THE DOWNWASH DISTRIBUTION BEHIND A DELTA WING AT SUPERSONIC SPEEDS

Thesis by

George M. Palmer, Jr.

In Partial Fulfillment of the Requirements for the Degree of
Aeronautical Engineer.

California Institute of Technology
Pasadena, California
June, 1947

ACKNOWLEDGMENT

The author wishes to thank Dr. P. A. Lagerstrom for his assistance and guidance in the work to be presented.

TABLE OF CONTENTS

I. INTRODUCTION

At the present date, very little information exists as to the downwash and sidewash in the wake of lifting surfaces for supersonic missiles and airplanes. To the author's knowledge, Reference 1 is the first paper to develop the theory and carry out numerical calculations for the downwash and sidewash behind wings of practical interest. However, the paper only contained brief indications of how to treat the very interesting triangular planform known as the delta wing.

It is the purpose of this paper to extend the basic idea for the determination of downwash as used in Reference 1, to the delta wing, and to carry out numerical calculations for a specific case. Although it is possible to develop expressions for the downwash and the sidewash at all points within the Mach cone and in the wake of the wing, see Reference 1, it was not carried out here. This paper applies only to the downwash field in the plane of the wing. Another restriction is that the delta wing half angle be less than the Mach angle.

II. SUMMARY

In Part III, the notation to be used is presented and explained.

In Part IV, the method of superposition is discussed in detail. Various assumptions of linearization pertaining to the delta wing and the conditions of pressure and velocity on the wing are presented. From this, the underlying boundary condition for superposition is discussed. This boundary condition is that the pressure perturbation must be zero behind the delta wing in the plane of the wing. Two basic types of constant lift distribution wings are discussed, and from them, the auxiliary constant lift distribution wing is developed. An infinite number of auxiliary wings of various shapes are superimposed upon a delta wing of infinite chord in such a manner as to satisfy the above basic boundary condition. Since the downwash field of each superimposed wing is known, the field for the resulting finite delta wing may be computed. The resulting expression contains the lift distribution and the slope of the lift distribution on the original delta wing.

Part V presents the expressions for the lift distribution and slope of the lift distribution on the original delta wing. They are then substituted in the general expression for the downwash developed in part IV.

In Part VI, the expression for the downwash at the trailing edge is developed. Here, an entirely different approach to the problem is used. The boundary condition is satisfied by means of velocity change through the Mach waves from the trailing edge of the delta wing, and not by the subterfuge of adding and subtracting an infinite number of constant lift wings as was done in Part IV.

Part VII contains the development of the expression for the downwash at infinity from the complete general expression developed in Part V and another more simple form developed in Reference 1 by complex variables.

In Part VIII, the downwash is determined for the following special case:

Mach number = $\sqrt{2}$

Delta wing half angle = 30°

Chord = 1

The downwash distribution in a spanwise direction was determined for three chordwise stations, viz. the trailing edge, one chord length behind the trailing edge, and at infinity. The lack of time prevented calculation of the downwash at other chordwise stations. The stations chosen were considered the most significant. Calculations for stations between the trailing edge and infinity involve graphical integrations in which the function goes to infinity at definite points on the span. A method of evaluating these infinities is presented in this section.

Part IX contains the results of the calculations. A table of numerical calculations and the corresponding integral plot for a single ray angle is given for illustrative purposes. Three graphs of the downwash parameter at the selected chordwise stations are given in this section.

In Part X, the results of the numerical calculations are discussed. It is pointed out that the results were somewhat disappointing, as the downwash distribution at one chord length behind the trailing edge does not appear to be compatible with the distribution at the trailing edge and at infinity. There is need for calculations at other chordwise stations in order that a more complete picture of the downwash in the wake be obtained. An experimental survey of the wake characteristics of this particular wing would be very desirable as a means of checking the theory.

III. NOTATION

The system of coordinates and symbols is to be the same as that used in Reference 1 wherever possible. As pointed out in Reference 1, this system was chosen because of its adaptability to conical flow problems.

It can be seen in Figure 1 that the delta wing, defined by the semi-vertex angle τ_0, lies in the x_1x_3-plane, and that the free stream velocity, $\bar{u}_\infty$ is directed along the x_3-axis. The x_2-axis is vertically upwards from the x_1x_3-plane. The semi-vertex angle μ is the Mach angle, and the angle τ is the angle of an arbitrary ray emanating from the vertex and lying in the plane of the wing.

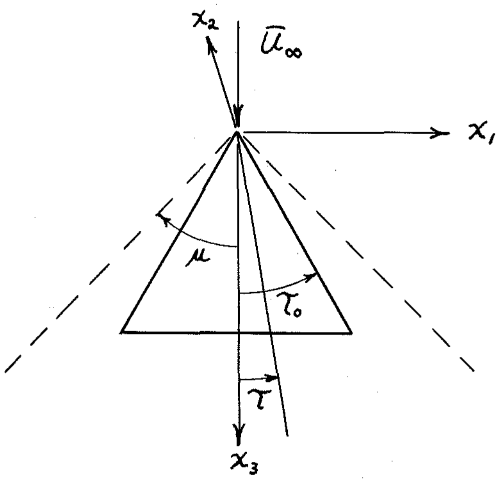


Figure 1

Basic Coordinate System

In the theory, the wing is actually at an angle of attack and could not lie in the x_1x_3-plane; but as linearization holds throughout, this difference is negligible.

The following is a list of symbols used:

$b = \dfrac{\tan \beta}{\tan \mu}$

$B = \dfrac{1-\sqrt{1-b^2}}{b}$

C = Chord of the wing

G = Downwash function

$m = \tan \mu = \dfrac{1}{\sqrt{M^2-1}}$

M = Free stream Mach number

$t_\phi = \dfrac{\tan \phi}{\tan \mu}$

$t_\psi = \frac{\tan\psi}{\tan\mu}$

$T = \frac{1-\sqrt{1-t^2}}{t}$, where $t = t_\phi$ or t_ψ

$\bar{U}_\infty$ = Free stream velocity

u_2 = Perturbation velocity in the x_2-direction (Upwash)

u_3 = Perturbation velocity in the x_3-direction

α = angle of attack

$\beta = \arctan \frac{x_1}{x_3}$ (angle of the side edge of any superimposed wing with the x_3-direction)

μ = Mach angle = arctan m = $\arctan \frac{1}{\sqrt{M^2-1}}$

ϕ = Angle formed at the apex of any super-imposed wing to the left of the x_3-axes by the x_3-direction and the line joining the point in the wake which is being investigated.

ψ = Similar to ϕ , but for super-imposed wings to the right of the x_3-axes.

$\gamma = \arctan \frac{x_1}{x_3}$

IV METHOD OF SUPERPOSITION

The problem at hand is to determine the downwash at an arbitrary point in the plane of and in the wake of a delta wing. The wing under consideration is one of triangular planform and finite chord length C. (See Figure 7). Since this is a linearized theory, the delta wing is assumed to be of zero thickness and flat, i.e. the angle of attack $\propto$ is assumed to be constant everywhere on the wing. This theoretical wing moving at a velocity $\bar{\mathcal{U}}_\infty$ and at a small angle of attack will induce a flow field which will produce a lifting force on the wing. This induced field also produces a downwash.

The lifting force is the result of a pressure differential brought about by the induced velocity u_3. (See Reference 1, Chapter 4, for a discussion of the antisymetric properties of u_3.) This perturbation velocity (u_3) distribution, i.e. the lift distribution, is conical in nature and will be determined later from Reference 2. The downwash, i.e. $-u_2$, on the wing, is constant and is equal to $\propto \bar{\mathcal{U}}_\infty$.

Behind the wing; however, the conditions are quite different. Since there is no physical surface to support a pressure differential, then

$$u_3 \equiv 0$$

This is the boundary condition that must be satisfied in order that any expression for the downwash behind the wing be valid. On the other hand, u_2 is completely unrestricted and it is the object of this investigation to determine an expression for the downwash distribution behind and in the plane of the wing. (i.e., for $x_2 = 0$ and for $C \leq x_3 \leq \infty$, See Figure 1)

As in Reference 1, this will be done by the superposition of an infinite number of auxiliary wings of constant lift distribution in such a manner as to satisfy the boundary conditions. These auxiliary wings will be formed by the superposition of two basic types of constant pressure wings.

The first basic type is the infinite flat plate or two-dimensional wing of zero thickness. It is a known fact that everywhere on the upper surface of the wing

$$\left.\begin{aligned} u_3 &= \alpha \bar{U}_\infty m \qquad &(a) \\ u_2 &= -\alpha \bar{U}_\infty \qquad &(b) \end{aligned}\right\} \quad (1)$$

where $m = \tan\mu = \dfrac{1}{\sqrt{M^2-1}}$, see Figure 1

The second basic type is the conical wing of constant lift distribution as shown in Figure 2. This wing is also of zero thickness; but as will be shown, it is not completely flat. The section of the wing which is outside the Mach cone from the apex A behaves exactly as the first type or two-dimensional wing. In this region-bounded by the Mach cone from A, the upper surface of the wing, and the planer Mach wave from the leading edge -- the induced velocities u_3 and u_2 are constant and are determined by equations (1a) and (1b) respectively. At any point in the x_1x_3 plane not on the surface of the wing, $u_3 = 0$. Furthermore, for any point in the x_1x_3-plane which is not on the surface and is outside the Mach cone from A, $u_2 = 0$ also. By definition, the u_3 on the surface of the wing within the Mach cone from A must be constant and equal to ($\alpha \bar{U}_\infty m$) also.

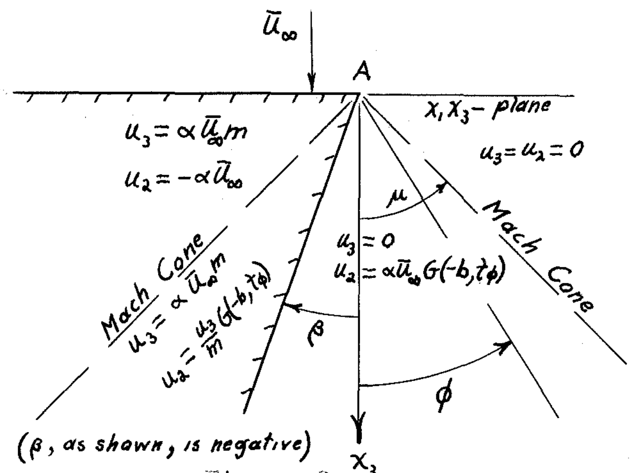


Figure 2
Conical Wing of Constant Pressure Distribution

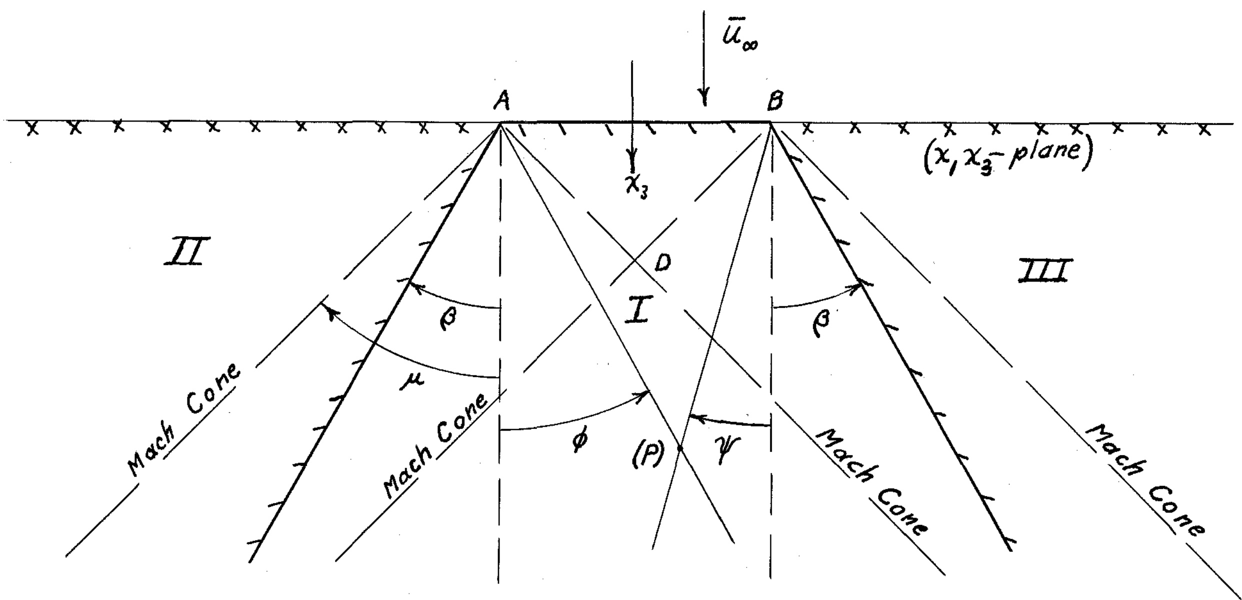
In order for $u_3 = \alpha \bar{U}_\infty m$ in this region, u_2 becomes a complicated function of the Mach number and the geometry. From Reference 1, equation (8b), the basic expression for the u_2 perturbation within the Mach cone is as follows:

$$u_2 = \frac{u_3}{m\pi}\left(-\frac{1}{b}\ln|T| + 2\arctan T + \frac{1-B^2}{2B}\ln\left|\frac{T+B}{1+BT}\right| - \frac{\pi}{2}\right)$$
$$= \frac{u_3}{m} G(-b, t_\phi) \qquad (2)$$

where $\quad b = \frac{\tan|\beta|}{m}$, $\quad B = \frac{1-\sqrt{1-b^2}}{b}$, $\quad T = \frac{1-\sqrt{1-t_\phi^2}}{t_\phi}$, $\quad t_\phi = \frac{\tan\phi}{m}$

It follows from equation 2 that as u_2 is not constant on the surface of the wing within the Mach cone the local angle of attack must vary. Therefore, for u_3 = a constant, the surface within the Mach cone must be curved in order that equations (1b) and (2) are compatible.

The auxiliary wings can now be formed by the superposition of these two basic types. Figure 3 is a diagram of the superposition to be used.



(as shown, both angles (β) are negative; whereas, ϕ and ψ are both positive)

Figure 3
Auxiliary Constant Lift Wing

Wing I is the first type, two-dimensional, with constant lift ($u_3 = \alpha\bar{u}_\infty m$) and constant upwash ($u_2 = -\alpha\bar{u}_\infty$).

Wing II is the second type and extends from the apex at A to $x_1 = -\infty$. This wing has the same constant lift as wing I, i.e. $u_3 = \alpha \bar{U}_\infty m$. Wing III extends from the apex at B to $x_1 = +\infty$ and is a mirror image of Wing II.

If wings II and III are subtracted from wing I, it is easily seen that $u_3 = 0$ everywhere except on the uncovered portion of wing I. In this region it is still constant and $= \alpha \bar{U}_\infty m$.

It is also seen that $u_2 = 0$ everywhere in the x_1x_3-plane except in the region bounded by the left trace of the Mach cone emanating from apex A, the right trace of the Mach cone emanating from apex B, and the triangular area ABD. The velocity u_2 in this region is obtained by subtracting the u_2 velocities due to wings II and III from that due to wing I. For an arbitrary point (P), $u_{2(P)}$ is obtained as follows:

$u_{2(P)_I}$ due to wing I $= -\alpha \bar{U}_\infty$

$u_{2(P)_{II}}$ " " " II $= \frac{u_3}{m} G(-b, t_\phi)$

$u_{2(P)_{III}}$ " " " III $= \frac{u_3}{m} G(-b, t_\psi)$

hence

$$u_{2(P)} = u_{2(P)_I} - u_{2(P)_{II}} - u_{2(P)_{III}} = \left[-\alpha \bar{U}_\infty - \frac{u_3}{m} G(-b, t_\phi) - \frac{u_3}{m} G(-b, t_\psi) \right] \qquad (3)$$

where

$$t_\psi = \frac{\tan \psi}{m}$$

In the region ABD, $G(-b, t_\phi)$ and $G(-b, t_\psi)$ are both zero; so, $u_{2(P)_{ABD}} = -\alpha \bar{U}_\infty$. The distance AB is arbitrary at present, but the x_3-axis always bisects it.

Now that u_3 and u_2 are known for any point on the auxiliary wing of Figure 3, it is possible to superimpose a number of these wings in such a manner as to satisfy the boundary condition for u_3 and obtain the downwash behind a delta wing of chord C.

First, consider a delta wing with infinite chord which has conical distribution of lift, i.e. u_3. The cross-section of the u_3 distribution for any x_1x_2-plane is shown in Figure 4. The actual distribution across the delta wing will be determined later, but for the purpose of presenting the superposition principle, this simplified distribution will be used.

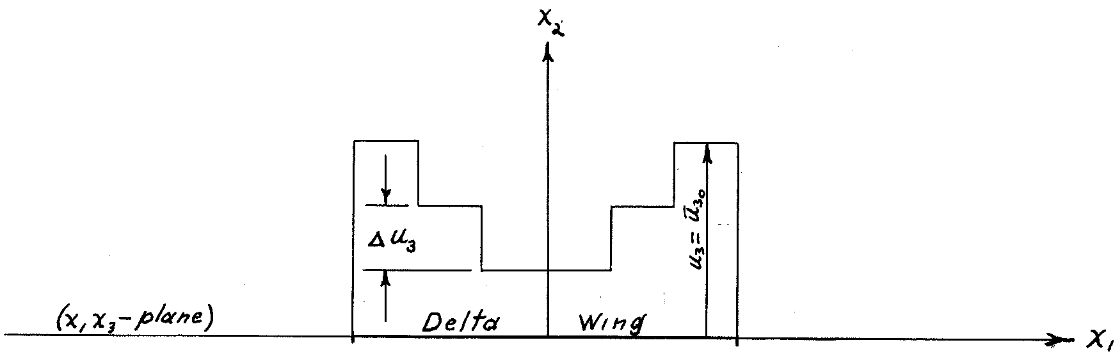


Figure 4
Simplified Lift Distribution on Delta Wing

By superimposing a single auxiliary wing of $u_3 = -\bar{u}_{3_0}$ at a distance C from the vertex of this infinite delta wing, the lift distribution immediately behind C and extending to ∞ in the x_3 direction will be as shown in Figure 5. It is clear that adding a wing of $u_3 = -\bar{u}_{3_0}$ is the same as subtracting one of $u_3 = +\bar{u}_{3_0}$.

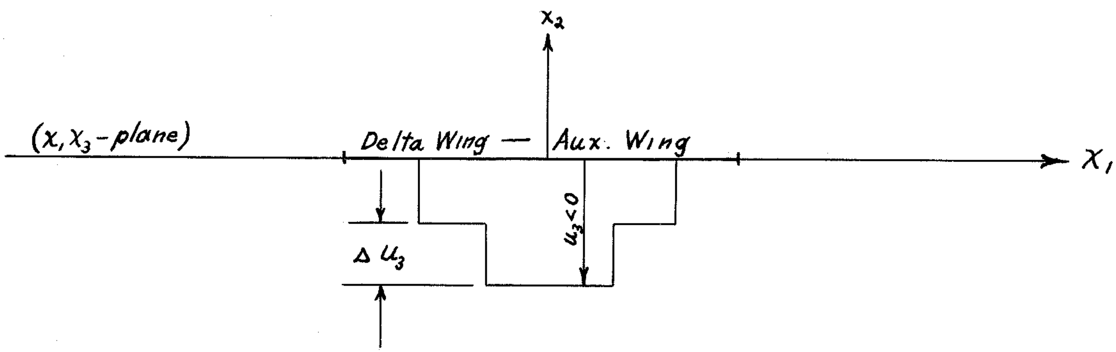


Figure 5
Simplified Lift Distribution of Delta Wing Minus One Auxiliary Wing

This remaining lift can be cancelled by adding two, and in the general case (n), auxiliary wings of $u_3 = \Delta u_3$ and having increasing angles (β).

A planform view of this superposition for $n = 2$ is shown in Figure 6.

For simplicity, consider the point P_1 on the x_3 axes and at $x_3 = 2C$. For this case, $t_\phi = t_\psi = t_\beta$. It is easily seen from Figures 4 and 5 that anywhere within the superposition

$$u_3 = u_{3(Inf.\Delta)} - \bar{u}_{3_0} + \Delta u_{3(\beta_1)} + \Delta u_{3(\beta_2)} = 0$$

Therefore, by combining the u_2's for the various wings in an algebraically similar manner the u_2 at point P_1 is

$$u_{2(P_1)} = -\alpha\bar{u}_\infty - \frac{\bar{u}_{3_0}}{m}\left[-1 - 2G(-b_0, t_{\beta_0})\right] + \frac{\Delta u_3}{m}(\beta_1)\left[-1 - 2G(-b_1, t_{(\beta_1)})\right] + \frac{\Delta u_3}{m}(\beta_2)\left[-1 - 2G(-b_2, t_{\beta_2})\right]$$

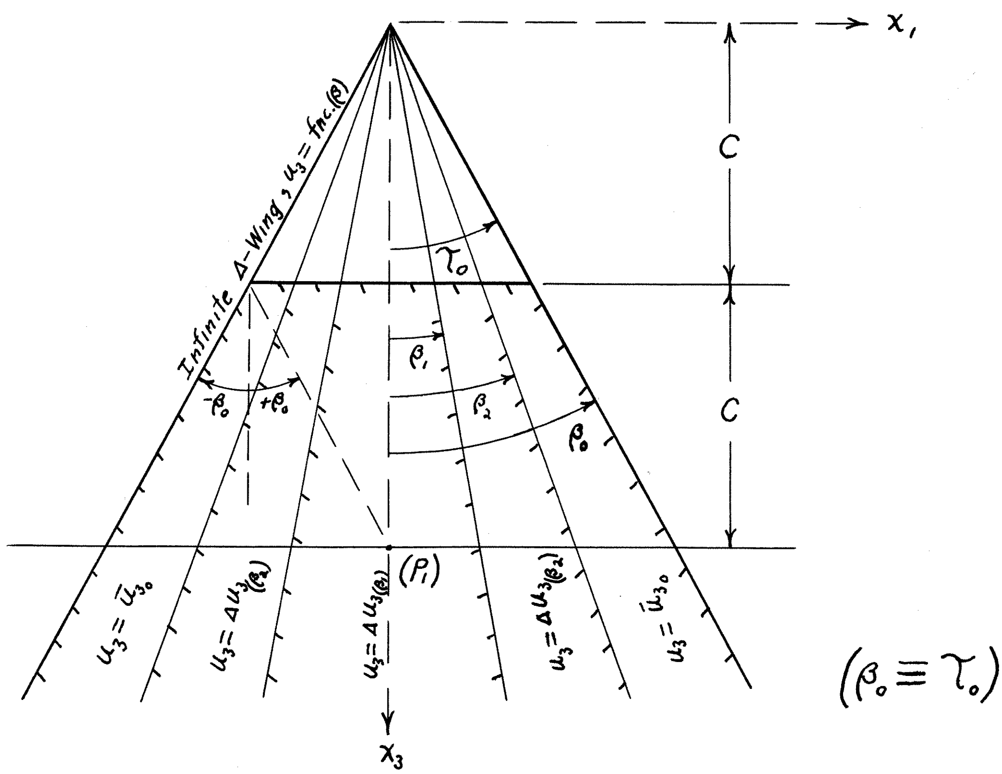


Figure 6
Planform View of the Superposition

By assuming $\Delta u_{3(\beta_n)} =$ constant $= \Delta\bar{u}_3$, it is possible to extend the above expression to the following general case:

$$u_{2(P_1)} = -\alpha\bar{u}_\infty - \frac{\bar{u}_{3_0}}{m}\left[-1 - 2G(-b_0, t_{\beta_0})\right] + \sum_{b=\Delta\tan\beta}^{b=\tan\beta_0} \frac{\Delta\bar{u}_3}{m}\left[-1 - 2G(-b, t_\beta)\right]$$

but

$$\Delta \bar{u}_3 \equiv \frac{\Delta \bar{u}_3}{\Delta \tan\beta} \, \Delta \tan\beta = \frac{\Delta \bar{u}_3}{\Delta b} \, \Delta b$$

then

$$u_{2_{(P)_1}} = -\alpha \bar{U}_\infty - \frac{\bar{u}_{3_0}}{m}\left[-1 - 2G(-b_0, t_{\beta_0})\right] + \sum_{\Delta b}^{b_0} \frac{\Delta \bar{u}_3}{m \, \Delta b}\left[-1 - 2G(-b, t_\beta)\right] \Delta b$$

Letting $\Delta b \to 0$, the complete expression for downwash at one chord length behind and on the center line of the delta wing is

$$u_{2_{(P)_1}} = -\alpha \bar{U}_\infty - \frac{\bar{u}_{3_0}}{m}\left[-1 - 2G(-b_0, t_{\beta_0})\right] + \int_0^{b_0} \frac{du_3}{m \, db}\left[-1 - 2G(-b, t_\beta)\right] db \qquad (4)$$

It is now desirable to extend the expression above to the completely general case. Figure 7 shows the rather complicated relationship between the point P under consideration and the superimposed auxiliary wings.

It can be seen that

$$\left.\begin{aligned} t_\phi &= \frac{\tan\phi}{\tan\mu} = \frac{a + e}{m(x_3 - C)} = \frac{C \tan\beta + x_3 \tan\tau}{m(x_3 - C)} & \quad (a) \\ t_\psi &= \frac{\tan\psi}{\tan\mu} = \frac{a - e}{m(x_3 - C)} = \frac{C \tan\beta - x_3 \tan\tau}{m(x_3 - C)} & \quad (b) \end{aligned}\right\} \qquad (5)$$

Hence from equations 1, 4, and 5

$$\frac{u_{2(P)}}{\alpha \bar{U}_\infty} = -1 - \frac{\bar{u}_{3_0}}{m \alpha \bar{U}_\infty}\left[-1 - G(-b_0, t_{\phi_0}) - G(-b_0, t_{\psi_0})\right]$$

$$+ \frac{1}{m \alpha \bar{U}_\infty} \int_0^{b_0} \frac{du_3}{db}\left[-1 - G(-b, t_\phi) - G(-b, t_\psi)\right] db \qquad (6)$$

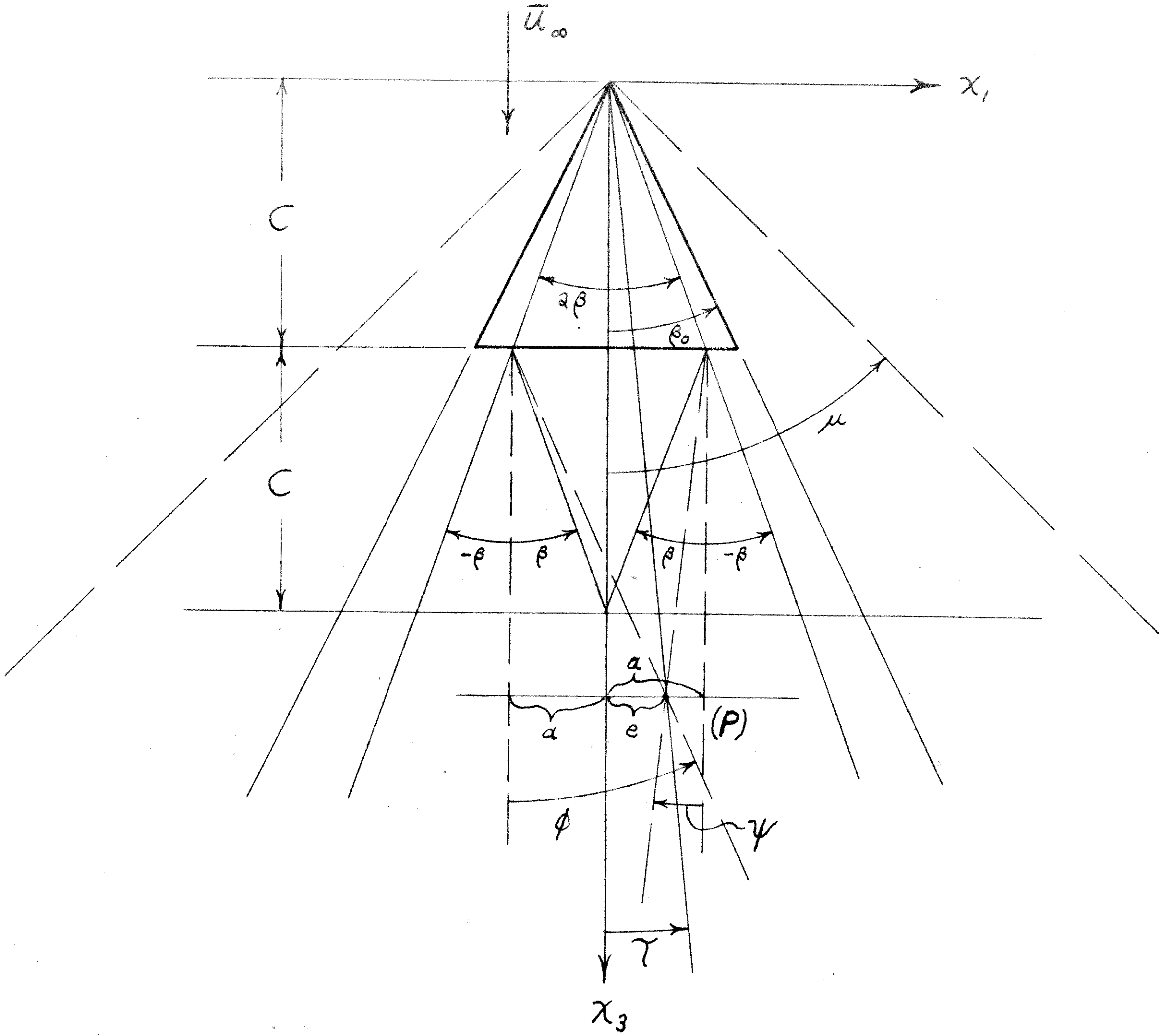


Figure 7

Parameters Relating to the Delta Wing

V. THE LIFT DISTRIBUTION ON THE ORIGINAL DELTA WING AND THE GENERAL DOWNWASH EXPRESSION

Before equation 6 can be developed further, it is necessary to obtain explicit expressions for u_{3_0} and $\dfrac{du_3}{db}$.

From Reference 2, equation 45,

$$u_3 = \frac{\alpha \bar{U}_\infty \tan\beta_0}{E(k')\sqrt{1-r^2}}$$

$$\text{where} \quad r = \frac{\tan\beta}{\tan\beta_0} = \frac{b}{b_0}$$

$$k = \sqrt{M^2-1}\,\tan\beta_0 = b_0$$

$$k' = \sqrt{1-k^2} = \sqrt{1-b_0^2}$$

and E(k') is the complete elliptic integral of the second kind having a modulus k'.

Therefore

$$u_3 = \frac{\alpha \bar{U}_\infty m b_0}{E(k')\sqrt{1-\left(\frac{b}{b_0}\right)^2}} \qquad (7)$$

For simplicity, let

$$u_3 = A\bar{U}_\infty \left(1-\frac{b^2}{b_0^2}\right)^{-\frac{1}{2}}$$

Hence

$$du_3 = A\bar{U}_\infty \left(-\tfrac{1}{2}\right)\left(1-\frac{b^2}{b_0^2}\right)^{-\frac{3}{2}}\left(-\frac{2b}{b_0^2}\right) db$$

$$\frac{du_3}{db} = \frac{A\bar{U}_\infty\, b/b_0^2}{\left(1-\frac{b^2}{b_0^2}\right)^{3/2}} = \frac{\alpha \bar{U}_\infty m\, b/b_0}{E(k')\left(1-\frac{b^2}{b_0^2}\right)^{3/2}} \qquad (8)$$

Substituting the expression for $\frac{du_3}{db}$ in equation 6, the following is obtained:

$$\frac{u_2(P)}{\alpha \bar{U}_\infty} = -1 - \frac{\bar{u}_{3_0}}{m\alpha \bar{U}_\infty}\left[-1 - G(-b_0, t_{\phi_0}) - G(-b_0, t_{\psi_0})\right]$$

$$+ \frac{1}{b_0 E(\sqrt{1-k_0^2})}\int_0^{b_0} \frac{b\left[-1 - G(-b, t_\phi) - G(-b, t_\psi)\right]}{\left[1-(b/b_0)^2\right]^{3/2}}\, db$$

Equations 7 and 8 show that for $b = b_0$, u_{3_0} and $\left(\frac{du_3}{db}\right)_0$ are infinite. This being the case, it is impossible to determine the downwash at any point with the above expression.

Actually, an infinite lift at the outer edges of the delta wing is a physical impossibility; therefore, any assumption which modifies this infinite lift would be justifiable by consideration of the viscosity.

It was decided to use the arbitrary value of

$$\left(\frac{u_3}{\bar{U}_\infty}\right)_{max} = 10\left(\frac{u_3}{\bar{U}_\infty}\right)_{min}$$

Let β_n = the angle of the $10\left(\frac{u_3}{\bar{U}_\infty}\right)_{min}$-ray, see Figure 8.

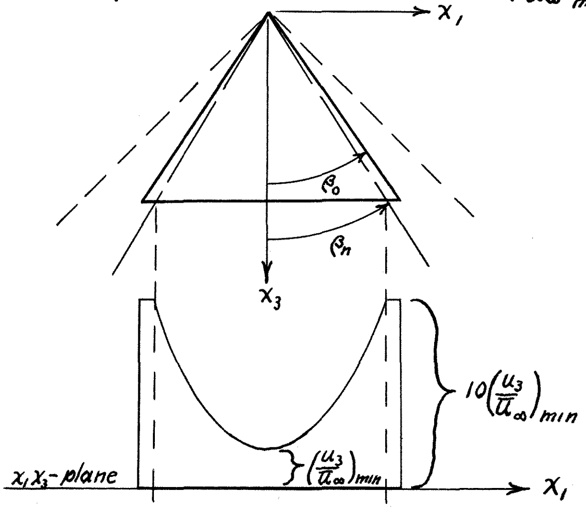


Figure 8
Modified Lift Distribution

$\left(\frac{u_3}{\bar{U}_\infty}\right)_{min}$ occurs at the center line or for $b = 0$. Therefore,

$$\left(\frac{u_3}{\bar{U}_\infty}\right)_{min} = \frac{\alpha m b_0}{E(K')}$$

hence

$$\frac{\left(\frac{u_3}{\bar{U}_\infty}\right)_{max}}{\left(\frac{u_3}{\bar{U}_\infty}\right)_{min}} = 10 = \frac{1}{\sqrt{1-\left(\frac{b_n}{b_0}\right)^2}}$$

or

$$b_n = \sqrt{.99}\, b_0 = (.995) b_0 \quad (9)$$

It will now be possible to obtain finite values for the downwash at P. The expression becomes

$$\frac{u_2}{\alpha U_\infty}(P) = -1 - \frac{b_0}{E(\sqrt{1-b_0^2})\sqrt{.01}}\left[-1 - G(-b_0, t_{\phi_0}) - G(-b_0, t_{\psi_0})\right]$$

$$+ \frac{1}{b_0 E(\sqrt{1-b_0^2})}\int_0^{b_n} \frac{b\left[-1 - G(-b, t_\phi) - G(-b, t_\psi)\right]}{\left[1-(b/b_0)^2\right]^{3/2}}\, db \qquad (10)$$

Integration of this equation appears to be impossible except by graphical means.

VI. THE DOWNWASH AT THE TRAILING EDGE

The downwash at the trailing edge can be obtained very easily by considering the velocity change through the Mach waves at the trailing edge. Here again, the boundary condition to satisfy is that the u_3 perturbation be zero behind the wing.

It can be seen in Figure 9 that if

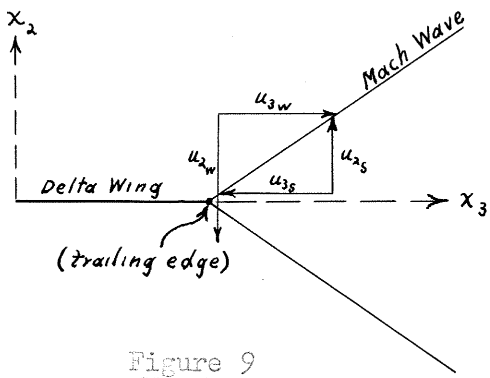


Figure 9
Mach Waves at Trailing Edge

$$u_{3_{(T.E.)}} = u_{3_w} + u_{3_s} = 0$$

then

$$u_{2_{(T.E.)}} = u_{2_w} + u_{2_s}$$

Since

$$u_{2_w} = -\alpha \bar{U}_\infty$$

and

$$u_{2_s} = -\frac{u_{3_s}}{m} = \frac{u_{3_w}}{m} = \frac{\alpha \bar{U}_\infty b_0}{E(k)\sqrt{1-(b/b_0)^2}}$$

from equation 7

Then

$$\frac{u_{2(T.E.)}}{\alpha \bar{U}_\infty} = \left[\frac{b_0}{E\left(\sqrt{1-b_0^2}\right)\sqrt{1-(b/b_0)^2}} - 1\right] \qquad (11)$$

Equation 11 is the relationship for the downwash at the trailing edge for the completely general case.

VII THE DOWNWASH AT INFINITY

It is desirable for comparative reasons to know the downwash distribution at infinity. This can be obtained by letting $X_3 \rightarrow \infty$ in equation 5.

For

$$t_\phi = \frac{\tan \tau}{m}$$

$$t_\psi = \frac{-\tan \tau}{m}$$

Therefore,

$$T_\phi = -T_\psi \quad \text{or} \; |T_\phi| = |T_\psi|$$

This fact greatly simplifies the integral in equation 10. Using the expanded form of equation 2 together with the knowledge that $|T_\phi| = |T_\psi|$, the integral becomes

$$\int_0^{b_n} \frac{\tan\beta}{\left[1-\left(\frac{\tan\beta}{\tan\beta_0}\right)^2\right]^{3/2}} \left\{-1-\frac{1}{\pi}\left[\frac{\ln|T_\phi|}{-\frac{\tan\beta}{m}} + \frac{\ln|T_\phi|}{-\frac{\tan\beta}{m}} + 2\arctan(T_\phi) + 2\arctan(-T_\phi) + \frac{1-B^2}{2B}\ln\left|\frac{T_\phi+B}{1+T_\phi B}\right| + \frac{1-B^2}{2B}\ln\left|\frac{-T_\phi+B}{1-T_\phi B}\right| - \pi\right]\right\} d(\tan\beta)$$

This simplifies to

$$\frac{1}{\pi}\int_0^{b_n} \frac{\tan\beta}{\left[1-\left(\frac{\tan\beta}{\tan\beta_0}\right)^2\right]^{3/2}} \left[\frac{2\ln\left(\frac{1}{T_\phi}\right)}{\frac{\tan\beta}{m}} + \frac{1-B^2}{2B}\ln\left|\frac{B^2-T_\phi^2}{1-(T_\phi B)^2}\right|\right] d(\tan\beta)$$

where T_ϕ = constant for a particular point at ∞ . An extended attempt was made to integrate this expression, but the second term proved to be extremely difficult. It was decided that this also would have to be evaluated graphically.

The complete relationship is as follows:

$$\frac{u_{2(\infty)}}{\alpha \bar{U}_\infty} = -1 - \frac{b_o}{\pi E\left(\sqrt{1-b_o^2}\right)\sqrt{.01}}\left[\frac{2\ln\left(\frac{1}{T_\phi}\right)}{\frac{\tan\beta_o}{m}} + \frac{1-B_o^2}{2B_o}\ln\left(\frac{B_o^2 - T_\phi^2}{1-(T_\phi B_o)^2}\right)\right]$$

$$+ \frac{1}{b_o \pi E\left(\sqrt{1-b_o^2}\right)}\int_0^{b_n} \frac{\tan\beta}{\left[1-\left(\frac{\tan\beta}{\tan\beta_o}\right)^2\right]^{3/2}}\left[\frac{2\ln\left(\frac{1}{T_\phi}\right)}{\frac{\tan\beta}{m}} + \frac{1-B^2}{2B}\ln\left(\frac{B^2-T_\phi^2}{1-(T_\phi B)^2}\right)\right] d(\tan\beta)$$

(12)

A much simpler expression for the downwash at infinity was developed in Reference 1 by a complex variable analysis of the sidewash at the trailing edge.

From equations 49 in Reference 1, the downwash becomes

$$\left.\begin{aligned}
\frac{u_{2(\infty)}}{\alpha \bar{U}_\infty} &= \frac{-1}{E\left(\sqrt{1-b_o^2}\right)} && \text{for } |\gamma| < \beta_o \\
\frac{u_{2(\infty)}}{\alpha \bar{U}_\infty} &= \frac{1}{E\left(\sqrt{1-b_o^2}\right)}\left[\frac{\frac{\tan\gamma}{\tan\beta_o}}{\sqrt{\left(\frac{\tan\gamma}{\tan\beta_o}\right)^2 - 1}} - 1\right] && \text{for } |\gamma| > \beta_o
\end{aligned}\right\} \quad (13)$$

where $\tan\gamma = \frac{x_1}{c}$

It should be pointed out that since $\tan\tau = \frac{x_1}{c}$, the points at $x_3 = \infty$ are not conical from the vertex; but are projections parallel to the x_3-axis of points at $x_3 = c$, which in turn are located by $\tan\tau$.

Equation 12 and 13 are relationships for the downwash at infinity for the completely general case.

VIII THE DETERMINATION OF THE DOWNWASH BEHIND A SPECIFIC DELTA WING

Numerical calculations of the downwash are to be carried out for various points at the trailing edge, at an infinite distance behind the trailing edge, and at one chord length behind the trailing edge.

The following simple case is to be used:

$$M = \sqrt{2}$$

$$\tau_0 \equiv \beta_0 = 30^\circ$$

$$C = 1$$

Using the above values, the following are obtained:

$$m = 1$$

$$\mu = 45^\circ$$

$$b = \tan\beta$$

$$b_0 = .577$$

$$b_n = .574 \quad \text{, eq. 9}$$

$$\beta_n = 29.85^\circ$$

$$\left.\begin{aligned} t_\phi &= \frac{\tan\beta + x_3 \tan\tau}{(x_3 - 1)} \\ t_\psi &= \frac{\tan\beta - x_3 \tan\tau}{(x_3 - 1)} \end{aligned}\right\} \text{eq. 5}$$

$$E(k') = 1.261 \quad \text{, see part V}$$

From equation 11 and the values just determined, the downwash at the trailing edge is obtained from

$$\frac{u_{3(T.E.)}}{\alpha U_\infty} = \left[\frac{.457}{\sqrt{1 - \left(\frac{b}{.577}\right)^2}} - 1\right] \tag{14}$$

where $b = \tan\beta \equiv \tan\tau$

Graph II is a plot of this equation versus $\frac{\tan \gamma}{\tan \beta_0}$ for $0 \leq \gamma \leq 30^\circ$

The simplified expression, equation 13, for the downwash at infinity will be used. Substituting for the known constants, the following are obtained:

$$\left.\begin{aligned} \frac{u_{2(\infty)}}{\alpha U_\infty} &= \frac{-1}{1.261} \quad , \quad |\gamma| < \beta_0 && (a) \\ \frac{u_{2(\infty)}}{\alpha U_\infty} &= \frac{1}{1.261}\left[\frac{\tan\gamma / \tan\beta_0}{\sqrt{\left(\frac{\tan\gamma}{\tan\beta_0}\right)^2 - 1}} - 1\right] \quad , \quad |\gamma| > \beta_0 && (b) \end{aligned}\right\} \qquad (15)$$

Graph IV is a plot of this equation versus $\frac{\tan \gamma}{\tan \beta_0}$ for $0 \leq \gamma \leq \infty$

As shown in parts IV and V, the downwash at a finite distance behind the wing is much more difficult to evaluate.

Substituting the known constants into equation 10, the following is obtained:

$$\frac{u_{2(P)}}{\alpha U_\infty} = -1 - 4.575\left[-1 - G(-.577, t_{\phi_{30^\circ}}) - G(-.577, t_{\psi_{30^\circ}})\right]$$

$$+ 1.373 \int_0^{.574} \frac{b\left[-1 - G(-b, t_\phi) - G(-b, t_\psi)\right]}{\left[1 - \left(\frac{b}{.577}\right)^2\right]^{3/2}} \, db \qquad (16)$$

where

$G(-b, t)$ is defined in equation 2,

and for $x_3 = 2C = 2$

$t_\phi = \tan\beta + 2\tan\gamma$

$t_\psi = \tan\beta - 2\tan\gamma$

The ray angle (γ) is the independent parameter, and calculations are carried out for as many angles as desired. The downwash for this specific case was determined at seven different ray angles. viz $\gamma = 0^\circ, 5^\circ, 10^\circ, 15^\circ, 20^\circ, 25^\circ$, and 29.85°. As the downwash is symmetrical about the x_3-axis, only positive ray angles need be considered.

The determination of a specific $G(-b,t)$ is a long and involved process; and as there are a large number of them to be evaluated for any one ray angle, a chart of $G(b,t)$ versus t with b as a parameter was used. This chart was used in the numerical calculations of Reference 1.

Calculations were then completed by means of tables, see Table I.

By plotting the final value obtained in the table versus b , see Graph I, the value of the integral for the specific ray angle can be determined by the use of a planimeter.

Graph I shows that an infinity occurs within the integral of equation 16. It can be seen from equation 2 that an infinity will occur when $T = -B$, *i.e.* $t = -b$.

Since

$$t_\psi = t \tan\beta - 2 t \tan\tau$$

it is seen that for $\beta = \tau$, $t_\psi = -b$. Therefore, the infinity occurs whenever the angle of one of the superimposed wings corresponds to the ray angle being investigated. This is an inherent difficulty and the infinity occurs at the particular ray angle for all plots similar to Graph I.

As suggested in Reference 1, this infinity may be removed by the subterfuge of adding and subtracting a suitable function which also goes to infinity at the particular ray angle. Since the infinity is quite localized, this artifice need be applied only over a small range of angles about the particular ray angle.

The infinities were removed as follows:

$$\int_{b_1}^{b_2} f(b,t)\frac{du_3}{db}\,db = \int_{b_1}^{b_2}\left[f(b,t)\frac{du_3}{db} - h(b,\tau)\right]db + \int_{b_1}^{b_2} h(b,\tau)\,db$$

where

τ = constant for a particular graph

and

$$h(b,\tau) = \frac{-1}{\pi \tan\tau}\left(\frac{du_3}{db}\right)_{b=\tan\tau} \ln\left|\tan\tau - \tan\beta\right|$$

$$= \frac{\alpha \bar{U}_\infty}{E(k')\tan\beta_0} \frac{-1}{\pi\left(1-\frac{\tan^2\tau}{\tan^2\beta_0}\right)^{3/2}} \ln\left|\tan\tau - \tan\beta\right|$$

Therefore

$$\int_{b_1}^{b_2} h(b,\tau)\,db = \alpha \bar{U}_\infty k \int_{\tan\beta_1}^{\tan\beta_2} \ln\left|\tan\tau - \tan\beta\right| d(\tan\beta)$$

$$= \frac{\alpha \bar{U}_\infty}{E(k')\tan\beta_0} \frac{1}{\pi\left(1-\frac{\tan^2\tau}{\tan^2\beta_0}\right)^{3/2}} \Big[(\tan\tau - \tan\beta)\ln\left|\tan\tau-\tan\beta\right| - (\tan\tau - \tan\beta)\Big]_{\tan\beta_1}^{\tan\beta_2}$$

Finally

$$\int_{b_1}^{b_2} f(b,t)\frac{du_3}{db}\,db = \int_{\tan\beta_1}^{\tan\beta_2} \left\{ \frac{\tan\beta}{\left(1-\frac{\tan^2\beta}{\tan^2\beta_0}\right)^{3/2}}\Big[-1 - G(-b,t_\phi) - G(-b,t_\psi)\Big] + \frac{1}{\pi\left(1-\frac{\tan^2\tau}{\tan^2\beta_0}\right)^{3/2}} \ln\left|\tan\tau - \tan\beta\right| \right\} d(\tan\beta)$$

$$+ \frac{1}{\pi\left(1-\frac{\tan^2\tau}{\tan^2\beta_0}\right)^{3/2}} \Big[(\tan\tau - \tan\beta)\ln\left|\tan\tau - \tan\beta\right| - (\tan\tau - \tan\beta)\Big]_{\tan\beta_1}^{\tan\beta_2}$$

A plot of this is shown on Graph I. The actual integrations were carried out on more expanded plots than the sample shown here.

Having the value of the integral, the downwash for the particular ray angle can now be determined, see Table I.

Graph III is a plot of equation 16 versus $\frac{\tan \gamma}{\tan \beta_o}$ for $0 \leq \gamma \leq 30^\circ$

Time did not permit the evaluation of equation 16 in the region of $\gamma > 30^\circ$ however, a qualitative plot was easily obtained from two known facts. First, the downwash must also approach infinity as $\gamma \longrightarrow \beta_o$. Second, the downwash must be zero for all points which fall to the right of the intersection Q of the Mach cone from the apex B of the maximum auxiliary wing and the line (x_1, 0, j), see Figure 8. It can be seen from Figure 8 that the maximum ray angle which need be investigated (γ_Q) is as follows:

$$\tan \gamma_Q = \frac{C \tan \beta_o + (j - C) \tan \mu}{j} \tag{17}$$

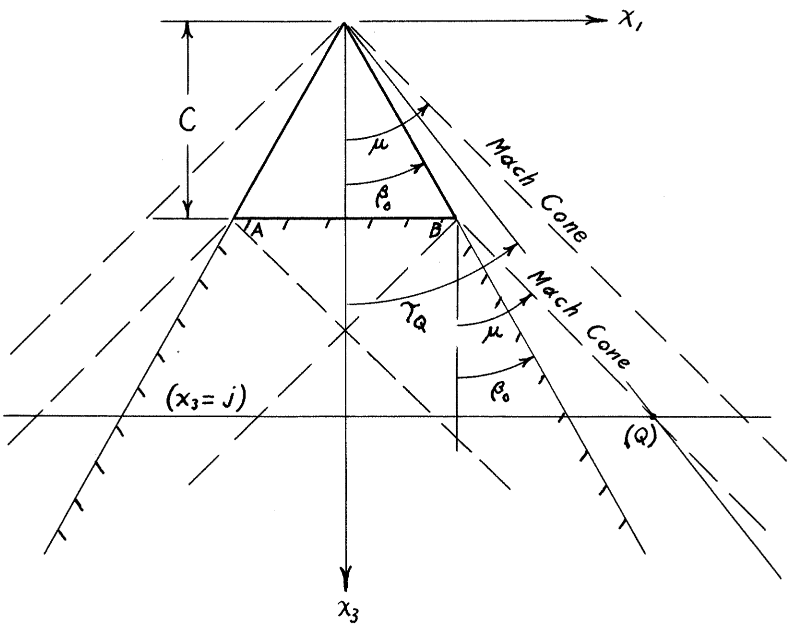


Figure 10
Point of Zero Downwash for

For the particular case being investigated, equation 17 reduces to

$$\tan \gamma_Q = \frac{\tan 30^\circ + \tan 45^\circ}{2} = .788$$

therefore

$$\gamma_Q = 38.25^\circ$$

and

$$\frac{\tan \tau_Q}{\tan \beta_0} = 1.365$$

By fairing a curve which is zero at $\frac{\tan \tau_Q}{\tan \beta_0} = 1.365$ and which approaches $\frac{\tan \tau}{\tan \beta_0} = 1.0$ as an asymptote, the downwash parameter is obtained qualitatively, see Graph III.

TABLE I

For $\tau = 5^\circ$
$\tan\phi = \tan\beta + .175$
$\tan\psi = \tan\beta - .175$

$\tan\beta$	$\tan\phi$	$\tan\psi$	(1) $G(-b,t_\phi)$	(2) $G(-b,t_\psi)$	(3) (1)+(2)	(4) -(1)-(3)	(5) Kdu_3/db	(6) (4)x(5) $f(b,t)$
0	.175	-.175	1.350	-2.315	- .965	- .035	0	0
.05	.225	-.125	.827	-3.555	-2.728	+1.728	.051	+ .087
.07	.245	-.105	.870	-5.170	-4.300	+3.300	.074	+ .244
.09	.265	-.085	.623	-10.260	-9.637	+8.637	.095	+ .820
.10	.275	-.075	.552	-4.200	-3.648	+2.648	.105	+ .277
.15	.325	-.025	.396	+2.840	+3.236	-4.236	.167	- .706
.20	.375	+.025	.294	3.085	3.379	-4.379	.242	- 1.062
.25	.425	.075	.225	1.500	1.725	-2.725	.342	- .931
.30	.475	.125	.168	.925	1.093	-2.093	.481	- 1.009
.35	.525	.175	.120	.631	.751	-1.751	.696	- 1.220
.40	.575	.225	.090	.461	.551	-1.551	1.066	- 1.656
.45	.625	.275	.068	.344	.412	-1.412	1.838	- 2.595
.50	.675	.325	.048	.261	.309	-1.309	4.000	- 5.240
.55	.725	.375	.031	.200	.231	-1.231	19.65	-24.20
.574	.749	.399	.030	.176	.206	-1.206	410.0	-495.0
.577	.752	.402	.029	.174	.203	-1.203	∞	$-\infty$

(1) $\tan\beta$	(2) $\tan\tau$	(3) $/(2)-(1)/$	(4) $ln(3)$	(5) $\left(\tan\tau/.577\right)^2$	(6) $(1-(5))^{3/2}$	(7) $-(\pi(6))^{-1}$	(8) (4)x(7)	(9) (6)above	(10) (9)-(8)
.05	.0875	.0375	-3.282	.0230	.9658	-.329	+1.081	.087	-.994
.07	↓	.0175	-4.050	↓	↓	↓	+1.332	.244	-1.088
.09	↓	.0025	-6.000	↓	↓	↓	+1.975	.820	-1.155
.10	↓	.0125	-4.380	↓	↓	↓	+1.441	.277	-1.164

$$\int_{.05}^{.1} h(b,\tau) = +.075$$

$$\int_0^{.574} f(b,t)\frac{du_3}{db}\,db = \int_0^{.05} f(b,t)\frac{du_3}{db}\,db + \int_{.05}^{.1}\left[f(b,t)\frac{du_3}{db} - h(b,\tau)\,db\right]db + \int_{.1}^{.574} f(b,t)\frac{du_3}{db}\,db + \int_{.05}^{.1} h(b,\tau)\,db$$

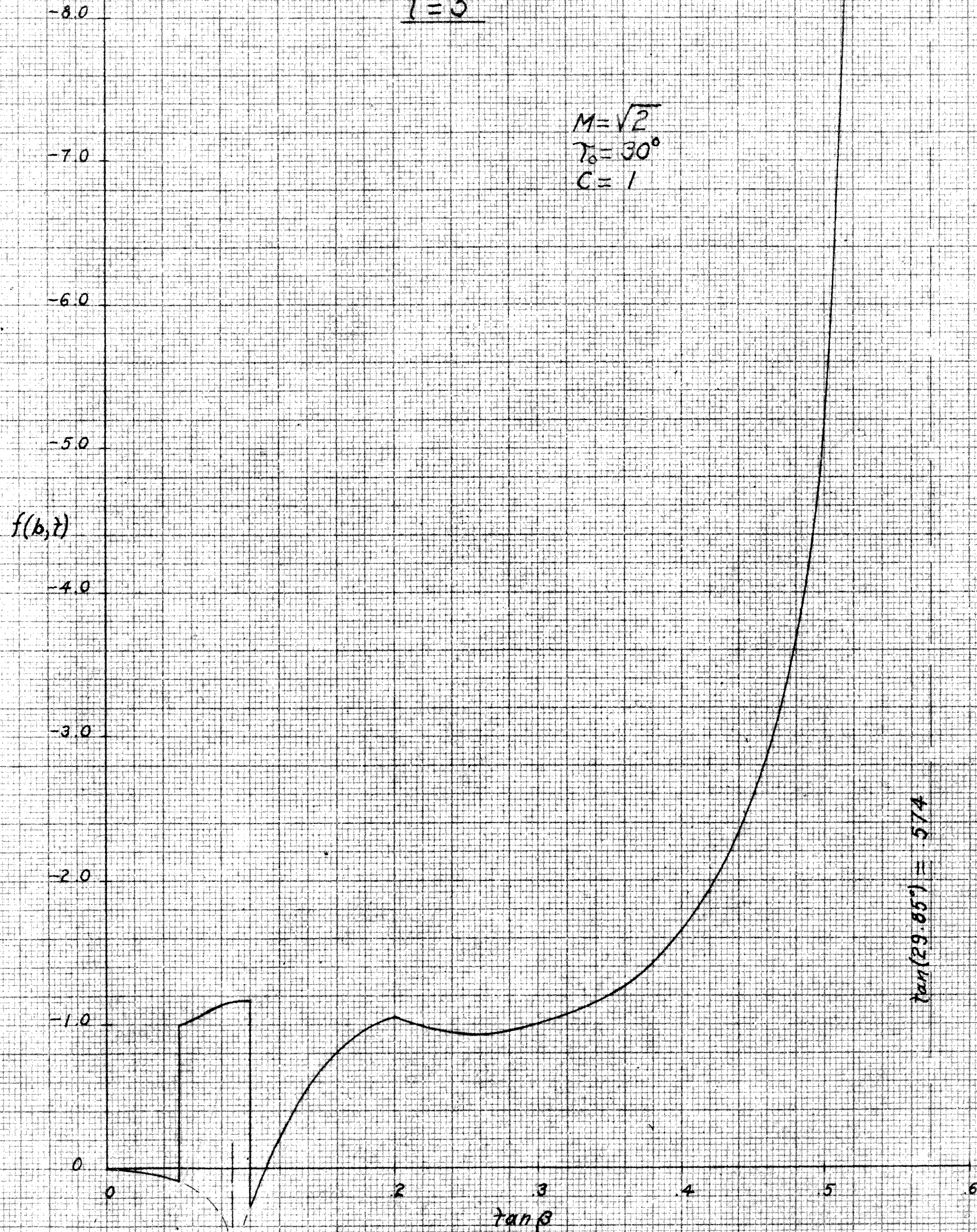
$$= -3.748 + .075 = -3.673$$

$$\frac{u_2(P)}{\alpha \bar{u}_\infty} = -1 - 4.573(-1.203) + 1.373(-3.673) = -.540$$

GRAPH I

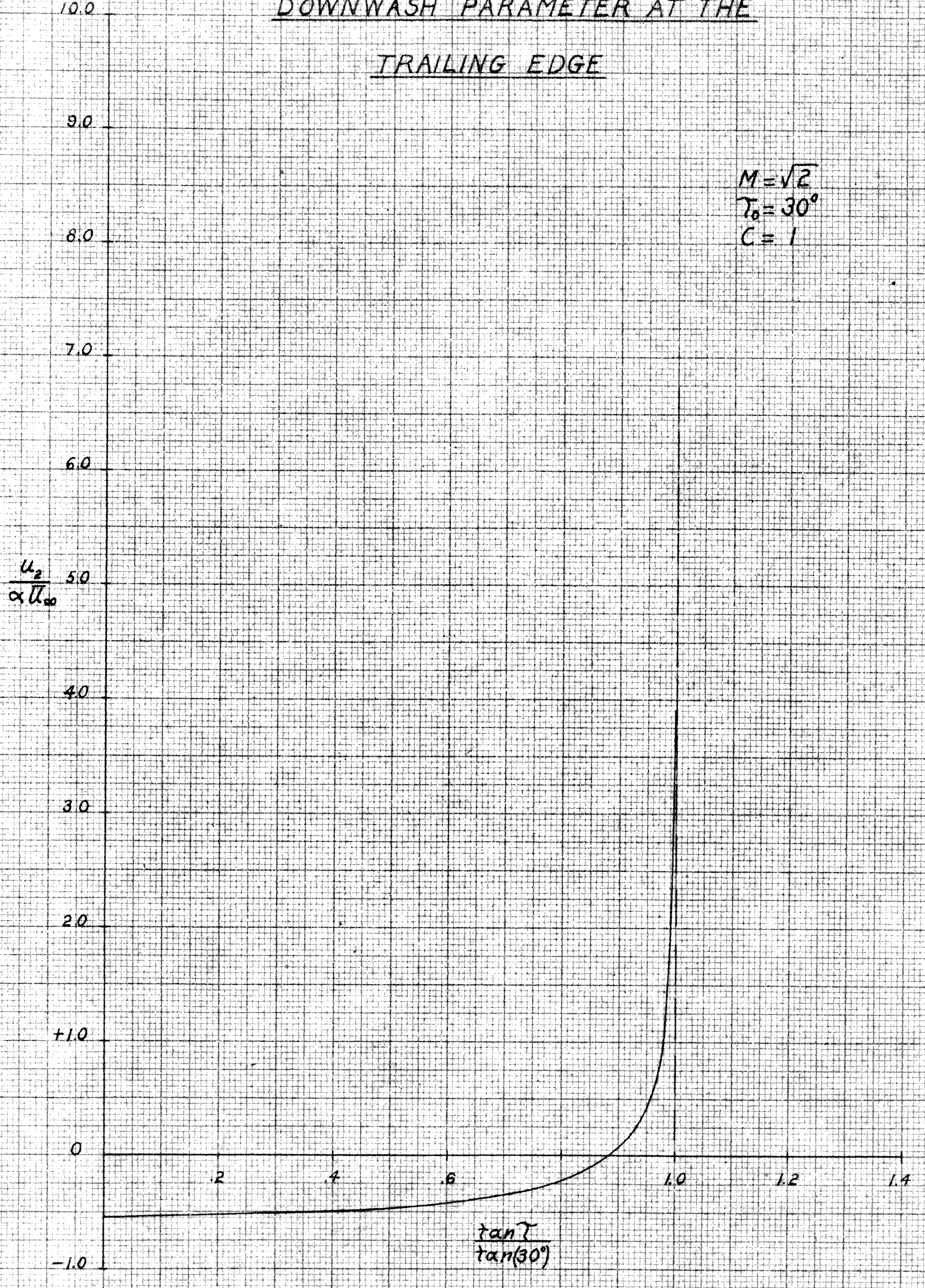
GRAPHICAL INTEGRATION PLOT

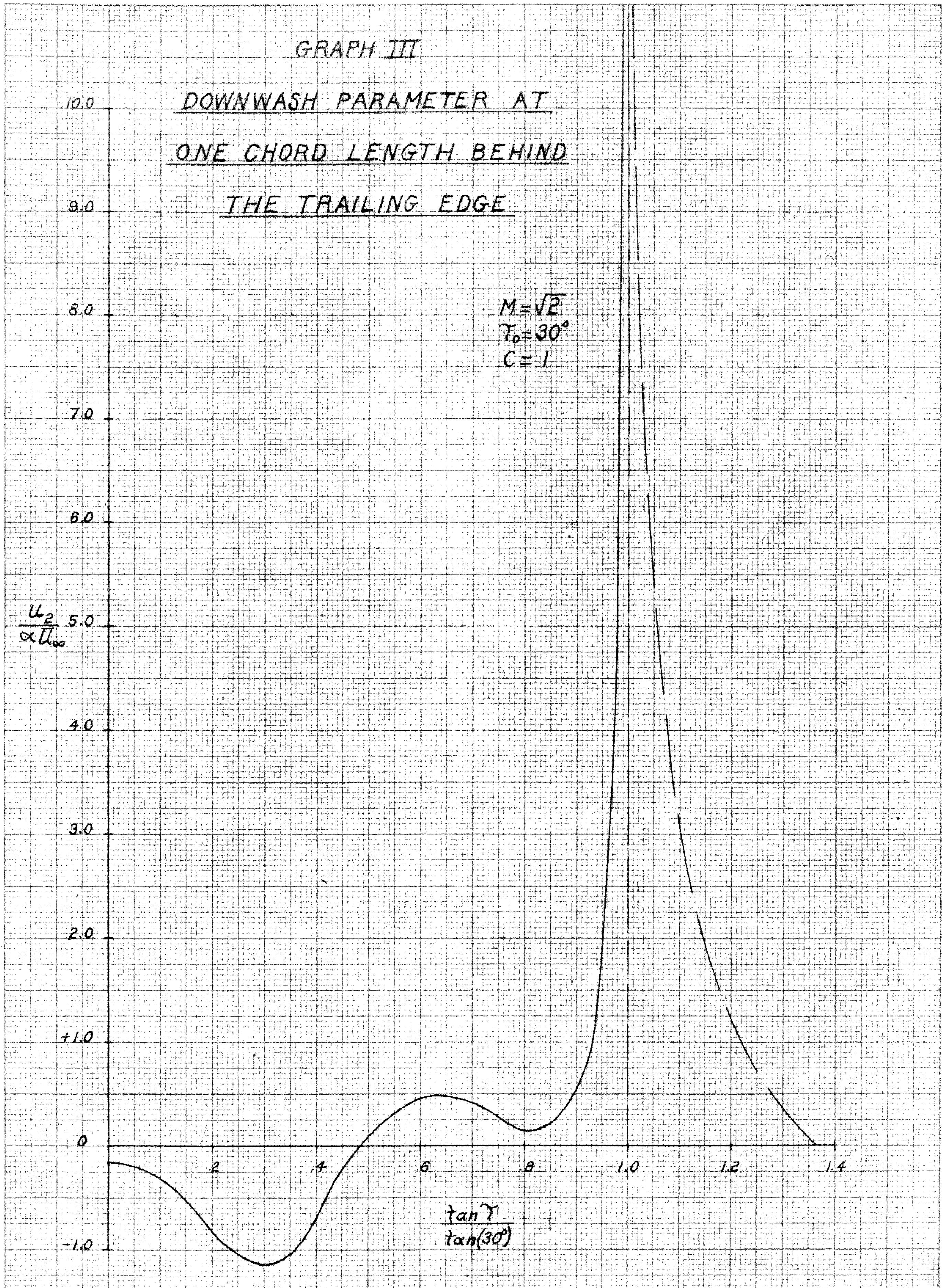
$\tau = 5°$

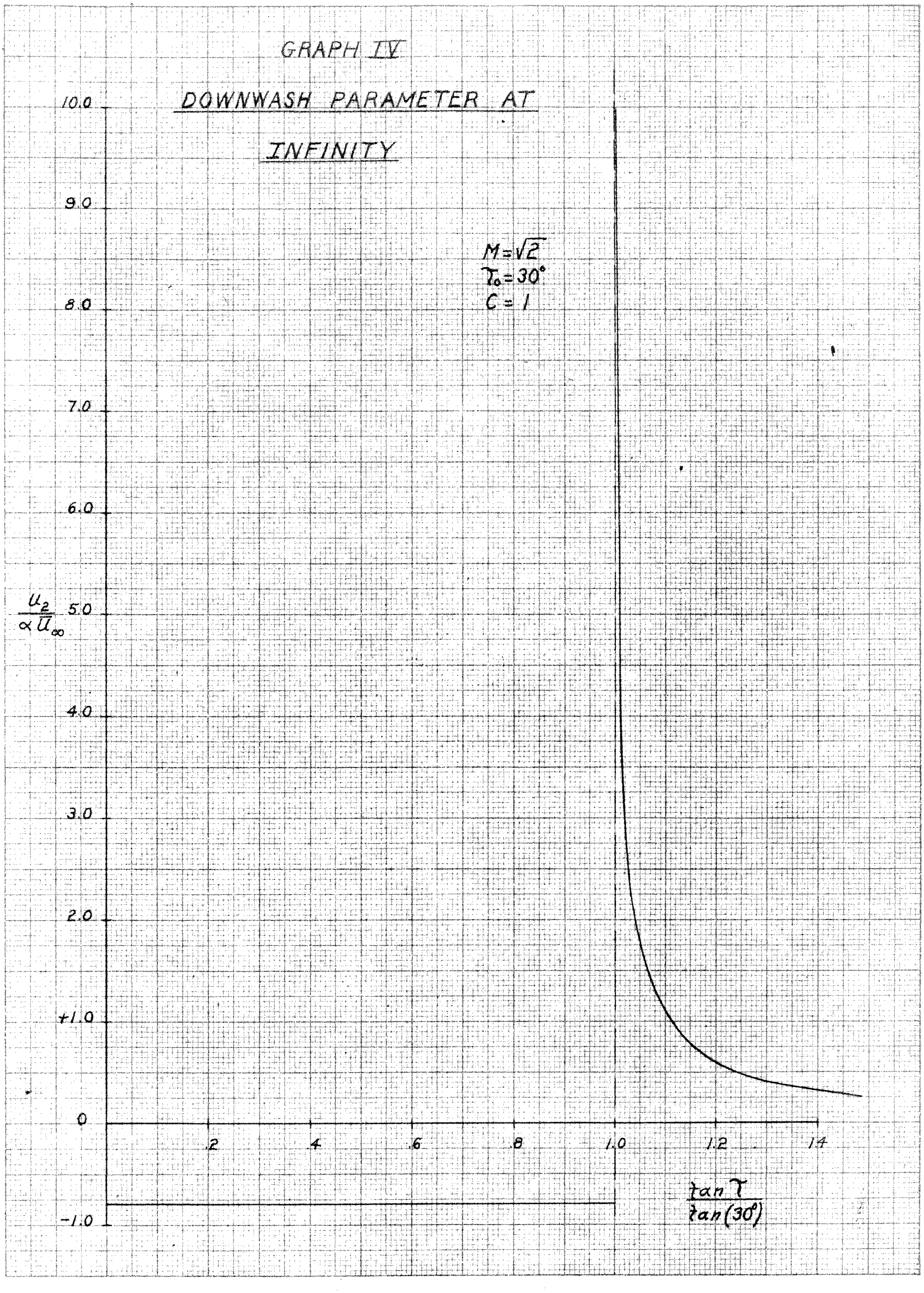


GRAPH II

DOWNWASH PARAMETER AT THE TRAILING EDGE




GRAPH III
DOWNWASH PARAMETER AT
ONE CHORD LENGTH BEHIND
THE TRAILING EDGE
$M=\sqrt{2}$
$\gamma_0=30°$
$C=1$
$\frac{u_2}{\alpha U_\infty}$
10.0
9.0
8.0
7.0
6.0
5.0
4.0
3.0
2.0
+1.0
0
-1.0
.2
.4
.6
.8
1.0
1.2
1.4
$\frac{\tan\gamma}{\tan(30°)}$


GRAPH IV
DOWNWASH PARAMETER AT INFINITY
$M=\sqrt{2}$
$\tau_0=30°$
$C=1$
$\frac{u_2}{\alpha \bar{U}_\infty}$
10.0
9.0
8.0
7.0
6.0
5.0
4.0
3.0
2.0
+1.0
0
-1.0
.2
.4
.6
.8
1.0
1.2
1.4
$\frac{\tan \tau}{\tan(30°)}$

X. DISCUSSION OF THE RESULTS

The results of the calculations are somewhat disappointing. Although the downwash distribution was determined by three different mathematical approaches for the three different chordwise stations, it seems that the results should be a little more consistent. Graph II shows that the distribution across the trailing edge is quite uniform. The same is true at infinity, see Graph IV. The results at these two stations are quite logical and can be accepted as being correct within the limitations of linearization. For the one chord length station; however, the results were quite different, see Graph III. Here, the distribution is quite erratic and does not appear to be compatible with the other stations. No particular reason for this behavior could be found. The desirability of having other chordwise stations for a more complete picture is quite evident. Qualitatively, the results for the one chord length station are correct in that the values as a whole are of the right order of magnitude and that the downwash approaches infinity at the wing tip ray. It is entirely possible that to the first order, the results are correct.

The best means of validating an expression is to compare experimental results with the predictions of the theory. The desirability of having the results of an experimental wake survey for the specific case is also evident.

XI. LIST OF REFERENCES

1. P. A. Lagerstrom, M. E. Graham, "Downwash and Sidewash Induced by Three Dimensional Lifting Wings in Supersonic Flow," Douglas Report No. SM-13007, 1947

2. H. J. Stewart, "The Lift of a Delta Wing at Supersonic Speeds," Quarterly of Applied Mathematics, October, 1946.