find

$$\begin{array}{rllllll} e^{i\nu\pi/2} & (a_l & +ib_l & +A_l & +iB_l) & = 1 \\ e^{-i\nu\pi/2} & (a_l & -ib_l & +A_l & -iB_l) & = (-1)^l + (f_l + g_l)e^{i\pi/4}\sqrt{2\pi k} \\ e^{i(\nu+1)\pi/2} & (ia_l & -b_l & -iA_l & +B_l) & = -1 \\ e^{-i(\nu+1)\pi/2} & (ia_l & +b_l & -iA_l & -B_l) & = (-1)^l + (f_l - g_l)e^{i\pi/4}\sqrt{2\pi k}, \end{array} \tag{28}$$

from which we deduce

$$\begin{aligned} A_l &= -iB_l \\ a_l &= -ib_l + e^{-i\nu\pi/2} \\ g_l &= e^{-i(\pi/4+\nu\pi/2)}\sqrt{\frac{1}{2\pi k}}(-2iB_l). \end{aligned} \tag{29}$$

(ii) For $-1 < \nu < 0$, taking $Z_\nu^1 = J_\nu$ and $Z_\nu^2 = J_{-\nu}$, the external wave function is

$$\begin{pmatrix} (a_l J_\nu & + & b_l J_{-\nu} & + & A_l J_\nu & + & B_l J_{-\nu})e^{il\theta} \\ i(a_l J_{\nu+1} & - & b_l J_{-\nu-1} & - & A_l J_{\nu+1} & + & B_l J_{-\nu-1})e^{i(l+1)\theta} \\ (a_l B^+ J_\nu & + & b_l B^+ J_{-\nu} & + & A_l B^- J_\nu & + & B_l B^- J_{-\nu})e^{i(l+1)\theta} \\ i(a_l B^+ J_{\nu+1} & - & b_l B^+ J_{-\nu-1} & - & A_l B^- J_{\nu+1} & + & B_l B^- J_{-\nu-1})e^{i(l+2)\theta} \end{pmatrix} \tag{30}$$

We proceed as before and find

$$\begin{aligned} A_l &= -e^{-i\nu\pi}B_l \\ a_l &= -e^{-i\nu\pi}b_l + e^{-i\nu\pi/2} \\ g_l &= \sqrt{\frac{1}{2\pi k}}e^{-i(\nu\pi/2+\pi/4)}2i\sin(\nu\pi)B_l. \end{aligned} \tag{31}$$

Note that

$$\left.\frac{d\sigma}{d\theta}\right|_{+\to-} = \sum_l |g_l|^2. \tag{32}$$

(d) Matching at $r = R$

We have obtained the solutions inside and outside the core in (a) and (b). Now we match them at $r = R$. Because of the difference in the masses in the two regions and the discontinuous distribution of the string flux, there is a discontinuity in the first derivatives of the wave functions. Nevertheless, the wave functions themselves are continuous at $r = R$. This is the matching condition that we will use.[5,6] Once again there are two cases.

(i) $\nu \geq 0$ or $\nu \leq -1$: Substituting (29) into (26) and matching it with the internal solution in eqn.(19), we obtain

$$\begin{array}{ccccccc} -iH_\nu b_l & - & iH_\nu B_l & = & J_l c_l & - & e^{-i\nu\pi/2} J_\nu \\ -iH_{\nu+1} b_l & + & iH_{\nu+1} B_l & = & J_{l+1} c_l & - & e^{-i\nu\pi/2} J_{\nu+1} \\ -iB^+ H_\nu b_l & - & iB^- H_\nu B_l & = & J_{l+1} d_l & - & e^{-i\nu\pi/2} B^+ J_\nu \\ -iB^+ H_{\nu+1} b_l & + & iB^- H_{\nu+1} B_l & = & J_{l+2} d_l & - & e^{-i\nu\pi/2} B^+ J_{\nu+1}. \end{array} \tag{33}$$

In deriving the above equations, we have used the definition of outgoing Hankel function: $H_\mu = J_\mu + iN_\mu$. Solving eqn.(33), we find

$$B_l = \frac{\Delta_B}{\Delta} \tag{34}$$

where

$$\Delta_B = B^+ e^{-i\nu\pi/2} (\frac{2}{\pi k R})(J_{l+1}^2 - J_l J_{l+2}) \tag{35}$$

and

$$\Delta = (B^- - B^+) J_{l+1} (J_l H_{\nu+1}^2 - J_{l+2} H_\nu^2) - (B^- + B^+)(J_{l+1}^2 - J_l J_{l+2}) H_\nu H_{\nu+1} \tag{36}$$

where H_ν is outgoing Hankel functions. Use has been made of the Wronskian formula $J_{\nu+1}(x) N_\nu(x) - J_\nu(x) N_{\nu+1}(x) = \frac{2}{\pi x}$ in the derivation of eqn.(35).

Now we consider the regimé $\omega \sim k \sim m$ and $kR \ll 1$ and perform small kR approximation:

$$J_\mu \sim O([kR]^\mu), \qquad H_\mu, N_\mu \sim O([kR]^{-|\mu|}). \tag{37}$$

It is straightforward, but tedious to show that

$$B_l \propto \begin{cases} (kR)^{2\nu+2} & \nu \geq 0, l \geq 0 \\ (kR)^{2\nu} & \nu \geq 0, l < 0 \\ (kR)^{-2\nu-2} & \nu \leq -1, l \geq -1 \\ (kR)^{-2\nu} & \nu \leq -1, l < -1. \end{cases} \tag{38}$$

Note that the cross section may still be logarithmically suppressed when the exponent in the suppression factor appears to be zero.

(ii) $-1 < \nu < 0$

Substituting eqn.(31) into (30) and matching it with the internal solution in eqn.(19), we obtain the following equations.

$$\begin{array}{lcccll} P(& b_l & + & B_l &) = J_l c_l & - \; e^{-i\nu\pi/2} J_\nu \\ Q(& b_l & - & B_l &) = J_{l+1} c_l & - \; e^{-i\nu\pi/2} J_{\nu+1} \\ P(& B^+ b_l & + & B^- B_l &) = J_{l+1} d_l & - \; e^{-i\nu\pi/2} B^+ J_\nu \\ Q(& B^+ b_l & - & B^- B_l &) = J_{l+2} d_l & - \; e^{-i\nu\pi/2} B^+ J_{\nu+1} \end{array} , \tag{39}$$

where P denotes $-e^{-i\nu\pi} J_\nu + J_{-\nu}$ and Q denotes $-e^{-i\nu\pi} J_{\nu+1} - J_{-\nu-1}$. Solving (39), we find

$$B_l = \frac{\Delta'_B}{\Delta'} \tag{40}$$

where

$$\Delta'_B = \frac{2B^+ \sin(\nu\pi) e^{-i\nu\pi/2}}{\pi k R} (J_l J_{l+2} - J_{l+1}^2) \tag{41}$$

and

$$\Delta' = (B^- - B^+) J_{l+1} (J_{l+2} P^2 - J_l Q^2) + (B^- + B^+)(J_{l+1}^2 - J_l J_{l+2}) PQ. \tag{42}$$

Use has also been made of the Wronskian formula $J_\nu J_{-\nu-1} + J_{-\nu} J_{\nu+1} = -\frac{2\sin(\nu\pi)}{\pi x}$ in deriving eqn.(41).

We consider the regimé $\omega \sim k \sim m$ and $kR \ll 1$. A straightforward calculation shows that

$$B_l \propto \begin{cases} (kR)^{2\nu+2} & l \geq 0 \\ (kR)^{-2\nu} & l \leq -2 \\ 1 & l = -1 \; . \end{cases} \tag{43}$$

We see immediately that when the last case occurs, the mode $l = -1$ $(-1 < \nu < 0)$ dominates the contribution from all other modes, and is of order 1. In that case, the helicity flip process is maximally enhanced with a cross section of order $1/m$, where m is the mass of the incoming particle. Recalling that $\nu = l - \alpha_R$, we see that this occurs precisely when $-1 < \alpha_R < 0$ (and thus $0 < \alpha_L < 1$.) For electrons $\alpha_R = -2\sin^2\theta_w$ and the condition reduces to $0 < \sin^2\theta_w < 1/2$.

Let us consider the changes in the helicity flip cross section as $\sin^2\theta_w$ increases from 0 to 1 in the regimé $\omega \sim k \sim m$ and $kR \ll 1$.

(1) $\sin^2\theta_w = 0$ $(\alpha_R = 0)$

(2) $0 < \sin^2\theta_w < \frac{1}{2}$ $(-1 < \alpha_R < 0)$

(3) $\sin^2\theta_w = \frac{1}{2}$ $(\alpha_R = -1)$

(4) $\frac{1}{2} < \sin^2\theta_w < \frac{3}{4}$ $(-1.5 < \alpha_R < -1)$

(5) $\sin^2\theta_w = \frac{3}{4}$ $(\alpha_R = -1.5)$

(6) $\frac{3}{4} < \sin^2\theta_w < 1$ $(-2 < \alpha_R < -1.5)$

(7) $\sin^2\theta_w = 1$ $(\alpha_R = -2)$

Before embarking on a discussion about the various cases for electrons, we would like to remark that the results for down quarks are similar. It is still true that $\alpha_L = \alpha_R + 1$. The only difference is that $\alpha_R = -\frac{2}{3}\sin^2\theta_w$ for d quarks. Therefore, there are just two cases. If $\sin^2\theta_w = 0$, the result is the same as in case (1) for electrons and the helicity flip scattering has, up to normalization, an Everett's cross section. (Cf. case (1) below.) If $0 < \sin^2\theta_w \leq 1$, there is a maximal enhancement and the cross section per unit length $\sim 1/m_d$. (Cf. case (2) below.) Now we turn to electrons.

(1) For $\sin^2\theta_w = 0$ $(\alpha_R = 0)$, the $l = \nu = -1$ mode dominates and from eqns. (29), (32) and (34)-(36), the differential cross section per unit length

$$\frac{d\sigma}{d\theta} \sim \frac{1}{k\ln^2(kR)}, \tag{44}$$

which is, up to normalization, the cross section obtained by Everett[19] for the scattering of scalar particles off cosmic strings with integral magnetic fluxes.

(2) For $0 < \sin^2\theta_w < \frac{1}{2}$ $(-1 < \alpha_R < 0)$, from eqns. (31), (32) and (43), the electron helicity flip process has a differential cross section

$$\frac{d\sigma}{d\theta} = O(m_e^{-1}) \tag{45}$$

which is dominated by the mode $-1 < \nu < 0$ $(l = -1)$ and is thus independent of angle. In this case, the helicity flip process remains unsuppressed as $R \to \infty$ with

k held fixed. Note that this maximal amplification occurs for a continuous range of values of the parameter $\sin^2\theta_w$. This is in contrast with an analogous calculation on baryon number violating processes due to cosmic strings which exhibit unsuppressed cross section for only discrete values of fluxes.[5,6]

(3) For $\sin^2\theta_w = \frac{1}{2}$ ($\alpha_R = -1$), the $l = -1, \nu = 0$ mode swamps contributions from all other modes. Eqns. (29), (32) and (34)-(36) together implies that the cross section is of the same order as in case (1).

(4) For $\frac{1}{2} < \sin^2\theta_w < \frac{3}{4}$ ($-1.5 < \alpha_R < -1$), the dominant mode is $0 < \nu < 0.5$ ($l = -1$). From eqns. (29), (32) and (38), the differential cross section is given by

$$\frac{d\sigma}{d\theta} \sim k^{-1}(kR)^{4\nu} = k^{-1}(kR)^{4(2\sin^2\theta_w - 1)}, \tag{46}$$

(5) For $\sin^2\theta_w = \frac{3}{4}$ ($\alpha_R = -1.5$), the two modes $l = -1$ and -2 give comparable contributions and we obtain from eqns. (29), (31), (32), (38) and (43) that

$$\frac{d\sigma}{d\theta} \sim \frac{1}{k(kR)^2}|1 + Ce^{i\theta}|^2. \tag{47}$$

(6) For $\frac{3}{4} < \sin^2\theta_w < 1$ ($-2 < \alpha_R < -1.5$), we need to consider the contribution from the $l = -2$ mode only and obtain from eqns. (31), (32) and (43) that

$$\frac{d\sigma}{d\theta} \sim k^{-1}(kR)^{8(1-\sin^2\theta_w)}, \tag{48}$$

(7) For $\sin^2\theta_w = 1$ ($\alpha_R = -2$), the $l = -2, \nu = 0$ mode will dominate and the differential cross section can be deduced from eqns. (29), (32) and (34)-(36):

$$\frac{d\sigma}{d\theta} \sim \frac{1}{k\ln^4(kR)}. \tag{49}$$

Note that the exponent of the logarithmic term is four, whereas in cases (1) and (3) it is two. We note on passing that case (7) corresponds to semi-local strings, where the $SU(2)$ gauge charge is set to zero.

The most prominent feature of our result is the presence of a plateau: For $\sin^2\theta_w$ between 0 and 1/2, we have maximal enhancement. Is there any heuristic way of understanding its origin? In Ref. 6, the cross section for a baryon number violating process was derived using first order perturbation theory in quantum field theories. The transition matrix element between an initial state $|\psi\rangle$ and a final state $|\psi'\rangle$ is given by $\mathcal{A} = \langle\psi'|\int d^4x\mathcal{L}_I(x)\,|\psi\rangle$. The computation is divided into two steps. Firstly, we evaluate $\mathcal{A}$ using free fermion spinors, resulting in the "geometric" cross section. In the second step, we solve the Dirac equation with the appropriate boundary conditions to determine the amplitude of the spinor at the core radius R and define the amplification factor A as the ratio of the amplitude of the actual spinor to that of a free spinor. Since the cross section is proportional to $\mathcal{A}^2$ and since $\mathcal{A}$ involves two spinors, the catalysis cross section is enhanced by a factor A^4 over the geometric cross section. Now we attempt a similar discussion for helicity flip due to electroweak strings. The difficulty of such an approach lies in the decomposition of the Hamiltonian into helicity conserving and helicity violating parts. The point is that the Yukawa coupling between the Higgs field and electrons, apart from giving rise to helicity violation, also makes electrons massive and it seems difficult to separate these two effects. The simplest way out is to consider another object instead, namely the commutator of the helicity operator with the Hamiltonian. This object is clearly proportional to the transition matrix element that we are interested in.⋆ From eqn.(11) we see that this commutator couples the first component of a spinor with the fourth and the second with the third, etc. In the same spirit as in Ref.6, we compute the coefficients a_l, b_l, A_l and B_l of our wave function. For the region $0 < \sin^2\theta_w < 1/2$, we find that the mode $-1 < \nu < 0$ have all coefficients of order unity. It is a simple matter to check that, at the core radius, the first and third components of the initial (+ helicity) wave function are enhanced by a factor $(kR)^\nu$ and the second and fourth by $(kR)^{-\nu-1}$. A similar analysis holds for the final (− helicity) state. Now we have the interesting result that all components are amplified by factors very sensitive to the fractional flux of the string, but the first and third components have a different amplification factor from that of the second and fourth such that when we take the product of the amplification factors, we get an enhancement factor of $(kR)^{-1}$, which is independent of α_R. This is the origin of the plateau.

⋆ We thank Ming Lu and Piljin Yi for helpful discussions about this.

7.4 Concluding Remarks

(1) We work in the regimé $\omega \sim k \sim m_e$ and $kR \ll 1$. Using a partial wave decomposition, we show that for $0 < \sin^2\theta_w < 1/2$, electrons scattering off an electroweak string have a helicity flip cross section (per unit length) of order m_e^{-1}. This huge cross section is due to an amplification of the fermionic wave function at the core. Within this region of the parameter space, it is found that one partial wave (the mode $-1 < \nu < 0$) dominates the contributions from all other modes, giving an angle independent differential cross section (per unit length) $\propto m_e^{-1}$.

(2) Whereas baryon number violating processes are maximally enhanced only for discrete values of the fractional flux, our results show that electroweak strings have maximal amplified helicity flip scattering amplitude for a continuous region of the parameter space $0 < \sin^2\theta_w < 1/2$. This is due to the asymmetry between left and right and a subsequent delicate cancellation of the dependence of the overall amplification factor on the Weinberg angle: We consider the commutator between the helicity operator and the Hamiltonian. This commutator gives a coupling between the first and fourth components as well as between the second and third components of the spinor. By computing the coefficients of the wave function, one observes that the first and third components are amplified at a factor which is very sensitive to the string flux and is different from the amplification factor for the second and fourth components. However, when we take the product of these two amplification factors to obtain the total amplification factor, we find it to be independent of the string flux, thus resulting in maximal enhancement for a continuous region of the parameter space.

(3) The case $0 \le \sin^2\theta_w \le 1/2$ has also been discussed in a revised version of Ref. 14. Can one rederive their results from our discussion? The answer is affirmative. For $0 < \sin^2\theta_w < 1/2$ and $\omega R,\ kR \ll 1$, one can deduce from eqns. (31), (32) and (40)-(42) that

$$\frac{d\sigma}{d\theta} \approx \frac{2}{\pi k}\left(\frac{m^2}{\omega(\omega+k)}\right)^2 \sin^2\pi\alpha_R. \tag{50}$$

(Here we have reinstated the mild $\sin^2\pi\alpha_R$ dependence that we have ignored in section 3.) In the limit $k \ll m$, this gives

$$\frac{d\sigma}{d\theta} \approx \frac{2}{\pi k}\sin^2\pi\alpha_R. \tag{51}$$

In the opposite limit $k \gg m$,

$$\frac{d\sigma}{d\theta} \approx \frac{1}{2\pi k} \left(\frac{m}{k}\right)^4 \sin^2 \pi\alpha_R. \tag{52}$$

We note that the vanishing of the differential cross section in the massless limit can be deduced directly from eqn.(11). Similarly, for $\sin^2\theta_w = 0$ or $1/2$, one deduces from eqns.(29), (32) and (34)-(36) that

$$\frac{d\sigma}{d\theta} \approx \frac{\pi}{8k} \left(\frac{m^2}{\omega(\omega+k)}\right)^2 \frac{1}{\ln^2 kR} . \tag{53}$$

For $k \ll m$, this gives

$$\frac{d\sigma}{d\theta} \approx \frac{\pi}{8k} \frac{1}{\ln^2 kR} . \tag{54}$$

For $k \gg m$, this gives

$$\frac{d\sigma}{d\theta} \approx \frac{\pi}{32k} \left(\frac{m}{k}\right)^4 \frac{1}{\ln^2 kR} . \tag{55}$$

These results are in good agreement with those of Ref. 14. (The authors gave zero as their final answer to the case $k \gg m$, but it is clear from their arguments that they had neglected $(k/m)^4$ terms.)

(4) The cross section for other values of θ_w are also computed. In particular, a semi-local string is none other than an electroweak string with $\sin^2\theta_w = 1$.

(5) For axion strings, the covariant derivatives in eqn.(11) should be replaced by partial derivatives. Therefore, helicity is violated outside the core and it makes no sense to talk about helicity conserving and helicity flip cross sections in this context.

(6) It would be interesting to investigate the relevance of this work to electroweak baryogenesis. Such scenarios are highly testible.

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Summary

Let us summarize our main results. Non-Abelian vortices have exotic properties. They obey braid statistics, rather than the conventional fermion and boson statistics. The exclusive cross section for vortex-vortex scattering is typically a multivalued function of the scattering angle. There can be an exchange contribution to the vortex-vortex scattering amplitude that adds coherently with the direct amplitude, even if the two vortices have distinct initial quantum numbers. Furthermore, a vortex-antivortex pair is capable of carrying charges without an apparent source. This unlocalizable charge has been dubbed "Cheshire charge." The mechanism for its existence and topological transfer was discussed in Chapter 3. The measurements of charges and fluxes in a wormhole background are also considered. We show how the charges carried by a wormhole mouth appear to be conserved when a charged particle traverses a wormhole from the perspective of an observer who is unable to resolve the small size of the wormhole mouth. We also work on non-Abelian Chern-Simons particles and electroweak strings.

All these sound novel, but how do they relate to phenomenology? We would like to argue that those exotic properties described above are natural consequences of non-Abelian gauge theories and it is conceivable that many of them do occur in nature. For instance, our discussion on the topological transfer of magnetic charges between a string loop and a monopole applies to nematic liquid crystals. Particles obeying non-Abelian statistics ("nonabelions") have been proposed independently by condensed matter theorists. An important question, then, is how would such objects be recognized in laboratory experiments? Much remains to be explored in the many-body physics of nonabelions.

Another interesting recent development of the subject is the global analog of the Aharonov-Bohm effect. This can have far-reaching phenomenological implications. So far such an analogous effect has been carefully discussed only for the case of a boson moving in a background of a global Z_2 string and is not without controversy. It remains to be generalized to other patterns of symmetry breaking and to fermions. These are highly non-trivial problems: generalization to other symmetry groups forces one to deal with nonlinear differential equations whereas the equations of motion for fermions contain commutator terms. Also, Patrick McGraw has recently investigated the global analog of Cheshire charge. One potential problem in such a formulation which remains to be addressed in the future is the radiation of charged Goldstone bosons. A global vortex-antivortex pair can, in principle, radiate away its charges by the emission of charged Goldstone bosons. To make sense of Cheshire charge, one must ensure that the radiation process occurs at an insignificant rate in the time scale that one is considering. What exactly this requirement amounts to is something to be worked out.

In our opinion, the greatest promise lies in the detailed study of the phenomenology of non-Abelian cosmic and electroweak strings. Owing to their topological interactions, they are expected to evolve and interact with charged particles in a manner qualitatively different from the widely studied $U(1)$ strings. An important question to ask is whether they would be a good candidate for the formation of the large scale structures in the early universe. Scenarios of baryogenesis due to electroweak strings are also interesting because they are highly testable.

From a theoretical point of view, the many body theory of non-Abelian vortices and a fully second quantized theory of non-Abelian vortices are very exciting fundamental problems in field theory. Immense courage and deep insight would be needed for making progress in these areas.