INTEGRAL METHODS APPLIED TO A COMPRESSIBLE BOUNDARY LAYER ON AN INSULATED FLAT PLATE PARALLEL TO THE WIND

Thesis by

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TABLE OF SYMBOLS

c_p	specific heat at constant pressure
T	temperature
u	velocity component parallel to plate
v	velocity component normal to plate
w	resultant velocity
y	coordinate normal to plate
y_1	thickness of boundary layer
δ	thickness of laminar sublayer
ρ	density
μ	viscosity
ν	kinematic viscosity
τ	shearing stress
λ	coefficient of heat conduction
γ	ratio of specific heat at constant pressure to that at constant volume
M	Mach number
R	Reynolds number
η	y/y_1
D	drag per unit span
l	chord

Subscripts

o	denotes conditions in free stream
w	denotes conditions at the plate
δ	denotes condition at the edge of the laminar sublayer

SUMMARY

The effect of compressibility upon the boundary layer thickness and wall shearing stress on an insulated flat plate parallel to the wind is investigated. Integral methods are used to investigate both laminar and turbulent boundary layers.

In order to simplify the calculations, only the limiting cases of small Mach numbers and very large Mach numbers are investigated in the laminar case. In the turbulent case only small Mach numbers are considered due to the present lack of understanding of supersonic turbulence.

It is found that in the subsonic laminar boundary layer, the wall shear stress is approximately independent of Mach number while the boundary layer thickness increases with Mach number. For large Mach numbers in the laminar case it is found that the wall shear stress is proportional to $M_0^{-.24}$ and the boundary layer thickness to $M_0^{1.76}$.

In the turbulent case, where only small Mach numbers are investigated, both the shear stress and boundary layer thickness are found to decrease with Mach number.

INTRODUCTION

The problem of the laminar boundary layer in compressible flow has been thoroughly investigated. Karman and Tsien have solved the differential equations by successive approximations under the assumption that the Prandtl number is equal to one. They found that the velocity profile was approximately linear.

Emmons and Brainerd have extended the solution to include the effect of varying the Prandtl number. Eckert and Drewitz have made a similar solution to the problem.

All of the above solutions are obtained by actually solving the differential equations. This is possible only in the case of laminar flow, where there is a known relation between shearing stress and mean velocity. In turbulent flow, the corresponding relation is as yet unknown. Until such a relationship is discovered there can be no complete solution of the turbulent case. In order to obtain some information about the boundary layers in general, von Karman has devised an alternative approach. This approach is known as the integral method, and it is essentially a generalization of this method which is used here.

As originally set forth by von Karman, the integral method consisted of expressing the wall shearing stress in two ways. First as

$$\tau_w = \frac{d}{dx}\int_0^\infty \rho U(U_0 - U)\, dy$$

and second as $\tau_w = \mu \left(\frac{\partial U}{\partial y}\right)_w$.

The assumption was then made that the velocity profiles at various stations downstream are similar to one another.

If this is true, the velocity parallel to the plate, U, may be written as $U = U_0 f(\eta)$ where $\eta = y/y_1$. This allows τ_w to be expressed as

$$\tau_w = \rho U_0^2 \frac{dy_1}{dx} \int_0^1 f(1-f)\, d\eta = \rho U_0^2 \frac{dy_1}{dx} A$$

or

$$\tau_w = \mu \frac{U_0}{y_1} \left(\frac{df}{d\eta}\right)_{\eta=0} = \mu \frac{U_0}{y_1} B$$

Both A and B are dimensionless constants, so that combination of the two expressions for τ_w leads to equations for y_1 and τ_w in terms of these constants. Thus the general nature of the variation of y_1 , and τ_w with the distance downstream can be found without knowledge of the velocity profile or of the relation between shear stress and velocity. Further, if some empirical expression for the velocity profile is assumed, the constants A and B may be computed.

In the case of an incompressible turbulent boundary layer, a function $f = \eta^{\frac{1}{7}}$ represents the velocity profile over the greater part of the boundary layer with reasonable accuracy. Close to the wall, however, the flow will be laminar, and the velocity profile becomes approximately linear. Further, such a function gives an infinite slope to the velocity profile at the wall and consequently the wall shear stress calculated from it will be infinite. It appears, then, that an alternative method for calculating the shear at the wall is necessary. Liepmann has suggested that it may be calculated by assuming the turbulence level outside the sublayer to be proportional to the velocity at the edge of the sublayer. From this, together with the assumption that the velocity profile in the sublayer is linear, he deduces that the Reynolds number of transition from laminar to turbulent flow based on the sublayer thickness is independent of the distance downstream. This gives him enough information to compute τ_w. He also computes τ_w from

$$\tau_w = \rho U_0^2 \frac{dy_1}{dx} \int_0^1 f(1-f)\, d\eta$$

just as von Karman does. This result will be reasonably accurate even though the expression $f = \eta^{\frac{1}{7}}$ is incorrect for small η's. He has applied this method to a turbulent boundary layer in incompressible flow, although his work has not yet been published.

Loitsianskii has proposed a somewhat more elegant integral method and has applied it to the laminar boundary layer. He starts by integrating the differential equations of motion term by term. He points out that an infinite set of such integral equations may be obtained by multiplying the differential equation by various powers of U before integrating. Solving this infinite set of integral equations is equivalent to solving the differential equation of motion. In practice, however, he uses only one of the equations together with the idea of similarity to obtain expressions for y_1 and τ_w. He applies the method to the laminar boundary layer only, but states that it can be extended to cover the turbulent case as well.

Neither von Karman nor Loitsianskii have applied their methods to compressible flow. The purpose of this paper is to investigate both laminar and turbulent boundary layer on an insulated flat plate parallel to the wind in compressible flow. In the first part the laminar boundary layer will be investigated by a method somewhat similar to Loitsianskii's. In the compressible case, however, there is an equation of thermal energy to be integrated, as well as the mechanical equation of motion. It is found that these equations, together with the equation of continuity, can be combined, after integration, into one integral equation. The second part of this paper deals with the turbulent boundary layer in compressible flow, and in this part Liepmann and Laufer's method is applied. The solution is of such a nature that the laminar boundary layer is given as a special case of the more general solution.

Throughout this paper it is assumed that the Prandtl number is equal to one. This is equivalent to assuming that widths of the thermal and kinematic boundary layers are equal. An examination of the results of Emmons and Brainerd indicate that the error involved here is small.

PART I

LAMINAR BOUNDARY LAYER IN COMPRESSIBLE FLOW

After the usual boundary layer approximations are made, the equations governing compressible laminar boundary layer flow become:

$$\frac{\partial(\rho U)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \quad (1) \text{ Continuity}$$

$$\rho U \frac{\partial U}{\partial x} + \rho v \frac{\partial U}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial U}{\partial y}\right) \quad (2) \text{ Momentum}$$

$$\rho U \frac{\partial}{\partial x}(c_p T) + \rho v \frac{\partial}{\partial y}(c_p T) = \frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right) + \mu\left(\frac{\partial U}{\partial y}\right)^2 \quad (3) \text{ Thermal Energy}$$

Since the pressure is approximately constant,

$$\rho = \rho_0 \frac{T_0}{T} \quad (4)$$

With regard to the variation of c_p, μ, and λ, the following assumptions will be made throughout this paper:

1. c_p is constant.
2. λ and μ are functions of temperature only, and are taken to be the same function of temperature in order to make the Prandtl number, $\frac{c_p \mu}{\lambda}$, a constant.
3. In accordance with experimental results, the following relations are used:

   $$\mu = \mu_0 \left(\frac{T}{T_0}\right)^n \qquad \lambda = \lambda_0 \left(\frac{T}{T_0}\right)^n \quad (5)$$

   where n is approximately .76.

Only the case in which no heat is transferred to or from the wall will be considered. For this case, the energy content per unit mass, $\frac{w^2}{2} + c_p T$, is constant throughout the flow field. Consequently:

$$\frac{T_w}{T_0} = 1 + \frac{\gamma - 1}{2} M_0^2 \quad (6)$$

Integrating the continuity equation (1) gives:

$$\int_0^{y_1} \frac{\partial(\rho U)}{\partial x}\, dy + \rho v \Big]_0^{y_1} = 0$$

By Leibniz's rule this becomes:

$$\frac{\partial}{\partial x}\int_0^{y_1} \rho U\, dy + \rho_0 U_0 \frac{\partial y_1}{\partial x} + \rho_0 v_0 = 0$$

Expressing the density in terms of the temperature by (4) leads

to:

$$\frac{\partial}{\partial x}\int_0^{y_1} \frac{U}{T}\, dy + \frac{U_0}{T_0}\frac{\partial y_1}{\partial x} + \frac{v_0}{T_0} = 0 \qquad (7)$$

The momentum equation (2) may be written as

$$\frac{\partial(\rho U^2)}{\partial x} - U\frac{\partial(\rho U)}{\partial x} + \frac{\partial(\rho v U)}{\partial y} - U\frac{\partial(\rho v)}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial U}{\partial y}\right)$$

By continuity the 2nd and 4th terms cancel. Integrating and

applying Leibniz rule gives

$$\frac{\partial}{\partial x}\int_0^{y_1} \rho U^2 dy + \rho_0 v_0 U_0 + \mu_w \left(\frac{\partial U}{\partial y}\right)_w - \rho_0 U_0^2 \frac{\partial y_1}{\partial x} = 0$$

Finally substituting expressions (4), (5), and (6) gives

$$\frac{\partial}{\partial x}\int_0^{y_1} \frac{U^2}{T} dy + \frac{v_0 U_0}{T_0} + \frac{\nu_0}{T_0}\left(1 + \frac{\gamma-1}{2}M_0^2\right)^n \left(\frac{\partial U}{\partial y}\right)_w - \frac{U_0^2}{T_0}\frac{\partial y_1}{\partial x} = 0 \qquad (8)$$

The energy equation may be written as

$$\frac{\partial}{\partial x}(c_p T \rho U) - c_p T \frac{\partial}{\partial x}(\rho U) + \frac{\partial}{\partial y}(c_p T \rho v) + c_p T \frac{\partial}{\partial x}(\rho U) = \frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right) + \mu \left(\frac{\partial U}{\partial y}\right)^2$$

Again the 2nd and 4th terms cancel by continuity. Integrating

and using Leibniz's rule

$$\frac{\partial}{\partial x}\int_0^{y_1} c_p T \rho U\, dy + c_p T_0 \rho_0 U_0 \frac{\partial y_1}{\partial x} + c_p T_0 \rho_0 U_0 v_0 = \lambda \frac{\partial T}{\partial y}\Big]_0^{y_1} + \int_0^{y_1} \mu \left(\frac{\partial U}{\partial y}\right)^2 dy$$

The 1st term on the R.H.S. is zero in this case since

no heat is transferred to or from the wall.

Now if equations (4) and (5) are introduced, the energy equation

may be written as

$$\frac{\partial}{\partial x}\int_0^{y_1} U\, dy + U_0 \frac{\partial y_1}{\partial x} + v_0 = \frac{\nu_0}{c_p T_0}\int_0^{y_1} \left(\frac{T}{T_0}\right)^n \left(\frac{\partial U}{\partial y}\right)^2 dy \qquad (9)$$

The quantity v_0 may be eliminated from the energy and

momentum equations, (8) and (9) by use of the continuity equation

(7).

Combining continuity and momentum in this way gives

$$\frac{\partial}{\partial x}\int_0^{y_1}\frac{U^2}{T}dy - U_0\frac{\partial}{\partial x}\int_0^{y_1}\frac{U}{T}dy + \frac{\nu_0}{T_0}\left(1+\frac{\gamma-1}{2}M_0^2\right)^n\left(\frac{\partial U}{\partial y}\right)_w = 0 \qquad (10)$$

Likewise eliminating between continuity and energy gives:

$$\frac{\partial}{\partial x}\int_0^{y_1}U\,dy - T_0\frac{\partial}{\partial x}\int_0^{y_1}\frac{U}{T}dy = \frac{\nu_0}{c_pT_0}\int_0^{y_1}\left(\frac{T}{T_0}\right)^n\left(\frac{\partial U}{\partial y}\right)^2 dy \qquad (11)$$

Finally (10) and (11) can be combined by eliminating $\frac{\partial}{\partial x}\int_0^{y_1}\frac{U}{T}dy$ between them. This gives

$$\frac{\partial}{\partial x}\int_0^{y_1}U\,dy - \frac{T_0}{U_0}\frac{\partial}{\partial x}\int_0^{y_1}\frac{U^2}{T}dy = \frac{\nu_0}{U_0}\left(\frac{\partial U}{\partial y}\right)_w\left(1+\frac{\gamma-1}{2}M_0^2\right)^n + \frac{\nu_0}{c_pT_0}\int_0^{y_1}\left(\frac{T}{T_0}\right)^n\left(\frac{\partial U}{\partial y}\right)^2 dy \qquad (12)$$

At this point, the similarity conditions are introduced.

Since (12) involves only and as dependent variables, it is necessary to make only two similarity assumptions. They are:

$$U = U_0\,f(\eta) \qquad\qquad T = (T_w - T_0)\,h(\eta) + T_0$$

where $\eta = \frac{y}{y_1}$

The value of η for a given $\frac{U}{U_0}$ or $\frac{T}{T_0}$ is assumed to be a function of neither x nor M_0, while y_1, is a function of both.

By use of (6), the second similarity equation may be expressed as

$$\frac{T}{T_0} = 1 + \frac{\gamma-1}{2}M_0^2\,h(\eta)$$

When the similarity expressions are substituted in (12), there results:

$$\frac{U}{\nu_0}\frac{\partial y_1}{\partial x}\int_0^1 f\,d\eta + \frac{U_0}{\nu_0}\frac{\partial}{\partial x}\,y_1\int_0^1\frac{f^2}{1+\frac{\gamma-1}{2}M_0^2h}d\eta$$
$$= \frac{1}{y_1}\left[2\,\frac{(\gamma-1)}{2}M_0^2\int_0^1\left(1+\frac{\gamma-1}{2}M_0^2h\right)f'^2d\eta - \left(1+\frac{\gamma-1}{2}M_0^2\right)^n f'_w\right] \qquad (13)$$

Because of the awkward form of the 2nd and 3rd integrals, it has been decided to investigate only those cases in which M_0 is very large or very small.

1st case— M_0 large:

To investigate this case, (13) is rewritten as:

$$\frac{U_0}{V_0}\frac{\partial y_1}{\partial x}\int_0^1 f\,d\eta + \frac{U_0}{V_0}\frac{\partial}{\partial x}\,y_1\int_0^1 \frac{f^2}{1+\frac{\gamma-1}{2}M_0^2 h}\,d\eta = \frac{2}{y_1}\left(\frac{\gamma-1}{2}M_0^2\right)^{n+1}\int_0^1 h^n f'^2\,d\eta - \frac{1}{y_1}\left(\frac{\gamma-1}{2}M_0^2\right)^n f_w'$$

In modifying the R.H.S., the approximation was made that, for sufficiently large M_0,

$$\int_0^1\left(1+\frac{\gamma-1}{2}M_0^2 h\right)^n f'^2\,d\eta \longrightarrow \left(\frac{\gamma-1}{2}M_0^2\right)^n\int_0^1 h^n f'^2\,d\eta$$

This is obviously true if h does not approach zero. At the outer edge of the wake, however, $h \to 0$ and $\frac{\gamma-1}{2}M_0^2 h$ is no longer much greater than one. However, at the edge of the wake, f'^2 also approaches zero so that the contribution of the terms in which $h \to 0$ to the value of the entire integral is very small. Consequently, the above relation between entire integrals is justified.

The integral in the 2nd term presents a more serious problem. One would feel that it might approach $\frac{1}{\frac{\gamma-1}{2}M_0^2}\int_0^1 \frac{f^2}{h}\,d\eta$. However, this is not immediately obvious because when $h \to 0$, $\frac{\gamma-1}{2}M_0^2 h$ is not large compared to one, and since the numerator of the integrand approaches one as approaches zero, it is not obvious that the contribution to the integral is small. To investigate this, the integral is differentiated with respect to M_0. Denoting the integral by I is seen that:

$$\frac{dI}{dM_0} = \int_0^1 \frac{-2f^2\,\frac{\gamma-1}{2}M_0 h}{\left(1+\frac{\gamma-1}{2}M_0^2 h\right)^2}\,d\eta$$

As M_0 becomes large, it is apparent that this approaches

$$\frac{1}{\frac{\gamma-1}{2}M_0^3}\int_0^1 -\frac{2f^2}{h}\,d\eta$$

The case in which $h \to 0$ is not serious here because the presence of h in the numerator insures that the contribution to total integral under this condition is small. Reintegrating the approximate expression for the integral gives

$$I_{M_0} \longrightarrow \frac{1}{\frac{\gamma-1}{2} M_0^2} \int_0^1 \frac{f^2}{h} d\eta$$

as was suspected.

So, as M_0 becomes arbitrarily large, equation (13) assumes this form:

$$\frac{U_0}{\nu_0} \frac{\partial y_1}{\partial x}\left(1 + \frac{\alpha}{\frac{\gamma-1}{2} M_0^2}\right) \sim \frac{1}{y_1}\left(\frac{\gamma-1}{2} M_0^2\right)^{n+1}\left(1 + \frac{\beta}{\frac{\gamma-1}{2} M_0^2}\right)$$

where α and β are constants. Again if M_0 is large enough $\frac{\alpha}{\frac{\gamma-1}{2} M_0^2}$ and $\frac{\beta}{\frac{\gamma-1}{2} M_0^2}$ will be small compared to 1 and can be neglected. Consequently:

$$\frac{U_0}{\nu_0} \frac{\partial y_1}{\partial x} \sim \frac{1}{y_1}\left(\frac{\gamma-1}{2} M_0^2\right)^{n+1}$$

which on integration yields

$$y_1 \sim \left(\frac{\gamma-1}{2} M_0^2\right)^{\frac{n+1}{2}} \sqrt{\frac{\nu_0 x}{U_0}}$$

The constant integration is found to be zero, since when $x = 0$, $y = 0$ for all M_0's.

Finally, for large Mach numbers

$$y_1 \sim M_0^{n+1} \sqrt{\frac{\nu_0 x}{U_0}} \tag{14}$$

To compute the shear on the wall, equation (10) is solved for $\left(\frac{\partial u}{\partial y}\right)_w$.

When this is combined with (5) and the similarity conditions there is obtained

$$\tau_w = \mu_w \left(\frac{\partial u}{\partial y}\right)_w = \rho_0 U_0^2 \frac{\partial}{\partial x}\left(y_1 \int_0^1 \frac{f}{1 + \frac{\gamma-1}{2} M_0^2 h} d\eta - y_1 \int_0^1 \frac{f^2}{1 + \frac{\gamma-1}{2} M_0^2 h} d\eta\right) \tag{15}$$

For large Mach numbers this gives:

$$\frac{\tau_w}{\rho_0 U_0^2} \sim \frac{\partial y_1}{\partial x} \frac{1}{\frac{\gamma-1}{2} M_0^2} \tag{16}$$

Using equation (14) for y_1, τ_w may now be computed by differentiation. This gives

$$\frac{\tau_w}{\rho_0 U_0^2} \sim M_0^{n-1}\sqrt{\frac{\nu_0}{U_0 x}} \sim \frac{M_0^{n-1}}{R_x} \qquad (17)$$

Since n is approximately .76, the results for large Mach numbers become:

$$y_1 \sim M_0^{1.76}\sqrt{\frac{\nu_0 x}{U_0}}$$

$$\frac{\tau_w}{\rho_0 U_0^2} \sim M_0^{-.24}\sqrt{\frac{\nu_0}{U_0 x}}$$

To check the result for y_1, Karman and Tsien's results are replotted against $y/M_0^{1.76}\sqrt{\frac{\nu_0 x}{U_0}}$ in Fig. II. It can be seen that the curves for $M_0 = 8$ and 10 are substantially identical. The original Karman and Tsien results are shown in Figs. I and II.

2nd case—— M_0 small.

In this case, equation (13) will be modified by expanding $\left(1+\frac{\gamma-1}{2}M_0^2 h\right)^{-1}$ and $\left(1+\frac{\gamma-1}{2}M_0^2 h\right)^{n}$ and neglecting squares and higher powers of $\frac{\gamma-1}{2}M_0^2 h$. It should be remembered that h is never greater than one. To this approximation (13) may be written as

$$\frac{U_0}{\nu_0}\left[\frac{\partial y_1}{\partial x}\int_0^1 f\,d\eta + \frac{\partial}{\partial x}\left(y_1\int_0^1 f^2 d\eta - y_1\int_0^1 \frac{\gamma-1}{2}M_0^2 f^2 h\,d\eta\right)\right] = \frac{1}{y_1}\left[\frac{2(\gamma-1)}{2}M_0^2\int_0^1 f'^2 d\eta - f_w'\left(1+n\frac{(\gamma-1)}{2}M_0^2\right)\right]$$

From which, if integrals of functions of η are considered constants, there results:

$$\frac{U_0}{\nu_0}\frac{\partial y_1}{\partial x}\left(1+\Omega\frac{\gamma-1}{2}M_0^2\right) \sim \frac{1}{y_1}\left(1+\Omega'\frac{\gamma-1}{2}M_0^2\right)$$

or

$$\frac{U_0}{\nu_0}\frac{\partial y_1}{\partial x} \sim \frac{1}{y_1}\left(1+\omega'\frac{\gamma-1}{2}M_0^2\right)$$

Integrating

$$y_1 \sim \sqrt{\frac{\nu_0 x}{U_0}}\left(1+\omega\frac{\gamma-1}{2}M_0^2\right) \qquad (18)$$

To compute τ_w , equation (10) is solved for $\left(\frac{\partial u}{\partial y}\right)_w$.

Then τ_w is computed as $\mu_w\left(\frac{\partial u}{\partial y}\right)_w$. The result for small Mach numbers is:

$$\frac{\tau_w}{\rho_0 U_0^2} = \frac{\partial y_1}{\partial x}\int_0^1\left(1-\frac{\gamma-1}{2}M_0^2 h\right)\left(f-f^2\right)d\eta$$

From which

$$\frac{\tau_w}{\rho_0 U_0^2} \sim \frac{\partial y_1}{\partial x}\left(1-\epsilon'\frac{\gamma-1}{2}M_0^2\right)$$

Using the value of y_1 given in (18) and differentiating, the shear stress is found to be

$$\frac{\tau_w}{\rho_0 U_0^2} \sim \sqrt{\frac{\nu_0}{U_0 x}}\left(1+\epsilon\frac{\gamma-1}{2}M_0^2\right) \tag{19}$$

As yet the sign of ϵ is unknown, so that it is not known whether τ_w increases or decreases with Mach number.

PART II

In Part II an attempt will be made to find y_1 and τ_w as functions of x and M_0 in the case where the boundary layer is turbulent. In addition to this, it will be possible to arrive at approximate values of the unknown constants in (18) and (19).

Again it will be necessary to assume similarity with respect to x and M_0. Because of the presence of the laminar sublayer, complete similarity across a turbulent boundary layer does not exist. This difficulty will be handled in the manner suggested by Liepmann and Laufer. That is, when τ_w is to be calculated from the integral of momentum defect, similarity across the entire layer will be assumed, but when it is calculated from $\mu_w\left(\frac{\partial u}{\partial y}\right)_w$, a linear velocity profile in the laminar sublayer will be assumed. Their result that the Reynolds number of transition from laminar to turbulent flow is independent of distance downstream will be retained. In addition it will be assumed that this Reynolds number is independent of Mach number, for small Mach numbers at least.

Whether or not it is allowable to assume similarity with respect to Mach number in turbulent flow is unknown at present and can be determined only by experiment. However, the fact that similarity with respect to Mach number was found to exist for small Mach numbers in laminar flow, makes it seem possible that the same may be true in turbulent flow.

In addition to the assumption of similarity, a definite form for the variation of velocity and temperature distributions is assumed. Namely, velocity and temperature are expressed by:

$$\frac{U}{U_0} = \eta^r \qquad \frac{T}{T_0} = 1 + \frac{\gamma-1}{2} M_0^2 \left(1 - \eta^p\right) \tag{20}$$

The above assumption of similarity is equivalent to assuming that r and p are independent of x and M_0. Preston has found experimentally that r varies slowly with R_x, but this variation is neglected here. No information was found on the variation of r with M_0 or of p with either x or M_0. The calculation is restricted to subsonic Mach numbers in that $\left(\frac{\gamma-1}{2} M_0^2\right)^2$ and higher powers is neglected and also because the present lack of knowledge of supersonic turbulence makes any analysis such as this seem groundless.

The first step is to compute τ_w from the distance rate of change of the momentum defect.

$$\tau_w = \rho_0 U_0^2 \frac{\partial}{\partial x} \int_0^\infty \frac{\rho}{\rho_0} \frac{U}{U_0}\left(1 - \frac{U}{U_0}\right) dy$$

Introducing similarity this becomes

$$\tau_w = \rho_0 U_0^2 \frac{\partial}{\partial x} \int_0^1 \frac{\left(\eta^r - \eta^{2r}\right) y_1}{1 + \frac{\gamma-1}{2} M_0^2 \left(1 - \eta^p\right)} d\eta$$

which upon integration gives:

$$\tau_w = \rho_0 U_0^2 \frac{\partial y_1}{\partial x} \left(\alpha + \beta \frac{\gamma-1}{2} M_0^2\right) \tag{21}$$

where:

$$\alpha = \frac{1}{r+1} - \frac{1}{2r+1}$$

$$\beta = \frac{1}{2r+1} - \frac{1}{r+1} + \frac{1}{r+p+1} - \frac{1}{2r+p+1}$$

is now calculated from $\mu_w \frac{U_\delta}{\delta}$ by assuming a linear velocity distribution in the laminar sublayer. To do this, use is made of the fact that R_δ is taken to be constant. This gives

$$\frac{\delta U_\delta}{\nu_\delta} = R_\delta = \text{CONST.}$$

Introducing

$$U_\delta = U_0 \left(\frac{\delta}{y_1}\right)^r$$

and $$\nu_\delta = \nu_0 \left(\frac{T_\delta}{T_0}\right)^{n+1}$$

there is obtained

$$\left(\frac{\delta}{y_1}\right)^{r+1} = \frac{R_\delta}{R_{y_1}} \left(\frac{T_\delta}{T_0}\right)^{n+1}$$

Using (20) to express $\frac{T_\delta}{T_0}$ and neglecting terms of $\frac{\delta}{y_1} \frac{\gamma-1}{2} M_0^2$ this becomes

$$\left(\frac{\delta}{y_1}\right)^{r+1} = \frac{R_\delta}{R_{y_1}} \left[1 + (n+1) \frac{\gamma-1}{2} M_0^2\right]$$

or

$$\frac{\delta}{y_1} = \frac{R_\delta^{\frac{1}{r+1}}}{R_{y_1}^{\frac{1}{r+1}}} \left[1 + \frac{n+1}{r+1} \frac{\gamma-1}{2} M_0^2\right] \qquad (22)$$

Everything necessary to calculate τ_w is now available. From the definition of R_δ, it is seen that

$$\frac{U_\delta}{\delta} = \frac{\nu_\delta R_\delta}{\delta^2}$$

The value of δ is obtained from (22). Also $\mu_w = \mu_0 \left(\frac{T_w}{T_0}\right)^n$

After combining and simplifying there results:

$$\tau_w = \mu_w \frac{U_\delta}{\delta} = \frac{\mu_0 \nu_0^{\frac{r-1}{r+1}}}{R_\delta^{\frac{1-r}{1+r}}} U_0^{\frac{2}{r+1}} y_1^{-\frac{2}{r+1}} \left[1 + \frac{2rn + r - 1}{r+1} \frac{\gamma-1}{2} M_0^2\right] \qquad (23)$$

The two expressions for τ_w given in (21) and (23) are equated and an expression relating y_1 to X and M_0 results. When solved this leads to

$$y_1 = \left(\frac{3r+1}{r+1}\right)^{\frac{r+1}{3r+1}} \left(\frac{\nu_0}{U_0}\right)^{\frac{2r}{3r+1}} \frac{X^{\frac{r+1}{3r+1}}}{R_\delta^{\frac{1-r}{3r+1}}} \frac{1}{\alpha^{\frac{r+1}{3r+1}}} \left[1 + \frac{r+1}{3r+1}\left(\frac{2rn + r - 1}{r+1} - \frac{\beta}{\alpha}\right) \frac{\gamma-1}{2} M_0^2\right] \qquad (24)$$

τ_w is now obtained by returning to (21) or (23) and substituting (24) to eliminate y_1. This gives

$$\frac{\tau_w}{\rho_0 U_0^2} = \frac{1}{R_\delta^{\frac{1-r}{3r+1}}} \left(\frac{\nu_0}{U_0 X}\right)^{\frac{2r}{3r+1}} \left(\frac{r+1}{3r+1}\right)^{\frac{2r}{3r+1}} \alpha^{\frac{2r}{3r+1}} \left[1 + \left\{\left(\frac{r+1}{3r+1}\right) \frac{2rn + r - 1}{r+1} + \frac{2r\frac{\beta}{\alpha}}{3r+1}\right\} \frac{\gamma-1}{2} M_0^2\right] \qquad (25)$$

Equations (24) and (25) are the desired results. The laminar boundary layer may be investigated as a special case of these equations. In this case $\frac{U^2}{2} + c_p T$ is constant throughout the field and from this relation it may easily be shown that $P = 2r$.

If the velocity distribution is taken to be approximately linear in the laminar case, it can be said that $k=1$ and $P=2$. Substituting these into (24) and (25) gives

$$y_1 = 2\sqrt{3}\sqrt{\frac{\nu_0 x}{U_0}}\left(1+.146\,M_0^2\right)$$

and

$$\frac{\tau_w}{\rho_0 U_0^2} = \frac{1}{2\sqrt{3}}\sqrt{\frac{\nu_0}{U_0 x}}\left(1+.006\,M_0^2\right)$$

In order to check this expression for y_1, the results of Karman and Tsien which are shown in Fig. I have been replotted against a new coordinate defined as $\eta = \frac{y}{y_1}$. The effect of this change in parameter is shown in Figs. III and IV. It is apparent that there is similarity with respect to y_1 over the greater portion of the boundary layer for Mach number as high as two.

It is interesting to note, in connection with the expression for $\tau_w/\rho_0 U_0^2$, that in a laminar boundary layer, the wall shearing stress is almost independent of Mach number, for small Mach numbers.

Some interesting comparisons can be made with results obtained by other people. If the shear stress equation is integrated over a plate of length l, one obtains

$$\frac{D}{\frac{\rho_0 U_0^2}{2}\,l} = \frac{1.155}{\sqrt{R_l}}\left(1+.006\,M_0^2\right)$$

This can be compared with von Karman's approximate result which is based on the assumption of a linear velocity profile. A plot of his result is given in Fig. V. It is seen that the two expressions agree for $M_0=0$ and that both indicate that the shear stress is approximately independent of Mach number for small Mach numbers. The approximation for large Mach numbers is also shown in Fig. V. Since this result did not depend on an assumed velocity profile, it is compared with the more exact results of Karman and Tsien, instead of Karman's first approximation. The undetermined constant of proportionality is chosen arbitrarily so that the curve passes through 1.0 at $M_0 = 10$.

The expression for $\frac{y_1}{x}$ may be compared with the results of Eckert and Drewitz. They find that there is approximate similarity with respect to the parameter $y/\sqrt{\frac{\nu_w x}{U_0}}$, where ν_w is the value of ν at the wall. Using (4) and (5) it is easily seen that, for small Mach numbers

$$\sqrt{\frac{\nu_w x}{U_0}} = \sqrt{\frac{\nu_0 x}{U_0}}\left(1+\frac{n+1}{2}\,\frac{\gamma-1}{2}M_0^2\right) = \sqrt{\frac{\nu_0 x}{U_0}}\left(1+.176\,M_0^2\right)$$

which compares with $\sqrt{\frac{\nu_0 x}{U_0}}\left(1+.146\,M_0^2\right)$, the approximate result obtained here. For large Mach numbers

$$\sqrt{\frac{\nu_w x}{U_0}} \longrightarrow \left(\frac{\gamma-1}{2}\right)^{\frac{n+1}{2}} M_0^{n+1}$$

which agrees with the result obtained for large Mach numbers in Part I.

Lack of knowledge of the value of p in turbulent flow hinders the investigation of the turbulent boundary layer. However, if the turbulent energy is relatively small compared to the enthalpy and the energy of the mean velocity, one might still take p roughly equal to $2r$. The value of r in incompressible turbulent boundary layers has been found experimentally to be about $\frac{1}{7}$. If $r=\frac{1}{7}$ and $p=\frac{2}{7}$ are substituted in (24) and (25), there results

$$\frac{y_1}{x} = 7.76\left(\frac{\nu_0}{U_0 x}\right)^{\frac{1}{5}}\frac{1}{R_s^{3/4}}\left(1-.0354\,M_0^2\right)$$

$$\frac{\tau_w}{\rho_0 U_0^2} = .600\left(\frac{\nu_0}{U_0 x}\right)^{\frac{1}{5}}\frac{1}{R_s^{3/4}}\left(1-.1034\,M_0^2\right)$$

These indicate a decrease of τ_w & $\frac{y_1}{x}$ with Mach number for a turbulent boundary layer.

As a partial check to see whether R_δ may be assumed constant, both of the above formulas may be compared with Prandtl's incompressible results. By means of these formulas R_δ may be back computed from Prandtl's formulas for wall shearing stress and boundary layer thickness. In order to compare the above shear stress expression with Prandtl's results, it is necessary to integrate it with respect to X and thus obtain an expression for drag per unit span of a flat plate parallel to the wind. If this is done , it is easily shown that:

$$\frac{D}{\frac{\rho_0 U_0^2}{2} l} = \frac{5}{2} \frac{.600}{R_\delta^{3/4}} \left(\frac{\nu_0}{U_0 l}\right)^{\frac{1}{5}} (1 - .1034 M_0^2) \qquad (26)$$

Prandtl's equations are

$$\frac{D}{\frac{\rho_0 U_0^2}{2} l} = \frac{.072}{R_l^{1/5}} \qquad \frac{y_1}{x} = \frac{.37}{R_x^{1/5}}$$

Comparing these with the above expressions it is seen that

$$\frac{5}{2} \frac{.600}{R_\delta^{3/4}} = .072 \qquad \frac{7.75}{R_\delta^{3/4}} = .37$$

When solved for $R_\delta^{3/4}$ these give:

$$R_\delta^{3/4} = 20.8 \qquad R_\delta^{3/4} = 20.9$$

If Prandtl's results are regarded as being at least good approximations to experimental data, it appears that the assumption that R_δ is constant is justified in incompressible flow. Whether this is also true in compressible flow is unknown at present.

To see the nature of the turbulent result, the drag coefficients calculated from (26) have been plotted in Fig. VI.

CONCLUSIONS

For a laminar boundary layer in an insulated flat plate parallel to the wind in compressible flow, the following results have been obtained from this analysis:

1) The boundary layer thickness at a given location on the plate increases with Mach number in a manner proportional to $(1 + .176\,M_0^2)$ for Mach numbers less than one.

2) The wall shear stress coefficient is approximately independent of Mach number for small Mach numbers.

3) For large Mach numbers, the thickness of the boundary layer at a given location increases in a manner proportional to $M_0^{1.76}$.

4) For large Mach numbers, the wall shear stress coefficient at a given location decreases with Mach number in a manner proportional to $M_0^{-.24}$.

5) At a given Mach number, the ratio of the thickness of the layer to the distance downstream from the nose, increases with distance downstream in a manner proportional to the Blausins parameter, $\sqrt{\frac{\nu_0 x}{U_0}}$.

6) At a given Mach number, the wall shear stress coefficient decreases with distance downstream in a manner inversely proportional to the Blausins parameter.

For the turbulent compressible boundary layer, the following approximate results were obtained, for subsonic Mach numbers:

1) The boundary layer thickness at a given location decreases with Mach number in a manner proportional to $1 - .035\,M_0^2$

2) The wall shear stress coefficient at a given location decreases with Mach number in a manner proportional to $1-.1034\,M_0^2$

3) At a given Mach number, the ratio of the thickness of the boundary layer to the distance downstream of the nose increases with distance downstream in a manner proportional to $\left(\frac{U_0 x}{\nu_0}\right)^{\frac{1}{5}}$

4) At a given Mach number, the wall shear stress coefficient decreases with distance downstream in a manner proportional to $\left(\frac{\nu_0 x}{U_0}\right)^{-\frac{1}{5}}$

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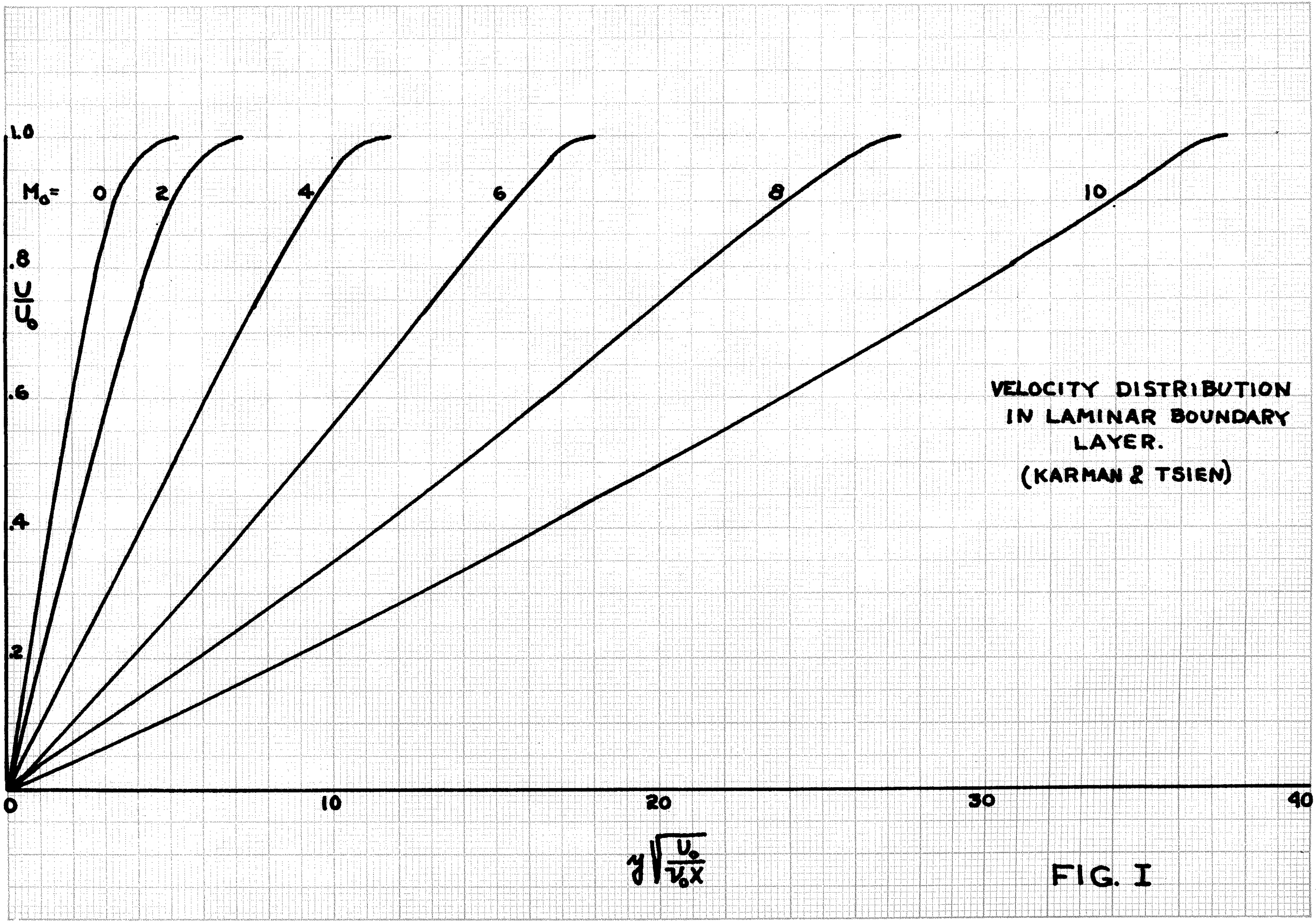
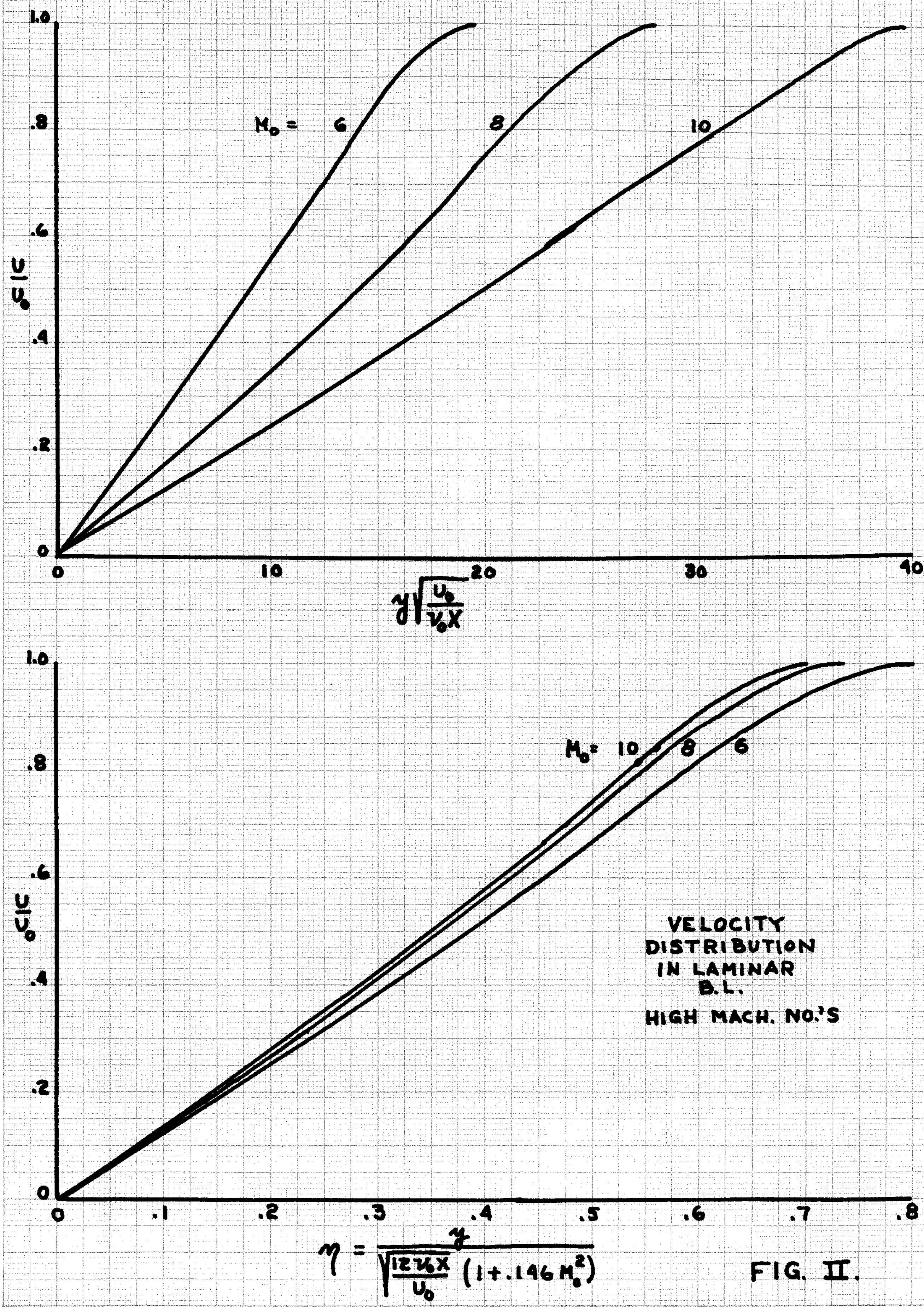


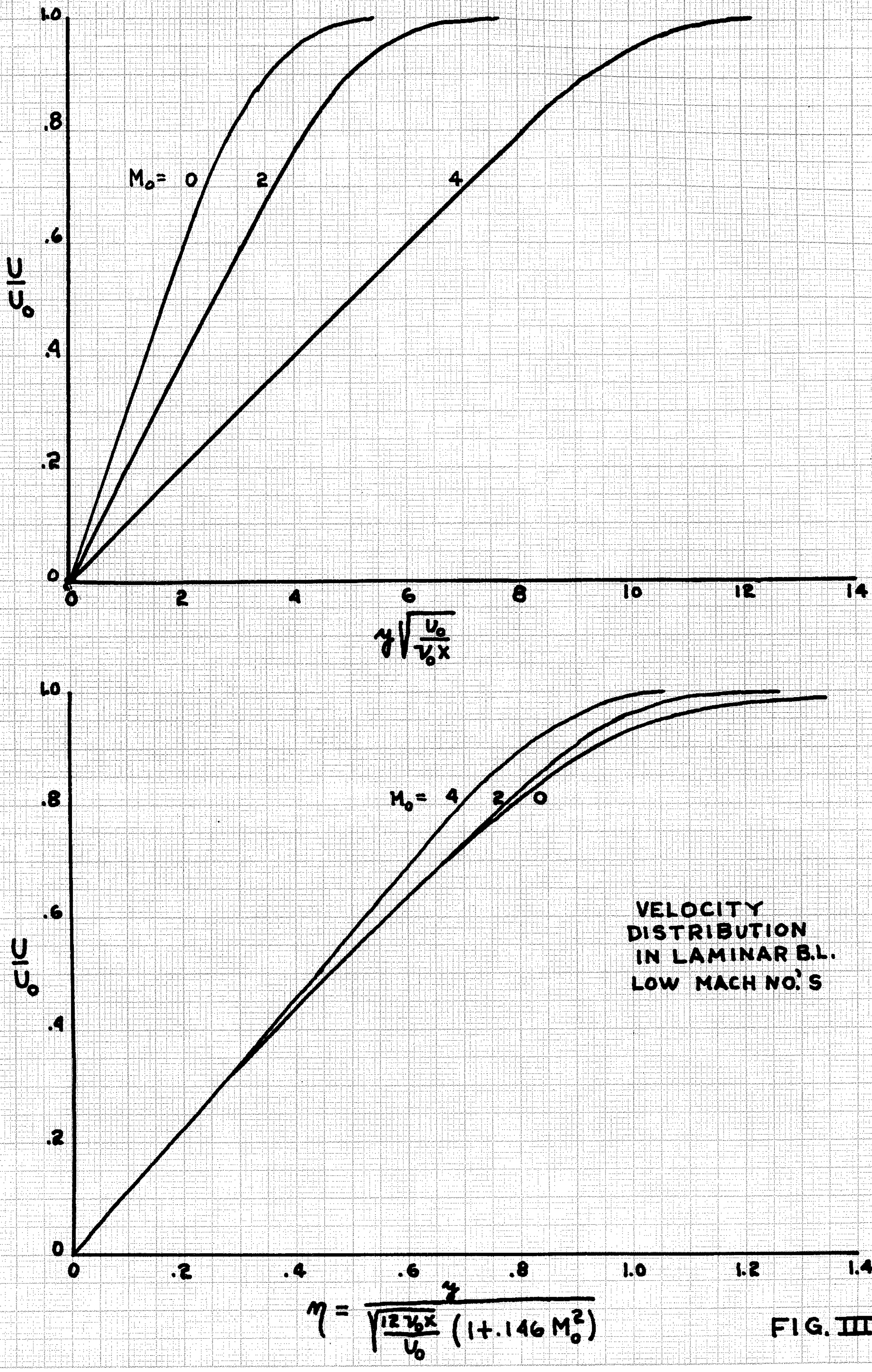
1.0
.8
$\frac{U}{U_0}$
.6
.4
.2
M_0= 0
2
4
6
8
10
0
10
20
30
40
$y\sqrt{\frac{U_0}{\nu_0 X}}$
VELOCITY DISTRIBUTION
IN LAMINAR BOUNDARY
LAYER.
(KARMAN & TSIEN)

FIG. I



1.0
.8
.6
.4
.2
0
$\frac{U}{U_0}$
$M_0 =$ 6
8
10
0
10
20
30
40
$y\sqrt{\frac{U_0}{\nu_0 X}}$
$M_0 =$ 10
8
6
VELOCITY
DISTRIBUTION
IN LAMINAR
B.L.
HIGH MACH. NO.'S
0
.1
.2
.3
.4
.5
.6
.7
.8
$\eta = \frac{y}{\sqrt{\frac{12\nu_0 X}{U_0}}\left(1+.146 M_0^2\right)}$

FIG. II.



1.0
.8
.6
.4
.2
0
$\frac{U}{U_0}$
$M_0 = 0 \quad 2 \quad 4$
0
2
4
6
8
10
12
14
$y\sqrt{\frac{U_0}{\nu_0 x}}$
$M_0 = 4 \quad 2 \quad 0$
0
.2
.4
.6
.8
1.0
1.2
1.4
$\eta = \frac{y}{\sqrt{\frac{12 \nu_0 x}{U_0}}\left(1 + .146 M_0^2\right)}$
VELOCITY
DISTRIBUTION
IN LAMINAR B.L.
LOW MACH NO.'S

FIG. III

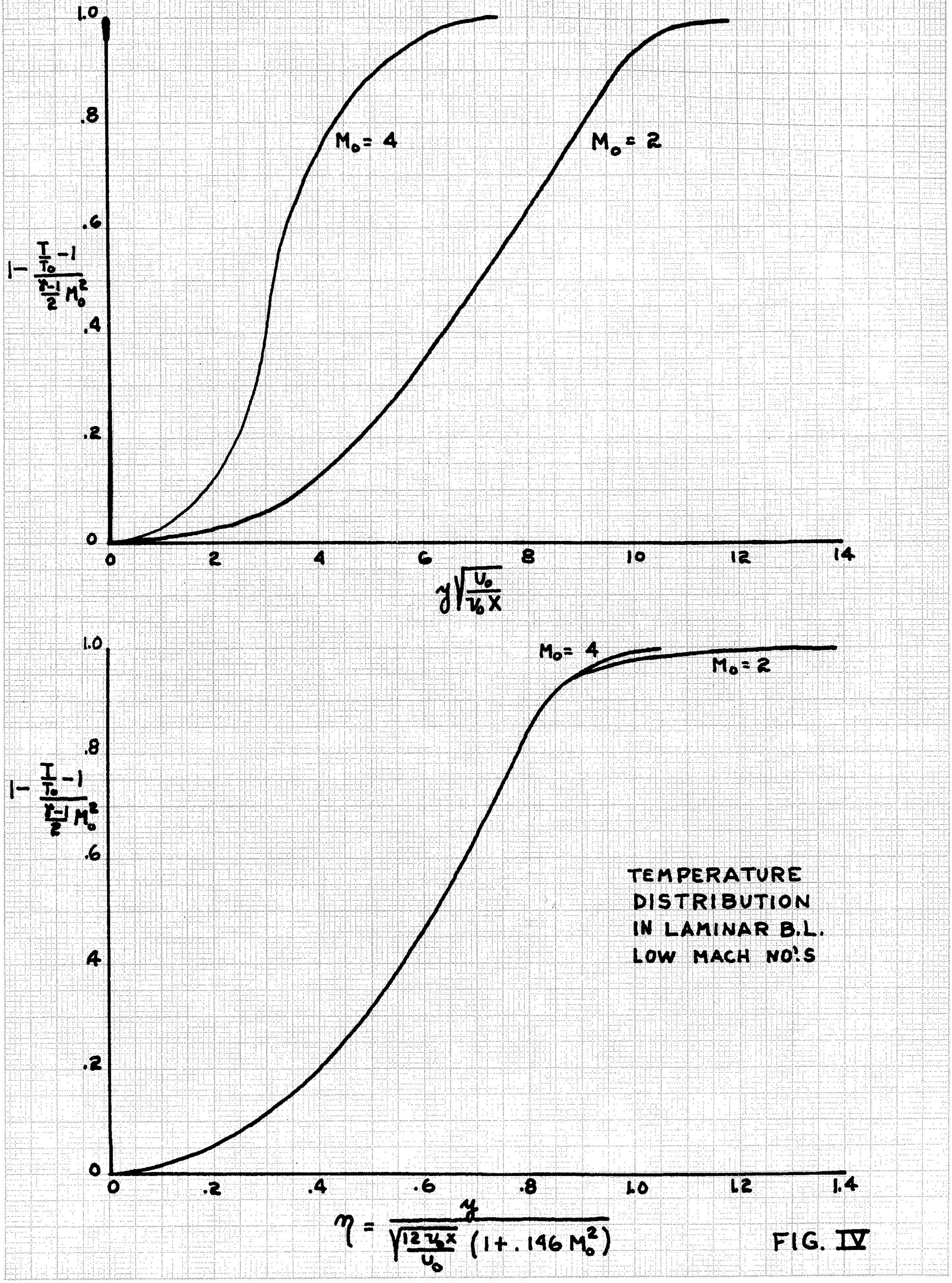
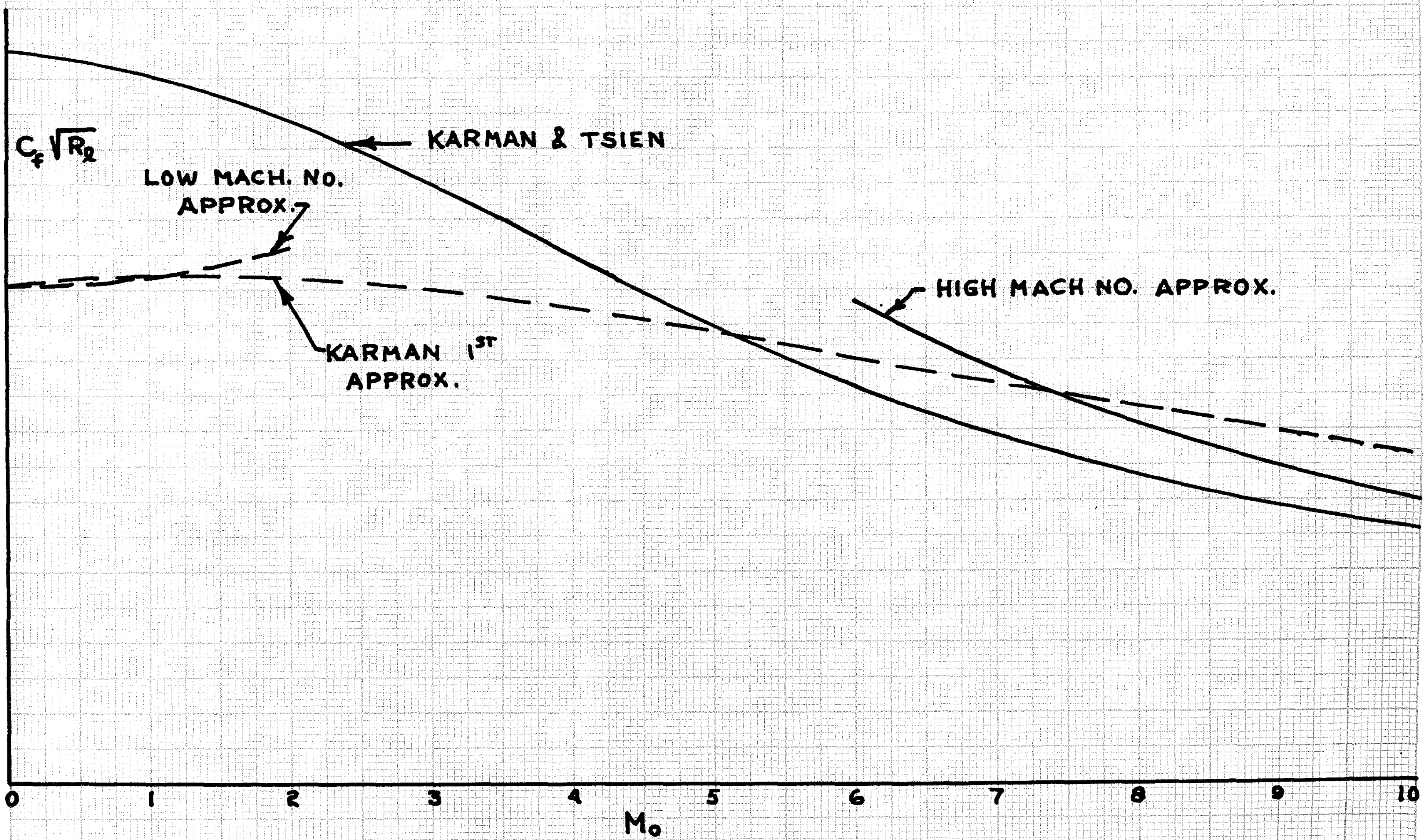
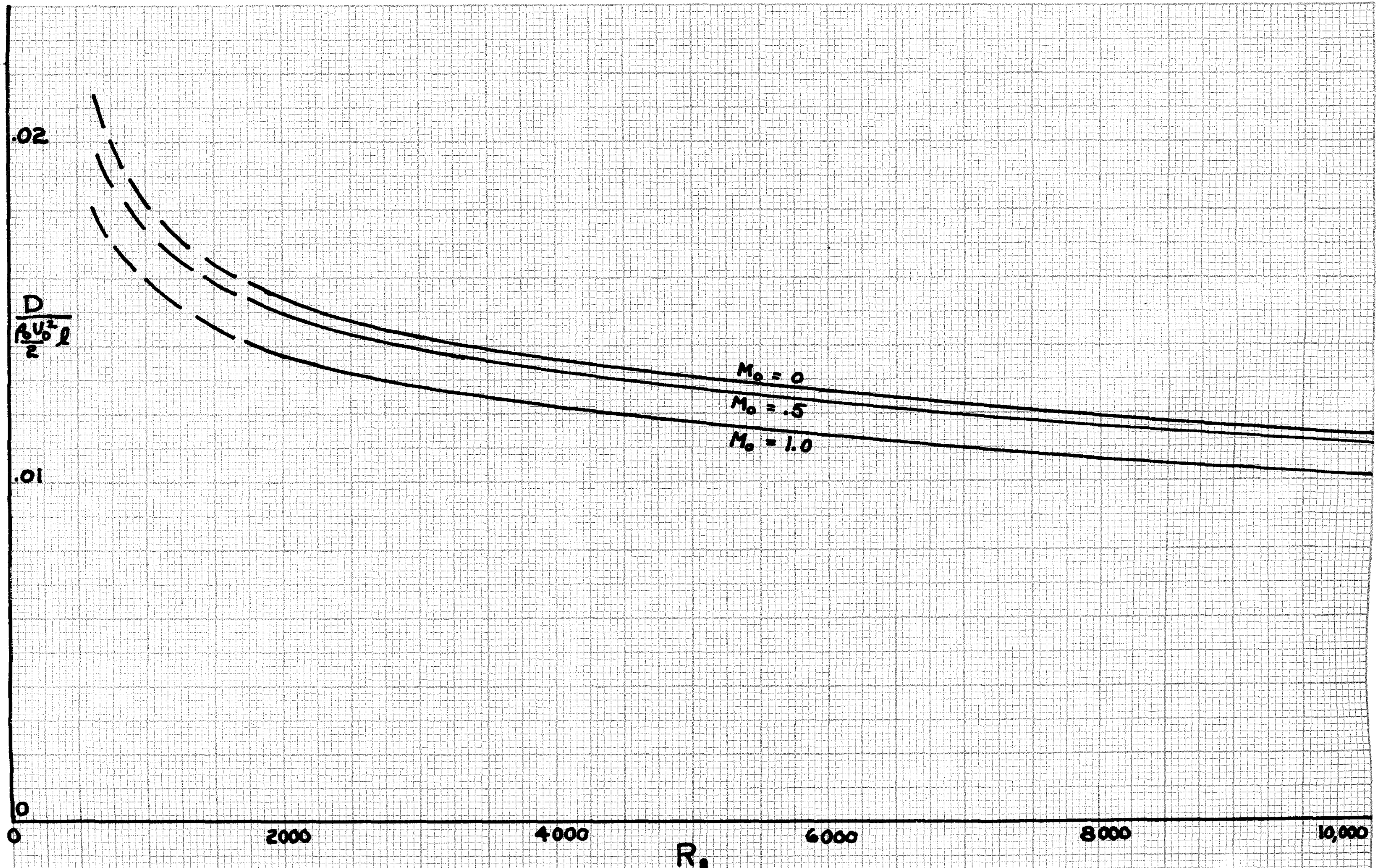

1.0
.8
.6
.4
.2
0
$1-\frac{\frac{T}{T_0}-1}{\frac{\gamma-1}{2}M_0^2}$
$M_0 = 4$
$M_0 = 2$
0 2 4 6 8 10 12 14
$y\sqrt{\frac{U_0}{\nu_0 X}}$
1.0
.8
.6
4
.2
0
$1-\frac{\frac{T}{T_0}-1}{\frac{\gamma-1}{2}M_0^2}$
$M_0 = 4$
$M_0 = 2$
TEMPERATURE
DISTRIBUTION
IN LAMINAR B.L.
LOW MACH NO.S
0 .2 .4 .6 .8 1.0 1.2 1.4
$\eta = \frac{y}{\sqrt{\frac{12\nu_0 x}{U_0}}\,(1+.146M_0^2)}$

FIG. IV



COMPARISON OF VARIOUS SHEAR FORMULAS.

FIG. V



DRAG COEFFICIENTS FOR FLAT PLATES WITH TURBULENT BOUNDARY LAYERS

FIG. VI.