Branes, Brane Actions and Applications to Field Theory

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Abstract

This thesis describes the construction of supersymmetric world-volume actions for various kinds of extended objects that appear in string theory, the so-called p-branes, D-branes and M-branes. We also present an application of branes to computing the spectrum of a conformal field theory in the context of the AdS-CFT correspondence.

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Chapter 1

Introduction

When in 1984 Green and Schwarz [1] discovered the first quantum mechanically consistent theory of strings there was high hope that one will be able to formulate quantum gravity in terms of superstrings only. The advent of the "second superstring revolution"[1], however, showed the importance of higher dimensional objects, in particular of supersymmetric branes. We will try to introduce the reader to what, by now, is a huge subject and we will present - sometimes in detail - various approaches to building supersymmetric world-volume actions for various kind of branes. This work is organized as follows: in the introduction, we explain why branes were considered much less promising than strings and then we present the means by which they finally gained the status of essential ingredients for string theory. The second chapter summarizes various formalisms for superbrane actions, treating separately the so-called p-branes, D-branes and M-branes. The third chapter presents in detail the flat-space action for super-D-branes, following the work of [3]. The fourth chapter deals with the M-theory fivebrane as presented in [4]. Chapter five introduces the AdS-CFT correspondence and describes in detail an application to studying the spectrum of a conformal field theory [5]. We end with conclusions.

1.1 Strings and Branes

Field theories have been extremely successful in describing all interactions except gravity. The technical obstacle in writing a field theory of the graviton is non-renormalizability, which basically means bad ultraviolet behavior. This bad behavior is eliminated if one replaces point-like particles by minute[2] strings. This is what string

[1] *i.e.*, the discovery of D-branes, string dualities and M-theory, which all happened roughly in 1995.

[2] Of the order of 10^{-33} cm.

theory is about.

The classical bosonic string can be regarded as a generalization of the classical relativistic particle (see [6]). The action for the latter is proportional to the invariant length of the particle world-line

$$S = -m \int d\sigma^0 \sqrt{-g_{mn}(X) \frac{dX^m}{d\sigma^0} \frac{dX^n}{d\sigma^0}} \tag{1.1}$$

where m is the mass of the particle, $X^m(\sigma^0)$ are the spacetime coordinates (roman indices run from 0 to $d-1$, where d is the dimension of spacetime), $g_{mn}(X)$ is the metric[3], and σ^0 parametrizes the world-line. This can be generalized to a p-dimensional object by enlarging the set of world-volume coordinates to $\{\sigma^\mu\}$ (greek indices run from 0 to p) and using the formula for the volume swept out by the object

$$S = -T_{p+1} \int d^{p+1}\sigma \sqrt{-\det\, G_{\mu\nu}} \tag{1.2}$$

where T_{p+1} is a constant (the "tension" of the brane) and $G_{\mu\nu}$ is the metric induced on the world-volume

$$G_{\mu\nu} = g_{mn}(X)\; \partial_\mu X^m \partial_\nu X^n \tag{1.3}$$

We see that $p = 0$ gives back the action (1.1), while $p = 1$ is the so-called "Nambu-Goto" action for the string. Similarly, $p = 2$ describes the membrane, and for generic p one gets[4] the so-called "p-branes," or simply "branes."

The action (1.2) can be regarded as a $(p+1)$-dimensional field theory on the world-volume swept out by the brane. From this perspective, spacetime coordinates X^m are just a set of d scalar fields. Since $g_{mn}(X)$ is an arbitrary function of the X's, we have an interacting theory with arbitrarily high powers of the fields in the Lagrangian. Such a theory is power-counting non-renormalizable unless the mass dimension of the scalar fields is non-positive, *i.e.*, unless $p \leq 1$. Consequently, for $p > 1$, we're back to where we started! This is why for a long time branes were considered less fundamental than strings and of little use in building a quantum theory of gravity. We can choose to ban the branes as inconsistent fundamental objects and concentrate on the study

[3]We use the "mostly plus" convention, in which a time-like interval is negative, which explains the minus sign under the square root.

[4]This action was first proposed by Dirac in [7].

of strings only. However, as we will see in the following two sections, branes reappear in various ways in the context of the theory of strings.

1.2 Supergravity and Branes

One way to encounter higher dimensional branes in string theory is by looking at its low energy limit, which is (super)gravity theory. Supergravities (theories of gravity with supersymmetry) were studied extensively in the '80's as a separate subject which – while keeping in mind the string theoretical motivation – will be our context in this section.

The bosonic fields of typical supergravities are the metric (g_{mn}), a scalar field called the dilaton (ϕ) and antisymmetric tensors of various ranks ($A_{[p+1]}$). Supersymmetry requires an equal number of (physical) components in fermionic fields. We can consistently truncate the theory by setting the fermionic fields to zero, which leads to actions of the type

$$S = \int d^D X \sqrt{-g} \left(R - \frac{1}{2} \nabla_m \phi \nabla^m \phi - \frac{1}{2(p+2)!} e^{a\phi} F_{[p+2]}^2 \right) \tag{1.4}$$

Here a is a parameter controlling the interaction of the dilaton ϕ with the field strength $F_{[p+2]} = dA_{[p+1]}$. This action leads to the following set of equations of motion:

$$\begin{aligned} R_{mn} &= \frac{1}{2} \partial_m \phi \partial_n \phi + S_{mn} \\ S_{mn} &= \frac{1}{2(p+1)!} e^{a\phi} \left(F_{mq_0 \ldots q_p} F_n{}^{q_0 \ldots q_p} - \frac{p+1}{(p+2)(D-2)} F_{[p+2]}^2 \, g_{mn} \right) \\ 0 &= \nabla_m (e^{a\phi} F^{mq_0 \ldots q_p}) \\ \Box \phi &= \frac{a}{2(p+2)!} e^{a\phi} F_{[p+2]}^2 \end{aligned} \tag{1.5}$$

Our purpose in this section is to look for classical field configurations that solve these equations and have the symmetries of an infinite hyperplanar brane[5]. Therefore, we split the coordinates as follows: $X^m = (x^\mu, y^i)$, where x^μ ($\mu = 0, \ldots, p$) denote coordinates along the brane and y^i ($i = p+1, \ldots, D$) denote transverse coordinates.

[5]No actual brane, in the sense of the previous section, is introduced yet. However, we will refer to the defining hyperplane as the "brane."

Translational symmetry along the brane and rotational symmetry transverse to the brane require that all fields depend only on the transverse distance $r = \sqrt{y^i y^i}$. Our ansatz for the metric is then

$$ds^2 = e^{2A(r)}dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B(r)}dy^i dy^j \delta_{ij} \tag{1.6}$$

For the scalar field the ansatz is simply $\phi(X^m) = \phi(r)$. For the gauge field we have two possibilities: the first is to couple the brane "electrically"[6], *i.e.*, $A^{(el)}_{\mu_0 \dots \mu_p} = \epsilon_{\mu_0 \dots \mu_p} e^{C(r)}$ (other components zero), leading to a p-brane. In terms of the field strength the ansatz is:

$$F^{(el)}_{i\mu_0 \dots \mu_p} = \epsilon_{\mu_0 \dots \mu_p} \partial_i e^{C(r)}, \quad \text{other components zero} \tag{1.7}$$

The second possibility is a "magnetic" coupling, in which only transverse components of the field strength are non-zero, leading to a $(D-p-4)$-brane:

$$F^{(mag)}_{i_1 \dots i_{p+2}} = \lambda \epsilon_{i_1 \dots i_{p+2} j} \frac{y^j}{r^{p+3}}, \quad \text{others zero} \tag{1.8}$$

In the magnetic ansatz, the Bianchi identity $dF_{[p+2]} = 0$ constrains the r dependence, leaving only the freedom of an overall constant λ.

Solving the equations of motion (1.5) with these ansätze is still no easy task. However, since we are mostly interested in supersymmetric branes, we can further constrain the bosonic fields, because the supersymmetry variation of the gravitino has to vanish[7]. The extra constraints are linear relations among the functions $A(r), B(r), C(r)$ which leave us with a single arbitrary function in the ansatz[8]. If we denote by d the spacetime dimension of the brane and by $\tilde{d}$ the dimension of the dual brane[9], we arrive at the solution [10, 11]:

$$\begin{aligned} ds^2 &= H^{\frac{-4\tilde{d}}{\Delta(D-2)}} dx^\mu dx^\nu \eta_{\mu\nu} + H^{\frac{4d}{\Delta(D-2)}} dy^i dy^i \\ e^\phi &= H^{\pm\frac{2a}{\Delta}}, \quad \text{with } +(-) \text{ for the electric (magnetic) ansatz} \\ H(r) &= 1 + \frac{k}{r^{\tilde{d}}} \end{aligned} \tag{1.9}$$

[6]By analogy with the electric coupling of a point particle to the Maxwell field.
[7]For details see [8].
[8]This calculation is presented in detail in [9].
[9]For an electric brane $d = p + 1$, $\tilde{d} = D - p - 3$, and vice-versa for a magnetic brane.

where k is a parameter that sets the mass scale of the solution[10] and

$$\Delta = a^2 + \frac{2d\tilde{d}}{D-2} \tag{1.10}$$

In the electric case we also have

$$e^{C(r)} = \frac{2}{\sqrt{\Delta}} H^{-1} \tag{1.11}$$

while in the magnetic case

$$\lambda = \frac{2\tilde{d}}{\sqrt{\Delta}} k \tag{1.12}$$

A simple example is the membrane ($p = 2$) of $D = 11$ supergravity. The bosonic action of the latter reads [12]

$$S = \int d^{11}X\sqrt{-g}\left(R - \frac{1}{48}F_{[4]}^2\right) + \int \frac{1}{6} F_{[4]} \wedge F_{[4]} \wedge A_{[3]} \tag{1.13}$$

This is of the form (1.4) except for the FFA Chern-Simons term. However, it is easy to check that for both the "electric" and the "magnetic" ansätze (1.7,1.8) this term does not contribute to the equations of motion, so it can be safely ignored. Moreover, there is no scalar field present[11], so we can set $a = 0$. Then (1.10) gives $\Delta = 4$. Plugging in (1.9) we get the "electric" solution

$$\begin{aligned} ds^2 &= \left(1 + \frac{k}{r^6}\right)^{-2/3} dx^\mu dx^\nu \eta_{\mu\nu} + \left(1 + \frac{k}{r^6}\right)^{1/3} dy^m dy^m \\ A_{\mu\nu\lambda} &= \epsilon_{\mu\nu\lambda}\left(1 + \frac{k}{r^6}\right)^{-1}, \qquad \text{other components zero} \end{aligned} \tag{1.14}$$

This solution is similar to the Reissner-Nordstrom black-hole, having a degenerate[12] horizon at $r = 0$. Although the horizon looks singular in these coordinates, physical quantities (curvature, field strength) are finite at $r = 0$. In fact, the above coordinates do not cover spacetime completely, as one can see by setting $r = (\tilde{r}^6 - k)^{1/6}$. In these

[10] k must be positive to avoid having naked singularities at finite r.

[11] The dilaton is characteristic of string theories; strictly speaking, eleven-dimensional supergravity is not the low-energy limit of a string theory, but of the so-called "M-theory" (see also remark at the end of this section).

[12] *i.e.*, inner and outer horizon coinciding. Inside such a horizon the t axis remains time-like.

"Schwarzschild-type" coordinates the metric becomes

$$ds^2 = \left(1 - \frac{k}{\tilde{r}^6}\right)^{2/3} dx^\mu dx_\mu + \left(1 - \frac{k}{\tilde{r}^6}\right)^{-2} d\tilde{r}^2 + \tilde{r}^2 d\Omega_7^2 \tag{1.15}$$

The horizon is now at $\tilde{r} = k^{1/6}$ and we have added the interior region. There is a physical singularity at $\tilde{r} = 0$ to which we will come back shortly. The interesting property [13] of this solution is that it interpolates between two vacua of eleven-dimensional supergravity: the Minkowski vacuum (far away) and the Anti-de-Sitter (AdS) vacuum (near the horizon). To see this explicitly, it is useful to introduce the "interpolating coordinates" [13] by further setting $\tilde{r} = k^{1/6}(1 - R^3)^{1/6}$ which in terms of the original coordinates corresponds to setting

$$r = k^{1/6} \frac{R^{1/2}}{(1 - R^3)^{1/6}} \tag{1.16}$$

In these coordinates the metric becomes

$$ds^2 = R^2(1 - R^3)^{-2/3} dx^\mu dx_\mu + \frac{k^{1/3}}{4} R^{-2}(1 - R^3)^{1/3} dR^2 + k^{1/3}(1 - R^3)^{1/3} d\Omega_7^2 \tag{1.17}$$

The horizon is brought back to $R = 0$, the interior region is now $R < 0$, and the singularity is infinitely far away "down the throat" at $R = -\infty$. The near horizon geometry is obtained by neglecting the R^3 terms in the above equation, which gives

$$ds^2 = R^2 dx^\mu dx_\mu + \frac{k^{1/3}}{4} R^{-2} dR^2 + k^{1/3} d\Omega_7^2 \tag{1.18}$$

This is the metric of $AdS_4 \times S^7$ (see Appendix A of chapter 5). In the last chapter we will see how this leads to the remarkable AdS-CFT correspondence.

To conclude this section, let us return to the physical meaning of the singularity at $\tilde{r} = 0$. Unlike the Schwarzschild singularity, this singularity is *timelike, i.e.,* a delta function source. A more careful analysis [14] shows that this is precisely the "Dirac-Nambu-Goto" membrane that we mentioned in the previous section. How are we to intrepret the existence of such an object at the quantum level, given the non-renormalizability of the membrane action? Eleven-dimensional supergravity is also non-renormalizable, but we have reason to believe since 1995 that there is a well-defined eleven-dimensional quantum theory [16], called "M-theory," which has eleven-dimensional supergravity as its low energy limit. We do not know what the "fundamental" formulation of M-theory is, but we do know that once we compactify one dimension it reduces to Type IIA string theory. In fact, the fundamental string of Type IIA is just a wrapped membrane [15]! We will come back to this in section 2.4.1.

1.3 D-branes

The real breakthrough for branes in string theory came in 1995, when D-branes were understood [17]. In this section we attempt a very brief introduction to this huge subject oriented towards the determination of the world-volume action of a D-brane. For a thorough introduction the reader is referred to [18].

The first step in solving a string theory is to find classical solutions of equation (1.2) for $p = 1$ in flat background[13]:

$$S = -T_2 \int d^2\sigma \sqrt{-\det\, G_{\mu\nu}} \tag{1.19}$$

A useful trick is to write down a quadratic form of the action, by introducing a world-sheet metric $\gamma_{\mu\nu}$ [19]:

$$S = -\frac{T_2}{2} \int d^2\sigma \sqrt{-\det\, \gamma_{\mu\nu}}\; \gamma^{\sigma\tau} G_{\sigma\tau} \tag{1.20}$$

Solving the γ equation of motion one finds that $\gamma_{\mu\nu} = G_{\mu\nu}$ and one recovers the action (1.19). The two actions being classically equivalent, we do not expect extra degrees of freedom coming from $\gamma_{\mu\nu}$. In fact, the quadratic action has both reparametrization invariance (general coordinate invariance on the world-sheet) and local rescaling invariance (Weyl symmetry). Together they allow (at least locally) the gauge choice $\gamma_{\mu\nu} = \eta_{\mu\nu}$. The action (1.20) takes the simple form

$$S = -\frac{T_2}{2} \int d^2\sigma\; \partial_\mu X^m \partial^\mu X_m \tag{1.21}$$

[13] *i.e.*, $g_{mn} = \eta_{mn}$, the Minkowski metric.

with wave-like equations of motion

$$\partial_\mu \partial^\mu X^m = 0 \tag{1.22}$$

These equations have to be supplemented by appropriate boundary conditions. For closed strings the world-sheet is a cylinder, so there are no actual boundaries; in other words all fields are periodic in the space-like world-sheet coordinate $\sigma^1 \equiv \sigma$. For open strings, however, the world-sheet is a time-like infinite strip and we need to be more careful. The time-like coordinate $\sigma^0 \equiv \tau$ still runs from $-\infty$ to ∞, but the space-like coordinate σ is bounded, and we will choose the limits to be 0 and π.

The variation of the action is

$$\begin{aligned}\delta S &= -T_2 \int d^2\sigma \; \partial_\mu(\delta X^m)\partial^\mu X_m \\ &= T_2 \int d^2\sigma \; \delta X^m \partial_\mu \partial^\mu X_m - T_2 \int d\tau \;\; (\delta X^m X_m') \,|_{\sigma=0}^{\sigma=\pi}\end{aligned} \tag{1.23}$$

where the prime indicates a derivative with respect to the space-like coordinate σ. The first term vanishes by the equation of motion, while the boundary term vanishes if one of the following conditions is imposed at the ends of the string:

a. **Neumann boundary conditions** – $X_m' = 0$, *i.e.*, ends free;

b. **Dirichlet boundary conditions** – $\delta X^m = 0$, *i.e.*, ends fixed.

Neumann boundary conditions were favored until 1995, because there was no need to introduce another object to fix the ends of the string. In fact, this object exists and is called a D-brane. We will now show that it is a necessary ingredient in a theory of open strings.

The tool we need is T-duality. For pedagogical purposes it is convenient to start with the theory of closed strings in D spacetime dimensions. To simplify the formulas we introduce a fundamental length, which we then set to one:

$$1/\sqrt{\pi T_2} \equiv \sqrt{2\alpha'} \equiv l = 1 \tag{1.24}$$

where in passing we have introduced the quantity α', which is called the Regge slope. The general solution of the wave equation (1.22) is the sum of a left-moving and a right-moving part:

$$X_R^m(\tau-\sigma) = \frac{x^m}{2} + \alpha_0(\tau-\sigma) + \sum_{k\neq 0} \frac{\alpha_k^m}{k} e^{ik(\tau-\sigma)}$$
$$X_L^m(\tau+\sigma) = \frac{x^m}{2} + \tilde{\alpha}_0(\tau+\sigma) + \sum_{k\neq 0} \frac{\tilde{\alpha}_k^m}{k} e^{ik(\tau+\sigma)} \qquad (1.25)$$

Single-valuedness implies $\alpha_0^m = \tilde{\alpha}_0^m = p^m/2$. Here x^m and p^m are the coordinates and momenta of the center of mass[14].

Consider now compactifying a direction, say $X^{(D-1)}$, on a circle of radius R. The $(D-1)^{th}$ momentum is then quantized, $p^{(D-1)} = n/R$. On the other hand the string can wind around the compact direction

$$X^{(D-1)}(\tau, 2\pi) = X^{(D-1)}(\tau, 0) + 2\pi w R \qquad (1.26)$$

where the integer w is the winding number. These conditions give

$$\tilde{\alpha}_0^{(D-1)} + \alpha_0^{(D-1)} = n/R$$
$$\tilde{\alpha}_0^{(D-1)} - \alpha_0^{(D-1)} = wR \qquad (1.27)$$

These formulas are symmetrical under

$$R \leftrightarrow 1/R$$
$$n \leftrightarrow w \qquad (1.28)$$
$$\alpha_0^{(D-1)} \leftrightarrow -\alpha_0^{(D-1)}$$

This transformation, called "T-duality" [20, 21] relates a compactification on a circle of radius R to a compactification on a circle of radius $1/R$ with winding and momentum interchanged and a "parity" flip on the right movers only[15]. We have seen that the spectrum is unaffected, but it can be shown [22] that all interactions are the same, order by order in perturbation theory. This physically identifies the two compactifications (see discussion in [18]).

[14] The splitting of x^m into two equal parts is, of course, just a convention.

[15] We need to use the coordinate $X'^{(D-1)} \equiv X_L^{(D-1)} - X_R^{(D-1)}$ to describe the T-dual theory.

In the case of open strings the situation is still more interesting. Winding modes do not exist in this case, but one can easily check that Neumann boundary conditions for $X^{(D-1)}$ correspond to Dirichlet boundary conditions for $X'^{(D-1)}$. This means that in the T-dual theory we have a D-brane on which the dual string ends. The brane fills all directions but the compactified one, so it is a $(D-2)$-brane. Repeating the procedure we can get Dp-branes of arbitrary dimension $p+1$.

Studying the massless spectrum of open strings ending on a Dp-brane, we find a $(p+1)$-dimensional U(1) gauge field A_μ along the brane and $D-p-1$ scalars transverse to the brane. The scalars parametrize transverse fluctuations, which means that D-branes are dynamical objects. These scalar fields correspond to gauge field components of T-dual open string theories, the precise relationship being[16]

$$X'^i = 2\pi\alpha' A^i \tag{1.29}$$

We conclude this brief introduction to D-branes by computing the world-volume action. Consider a planar Dp-brane with constant gauge field $F_{\mu\nu}$. We can Lorentz transform it to bring $F_{\mu\nu}$ in canonical (two-by-two block-diagonal) form. We now concentrate on a two-by-two block, and call its coordinates X^1 and X^2. We choose the gauge $A_2 = X^1 F_{12}$, and then T-dualize along X^2. The second coordinate becomes transverse to the brane, and equation (1.29) implies that the brane is tilted,

$$X'^2 = 2\pi\alpha' X^1 F_{12} \tag{1.30}$$

Since the gauge field was dualized, the contribution to the action from directions 1, 2 is proportional to the length of the remaining D-string,

$$\int dX^1\ \sqrt{1+(\partial_1 X'^2)^2} = \int dX^1\ \sqrt{1+(2\pi\alpha' F_{12})^2} \tag{1.31}$$

Putting the blocks together and restoring Lorentz covariance, we arrive at the Born-Infeld action [23]

$$S = -T\int d^{p+1}\sigma\sqrt{-\det\,(G_{\mu\nu}+2\pi\alpha' F_{\mu\nu})} \tag{1.32}$$

[16]For details see [18].

For a more general (*i.e.*, curved, gauge field not constant) D-brane, this formula gives the leading contribution as long as fields do not vary appreciably on the scale $l \equiv \sqrt{2\alpha'}$. For the rest of the thesis, we will set $2\pi\alpha' = 1$.

Chapter 2

Supersymmetric Brane Actions

In this chapter we will review the construction of world-volume supersymmetric actions for various kind of branes, restricting our analysis to trivial[1] background, using the formalism of [24, 3, 4]. We will see that for p-branes this is a relatively simple task, while for D-branes and especially for the $M5$-brane it becomes significantly more complicated. This partly explains why the latter were first written down almost a decade after the former. In fact, the action for the five-brane of M-theory is a challenge even at the bosonic level; therefore, we will postpone its supersymmetrization until chapter 4.

2.1 Supersymmetric p-branes

The first step in finding the supersymmetric extension of the p-brane action (1.2) is to replace the spacetime coordinates X^m by superspace coordinates

$$Z^M = (X^m, \theta^\alpha) \tag{2.1}$$

As discussed above, we assume that superspace is flat. Thus, in particular, the Dirac gamma matrices for the appropriate number of dimensions obey

$$\{\Gamma^m, \Gamma^n\} = 2\eta^{mn}, \quad \text{where } \eta = (- + + \ldots +) \tag{2.2}$$

The coordinate supersymmetry transformations are

$$\delta_\epsilon \theta = \epsilon, \qquad \delta_\epsilon X^m = \bar{\epsilon}\Gamma^m \theta \tag{2.3}$$

[1] *i.e.*, flat space and vanishing antisymmetric background fields.

where ϵ is a constant infinitesimal spinor. We want to build the Lagrangian out of supersymmetric quantities. These are $\partial_\mu \theta$ and

$$\Pi_\mu^m = \partial_\mu X^m - \bar{\theta}\Gamma^m \partial_\mu \theta \tag{2.4}$$

The supersymmetric generalization of the induced world-volume metric (1.3) is

$$G_{\mu\nu} = \eta_{mn} \Pi_\mu^m \Pi_\nu^n \tag{2.5}$$

Many subsequent formulas are written more concisely using differential form notation. In doing this one has to be careful about minus signs when Grassmann variables appear. The basic rule that we use is that $d = d\sigma^\mu \partial_\mu$ and $d\sigma^\mu$ is regarded as an odd element of the Grassmann algebra. Thus, for example,

$$d\theta = d\sigma^\mu \partial_\mu \theta = -\partial_\mu \theta d\sigma^\mu \tag{2.6}$$

There are various possible conventions that would be consistent. Ours, while perhaps not the most common, is convenient. Taking θ to anticommute with $d\sigma^\mu$ allows us to keep track of just one (combined) grading instead of two[2]. In this notation, the two supersymmetric one-forms are $d\theta$ and $\Pi^m = dX^m + \bar{\theta}\Gamma^m d\theta$. We also have

$$d\Pi^m = d\bar{\theta}\Gamma^m d\theta \tag{2.7}$$

Wedge products are always implicit in our formulas.

The generalization of the "Dirac" action (1.2) is

$$S_1 = -T_{p+1} \int d^{p+1}\sigma \sqrt{-\mathrm{det} G_{\mu\nu}} \tag{2.8}$$

This is not the full action, though. To find other supersymmetric terms that can enter the Lagrangian, it is useful to consider the following rescaling[3] [24]

$$X^m \to \lambda X^m \quad \theta \to \lambda^{\frac{1}{2}} \theta^\alpha \quad \Pi^m \to \lambda \Pi^m \tag{2.9}$$

[2] For example, $d\theta$ is always even.

[3] This is just a convenient way to do dimensional analysis.

The action (2.8) scales as

$$S_1 \rightarrow \lambda^{(p+1)} S_1 \tag{2.10}$$

All terms that scale with higher power of λ cannot be relevant for the fundamental p-brane, which is assumed to be a structureless object. We can consider, however, adding a "Wess-Zumino" term that scales like (2.10). This term is supersymmetric up to a total derivative and can be conveniently expressed [25] in terms of a closed $(p+2)$ form in superspace

$$I_{p+2} = d\Omega_{p+1}, \qquad S_2 = T_{p+1} \int \Omega_{p+1} \equiv T_{p+1} \int d^{p+1}\sigma \; \epsilon^{\mu_0 \ldots \mu_p} \Omega_{\mu_0 \ldots \mu_p}{}^4 \tag{2.11}$$

This has the advantage that I_{p+2} is typically a much simpler and more symmetrical expression than Ω_{p+1} and it can be constructed entirely from the supersymmetry invariants $d\theta$ and Π^m. Thus the variation $\delta_\epsilon \Omega_{p+1}$ is closed and – assuming trivial cohomology – also exact, which implies the invariance of S_2 up to an unimportant[5] boundary term.

In order to obtain the correct scaling (2.10) we need an I_{p+2} of the form

$$I_{p+2} = (-1)^{p+1} \frac{1}{p!} \Pi^{m_1} \ldots \Pi^{m_p} (d\bar{\theta} \Gamma_{m_1 \ldots m_p} d\theta) \tag{2.12}$$

where the coefficient is written down in anticipation of the kappa symmetry analysis (see next section). This is nonvanishing only if $(\Gamma_0 \Gamma_{m_1 \ldots m_p})_{\alpha\beta}$ is symmetric in spinor indices, which restricts the values of d and p. Furthermore, since I_{p+2} is closed we need to have

$$(d\bar{\theta} \Gamma^{m_1} d\theta)(d\bar{\theta} \Gamma_{m_1 \ldots m_p} d\theta) = 0 \tag{2.13}$$

This is a generalization of the well-known identity

$$\Gamma^m d\theta (d\bar{\theta} \Gamma_m d\theta) = 0 \tag{2.14}$$

[4]Strictly speaking, the Ω appearing in S_2 is the pullback to the world-volume of the superspace form, but we do not make a notational distinction between superspace quantities and their world-volume pullbacks.

[5]The boundary term vanishes for cases of interest, which include infinite branes or branes wrapped around cycles.

which is true in $d = 3, 4, 6$ and 10 dimensions, and is a necessary condition for the existence of super Yang-Mills theories in precisely these dimensions. In our context, it implies (and is actually equivalent to) the $p = 1$ case of (2.13), which corresponds to the superstring of Green and Schwarz [1].

Analyzing (2.13) we find the necessary and sufficient condition[6]

$$d - p - 1 = \frac{n}{4} \equiv \frac{mN}{4} \qquad (p > 1) \tag{2.15}$$

where m is the real dimension of the (possibly chiral) spinor representation in d dimensions (given in table 2.2 below) and N the number of supersymmetries. In the next section we will see that (2.15) is precisely the condition for world-volume supersymmetry. It gives the so called "old brane-scan," which is represented in table 2.1.

2.2 Local Kappa Symmetry

Now that we have the two terms that have the correct scaling (2.10), let us see how they link together into the supersymmetric p-brane action. In order to do this, we need to understand the symmetries of the latter. Both S_1 and S_2 are manifestly invariant under world-volume general coordinate transformations and spacetime supersymmetry. What about world-volume supersymmetry? The simplest check is to count world-volume bosonic and fermionic physical degrees of freedom and see if they match.

To get rid of unphysical degrees of freedom, we go to the so-called "static gauge"

$$X^\mu = \sigma^\mu, \qquad \mu = 0, \ldots, p \tag{2.16}$$

This fixes general coordinate symmetry and leaves $d-p-1$ bosonic degrees of freedom $\phi^i = X^i$, $i = p+1, \ldots d$. On the other hand, the number of world-volume fermionic degrees of freedom is naively given by the following table:

[6] For details see [24].

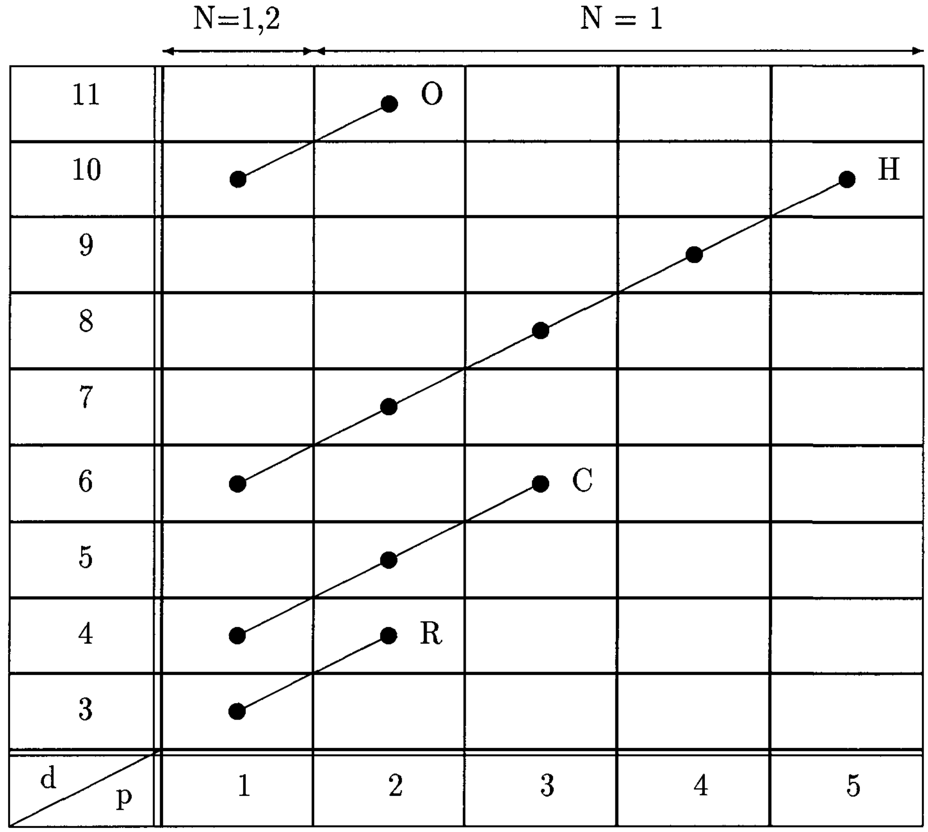

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Table 2.1: The "brane-scan," composed of four sequences usually labelled R,C,H,O, by analogy with the real, complex, quaternionic and octonionic fields.

We can easily check that the number of world-volume bosonic and fermionic degrees of freedom do not match. In fact, for the values of d and p that appear in the brane scan, the number of fermionic degrees of freedom seems to equal twice the number of bosonic degrees of freedom. This situation is familiar from the superparticle [26] and the Green-Schwarz string [1], and the resolution turns out to be the same: there is a local fermionic symmetry, called "kappa symmetry," which makes half of the fermionic degrees of freedom unphysical. Once we fix it, we are left with a world-volume supersymmetric theory. Indeed, imposing the Dirac equation leaves us with $n/2$ "on-shell" fermionic degrees of freedom which are further reduced to $n/4$ by fixing kappa symmetry. These have to match the $d-p-1$ bosonic degrees of freedom, which is precisely condition (2.15).

We will now establish kappa symmetry of the action $S_1 + S_2$. Leaving the kappa

=

Dimension d	11	10	9	8	7	6	5	4	3	2
Minimal Spinor m	32	16	16	16	16	8	8	4	2	1

(2.17)

Table 2.2: Minimal real dimension of (possibly chiral) spinors in d dimensions.

symmetry variation $\delta\theta$ undetermined for the moment, we require that

$$\delta X^m = \bar{\theta}\Gamma^m \delta\theta = -\delta\bar{\theta}\Gamma^m\theta \tag{2.18}$$

It follows that

$$\delta\Pi^m_\mu = -2\delta\bar{\theta}\Gamma^m\partial_\mu\theta \tag{2.19}$$

Equivalently, one can write

$$\delta\Pi^m = 2\delta\bar{\theta}\Gamma^m d\theta \tag{2.20}$$

Another useful definition is the "induced γ matrix"

$$\gamma_\mu \equiv \Pi^m_\mu \Gamma_m \tag{2.21}$$

Note that $\{\gamma_\mu, \gamma_\nu\} = 2G_{\mu\nu}$. These formulas imply that

$$\delta G_{\mu\nu} = -2\delta\bar{\theta}(\gamma_\mu\partial_\nu + \gamma_\nu\partial_\mu)\theta \tag{2.22}$$

Now let's consider a kappa transformation of S_1 using

$$\delta L_1 = \delta\left(-\sqrt{-\det G}\right) = -\frac{1}{2}\sqrt{-\det G}\ \mathrm{tr}(G^{-1}\delta G) = 2\sqrt{-\det G}\ \delta\bar{\theta}\gamma_\mu G^{\mu\nu}\partial_\nu\theta \tag{2.23}$$

We will set up the proof so that it easily generalizes to the case of D-branes. Proceed by rewriting this variation in the form

$$\delta L_1 = 2\delta\bar{\theta}\gamma^{(p)}T^\nu_{(p)}\partial_\nu\theta \tag{2.24}$$

where

$$(\gamma^{(p)})^2 = 1 \tag{2.25}$$

It is not yet obvious that this is possible. The proof that it is, and the simultaneous determination of $\gamma^{(p)}$ and $T^{\nu}_{(p)}$, will emerge as we proceed.

It is useful to define

$$\rho^{(p)} = \sqrt{-\det G}\gamma^{(p)} \tag{2.26}$$

so that eq. (2.25) becomes

$$(\rho^{(p)})^2 = -\det G \tag{2.27}$$

The requirement

$$\sqrt{-\det G}\ \gamma_\mu G^{\mu\nu} = \gamma^{(p)} T^{\nu}_{(p)} \tag{2.28}$$

can then be recast in the more convenient form

$$\rho^{(p)}\gamma_\mu = T^{\nu}_{(p)} G_{\nu\mu} \tag{2.29}$$

It is also useful to represent $\rho^{(p)}$ in terms of an antisymmetric tensor

$$\rho^{(p)} = \frac{1}{(p+1)!}\epsilon^{\mu_1\mu_2\ldots\mu_{p+1}}\rho_{\mu_1\mu_2\ldots\mu_{p+1}} \tag{2.30}$$

or, equivalently, by a $(p+1)$-form

$$\rho_{p+1} = \frac{1}{(p+1)!}\rho_{\mu_1\mu_2\ldots\mu_{p+1}} d\sigma^{\mu_1} d\sigma^{\mu_2} \ldots d\sigma^{\mu_{p+1}} \tag{2.31}$$

Similarly, the vector $T^{\nu}_{(p)}$ can be represented by an antisymmetric tensor

$$T^{\nu}_{(p)} = \frac{1}{p!}\epsilon^{\nu_1\nu_2\ldots\nu_p\nu} T_{\nu_1\nu_2\ldots\nu_p} \tag{2.32}$$

or, equivalently, by a p-form

$$T_p = \frac{1}{p!} T_{\nu_1\nu_2\ldots\nu_p} d\sigma^{\nu_1} \ldots d\sigma^{\nu_p} \tag{2.33}$$

In order to achieve kappa symmetry, we require that

$$\delta L_2 = 2\delta\bar{\theta} T^{\nu}_{(p)} \partial_\nu \theta \tag{2.34}$$

so that adding eq. (2.24) gives

$$\delta(L_1 + L_2) = 2\delta\bar{\theta}(1 + \gamma^{(p)}) T^{\nu}_{(p)} \partial_\nu \theta \tag{2.35}$$

Since $\frac{1}{2}(1 \pm \gamma^{(p)})$ are projection operators, $\delta\bar{\theta} = \bar{\kappa}(1 - \gamma^{(p)})$ gives the desired symmetry. In terms of differential forms, the kappa variation of S_2 is

$$\delta S_2 = 2(-1)^{p+1} \int \delta\bar{\theta} T_p d\theta = \delta \int \Omega_{p+1} \tag{2.36}$$

The preceding formula and the definition $I_{p+2} = d\Omega_{p+1}$ imply that

$$\delta I_{p+2} = 2(-1)^{p+1} d\Big(\delta\bar{\theta} T_p d\theta\Big) = 2(-1)^{p+1}\Big(\delta d\bar{\theta} T_p d\theta - \delta\bar{\theta} dT_p d\theta\Big) \tag{2.37}$$

This equation is solved by

$$I_{p+2} = (-1)^{p+1} d\bar{\theta} T_p d\theta \tag{2.38}$$

if we can show that

$$d\bar{\theta}\delta T_p d\theta + 2\delta\bar{\theta} dT_p d\theta = 0 \tag{2.39}$$

A corollary of this identity is the closure condition $dI_{p+2} = d\bar{\theta} dT_p d\theta = 0$.

If we define the matrix-valued one-form

$$\not{\psi} \equiv \gamma_\mu d\sigma^\mu = \Pi^m \Gamma_m \tag{2.40}$$

then the solution of eqs. (2.27) and (2.29) can be written compactly as

$$\rho_{p+1} = \frac{1}{(p+1)!} \not{\psi}^{p+1} \quad \text{and} \quad T_p = \frac{1}{p!} \not{\psi}^p \tag{2.41}$$

Indeed, one can easily check that (2.39) is also satisfied. We therefore recover (2.12), confirming its coefficient. This completes the proof of kappa symmetry and the construction of super p-brane actions.

2.3 Supersymmetric D-brane Actions

We can repeat the above procedure for the case of D-branes. Some of the proofs are more involved, and we postpone them until chapter 3. However, we briefly list the results here, because the following section assumes some familiarity with them.

We will study the super Dp-branes of type IIA (p even) and IIB (p odd) string theory. These theories both have $d = 10$, so we need 32×32 Dirac matrices Γ^m. We can also define $\Gamma_{11} = \Gamma_0\Gamma_1 \ldots \Gamma_9$, which satisfies $\{\Gamma_{11}, \Gamma^m\} = 0$ and $(\Gamma_{11})^2 = 1$. Type IIA has a 32 component Majorana spinor θ which can be decomposed as $\theta = \theta_1 + \theta_2$, where

$$\theta_1 = \frac{1}{2}(1 + \Gamma_{11})\theta, \qquad \theta_2 = \frac{1}{2}(1 - \Gamma_{11})\theta \tag{2.42}$$

These are Majorana–Weyl spinors of opposite chirality. For Type IIB, there are two Majorana–Weyl spinors θ_α ($\alpha = 1, 2$) of the same chirality. The index α will not be displayed explicitly. The group that naturally acts on it is SL(2,R), whose generators we denote by Pauli matrices τ_1, τ_3. (We will mostly avoid using $i\tau_2$, which corresponds to the compact generator.)

Once again, the world-volume action is given by a sum of two terms, $S = S_1 + S_2$. The first term is the supersymmetric generalization of the Dirac-Born-Infeld action (1.32):

$$S_1 = -\int d^{p+1}\sigma\sqrt{-\det(G_{\mu\nu} + \mathcal{F}_{\mu\nu})} \tag{2.43}$$

where for convenience we have set the tension to one. The supersymmetrization of the field strength in the above is given by

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu} - b_{\mu\nu} \tag{2.44}$$

where b is a two form written in terms of fermions only, whose supersymmetric variation is *exact*, thus allowing one to define the supersymmetry variation of A such that $\mathcal{F}$ is invariant[7]. The second term is a "Wess-Zumino" term similar to (2.11). Using

[7] For details see section 3.1.

the notation of the previous section, we find that for p even, corresponding to type IIA, we can introduce the following formal sums of differential forms:

$$\rho_A = \sum_{p=\text{even}} \rho_{p+1} \quad \text{and} \quad T_A = \sum_{p=\text{even}} T_p \tag{2.45}$$

Then the solution is described by the formulas

$$\rho_A = e^{\mathcal{F}} S_A(\psi) \quad \text{and} \quad T_A = e^{\mathcal{F}} C_A(\psi) \tag{2.46}$$

where

$$S_A(\psi) = \Gamma_{11}\psi + \frac{1}{3!}\psi^3 + \frac{1}{5!}\Gamma_{11}\psi^5 + \frac{1}{7!}\psi^7 + \dots \tag{2.47}$$

$$C_A(\psi) = \Gamma_{11} + \frac{1}{2!}\psi^2 + \frac{1}{4!}\Gamma_{11}\psi^4 + \frac{1}{6!}\psi^6 + \dots \tag{2.48}$$

Thus,

$$\rho_1 = \Gamma_{11}\psi, \quad \rho_3 = \frac{1}{6}\psi^3 + \mathcal{F}\Gamma_{11}\psi, \quad \rho_5 = \frac{1}{120}\Gamma_{11}\psi^5 + \frac{1}{6}\mathcal{F}\psi^3 + \frac{1}{2}\mathcal{F}^2\Gamma_{11}\psi, \quad \text{etc.} \tag{2.49}$$

and

$$T_0 = \Gamma_{11}, \quad T_2 = \frac{1}{2}\psi^2 + \mathcal{F}\Gamma_{11}, \quad T_4 = \frac{1}{24}\Gamma_{11}\psi^4 + \frac{1}{2}\mathcal{F}\psi^2 + \frac{1}{2}\mathcal{F}^2\Gamma_{11}, \quad \text{etc.} \tag{2.50}$$

Separating positive chirality (θ_1) and negative chirality (θ_2) subspaces, ρ_A and T_A can be rewritten as 2×2 matrices

$$\rho_A = e^{\mathcal{F}} \begin{pmatrix} 0 & \sinh\psi \\ -\sin\psi & 0 \end{pmatrix} \tag{2.51}$$

and

$$T_A = e^{\mathcal{F}} \begin{pmatrix} \cosh\psi & 0 \\ 0 & -\cos\psi \end{pmatrix} \tag{2.52}$$

The solution for p odd (Type IIB) is very similar. In this case we define (the subscript B denotes IIB)

$$\rho_B = \sum_{p=\mathrm{odd}} \rho_{p+1} \quad \text{and} \quad T_B = \sum_{p=\mathrm{odd}} T_p \tag{2.53}$$

The solution is given by

$$\rho_B = e^{\mathcal{F}} C_B(\psi)\tau_1 \quad \text{and} \quad T_B = e^{\mathcal{F}} S_B(\psi)\tau_1 \tag{2.54}$$

where

$$S_B(\psi) = \psi + \frac{1}{3!}\tau_3\psi^3 + \frac{1}{5!}\psi^5 + \frac{1}{7!}\tau_3\psi^7 + \dots \tag{2.55}$$

$$C_B(\psi) = \tau_3 + \frac{1}{2!}\psi^2 + \frac{1}{4!}\tau_3\psi^4 + \frac{1}{6!}\psi^6 + \dots \tag{2.56}$$

Thus

$$\rho_2 = \frac{1}{2}\tau_1\psi^2 + i\tau_2\mathcal{F}, \quad \rho_4 = \frac{1}{24}i\tau_2\psi^4 + \frac{1}{2}\tau_1\mathcal{F}\psi^2 + \frac{1}{2}i\tau_2\mathcal{F}^2, \quad \text{etc.} \tag{2.57}$$

and

$$T_1 = \tau_1\psi, \quad T_3 = \frac{1}{6}i\tau_2\psi^3 + \tau_1\mathcal{F}\psi, \quad \text{etc.} \tag{2.58}$$

Displaying the Pauli matrices explicitly, ρ_B and T_B can be rewritten as 2×2 matrices

$$\rho_B = e^{\mathcal{F}} \begin{pmatrix} 0 & \cosh\psi \\ -\cos\psi & 0 \end{pmatrix} \tag{2.59}$$

and

$$T_B = e^{\mathcal{F}} \begin{pmatrix} 0 & \sinh\psi \\ \sin\psi & 0 \end{pmatrix} \tag{2.60}$$

The proofs for the above formulas will be given in chapter 3.

2.4 Super M-branes

As we have seen in the introduction, eleven-dimensional supergravity (and hence M-theory at least in the low energy limit) admits two types of branes: an “electric” membrane and a "magnetic" fivebrane. The membrane is a regular p-brane and thus appeared in the brane scan as $d = 11, p = 2$. We will write down its action explicitly in the following subsection and show that double dimensional reduction leads [15] to the type IIA superstring, while world-volume duality [36, 37] relates it to the $D2$-brane. The rest of this section is devoted to the construction of the bosonic world-volume action for the fivebrane, which in itself is a nontrivial task.

2.4.1 The Super-membrane

Since we want to study the double-dimensional reduction of the membrane to the fundamental string of type IIA, it is useful to set up a notation which distinguishes eleven-dimensional indices from their ten-dimensional counterparts. We will therefore denote the eleven-dimensional coordinates by X^M (capital index). The index M takes the values $M = 0, 1, \ldots, 9, 11$. Skipping $M = 10$ may seem a bit peculiar, but then X^{11} is the eleventh dimension. Also, the Dirac matrix $\Gamma_{11} = \Gamma_0\Gamma_1 \ldots \Gamma_9$, which appears in ten dimensions as a chirality operator, is precisely the matrix we associate with the eleventh dimension. The fermionic coordinates θ are 32-component Majorana spinors. We also denote membrane indices by $\hat{\mu}, \hat{\nu}, \ldots = 0, 1, 2$ reserving un-hatted indices for the string $(\mu, \nu, \ldots = 0, 1)$. The ten-dimensional type IIA metric is denoted by $g_{\mu\nu}$.

The "Dirac" term in the action is given by (2.8):

$$S_1 = -T_3 \int d^3\sigma \sqrt{-\det G_{\hat{\mu}\hat{\nu}}} \tag{2.61}$$

while the Wess-Zumino term is characterized by the form (2.12):

$$I_4 = -\frac{1}{2}(d\bar{\theta}\Gamma_{MN}d\theta)\Pi^M\Pi^N \tag{2.62}$$

A simple calculation based upon the gamma-matrix identity (2.13) shows that the corresponding Wess-Zumino action is

$$S_2 = -\int \bar{\theta}\Gamma_{MN}d\theta \left(\frac{1}{2}\Pi^M\Pi^N - \frac{1}{2}\bar{\theta}\Gamma^M d\theta \Pi^N - \frac{1}{6}\bar{\theta}\Gamma^M d\theta\ \bar{\theta}\Gamma^N d\theta\right) \tag{2.63}$$

Consider now compactifying the eleventh dimension and wrapping the membrane once around the compact dimension. For simplicity, we take the compact direction to be a unit circle[8]. If σ^2 runs from 0 to 2π, parametrizing the compact direction, we need to identify

$$X^{11} = \sigma^2 \tag{2.64}$$

Dropping then all implicit σ^2 dependence amounts to extracting the zeroth Fourier mode and gives the so-called "double-dimensionally reduced" action. Defining $C_1 = \bar{\theta}\Gamma^{11}d\theta$ (in components $C_\mu = -\bar{\theta}\Gamma^{11}\partial_\mu\theta$), the determinant of the metric decomposes as follows:

$$\det G_{\hat{\mu}\hat{\nu}} = \det \left(\begin{array}{c|c} g_{\mu\nu} + C_\mu C_\nu & C_\mu \\ \hline C_\nu & 1 \end{array}\right) = \det \left(\begin{array}{c|c} g_{\mu\nu} & C_\mu \\ \hline 0 & 1 \end{array}\right) = \det g_{\mu\nu} \tag{2.65}$$

Therefore, the first term (2.61) in the membrane action reduces trivially to the "Nambu-Goto" action of the string, with tension $T_2 = 2\pi T_3$, as expected for a membrane wrapped around a unit circle. For the "Wess-Zumino" terms, we note that since we can write Ω_3 in the form $\Omega_3 = d\sigma^2\Omega_2$, we also have

$$I_4 \equiv \hat{d}\Omega_3 = d\Omega_2 d\sigma^2 \equiv I_3 d\sigma^2 \tag{2.66}$$

Therefore, using (2.62) we easily identify

$$\begin{aligned} I_3 &= (d\bar{\theta}\Gamma_{11}\Gamma_n d\theta)\Pi^n \\ &= (d\bar{\theta}_1\Gamma_n d\theta_1)\Pi^n - (d\bar{\theta}_2\Gamma_n d\theta_2)\Pi^n \end{aligned} \tag{2.67}$$

where in the second line we decomposed the Majorana fermions into their Majorana-Weyl constituents. (See definition (2.42) above.) This is the "Wess-Zumino" form that characterizes the Green-Schwarz string. (See, for example, page 253 of [6].)

We have shown that a wrapped membrane is a string. A second interesting possi-

[8]this corresponds to a vanishing dilaton in the resulting type IIA string theory.

bility is to exchange the eleventh coordinate with a vector field A_μ, using world-volume duality. This leads to the $D2$-brane action. This is actually how the first supersymmetric world-volume action for a D-brane was written down [36]. One way to perform the duality is by adding to the action a Lagrange multiplier term[9]

$$S_D = -\int d^3\sigma\sqrt{-\det G_{\mu\nu}} + \int \Omega_3(\Pi^{11}, X^m, \theta) + \int d^3\sigma \Lambda^\mu(\Pi^{11}_\mu - \partial_\mu X^{11} - C_\mu) \quad (2.68)$$

and treat Π^{11} as an independent field. The X^{11} field equation implies that Λ^μ is divergence-free, thus it can be written as the curl of a vector field A_ρ:

$$\Lambda^\mu = \epsilon^{\mu\nu\rho}\partial_\nu A_\rho = \frac{1}{2}\epsilon^{\mu\nu\rho}F_{\nu\rho} \quad (2.69)$$

When we plug the above in the action the X^{11} term drops out. We still need to get rid of Π^{11}. To organize the calculations it is useful to identify quantities relevant to the dual $D2$-brane action. First of all, we expect to encounter the two-form b which makes F supersymmetric (2.44). We define the closed (and exact) form[10]

$$db \equiv -d\bar{\theta}\Gamma_{11}\Gamma_m d\theta \Pi^m \quad (2.70)$$

and postpone to section 3.1 the explicit determination of b. For the pupose of this calculation, we only need to recall that the supersymmetrized version of F is (2.44) $\mathcal{F} = F - b$.

Secondly, the "Wess-Zumino" term of the $D2$-brane action is characterized by (see (2.38, 2.50))

$$I_{4D} = -d\bar{\theta}\left(\frac{1}{2}\psi\!\!\!/^2 + \mathcal{F}\Gamma_{11}\right)d\theta = -d(C_3 + C_1\mathcal{F}) \equiv d\Omega_{3D} \quad (2.71)$$

with $\psi\!\!\!/ \equiv \Gamma^m\Pi_m$. Since $dC_1 = d\bar{\theta}\Gamma_{11}d\theta$ we get the relation

$$dC_3 + C_1 db = \frac{1}{2}d\bar{\theta}\psi\!\!\!/^2 d\theta \quad (2.72)$$

[9]The tension is set to one for convenience. Also, since all world-volume indices are three-dimensional, we drop the hats.

[10]Up to a minus sign, this is the "Wess-Zumino" form I_3 for the Green-Schwarz string; see 2.67.

Coming back to the membrane "Wess-Zumino" form (2.62)

$$I_4 = -\frac{1}{2}(d\bar{\theta}\Gamma_{mn}d\theta)\Pi^m\Pi^n + (d\bar{\theta}\Gamma_{11}\Gamma_n d\theta)\Pi^n\Pi^{11} \tag{2.73}$$

we see that

$$d(C_3 + bdX^{11}) = \frac{1}{2}d\bar{\theta}\not{\psi}^2 d\theta - \Pi^{11}db = -I_4 \tag{2.74}$$

which gives

$$\Omega_3(\Pi^{11}, X^m, \theta) = -b\;\Pi^{11} - C_3(X^m, \theta) + bC_1(\theta) \tag{2.75}$$

Since

$$\det\, G_{\mu\nu} = \det(g_{\mu\nu} + \Pi^{11}_\mu \Pi^{11}_\nu) = (\det\, g)(1 + \Pi^{11}_\mu g^{\mu\nu}\Pi^{11}_\nu) \equiv (\det\, g)(1+x) \tag{2.76}$$

the Π^{11}_μ equation of motion takes the form

$$\sqrt{\frac{-\det\, g}{1+x}}\; g^{\mu\nu}\Pi^{11}_\nu = \frac{1}{2}\epsilon^{\mu\nu\rho}\mathcal{F}_{\nu\rho} \tag{2.77}$$

Consequently

$$\begin{aligned} \frac{1}{2}\epsilon^{\mu\nu\rho}\Pi^{11}_\mu \mathcal{F}_{\nu\rho} &= x\sqrt{\frac{-\det\, g}{1+x}} \\ \mathrm{tr}\; \mathcal{F}g^{-1}\mathcal{F}g^{-1} &= 2\Pi^{11}_\mu G^{\mu\nu}\Pi^{11}_\nu = \frac{2x}{1+x} \end{aligned} \tag{2.78}$$

Collecting these results we get the action

$$\begin{aligned} S_D &= -\int d^3\sigma\sqrt{(-\det\, g)(1+x)} + \int \Big(\mathcal{F}(\Pi^{11} - C_1) - C_3\Big) \\ &= -\int d^3\sigma\sqrt{\frac{(-\det\, g)}{1+x}} - \int (C_3 + C_1\mathcal{F}) \\ &= -\int d^3\sigma\sqrt{(-\det\, g)\left(1 - \frac{1}{2}\mathrm{tr}\; \mathcal{F}g^{-1}\mathcal{F}g^{-1}\right)} + \int \Omega_{3D} \\ &= -\int d^3\sigma\sqrt{-\det(g+\mathcal{F})} + \int \Omega_{3D} \end{aligned} \tag{2.79}$$

The last step, valid for three by three matrices, can be easily checked in the diagonal basis. As expected, the last line is the supersymmetric $D2$-brane action [41, 36, 42].

2.4.2 The Self-Dual Antisymmetric Tensor in Six Dimensions

Consider a $D4$-brane of type IIA. If we take the strong coupling limit, type IIA develops a circular extra dimension. The resulting brane must be a five-brane wrapped on this extra dimension, since there are no four-branes in M-theory. It is now easy to guess the world-volume field content of the five-brane: besides the $5 = 11 - 6$ scalars parametrizing transverse motion, there must be a two-index antisymmetric tensor $B_{\mu\nu}$ which corresponds to the gauge field A_μ on the $D4$-brane. More careful consideration, as we will see shortly, requires this two-form to be self-dual, so that we get precisely the bosonic fields of the so-called $\mathcal{N} = (2,0)$[11] supersymmetric tensor multiplet.

For reasons that will become clear in a moment, we use σ^μ, $\mu = 0,\ldots,4$, to specify a set of five-dimensional coordinates, while the full set of six world-volume dimensions is parametrized by $\sigma^{\hat\mu}$, $\hat\mu = 0,\ldots,5$. Define the field strength and its dual by

$$H_{\hat\mu\hat\nu\hat\rho} = \partial_{\hat\mu} B_{\hat\nu\hat\rho} + \partial_{\hat\nu} B_{\hat\rho\hat\nu} + \partial_{\hat\rho} B_{\hat\mu\hat\nu} \tag{2.80}$$

$$\tilde H^{\hat\mu\hat\nu\hat\rho} = \frac{1}{6}\epsilon^{\hat\mu\hat\nu\hat\rho\lambda\sigma\tau} H_{\lambda\sigma\tau} \tag{2.81}$$

It is instructive to consider first the free theory. The equation of motion is then the self-duality condition $H = \tilde H$. There is no way, though, to write down a Lorentz-covariant action in terms of $B_{\hat\mu\hat\nu}$[12]. This happens because the quadratic action H^2 describes the propagation of both self-dual and anti-self dual degrees of freedom, and there is no covariant way to drop the latter while keeping the former [27].

One way around this problem is to drop manifest covariance by choosing a preferred direction, say σ^5 [32]. Under $\sigma^{\hat\mu} = (\sigma^\mu, \sigma^5)$, $B_{\hat\mu\hat\nu}$ decomposes into $B_{\mu\nu}$, and $A_\mu \equiv B_{\mu 5}$ with field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Then $H_{\hat\mu\hat\nu\hat\rho}$ decomposes into $H_{\mu\nu\rho}$ and

$$\mathcal{F}_{\mu\nu} \equiv H_{\mu\nu 5} = F_{\mu\nu} + \partial_5 B_{\mu\nu} \tag{2.82}$$

If we define

$$\tilde H^{\mu\nu} = \frac{1}{6}\epsilon^{\mu\nu\rho\lambda\sigma} H_{\rho\lambda\sigma} \tag{2.83}$$

[11] The multiplet also contains two spinors of same chirality.

[12] A formalism involving an auxiliary field will be described in section 2.4.4.

the self-duality equation becomes

$$\tilde{H}_{\mu\nu} = \mathcal{F}_{\mu\nu} \tag{2.84}$$

This is a first order equation, which is hard to get from a Lagrangian. We can get a second order field equation by using the Bianchi identity for $F_{\lambda\sigma}$:

$$\epsilon^{\mu\nu\rho\lambda\sigma}\partial_\rho(\tilde{H}_{\lambda\sigma} - \partial_5 B_{\lambda\sigma}) = 0 \tag{2.85}$$

This field equation follows from the action [28, 29, 30]

$$S = \frac{1}{2}\int d^6\sigma(\tilde{H}^{\mu\nu}\partial_5 B_{\mu\nu} - \tilde{H}^{\mu\nu}\tilde{H}_{\mu\nu}) \tag{2.86}$$

Although this action has only five-dimensional Lorenz covariance, we know that it must be invariant under the full six-dimensional Lorentz group, since it encodes the free propagation described by the self-duality equation. This can be checked explicitly (see [32]).

The formalism above allows an easy generalization to the interacting theory. We will present the results, referring the reader to [32] for details. Define

$$\begin{aligned} z_1 &= \frac{1}{2}\mathrm{tr}(\tilde{H}^2) \\ z_2 &= \frac{1}{4}\mathrm{tr}(\tilde{H}^4) \end{aligned} \tag{2.87}$$

Six-dimensional Lorentz invariance then restricts the form of the action to the class

$$S = \frac{1}{2}\int d^6\sigma(\tilde{H}^{\mu\nu}\partial_5 B_{\mu\nu} + f(z_1, z_2)) \tag{2.88}$$

The function $f(z_1, z_2)$ is subject to the nonlinear partial differential equation [13]

$$f_1^2 + z_1 f_1 f_2 + (\frac{1}{2}z_1^2 - z_2)f_2^2 = 1 \tag{2.89}$$

where $f_i \equiv \frac{\partial f}{\partial z_i}, i = 1, 2$. In the next section we will see that the self-interacting theory that lives on the five-brane has an action of this type.

2.4.3 Fivebrane Action Without Manifest Covariance

To proceed with the study of the five-brane action, we need to introduce the induced metric $G_{\hat{\mu}\hat{\nu}}$. The corresponding 5d pieces are $G_{\mu\nu}, G_{\mu 5}$, and G_{55}. Let G denote the 6d determinant ($G = \det G_{\hat{\mu}\hat{\nu}}$) and G_5 the 5d determinant. Upper indices denote components of the inverse 6d metric $G^{\hat{\mu}\hat{\nu}}$. The ϵ symbols are purely numerical with $\epsilon^{01234} = 1$ and $\epsilon^{\mu\nu\rho\lambda\sigma} = -\epsilon_{\mu\nu\rho\lambda\sigma}$. A useful relation is $G_5 = GG^{55}$. All formulas will be written with manifest 5d general coordinate invariance.

The self duality condition

$$H_{\hat{\mu}\hat{\nu}\hat{\rho}} = \frac{1}{6\sqrt{-G}} G_{\hat{\mu}\hat{\mu}'} G_{\hat{\nu}\hat{\nu}'} G_{\hat{\rho}\hat{\rho}'} \epsilon^{\hat{\mu}'\hat{\nu}'\hat{\rho}'\hat{\lambda}\hat{\sigma}\hat{\eta}} H_{\hat{\lambda}\hat{\sigma}\hat{\eta}} \tag{2.90}$$

together with the definition of $\tilde{H}$ (2.83), give

$$H_{\mu\nu\rho} = \frac{1}{2\sqrt{-G}} G_{\mu\mu'} G_{\nu\nu'} G_{\rho\rho'} \epsilon^{\mu'\nu'\rho'\lambda\xi} H_{\lambda\xi 5} - \frac{3}{\sqrt{-G}} G_{[\mu 5} G_{\nu\nu'} G_{\rho]\rho'} \tilde{H}^{\nu'\rho'} \tag{2.91}$$

If we multiply this by $\frac{1}{6}\epsilon^{\mu\nu\rho\sigma\tau}$ and use the definition of the determinant in the form

$$\epsilon^{\mu\nu\rho\sigma\tau} G_{\mu\mu'} G_{\nu\nu'} G_{\rho\rho'} G_{\sigma\sigma'} G_{\tau\tau'} = -G_5 \epsilon_{\mu'\nu'\rho'\sigma'\tau'} \tag{2.92}$$

we get

$$H_{\mu\nu 5} = K_{\mu\nu}(G, H) = K^{(1)}_{\mu\nu} + K^{(\epsilon)}_{\mu\nu} \tag{2.93}$$

where

$$\begin{aligned} K^{(1)}_{\mu\nu} &= \frac{\sqrt{-G}}{(-G_5)} (G\tilde{H}G)_{\mu\nu} \\ K^{(\epsilon)}_{\mu\nu} &= \epsilon_{\mu\nu\rho\lambda\sigma} \frac{G^{5\rho}}{2G^{55}} \tilde{H}^{\lambda\sigma} \end{aligned} \tag{2.94}$$

On the other hand, since

$$H_{\mu\nu 5} = \partial_5 B_{\mu\nu} + \partial_\mu B_{\nu 5} - \partial_\nu B_{\mu 5} \tag{2.95}$$

taking a curl eliminates $B_{\mu 5}$ leaving a second-order field equation that involves $B_{\mu\nu}$

only:

$$\frac{1}{2}\epsilon^{\mu\nu\rho\lambda\sigma}\partial_\rho K_{\lambda\sigma} = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda\sigma}\partial_5\partial_\rho B_{\lambda\sigma} = \partial_5\tilde{H}^{\mu\nu} \tag{2.96}$$

The R.H.S. of this equation clearly comes from the Lagrangian

$$L_2 = -\frac{1}{4}\tilde{H}^{\mu\nu}\partial_5 B_{\mu\nu} \tag{2.97}$$

In order to find the Lagrangian L' that gives the L.H.S. of (2.96), we use

$$\frac{\delta L'}{\delta B_{\mu\nu}} = \frac{\delta\tilde{H}^{\lambda\sigma}}{\delta B_{\mu\nu}}\frac{\delta L'}{\delta\tilde{H}^{\lambda\sigma}} = \frac{1}{2}\epsilon^{\lambda\sigma\rho\mu\nu}\partial_\rho\left(\frac{\delta L'}{\delta\tilde{H}^{\lambda\sigma}}\right) \tag{2.98}$$

which leads to the identification

$$K_{\mu\nu} = -\frac{\delta L'}{\delta\tilde{H}^{\mu\nu}} \tag{2.99}$$

We thus arrive at the Lagrangian first presented in [29]: $L = L' + L_2 = L_1 + L_2 + L_3$, where[13]

$$\begin{aligned} L_1 &= \frac{\sqrt{-G}}{2(-G_5)}\mathrm{tr}(G\tilde{H}G\tilde{H}) \\ L_3 &= \frac{1}{8}\epsilon_{\mu\nu\rho\lambda\sigma}\frac{G^{5\rho}}{G^{55}}\tilde{H}^{\mu\nu}\tilde{H}^{\lambda\sigma} \end{aligned} \tag{2.100}$$

The generalization to the interacting theory is natural in this setup. All we need to change is L_1 which becomes

$$L_1 = \sqrt{-G}f(z_1, z_2) \tag{2.101}$$

L_2 and L_3 remain the same. The z variables are straightforward generalizations of (2.87):

$$\begin{aligned} z_1 &= \frac{\mathrm{tr}(G\tilde{H}G\tilde{H})}{2(-G_5)} \\ z_2 &= \frac{\mathrm{tr}(G\tilde{H}G\tilde{H}G\tilde{H}G\tilde{H})}{4(-G_5)^2} \end{aligned} \tag{2.102}$$

[13]The formula given in ref. [31] has been rescaled by an overall factor of $-1/2$.

The trace only involves 5d indices:

$$\mathrm{tr}(G\tilde{H}G\tilde{H}) = G_{\mu\nu}\tilde{H}^{\nu\rho}G_{\rho\lambda}\tilde{H}^{\lambda\mu} \tag{2.103}$$

Six-dimensional coordinate invariance is not guaranteed in this formalism and needs to be checked explicitly. We denote the infinitesimal parameters of general coordinate transformations by $\xi^{\hat{\mu}} = (\xi^\mu, \xi)$. Since 5d general coordinate invariance is manifest, we focus on the ξ transformations only. The metric transforms in the standard way

$$\delta_\xi G_{\hat{\mu}\hat{\nu}} = \xi\partial_5 G_{\hat{\mu}\hat{\nu}} + \partial_{\hat{\mu}}\xi G_{5\hat{\nu}} + \partial_{\hat{\nu}}\xi G_{\hat{\mu}5} \tag{2.104}$$

Naively, $\delta_\xi B_{\mu\nu} = \xi H_{5\mu\nu}$, but $H_{5\mu\nu}$ is not a field in this formalism. The natural guess is to replace it with

$$K_{\mu\nu} = -\frac{\partial L'}{\partial \tilde{H}^{\mu\nu}} = K^{(1)}_{\mu\nu} f_1 + K^{(2)}_{\mu\nu} f_2 + K^{(\epsilon)}_{\mu\nu} \tag{2.105}$$

which is the value of $H_{5\mu\nu}$ given by the equation of motion (2.93). Therefore, we postulate the transformation law

$$\delta_\xi B_{\mu\nu} = \xi K_{\mu\nu} \tag{2.106}$$

which includes the "nonlinear contribution"

$$K^{(2)}_{\mu\nu} = \frac{\sqrt{-G}}{(-G_5)^2}(G\tilde{H}G\tilde{H}G\tilde{H}G)_{\mu\nu} \tag{2.107}$$

Assembling the results given above, ref. [31] showed that the required general coordinate transformation symmetry is achieved if, and only if, the function f satisfies the same equation (2.89) which was required by Lorentz invariance:

$$f_1^2 + z_1 f_1 f_2 + (\frac{1}{2}z_1^2 - z_2)f_2^2 = 1 \tag{2.108}$$

As discussed in [32], this equation has many solutions, but the one of relevance to the

M theory five-brane is

$$f = 2\sqrt{1 + z_1 + \frac{1}{2}z_1^2 - z_2} \tag{2.109}$$

For this choice L_1 can be re-expressed in the Born–Infeld form

$$L_1 = -\sqrt{-\det\left(G_{\hat{\mu}\hat{\nu}} + iG_{\hat{\mu}\rho}G_{\hat{\nu}\lambda}\tilde{H}^{\rho\lambda}/\sqrt{-G_5}\right)} \tag{2.110}$$

This expression is real, despite the factor of i, because it is an even function of $\tilde{H}$. Eliminating the factor of i would correspond to replacing z_1 by $-z_1$, which also solves the differential equation. However, it is essential for the five-brane application that the phases be chosen as shown.

2.4.4 Fivebrane Action in the PST Formulation

A manifestly covariant version of the above was obtained by Pasti, Sorokin and Tonin [33] using techniques developed in [34]. Instead of choosing a special direction, they introduce an auxiliary scalar field a. This so-called PST formulation has new gauge symmetries (described below) that allow one to choose the gauge $B_{\mu 5} = 0$, $a = \sigma^5$ (and hence $\partial_{\hat{\mu}} a = \delta^5_{\hat{\mu}}$). In this gauge, the covariant PST formulas reduce to those of the previous section.

Equation (2.110) expressed L_1 in terms of the determinant of the 6×6 matrix

$$M_{\hat{\mu}\hat{\nu}} = G_{\hat{\mu}\hat{\nu}} + i\frac{G_{\hat{\mu}\rho}G_{\hat{\nu}\lambda}}{\sqrt{-GG^{55}}}\tilde{H}^{\rho\lambda} \tag{2.111}$$

In the PST approach this is extended to the manifestly covariant form

$$M^{\text{cov.}}_{\hat{\mu}\hat{\nu}} = G_{\hat{\mu}\hat{\nu}} + i\frac{G_{\hat{\mu}\hat{\rho}}G_{\hat{\nu}\hat{\lambda}}}{\sqrt{-G(\partial a)^2}}\tilde{H}^{\hat{\rho}\hat{\lambda}}_{\text{cov.}} \tag{2.112}$$

The quantity

$$(\partial a)^2 = G^{\hat{\mu}\hat{\nu}}\partial_{\hat{\mu}} a \partial_{\hat{\nu}} a \tag{2.113}$$

reduces to G^{55} upon setting $\partial_{\hat{\mu}} a = \delta^5_{\hat{\mu}}$, and

$$\tilde{H}^{\hat{\rho}\hat{\lambda}}_{\text{cov.}} \equiv \frac{1}{6}\epsilon^{\hat{\rho}\hat{\lambda}\hat{\mu}\hat{\nu}\hat{\sigma}\hat{\tau}} H_{\hat{\mu}\hat{\nu}\hat{\sigma}}\partial_{\hat{\tau}} a \tag{2.114}$$

reduces to $\tilde{H}^{\rho\lambda}$. Thus $M^{\text{cov.}}_{\hat{\mu}\hat{\nu}}$ replaces $M_{\hat{\mu}\hat{\nu}}$ in L_1. Furthermore, the expression

$$L' = -\frac{1}{4(\partial a)^2}\tilde{H}^{\hat{\mu}\hat{\nu}}_{\text{cov.}}H_{\hat{\mu}\hat{\nu}\hat{\rho}}G^{\hat{\rho}\hat{\lambda}}\partial_{\hat{\lambda}}a \qquad (2.115)$$

which transforms under general coordinate transformations as a scalar density, reduces to $L_2 + L_3$ upon gauge fixing. It is interesting that L_2 and L_3 are unified in this formulation.

Let us now describe the new gauge symmetries of ref. [33]. Since degrees of freedom a and $B_{\mu 5}$ have been added, corresponding gauge symmetries are required. One of them is

$$\delta B_{\hat{\mu}\hat{\nu}} = 2\phi_{[\hat{\mu}}\partial_{\hat{\nu}]}a \qquad (2.116)$$

where $\phi_{\hat{\mu}}$ are infinitesimal parameters, and the other fields do not vary. In terms of differential forms, this implies $\delta H = d\phi da$. $\tilde{H}^{\hat{\rho}\hat{\lambda}}_{\text{cov.}}$ is invariant under this transformation, since it corresponds to the dual of Hda, but $dada = 0$. Thus the covariant version of L_1 is invariant under this transformation. The variation of L', on the other hand, is a total derivative.

The second local symmetry involves an infinitesimal scalar parameter φ. The transformation rules are $\delta G_{\hat{\mu}\hat{\nu}} = 0, \delta a = \varphi$, and

$$\delta B_{\hat{\mu}\hat{\nu}} = \frac{1}{(\partial a)^2}\varphi H_{\hat{\mu}\hat{\nu}\hat{\rho}}G^{\hat{\rho}\hat{\lambda}}\partial_{\hat{\lambda}}a + \varphi V_{\hat{\mu}\hat{\nu}} \qquad (2.117)$$

where the quantity $V_{\hat{\mu}\hat{\nu}}$ is to be determined. This transformation is just as complicated as the non-manifest general coordinate transformation in the non-covariant formalism. Rather than derive it from scratch, let's see what is required to agree with the previous formulas after gauge fixing. In other words, we fix the gauge $\partial_{\hat{\mu}}a = \delta^5_{\hat{\mu}}$ and $B_{\mu 5} = 0$, and figure out what the resulting ξ transformations are. We need

$$\delta a = \varphi + \xi\partial_5 a = \varphi + \xi = 0 \qquad (2.118)$$

which tells us that $\varphi = -\xi$. Then

$$\delta_\xi B_{\mu\nu} \;=\; \frac{1}{(\partial a)^2}\varphi H_{\mu\nu\hat{\rho}}G^{\hat{\rho}\hat{\lambda}}\partial_{\hat{\lambda}}a + \varphi V_{\mu\nu} + \xi H_{5\mu\nu}$$

$$= -\xi\left(\frac{G^{\rho 5}}{G^{55}}H_{\mu\nu\rho} + V_{\mu\nu}\right) = \xi(K^{(\epsilon)}_{\mu\nu} - V_{\mu\nu}) \qquad (2.119)$$

Thus, comparing with eqs. (2.106) and (2.105), we need the covariant definition

$$V_{\hat{\mu}\hat{\nu}} = -2\frac{\partial L_1}{\partial \tilde{H}^{\hat{\mu}\hat{\nu}}_{\text{cov.}}} \qquad (2.120)$$

to achieve agreement with our previous results.

To summarize, we have learned that the covariant PST formulation has new gauge transformations, and one of them encodes the complications that end up in general coordinate invariance after gauge fixing. Thus this formalism is not simpler than the non-covariant one. However, it is more symmetrical, and it does raise new questions, such as whether there are other gauge choices that are worth exploring.

Chapter 3

D-Brane Actions in Flat Spacetime

In this section we will present in detail the construction of supersymmetric D-brane actions in a trivial supergravity background, following [3]. A preview of the construction was already given in section 2.3.

3.1 Gauge-Invariant D-Brane Actions

We have already seen that the world-volume action is the sum of a "Dirac-Born-Infeld" (1.32) and a "Wess-Zumino" (2.11) term. They are manifestly invariant under $(p+1)$-dimensional general coordinate transformations and supersymmetry. The counting of bosonic degrees of freedom involves now, besides the $9-p$ transverse scalars ϕ_i, the $p-1$ physical degrees of freedom from the gauge field A_μ, for a total of 8 bosonic modes. We therefore need kappa symmetry to eliminate half of the 32 fermionic coordinates of type II supergravity. The "on-shell" condition further cuts the number by two, leaving 8 physical fermionic modes that match the bosonic modes giving a world-volume supersymmetric theory. We will see in detail how this works in section 3.2.

To proceed with the analysis of the symmetries, we need to supplement the formulas for the p-branes with formulas for the variation of the gauge field A. In fact, in section 2.4.1 we already found the full supersymmetric action of the $D2$-brane (2.79) by dualizing the eleventh coordinate of the $M2$-brane action. It can be used to guess the supersymmetric version of the field strength, $\mathcal{F} = F - b$. Indeed, from (2.70) we have

$$db = d\bar{\theta}\Gamma_{11}\Gamma_m d\theta\Pi^m \tag{3.1}$$

which is manifestly supersymmetric. A straightforward application of (2.13) gives

$$b = -\bar{\theta}\Gamma_{11}\Gamma_m d\theta \left(dX^m + \frac{1}{2}\bar{\theta}\Gamma^m d\theta \right) \tag{3.2}$$

In components the formula for b becomes

$$b_{\mu\nu} = \bar{\theta}\Gamma_{11}\Gamma_m \partial_\mu \theta \left(\partial_\nu X^m - \frac{1}{2}\bar{\theta}\Gamma^m \partial_\nu \theta \right) - (\mu \leftrightarrow \nu) \tag{3.3}$$

This is the formula for p even. When p is odd, Γ_{11} is replaced by τ_3. As we discussed in section 2.3, the crucial feature of this choice of b is that $\delta_\epsilon b$ is an exact differential form. This implies that $\mathcal{F}$ is supersymmetric for an appropriate choice of $\delta_\epsilon A$. To be explicit, using eq. (2.3)

$$\delta_\epsilon b = -\bar{\epsilon}\Gamma_{11}\Gamma_m d\theta \left(dX^m + \frac{1}{2}\bar{\theta}\Gamma^m d\theta \right) + \frac{1}{2}\bar{\theta}\Gamma_{11}\Gamma_m d\theta \bar{\epsilon}\Gamma^m d\theta \tag{3.4}$$

This must be an exact differential form, because db is supersymmetric. To prove it explicitly, we substitute $\theta = \theta_1 + \theta_2$, using eq. (2.42):

$$\delta_\epsilon b = (\bar{\epsilon}_1 \Gamma_m d\theta_1 - \bar{\epsilon}_2 \Gamma_m d\theta_2) dX^m + \bar{\epsilon}_1 \Gamma_m d\theta_1 \bar{\theta}_1 \Gamma^m d\theta_1 - \bar{\epsilon}_2 \Gamma_m d\theta_2 \bar{\theta}_2 \Gamma^m d\theta_2 \tag{3.5}$$

The next step is to use the following fundamental identity[1] which is valid for any three Majorana–Weyl spinors $\lambda_1, \lambda_2, \lambda_3$ of the same chirality,

$$\Gamma_m \lambda_1 \bar{\lambda}_2 \Gamma^m \lambda_3 + \Gamma_m \lambda_2 \bar{\lambda}_3 \Gamma^m \lambda_1 + \Gamma_m \lambda_3 \bar{\lambda}_1 \Gamma^m \lambda_2 = 0 \tag{3.6}$$

This formula is valid regardless of whether each of the λ's is an even element or an odd element of the Grassmann algebra. (Note that θ is odd and $d\theta$ is even.) It implies, in particular, that

$$\bar{\epsilon}_1 \Gamma_m d\theta_1 \bar{\theta}_1 \Gamma^m d\theta_1 = -\frac{1}{2}\bar{\epsilon}_1 \Gamma_m \theta_1 d\bar{\theta}_1 \Gamma^m d\theta_1 = -\frac{1}{3} d\left(\bar{\epsilon}_1 \Gamma_m \theta_1 \bar{\theta}_1 \Gamma^m d\theta_1 \right) \tag{3.7}$$

[1]This is basically the same as (2.14), which was crucial for the existence of the Green-Schwarz string.

Thus $\delta_\epsilon \mathcal{F} = 0$ if we take

$$\delta_\epsilon A = \bar{\epsilon}\Gamma_{11}\Gamma_m \theta dX^m + \frac{1}{6}(\bar{\epsilon}\Gamma_{11}\Gamma_m\theta\bar{\theta}\Gamma^m d\theta + \bar{\epsilon}\Gamma_m\theta\bar{\theta}\Gamma_{11}\Gamma^m d\theta) \tag{3.8}$$

To determine the kappa variation of the gauge field one first uses eqs. (3.2) and (2.18) to compute

$$\delta b = -2\delta\bar{\theta}\Gamma_{11}\Gamma_m d\theta\Pi^m + d\Big(- \delta\bar{\theta}\Gamma_{11}\Gamma_m\theta\Pi^m + \frac{1}{2}\delta\bar{\theta}\Gamma_{11}\Gamma_m\theta\bar{\theta}\Gamma^m d\theta - \frac{1}{2}\delta\bar{\theta}\Gamma^m\theta\bar{\theta}\Gamma_{11}\Gamma_m d\theta\Big) \tag{3.9}$$

Then, to obtain a relatively simple result for $\delta\mathcal{F}$, let us require that

$$\delta A = -\delta\bar{\theta}\Gamma_{11}\Gamma_m\theta\Pi^m + \frac{1}{2}\delta\bar{\theta}\Gamma_{11}\Gamma_m\theta\bar{\theta}\Gamma^m d\theta - \frac{1}{2}\delta\bar{\theta}\Gamma^m\theta\bar{\theta}\Gamma_{11}\Gamma_m d\theta \tag{3.10}$$

The variation of $\mathcal{F}$ under a kappa transformation is then

$$\delta\mathcal{F} = 2\delta\bar{\theta}\Gamma_{11}\Gamma_m d\theta\Pi^m \tag{3.11}$$

or in terms of components

$$\delta\mathcal{F}_{\mu\nu} = 2\delta\bar{\theta}\Gamma_{11}(\gamma_\mu\partial_\nu - \gamma_\nu\partial_\mu)\theta \tag{3.12}$$

These are the formulas for the Type IIA case (p even). In the Type IIB case (p odd), one should make the replacement $\Gamma_{11} \to \tau_3$. (Γ_{11} must not be anticommuted past another Γ matrix before making this substitution!) The normalization of the two-form b in the preceding section was chosen so that the formula for $\delta\mathcal{F}$ obtained in this way would combine nicely with the formula for δG in eq. (2.22).

Now we are ready to establish kappa symmetry of the action by the procedure of section 2.2 and prove the results stated in section 2.3. Consider a kappa transformation of S_1 using

$$\delta L_1 = \delta\left(-\sqrt{-\det(G+\mathcal{F})}\right) = -\frac{1}{2}\sqrt{-\det(G+\mathcal{F})}\mathrm{tr}[(G+\mathcal{F})^{-1}(\delta G + \delta\mathcal{F})] \tag{3.13}$$

Inserting the variations $\delta G_{\mu\nu}$ and $\delta\mathcal{F}_{\mu\nu}$ given in eqs. (2.22) and (3.12) gives

$$\delta L_1 = 2\sqrt{-\det(G+\mathcal{F})}\delta\bar{\theta}\gamma_\mu\{(G-\mathcal{F}\Gamma_{11})^{-1}\}^{\mu\nu}\partial_\nu\theta \tag{3.14}$$

For p odd Γ_{11} is replaced this time by $-\tau_3$, since it has been moved past γ_μ. Again the key step is to rewrite this in the form (2.24).

This time we define

$$\rho^{(p)} = \sqrt{-\det(G+\mathcal{F})}\gamma^{(p)} \tag{3.15}$$

so that eq. (2.25) becomes

$$(\rho^{(p)})^2 = -\det(G+\mathcal{F}) \tag{3.16}$$

and we can recast the requirement

$$\sqrt{-\det(G+\mathcal{F})}\gamma_\mu\{(G-\mathcal{F}\Gamma_{11})^{-1}\}^{\mu\nu} = \gamma^{(p)}T^\nu_{(p)} \tag{3.17}$$

in the more convenient form

$$\rho^{(p)}\gamma_\mu = T^\nu_{(p)}(G-\mathcal{F}\Gamma_{11})_{\nu\mu} \tag{3.18}$$

Once we established this notation, the formulas are identical to those in section 2.2. To prove the results of section 2.3 we need to check that they satisfy equations (3.16,3.18,2.39). This is done in Appendices A,B,C of this section.

3.2 Super D-Branes in Static Gauge

In section 2.2 we talked about how one gets rid of unphysical bosonic degrees of freedom by going to the static gauge (2.16). Now that we understand kappa symmetry we can use it to eliminate unphysical fermionic degrees of freedom. A natural kappa gauge choice that leads to surprisingly simple and tractable formulas is to set one of the two Majorana–Weyl spinors that comprise θ equal to zero. Specifically, in the IIA case we set $\theta_2 = 0$. The surviving Majorana–Weyl spinor, θ_1, is then renamed

λ. We could do the same thing for the IIB case. However, for the conventions that have been introduced in the preceding sections, it is more convenient to set $\theta_1 = 0$ and $\theta_2 = \lambda$ in the IIB case. This will result in formulas that take the same form in both cases.

After gauge fixing, λ becomes a world-volume spinor, even though the θ's were originally world-volume scalars, as a consequence of the identification of X^μ with σ^μ. When $p \leq 5$, the gauge-fixed world-volume theory has extended supersymmetry. In this case the 32-component Majorana–Weyl spinor λ actually represents a set of minimal spinors. However, we find it convenient to leave it alone rather than to decompose it into pieces.

To see how our proposed gauge choice works, let us consider the global supersymmetry transformations with parameter ϵ, which we now decompose into two parts called ϵ_1 and ϵ_2. In the IIA case ϵ_1 and ϵ_2 have opposite chirality, while in the IIB case they have the same chirality. Nonetheless, all the formulas that follow are valid for both cases (unless otherwise indicated). Since supersymmetry transformations move the variables out of the gauge, it is necessary to add compensating general coordinate and kappa transformations that restore the gauge[2]

$$\begin{aligned} \delta\bar{\theta} &= \bar{\epsilon} + \bar{\kappa}(1 - \gamma^{(p)}) + \xi^\mu \partial_\mu \bar{\theta} \\ \delta X^m &= \bar{\epsilon}\Gamma^m \theta - \bar{\kappa}(1 - \gamma^{(p)})\Gamma^m \theta + \xi^\mu \partial_\mu X^m \end{aligned} \tag{3.19}$$

From the structure of ρ_A and ρ_B displayed in eqs. (2.51) and (2.59), one sees that the matrix $\gamma^{(p)}$ is off-diagonal in both the IIA and IIB cases. Thus we write it in the block form

$$\gamma^{(p)} = \begin{pmatrix} 0 & \zeta^{(p)} \\ \tilde{\zeta}^{(p)} & 0 \end{pmatrix} \tag{3.20}$$

The equation $[\gamma^{(p)}]^2 = 1$ then becomes

$$\zeta^{(p)}\tilde{\zeta}^{(p)} = \tilde{\zeta}^{(p)}\zeta^{(p)} = 1 \tag{3.21}$$

There is no reason that the square of $\zeta^{(p)}$ should be anything simple. In this notation,

[2]Exactly the same sort of reasoning can be used to argue that the gauge choice is consistent in the first place.

the requirements $\delta\bar{\theta}_2 = 0$ and $\delta X^\mu = 0$ in the IIA case become

$$\begin{aligned} 0 &= \bar{\epsilon}_2 + \bar{\kappa}_2 - \bar{\kappa}_1 \tilde{\zeta}^{(p)} \\ 0 &= \bar{\epsilon}_1 \Gamma^\mu \lambda - (\bar{\kappa}_1 - \bar{\kappa}_2 \zeta^{(p)}) \Gamma^\mu \lambda + \xi^\mu \end{aligned} \quad (3.22)$$

The above equations are solved by

$$\bar{\kappa}_1 - \bar{\kappa}_2 \zeta^{(p)} = \bar{\epsilon}_2 \zeta^{(p)} \quad (3.23)$$

$$\xi^\mu = (\bar{\epsilon}_2 \zeta^{(p)} - \bar{\epsilon}_1) \Gamma^\mu \lambda \quad (3.24)$$

For these choices the total supersymmetry transformations of the fields that remain in the gauge-fixed theory are

$$\begin{aligned} \Delta\bar{\lambda} &= \bar{\epsilon}_1 + \bar{\epsilon}_2 \zeta^{(p)} + \xi^\mu \partial_\mu \bar{\lambda} \\ \Delta\phi^i &= (\bar{\epsilon}_1 - \bar{\epsilon}_2 \zeta^{(p)}) \Gamma^i \lambda + \xi^\mu \partial_\mu \phi^i \\ \Delta A_\mu &= (\bar{\epsilon}_2 \zeta^{(p)} - \bar{\epsilon}_1)(\Gamma_\mu + \Gamma_i \partial_\mu \phi^i) \lambda \\ &\quad + (\frac{1}{3}\bar{\epsilon}_1 - \bar{\epsilon}_2 \zeta^{(p)}) \Gamma_m \lambda \bar{\lambda} \Gamma^m \partial_\mu \lambda + \xi^\rho \partial_\rho A_\mu + \partial_\mu \xi^\rho A_\rho \end{aligned} \quad (3.25)$$

Note that the index m is a 10d index, which includes both μ and i values. The parameter ξ^μ in these equations is understood to take the value given in eq. (3.24).

In the IIB case, one finds exactly the same set of gauge-fixed supersymmetry transformations except that the labels 1 and 2 on ϵ and κ are interchanged. Since there is no fundamental distinction between them, anyway, we simply interchange these labels in the IIB case, so that the formulas then look identical to those of the IIA case. Thus (3.25) describes the symmetry transformations of our gauge-fixed theory for all values of p from 0 to 9.

Now let's look at the actions that result from imposing the gauge choices on the gauge-invariant D-brane actions. Recall that the Wess–Zumino term S_2 is characterized by the $(p+2)$-form $I_{p+2} = \pm d\bar{\theta} T_p d\theta$. The crucial fact is that T_p connects θ_1 and θ_2 in both the IIA and IIB cases, so that $I_{p+2} \propto d\bar{\theta}_2(\ldots)d\theta_1$, which vanishes for $\theta_1 = 0$

or $\theta_2 = 0$. Therefore, only S_1 contributes to the gauge fixed action. The result is

$$S^{(p)} = -\int d^{p+1}\sigma\sqrt{-\det M^{(p)}} \tag{3.26}$$

where $M^{(p)}_{\mu\nu} = G^{(p)}_{\mu\nu} + \mathcal{F}^{(p)}_{\mu\nu}$, and

$$G^{(p)}_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu\phi^i\partial_\nu\phi^i - \bar{\lambda}(\Gamma_\mu + \Gamma_i\partial_\mu\phi^i)\partial_\nu\lambda - \bar{\lambda}(\Gamma_\nu + \Gamma_i\partial_\nu\phi^i)\partial_\mu\lambda + \bar{\lambda}\Gamma^m\partial_\mu\lambda\bar{\lambda}\Gamma_m\partial_\nu\lambda \tag{3.27}$$

$$\mathcal{F}^{(p)}_{\mu\nu} = F_{\mu\nu} - \bar{\lambda}(\Gamma_\mu + \Gamma_i\partial_\mu\phi^i)\partial_\nu\lambda + \bar{\lambda}(\Gamma_\nu + \Gamma_i\partial_\nu\phi^i)\partial_\mu\lambda \tag{3.28}$$

Thus

$$M^{(p)}_{\mu\nu} = \eta_{\mu\nu} + F_{\mu\nu} + \partial_\mu\phi^i\partial_\nu\phi^i - 2\bar{\lambda}(\Gamma_\mu + \Gamma_i\partial_\mu\phi^i)\partial_\nu\lambda + \bar{\lambda}\Gamma^m\partial_\mu\lambda\bar{\lambda}\Gamma_m\partial_\nu\lambda \tag{3.29}$$

If we have made no errors, the action $S^{(p)}$ in eq. (3.26) should be invariant under the ϵ_1 and ϵ_2 transformations given in (3.25). However, as a check of both our reasoning and our mathematics, it is a good idea to check this explicitly. In Appendix D we show in detail that the variation of the integrand of $S^{(p)}$ is a total derivative, as required. Thus we are confident that the action and the symmetry transformations are correct.

The action $S^{(p)}$, which describes a p-brane in a flat space-time background, is (maximally) supersymmetric Born–Infeld theory. In particular, for the case $p = 9$, we obtain supersymmetric Born–Infeld theory in 10d. In this case there are no transverse coordinates ϕ^i and the target space index m can be replaced by a Greek world-volume index. The resulting formula is given by

$$M^{(9)}_{\mu\nu} = \eta_{\mu\nu} + F_{\mu\nu} - 2\bar{\lambda}\Gamma_\mu\partial_\nu\lambda + \bar{\lambda}\Gamma^\rho\partial_\mu\lambda\bar{\lambda}\Gamma_\rho\partial_\nu\lambda \tag{3.30}$$

The supersymmetries of $S^{(9)}$ are given by

$$\Delta\bar{\lambda} = \bar{\epsilon}_1 + \bar{\epsilon}_2\zeta^{(9)} + \xi^\mu\partial_\mu\bar{\lambda} \tag{3.31}$$

$$\Delta A_\mu = (\bar{\epsilon}_2\zeta^{(9)} - \bar{\epsilon}_1)\Gamma_\mu\lambda + (\frac{1}{3}\bar{\epsilon}_1 - \bar{\epsilon}_2\zeta^{(9)})\Gamma_\rho\lambda\bar{\lambda}\Gamma^\rho\partial_\mu\lambda + \xi^\rho\partial_\rho A_\mu + \partial_\mu\xi^\rho A_\rho \tag{3.32}$$

where

$$\xi^\mu = (\bar{\epsilon}_2 \zeta^{(9)} - \bar{\epsilon}_1)\Gamma^\mu \lambda \tag{3.33}$$

The transformation with parameter ϵ_1 describes the supersymmetries of Volkov–Akulov type, which are broken by the presence of the D-brane. The inhomogeneity of the ϵ_1 transformation of λ shows that it is the associated Goldstone fermion. The unbroken supersymmetries should give no inhomogeneous terms, so they must be given by a combined transformation with $\epsilon_1 = \epsilon_2$.

3.3 Dimensional Reduction

The set of gauge-fixed D-brane actions $S^{(p)}$ given in eq. (3.26) are related to one another by straightforward dimensional reduction. In particular, this means that starting with $S^{(9)}$ and dropping the dependence on $9-p$ of the world-volume coordinates gives the action $S^{(p)}$. With our conventions one must identify the $9-p$ scalar components of A as $A_i = -\phi^i$. (For other conventions this equation could have a plus sign.)

To demonstrate the claim given above, let us consider the dimensional reduction from $S^{(p)}$ to $S^{(p-1)}$, so that the general case is implied by induction. Setting all σ^p derivatives to zero and $A_p = -\phi^p$, we can write $M^{(p)}$ in block form as

$$M^{(p)}_{\hat{\mu}\hat{\nu}} = \left(\begin{array}{c|c} \begin{array}{l} \eta_{\mu\nu} + F_{\mu\nu} + \partial_\mu\phi^i\partial_\nu\phi^i \\ -2\bar{\lambda}(\Gamma_\mu + \Gamma_i\partial_\mu\phi^i)\partial_\nu\lambda \\ +\bar{\lambda}\Gamma^m\partial_\mu\lambda\bar{\lambda}\Gamma_m\partial_\nu\lambda \end{array} & -\partial_\mu\phi^p \\ \hline -2\bar{\lambda}\Gamma_p\partial_\nu\lambda + \partial_\nu\phi^p & 1 \end{array}\right) \tag{3.34}$$

It is understood here that $\hat{\mu} = (\mu, p)$ and the last row and column correspond to $\hat{\mu} = p$ and $\hat{\nu} = p$, respectively. Also, the index i is summed from $p+1$ to 9. All that matters in the action is the determinant of $-M^{(p)}$, so we may add multiples of the last row to the other rows. Doing this with a factor of $\partial_\mu\phi^p$, so as to create zeros in the upper right corner, one obtains precisely the desired matrix $M^{(p-1)}_{\mu\nu}$ in the upper left block. The sum on i now goes from p to 9. This proves the compatibility of the formulas under dimensional reduction.

The supersymmetry transformation formulas have the same compatibility under dimensional reduction. However, this is a little more work to prove, because one needs to know a formula for the dimensional reduction of $\zeta^{(p)}$. In Appendix D we prove that upon dimensional reduction from p to $p-1$ (as above)

$$\zeta^{(p)} \to (-1)^p \Gamma_p \zeta^{(p-1)} \tag{3.35}$$

This implies that the supersymmetry transformation formulas in (3.25) retain their form upon dimensional reduction for the identifications $\bar{\epsilon}_1^{(p)} = \bar{\epsilon}_1^{(p-1)}$ and $\bar{\epsilon}_2^{(p)} = (-1)^p \bar{\epsilon}_2^{(p-1)} \Gamma_p$. The unbroken supersymmetry after dimensional reduction is given by a combined transformation with $\epsilon_1^{(p)} = \pm \Gamma_{p+1} \ldots \Gamma_9 \epsilon_2^{(p)}$. (Some care is required to determine the sign in each case.)

Appendix A: The Proof of Equation (3.16)

We wish to prove that the expressions we have found for $\rho^{(p)}$ satisfy

$$(\rho^{(p)})^2 = -\det(G + \mathcal{F}).$$

The proof is somewhat simpler for the special case $G_{\mu\nu} = \eta_{\mu\nu}$, where η is the flat Minkowski metric with signature $(- + \cdots +)$ in $p+1$ dimensions. General covariance considerations imply that if the formula is true in this case, then it is true in general. This can be proved, for example, by introducing a vielbein to relate base space and tangent space coordinates. In the tangent space coordinates $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$.

The IIA Case: $p = 2k,\ k = 0, \ldots, 4$

We have defined a $(p+1)$-form

$$\rho_{p+1} = \frac{1}{(p+1)!} \rho_{\mu_1 \ldots \mu_{p+1}} d\sigma^{\mu_1} \ldots d\sigma^{\mu_{p+1}} \tag{3.36}$$

and represented $\rho^{(p)}$ as:

$$\rho^{(p)} = \frac{1}{(p+1)!}\epsilon^{\mu_1\cdots\mu_{p+1}}\rho_{\mu_1\ldots\mu_{p+1}} \tag{3.37}$$

For $p = 2k$ it follows from eqs. (2.45-2.47) that

$$\rho_{2k+1} = \sum_{n=0}^{k} \Gamma_{11}^{(k-n+1)} \frac{\mathcal{F}^n}{n!} \frac{\not{\psi}^{2(k-n)+1}}{(2(k-n)+1)!} \tag{3.38}$$

The expression can be most easily examined by choosing a canonical basis for $\mathcal{F}$. The point is that both sides of the equation we are attempting to prove are Lorentz invariant, and a $(p+1)$-dimensional Lorentz transformation can bring $\mathcal{F}$ to the special form

$$\mathcal{F} = \sum_{i=1}^{k} \Lambda_i \, d\sigma^{2i-1} \wedge d\sigma^{2i} \tag{3.39}$$

Since p is even, there is necessarily a row and a column of zeroes, which we have chosen to associate with the time direction, thereby making $\mathcal{F}$ purely magnetic. The argument works the same if there are electric components. In this basis, defining $\gamma_i^{[2]} \equiv \gamma_{2i-1}\gamma_{2i}$,

$$\rho^{(2k)} = \sum_{n=0}^{k} \sum_{\substack{i_1<\ldots<i_n \\ i_{n+1}<\ldots<i_k}} \Gamma_{11}^{(k-n+1)} \Lambda_{i_1}\cdots\Lambda_{i_n}\gamma_0\gamma_{i_{n+1}}^{[2]}\cdots\gamma_{i_k}^{[2]} \tag{3.40}$$

where $(i_1,\ldots,i_k)$ is a permutation of the numbers $(1,\ldots,k)$. This can be rewritten in the much more transparent form

$$\rho^{(2k)} = \Gamma_{11}\gamma_0 \prod_{i=1}^{k} (\Lambda_i + \Gamma_{11}\gamma_i^{[2]}) \tag{3.41}$$

As all the $\gamma^{[2]}$'s commute with one another and with Γ_{11}, whereas γ_0 and Γ_{11} anti-commute:

$$\begin{aligned} (\rho^{(2k)})^2 &= \prod_{i=1}^{k}(\Lambda_i - \Gamma_{11}\gamma_i^{[2]})(\Lambda_i + \Gamma_{11}\gamma_i^{[2]}) \\ &= \prod_{i=1}^{k}(1 + \Lambda_i^2) \end{aligned} \tag{3.42}$$

Therefore,

$$(\rho^{(2k)})^2 = -\det(G+\mathcal{F}) \tag{3.43}$$

in this basis. This completes the proof for p even.

The IIB Case: $p = 2k+1,\ k = 0,\ldots,4$

The proof for p odd is almost identical, and can be made very brief. One difference is that $\mathcal{F}$ has an even number of row and columns, so the canonical form has no rows or columns of zeroes. Thus, in canonical basis, $\mathcal{F}$ contains both electric (Λ_0) and magnetic (Λ_i, $i = 1,\ldots,k$) components

$$\mathcal{F} = \sum_{i=0}^{k} \Lambda_i \, d\sigma^{2i} \wedge d\sigma^{2i+1} \tag{3.44}$$

This time it is convenient to define $\gamma_i^{[2]} \equiv \gamma_{2i}\gamma_{2i+1}$. Then, using eqs. (2.53-2.56), one can show that the counterpart of eq. (3.41) is

$$\rho^{(2k+1)} = \tau_3\tau_1 \prod_{i=0}^{k} (\Lambda_i - \tau_3\gamma_i^{[2]}) \tag{3.45}$$

The square of this also gives the desired determinant.

Appendix B: The Proof of Equation (3.18)

We wish to prove the IIA identity

$$\rho^{(p)}\gamma_\mu = T^\nu_{(p)}(G - \mathcal{F}\Gamma_{11})_{\nu\mu} \tag{3.46}$$

The proof is the quickest in the differential form representation. By definitions (2.30-2.31) and (2.32-2.33), eq. (3.46) is equivalent to:

$$\rho_{p+1}\gamma_\mu = T_p d\sigma^\nu (G - \mathcal{F}\Gamma_{11})_{\nu\mu} \tag{3.47}$$

This allows us to combine the formulas for all even p's. Using

$$\rho_A = \sum_{p\,even} \rho_{p+1} = e^{\mathcal{F}} S_A(\psi) \quad \text{and} \quad T_A = \sum_{p\,even} T_p = e^{\mathcal{F}} C_A(\psi) \tag{3.48}$$

we get

$$\rho_A \gamma_\mu = T_A d\sigma^\nu (G - \mathcal{F}\Gamma_{11})_{\nu\mu} \tag{3.49}$$

The key to the proof is the relation[3]

$$\frac{\psi^n}{n!}\gamma_\mu = \frac{\psi^{n-1}}{(n-1)!} d\sigma^\nu G_{\nu\mu} + \frac{(-1)^n}{(n+1)!} i_{e_\mu}(\psi^{n+1}) \tag{3.50}$$

where i_{e_μ} denotes the interior product operator induced by $e_\mu = \frac{\partial}{\partial\sigma^\mu}$. This is a consequence of the definition of i_X,

$$i_X\omega = \frac{1}{n!}\sum_{s=1}^{n}(-1)^{s-1} X^{\mu_s}\omega_{\mu_1\ldots\mu_s\ldots\mu_n} d\sigma^{\mu_1}\ldots d\sigma^{\mu_{s-1}} d\sigma^{\mu_{s+1}}\ldots d\sigma^{\mu_n} \tag{3.51}$$

for an n-form ω and a vector field X. Using eq. (3.50), it follows directly that

$$\rho_A \gamma_\mu = T_A d\sigma^\nu G_{\nu\mu} - e^{\mathcal{F}} i_{e_\mu}(C_A(\psi))\Gamma_{11} \tag{3.52}$$

It must be kept in mind that eq. (3.52) is a set of equations relating differential forms of order $p+1$, the dimension of the world-volume. As $e^{\mathcal{F}} C_A(\psi)$ is a $p+2$ form, and therefore vanishes, we have

$$e^{\mathcal{F}} i_{e_\mu}(C_A(\psi))\Gamma_{11} = -i_{e_\mu}(e^{\mathcal{F}}) C_A(\psi)\Gamma_{11} = -T_A d\sigma^\nu \mathcal{F}_{\mu\nu}\Gamma_{11} \tag{3.53}$$

This gives the second term on the RHS of eq. (3.49) and completes the proof for the IIA case.

The proof for IIB is similar, except that τ_3 anticommutes with $S_B(\psi)$ which introduces an extra minus sign in the second term. This is precisely what is needed,

[3] This is equivalent to $\gamma_{\mu_1\ldots\mu_n}\gamma_\mu = n\gamma_{[\mu_1\ldots\mu_{p+1}} G_{\mu_n]\mu} + \gamma_{\mu_1\ldots\mu_n\mu}$.

because the IIB version of eq. (3.46) is

$$\rho^{(p)}\gamma_\mu = T^\nu_{(p)}(G + \mathcal{F}\tau_3)_{\nu\mu} \tag{3.54}$$

Appendix C: The Proof of Equation (2.39)

We wish to prove the identity

$$d\bar{\theta}\delta T_p d\theta + 2\delta\bar{\theta} dT_p d\theta = 0 \tag{3.55}$$

We start with the IIA case. Summing over all even values of p gives

$$d\bar{\theta}\delta T_A d\theta + 2\delta\bar{\theta} dT_A d\theta = 0 \tag{3.56}$$

where T_A is given in eqs. (2.45-2.48). We evaluate this using

$$\delta T_A = \delta[e^{\mathcal{F}} C_A(\psi)] = e^{\mathcal{F}}(\delta\mathcal{F} C_A + \delta C_A) \tag{3.57}$$

$$\delta C_A = \Gamma_{11}[\delta\psi S_A] = 2\delta\bar{\theta}\Gamma^m d\theta \Gamma_{11}[\Gamma_m S_A] \tag{3.58}$$

and

$$\delta\mathcal{F} = 2\delta\bar{\theta}\Gamma_{11}\Gamma_{m_1} d\theta \Pi^{m_1} \tag{3.59}$$

where the brackets denote antisymmetrization of all enclosed Γ matrices. Dropping an overall factor of $2e^{\mathcal{F}}$ and collecting the coefficient of $2k+1$ Π's, we can write the contribution to $d\bar{\theta}\delta T_A d\theta$ as

$$\frac{1}{2k!}\delta\bar{\theta}\Gamma_{11}\Gamma_{[m_1} d\theta d\bar{\theta}\Gamma_{11}^{k+1}\Gamma_{m_2\ldots m_{2k+1}]} d\theta + \frac{1}{(2k+1)!}\delta\bar{\theta}\Gamma^m d\theta d\bar{\theta}\Gamma_{11}^k \Gamma_{mm_1\ldots m_{2k+1}} d\theta \tag{3.60}$$

The contribution from $2\delta\bar{\theta} dT_A d\theta$ has precisely the same form, except that the δ's appear in the second factor of each term. Both of the terms in (3.60) have the structure $\delta\bar{\theta} X d\theta d\bar{\theta} Y d\theta$, which involves $X_{\alpha(\beta}Y_{\gamma\delta)}$. However, when $d\bar{\theta} X d\theta \delta\bar{\theta} Y d\theta$ is added, the totally symmetric combination $X_{(\alpha\beta}Y_{\gamma\delta)}$ is formed. This implies that it is sufficient to prove the vanishing of the sum of the two terms above with $\delta\theta$ replaced by $d\theta$

$$(2k+1)d\bar{\theta}\Gamma_{11}\Gamma_{[m_1}d\theta d\bar{\theta}\Gamma_{11}^{k+1}\Gamma_{m_2\ldots m_{2k+1}]}d\theta + d\bar{\theta}\Gamma^m d\theta d\bar{\theta}\Gamma_{11}^k\Gamma_{mm_1\ldots m_{2k+1}}d\theta \qquad (3.61)$$

since this enforces total symmetrization. This is an identity we need anyway, to prove closure of the Wess-Zumino forms I_{p+2}.

The next step is to transform the second term in (3.61) using the formula

$$\Gamma_m\Gamma_{m_1\ldots m_{2k+1}} = \Gamma_{mm_1\ldots m_{2k+1}} + (2k+1)\eta_{m[m_1}\Gamma_{m_2\ldots m_{2k+1}]} \qquad (3.62)$$

The two terms on the RHS of this formula have opposite symmetry in spinor indices, so only one of them survives when sandwiched in between $d\theta$'s. In the present (IIA) case, it is the first one that survives, which means that we can pull out a factor Γ_m from the antisymmetrized product for free. Next, eq. (3.6) for Majorana-Weyl spinors in 10 dimensions implies that

$$d\bar{\theta}\Gamma^m d\theta d\bar{\theta}\Gamma_{11}\Gamma_m\Gamma_{m_1\ldots m_{2k+1}}\Gamma_{11}^{k+1}d\theta = -d\bar{\theta}\Gamma_{11}\Gamma^m d\theta d\bar{\theta}\Gamma_m\Gamma_{m_1\ldots m_{2k+1}}\Gamma_{11}^{k+1}d\theta \qquad (3.63)$$

We now use eq. (3.62) a second time. This time only the second term on the RHS survives, because we have removed a Γ_{11}, which reverses the symmetry. This leaves the negative of the first term in eq. (3.61), and thus the proof is complete.

The IIB proof is essentially the same.

Appendix D: Dimensional Reduction of $\zeta^{(p)}$

As indicated in eq. (3.20), $\zeta^{(p)}$ is the (12) block of $\gamma^{(p)}$. Moreover $\gamma^{(p)}$ is related to $\rho^{(p)}$ by $\gamma^{(p)} = (-\det M^{(p)})^{-1/2}\rho^{(p)}$. We showed in sect. (4.3) that, dropping the dependence on one coordinate (σ^p, say), $M^{(p)}$ reduces to $M^{(p-1)}$. Therefore, we need to study the dimensional reduction of the (12) block of $\rho^{(p)}$. However, $\rho^{(p)}$ is conveniently described by a $(p+1)$-form, as explained in sect. (3.2), and these are conveniently summed to give ρ_A and ρ_B. Their (12) blocks, given in eqs. (2.51) and

(2.59), can be combined to give

$$\rho = e^{\mathcal{F}+\psi} \tag{3.64}$$

The $(p+1)$-form part of this, evaluated in the static gauge, determines $\zeta^{(p)}$ for all p.

Now we evaluate $\mathcal{F}^{(p)} = F^{(p)} - b^{(p)}$ and $\psi^{(p)}$ in the static gauge. With the conventions described in the text, one obtains in both the IIA and IIB cases

$$\begin{aligned} b^{(p)} &= \bar{\lambda}\Gamma_\rho d\lambda d\sigma^\rho + \bar{\lambda}\Gamma_i d\lambda d\phi^i \\ \psi^{(p)} &= \Gamma_\rho d\sigma^\rho + \Gamma_i d\phi^i + \Gamma_m \bar{\lambda}\Gamma^m d\lambda \end{aligned} \tag{3.65}$$

where ρ runs from 0 to p, i runs from $p+1$ to 9, and m runs from 0 to 9. Upon dimensional reduction, dropping σ^p derivatives,

$$\begin{aligned} \mathcal{F}^{(p)} &\to \mathcal{F}^{(p-1)} + \bar{\lambda}\Gamma_p d\lambda d\phi^p - (d\phi^p + \bar{\lambda}\Gamma_p d\lambda) d\sigma^p \\ \psi^{(p)} &\to \psi^{(p-1)} - \Gamma_p d\phi^p + \Gamma_p d\sigma^p \end{aligned} \tag{3.66}$$

The next step is to see what these imply for $\exp \mathcal{F}^{(p)}$ and $\exp \psi^{(p)}$. Since the extra terms in the reduction of $\mathcal{F}^{(p)}$ square to zero,

$$e^{\mathcal{F}^{(p)}} \to (1 + \bar{\lambda}\Gamma_p d\lambda d\phi^p - (d\phi^p + \bar{\lambda}\Gamma_p d\lambda) d\sigma^p) e^{\mathcal{F}^{(p-1)}} \tag{3.67}$$

The extra terms in $\psi^{(p)}$ also square to zero, but care is required since they do not commute with $\psi^{(p-1)}$. The general formula that applies to such a case is that the part of $\exp(X+Y)$ that is linear in Y is given by $(Y + \frac{1}{2}[X,Y] + \frac{1}{6}[X,[X,Y]] + \ldots)e^X$. Using this, we find that

$$e^{\psi^{(p)}} \to (1 + (\Gamma^p + d\phi^p + \bar{\lambda}\Gamma^p d\lambda) d\sigma^p - (\Gamma^p + \bar{\lambda}\Gamma^p d\lambda) d\phi^p) e^{\psi^{(p-1)}} \tag{3.68}$$

We now require the part of the dimensional reduction of $\exp(\mathcal{F}^{(p)} + \psi^{(p)})$ that is proportional to $d\sigma^p$. Several terms cancel and we obtain

$$e^{\mathcal{F}^{(p)}+\psi^{(p)}} \to \Gamma_p d\sigma^p e^{\mathcal{F}^{(p-1)}+\psi^{(p-1)}} = (-1)^p \Gamma_p e^{\mathcal{F}^{(p-1)}+\psi^{(p-1)}} d\sigma^p \tag{3.69}$$

This result implies that

$$\zeta^{(p)} \to (-1)^p \Gamma_p \zeta^{(p-1)} \tag{3.70}$$

as asserted in the text.

Chapter 4

M-theory Fivebrane Action

In this chapter we will present in detail the super-fivebrane action in flat background along the lines of [4]. We start by describing the supersymmetric version of the action given in section 2.4.3, then we check the general coordinate invariance of the action, we prove kappa symmetry, we write down the PST covariant version of our formulas and finally we perform the double dimensional reduction to the $D4$-brane of type IIA.

4.1 Supersymmetrization of the Non-Covariant Action

Following the same procedure we used for p-branes and D-branes, we introduce superspace coordinates X^M [1] and θ with supersymmetry transformations

$$\delta\theta = \epsilon \quad \text{and} \quad \delta X^M = \bar{\epsilon}\Gamma^M\theta \tag{4.1}$$

The index M takes the values $M = 0, 1, \ldots, 9, 11$. Skipping $M = 10$ may seem a bit peculiar, but then X^{11} is the eleventh dimension. Also, the Dirac matrix $\Gamma_{11} = \Gamma_0\Gamma_1 \ldots \Gamma_9$, which appears in ten dimensions as a chirality operator, is precisely the matrix we associate with the eleventh dimension. The spinors ϵ and θ are 32-component Majorana spinors. The Dirac algebra is

$$\{\Gamma_M, \Gamma_N\} = 2\eta_{MN} \tag{4.2}$$

where η_{MN} is the 11d Lorentz metric with signature $(- + + \ldots +)$.

As in other supersymmetric p-brane theories, two supersymmetric quantities are

[1] We use capital letters for eleven dimensional space-time indices.

$\partial_{\hat{\mu}}\theta$ and

$$\Pi^M_{\hat{\mu}} = \partial_{\hat{\mu}} X^M - \bar{\theta}\Gamma^M \partial_{\hat{\mu}}\theta \tag{4.3}$$

The appropriate choice for the world-volume metric is then the supersymmetric quantity

$$G_{\hat{\mu}\hat{\nu}} = \eta_{MN}\Pi^M_{\hat{\mu}}\Pi^N_{\hat{\nu}} \tag{4.4}$$

Taking θ and X^M to be scalars under world-volume general coordinate transformations, $G_{\hat{\mu}\hat{\nu}}$ transforms in the standard way.

In addition, we require an appropriate supersymmetric extension of $H = dB$, which we write as

$$\mathcal{H}_{\mu\nu\rho} = H_{\mu\nu\rho} - b_{\mu\nu\rho} \tag{4.5}$$

or, in terms of differential forms, $\mathcal{H} = H - b_3$. The idea is to choose a b_3 whose supersymmetry variation is exact, so that it can be cancelled by an appropriate variation of B. The appropriate choice turns out to be

$$b_3 = \frac{1}{6} b_{\mu\nu\rho} d\sigma^\mu d\sigma^\nu d\sigma^\rho = \frac{1}{2}\bar{\theta}\Gamma_{MN} d\theta (dX^M dX^N + dX^M \bar{\theta}\Gamma^N d\theta + \frac{1}{3}\bar{\theta}\Gamma^M d\theta \bar{\theta}\Gamma^N d\theta) \tag{4.6}$$

Varying this, using $\delta_\epsilon\theta = \epsilon$ and $\delta_\epsilon X^M = \bar{\epsilon}\Gamma^M\theta$, one finds that $\mathcal{H}$ is invariant for the choice

$$\delta_\epsilon B = -\frac{1}{2}\bar{\epsilon}\Gamma_{MN}\theta(dX^M dX^N + \frac{2}{3}\bar{\theta}\Gamma^M d\theta dX^N + \frac{1}{15}\bar{\theta}\Gamma^M d\theta\bar{\theta}\Gamma^N d\theta)$$
$$-\frac{1}{6}\bar{\epsilon}\Gamma_M\theta\bar{\theta}\Gamma_{MN} d\theta (dX^N + \frac{1}{5}\bar{\theta}\Gamma^N d\theta) \tag{4.7}$$

A useful (and standard) identity that has been used in deriving this result is

$$d\bar{\theta}\Gamma^M d\theta d\bar{\theta}\Gamma_{MN} + d\bar{\theta}\Gamma_{MN} d\theta d\bar{\theta}\Gamma^M = 0 \tag{4.8}$$

The overall normalization of b_3 and $\delta_\epsilon B$ could be scaled arbitrarily (including zero) as far as the present reasoning is concerned. The specific choice that has been made is the one that will be required later. We also note, for future reference, that

$$d\mathcal{H} = -db_3 = -\frac{1}{2} d\bar{\theta}\Gamma_{MN} d\theta \Pi^M \Pi^N = -\frac{1}{2} d\bar{\theta}\psi_5^2 d\theta \tag{4.9}$$

where we have introduced the matrix valued one-form

$$\psi_5 = \Gamma_M \Pi^M_\mu d\sigma^\mu \tag{4.10}$$

With these choices for $G_{\hat{\mu}\hat{\nu}}$ and $\mathcal{H}$, we can now write down extensions of L_1 and L_3 that have manifest 11d super-Poincaré symmetry:

$$\begin{aligned} L_1 &= -\sqrt{-G}\sqrt{1 + z_1 + \frac{1}{2}z_1^2 - z_2} \\ L_3 &= \frac{1}{8}\epsilon_{\mu\nu\rho\lambda\sigma}\frac{G^{5\rho}}{G^{55}}\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\lambda\sigma} \end{aligned} \tag{4.11}$$

where z_1 and z_2 are now formed from $\mathcal{H}$ instead of H.

The next step is to construct a supersymmetric extension of L_2. This term is the Wess–Zumino term, which can be represented as the integral of a closed 7-form I_7 over a region that has the 6d world-volume M_6 as its boundary. In other words,

$$S_2 = \int_{M_7} I_7 = \int_{M_6} \Omega_6 \tag{4.12}$$

where $I_7 = d\Omega_6$ and $M_6 = \partial M_7$. The appropriate expression for I_7 that reproduces L_2 of the purely bosonic theory is

$$I_7^{(B)} = -\frac{1}{2}HdH = \frac{1}{2}H\partial_5 H d\sigma^5 \tag{4.13}$$

To understand this properly, there is a point that needs to be stressed. Namely, in adding a formal 7th dimension, the extra dimension is required to enter symmetrically with the first five. There continues to be one preferred direction, σ^5, that is treated specially. Correspondingly, in writing $M_6 = \partial M_7$, the boundary operator should not act on the σ^5 direction. In other words, M_7 should have no σ^5 = constant faces. It should also be noted that this M theory five-brane theory action has a Wess–Zumino term that survives even for the bosonic truncation in a flat space-time background. However, as we will see in the next section, this feature is particular to the non-covariant formulation and is not shared by the PST formulation in which the pieces of the action are arranged somewhat differently.

To complete the construction of L_2 we must now supersymmetrize $I_7^{(B)}$. The term $\frac{1}{2}\mathcal{H}\partial_5\mathcal{H}d\sigma^5$ achieves this, of course, but it is no longer closed. Additional terms should be added such that $dI_7 = 0$, up to a total derivative in the σ^5 direction. The result that we find is

$$I_7 = \frac{1}{2}\mathcal{H}\partial_5\mathcal{H}d\sigma^5 - \frac{1}{2}\mathcal{H}d\bar{\theta}\hat{\psi}^2 d\theta - \frac{1}{120}d\bar{\theta}\hat{\psi}^5 d\theta \qquad (4.14)$$

where

$$\hat{\psi} = \Gamma_M \Pi^M_{\hat{\mu}} d\sigma^{\hat{\mu}} = \psi_5 + \Gamma_M \Pi^M_5 d\sigma^5 \qquad (4.15)$$

When interpreting the 4-form $d\theta\hat{\psi}^2 d\theta$ and the 7-form $d\theta\hat{\psi}^5 d\theta$ it must be understood that one of the derivatives is required to be in the σ^5 direction. The proof that dI_7 is a total σ^5 derivative is reasonably straightforward using the identity (4.8) as well as

$$\frac{1}{6}(d\bar{\theta}\Gamma_{MNPQR}d\theta d\bar{\theta}\Gamma^R + d\bar{\theta}\Gamma^R d\theta d\bar{\theta}\Gamma_{MNPQR}) = d\bar{\theta}\Gamma_{[MN}d\theta d\bar{\theta}\Gamma_{PQ]} \qquad (4.16)$$

Since I_7 is manifestly supersymmetric, it is guaranteed that Ω_6 is invariant up to a total derivative under a supersymmetry transformation. For most purposes an explicit formula for L_2 is not required. Here we will simply report that

$$L_2 = -\frac{1}{4}\tilde{H}^{\mu\nu}(\partial_5 B_{\mu\nu} - 2b_{\mu\nu}) + \text{ terms indep. of } B \qquad (4.17)$$

where $b_2 = \frac{1}{2}b_{\mu\nu}d\sigma^\mu d\sigma^\nu$ is given by[2]

$$\begin{aligned} b_2 \;=\; & -\frac{1}{2}\bar{\theta}\Gamma_{MN}\partial_5\theta(dX^M dX^N + dX^M\bar{\theta}\Gamma^N d\theta + \frac{1}{3}d\bar{\theta}\Gamma^M d\theta d\bar{\theta}\Gamma^N d\theta) \qquad (4.18)\\ & +\frac{1}{2}\bar{\theta}\Gamma_{MN}d\theta(2dX^M\partial_5 X^N - \partial_5 X^M\bar{\theta}\Gamma^N d\theta - dX^M\bar{\theta}\Gamma^N\partial_5\theta - \frac{2}{3}\bar{\theta}\Gamma^M d\theta\bar{\theta}\Gamma^N\partial_5\theta) \end{aligned}$$

Knowing this much of L_2 is sufficient to obtain the $B_{\mu\nu}$ equation of motion.

4.2 General Coordinate Invariance

We should now check whether the general coordinate invariance of the bosonic theory in section 2.4.3 continues to hold after adding terms depending on θ in the way that we

[2]This expression is equal to $b_{\mu\nu5}$, where $b_{\hat{\mu}\hat{\nu}\hat{\rho}}$ is the covariant extension of the expression given in eq. (4.6).

have described. As in the bosonic case, general coordinate invariance in five directions is manifest, so only the transformation in the σ^5 direction needs to be checked. The coordinates X^M and θ transform as scalars, *i.e.*,

$$\delta_\xi X^M = \xi \partial_5 X^M \quad \text{and} \quad \delta_\xi \theta = \xi \partial_5 \theta \tag{4.19}$$

which implies that $G_{\hat{\mu}\hat{\nu}}$ transforms as in eq. (2.104). To specify the proper transformation law for $B_{\mu\nu}$, we should first examine its equation of motion. Using eq. (4.17), this is

$$\epsilon^{\mu\nu\rho\lambda\sigma} \partial_\rho \left(K_{\lambda\sigma} - \partial_5 B_{\lambda\sigma} + b_{\lambda\sigma} \right) = 0 \tag{4.20}$$

The formula for $K_{\mu\nu}$ is as given in eqs. (2.105) and (2.107), except that now L_1 and L_3 of the supersymmetrized theory should be used. This simply amounts to replacing H by $\mathcal{H}$ and using the supersymmetric expression for $G_{\hat{\mu}\hat{\nu}}$. By the reasoning explained in ref. [31], the B equation of motion suggests that the appropriate transformation formula, generalizing eq. (2.106), is

$$\delta_\xi B_{\mu\nu} = \xi (K_{\mu\nu} + b_{\mu\nu}) \tag{4.21}$$

To determine $\delta_\xi \mathcal{H}$, one first computes that

$$\delta_\xi b_3 = \xi \partial_5 b_3 + b_2 d\xi \tag{4.22}$$

It follows that

$$\delta_\xi \mathcal{H} = d(\delta_\xi B) - \xi \partial_5 b_3 - b_2 d\xi = d(\xi K) - \xi Z_3 \tag{4.23}$$

where

$$Z_3 = \partial_5 b_3 - d b_2 \tag{4.24}$$

This can be made manifestly supersymmetric by noting that

$$Z_3 d\sigma^5 = (\partial_5 b_3 - d b_2) d\sigma^5 = -\frac{1}{2} d\bar{\theta} \hat{\psi}^2 d\theta \tag{4.25}$$

The 4-form on the right-hand side of this equation is required to contain one σ^5 derivative.

The important point is that the Z_3 term in $\delta_\xi \mathcal{H}$ has no counterpart in the bosonic theory, so general coordinate invariance of the supersymmetric theory is not an immediate consequence of the corresponding symmetry of the bosonic theory. Let us examine next the part of $\delta_\xi(L_1 + L_3)$ that arises from varying $\mathcal{H}$, but not G. It is

$$\delta_\xi \tilde{\mathcal{H}}^{\mu\nu} \frac{\partial(L_1 + L_3)}{\partial \tilde{\mathcal{H}}^{\mu\nu}} = \frac{1}{2} \delta_\xi \tilde{\mathcal{H}}^{\mu\nu} K_{\mu\nu} \tag{4.26}$$

This is conveniently characterized by the 5-form

$$(d(\xi K) - \xi Z_3) K \sim -\xi K (dK + Z_3) \tag{4.27}$$

where $\sim$ means that a total derivative has been dropped.

Consider now the ξ transformation of L_2. A portion of L_2 was given in eq. (4.17). Representing this as a 5-form and using

$$\delta_\xi b_2 = \partial_5(\xi b_2) \tag{4.28}$$

one obtains

$$\begin{aligned} \delta_\xi L_2 &= -(\partial_5 B - b_2) d(\xi(K + b_2)) + H \partial_5(\xi b_2) + \ldots \\ &\sim \xi K(\partial_5 \mathcal{H} + Z_3) + \frac{1}{2} b_2^2 d\xi + \ldots \end{aligned} \tag{4.29}$$

where the dots are the contribution from varying the H independent terms in L_2. The $\ldots$ terms precisely cancel the b_2^2 term, leaving

$$\delta_\xi L_2 \sim \xi K(\partial_5 \mathcal{H} + Z_3) \tag{4.30}$$

The demonstration that the $\ldots$ terms contribute $-\frac{1}{2} b_2^2 d\xi$ can be made as follows. The first two terms in eq. (4.14) contribute the non-H pieces

$$\frac{1}{2} b_3 \partial_5 b_3 d\sigma^5 + \frac{1}{2} b_3 d\bar{\theta} \hat{\psi}^2 d\theta \tag{4.31}$$

which has a non-trivial ξ transformation, because of the asymmetric way in which the σ^5 direction appears. The variation is easy to compute, and can be expressed as the exterior derivative of $-\frac{1}{2}b_2^2 d\xi$, which implies that this contributes the required variation of L_2.

Combining eq. (4.30) with eq. (4.27) leaves

$$\delta_{\mathcal{H}}(L_1 + L_3) + \delta_\xi L_2 \sim \xi K(\partial_5 \mathcal{H} - dK) \tag{4.32}$$

This must now be combined with the terms arising from varying $G_{\hat{\mu}\hat{\nu}}$ in L_1 and L_3. However, at this point all terms whose structure is peculiar to the supersymmetric theory have cancelled. The rest of the calculation is identical to that for the bosonic theory given in ref. [31] and, therefore, need not be repeated here.

4.3 Proof of Kappa Symmetry

As with all other super p-branes of maximally supersymmetric theories, the world-volume theory should have 8 bosonic and 8 fermionic physical degrees of freedom. This requires, in particular, the existence of a local fermionic symmetry (called kappa) that eliminates half of the components of θ. Despite the lack of manifest general coordinate invariance, the analysis of kappa symmetry for the M theory five-brane is very similar to that of other super p-branes. As usual, we require that

$$\delta\bar{\theta} = \bar{\kappa}(1 - \gamma) \tag{4.33}$$

where $\kappa(\sigma)$ is an arbitrary Majorana spinor and γ is a quantity (to be determined) whose square is the unit matrix. This implies that $\frac{1}{2}(1 - \gamma)$ is a projection operator, and half of the components of θ can be gauged away. In addition, just as for all other super p-branes, we require that

$$\delta X^M = -\delta\bar{\theta}\Gamma^M\theta \tag{4.34}$$

so that

$$\delta\Pi_{\hat{\mu}}^M = -2\delta\bar{\theta}\Gamma^M\partial_{\hat{\mu}}\theta \tag{4.35}$$

As in our other work [2, 3], we introduce the induced γ matrix

$$\gamma_{\hat{\mu}} = \Pi^M_{\hat{\mu}} \Gamma_M \tag{4.36}$$

which satisfies

$$\{\gamma_{\hat{\mu}}, \gamma_{\hat{\nu}}\} = 2G_{\hat{\mu}\hat{\nu}} \tag{4.37}$$

In this notation, the kappa variation of the metric is

$$\delta G_{\hat{\mu}\hat{\nu}} = -2\delta\bar{\theta}(\gamma_{\hat{\mu}}\partial_{\hat{\nu}} + \gamma_{\hat{\nu}}\partial_{\hat{\mu}})\theta \tag{4.38}$$

Before we can examine the symmetry of our theory, we must also specify the kappa variation of $B_{\mu\nu}$. This works in a way that is analogous to that of the world-volume gauge field for D-branes. Specifically, for the choice

$$\delta B = \frac{1}{2}\delta\bar{\theta}\Gamma_{MN}\theta(dX^M dX^N + \bar{\theta}\Gamma^M d\theta dX^N + \frac{1}{3}\bar{\theta}\Gamma^M d\theta\bar{\theta}\Gamma^N d\theta)$$

$$+\frac{1}{2}\delta\bar{\theta}\Gamma^M\theta\bar{\theta}\Gamma_{MN}d\theta(dX^N + \frac{1}{3}\bar{\theta}\Gamma^N d\theta) \tag{4.39}$$

we find that most of the terms in $\delta\mathcal{H}$ cancel leaving

$$\delta\mathcal{H}_{\mu\nu\rho} = 6\delta\bar{\theta}\gamma_{[\mu\nu}\partial_{\rho]}\theta \tag{4.40}$$

or, equivalently,

$$\delta\tilde{\mathcal{H}}^{\mu\nu} = \epsilon^{\mu\nu\rho\lambda\sigma}\delta\bar{\theta}\gamma_{\rho\lambda}\partial_\sigma\theta \tag{4.41}$$

Since we now have the complete theory and all the field transformations, it is just a matter of computation to check the symmetry.

Before plunging into the details of the calculation, it is helpful to sketch the general strategy that will be employed. It turns out to be convenient to consider L_2 and L_3 together and to write their kappa variation in the form

$$\delta(L_2 + L_3) = \frac{1}{2}\delta\bar{\theta}T^{\hat{\mu}}\partial_{\hat{\mu}}\theta \tag{4.42}$$

The variation of L_1 is represented in a similar manner:

$$\delta L_1 = -\frac{1}{2L_1}\delta\bar{\theta}U^{\hat{\mu}}\partial_{\hat{\mu}}\theta \tag{4.43}$$

Then, in order that $\delta\bar{\theta} = \bar{\kappa}(1-\gamma)$ should be a symmetry, we require that altogether

$$\delta(L_1 + L_2 + L_3) = \frac{1}{2}\delta\bar{\theta}(1+\gamma)T^{\hat{\mu}}\partial_{\hat{\mu}}\theta \tag{4.44}$$

which is achieved if

$$U^{\hat{\mu}} = \rho T^{\hat{\mu}} \tag{4.45}$$

where

$$\rho = -\gamma L_1 = \gamma\sqrt{-G}\sqrt{1 + z_1 + \frac{1}{2}z_1^2 - z_2} \tag{4.46}$$

This implies that

$$\rho^2 = -G(1 + z_1 + \frac{1}{2}z_1^2 - z_2) \tag{4.47}$$

We must vary the Lagrangian to find $T^{\hat{\mu}}$ and $U^{\hat{\mu}}$, and then determine ρ with the proper square and show that $U^{\hat{\mu}} = \rho T^{\hat{\mu}}$. This is all straightforward, but it needs to be done carefully.

Since the σ^5 direction appears asymmetrically in the Lagrangian, the analysis of $U^{\hat{\mu}} = \rho T^{\hat{\mu}}$ is naturally split into two separate problems, corresponding to $\hat{\mu} = 5$ and $\hat{\mu} \neq 5$. The $\hat{\mu} = 5$ case is the easier of the two, so let us begin with that. We must examine where we can get $\partial_5\theta$'s. The variations of $B_{\mu\nu}$ and $G_{\mu\nu}$ do not give any. Therefore, in varying L_1, the variations of z_1 and z_2 do not contribute. The only contribution comes from

$$\delta\sqrt{-G} = -2\sqrt{-G}\delta\bar{\theta}\gamma^{\hat{\mu}}\partial_{\hat{\mu}}\theta \tag{4.48}$$

where, of course, $\gamma^{\hat{\mu}} = G^{\hat{\mu}\hat{\nu}}\gamma_{\hat{\nu}}$. Thus

$$U^5 = -4\rho^2\gamma^5 \tag{4.49}$$

To determine T^5 we must vary $L_2 + L_3$. Using the identity

$$\delta\left(\frac{G^{5\rho}}{G^{55}}\right) = 2\frac{G_5^{\eta\rho}G^{5\hat{\mu}}}{G^{55}}\delta\bar{\theta}(\gamma_{\hat{\mu}}\partial_\eta + \gamma_\eta\partial_{\hat{\mu}})\theta \tag{4.50}$$

the relevant piece of δL_3 is

$$\frac{1}{4}\epsilon_{\mu\nu\rho\lambda\sigma}G_5^{\eta\rho}\delta\bar{\theta}\gamma_\eta\partial_5\theta\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\lambda\sigma} \tag{4.51}$$

which contributes

$$T_2^5 = \frac{1}{2}\epsilon_{\mu\nu\rho\lambda\sigma}G_5^{\eta\rho}\gamma_\eta\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\lambda\sigma} \tag{4.52}$$

to T^5. (The subscript on T represents the power of $\mathcal{H}$.)

The variation of the Wess–Zumino term S_2 is

$$\delta S_2 = \int(\mathcal{H}\delta\bar{\theta}\hat{\psi}^2 d\theta - \frac{1}{60}\delta\bar{\theta}\hat{\psi}^5 d\theta) \tag{4.53}$$

a result that is obtained by expressing δI_7 as a total differential. This determines $T_0^5 + T_1^5$, with

$$T_0^5 = -\frac{1}{30}\epsilon^{\mu_1\ldots\mu_5}\gamma_{\mu_1\ldots\mu_5} = -4\bar{\gamma}\gamma^5 \tag{4.54}$$

where we have introduced

$$\bar{\gamma} = \gamma_{012345} \tag{4.55}$$

which satisfies $(\bar{\gamma})^2 = -G$. The $\mathcal{H}$ linear term is

$$T_1^5 = -2\tilde{\mathcal{H}}^{\mu\nu}\gamma_{\mu\nu} \tag{4.56}$$

Combining these results with

$$U^5 = -4\rho^2\gamma^5 = \rho T^5 \tag{4.57}$$

we infer that $T^5 = -4\rho\gamma^5$, where

$$\rho = \bar{\gamma} + \frac{1}{2G^{55}}\tilde{\mathcal{H}}^{\nu\rho}\gamma_{\nu\rho}\gamma^5 - \frac{1}{8G^{55}}\epsilon_{\mu\nu\rho\lambda\sigma}\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\rho\lambda}\gamma^{\sigma 5} \tag{4.58}$$

To obtain the $\mathcal{H}^2$ term we have used the identity

$$G_5^{\eta\sigma}\gamma_\eta = \gamma^\sigma - \frac{G^{\sigma 5}}{G^{55}}\gamma^5 \tag{4.59}$$

from which it follows that

$$G_5^{\eta\sigma}\gamma_\eta\gamma^5 = \gamma^{\sigma 5} \tag{4.60}$$

If our reasoning is correct, this expression for ρ should have the square given in eq. (4.47). For the proof of this fact we refer the reader to Appendix A of [4].

To complete the proof of kappa symmetry, we must find U^μ and T^μ and show that $U^\mu = \rho T^\mu$. Separating powers of $\mathcal{H}$, as above, the variation of L_2 contributes to T_0^μ and T_1^μ while the variation of L_3 contributes to T_1^μ and T_2^μ. Altogether, we find that

$$\begin{aligned}
T_0^\mu &= -4\bar{\gamma}\gamma^\mu \\
T_1^\mu &= -\frac{2}{G^{55}}(G^{5\mu}\tilde{\mathcal{H}}^{\nu\rho}\gamma_{\nu\rho} + 2\tilde{\mathcal{H}}^{\mu\nu}\gamma_\nu\gamma^5) \\
T_2^\mu &= \frac{1}{2G^{55}}\epsilon_{\eta\nu\rho\lambda\sigma}\tilde{\mathcal{H}}^{\nu\rho}\tilde{\mathcal{H}}^{\lambda\sigma}(G^{5\mu}G_5^{\eta\zeta}\gamma_\zeta + G_5^{\mu\eta}\gamma^5)
\end{aligned} \tag{4.61}$$

The variation of L_1 determines $U^\mu = \sum_{n=0}^4 U_n^\mu$, where

$$\begin{aligned}
U_0^\mu &= 4G\gamma^\mu \\
U_1^\mu &= -\frac{1}{G^{55}}\epsilon^{\mu\nu\rho\lambda\sigma}\gamma_{\lambda\sigma}(G\tilde{\mathcal{H}}G)_{\nu\rho} \\
U_2^\mu &= -\frac{4}{G^{55}}\gamma_\nu(\tilde{\mathcal{H}}G\tilde{\mathcal{H}})^{\mu\nu} - \frac{2}{(G^{55})^2}G^{5\mu}\gamma^5\mathrm{tr}(G\tilde{\mathcal{H}}G\tilde{\mathcal{H}}) \\
U_3^\mu &= \frac{1}{G(G^{55})^2}\epsilon^{\mu\nu\rho\lambda\sigma}\gamma_{\lambda\sigma}\left(\frac{1}{2}(G\tilde{\mathcal{H}}G)_{\nu\rho}\mathrm{tr}(G\tilde{\mathcal{H}}G\tilde{\mathcal{H}}) - (G\tilde{\mathcal{H}}G\tilde{\mathcal{H}}G\tilde{\mathcal{H}}G)_{\nu\rho}\right) \\
U_4^\mu &= \frac{4}{G(G^{55})^2}\gamma_\nu\left(\frac{1}{2}(\tilde{\mathcal{H}}G\tilde{\mathcal{H}})^{\mu\nu}\mathrm{tr}(G\mathcal{H}G\mathcal{H}) - (\tilde{\mathcal{H}}G\tilde{\mathcal{H}}G\tilde{\mathcal{H}}G\tilde{\mathcal{H}})^{\mu\nu}\right) \\
&+ \frac{2}{G(G^{55})^2}\left(G^{\mu 5}\gamma^5 - \frac{1}{2}G^{55}\gamma^\mu\right)\left(\frac{1}{2}(\mathrm{tr}(G\tilde{\mathcal{H}}G\tilde{\mathcal{H}}))^2 - \mathrm{tr}(G\mathcal{H}G\mathcal{H}G\mathcal{H}G\mathcal{H})\right)
\end{aligned} \tag{4.62}$$

The demonstration that $U^\mu = \rho T^\mu$ is presented in Appendix B of [4].

In conclusion, we have shown that the theory specified by $L_1 + L_2 + L_3$ has all the desired symmetries: global 11d super–Poincaré symmetry, general coordinate invariance, and local kappa symmetry.

4.4 Supersymmetric Theory in the PST Formulation

The supersymmetric theory that we have just presented can be recast in a manifestly general covariant form, using the PST formalism, just as we did for the bosonic theory in sect. 2.2. In order to keep the notation from being too cumbersome, in this section (and only in this section) indices μ, ν, etc., take six values, (*i.e.*, we drop the hats used until now). Also the label "cov." is dropped. Thus, upon supersymmetrization, eq. (2.112), for example, becomes

$$M_{\mu\nu} = G_{\mu\nu} + i\frac{G_{\mu\rho}G_{\nu\lambda}}{\sqrt{-G(\partial a)^2}}\tilde{\mathcal{H}}^{\rho\lambda} \tag{4.63}$$

where

$$\tilde{\mathcal{H}}^{\rho\lambda} = \frac{1}{6}\epsilon^{\rho\lambda\mu\nu\sigma\tau}\mathcal{H}_{\mu\nu\sigma}\partial_\tau a \tag{4.64}$$

Also, $G_{\mu\nu}$ is constructed as in eqs. (4.3) and (4.4), and $\mathcal{H} = H - b_3$ is extended to six dimensions. In this notation the supersymmetric theory is given by $L = L_1 + L' + L_{WZ}$, where

$$\begin{aligned} L_1 &= -\sqrt{-\det M_{\mu\nu}} \\ L' &= -\frac{1}{4(\partial a)^2}\tilde{\mathcal{H}}^{\mu\nu}\mathcal{H}_{\mu\nu\rho}G^{\rho\lambda}\partial_\lambda a \\ S_{WZ} &= \int \Omega_6 \end{aligned} \tag{4.65}$$

L_1 can again be recast in the form

$$L_1 = -\sqrt{-G}\sqrt{1 + z_1 + \frac{1}{2}z_1^2 - z_2} \tag{4.66}$$

where now z_1 and z_2 are the obvious covariant counterparts of those in eq. (2.102). The Wess–Zumino term is again characterized by a seven-form $I_7 = d\Omega_6$, where now

$$I_7 = -\frac{1}{4}\mathcal{H}d\bar{\theta}\hat{\psi}^2 d\theta - \frac{1}{120}d\bar{\theta}\hat{\psi}^5 d\theta \tag{4.67}$$

It is easy to check that $dI_7 = 0$ using eqs. (4.8) and (4.16). Global ϵ supersymmetry and local reparametrization symmetry are manifest in these formulas. Note that neither the metric $G_{\mu\nu}$ nor the scalar field a occur in L_{WZ}.

When one chooses the gauge $a = \sigma^5$ and $B_{\mu 5} = 0$, the Lagrangian given above reduces to the one in sect. 3. The way this happens is somewhat non-trivial. The point is that L' reduces to L_3 and a portion of the non-covariant Wess–Zumino term L_2. Specifically, in the gauge-fixed theory the sum over the index ρ in the formula for L' can be separated into $\rho = 5$ and $\rho \neq 5$ terms. The $\rho \neq 5$ term accounts for L_3 of the gauge-fixed theory, while the $\rho = 5$ term accounts for the $\mathcal{H}^2$ piece of L_2 and a portion of the $\mathcal{H}$ piece. In particular, this accounts for why the coefficient of the $\mathcal{H}$ linear term in eq. (4.67) differs from that in eq. (4.14).

The proof of kappa symmetry in the PST formulation works as before (with $\delta a = 0$), so we will not repeat the argument. The covariant extension of eq. (4.58) is

$$\rho = \bar{\gamma} + \frac{1}{2(\partial a)^2}\tilde{\mathcal{H}}^{\nu\rho}\gamma_{\nu\rho}\gamma^{\lambda}\partial_{\lambda}a - \frac{1}{16(\partial a)^2}\epsilon_{\mu\nu\rho\lambda\sigma\tau}\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\rho\lambda}\gamma^{\sigma\tau} \tag{4.68}$$

The demonstration that $\rho^2 = -\det M_{\mu\nu}$ is essentially the same as in Appendix A. The covariant formula for $T^{\mu} = T_0^{\mu} + T_1^{\mu} + T_2^{\mu}$ is given by

$$\begin{aligned} T_0^{\mu} &= -4\bar{\gamma}\gamma^{\mu} \\ T_1^{\mu} &= -\frac{2}{(\partial a)^2}\tilde{\mathcal{H}}^{\nu\rho}(\gamma_{\nu\rho}G^{\mu\lambda} - 2\delta^{\mu}_{\rho}\gamma_{\nu}\gamma^{\lambda})\partial_{\lambda}a \\ T_2^{\mu} &= -\frac{1}{(\partial a)^2}\tilde{\mathcal{H}}^{\eta\nu}\mathcal{H}_{\eta\nu\rho}(\gamma^{\rho}G^{\lambda\mu} + \gamma^{\lambda}G^{\rho\mu})\partial_{\lambda}a \\ &\quad + \frac{2}{[(\partial a)^2]^2}\tilde{\mathcal{H}}^{\eta\nu}\mathcal{H}_{\eta\nu\rho}G^{\rho\lambda}\partial_{\lambda}a\gamma^{\sigma}\partial_{\sigma}aG^{\mu\zeta}\partial_{\zeta}a \end{aligned} \tag{4.69}$$

In the $B_{\mu 5} = 0$, $a = \sigma^5$ gauge, these expressions reduce to the formulas T^5 and T^{μ} given in eqs. (4.52), (4.54), (4.56), and (4.61). The proof of kappa symmetry works essentially the same as before.

4.5 Double Dimensional Reduction

We are now ready to apply to the five-brane the procedure that was used in section 2.4.1 to recover the superstring from a wrapped $M2$-brane. In fact, we will need to use both double-dimensional reduction and world-volume duality to get to the usual $D4$-brane action. The calculations are significantly more complicated, but the principles remain the same.

The first step is double-dimensional reduction, which leads to a dual formulation of the $D4$-brane action in terms of an antisymmetric tensor field. As explained in section 2.4.1, we identify

$$X^{11} = \sigma^5 \tag{4.70}$$

and then drop all dependence on σ^5, *i.e.*, extract the zeroth Fourier mode. This corresponds to a compactification on a unit circle with the fivebrane wrapping once around the compact eleventh dimension. If we denote by $g_{\mu\nu}$ the ten-dimensional type IIA metric, then

$$G_5 \equiv \det\, G_{\mu\nu} = \det\,(g_{\mu\nu} + C_\mu C_\nu) = (\det\, g)(1 + C_\mu g^{\mu\nu} C_\nu) \equiv g\ (1 + C^2) \tag{4.71}$$

where $C_\mu = -\bar{\theta}\Gamma^{11}\partial_\mu\theta$ as before and, as in (2.65)

$$G \equiv \det\, G_{\hat{\mu}\hat{\nu}} = \det\left(\begin{array}{c|c} g_{\mu\nu} + C_\mu C_\nu & C_\mu \\ \hline C_\nu & 1 \end{array}\right) = \det\left(\begin{array}{c|c} g_{\mu\nu} & C_\mu \\ \hline 0 & 1 \end{array}\right) = g \tag{4.72}$$

From (4.9) we see that db_3 reduces as

$$\begin{aligned} db_3 &= \frac{1}{2} d\bar{\theta}\psi_5^2 d\theta \\ &= \frac{1}{2} d\bar{\theta}\psi^2 d\theta - d\bar{\theta}\Gamma_{11}\psi d\theta C_1 \\ &= dC_3 \end{aligned} \tag{4.73}$$

where $\psi \equiv \Gamma^m \Pi_m$ is the ten-dimensional gamma-matrix combination that appears in the action of the $D4$-brane. The last step follows from (2.70) and (2.72). Thus b_3 reduces to C_3, which appeared in the $D2$-brane "Wess-Zumino" term, and – as we will see below – is also part of the $D4$-brane action. The five-brane "Wess-Zumino" form (4.14) becomes, by analogy with (2.66),

$$I_7 \equiv I_6 d\sigma^5 = \left(-\frac{1}{4!}d\bar{\theta}\Gamma_{11}\psi^4 d\theta + \mathcal{H}d\bar{\theta}\Gamma_{11}\psi d\theta\right) d\sigma^5 \tag{4.74}$$

The "Wess-Zumino" form of the $D4$-brane, given by (2.38, 2.50), is

$$I_{6D} = -d\bar{\theta}\left(\frac{1}{4!}\Gamma_{11}\psi^4 + \frac{1}{2}\mathcal{F}\psi^2 + \frac{1}{2}\mathcal{F}^2\Gamma_{11}\right) d\theta = -d(C_5 + C_3\mathcal{F} + \frac{1}{2}C_1\mathcal{F}^2) \equiv d\Omega_{5D} \tag{4.75}$$

Thus we recover (2.72) and we also get

$$dC_5 + C_3 db = \frac{1}{4!}d\bar{\theta}\Gamma_{11}\psi^4 d\theta \tag{4.76}$$

We can therefore reorganize I_6 as

$$I_6 = -dC_5 - C_3 db - (H - C_3)db = -d(C_5 - Hb) \equiv d\Omega_5 \tag{4.77}$$

This is consistent with our earlier assertion (4.17), because the two form b_2 that appeared in 4.18 clearly reduces to b.

Thus far we found the reduction of L_2. By manipulations similar to (4.72) we determine that L_1 reduces to

$$\begin{aligned} L_1 &= -\sqrt{-\det\left(g_{\mu\nu} + i\frac{g_{\mu\rho}g_{\nu\lambda}\tilde{\mathcal{H}}^{\rho\lambda}}{\sqrt{-g(1+C^2)}} + Y_\mu Y_\nu\right)} \\ Y_\mu &\equiv i\frac{g_{\mu\rho}\tilde{\mathcal{H}}^{\rho\lambda}C_\lambda}{\sqrt{-g(1+C^2)}} \end{aligned} \tag{4.78}$$

For the reduction of L_3 we have

$$\begin{aligned} \frac{G^{5\mu}}{G^{55}} &= -\frac{g\, g^{\mu\rho}C_\rho}{G_5} = -\frac{g^{\mu\rho}C_\rho}{(1+C^2)}, \quad \text{so that} \\ L_3 &= -\frac{1}{8}\epsilon_{\mu\nu\rho\lambda\sigma}g^{\rho\tau}C_\tau\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\lambda\sigma} \end{aligned} \tag{4.79}$$

This completes the first step.

The second step is to relate the Lagrangian (4.78) to the usual $D4$-brane Lagrangian by using world-volume duality. This is achieved by subtracting the Lagrange multiplier

$$\frac{1}{6}\Lambda^{\mu\nu\rho}(H_{\mu\nu\rho} - \partial_\mu B_{\nu\rho}) \tag{4.80}$$

and treating H as an independent field. The B equation of motion leads to the requirement

$$\Lambda^{\mu\nu\rho} = \epsilon^{\mu\nu\rho\sigma\lambda}\partial_\sigma A_\lambda = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma\lambda}F_{\sigma\lambda} \tag{4.81}$$

which in turn eliminates B from the action.

At this point it is crucial to choose a good strategy for proceeding with the calculation, otherwise the equations quickly become intractable. We therefore express everything in terms of the five-dimensional metric G_5 and then use general coordinate invariance to set it to the flat metric η. After we invert the equations of motion, we should remember to revert to the proper IIA metric g. Using $g = G_5(1 - C_\mu (G_5^{-1})^{\mu\nu} C_\nu)$ we get the following action:

$$\begin{aligned} S_D &= -\int d^5\sigma \left(\sqrt{(1-C^2)\left(-\det\,(\eta_{\mu\nu} + i\tilde{\mathcal{H}}_{\mu\nu})\right)} + \frac{1}{8}\epsilon_{\mu\nu\rho\lambda\sigma}C^\rho\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\lambda\sigma}\right) \\ &\quad - \int (C_5 - Hb + HF) \\ &= -\int d^5\sigma \left(\sqrt{-\det\,(\eta_{\mu\nu} + i\tilde{\mathcal{H}}_{\mu\nu})} + \frac{1}{2}\tilde{\mathcal{H}}^{\mu\nu}\mathcal{F}_{\mu\nu} + \frac{1}{8}\epsilon_{\mu\nu\rho\lambda\sigma}C^\rho\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\lambda\sigma}\right) \\ &\quad - \int (C_5 + \mathcal{F}C_3) \end{aligned} \tag{4.82}$$

Unlike the case of the membrane, the calculations are too complicated to be done in a manifestly Lorentz invariant fashion. Instead, we Lorentz transform to the basis in which

$$\tilde{\mathcal{H}} = \left(\begin{array}{c|cc|cc} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & h_+ & 0 & 0 \\ 0 & -h_+ & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & h_- \\ 0 & 0 & 0 & -h_- & 0 \end{array}\right) \tag{4.83}$$

Without altering the canonical form of $\tilde{\mathcal{H}}$, we can further rotate in the $1-2$ and

$3-4$ directions to bring C_μ to the form $(c, c_+, 0, c_-, 0)$. We will specialize to the case $c_\pm = 0$ to keep the formulas more readable, but the general proof follows the same steps.

The calculation in this special case is perfectly analogous to the bosonic calculation with constant axionic background [43]. Since F assumes the same canonical form (4.83), the $\tilde{\mathcal{H}}$-dependent part of the Lagrangian is

$$
\begin{aligned}
L'_D &= -\sqrt{1+c^2}\sqrt{-\det\left(\begin{array}{c|cc|cc} -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & ih_+ & 0 & 0 \\ 0 & -ih_+ & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & ih_- \\ 0 & 0 & 0 & -h_- & 1 \end{array}\right)} - h_+f_+ - h_-f_- - ch_+h_- \\
&= -\sqrt{(1+c^2)(1-h_+^2)(1-h_-^2)} - h_+f_+ - h_-f_- - ch_+h_- \qquad (4.84)
\end{aligned}
$$

The $h_\pm$ equations of motion read

$$
f_\pm = h_\pm \sqrt{\frac{(1+c^2)(1-h_\mp^2)}{1-h_\pm^2}} - ch_\pm \qquad (4.85)
$$

which lead to

$$
\begin{aligned}
(f_+ + ch_-)(f_- + ch_+) &= (1+c^2)h_+h_-, \quad \text{so that} \\
c(f_+h_+ + f_-h_-) &= h_+h_- - f_+f_- \qquad (4.86)
\end{aligned}
$$

We can now use the symmetry of the formulas to invert the equations of motion:

$$
(h_+ - cf_-)(h_- - cf_+) = f_+f_-(1+c^2), \quad \text{so that} \qquad (4.87)
$$

$$
h_\pm = f_\pm \sqrt{\frac{(1+c^2)(1+f_\mp^2)}{1+f_\pm^2}} + cf_\mp \qquad (4.88)
$$

Plugging these back into the Lagrangian (4.84) and doing straightforward algebra, we get

$$L'_D = -\sqrt{(1+c^2)(1+f_+^2)(1+f_-^2)} - cf_+f_- \tag{4.89}$$

Since

$$\begin{aligned} \det\,(g_{\mu\nu}+\mathcal{F}_{\mu\nu}) &= \det\,(\eta_{\mu\nu}+\mathcal{F}_{\mu\nu}-C_\mu C_\nu) \\ &= \det\,(\eta_{\mu\nu}+\mathcal{F}_{\mu\nu})\left(1-C_\mu(\eta+\mathcal{F})^{-1})^{\mu\nu}C_\nu)\right) \\ &= -(1+c^2)(1+f_+^2)(1+f_-^2) \end{aligned} \tag{4.90}$$

L'_D becomes

$$L'_D = -\sqrt{-\det\,(g_{\mu\nu}+\mathcal{F}_{\mu\nu})} - \frac{1}{8}\epsilon_{\mu\nu\rho\lambda\sigma}C^\rho\mathcal{F}^{\mu\nu}\mathcal{F}^{\lambda\sigma} \tag{4.91}$$

which, plugged into (4.82), is precisely the standard $D4$-brane action (cf. 4.75). This completes the second step. We have thus shown explicitly that the double-dimensional reduction of the M-theory five-brane is the type IIA $D4$-brane.

Chapter 5

The AdS-CFT Correspondence and the Spectrum of Nonlagrangian CFT's

In this chapter we will present an application of brane systems, namely determining the operator spectrum of a superconformal theory with no known Lagrangian description. This is done by means of the AdS-CFT correspondence, which is described in section 5.1. A few propeties of the Anti-de Sitter spaces and of superconformal theories are relegated to appendices. Section 5.2 describes the specific system of five-branes under study. Section 5.3 shows how to build representations of the relevant supergroup. The last two sections contain the results.

5.1 Introduction to the AdS-CFT Correspondence

The so-called AdS-CFT correspondence which was proposed by Maldacena at the end of 1997 [45] is yet another example of branes playing an important rôle in what appeared to be a string theory. Indeed, it is a concrete realization, using branes, of 't Hooft's speculation [44] that $SU(N)$ gauge theories in the large N limit seem to be behave as a string.

To see how this works, we study a system of N parallel $D3$-branes of type IIB string theory. If none of them coincide, the only massless states are those described in section 1.3, coming from open strings that begin and end on the same D-brane. The unbroken gauge group is therefore $U(1)^N$. The ground-states of strings that connect different branes have masses proportional to the separation between the branes, and become massless only in the limit when the branes coincide. Therefore, when all D-branes coincide, the unbroken gauge group is enlarged to $U(N)$ [16]. The conjecture based upon this system is the following: $\mathcal{N} = 4, D = 4$ S(uper)Y(ang)M(ills) with gauge group $SU(N)$ is equivalent to type IIB superstrings propagating on an $AdS_5 \times$

S^5 background, where the flux of the five-form self-dual field strength[1] through S^5 equals N.

We can justify this identification along the lines of chapter 3.1 in [46], by considering the low energy limit of two descriptions of the $D3$-brane system. These descriptions are well understood in opposite regimes, parametrized by $g_s N$, where g_s is the string coupling constant. If $g_s N \ll 1$, the D-brane picture with open string ending on the branes and closed strings moving in the bulk is very accurate. In the low energy limit the states coming from the two types of strings decouple (see [46]), so that we get type IIB supergravity in the bulk decoupled from $SU(N)$[2] SYM on the branes.

On the other hand, if $g_s N \gg 1$, we can use the supergravity description, along the lines of section 1.2. Much as we have seen in the case of the eleven-dimensional membrane, the supergravity solution corresponding to the $D3$-brane system interpolates between flat space at infinity and $AdS_5 \times S^5$ near the horizon [13]. For an observer at infinity, there will be two kinds of low energy modes: large wavelength massless excitations of supergravity in flat space, and *all* excitations which live close enough to the horizon. The latter appear to have low energy because of the large redshift factor. It can be shown [47, 46] that in the low energy limit these two types of modes decouple, because both the absorption and the emission of the "black" $D3$-brane system vanish. We have once again two decoupled sectors, one of which is low energy supergravity in flat space. Identifying the remaining sectors leads to the conjecture presented above.

This conjecture is sometimes called a duality because it relates an untractable regime of a theory to a completely understood regime of the dual theory (and vice-versa). Various checks have been performed using properties that are regime independent (for example, protected by supersymmetry), and they all confirmed the identification. In this chapter we will not perform such a verification, instead we will use the correspondence to describe a theory which was previously intractable.

---

[1]Type IIB has antisymmetric tensors of even rank, including a four-form with self-dual field strength.

[2]There is a subtlety involving an extra U(1) which we will ignore at the level of the present discussion.

5.2 The Five-brane System

Our example is based upon a different instance of the AdS-CFT conjecture, which corresponds to a stack of N parallel five-branes in M-theory. For $N = 1$ we wrote down the world-volume action in the previous chapter. The low-energy limit is a freely propagating tensor multiplet, described by a $\mathcal{N} = (2,0)$ super-conformal theory. Although for $N > 1$ there is no Lagrangian description, it is believed that an infrared $(2,0)$ interacting superconformal fixed point still exists [50]. One can use the AdS-CFT correspondence to extract information about this theory.

The five-brane is a "magnetic" solution in the sense of (1.8). Recalling that $\Delta = 4$, (1.12) gives $\lambda = 3k$. Using also equation (1.9) we get

$$\begin{aligned} ds^2 &= \left(1+\frac{k}{r^3}\right)^{-1/3} dx^\mu dx^\nu \eta_{\mu\nu} + \left(1+\frac{k}{r^3}\right)^{2/3} dy^i dy^i \\ F_{ijkl} &= 3k\epsilon_{ijkls}\frac{y^s}{r^5}, \qquad \text{other components zero} \end{aligned} \tag{5.1}$$

The analysis of the solution is very similar to the one for the membrane. The "interpolating coordinates" [13] are given by the change of variables

$$r = k^{\frac{1}{3}} R^2/(1-R^6)^{\frac{1}{3}} \tag{5.2}$$

In these coordinates the metric becomes

$$ds^2 = R^2 dx^\mu dx_\mu + 4k^{2/3}R^{-2}(1-R^6)^{-8/3}dR^2 + k^{2/3}(1-R^6)^{-2/3}d\Omega_4^2 \tag{5.3}$$

Near the horizon at $R = 0$ the metric becomes that of $AdS_7 \times S^4$:

$$ds^2 = R^2 dx^\mu dx_\mu + 4k^{2/3}R^{-2}dR^2 + k^{2/3}d\Omega_4^2 \tag{5.4}$$

Comparing with equation (5.31) we see[3] that $\mathcal{R}_{AdS} = 2\mathcal{R}_{S^4} = 2k^{1/3}$. Since there are N units of "magnetic" flux through S^4 ($\int_{S^4} F_{[4]} = N$), we also infer

$$k = N/8\pi^2 = \pi N l_p^3 \tag{5.5}$$

[3]The x^μ's need to be rescaled.

where the last equality follows from the fact that we set to one the constant[4] in front of the gravitational action (1.13). The radius in Planck units is therefore of order $N^{1/3}$, and the supergravity approximation is valid for large N.

The AdS-CFT correspondence, as explained in [48, 49], identifies supergravity fields with operators of the large N interacting conformal theory that lives on the boundary of the Anti-de Sitter space. The supergroup for both theories is $OSp(6,2|4)$. Therefore, we expect the short multiplets on both sides to match. This is the basic strategy for computing the spectrum of chiral primary[5] operators in the $(2,0)$ theory, which was done in [52, 53, 54, 55].

One can consider a configuration with less super(conformal) symmetry, taking the quotient (*i.e.*, orbifolding) with respect to a Z_2 symmetry. The symmetry in question is reflection of a transverse coordinate, which we choose to be $X^{11} \equiv y^5$. On the M-theory side, this is known as the Hořava-Witten construction [59]. Cancellation of anomalies requires adding an E_8 gauge field living on the ten-dimensional fixed plane $X^{11} = 0$ [6]. Since the orbifold preserves only half the supersymmetry, the low energy theory living on the stack of five-branes is the $(1,0)$ six-dimensional superconformal theory with E_8 global symmetry. The correspondence identifies this theory with M-theory on $AdS_7 \times S^4/Z_2$ with $N/2$ units of magnetic flux on S^4/Z_2.

If we regard this as an S^4/Z_2 compactification, we get two kinds of Kaluza-Klein (KK) modes. The first kind is given by KK modes of M-theory on S^4 which survive the Z_2 projection. These modes carry no E_8 quantum numbers and hence couple only to E_8-neutral operators on the boundary of AdS_7. The second kind is given by KK modes on the fixed S^3 which have adjoint E_8 quantum numbers and couple to charged operators on the AdS_7 boundary, in a manner similar to the one described in [56].

The last two sections describe these two kinds of Kaluza-Klein modes, following [5]. Before doing that, we need to review the group theory of $OSp(6,2|2)$.

[4]This constant should be $\frac{1}{4\pi^2 G_N} \equiv \frac{1}{4\pi^2 l_p^9}$, where l_p is the eleven-dimensional Planck length.

[5]For terminology see Appendix B.

[6]This is also the location of the stack of five-branes.

5.3 Oscillator Construction of $OSp(6,2|2)$ Short Multiplets

The supergroup $OSp(6,2|2)$ is related to the isometry group of $AdS_7 \times S^4/Z_2$. The AdS piece has isometry group $SO(6,2)$, same as the conformal group for six-dimensional Minkowski space. The isometry group $SO(5)$ of the original S^4 is broken to $SO(4) = SU(2)_R \times SU(2)_L$ by the Z_2 projection. Only one of the $SU(2)$'s, which we choose to be $SU(2)_R$, is part of the superconformal group $OSp(6,2|2)$.

The first step is to construct the short multiplets of $OSp(6,2|2)$ which will be relevant to our problem. We do this because any KK modes which we fit into one of these short multiplets will have mass eigenvalues fixed by the $OSp(6,2|2)$ group theory, making explicit calculations unnecessary. We follow the methods of Günaydin et al. [58] for constructing multiplets of $OSp(6,2|4)$ with just a slight modification to allow for the reduced R-symmetry[7].

The oscillator method, developed in [60, 61], is useful for building representations of a noncompact group starting from representations of the maximal compact subgroup. One introduces creation and annihilation operators in the fundamental representation of this subgroup and then the generators of the full group are bilinears in these oscillators. If one chooses the right number of oscillators, one gets a three graded (also called Jordan) decomposition of the algebra of the form $L = L^- \oplus L^0 \oplus L^+$, where L^0 is the compact subgroup. We will see below on our particular example how to use $L^\pm$ to build representations of the full group based upon representations of the maximally compact subgroup.

This method can be extended to supergroups with the understanding that we build upon the maximally compact sub-supergroup. For $OSp(6,2|2)$, this is $U(4|1)$. Indeed, the bosonic subgroup of $OSp(6,2|2)$ is $SO(6,2) \times USp(2)$. The maximally compact subgroup of $SO(6,2)$ is $SO(6) \times SO(2) \sim SU(4) \times U(1) \sim U(4)$. The R-symmetry group $USp(2) \sim SU(2)$ is compact, but in order to get the bosonic subgroup of a supergroup we need to start with a $U(1)$ subgroup.

Following the notation of [58], we denote the bosonic part of the $U(4|1)$ sub-supergroup $U(4)_B \times U(1)_F$. We next introduce p pairs of bosonic creation/annihilation

[7] See Appendix B.

operators $a^i(r) = (a_i(r))^\dagger$ and $b^i(r) = (b_i(r))^\dagger$ transforming in the 4 and $\bar{4}$ representation of $U(4)_B$ (with $r = 1, \ldots, p$), as well as a pair of fermionic creation/annihilation operators $\alpha^\dagger(r), \alpha(r)$ and $\beta^\dagger(r), \beta(r)$ with $U(1)_F$ charges $+\frac{1}{2}$ and $-\frac{1}{2}$ respectively. We organize these oscillators into column vectors:

$$\xi_A(r) = \begin{pmatrix} a_i(r) \\ \alpha(r) \end{pmatrix} \qquad \eta_A(r) = \begin{pmatrix} b_i(r) \\ \beta(r) \end{pmatrix}$$

$$\xi^A(r) = \begin{pmatrix} a^i(r) \\ \alpha^\dagger(r) \end{pmatrix} \quad \eta^A(r) = \begin{pmatrix} b^i(r) \\ \beta^\dagger(r) \end{pmatrix} \quad \begin{cases} r = 1, \ldots, p, \\ i, j, \ldots = 1, \ldots, 4 \end{cases} \tag{5.6}$$

($A = \cdot$ will represent the $U(1)_F$ "index") whose only non-zero commutation/anti-commutation relations can be represented symbolically as

$$\left\{\xi_A(r), \xi^B(s)\right] = \delta_A{}^B \delta^r_s \qquad \left\{\eta_A(r), \eta^B(s)\right] = \delta_A{}^B \delta^r_s \tag{5.7}$$

The Lie superalgebra can now be realized in terms of bilinears in ξ and η. They are given by

$$A_{AB} = \xi_A \cdot \eta_B - \eta_A \cdot \xi_B$$

$$A^{AB} = A^\dagger_{AB} = \eta^B \cdot \xi^A - \xi^B \cdot \eta^A \tag{5.8}$$

$$M^A{}_B = \xi^A \cdot \xi_B + (-1)^{deg(A)deg(B)} \eta_B \cdot \eta^A$$

where $deg(A) = 0$ for a bosonic index A and $deg(A) = 1$ for a fermionic index A. The even subalgebra $SO(6,2) \times SU(2)_R$ is generated by the elements $\{A_{ij}, A^{ij}, M^i{}_j\}$ and $\{A_{\cdot\cdot}, A^{\cdot\cdot}, M^\cdot{}_\cdot\}$.

As discussed above, this oscillator construction of generators naturally implements a Jordan decomposition of $OSp(6,2|2)$ with respect to the maximal compact subgroup $U(4|1)$ graded with the $U(1)$ generator, $Q = \frac{1}{2} M^A{}_A$:

$$L = L^- \oplus L^0 \oplus L^+ \tag{5.9}$$

The generators A_{AB} and A^{AB} correspond to the L^- and L^+ spaces, respectively. The generators $M^A{}_B$ of $U(4|1)$ give the L^0 space. We use this Jordan decomposition to generate unitary irreducible representations (UIR's) of $OSp(6,2|2)$. We start with a lowest weight state, $|\,\Omega\rangle$, in an irreducible representation of L^0 and annihilated by all the generators in L^-. The complete UIR is then generated using successive applications of generators in L^+ on $|\,\Omega\rangle$.

To understand the physical interpretation of states in a given UIR, it is convenient to further decompose representations of $U(4|1)$ into representations of $U(4)_B \times U(1)_F$. We label the $U(4|1)$ representations in terms of super-Young tableaux[8]. Their decompositions look like:

$$\begin{smallmatrix}\boxslash\\ \boxslash\end{smallmatrix} \rightarrow \left(\begin{smallmatrix}\square\\ \square\end{smallmatrix}, 1\right),\ (\square\,, \square\,),\ (1, \square\square\,) \tag{5.10}$$

As we discussed above, the group $U(4)_B \simeq Spin(6) \times U(1)_B$ is the maximal compact subgroup of $SO(6,2)$. $U(1)_B$ is generated by the charge $Q_B = \frac{1}{2}M^i{}_i = \frac{1}{2}(N_B + 4p)$. In the AdS/CFT duality, this charge corresponds to the AdS energy of a supergravity mode and to the dimension of its dual CFT operator. N_B is the bosonic number operator, and for a $U(4)$ representation is just the number of boxes in the corresponding Young tableau. For that given Young tableau, the $Spin(6)$ representation can be recovered by matching the $SU(4)$ indices with the appropriate $SU(4)$-invariant tensor. A few examples[9] are:

$$\underset{\text{scalar}}{|\,0\,\rangle}\,, \qquad \underset{\text{spinor}}{|\,\square\,\rangle}\,, \qquad \underset{A_{\alpha\beta\gamma}}{|\,\square\square\,\rangle}\,, \qquad \underset{\text{vector}}{\left|\,\begin{smallmatrix}\square\\ \square\end{smallmatrix}\,\right\rangle} \tag{5.11}$$

Finally, $U(1)_F$ is generated by $Q_F = \frac{1}{2}M^{\cdot}{}_{\cdot} = \frac{1}{2}(N_F - p)$, which measures $SU(2)_R$ spin.

Now that we have a better understanding of how to interpret states in representations of $U(4|1)$, let us quickly describe the action of the L^+ generators. In order to

[8] These super-Young tableaux are presented in detail within section 5 of [58].
[9] More details can be found at the end of section 2 of [58].

do this, we break up the set L^+ into the subsets:

$$L^+ = \left\{A^{AB}\right\} \rightarrow \begin{cases} B^+ = \{A^{ij}\} \\ Q^+ = \{A^{i\cdot}\} \\ F^+ = \{A^{\cdot\cdot}\} \end{cases} \tag{5.12}$$

Given a representation of $U(4)_B \times U(1)_F$, B^+ gives conformal descendants. In AdS these correspond to higher energy Fourier-like modes. F^+ generates the complete set of spin states in a given $SU(2)_R$ representation. Finally, Q^+ is the set of supersymmetry generators.

To illustrate how the oscillator construction works, we start with the simplest example

$$|\,\Omega\rangle = |\,0\rangle_p \tag{5.13}$$

the vacuum corresponding to p pairs of oscillators. This state is a scalar with dimension/energy equal to $2p$ and lowest $SU(2)_R$ spin component $-p/2$. With its F^+ descendants, $|\,\Omega\rangle$ fills out the $\mathbf{(p{+}1)}$ of $SU(2)_R$. Acting with Q gives the following supermultiplet of conformal primary states (lowest energy states):

$U(4)$	State	$SU(2)_R$	Δ
1	Scalar	$(\mathbf{p+1})$	$2p$
$\square$	Spinor	$\mathbf{p}$	$(2p+\frac{1}{2})$
$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	Vector	$(\mathbf{p-1})$	$(2p+1)$
$\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$	Spinor	$(\mathbf{p-2})$	$(2p+\frac{3}{2})$
$\begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix}$	Scalar	$(\mathbf{p-3})$	$(2p+2)$

(5.14)

For $p < 4$ we keep only those states with positive $SU(2)_R$ dimension. For readability's sake we always include the $Spin(6)$ representation labeled in terms of fields in AdS_7.

The highest spacetime spin in the multiplet (5.14) is the vector, indicating a short representation of the superalgebra. In fact, if we act with Q on the superconformal primary, the topmost scalar, its highest weight $SU(2)_R$ component is annihilated.

This means that we are dealing with a level one short multiplet in the formalism discussed by [62]. The superconformal primary state in this type of short multiplet (the topmost scalar) has dimension $\Delta = 4s$ where s is the $SU(2)_R$ spin. We can check this explicitly:

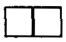
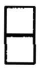
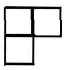
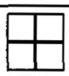
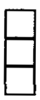
$$\Delta = Q_B | \, 0 \rangle_p = 2p = -4 \, Q_F | \, 0 \rangle_p = 4s \tag{5.15}$$

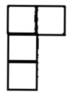
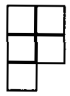
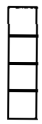
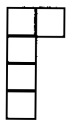
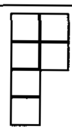
There is another short[10] multiplet of $OSp\,(6,2|\,2)$ that we will find useful, starting with the ground state

$$\xi^{[A} \xi^{B]} \, | \, 0 \rangle_p \tag{5.16}$$

This "ground state," with super-Young tableau given in (5.10), gives a vector of dimension $(2p+1)$, a spinor of dimension $(2p+\frac{1}{2})$ and a scalar of dimension $2p$. Since the scalar has lowest dimension, we will refer to it as the superconformal primary. The states in this multiplet are:

[10]Since it is longer than the previous multiplet, but shorter than the generic long multiplet, it is sometimes called a medium multiplet.

$U(4)$	State	$SU(2)_R$	Δ
0	Scalar	$(\mathbf{p}-\mathbf{1})$	$2p$
	Spinor	$\mathbf{p}$ $(\mathbf{p}-\mathbf{2})$	$(2p+\frac{1}{2})$
	3-Form	$(\mathbf{p}-\mathbf{1})$	$(2p+1)$
	Vector	$(\mathbf{p}+\mathbf{1})$ $(\mathbf{p}-\mathbf{1})$ $(\mathbf{p}-\mathbf{3})$	$(2p+1)$
	Gravitino	$\mathbf{p}$ $(\mathbf{p}-\mathbf{2})$	$(2p+\frac{3}{2})$
	Graviton	$(\mathbf{p}-\mathbf{1})$	$(2p+2)$
	Spinor	$\mathbf{p}$ $(\mathbf{p}-\mathbf{2})$ $(\mathbf{p}-\mathbf{4})$	$(2p+\frac{3}{2})$

$U(4)$	State	$SU(2)_R$	Δ
	2-Form	$(\mathbf{p}-\mathbf{1})$ $(\mathbf{p}-\mathbf{3})$	$(2p+2)$
	Gravitino	$(\mathbf{p}-\mathbf{2})$ $\mathbf{p}$	$(2p+\frac{5}{2})$
	Scalar	$(\mathbf{p}-\mathbf{1})$ $(\mathbf{p}-\mathbf{3})$ $(\mathbf{p}-\mathbf{5})$	$(2p+2)$
	Spinor	$(\mathbf{p}-\mathbf{2})$ $(\mathbf{p}-\mathbf{4})$	$(2p+\frac{5}{2})$
	Vector	$(\mathbf{p}-\mathbf{3})$	$(2p+3)$

(5.17)

Again, we only allow states with positive $SU(2)_R$ dimension. Even then, for $p<5$ some of the elements listed above are not in fact conformal primaries, but F^+ and B^+ descendants of other primaries. For example, when $p=2$ the vector in the **1** of $SU(2)_R$ is the F^+ descendant of the vector in the **3**. This odd behavior is related to the fact that this multiplet does not develop null states until level 3. The superconformal primary (scalar) for this short multiplet has $\Delta = 4s+4$. We know from [62] that short multiplets of $OSp(6,2|2)$ can have scalar superconformal primaries with

$$\Delta = 4s,\ 4s+2,\ 4s+4,\ 4s+6 \tag{5.18}$$

but only the ones described in this chapter will be relevant for the supergravity spectrum.

5.4 Kaluza-Klein Reduction of the Bulk Eleven-Dimensional Supergravity

Now that we have established some of the multiplet structure of $OSp(6,2|2)$ we can carry out our analysis of the bulk S^4 Kaluza-Klein modes which survive the Z_2 projection. The group theory for this process is simple. These KK modes come from S^4 harmonics in representations of $SO(5)$. Decomposing $SO(5) \to SU(2)_R \times SU(2)_L \times Z_2$, with $Z_2 = +/-$ for harmonics which are even/odd under $X^{11} \to -X^{11}$, we reduce all the eleven-dimensional supergravity fields on the even harmonics except for the 3-form[11]. This last field is reduced on odd harmonics since it flips sign under parity reversal. For convenience, we will describe the KK spectrum in terms of the dual CFT operators and apply the Z_2 projection on the CFT spectrum.

The S^4 KK spectrum derived in Ref. [63, 64] can be nicely organized in terms of dual operators using "place-holder" fields [65]. In [54] the short multiplet operator spectrum for the $(2,0)$ theory dual to $AdS_7 \times S^4$ was written in terms of place-holder scalar, spinor and tensor fields

$$\tilde{\phi},\ \tilde{\psi},\ \tilde{H} \tag{5.19}$$

taken to be in the adjoint of $U(N)$[12]. They transform in the **5**, **4**, and **1** representation of $SO(5)$[13]. Starting with a superconformal primary operator

$$\mathcal{O}_{0,0,p} = Tr\, \tilde{\phi}^p \tag{5.20}$$

(the power p is schematic, the $\tilde{\phi}$'s are actually in symmetric traceless representation of $SO(5)$) we build a complete supermultiplet of conformal primary operators

$$\mathcal{O}_{m,n,k} = Tr\, \tilde{H}^m \tilde{\psi}^n \tilde{\phi}^k, \quad m+n+k=p \tag{5.21}$$

Note that in the oscillator method used to generate superconformal multiplets of $OSp(6,2|4)$, these multiplets are built using the vacuum ground state with the same

[11]Here we refer to forms in index free notation.

[12]This $U(N)$ is not physical; the place-holder fields should be regarded as a useful bookkeeping device.

[13]We denote with a ˜ fields in $SO(5)$ R-symmetry representations.

number p of oscillator flavors (see [58]). The p=1 operators Tr $\tilde{H}$, Tr $\tilde{\psi}$, and Tr $\tilde{\phi}$ are the only operators in the Abelian part of $U(N)$ and correspond to the doubleton. They decouple from the theory. The $p = 2$ supermultiplet (also referred to as the massless multiplet) is very important, as it contains the R-symmetry currents, the super-currents and the stress-energy tensor, as well as relevant scalar operators.

To look at the operators dual to the KK modes of S^4/Z_2 we use a simple extension of the methods above. We split the place-holder fields into an even group and an odd group. The even group contains scalars and spinors

$$\begin{array}{ll} \phi & \text{in the } (\mathbf{2},\mathbf{2}) \\ \psi & \text{in the } (\mathbf{1},\mathbf{2}) \end{array} \tag{5.22}$$

where $SU(2)_R \times SU(2)_L$ is now the global symmetry. These place-holders transform in the anti-symmetric (**AS**) of $USp(N)$ (N must be even) and loosely correspond to fluctuations inside the end-of-the-world 9-brane. Since we have broken $U(N)$ to $USp(N)$, traces of place-holder fields include the symplectic matrix **J**. This means any number of these even place-holder fields can appear in a trace ($Tr[\mathbf{J}\cdot\mathbf{AS}] \neq 0$).

The odd group contains a scalar, spinors, and a self-dual 2-form

$$\begin{array}{ll} \rho & \text{in the } (\mathbf{1},\mathbf{1}) \\ \chi & \text{in the } (\mathbf{2},\mathbf{1}) \\ H & \text{in the } (\mathbf{1},\mathbf{1}) \end{array} \tag{5.23}$$

These place-holders transform in the adjoint of $USp(N)$, so only operators represented by traces with an *even* number of these survive the Z_2 projection due to the commutation relation of adjoint $USp(N)$ matrices with **J**. For a given p, the superconformal primaries $\mathcal{O}_{0,0,p}$ of the $\mathcal{N}=2$ algebra break up into separate superconformal primaries of the $\mathcal{N}=1$ algebra. The ones which survive the Z_2 projection can be schematically written as

$$Tr\,\phi^p,\; Tr\,\phi^{p-2}\rho^2,\; Tr\,\phi^{p-4}\rho^4,\; \ldots \tag{5.24}$$

The first of these primaries will transform in the **(p+1,p+1)** of $SU(2)_R \times SU(2)_L$

the next in the **(p−1,p−1)**, and so on. They all inherit the dimension of $\mathcal{O}_{0,0,p}$, so will have $\Delta = 2p$. Relating their R-symmetry spin, s, to this dimension we get the relations:

$$\Delta = 4s,\ \Delta = 4s + 4,\ \Delta = 4s + 8, \dots \tag{5.25}$$

For each p, the first operator in this series is a superconformal primary for a short multiplet with content as in (5.14). The second is an $\mathcal{N} = 1$ superconformal primary for a medium multiplet, with the primary fields listed in (5.17). The rest of the operators in (5.25) are superconformal primaries for *long* multiplets of $OSp(6, 2|\,2)$. This can be seen both from their dimension, scalars with $\Delta > 4s + 6$ are superconformal primaries for long multiplets only, and from the number of states in the multiplet. Their dimensions are valid only in the large N limit, since they can receive corrections of order 1/N when tree-level supergravity computations are no longer protected by supersymmetry.

For $p = 1$, only one superconformal operator can be written down, Tr ϕ. The conformal primaries in this multiplet are this scalar in the **(2,2)** of $SU(2)_R \times SU(2)_L$ and a spinor in the **(1,2)**. The anti-symmetric tensor of $USp(N)$ is reducible, its symplectic trace is a singlet and won't mix with other operators. This matches with our understanding of the $p = 1$ operators as dual to the "pure gauge" (*i.e.*, no physical degrees of freedom) AdS fields in the doubleton representation of $OSp(6, 2|\,2)$. They correspond to the center of mass motion of the M5-branes parallel to the 9-branes and decouple.

We would like to point out that a clear distinction should be drawn between place-holder fields and doubleton degrees of freedom. In the $\mathcal{N} = 2$ formalism, the doubleton and the place-holder multiplet carry the same quantum numbers, so an identification might be drawn. For $\mathcal{N} = 1$ only the even multiplet of place-holders yields a doubleton, the odd place-holders are necessary for computing the correct spectrum but can't appear alone in a trace. Thus, the pure-gauge doubleton degrees of freedom of the AdS background should really only be interpreted as decoupled center of mass degrees of freedom for the CFT and bear no direct connection to the place-holder fields.

The odd place-holders first play a role for the $p = 2$ operators. The $\mathcal{N} = 2$ superprimary operator Tr $\tilde{\phi}\tilde{\phi}$ splits into Tr $\phi\phi$, Tr $\phi\rho$, and Tr $\rho\rho$. Only the first and third of these survive the Z_2 projection. We list the results in the following table:

Type	Δ	$SO(5) \to SU(2)_R \times SU(2)_L \times Z_2$	Place-Holders
Scalar	4	$\mathbf{14} \to \ (\mathbf{3},\mathbf{3})_+ \ + \ (\mathbf{1},\mathbf{1})_+ \ + Z_2\text{-odd}$	$Tr\,\tilde\phi\tilde\phi \to \ Tr\,\phi\phi \ + \ Tr\,\rho\rho \ + Z_2\text{-odd}$
Spinor	$4\frac{1}{2}$	$\mathbf{16} \to \ (\mathbf{2},\mathbf{3})_+ \ + \ (\mathbf{2},\mathbf{1})_+ \ + Z_2\text{-odd}$	$Tr\,\tilde\phi\tilde\psi \to \ Tr\,\phi\psi \ + \ Tr\,\rho\chi \ + Z_2\text{-odd}$
Vector	5	$\mathbf{10} \to \ (\mathbf{1},\mathbf{3})_+ \ + \ (\mathbf{3},\mathbf{1})_+ \ + Z_2\text{-odd}$	$Tr\,\tilde\psi\tilde\psi \to \ Tr\,\psi\psi \ + \ Tr\,\chi\chi \ + Z_2\text{-odd}$
3-Form	5	$\mathbf{5} \to \ (\mathbf{1},\mathbf{1})_+ \ + Z_2\text{-odd}$	$Tr\,\tilde\phi\tilde H \to \ Tr\,\rho H \ + Z_2\text{-odd}$
Gravitino	$5\frac{1}{2}$	$\mathbf{4} \to \ (\mathbf{2},\mathbf{1})_+ \ + Z_2\text{-odd}$	$Tr\,\tilde\psi\tilde H \to \ Tr\,\chi H \ + Z_2\text{-odd}$
Graviton	6	$\mathbf{1} \to \ (\mathbf{1},\mathbf{1})_+ \ + Z_2\text{-odd}$	$Tr\,\tilde H\tilde H \to \ Tr\,HH \ + Z_2\text{-odd}$

(5.26)

From this table, we can see that we get two massless multiplets for $AdS_7 \times S^4/Z_2$. The first is a chiral multiplet and contains the $SU(2)_L$ gauge field. The second multiplet contains the $SU(2)_R$ gauge field, the gravitino, and the graviton. These fields couple to the R-symmetry current, the super-current and the stress-energy tensor of the boundary CFT respectively. The presence of the R-symmetry current and the stress-energy tensor in the same multiplet fixes the relation between R-spin, s, and the dimension, Δ, of the short multiplets superconformal primaries[14].

One can continue this procedure for higher p to get a complete classification of KK modes coming from the bulk which survive the Z_2 projection.

5.5 Kaluza-Klein Reduction of E_8-charged Modes

To get the mass spectrum of the Kaluza-Klein reduction of the E_8 twisted sector modes, one might expect to have to calculate the eigenvalues of the Laplace operator for the E_8 $\mathcal{N} = 1$ $D = 10$ vector multiplet reduced on a sphere. Luckily, group theory comes to our rescue again, in a fashion quite similar to that in the analysis of [56]. Upon reduction to AdS_7 the $D = 10$ vector multiplet can give only scalars, spinors, and vectors. This implies that its KK modes must fit into short multiplets

[14]The fact that the various short multiplets relate Δ and s differently is due to the appearance of null-states at different levels.

of type (5.14), since all other multiplets include a larger set of bosonic fields.

Looking at just the bosonic modes, we derive the complete Kaluza-Klein spectrum. Our only initial bosonic field is the $D = 10$ vector field. It can be reduced on the fixed S^3 using a scalar harmonic in the **(k,k)** of $SU(2)_R \times SU(2)_L$ or using a vector harmonic in the **(k+2,k)** or **(k,k+2)**. These harmonics give a bosonic spectrum of the form

$$\begin{array}{llllll} \text{Scalars} \rightarrow & (\mathbf{3},\mathbf{1}), & (\mathbf{4},\mathbf{2}), & (\mathbf{5},\mathbf{3}), & (\mathbf{6},\mathbf{4}), & \ldots \\ \text{Vectors} \rightarrow & (\mathbf{1},\mathbf{1}), & (\mathbf{2},\mathbf{2}), & (\mathbf{3},\mathbf{3}), & (\mathbf{4},\mathbf{4}), & \ldots \\ \text{Scalars} \rightarrow & & & (\mathbf{1},\mathbf{3}), & (\mathbf{2},\mathbf{4}), & \ldots \end{array} \tag{5.27}$$

The columns in this series fit easily into chiral multiplets with superconformal primary scalars in the **(k+2,k)**. The AdS energy/dimensions of all these states can be directly read off from (5.14). Just as in [56], scalars coming from the vector harmonics **(k+2,k)** and **(k,k+2)** have different AdS energies due to a subtlety involving the eigenvalues of the E_8 covariant derivative on S^3. Note that the first multiplet in the series contains the massless vector which carries the E_8 gauge symmetry and couples to the dual E_8 current algebra on the AdS_7 boundary CFT.

In the last two sections we got two infinite lists of Kaluza-Klein modes that fit into short (or medium) multiplets and therefore correspond to chiral primary operators of the $(1,0)$ theory. We have also determined the superconformal dimension of these operators using the group theory of $OSp(6,2|2)$. This shows that only a finite number of these operators are relevant ($\Delta < 6$) or marginal ($\Delta = 6$).

One could further use the AdS-CFT correspondence to compute correlation functions of these operators[15]. For a more detailed physical interpretation of the above results, we refer the reader to [5].

Appendix A: Anti-de Sitter Spaces

Consider a flat space with signature $(2, p+1)$:

$$ds^2 = -dX_0^2 - dX_{p+2}^2 + \sum_{i=1}^{p+1} dX_i^2 \tag{5.28}$$

[15]For the $(2,0)$ theory, this was done in [67].

The $p+1$-dimensional Anti-de Sitter space AdS_{p+2} is the hyperboloid

$$-X_0^2 - X_{p+2}^2 + \sum_{i=1}^{p+1} dX_i^2 = \mathcal{R}_{AdS}^2 \tag{5.29}$$

There are several useful representations of AdS_{p+2}, in particular the one which comes out from near horizon regions of branes, described by coordinates x^μ, R (as in (1.18), (5.4)). The required coordinate change is

$$\begin{aligned} X^0 &= \frac{1}{2R}\left(1 + R^2(\mathcal{R}_{AdS}^2 + x_\mu x^\mu)\right) \\ X^{p+2} &= Rx^0\mathcal{R}_{AdS} \\ X^i &= Rx^i\mathcal{R}_{AdS} \\ X^{p+1} &= \frac{1}{2R}\left(1 - R^2(\mathcal{R}_{AdS}^2 - x_\mu x^\mu)\right) \end{aligned} \tag{5.30}$$

The metric becomes

$$ds^2 = \mathcal{R}_{AdS}^2(R^2 dx_\mu dx^\mu + R^{-2}dR^2) \tag{5.31}$$

This is the only formulation of AdS that we used above. For a much more complete treatment of Anti-de Sitter spaces, we refer the reader to section 2.2 of [46].

Appendix B: Superconformal Symmetry

In this appendix we will try to give a brief overview of superconformal symmetry in order to clarify the terminology used in this chapter.

The conformal group is an extension of the Poincare group[16] (Lorentz M + translations P) which includes the dilatation D and special conformal transformations K (translations conjugated by inversion). This extension avoids the Coleman-Mandula theorem[17] because the latter restricts only the S-matrix of a theory, and scale invariant theories (which include conformal theories) do not have an S-matrix.

The second possible extension of the Poincare symmetry is supersymmetry. The corresponding algebra is a *graded* Lie algebra that includes fermionic generators Q

[16]We indicate the generators schematically, dropping indices.

[17]Which says that there is no Lie Algebra enlarging nontrivially the Poincare algebra of symmetries for the S-matrix of a field theory.

(hence avoiding the Coleman-Mandula theorem). One can combine the two extensions together to get the so-called superconformal symmetry. To achieve this one needs a second set of fermionic generators S, leading to the following nontrivial (anti)commutation relations:

$$\begin{array}{lll} [D,P]=-iP & [P,K]=2iM-2i\eta D & \{Q,Q\}=P \\ [D,K]=iK & [P,S]\sim Q & \{S,S\}\sim K \\ [D,Q]=-i/2\ Q & [K,Q]\sim S & \{Q,S\}\sim M+D+R \\ [D,S]=i/2\ S & & \end{array} \tag{5.32}$$

The commutation relations with M, not included in the above table, are those of space-time vectors for P, K, spinors for Q, S and scalars for D. We also did not include some coefficients because they depend on the number of dimensions and number of supersymmetries. In fact, Nahm classified the superconformal theories in [66], showing that they exist only in six or fewer space-time dimensions. Among these, we find the two six-dimensional theories with supergroups $OSp(6,2|\,2n)$, $\;n=1,2$, which we called above the $(n,0)$ theories[18].

The representations of superconformal algebras have multiplets of various lengths: the generic multiplets are called "long," the shorter multiplets "short" and (if it is the case) "medium." This phenomenon is closely related to the notion of superconformal chiral primary. A conformal primary is an operator of minimal dimension in a given Lorentz representation (scalar, spinor, vector). The scaling dimension is the eigenvalue of iD, so it is lowered by K. Since unitarity imposes lower bounds on the scaling dimension (see [62]), we infer that there must be a conformal primary that is annihilated by K. A superconformal primary is, by definition, an operator annihilated by K and S. If it is also annihilated by a combination of the Q's, it is called a "chiral primary" operator. Acting with Q's on a superconformal primary gives the conformal primaries of the multiplet, thus the superconformal multiplet will be shorter (containing less conformal multiplets) if we start from a chiral primary.

[18] The names encode the fact that supercharges have the same six-dimensional chirality, and the total number of supercharges is $8n$, where eight is the dimension of the generic six-dimensional spinor.

The generators "R" which appeared in the $\{Q, S\} \sim M + D + R$ anti-commutator above form the Lie algebra of the so-called "R-symmetry." For a chiral primary operator there is a combination of the Q's such that the anti-commutator applied to the operator vanishes. Therefore, the scaling dimension is determined by the R-symmetry and Lorentz quantum numbers. This explains why, in the last two sections of this chapter, we got the dimensions of chiral primaries using just group theory.

Conclusions

We have seen how multi-dimensional objects, called "branes," appear in the context of string theory. Among these, a prominent rôle is played by the "Dirichlet" branes and the M-branes (branes of M-theory), which we discussed at some length.

The main result of this thesis is the supersymmetrization of the world-volume action for D and M-branes. This was done by adding a "Wess-Zumino" term to the action and proving that an extra local fermionic symmetry, called "kappa-symmetry," exists, and that it removes unwanted fermionic degrees of freedom.

In the last chapter we did a short excursion into the rich field of the AdS-CFT correspondence, which is one of the many applications of branes that relates string theory to the more familiar field theory. The new result was the determination of the spectrum of a conformal theory which is intractable by field-theoretic methods because it does not have a Lagrangian description.

There are many other applications of branes to field theory and beyond that were not treated here. To name just two which attracted a lot of attention in the last few years, brane configurations were used to reproduce Seiberg-Witten theory [68, 69, 70] and the physics of black holes [71, 72] (and references therein). Branes became so important that many people believe that we need to reconsider the name of the subject, finding "string theory" misleading! In any case, this story is just at the begining, and we expect a lot of new insights on the fundamental theory of nature to come from the physics of branes.

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