ISENTROPIC PLANE WAVES IN MAGNETOHYDRODYNAMICS

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ABSTRACT

Plane waves propagating in a perfectly electrically conducting polytropic gas of otherwise uniform state in the presence of an arbitrarily oriented uniform magnetic field are studied; they correspond to plane simple waves in magnetohydrodynamics. Riemann invariants across finite amplitude waves in ordinary gasdynamics are generalized herein to take into account all possible magnetohydrodynamic effects. There exist totally seven types of waves, namely, contact surfaces, forward and backward facing transverse simple waves and forward and backward facing coupled (fast and slow) simple waves. But of these only coupled waves are genuinely nonlinear and receive most of our attention. The mathematical theory of simple waves is discussed first to give a general picture of the underlying structure of solutions. Contact surfaces and transverse simple wave solutions are given next with particular emphasis on the case of the contact surface adjacent to a vacuum region. An exact analytical solution of coupled waves for gases of arbitrary value of γ is obtained in terms of generalized Riemann invariants; some of these invariants are expressed in terms of definite integrals of a parameter $\alpha = a^2/c^2$. The invariant relations among several physical quantities are thus expressed in a parametric form. An alternative method of solving coupled waves by graphical means is proposed and some detailed calculations are presented. General properties of physical variables across coupled waves are mentioned. For the special case of gas in a purely transverse magnetic field, a scheme of solving arbitrary flow problems is discussed briefly. The corresponding case of coupled wave solutions is given in terms of a hypergeometric function. Finally, some examples are shown to illustrate the application of the solutions to actual physical problems.

Square wave speed ratios:

$$\alpha = \frac{a^2}{c^2}, \quad \beta_1 = \frac{b_1^2}{c^2}, \quad \beta_2 = \frac{b_2^2}{c^2}$$

and

$$\alpha^* = \frac{1}{\alpha}, \quad \beta_1^* = \frac{1}{\beta_1}$$

I = a subscript denoting physical quantities at initial state of the fluid

$\overline{}$ = the corresponding dimensionless variable defined as follows

$$\bar{\rho} = \frac{\rho}{\rho_I}, \quad \bar{P} = \frac{P}{P_I}, \quad \bar{B_2} = \frac{B_2}{B_1}, \quad \bar{B_2}' = \frac{B_2}{B_{2\,I}} = \frac{\bar{B_2}}{\bar{B_2}_I},$$

$$\bar{u}_1 = \frac{u_1}{a_I}, \quad \bar{u_2} = \frac{u_2}{a_I}, \quad \bar{a} = \frac{a}{a_I}, \quad \bar{b}_1 = \frac{b_1}{a_I}, \quad \bar{b}_2 = \frac{b_2}{a_I}, \quad \bar{c} = \frac{c}{a_I}$$

$$K_1 = \frac{b_{1\,I}}{a_I} = \frac{1}{a_I} \frac{B_1}{\sqrt{\mu \rho_I}} = \bar{b}_{1\,I}$$

θ = the angle between the magnetic field component in (x_1, x_2) plane and x_1-axis $= \tan^{-1} \frac{B_2}{B_1}$

sgn ( ) = sign of the quantity inside the bracket

| | = absolute value of the quantity inside it

|| || = determinant of the matrix inside it

$$H = H(\alpha; K_1, \alpha_I) = \left| \frac{\alpha_I}{\alpha_I - 1} \right|^{\gamma/(2-\gamma)} \mp \frac{\gamma K_1^2}{2-\gamma} \int_{\alpha_I}^{\alpha} \left| \frac{\zeta}{\zeta - 1} \right|^{2/(2-\gamma)} d\zeta$$

$\nu = \frac{2}{2-\gamma}$ = an integer

$-2s_1, 2r_1$ = longitudinal flow Riemann invariants for forward and backward facing coupled waves respectively

$-2s_2, 2r_2$ = transverse flow Riemann invariants for forward and backward facing coupled waves respectively

* = a subscript denoting physical quantities expressed as functions of α and β_1 for $K_1 = 1$

PRINCIPAL NOTATION

$\gamma = \frac{C_p}{C_v}$ = specific heat ratio or adiabatic exponent

n = number of degrees of freedom of gas molecules

x_1, x_2, x_3, t = space and time variables

P = pressure

ρ = density

S = entropy per unit mass

$\vec{u} = (u_1, u_2, u_3)$ = fluid velocity vector

$\vec{B} = (B_1, B_2, B_3)$ = magnetic field vector

$\vec{B}_t = B_2\vec{e_2} + B_3\vec{e_3}$ = transverse magnetic field vector in (x_2, x_3) plane

$\vec{E} = (E_1, E_2, E_3)$ = electric field vector (in fixed space coordinates)

$\vec{E}_t = E_2\vec{e_2} + E_3\vec{e_3}$ = transverse electric field vector in (x_2, x_3) plane

$\vec{f} = (f_1, f_2, f_3)$ = bodyforce per unit volume

$\vec{j} = (j_1, j_2, j_3)$ = current density vector

$a = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_S}$ = local speed of sound in ordinary gasdynamics

$b_1 = \frac{B_1}{\sqrt{\mu\rho}}$ = local Alfvén wave speed based on B_1

$b_2 = \frac{B_2}{\sqrt{\mu\rho}}$ = local Alfvén wave speed based on B_2

$b_3 = \frac{B_3}{\sqrt{\mu\rho}}$ = local Alfvén wave speed based on B_3

$b_t = \frac{B_t}{\sqrt{\mu\rho}}$ = local Alfvén wave speed based on B_t

ξ = parameter to characterize a particular phase of physical state

$U(\xi)$ = absolute phase velocity relative to fixed space coordinates

$c = U(\xi) - u_1$ = phase velocity relative to longitudinal velocity, u_1

φ = the angle between transverse magnetic field direction and x_2-axis

$k = \frac{B_3}{B_2} = \tan\varphi$

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I. INTRODUCTION

Plane MHD[†] waves in an arbitrarily oriented uniform magnetic field are investigated in this paper. The working medium is considered to be a continuum and the problem is treated from a macroscopic point of view. For ionized gases in the presence of magnetic fields, we have to assume not only the mean free path of gas particles to be much smaller than any pertinent characteristic length of the problem but also the magnetic field strength not very strong. Consequently, Larmor radii of ions are considerably larger than mean free path and collisions dominate the underlying interaction mechanism. The pressure can then be regarded isotropic throughout.[‡] Moreover, diffusion velocities of ions and electrons are assumed to be very small compared with mass velocity so that a treatment from one-fluid theory is justified.

In a study of the theory of MHD flow, cumbersome nonlinear terms arise from Maxwell stresses in addition to inertia stresses. For sufficiently high flow speeds the compressibility effects have also to be taken into account. The possibility of obtaining exact solutions of a general flow problem is remote and various approximate methods should be considered. Usually simplification is achieved at the expense of losing some essential features of the actual physical phenomenon. It is the object of the present work to study in detail finite amplitude MHD waves

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† MHD = magnetohydrodynamic, or magnetohydrodynamics.

‡ The corresponding simple waves in rarefied ionized gas with collisions playing negligible role has been investigated by Akhiezer, Polovin and Tsintsadge (1960) according to the Chew-Goldberger-Low approximation.

propagating in a fluid of otherwise uniform state. Disturbances of physical quantities are not small compared with those at initial state and the usual linearization procedure (acoustic approximation) is not justified. As a matter of fact, the basic principle of our investigation is to preserve all nonlinear terms of the governing equations and to seek an exact solution of the problem exhibiting the compressibility and MHD effects as well as their mutual interactions. Thus many powerful methods for solving linear equations find no application here. In order to avoid extra complexities due to geometry we have limited our attention to the study of one-dimensional unsteady waves for which all physical variables are functions of only one space variable, x_1, in addition to time, t. There is no transverse gradient in the x_2- and x_3-directions; or the state of fluid is homogeneous on all planes perpendicular to the x_1-direction. Wave fronts of any discontinuity are then necessarily of planar form and travel along x_1-direction. The problem is actually a generalization of the famous one-dimensional unsteady waves treated by Earnshaw and Riemann about a century ago. Due to the electrical conductivity of the fluid and the presence of magnetic fields, disturbances can be generated by electromagnetic means as well as by mechanical motion. These disturbances propagate as waves with finite speeds to other parts of the fluid. Thus wave motion governs the basic mechanism of fluid flow and is the most important phenomenon to be understood. These solutions are directly applicable to describe practical MHD flow in a channel of constant area, e.g. the generalized MHD shock tube flow for experimental studies of basic flow properties. It can also be used as an approximate solution of more general flows, e.g. those in a magnetic annular shock tube (MAST) developed by Patrick (1959). The prototypes of two- and three-dimensional steady flow are contained in the present solution; the insight gained here would be helpful in further investigations.

We consider only thermodynamically reversible flow so that there is no entropy production always. In other words, the entropy remains constant along each fluid particle path. Apart from the usual assumptions for a dissipationless fluid in ordinary gasdynamics (inviscid and non-heat-conducting, $\nu = 0$ and $k = 0$), we have to assume the electrical conductivity, σ, to be infinite also. This implies that the entropy produced by Joule heating, j^2/σ, vanishes and we are essentially dealing with an ideal fluid in MHD. Magnetic flux lines are then frozen in the medium, they are fixed to the same fluid particles always. The justification of assuming $\nu = 0$ and $k = 0$ for actual applications in ordinary gasdynamics is familiar; some remarks should be made about the proper significance of the assumption of infinite electrical conductivity. σ is finite indeed for most ionized gases. In addition to the system of nonlinear equations governing an ideal flow, a diffusion term (higher order spatial derivative) must enter in the problem. The general situation is similar to viscosity effects in ordinary gasdynamics. Nonlinear terms dominate the underlying motion but diffusion is important in certain regions of space and time. It is convenient to measure the effect of electrical conductivity by means of a dimensionless variable

$$\tau = \frac{t}{T^*}$$

where $T^* = 1/\mu\sigma V^2$ = the characteristic time and V can be any characteristic speed, e. g. the disturbance speed in the fluid. At $\tau >> 1$, an actual flow close to ideal flow can be achieved over large regions. Detailed calculations showing the asymptotic approach to ideal fluid motion have been given by Cole (1959).

The assumption of S = constant throughout the entire space has been tacitly avoided here; this permits the existence of contact surfaces across which an entropy jump is allowed. A typical example would be a flow in a shock tube. If gases on each side of the diaphragm have different pressures and densities initially, the entropies are not the same in general. They may even be of different kinds of gas. After the rupture of diaphragm, a contact surface separating gases on two sides exists and it moves with the same velocity as that of fluid in x_1-direction. The initial entropy jump is maintained throughout and an actual flow consisting of two separate isentropic flows of different entropies appears.

The flow speed is supposed to be much smaller than the speed of light so that no relativistic effects need be taken into account. Displacement currents may then be neglected throughout as usual. It can also be shown that electric forces are much smaller than magnetic forces and only the latter should be considered in the Lorentz force term that enters as a body force in ordinary gasdynamics. The gas is assumed to be isotropic, homogeneous, devoid of electric and magnetic polarization throughout. It is electrically neutral in the bulk though some weak local charge distribution may be built up owing to nonuniformity of electric fields (see e. g. Cowling, 1956). The permittivity, ϵ, and permeability, μ, are assumed to be constant in the entire fluid; they take the same values as those in free space. Rationalized MKSQ units will be used throughout. Then the flow problem can be formulated according to a simplified version of Lunquist's model (1952).

In order to facilitate detailed mathematical analysis, the working medium considered in this paper is a polytropic gas which implies that the adiabatic exponent, $\gamma = C_p/C_v$,[†] is a constant. The gas is also considered calorically perfect. The caloric equation of state reads

$$P = A\rho^{\gamma} \tag{1-1}$$

where the coefficient A is generally a function of entropy only. It follows from classical statistical mechanics that the value of γ is directly related to the number of degrees of freedom, n, of the particular molecular model of gas, that is

$$\gamma = \frac{n+2}{n} \tag{1-2}$$

γ ranges from 1 to $5/3$ for most gases and it is a fixed value after a certain gas or mixture is chosen.

After the general formulation of the problem, the flow is seen to be governed by a system of quasi-linear first order hyperbolic partial differential equations as functions of two independent variables. The fundamental mathematical nature of this system is well understood. Existence, uniqueness and differentiability of solutions of the corresponding initial value problem have been established by Friedrichs (1948, 1955) before the advent of MHD. Lax (1957) has made some further investigations on a special class of the general system satisfying conservation laws. A detailed proof of many important theorems characterizing

† Due to the extreme difference of mass in ions and electrons in ionized gases, γ can be calculated as due to ions only.

the solution, especially that of simple waves was given there. These fundamental studies place the physical problem on a sound mathematical ground and one is left to examine detailed structures of the solution.

Due to the important property of simple waves that they are always contiguous to a uniform state and due to their relative simplicity in mathematical analysis, they constitute the major portion of the present study. The general solution of one-dimensional unsteady MHD flow is not attempted here except that some remarks are made about the special case of flow in a purely transverse magnetic field. Simple wave solutions exist only in isentropic flows. The entire process of expansion before the interaction of waves takes place, the part of the compression process before the formation of shock wave, and contact surface between different streamlines are described completely by simple waves. These solutions, together with jump conditions for MHD shocks,[†] suffice to describe a wide variety of one-dimensional unsteady MHD flow problems. Flows involving interaction of waves have, however, to be determined by general solutions of the system. Since we are considering only flows of an ideal fluid which enable us to obtain exact solutions, the diffusion of shear by viscosity, that of temperature by heat conductivity and that of magnetic flux lines by electrical conductivity are neglected. Sharp discontinuities of wave fronts and contact surfaces persisting with time are admitted. Their detailed structures may be studied individually by including transport parameters in the analysis and by suitable "boundary" layers.

[†] Jump conditions of MHD shocks have been investigated by de Hoffmann, Teller, Helfer, Lüst and Friedrichs. An exhaustive treatment by Bazer and Ericson (1959) is given recently and an extensive bibliography can be found there.

Plane MHD simple waves[†] have been investigated previously by many eminent authors. In his masterly treatment of nonlinear wave motion in MHD, Friedrichs (1954) first illustrated a method of approach for obtaining solutions and pointed out the reducibility of the system to a single linear first order differential equation for coupled waves. Detailed solutions were not given there. Bazer (1958) has extended Friedrichs method to the solution of a shear flow discontinuity problem. Centered slow simple waves were discussed with particular emphasis on the gas of $\gamma = 5/3$ and an approximation was made to obtain the analytic solution. The problem has also been formulated by Akhiezer, Liubarsky and Polovin (1958) who made some general studies of the mathematical content of the governing equations and classified different types of simple waves. No final solution is available either. In the theoretical analysis of MAST flow, Kemp and Petschek formulated centered slow simple waves in terms of three simultaneous first order ordinary differential equations; a numerical integration by use of an electronic computer was carried out subsequently to obtain the solution. The special case of simple waves in a purely transverse magnetic field, has been investigated by Golitsyn (1959) and Mitchner (1959). Some linearized approximate solutions for plane waves with $B_1 \neq 0$ and $B_2 \neq 0$ were given by Grad (1959).

In the present work plane MHD simple waves are analyzed systematically from a general point of view. After the formulation of the system of governing differential equation in Section II, we first

[†] It may be noted that the corresponding simple waves in two-dimensional steady MHD flow have been treated by Golitsin (1958).

digress to a brief discussion of some basic properties of a corresponding general system for N dependent variables. This is helpful to demonstrate the essential structure of solutions. There exist totally seven different kinds of simple waves in one-dimensional unsteady MHD flow. Out of these only coupled fast and slow simple waves are genuinely nonlinear. They propagate with varying speed and the wave profile (the distribution of physical quantities) always distorts. The steepening up of compression waves due to nonlinear effects leads ultimately to the development of MHD shocks. The interplay of compressibility and MHD effects is most pronounced here and we shall devote most of our attention to these coupled waves to expound in full details the mechanism involved. Solutions for contact surfaces and transverse simple waves are relatively trivial;they are mentioned in Section III for the sake of completeness. Particular attention has been given to contact surfaces adjacent to a vacuum state because of its occurrence in actual problems. In Section IV we first give an analysis of the exact solution of coupled simple waves travelling in a gas of arbitrary value of γ. An alternative method of solving coupled waves by graphical means which is especially convenient for an immediate estimate of the outcome of a flow problem is discussed next. Section V discusses the special case of flow in a purely transverse magnetic field. For this case, the general solution of an arbitrary flow is shown to be governed by a single second order linear partial differential equation and the coupled simple wave solution degenerates to a much simpler form. In Section VI, some examples showing the application of our analytic as well as graphical solutions to physical problems are given. The present investigation is concluded in Section VII by several general remarks.

II. FORMULATION OF THE PROBLEM

With general assumptions stated in the previous Section, we are now able to formulate the physical problem in a mathematically amenable form. One-dimensional steady flow of an ideal fluid in ordinary gasdynamics is governed by

Continuity $$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_1)}{\partial x_1} = 0 \qquad (2\text{-}1a)$$

Momentum $$\rho\left(\frac{\partial \vec{u}}{\partial t} + u_1 \frac{\partial \vec{u}}{\partial x_1}\right) + \frac{\partial P}{\partial x_1} = \vec{f} \qquad (2\text{-}1b)$$

Isentropy $$\frac{\partial S}{\partial t} + u_1 \frac{\partial S}{\partial x_1} = 0 \qquad (2\text{-}1c)$$

together with the equation of state 1-1 where $\vec{f}$ is the body force of any mechanical origin. However, we have here $\vec{f}$ = Lorentz force = $\vec{j} \times \vec{B}$ in MHD with $\vec{j}$ and $\vec{B}$ being governed by Maxwell equations. Since electric current density flowing in a fluid must be finite, it follows from Ohm's law $\vec{j} = \vec{\sigma}(\vec{E} + \vec{u} \times \vec{B})$ that

$$\vec{E} = -\vec{u} \times \vec{B} \qquad (2\text{-}2)$$

for ideal fluids. Thus only an induced electric field resulting from the interaction of fluid motion and magnetic field exists. It is based on the aforementioned assumptions that Maxwell equations for one-dimensional space variation take the following simple form:

$$\frac{\partial \vec{B}}{\partial t} - \vec{e}_1 \times \frac{\partial}{\partial x_1}(\vec{u} \times \vec{B}) = 0 \qquad (2\text{-}3a)$$

$$\vec{j} = \frac{1}{\mu}\vec{e}_1 \times \frac{\partial \vec{B}}{\partial x_1} \qquad (2\text{-}3b)$$

$$\frac{\partial B_1}{\partial x_1} = 0 \qquad (2\text{-}3c)$$

$$\epsilon \frac{\partial(\vec{B} \times \vec{u})_1}{\partial x_1} = \omega \qquad (2\text{-}3d)$$

The charge density, ω, appears only in 2-3d and it can be calculated after knowing all other quantities. From 2-3b, the Lorentz force becomes

$$\vec{\Gamma} = \frac{1}{\mu} (\vec{e_1} \times \frac{\partial \vec{B}}{\partial x_1}) \times \vec{B} \qquad (2\text{-}4)$$

It is through this term that the original system of hydrodynamic equations 2-1 is coupled with the Maxwell equations and the interaction between electromagnetism and hydrodynamics enters. Substituting 2-4 into 2-1b and combining the induction equations 2-3a, c with 2-1, we get a complete system of equations for the description of one-dimensional unsteady flow of an ideal fluid in MHD.

It is easily seen from 2-3c and the x_1-component of 2-3a that

$$B_1 = \text{constant} \qquad (2\text{-}5)$$

throughout. This provides an essential simplification in later analysis. The constant value of B_1 corresponds to the minimum amount of magnetic field strength embedded in the fluid and it plays a prominent role in characterizing the actual flow problem.

Writing physical quantities in component form, we have from 2-3b the current density

$$\vec{j} = -\frac{1}{\mu} \frac{\partial B_3}{\partial x_1} \vec{e_2} + \frac{1}{\mu} \frac{\partial B_2}{\partial x_1} \vec{e_3} \qquad (2\text{-}6)$$

which flows on transverse plane. The Lorentz force is

$$\vec{T} = \left(-\frac{B_2}{\mu}\frac{\partial B_2}{\partial x_1} - \frac{B_3}{\mu}\frac{\partial B_3}{\partial x_1}\right)\vec{e_1} + \frac{B_1}{\mu}\frac{\partial B_2}{\partial x_1}\vec{e_2} + \frac{B_1}{\mu}\frac{\partial B_3}{\partial x_1}\vec{e_3} \qquad (2\text{-}7)$$

Then the governing equations written explicitly in component form read

Continuity $$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_1)}{\partial x_1} = 0 \qquad (2\text{-}8a)$$

x_1-momentum $$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1} + \frac{1}{\rho}\frac{\partial P}{\partial x_1} + \frac{B_2}{\mu\rho}\frac{\partial B_2}{\partial x_1} + \frac{B_3}{\mu\rho}\frac{\partial B_3}{\partial x_1} = 0 \qquad (2\text{-}8b)$$

x_2-momentum $$\frac{\partial u_2}{\partial t} + u_1\frac{\partial u_2}{\partial x_1} - \frac{B_1}{\mu\rho}\frac{\partial B_2}{\partial x_1} = 0 \qquad (2\text{-}8c)$$

x_3-momentum $$\frac{\partial u_3}{\partial t} + u_1\frac{\partial u_3}{\partial x_1} - \frac{B_1}{\mu\rho}\frac{\partial B_3}{\partial x_1} = 0 \qquad (2\text{-}8d)$$

x_2-induction $$\frac{\partial B_2}{\partial t} + \frac{\partial}{\partial x_1}(B_2 u_1 - B_1 u_2) = 0 \qquad (2\text{-}8e)$$

x_3-induction $$\frac{\partial B_3}{\partial t} + \frac{\partial}{\partial x_1}(B_3 u_1 - B_1 u_3) = 0 \qquad (2\text{-}8f)$$

Isentropy $$\frac{\partial S}{\partial t} + u_1\frac{\partial S}{\partial x_1} = 0 \qquad (2\text{-}8g)$$

together with the equation of state 1-1. These actually correspond to the inviscid case of Lunquist's model of MHD equations (Lunquist 1952). No characteristic length or time can be formed from variables in 2-8 and the flow is characterized only by various speeds which are defined as follows. The local speed of sound in ordinary gasdynamics is

$$a = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_s} = \sqrt{\gamma A \rho^{\gamma-1}} \qquad (2\text{-}9a)$$

It has no direct physical significance in MHD except serving as a reference of comparison between states of fluid flow with and without magnetic effects. Local Alfvén wave speeds based on different components of magnetic field strength are respectively

$$b_1 = \frac{B_1}{\sqrt{\mu\rho}} \tag{2-9b}$$

$$b_2 = \frac{B_2}{\sqrt{\mu\rho}} \tag{2-9c}$$

$$b_3 = \frac{B_3}{\sqrt{\mu\rho}} \tag{2-9d}$$

If we define $\vec{B}_t$ to be the magnetic field in transverse plane, i. e.

$$\vec{B}_t = B_2\vec{e}_2 + B_3\vec{e}_3 \tag{2-10a}$$

the corresponding Alfvén wave speed is

$$b_t = \frac{B_t}{\sqrt{\mu\rho}} = \sqrt{\frac{B_2^2 + B_3^2}{\mu\rho}} \tag{2-10b}$$

One may note that a characteristic length of the problem may, however, be formed from boundary conditions. It is sometimes more convenient to eliminate P in 2-8b by making use of the equation of state 1-1; we have then

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1} + \frac{a^2}{\rho}\frac{\partial\rho}{\partial x_1} + A'\rho^{\gamma-1}\frac{\partial S}{\partial x_1} + \frac{B_2}{\mu\rho}\frac{\partial B_2}{\partial x_1} + \frac{B_3}{\mu\rho}\frac{\partial B_3}{\partial x_1} = 0 \tag{2-8h}$$

where $A' = A'(S) = \frac{dA}{dS}$ is a known function of entropy. This, together with 2-8a, c, d, e, f, g, gives seven equations for seven unknowns, namely, ρ, u_1, u_2, u_3, B_2, B_3 and S. Coefficients of derivatives in these equations are functions of dependent variables only. The equations are a system of homogeneous first order quasilinear partial differential equations. Moreover, they can be shown to be of hyperbolic type. The general solution of this highly nonlinear system is very difficult if not impossible to obtain; we confine our interest here mainly to the investigation of a special class of simple wave equations defined by the requirement that all dependent variables are functions of one of them, which is in turn a function of x_1 and t. The mathematical consequence of this solution has the important physical property that a flow contiguous to a constant state is always described by simple waves. The simple waves are basic elements in constructing many flow problems and essential features of the interaction mechanism in compressible MHD may be revealed from a study of the solutions.

In view of acquiring a deeper understanding of the mathematical nature of governing equations and their simple wave solutions, we begin with a discussion of some general properties of a system of first order quasilinear hyperbolic equations for N dependent variables as functions of two independent variables. It has theoretical interest in itself and has been treated by Friedrichs (1948), Lax (1957), Akheizer, Liubarsky and Polovin (1958). Many important theorems about this general system have been established and proved. The simple wave solution may be regarded as a generalization of those in ordinary gasdynamics ($N = 2$). But a somewhat different point of view and formulation of the solution is presented here. Some basic features and their mutual connections that have not been pointed out specifically by previous authors will be shown below. They are believed to be essential in understanding the structure of simple waves in a general sense.

Let us denote $\vec{v} = (v_1, v_2, \ldots, v_N)$ to be a vector in N-dimensional space with components representing dependent variables of a physical system. The general expression of first order quasilinear partial differential equations as functions of two independent variables, x and y, is

$$L_i = X_{ij}\frac{\partial v_j}{\partial x} + Y_{ij}\frac{\partial v_j}{\partial y} + Z_i = 0 \qquad (i, j = 1, 2, \ldots N) \quad (2\text{-}11)$$

where X_{ij}, Y_{ij}, Z_i are functions of v_j, x and y in general. The summation sign has been omitted in favor of the convention that repeated indices signifies summation over all terms.

The characteristics of the system will be determined first. This corresponds to finding a linear combination of 2-11 for different i such that each of the dependent variables, v_j, has a derivative along the same direction in the (x-y) plane. Thus we write

$$L = \lambda_i L_i = \lambda_i\left(X_{ij}\frac{\partial v_j}{\partial x} + Y_{ij}\frac{\partial v_j}{\partial y} + Z_i\right) = 0 \qquad (2\text{-}12)$$

and require

$$\lambda_i X_{ij} : \lambda_i Y_{ij} = x_\sigma : y_\sigma \qquad (2\text{-}13a)$$

for each j where σ is a parameter along the characteristic direction. Or

$$\lambda_i(X_{ij}y_\sigma - Y_{ij}x_\sigma) = 0 \qquad j = 1, 2, \ldots, N \qquad (2\text{-}13b)$$

The necessary and sufficient condition for existence of non-trivial values of λ_i is

$$\|X_{ij}y_\sigma - Y_{ij}x_\sigma\| = 0 \tag{2-14a}$$

Let us define

$$V = \frac{dx}{dy} = \frac{x_\sigma}{y_\sigma}$$

to be the inverse of the slope of characteristics on the (x-y) plane and assume y_σ to be nowhere zero (i. e. there is no characteristics parallel to the x-axis). 2-14a becomes

$$\| X_{ij} - VY_{ij}\| = 0 \tag{2-14b}$$

This is an N-th degree algebraic equation for N roots of V, namely $V_1, V_2, \ldots V_N$ which are generally functions of v_j, x and y. If all V_j are real and distinct, 2-11 is a totally hyperbolic system and this will be our main concern here.

Simple wave solutions exist only for a restricted class of the general totally hyperbolic system 2-11 such that X_{ij} and Y_{ij} are functions of dependent variables, v_j, only and Z_i equals zero. The corresponding equations for $N = 2$ (e. g. the one-dimensional unsteady flow in ordinary gasdynamics) are called reducible (Courant and Friedrichs, 1948) since the dependent variable space and independent variable space have the same dimension and a simple hodograph transformation can be performed to reduce the governing equations to linear ones. In particular, its simple wave solution corresponds to

$$v_2 = v_2(v_1)$$

only. The well known fact that the Jacobian of transformation vanishes identically can be seen easily, because

$$J = \frac{\partial v_1}{\partial x}\frac{\partial v_2}{\partial y} - \frac{\partial v_1}{\partial y}\frac{\partial v_2}{\partial x} = \frac{dv_2}{dv_1}\left(\frac{\partial v_1}{\partial x}\frac{\partial v_1}{\partial y} - \frac{\partial v_1}{\partial y}\frac{\partial v_1}{\partial x}\right) = 0$$

However, for cases of $N > 2$, no general method can be applied to reduce the original system to linear equations.[†]

Let us now consider the simple wave solution of the general system 2-11 which is defined by

$$v_j = v_j(v_1) \tag{2-15}$$

2-11 can be written as

$$\left(X_{ij}\frac{\partial v_1}{\partial x} + Y_{ij}\frac{\partial v_1}{\partial y}\right)\frac{dv_j}{dv_1} = 0 \qquad i = 1, \ldots, N \tag{2-16a}$$

For non-trivial values of $\frac{dv_j}{dv_1}$ to exist we require

$$\left\| X_{ij}\frac{\partial v_1}{\partial x} + Y_{ij}\frac{\partial v_1}{\partial y} \right\| = 0 \tag{2-16b}$$

The derivative along a curve of constant v_1 in the $(x-y)$ plane is

$$dv_1 = \frac{\partial v_1}{\partial x}dx + \frac{\partial v_1}{\partial y}dy = 0$$

or we denote

$$U = \frac{dx}{dy} = -\frac{\partial v_1/\partial y}{\partial v_1/\partial x} = U(v_1) \tag{2-17}$$

[†] This is one of the reasons that a general solution of one-dimensional unsteady flow in MHD is very difficult to obtain.

Since U is constant for each given value of v_1, it is in fact the inverse of the slope of a straight line on the $(x-y)$ plane along which not only v_1 but also all $v_2, \ldots, v_N$ are constant. The physical state remains constant along this line which is thus termed a phase line. In N-dimensional dependent variable space each phase line corresponds to a point. Substituting 2-17 into 2-16a, b we get

$$(X_{ij} - UY_{ij})dv_j = 0 \tag{2-18}$$

and

$$||X_{ij} - UY_{ij}|| = 0 \tag{2-19}$$

for $\frac{\partial v_1}{\partial x} \neq 0$ in general. This gives N roots of U, namely $U_1, U_2, \ldots, U_N$ which are all different for a totally hyperbolic system. Obviously only one family of phase lines corresponding to one of U_j can exist in order to describe a region of varying state in the $(x-y)$ plane by simple waves. The choice of a particular one of them depends on boundary conditions of the problem. The notion of phases is most helpful in replacing the independent variables x and y for simple waves; it amounts to a reduction of the original system to ordinary differential equations.

A comparison of 2-14b and 2-19 shows that slopes of phase lines and characteristics are determined by the same equation. Only one family of characteristics coincides with phase lines of a given flow problem and consists of straight lines in the $(x-y)$ plane. The other N-1 characteristics are curved in general.

We may write for the k-th simple waves characterized by phases of $U = U_k$

$$(X_{ij} - U_k Y_{ij})dv_j = 0 \tag{2-20}$$

This is the reduced form of 2-11 and consists of N first order ordinary differential equations showing differential relations among various physical quantities through a continuous change of phases. First integrals of these equations, together with the use of initial and boundary conditions of the problem, give the simple wave solution. Since 2-20 is homogeneous, there exist only $N - 1$ linearly independent relations (assuming no degeneracy) among N physical variables for each value of U_k. These form a family of k-th Riemann invariants across simple waves. For any given system 2-11, there are totally N families of these invariant relations and only a pertinent one of them is sufficient to describe completely a flow governed by simple waves.

We shall now discuss the special case that $Y_{ij} = I_{ij}$ = the identity matrix. The corresponding system has a simple physical meaning in obeying conservation laws and has been investigated by Lax (1957). A somewhat different point of view illustrating several of its essential properties is given below.

2-14b has now the form

$$||X_{ij} - VI_{ij}|| = 0 \qquad (2\text{-}21)$$

2-18 becomes

$$(X_{ij} - UI_{ij})dv_j = 0 \qquad (2\text{-}22a)$$

where U is determined by the condition

$$||X_{ij} - UI_{ij}|| = 0 \qquad (2\text{-}22b)$$

and has N distinct values $U_1, U_2, \ldots, U_N$. For a k-th simple wave of

$U = U_k$, we have from 2-20 or 2-22a

$$(X_{ij} - U_k I_{ij}) dv_j = 0 \tag{2-23}$$

If we denote the right eigenvector of $\underline{\underline{X}}$ † in N-dimensional dependent variable space belonging to the eigenvalue W by $\vec{r} = (r_1, r_2, \ldots r_N)$, then

$$(X_{ij} - W I_{ij}) r_j = 0 \tag{2-24a}$$

where W is obtained from the equation

$$||X_{ij} - W I_{ij}|| = 0 \tag{2-24b}$$

and has N distinct values $W_1, W_2, \ldots, W_N$. Thus for a k-th right eigenvector $\vec{r}^{(k)} = [r_1^{(k)}, r_2^{(k)}, \ldots, r_N^{(k)}]$ belonging to the eigenvalue W_k, we have from 2-24a

$$(X_{ij} - W_k I_{ij}) r_j^{(k)} = 0 \tag{2-25}$$

It is apparent from 2-21, 2-22b and 2-24b that V_j, U_j, and W_j $(j = 1, 2, \ldots, N)$ are roots of the same equation, they have exactly the same values. For a flow governed by simple waves, there can exist only one value of U, say U_k, pertaining to a k-th simple wave; but the inverse of slopes of different families of characteristics are always eigenvalues of the operator $\underline{\underline{X}}$.

Since $U_k = W_k$ for a k-th simple wave, we obtain from 2-23 and 2-25 the following relation

† In contrast to the left eigenvector which is defined to be the eigenvector of the transpose of matrix $\underline{\underline{X}}$. (See Friedman, 1956)

$$\frac{dv_1}{r_1^{(k)}} = \frac{dv_2}{r_2^{(k)}} = \dots = \frac{dv_N}{r_N^{(k)}} = \text{constant} \tag{2-26}$$

This gives the complete set of equations for describing a simple wave flow from which Riemann invariants can be found easily, in principle. Let us denote $R^{(k)}(v_1, v_2, \dots, v_N)$ = constant to be the envelope of the k-th Riemann invariants in N-dimensional dependent variable space. It follows from 2-26† that

$$r_j^{(k)} \frac{\partial R^{(k)}}{\partial v_j} = 0 \tag{2-27a}$$

or equivalently

$$\vec{r}^{(k)} \cdot \text{grad } R^{(k)} = 0 \tag{2-27b}$$

for the k-th Riemann invariants. Since the gradient of $R^{(k)}$ is perpendicular to $\vec{r}^{(k)}$ in an N-dimensional space, there exist precisely N-1 linearly independent k-th Riemann invariants with the gradient of their envelope spanning the orthogonal complement of $\vec{r}^{(k)}$ (assume no degeneracy). This result agrees with what we obtained before. 2-27b can be considered as an alternative definition of Riemann invariants in the first place (Lax, 1957) while the definition of simple waves follows as a consequence afterwards.

A simple recipe (Friedrichs, 1954) can also be used to reduce the governing equations to ordinary differential equations. Several interesting properties of the solution are, however, concealed in carrying

†See e. g. Courant and Hilbert (1937), p. 25.

out this process. It follows from 2-17 that its general solution is (see fig. 1)

$$x - U(v_1)y = \xi(v_1) \tag{2-28}$$

where $\xi(v_1)$ is an arbitrary function of v_1 and can be regarded as a parameter to characterize the phase of a physical state. U takes the value of one of the roots of 2-22b for its corresponding type of simple wave. The specific one of v_j to be chosen as reference, i. e. v_1, on which other physical variables depend is judged solely from practical convenience.

In accordance with 2-28 the following simple relations exist

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{d}{d\xi} = \frac{1}{1+U'(\xi)y}\frac{d}{d\xi} = \frac{1}{1+U'(\xi)y}\frac{dv_1}{d\xi}\frac{d}{dv_1} \tag{2-29a}$$

$$\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{d}{d\xi} = \frac{-U(\xi)}{1+U'(\xi)y}\frac{d}{d\xi} = \frac{-U(\xi)}{1+U'(\xi)y}\frac{dv_1}{d\xi}\frac{d}{dv_1} \tag{2-29b}$$

For the special case of centered waves in which constant phase lines run through a fixed point, say the origin in the (x, y) plane, 2-28 becomes

$$x_1 = U(\xi)y \tag{2-30}$$

The partial derivatives assume the following form instead

$$\frac{\partial}{\partial x} = \frac{1}{U'(\xi)y}\frac{d}{d\xi} = \frac{1}{U'(\xi)y}\frac{dv_1}{d\xi}\frac{d}{dv_1} \tag{2-31a}$$

$$\frac{\partial}{\partial y} = \frac{-U(\xi)}{U'(\xi)y}\frac{d}{d\xi} = -\frac{U(\xi)}{U'(\xi)y}\frac{dv_1}{d\xi}\frac{d}{dv_1} \tag{2-31b}$$

2-30 corresponds to a self-similar flow and no characteristic length exists in the whole problem.

Since 2-11 must be homogeneous for existence of simple waves, upon substitution of either 2-29 or 2-31 into the system for partial derivatives, a common factor $1/(1+U'(\xi)y)(1/d\xi)$ or $1/(U'(\xi)y)(1/d\xi)$ appears in each term and can be eliminated. We arrive in either case at a system of ordinary differential equations describing the change of physical states across phases. It is equivalent to 2-22a exactly.

Returning now to our problem of plane simple waves in MHD and in the light of 2-8a, c, d, e, f, g, h, we may identify the dependent variables $v_1, v_2, \ldots, v_7$ by ρ, u_1, u_2, u_3, B_2, B_3, S, the independent variables x and y by x_1 and t respectively, $Y_{ij} = I_{ij}$ and

$$X_{ij} = \begin{pmatrix} u_1 & \rho & 0 & 0 & 0 & 0 & 0 \\ \frac{a^2}{\rho} & u_1 & 0 & 0 & \frac{B_2}{\mu\rho} & \frac{B_3}{\mu\rho} & A'\rho^{\gamma-1} \\ 0 & 0 & u_1 & 0 & -\frac{B_1}{\mu\rho} & 0 & 0 \\ 0 & 0 & 0 & u_1 & 0 & -\frac{B_1}{\mu\rho} & 0 \\ 0 & B_2 & -B_1 & 0 & u_1 & 0 & 0 \\ 0 & B_3 & 0 & -B_1 & 0 & u_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u_1 \end{pmatrix} \qquad (2\text{-}32)$$

We obtain either from 2-22a or by substituting 2-29 or 2-31 into 2-8 the following system of ordinary differential equations.†

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†The application of 2-26 to 2-8 to obtain an equivalent set of equations as 2-33 is given in Appendix A for reference.

$$-c\,d\rho + \rho\,du_1 = 0 \tag{2-33a}$$

$$-c\rho\,du_1 + \frac{a^2}{\rho}\,d\rho + \frac{B_2}{\mu}\,dB_2 + \frac{B_3}{\mu}\,dB_3 + A'\rho^{\gamma-1}dS = 0 \tag{2-33b}$$

$$-c\rho\,du_2 - \frac{B_1}{\mu}\,dB_2 = 0 \tag{2-33c}$$

$$-c\rho\,du_3 - \frac{B_1}{\mu}\,dB_3 = 0 \tag{2-33d}$$

$$-c\,dB_2 + B_2\,du_1 - B_1\,du_2 = 0 \tag{2-33e}$$

$$-c\,dB_3 + B_3\,du_1 - B_1\,du_3 = 0 \tag{2-33f}$$

$$-c\,dS = 0 \tag{2-33g}$$

where

$$c(\xi) = U(\xi) - u_1(\xi) \tag{2-34}$$

is the phase velocity relative to longitudinal flow velocity and can be either positive or negative corresponding to forward or backward facing simple waves respectively.

We may write 2-33b in another form by making use of the equation of state 1-1, then

$$-c\rho\,du_1 + dP + \frac{B_2}{\mu}dB_2 + \frac{B_3}{\mu}dB_3 = 0 \tag{2-35}$$

2-33 is still a system of highly nonlinear equations as it stands. c is determined by setting the determinant of 2-33 to be zero which actually corresponds to 2-21, 2-22b, or 2-24b. Thus we get the following equation written in terms of speeds only by virtue of 2-9 and 2-10

$$c(c^2 - b_l^2)[(c^2 - a^2)(c^2 - b_l^2) - c^2 b_t^2] = 0 \tag{2-36}$$

This gives seven distinct roots of c, each describing a particular type of simple wave. They may be classified into three different modes, namely

(a) $c = 0$ corresponding to contact surfaces. They are also called entropy waves due to the fact that entropy may take arbitrary values along different phase lines of contact surfaces.

(b) $c = \pm b_l$ corresponding to transverse simple waves. They are called hydromagnetic waves also and are essentially new features in MHD.

(c) $(c^2-a^2)(c^2-b_l^2) - c^2 b_t^2 = 0$

or

$$c = \pm \sqrt{\frac{a^2+b_l^2+b_t^2}{2} \pm \sqrt{\left(\frac{a^2+b_l^2+b_t^2}{2}\right)^2 - a^2 b_l^2}} \qquad (2\text{-}37)$$

corresponding to fast and slow coupled simple waves. They also have been called magnetoacoustic waves since ordinary sound waves appear as a limiting case in the absence of magnetic fields.

The detailed structure of these simple waves is discussed in subsequent sections.

III. CONTACT SURFACES AND TRANSVERSE SIMPLE WAVES

1. Contact Surfaces

When $c = 0$, the phase velocity is always equal to longitudinal fluid velocity. Since there is no relative motion between the phase and fluid particle path, these waves are more properly called contact surfaces. A sharp discontinuity of certain physical quantities is admissible here owing to the neglect of all real gas effects. One may see from 2-33g that ds = arbitrary, hence entropy may undergo any finite jump across contact surfaces. They differ from shock waves mainly in there being no mass flow of fluid across the discontinuity surface. Various properties of contact surfaces in MHD are given below, they may also be obtained by taking the limit of the general result of MHD shocks. Particular interest will be placed on the special kind of contact surface adjacent to a vacuum region. Some basic properties associated with the cavitation zone are also discussed.

Putting $c = 0$ in 2-33a, c, d, e, f, g and 2-35, we have

$$\rho \, du_1 = 0 \tag{3-1a}$$

$$dP + \frac{B_2}{\mu} dB_2 + \frac{B_3}{\mu} dB_3 = 0 \rightarrow d\left(P + \frac{B_t^2}{2\mu}\right) = 0 \tag{3-1b}$$

$$\frac{B_1}{\mu} dB_2 = 0 \tag{3-1c}$$

$$\frac{B_1}{\mu} dB_3 = 0 \tag{3-1d}$$

$$B_2 \, du_1 - B_1 \, du_2 = 0 \tag{3-1e}$$

$$B_3 \, du_1 - B_1 \, du_3 = 0 \tag{3-1f}$$

and dS = arbitrary. Due to the fact that a finite jump of S is allowed, this permits abrupt changes of other physical quantities across a contact surface which should be governed by jump conditions. The general method of Riemann invariants is not appropriate here. Jump conditions can be obtained simply by replacing d by $[\;]$ in 3-1. Thus the relation of discontinuities of different variables is seen clearly from 3-1. If $[\vec{B}_t] \neq 0$, there exists always a current sheet flowing along the discontinuity surface. This is seen upon integration of 2-26 with respect to x_1 together with B_1 = constant. The current per unit length in transverse direction is

$$\vec{I} = \vec{e}_1 \times [\vec{B}_t] \tag{3-2}$$

We recall the density is always related to entropy and pressure by the equation of state 1-1. Solutions for various cases of contact surface are trivial; we shall merely state the result as follows

(A) $\rho \neq 0$ on both sides.

Neither side of the contact surface consists of a vacuum region. It follows from 3-1a that u_1 is always continuous across the contact surface.

(i) $B_1 \neq 0$

Physical quantities $\vec{u}$, $\vec{B}$, P should remain the same in crossing a contact surface. No discontinuity in transverse velocity or transverse magnetic field can be preserved. The former differs radically from ordinary gasdynamics and it serves as a starting point for the problem of resolution of a shear flow discontinuity investigated by Bazer (1958).

Since $\vec{B}_t$ must be continuous across the discontinuity, there is no current flowing on it.

(ii) $B_1 = 0$

(a) $B_t \neq 0$ (purely transverse field case)

$\vec{u}_t$ can jump across the contact surface, hence the shear flow discontinuity is allowed in this case. The longitudinal velocity on each side of the surface remains always the same (see Section V). P and $\vec{B}_t$ may vary arbitrarily under the condition 3-1b

(b) $B_t = 0$ (ordinary gasdynamic case)

P is continuous and $\vec{u}_t$ can be completely arbitrary across the contact surface.

(B) $\rho = 0$ on one side.†

A cavitation zone exists on one side of the contact surface in which we have only field quantities governed by Maxwell equations. Since there are no material particles inside this zone, the current density and charge distribution vanish identically there. There is also no physical sense to talk about the continuity or jump of $\vec{u}$ across the contact surface here.

The reduced set of Maxwell equations 2-3 for empty space is

$$\frac{\partial \vec{B}}{\partial t} + \vec{e}_1 \times \frac{\partial \vec{E}}{\partial x_1} = 0 \qquad (3\text{-}3a)$$

$$\frac{1}{\mu} \vec{e}_1 \times \frac{\partial \vec{B}}{\partial x_1} = 0 \qquad (3\text{-}3b)$$

$$\frac{\partial B_1}{\partial x_1} = 0 \qquad (3\text{-}3c)$$

$$\epsilon \frac{\partial E_1}{\partial x_1} = 0 \qquad (3\text{-}3d)$$

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† The discussion stated here is a generalization of Bazer's result (1958).

together with the assumption of displacement current being equal to zero, i. e.

$$\epsilon \frac{\partial \vec{E}}{\partial t} = 0 \tag{3-3e}$$

which has already been used in 3-3b. 3-3e is justified on the grounds that the characteristic speed of the problem of interest is much less than the speed of light; this holds even for the restricted region of empty space here. One may see immediately from 3-3d, e that

$$E_1 = \text{constant} \tag{3-4a}$$

and

$$\vec{E_t} = \vec{E_t}(x_1) \tag{3-4b}$$

Similarly, one gets from 3-2a, c

$$B_1 = \text{constant} \tag{3-5a}$$

and from 3-3b that

$$\vec{B_t} = \vec{B_t}(t) \tag{3-5b}$$

Thus we may write 3-3a as

$$\frac{\partial \vec{B_t}}{\partial t} = \frac{\partial \vec{E_t}}{\partial x_1} \times \vec{e_1} = \vec{T} \tag{3-6}$$

where $\vec{T}$ is a constant vector independent of x_1 and t. Various cases of contact surfaces are discussed as follows.

(i) $B_1 \neq 0$

$\vec{B}$, $\vec{E} = -\vec{u} \times \vec{B}$ and P should be continuous across the contact surface on which there is no current flowing. Since $\rho = P = 0$ on one side of the contact surface and P is continuous, it follows from the equation of state that $\rho = 0$ on the other side of it too. There is no jump in entropy. All physical quantities are continuous across this type of contact surface which is the straight characteristic (phase line) in the $(x_1 - t)$ plane separating the vacuum region from the varying field. The latter can be shown to be composed of coupled simple waves. Because all physical quantities are kept constant along the separating characteristics, in particular $\vec{B}$ and $\vec{E}$, we conclude from 3-6 that

$$\vec{T} = 0 \tag{3-7a}$$

and

$$\vec{B}_t = (\vec{B}_t)_{cav.} = \text{a constant vector} \tag{3-7b}$$

$$\vec{E}_t = (\vec{E}_t)_{cav.} = \text{a constant vector} \tag{3-7c}$$

or equivalently we may say that $\vec{B}$ and $\vec{E}$ are constant vectors in the cavitation zone. The value of $\vec{E}_{cav.}$ is determined by

$$\vec{E}_{cav.} = \vec{E}_{esc.} = -\vec{u}_{esc.} \times \vec{B}_{esc.} \tag{3-8a}$$

where the subscript "esc." is the abbreviation of "escape" and denotes the corresponding values at the tail of the coupled simple waves in the continuum region. The escape velocity $\vec{u}_{esc.}$ and escape magnetic field strength $\vec{B}_{esc.}$ are finite; they can be obtained explicitly for a given problem (see Section IV). We know also

$$\vec{B}_{cav.} = \vec{B}_{esc.} \tag{3-8b}$$

(ii) $B_1 = 0$

(a) $B_t \neq 0$ (purely transverse field case)

P and $\vec{B}_t$ may jump arbitrarily across the contact surface except that they are subjected to the condition 3-1b. A jump of density is also allowed and the gas may in this case possess a finite density on the continuum side. If there is a transverse magnetic field $(\vec{B}_t)_{cav.}$ present in the vacuum zone initially, the balance of pressure across contact surface would be

$$\left(\frac{B_t^2}{2\mu}\right)_{cav.} = \left(\frac{B_t^2}{2\mu}\right)_c + P_c$$

where the subscript "c" denotes those quantities in continuum region. It can be shown that B_t is directly proportional to ρ in a purely transverse field and so is P, hence the gas in the continuum region can expand (or compress) only a finite amount such that its density attains a minimum (or maximum) possible value. (A fuller account of this will be given in Section V.) Moreover, $(\vec{B}_t)_{cav.}$ is not required to be parallel to $(\vec{B}_t)_c$. When the gas is allowed to expand to vacuum completely as in the case of a receding piston problem, a special kind of contact surface separating the cavitation zone from continuum region exists; $P = \rho = 0$ on both sides of it. $\vec{B}_t$ is continuous in magnitude but is not necessarily of the same direction on each side of the front.

(b) $B_t = 0$ (ordinary gasdynamic case)

P and ρ are zero on both sides of the contact surface. Since no change of state can take place across the contact surface here, its existence is unnecessary and it has no physical significance.

2. Transverse Simple Waves

When $c = \pm b_1$, the fluid velocity undergoes changes only in transverse direction through the wave region which is thus called a transverse simple wave. We assume $B_1 \neq 0$, $B_t \neq 0$ and give a general discussion first. It follows from 2-33g that whenever c is different from zero $dS = 0$ is required. In other words, simple wave motion causes no entropy change in a fluid always. The transverse simple waves play an excessive role in aligning the transverse magnetic field in its proper direction and is essential for solving a general problem in one-dimensional unsteady MHD flow.

Knowing $c = \pm b_1$ and combining either 2-33c and 2-33e or 2-33d and 2-33f, we get

$$du_1 = 0 \tag{3-8a}$$

Substituting this into 2-33a, we have

$$d\rho = 0 \tag{3-8b}$$

It follows from 2-33b that

$$d(B_2^2 + B_3^2) = d(B_t^2) = 0 \tag{3-8c}$$

Hence the density, longitudinal fluid velocity, transverse magnetic field strength as well as the wave speed $c = \pm b_1$ remain unchanged across transverse simple waves. $\vec{B}_t$ can, however, change in direction. We write B_2 and B_3 in parametric form as shown below

$$B_2 = B_t \cos \varphi \tag{3-9a}$$

$$B_3 = B_t \sin \varphi \tag{3-9b}$$

where $\varphi = \varphi(\xi)$ is defined to be the angle between the transverse magnetic field direction and the x_2-axis and is a function of phase only.

From 2-33c and 2-33d, we have

$$du_2 = \mp \frac{dB_2}{\sqrt{\mu\rho}}$$

and

$$du_3 = \mp \frac{dB_3}{\sqrt{\mu\rho}}$$

respectively. Upon integration of the above equations, we get

$$u_2 = \mp \frac{B_2}{\sqrt{\mu\rho}} + u_2' = \mp b_2 + u_2' \qquad (3\text{-}10a)$$

and

$$u_3 = \mp \frac{B_3}{\sqrt{\mu\rho}} + u_3' = \mp b_3 + u_3' \qquad (3\text{-}10b)$$

respectively where u_2' and u_3' are constants determined by initial conditions.

Alternatively, we may write 3-10 by making use of 3-9 as

$$u_2 = \mp \frac{B_t}{\sqrt{\mu\rho}} \cos\varphi + u_2' = \mp b_t \cos\varphi + u_2' \qquad (3\text{-}11a)$$

and

$$u_3 = \mp \frac{B_t}{\sqrt{\mu\rho}} \sin\varphi + u_3' = \mp b_t \sin\varphi + u_3' \qquad (3\text{-}11b)$$

Or

$$(u_2 - u_2')^2 + (u_3 - u_3')^2 = b_t^2 \qquad (3\text{-}12a)$$

and

$$\frac{u_3 - u_3'}{u_2 - u_2'} = \frac{B_3}{B_2} = \tan\varphi \qquad (3\text{-}12b)$$

The magnetic field and fluid velocity in the transverse plane rotate as the transverse simple wave propagates in the x_1-direction.

The six Riemann invariants through transverse simple waves may be enumerated easily, they are constant values of S, ρ, u_1, B_t together with 3-12a and 3-12b.

For the special case of $B_t = 0$ and $B_1 \neq 0$ (i.e. $c \neq 0$), we have from 3-11

$$u_2 = u_2' = \text{constant}$$

and

$$u_3 = u_3' = \text{constant}$$

Hence in this case the physical state of the fluid remains constant across the transverse simple wave whose existence is unnecessary. This situation does occur frequently, e.g. at the end of a coupled wave where the transverse magnetic field is switched off completely due to either excessive compression in slow waves or excessive expansion in fast waves.

IV. COUPLED FAST AND SLOW SIMPLE WAVES

The vanishing of the last bracket in 2-3b gives

$$(c^2 - a^2)(c^2 - b_l^2) = c^2 b_t^2 \qquad (4\text{-}1)$$

This determines speeds of coupled fast and slow simple waves in which the interaction between fluid motion and magnetic field has the most prominent effect. Both longitudinal and transverse fluid velocities vary across these waves. In order to give a general discussion of them, we assume $B_l \neq 0$ and $B_t \neq 0$ here. Otherwise coupled waves degenerate to the corresponding ones in a purely transverse magnetic field when $B_l = 0$ (see Section V) or to either ordinary sound waves or a special case of transverse simple waves when $B_t = 0$.

1. Analysis

4-1 yields the following four roots of c

$$c = \pm \sqrt{\frac{a^2+b_l^2+b_t^2}{2} + \sqrt{\left(\frac{a^2+b_l^2+b_t^2}{2}\right)^2 - a^2 b_l^2}} \qquad (4\text{-}2a)$$

and

$$c = \pm \sqrt{\frac{a^2+b_l^2+b_t^2}{2} - \sqrt{\left(\frac{a^2+b_l^2+b_t^2}{2}\right)^2 - a^2 b_l^2}} \qquad (4\text{-}2b)$$

Wave speeds in 4-2a are those of forward and backward facing fast waves respectively since we may see from 4-1 that

$$c^2 > a^2 \qquad \text{and} \qquad c^2 > b_l^2 \qquad (4\text{-}3a)$$

in this case. On the other hand, wave speeds in 4-2b are those of forward and backward facing slow waves respectively since

$$c^2 < a^2 \quad \text{and} \quad c^2 < b_1^2 \tag{4-3b}$$

in this case.

We eliminate du_2 from 2-33c, e and du_3 from 2-33d, f, then

$$(b_1^2 - c^2) dB_2 + cB_2 \, du_1 = 0 \tag{4-4a}$$

$$(b_1^2 - c^2) dB_3 + cB_3 \, du_1 = 0 \tag{4-4b}$$

Because $b_1 \neq c$ is implied in the assumption $B_t \neq 0$, the following relation exists

$$\frac{dB_2}{B_2} = \frac{dB_3}{B_3}$$

or

$$\frac{B_3}{B_2} = k = \text{constant} \tag{4-5a}$$

always in coupled waves. The transverse magnetic field strength is

$$B_t = B_2 (1 + k^2)^{1/2} \tag{4-5b}$$

One may always orient the coordinate system (x_2, x_3) in transverse plane in such a way that $B_3 = 0$ identically $(k \equiv 0)$. It follows from 2-33d that u_3 = constant whose value may also be reduced to zero by a suitable Galilean transformation of coordinates in $\vec{e}_3$ direction. 2-33d, f can then be left out of our discussion and the problem may, without loss of generality, be studied in the $(x_1 - x_2)$ plane only. 4-1 is now

$$(c^2 - a^2)(c^2 - b_1^2) = c^2 b_2^2 \tag{4-6}$$

Only four dependent variables, ρ, u_1, u_2, B_2 should be considered and the flow field governed by coupled waves is described by

$$-c\,d\rho + \rho\,du_1 = 0 \tag{4-7a}$$

$$-c\,\rho\,du_1 + a^2 d\rho + \frac{B_2}{\mu} dB_2 = 0 \tag{4-7b}$$

$$-c\,\rho\,du_2 - \frac{B_1}{\mu} dB_2 = 0 \tag{4-7c}$$

$$-c\,dB_2 + B_2\,du_1 - B_1\,du_2 = 0 \tag{4-7d}$$

No characteristic length or time can be formed from 4-7, the physical phenomenon is most conveniently described in terms of various speeds from which the fundamental character of the flow can be brought out clearly. Owing to c being different from zero in general, the flow is always isentropic through coupled waves. The local speed of sound, a, given by 2-9a is then a function of ρ only and characterizes the mechanical state of the fluid. Alfvén wave speeds based on longitudinal and transverse magnetic fields, i. e. b_1 and b_2, are defined in 2-9b, c and characterize magnetic field strengths in corresponding directions. A relation between a and b_1 can be obtained by eliminating ρ from 2-9a, b, it is

$$a^2 b_1^{2(\gamma-1)} = \gamma A \left(\frac{B_1^2}{\mu}\right)^{\gamma-1} \tag{4-8}$$

where the right-hand side is a known constant in the entire coupled wave region. We write 2-9a, b, c in differential form as follows

$$\frac{d\rho}{\rho} = \frac{2}{\gamma-1}\frac{da}{a} \tag{4-9a}$$

$$\frac{dB_1}{B_1} = \frac{db_1}{b_1} + \frac{1}{2}\frac{d\rho}{\rho} = \frac{db_1}{b_1} + \frac{1}{\gamma-1}\frac{da}{a} = 0 \qquad (4\text{-}9b)$$

$$\frac{dB_2}{B_2} = \frac{db_2}{b_2} + \frac{1}{2}\frac{d\rho}{\rho} = \frac{db_2}{b_2} + \frac{1}{\gamma-1}\frac{da}{a} \qquad (4\text{-}9c)$$

With the aid of 2-9 and 4-9, the governing equations, 4-7, may be expressed in terms of speeds, a, b_1, b_2, u_1, u_2, only. They are

$$\frac{2}{\gamma-1}\frac{c}{a}\,da - du_1 = 0 \qquad (4\text{-}10a)$$

$$-c\,du_1 + \frac{2}{\gamma-1}\,a\,da + b_2^2\left(\frac{db_2}{b_2} + \frac{1}{\gamma-1}\frac{da}{a}\right) = 0 \qquad (4\text{-}10b)$$

$$-c\,du_2 - b_1 b_2\left(\frac{db_2}{b_2} + \frac{1}{\gamma-1}\frac{da}{a}\right) = 0 \qquad (4\text{-}10c)$$

$$-c\,\frac{db_2}{b_2} - \frac{c}{\gamma-1}\frac{da}{a} + du_1 - \frac{b_1}{b_2}\,du_2 = 0 \qquad (4\text{-}10d)$$

We note here that a Galilean transformation in either x_1 or x_2 direction will not alter the structure of 4-10 except in the former case an appropriate change of phase velocity $U = u_1 + c$ should be made in the final solution. Since coefficients in front of differentials in 4-10 depend on wave speeds a, b_1, b_2, c only, the calculation can be simplified if du_1 and du_2 are eliminated. Then 4-10 are reduced to

$$\left[\,2(a^2-c^2) + b_2^2\right] d(a^2) + (\gamma-1)a^2 d(b_2^2) = 0 \qquad (4\text{-}11a)$$

$$(b_1^2 + c^2)b_2^2 d(a^2) + (\gamma-1)(b_1^2-c^2)a^2 d(b_2^2) = 0 \qquad (4\text{-}11b)$$

The phenomenon is thus described solely by wave speeds characterizing various aspects of the physical problem. 4-11 are homogeneous and a non-trivial solution exists only if the determinant of their coefficients vanishes. This gives

$$c^4 - (a^2+b_1^2+b_2^2)c^2 + a^2b_1^2 = 0 \qquad (4\text{-}12)$$

which is identical to 4-6, as it should be. 4-12 adds a constraint condition among wave speeds and only one of the differential relations in 4-11 is independent. Or equivalently we have from 4-11 one differential equation and one algebraic equation 4-12 at our disposal. These, combining with the other algebraic equation 4-8, provides three equations for four variables a, b_1, b_2, c. Either a or b_1 can be easily eliminated with the use of 4-8, however this has not been done in order to retain the rather symmetric form of expressing all variables in terms of square of wave speeds. The differential form of 4-8 has the desired form also, namely

$$b_1^2 d(a^2) + (\gamma-1)a^2 d(b_1^2) = 0 \qquad (4\text{-}13)$$

A close inspection of 4-11 and 4-13 reveals that they are invariant under a group of transformation (Birkhoff, 1950) in wave speeds, i.e.

$$a \rightarrow \Omega a,\ b_1 \rightarrow \Omega b_1,\ b_2 \rightarrow \Omega b_2,\ c \rightarrow \Omega c$$

where Ω can be any arbitrary constant. This permits the group theoretic consideration to apply and any new variables formed from a, b_1, b_2, c that are invariant as shown above can be used. We have then only three variables for three equations which infers the possibility of reducing the whole problem to the integration of a single first-order differential equation of two variables. Consequently, we introduce the following dimensionless variables

$$\alpha = \frac{a^2}{c^2} \qquad \beta_1 = \frac{b_1^2}{c^2} \qquad \beta_2 = \frac{b_2^2}{c^2} \tag{4-14}$$

The basic condition 4-6 or 4-12 for wave speeds becomes

$$(\alpha - 1)(\beta_1 - 1) = \beta_2 \tag{4-15}$$

Fast waves are bounded by

$$0 \leq \alpha < 1 \qquad \text{and} \qquad 0 \leq \beta_1 < 1 \tag{4-16a}$$

and slow waves are bounded by

$$\alpha > 1 \qquad \text{and} \qquad \beta_1 > 1 \tag{4-16b}$$

Making use of differential relations

$$\frac{d\alpha}{\alpha} = \frac{d(a^2)}{a^2} - \frac{d(c^2)}{c^2}$$

$$\frac{d\beta_1}{\beta_1} = \frac{d(b_1^2)}{b_1^2} - \frac{d(c^2)}{c^2}$$

$$\frac{d\beta_2}{\beta_2} = \frac{d(b_2^2)}{b_2^2} - \frac{d(c^2)}{c^2}$$

together with 4-11, 4-13, 4-14 and 4-15, we arrive at the following first order ordinary differential equations in any of the planes $(\alpha - \beta_1)$, $(\alpha - \beta_2)$ or $(\beta_1 - \beta_2)$

$$\frac{d\beta_1}{d\alpha} = \frac{2(\gamma-1) + \gamma\alpha(\beta_1 - 1)}{(2-\gamma)(\alpha-1)} \frac{\beta_1}{\alpha} \tag{4-17a}$$

$$\frac{d\beta_2}{d\alpha} = \frac{2(\alpha-1)[(\alpha-1)^2 + \beta_2(2\alpha-1)] + \gamma\alpha\beta_2^2}{(2-\gamma)\alpha(\alpha-1)^2} \tag{4-17b}$$

$$\frac{d\beta_2}{d\beta_1} = \frac{(\beta_1-1)(\beta_1+\beta_2-1)[\gamma(\beta_1-1)+2] + 2\beta_1\beta_2}{(\beta_1-1)[\gamma(\beta_1-1)(\beta_1+\beta_2-1)+2\beta_2]} \frac{\beta_2}{\beta_1} \tag{4-17c}$$

This is the condensed form of the basic structure of the problem and any one of the above equations provides a complete description of coupled simple waves. The solution of a flow problem is constructed by first solving any of the first order differential equations in 4-17 and satisfying its initial condition to get a functional relationship between the corresponding two square wave speed ratios. Riemann invariants among physical quantities, ρ, u_1, u_2, B_2, are then obtained from known equations and initial conditions, these together with the constant values of s, u_3 and B_3 constitute six Riemann invariants for coupled simple waves. So far the solution is expressed in terms of phases only and have to be transformed back to functions of x_1 and t by identifying various phases with given boundary conditions on the initial curve in the physical plane that initiates the fluid motion.

2. Analytic Solution

The analytic solution of coupled simple waves in the presence of an arbitrarily oriented magnetic field ($B_1 \neq 0$, $B_t \neq 0$) is given in this section. Although the choice of any one of 4-17 to study is completely arbitrary, it is found to be most convenient to investigate 4-17a and all of our later analysis will be based on this. A particularly convenient variable to use is $\alpha = a^2/c^2$ which has a simple physical meaning in describing the departure of plane simple wave speed in MHD from that in ordinary gasdynamics. All physical quantities can be expressed analytically as functions of α.

First of all we shall, as in many other physical problems, introduce the following dimensionless variables which, normalized with respect to initial physical quantities, are defined as

$$\bar{\rho} = \frac{\rho}{\rho_I} \qquad \bar{P} = \frac{P}{P_I} \qquad \bar{B}_2 = \frac{B_2}{B_1}$$
$$\bar{u}_1 = \frac{u_1}{a_I} \qquad \bar{u}_2 = \frac{u_2}{a_I} \qquad \bar{a} = \frac{a}{a_I} \tag{4-18}$$
$$\bar{b}_1 = \frac{b_1}{a_I} \qquad \bar{b}_2 = \frac{b_2}{a_I} \qquad \bar{c} = \frac{c}{a_I}$$

Since simple waves are always contiguous to an initial state, the initial state of the fluid has its fundamental importance in the problem. Let us introduce the following parameters, K_1 and $\bar{B}_{2_I}$ where K_1 is defined by the expression

$$K_1 = \frac{|b_{1_I}|}{a_I} = \frac{1}{a_I}\frac{|B_1|}{\sqrt{\mu\rho_I}} \tag{4-19a}$$

One may see from 4-18 that K_1 is identical with $|\bar{b}_{1_I}|$ in fact and it provides a measure of the relative importance of the imposed longitudinal magnetic field and the mechanical state of the fluid at the initial instant. K_1 plays an important role in the present study because B_1 is a constant in the entire flow. An alternative form of 4-19a may be written as

$$K_1 = \left[\frac{B_1^2}{2\mu} \frac{1}{P_I} \frac{2}{\gamma} \right]^{1/2} = \left[\frac{P_{m_1}}{P_I} \frac{2}{\gamma} \right]^{1/2} \qquad (4\text{-}19b)$$

where P_{m_1} is the part of magnetic pressure based on longitudinal magnetic field and remains constant always. Thus K_1 is also, apart from a numerical factor, directly related to the square root of the ratio of minimum magnetic pressure to gasdynamic pressure. It shows actually the relative importance of MHD effects compared with compressibility effects. The other parameter $\overline{B_{2_I}}$ is

$$\overline{B_{2_I}} = \frac{B_{2_I}}{B_1} = \tan \theta_I$$

where θ_I is the angle between initial magnetic field direction and x_1-axis in (x_1, x_2) plane. The initial state of fluid is completely characterized by these two parameters.

Now, we are in position to express dimensionless variables given in 4-18 in terms of a, β_1 and K_1 as follows. From 4-15

$$\overline{B_2} = \mathrm{sgn}(B_1 B_{2_I})\left(\frac{\beta_2}{\beta_1}\right)^{1/2} = \mathrm{sgn}\,(B_1 B_{2_I})\left[(a-1)\left(1-\frac{1}{\beta_1}\right)\right]^{1/2} \qquad (4\text{-}20)$$

Note the sign of $\overline{B_2}$ cannot change across coupled waves (see section IV-4). Since

$$\frac{\beta_1}{a} = \frac{b_1^2}{a^2} = \frac{\overline{b_1}^2}{\overline{a}^2} = \frac{K_1^2}{\overline{\rho}^\gamma} \qquad (4\text{-}21)$$

we have

$$\overline{\rho} = \left(\frac{K_1^2 a}{\beta_1}\right)^{1/\gamma} \qquad (4\text{-}22a)$$

and from 1-1

$$\overline{p} = \overline{\rho}^{\gamma} = \frac{K_1^2 \alpha}{\beta_1} \qquad (4\text{-}22b)$$

From 2-9a

$$\overline{a} = \overline{\rho}^{(\gamma-1)/2} = K_1^{(\gamma-1)/\gamma}\left(\frac{\alpha}{\beta_1}\right)^{(\gamma-1)/2\gamma} \qquad (4\text{-}22c)$$

From 2-9b

$$\overline{b_1} = \operatorname{sgn}(B_1)\frac{K_1}{\overline{\rho}^{1/2}} = \operatorname{sgn}(B_1)K_1^{(\gamma-1)/\gamma}\left(\frac{\beta_1}{\alpha}\right)^{1/2\gamma} \qquad (4\text{-}22d)$$

From 4-20

$$\overline{b_2} = \overline{b_1}\,\overline{B_2} = \operatorname{sgn}(B_1)\,K_1^{(\gamma-1)/\gamma}\left[\left(\frac{\beta_1}{\alpha}\right)^{1/\gamma}(\alpha-1)\left(1-\frac{1}{\beta_1}\right)\right]^{1/2} \qquad (4\text{-}22e)$$

From 4-14

$$|\overline{c}| = \frac{\overline{a}}{\alpha^{1/2}} = K_1^{(\gamma-1)/\gamma}\left[\frac{1}{\alpha\beta_1^{\gamma-1}}\right]^{1/2\gamma} \qquad (4\text{-}22f)$$

Flow velocities in longitudinal (x_1-) and transverse (x_2-) directions, u_1 and u_2, have to be obtained by first integrals. 4-10a leads to

$$d\overline{u_1} = \frac{\gamma-1}{2}\,\overline{c}\,\frac{d\overline{a}}{\overline{a}} = \overline{c}\,\frac{d\overline{\rho}}{\overline{\rho}} = \pm\frac{K_1^{(\gamma-1)/\gamma}}{\gamma}\left[\frac{1}{\alpha\beta_1^{\gamma-1}}\right]^{1/2\gamma}\left[\frac{d\alpha}{\alpha} - \frac{d\beta_1}{\beta_1}\right] \qquad (4\text{-}22g)$$

for forward and backward facing waves. $\overline{u_1}$ is determined upon integration of the above expression and satisfying an initial condition. Similarly for the transverse velocity, we first eliminate $\underline{a}$ from 4-10c, d, then

$$du_2 = \frac{b_1 b_2}{b_1^2 - c^2} du_1$$

or in dimensionless form

$$d\bar{u}_2 = \frac{\bar{b}_1 \bar{b}_2}{\bar{b}_1^2 - \bar{c}^2} d\bar{u}_1 = \text{sgn}(B_1 B_{2_I}) \left[\frac{(\alpha-1)\beta_1}{\beta_1 - 1} \right]^{1/2} d\bar{u}_1$$

$$= \pm \text{ sgn}(B_1 B_{2_I}) \frac{K_1^{(\gamma-1)/\gamma}}{\gamma} \left[\left(\frac{\alpha-1}{\beta_1 - 1}\right)\left(\frac{\beta_1}{\alpha}\right)^{1/\gamma} \right]^{1/2} \left(\frac{d\alpha}{\alpha} - \frac{d\beta_1}{\beta_1} \right) \qquad (4\text{-}22h)$$

for forward and backward facing waves. Thus $\bar{u}_2$ is obtained by performing the integration and satisfying an initial condition.

Now, the central problem lies in the solution of 4-17a. It can be reduced to a linear form by the introduction of the following variables

$$\alpha^* = \frac{1}{\alpha} \quad \text{and} \quad \beta_1^* = \frac{1}{\beta_1}$$

4-17a becomes

$$\frac{d\beta_1^*}{d\alpha^*} = \frac{[(2-\gamma)-2\alpha^*]\beta_1^* + \gamma}{(2-\gamma)\alpha^*(1-\alpha^*)} \qquad (4\text{-}23)$$

This is a standard linear ordinary differential equation of first order and can be integrated readily as

$$\beta_1^* = \alpha^* |1-\alpha^*|^{\gamma/(2-\gamma)} \left\{ C_1 \pm \frac{\gamma}{2-\gamma} \int^{\alpha^*} \frac{d\eta}{\eta^2 |1-\eta|^{2/(2-\gamma)}} \right\}$$

(4-24a)

where the upper sign (+) in the bracket refers to the case of slow waves $(0 \leq \alpha^* < 1)$ and the lower one (-) refers to that of fast waves $(\alpha^* > 1)$.

C_1 is a constant and it is determined by the initial condition that

$$\alpha^* = \alpha^*_I \quad \text{and} \quad \beta_1^* = \beta^*_{1I}$$

However from 4-21

$$\frac{\beta_1^*}{\alpha^*} = \frac{\alpha}{\beta_1} = \frac{a^2}{b_1^2} = \frac{\bar{\rho}^{\gamma}}{K_1^2}$$

and $\bar{\rho} = 1$ initially, hence

$$\beta^*_{1I} = \frac{\alpha^*_I}{K_1^2}$$

Thus 4-24a can be written as

$$\beta_1^* = \alpha^* |1-\alpha^*|^{\gamma/(2-\gamma)} \left\{ \frac{1}{K_1^2 |1-\alpha^*_I|^{\gamma/(2-\gamma)}} \pm \frac{\gamma}{2-\gamma} \int_{\alpha^*_I}^{\alpha^*} \frac{d\eta}{\eta^2 |1-\eta|^{2/(2-\gamma)}} \right\}$$

(4-24b)

This is the equation of a family of integral curves in $(\alpha^* - \beta_1^*)$ plane characterized by parameters K_I and α^*_I at the initial state. Expressing the solution in terms of original variables α and β_1, we get the equation for the family of integral curves in the $(\alpha - \beta_1)$ plane characterized by parameters K_1 and α_I as follows

$$\beta_1 = \frac{K_1^2 \alpha \left|\frac{\alpha}{\alpha - 1}\right|^{\gamma/(2-\gamma)}}{\left|\frac{\alpha_I}{\alpha_I - 1}\right|^{\gamma/(2-\gamma)} \mp \frac{\gamma K_1^2}{2-\gamma} \int_{\alpha_I}^{\alpha} \left|\frac{\zeta}{\zeta - 1}\right|^{2/(2-\gamma)} d\zeta} \qquad (4\text{-}25)$$

where the upper sign (-) in front of the integral refers to slow waves and the lower one (+) refers to fast waves. It is recalled that regions of physical interest in the $(\alpha-\beta_1)$ plane are bounded by $0 \leq \alpha < 1$ and $0 \leq \beta_1 < 1$ for fast waves together with $\alpha > 1$ and $\beta_1 > 1$ for slow waves. The parameters K_1 and α_I instead of K_1 and $\overline{B_2}_I$ have been used here to characterize the initial state where α_I is related to K_1 and $\overline{B_2}_I$ as follows. From 4-15

$$(\alpha-1)(1 - \frac{1}{\beta_1}) = \overline{B_2}^2$$

At initial state

$$\frac{\beta_{1I}}{\alpha_I} = K_1^2$$

so that

$$(\alpha_I - 1)(1 - \frac{1}{K_1^2 \alpha_I}) = \overline{B_2}_I^2$$

α_I is solved to be

$$\alpha_I = \frac{1}{2K_1^2} \left\{ 1 + K_1^2(1 + \overline{B_2}_I^2) \pm \sqrt{[1 + K_1^2(1 + \overline{B_2}_I^2)]^2 - 4K_1^2} \right\} \quad (4\text{-}26)$$

for slow and fast simple waves respectively. Therefore, there are always two values of α_I for any given initial state which, together with K_I, correspond to two points in the $(\alpha-\beta_1)$ plane where one is in fast wave region and the other in slow wave region . The selection of a suitable type of wave for the description of a given physical phenomenon is of course dependent on specific boundary conditions of the problem.

We remark here that the solution of 4-17b is known by use of 4-15 and 4-25, then

$$\beta_2 = (a-1)\left[\frac{K_1^2 a \left|\frac{a}{a-1}\right|^{\gamma/(2-\gamma)}}{\left|\frac{a_I}{a_I-1}\right|^{\gamma/(2-\gamma)} \mp \frac{\gamma K_1^2}{2-\gamma}\int_{a_I}^{a}\left|\frac{\zeta}{\zeta-1}\right|^{2/(2-\gamma)} d\zeta} - 1\right] \quad (4\text{-}27)$$

which can, however, be verified by direct substitution into 4-17b.

Let us introduce the following notation for abbreviation

$$H = H(a;K_1, a_I) = \left|\frac{a_I}{a_I-1}\right|^{\gamma/(2-\gamma)} \mp \frac{\gamma K_1^2}{2-\gamma}\int_{a_I}^{a}\left|\frac{\zeta}{\zeta-1}\right|^{2/(2-\gamma)} d\zeta \quad (4\text{-}28)$$

where the choice of sign is the same as before. Because of $a \neq 1$ and its being always a finite value for most cases of coupled simple waves (see Section IV 5), the definite integral in 4-28 is well behaved. Its value cannot generally be written out in an explicit form except for $2/(2-\gamma)$ being of integers. In that case, we denote $\nu = 2/(2-\gamma)$ for brevity; the definite integral consists of a sum of finite series (Appendix B) as follows.

(a) Slow waves ($a > 1$)

$$\int_{a_I}^{a}\left(\frac{\zeta}{\zeta-1}\right)^{\nu} d\zeta = (a-a_I) + \nu \ln\frac{a-1}{a_I-1} + \sum_{s=1}^{\nu-1}\frac{\nu!}{(1+s)!(\nu-s-1)!}\frac{1}{s}\left[\frac{1}{(a_I-1)^s} - \frac{1}{(a-1)^s}\right]$$

(4-29a)

(b) Fast waves ($0 \leq a < 1$)

$$\int_{a_I}^{a}\left(\frac{\zeta}{1-\zeta}\right)^{\nu} d\zeta = (-1)^{\nu}\left[(a-a_I) - \nu \ln\frac{1-a_I}{1-a}\right] + \sum_{s=1}^{\nu-1}(-1)^{\nu+s+1}\frac{\nu!}{(1+s)!(\nu-s-1)!}\frac{1}{s}\left[\frac{1}{(1-a)^s} - \frac{1}{(1-a_I)^s}\right]$$

(4-29b)

We are dealing mostly with monatomic gases of $\gamma = 5/3$ and $2/(2-\gamma) = 6$; 4-29 can thus be applied directly (Appendix B).

Using the simplified notation defined in 4-28, one is able to write 4-25 as

$$\beta_1 = \frac{K_1^2}{H} a \left| \frac{a}{a-1} \right|^{\gamma/(2-\gamma)} \qquad (4\text{-}30)$$

The transverse magnetic field strength expressed as a function of a is obtained from 4-20 as

$$\overline{B_2} = \operatorname{sgn}(B_1 B_{2_I}) \left\{ (a-1) \left[1 - \frac{H}{K_1^2 a} \left| \frac{a-1}{a} \right|^{\gamma/(2-\gamma)} \right] \right\}^{1/2} \qquad (4\text{-}31)$$

The density obtained from 4-22a is

$$\overline{\rho} = \left[\frac{K_1^2 a}{\beta_1} \right]^{1/\gamma} = H^{1/\gamma} \left| \frac{a-1}{a} \right|^{1/(2-\gamma)} \qquad (4\text{-}32a)$$

and the pressure

$$\overline{P} = H \left| \frac{a-1}{a} \right|^{\gamma/(2-\gamma)} \qquad (4\text{-}32b)$$

With the aid of 4-22c, d, e, f, the wave speeds $\overline{a}$, $\overline{b}_1$, $\overline{b}_2$, $\overline{c}$ can also be expressed as functions of a only. They are

$$\overline{a} = \left[H^{1/\gamma} \left| \frac{a-1}{a} \right|^{1/(2-\gamma)} \right]^{(\gamma-1)/2} \qquad (4\text{-}32c)$$

$$\overline{b}_1 = \operatorname{sgn}(B_1) \frac{K_1}{H^{1/2\gamma}} \left| \frac{a}{a-1} \right|^{1/2(2-\gamma)} \qquad (4\text{-}32d)$$

$$\overline{b}_2 = \operatorname{sgn}(B_{2_I}) \frac{K_1}{H^{1/2\gamma}} \left| \frac{a}{a-1} \right|^{1/2(2-\gamma)} \left\{ (a-1) \left[1 - \frac{H}{K_1^2 a} \left| \frac{a-1}{a} \right|^{\gamma/(2-\gamma)} \right] \right\}^{1/2}$$

$$(4\text{-}32e)$$

$$\overline{c} = \left[\frac{H^{(\gamma-1)/\gamma}}{a} \left| \frac{a-1}{a} \right|^{(\gamma-1)/(2-\gamma)} \right]^{1/2} \qquad (4\text{-}32f)$$

According to the general formalism of simple waves given in 2-15, we shall regard ρ as v_1 hereafter and express all physical variables as functions of ρ.

In principle, we may get a relation between $\overline{B_2}$ and $\overline{\rho}$ by eliminating a from 4-31 and 4-32a for fast and slow waves respectively; but a appears in a very complicated manner in these equations and an explicit analytic expression of $\overline{B_2} = \overline{B_2}(\overline{\rho})$ cannot be obtained in general. On the other hand, 4-31 and 4-32a may instead be considered as parametric equations describing the corresponding Riemann invariant relation between B_2 and ρ across coupled waves and their specific values depend on initial condition of the flow problem only. These will be called magnetic field Riemann invariants.

For the longitudinal flow velocity, we have from 4-30

$$\frac{da}{a} - \frac{d\beta_1}{\beta_1} = \frac{\gamma}{2-\gamma}\frac{da}{a(a-1)} + \frac{dH}{H} = \frac{\gamma}{2-\gamma}\left[\frac{1}{a(a-1)} \mp \frac{K_1^2}{H}\left|\frac{a}{a-1}\right|^{2/(2-\gamma)}\right]da \tag{4-33}$$

Substituting this and 4-30 into 4-22g and carrying out the integration, one obtains†

$$-2s_1 = \overline{u}_1 - \frac{1}{2-\gamma}\int_{a'}^{a} H^{(\gamma-1)/2\gamma}\left|\frac{\sigma-1}{\sigma}\right|^{(\gamma-1)/2(2-\gamma)}\left[\frac{1}{\sigma(\sigma-1)} \mp \frac{K_1^2}{H}\left|\frac{\sigma}{\sigma-1}\right|^{2/(2-\gamma)}\right]\frac{d\sigma}{\sigma^{1/2}} \tag{4-34a}$$

and

---

† The general form follows from that in ordinary gasdynamics given by Courant and Friedrichs (1948).

$$2r_1 = \bar{u}_1 + \frac{1}{2-\gamma}\int_{a'}^{a} H^{(\gamma-1)/2\gamma}\left|\frac{\sigma-1}{\sigma}\right|^{(\gamma-1)/2(2-\gamma)}\left[\frac{1}{\sigma(\sigma-1)} \mp \frac{K_1^2}{H}\left|\frac{\sigma}{\sigma-1}\right|^{2/(2-\gamma)}\right]\frac{d\sigma}{\sigma^{1/2}} \qquad (4\text{-}34b)$$

with

$$H = H(\sigma;K_1, a_I) = \left|\frac{a_I}{a_I-1}\right|^{\gamma/(2-\gamma)} \mp \frac{\gamma K_1^2}{2-\gamma}\int_{a_I}^{\sigma}\left|\frac{\zeta}{\zeta-1}\right|^{2/(2-\gamma)} d\zeta$$

for forward and backward facing coupled simple waves respectively. The values of s_1 and r_1 are determined by initial conditions of the problem and are constants for a given flow. Since a is obtained from 4-32a as a unique function of ρ for each of slow and fast simple waves, 4-34 gives the Riemann invariant relation between u_1 and ρ across coupled waves. Consequently we have in 4-32a and 4-34 the parametric equations of the longitudinal flow Riemann invariant. The lower limit of the definite integral in 4-34, a', can be an arbitrary constant. It is defined here in accord with the convention employed customarily in ordinary gasdynamics that the integral vanishes when density reaches its lowest possible value (say at $\bar{\rho} = 0$ in ordinary gasdynamics). It can be shown (see Section IV 4) that the minimum value of $\bar{\rho}$ in slow waves is $\bar{\rho} = 0$ and that in fast waves is given by the value of $\bar{\rho}$ at $\beta_1 = 1$. a' for different types of coupled waves are determined as follows.

(a) Fast waves ($1 > a \geq 0$)

a' is obtained by putting $\beta_1 = 1$ in 4-25 which gives

$$a'\left(\frac{a'}{1-a'}\right)^{\gamma/(2-\gamma)} - \frac{\gamma}{2-\gamma}\int_{a_I}^{a'}\left(\frac{\zeta}{1-\zeta}\right)^{2/(2-\gamma)} d\zeta = \frac{1}{K_1^2}\left(\frac{a_I}{1-a_I}\right)^{\gamma/(2-\gamma)} \qquad (4\text{-}35)$$

so $\alpha' = \alpha'(K_1, \alpha_I)$ is determined.

(b) Slow waves ($\alpha > 1$)

From 4-29a, $\bar{\rho} = 0$ when

$$\left|\frac{\alpha'-1}{\alpha'}\right|^{1/(2-\gamma)} H^{1/\gamma} = 0$$

(i) When $\alpha' = 1 = a'^2/c'^2$

$$a' = c' = 0 \qquad \text{at} \qquad \bar{\rho} = 0$$

This is a trivial solution and corresponds to the ordinary gas-dynamic case.

(ii) When $H = 0$, we have

$$\int_{\alpha_I}^{\alpha'} \left(\frac{\zeta}{\zeta-1}\right)^{2/(2-\gamma)} d\zeta = \frac{2-\gamma}{\gamma K_1^2} \left|\frac{\alpha_I}{\alpha_I-1}\right|^{\gamma/(2-\gamma)} \tag{4-36}$$

α' is determined implicitly here. It corresponds to the limiting value of α as gas expands completely to vacuum.

Let us now consider the transverse velocity of the fluid. Substituting 4-30 and 4-33 into 4-22h and carrying out the integration, we get

$$-2s_2 = \bar{u}_2 - \operatorname{sgn}(B_1 B_{2I}) \frac{K_1}{2-\gamma} \int_{\alpha'}^{\alpha} \frac{H^{(\gamma-1)/2\gamma} \left|\frac{\sigma-1}{\sigma}\right|^{1/2(2-\gamma)}}{\left|K_1^2 - \frac{H}{\sigma}\left|\frac{\sigma-1}{\sigma}\right|^{\gamma/(2-\gamma)}\right|^{1/2}} \left\{ \frac{1}{\sigma(\sigma-1)} \mp \frac{K_1^2}{H}\left|\frac{\sigma}{\sigma-1}\right|^{2/(2-\gamma)} \right\} d\sigma \tag{4-37a}$$

and

$$2r_2 = \bar{u}_2 + \operatorname{sgn}(B_1 B_{2I}) \frac{K_1}{2-\gamma} \int_{\alpha'}^{\alpha} \frac{H^{(\gamma-1)/2\gamma} \left|\frac{\sigma-1}{\sigma}\right|^{1/2(2-\gamma)}}{\left|K_1^2 - \frac{H}{\sigma}\left|\frac{\sigma-1}{\sigma}\right|^{\gamma/(2-\gamma)}\right|^{1/2}} \left\{ \frac{1}{\sigma(\sigma-1)} \mp \frac{K_1^2}{H}\left|\frac{\sigma}{\sigma-1}\right|^{2/(2-\gamma)} \right\} d\sigma \tag{4-37b}$$

for forward and backward facing coupled simple waves respectively. By the same reasoning as for the longitudinal flow Riemann invariant, we have now in 4-32a and 4-37 the parametric equations of the transverse flow Riemann invariant with s_2 and r_2 being constants determined by initial condition of the problem.

To sum up the result obtained above, we have six Riemann invariants for coupled simple waves consisting of magnetic field, longitudinal and transverse flow Riemann invariants derived from 4-32a combined with 4-31, 4-34, 4-37 respectively and constant values of S, u_3 and B_3.

The notion of escape speed in ordinary gasdynamics can be generalized here. It corresponds to the ultimate magnitude of fluid velocity, whose longitudinal component is in a direction opposite to that of wave propagation, in a continuous rarefaction process of the fluid up to a vacuum state. Obviously only slow waves can give rise to escape speeds, because the gas cannot generally be expanded to vacuum through fast waves. We have here two values of escape speeds corresponding to magnitudes of longitudinal and transverse escape velocities respectively. Let us denote

$$\ell_1(a) = \frac{1}{2-\gamma}\int_{a'}^{a} H^{(\gamma-1)/2\gamma}\left(\frac{\sigma-1}{\sigma}\right)^{(\gamma-1)/2(2-\gamma)}\left[\frac{1}{\sigma(\sigma-1)} - \frac{K_1^2}{H}\left(\frac{\sigma}{\sigma-1}\right)^{2/(2-\gamma)}\right]\frac{d\sigma}{\sigma^{1/2}} \qquad (4\text{-}38a)$$

and

$$\ell_2(a) = \operatorname{sgn}(B_1 B_{2I})\frac{K_1}{2-\gamma}\int_{a'}^{a}\frac{H^{(\gamma-1)/2\gamma}\left(\frac{\sigma-1}{\sigma}\right)^{1/2(2-\gamma)}}{\left[K_1^2-\frac{H}{\sigma}\left(\frac{\sigma-1}{\sigma}\right)^{\gamma/(2-\gamma)}\right]^{1/2}}\left\{\frac{1}{\sigma(\sigma-1)} - \frac{K_1^2}{H}\left(\frac{\sigma}{\sigma-1}\right)^{2/(2-\gamma)}\right\}d\sigma \qquad (4\text{-}38b)$$

for abbreviation with

$$H = \left(\frac{a_I}{a_I - 1}\right)^{\gamma/(2-\gamma)} - \frac{\gamma K_1^2}{2-\gamma} \int_{a_I}^{\sigma} \left(\frac{\zeta}{\zeta - 1}\right)^{2/(2-\gamma)} d\zeta$$

and also assume $u_1 = u_2 = 0$ initially. Then we get from 4-34 and 4-37

$$-2s_1 = \bar{u}_1 - \ell_1(a) = -\ell_1(a_I) = \bar{u}_{1\,esc} \tag{4-39a}$$

$$2r_1 = \bar{u}_1 + \ell_1(a) = \ell_1(a_I) = \bar{u}_{1\,esc} \tag{4-39b}$$

and

$$-2s_2 = \bar{u}_2 - \ell_2(a) = -\ell_2(a_I) = \bar{u}_{2\,esc} \tag{4-40a}$$

$$2r_2 = \bar{u}_2 + \ell_2(a) = \ell_2(a_I) = \bar{u}_{2\,esc} \tag{4-40b}$$

for forward and backward facing slow waves respectively. Hence escape speeds are

$$|\bar{u}_{1\,esc}| = \ell_1(a_I) \tag{4-41a}$$

$$|\bar{u}_{2\,esc}| = \ell_2(a_I) \tag{4-41b}$$

3. Procedures for Construction of Solution by Graphical Means

An alternative way of constructing coupled simple wave solutions by graphical means is discussed in this section. The variation of physical quantities as a wave progresses can be seen clearly from the graph. It has the advantage over the exact analytic solution of giving the result immediately as well as showing intuitively the basic structure of the problem. This is achieved at the expense of the precision of the solution but is particularly convenient for a quick estimate of the outcome in a practical case.

It is most convenient to investigate the graphical solution in $(\alpha-\beta_1)$ plane where the integral curves (trajectories) of the first order differential equation 4-17a can be analyzed precisely. These are solution curves of the problem and the choice of a particular one of them is determined by initial conditions.

Let us start with an examination of singular points of 4-17a which are located at (0, 0), (1, 0) and (1, 1) in $(\alpha-\beta_1)$ plane and are denoted by L, M, N respectively. The behavior of integral curves in the neighborhood of these points are studied by the usual method of local linearization. For γ in the range between 1 and 5/3 the result (see fig. 2) is given as follows.

(a) L(0, 0) is a nodal point. All trajectories except one lying on the β_1-axis are tangent to α-axis.

(b) M(1, 0) is a saddle point with two exceptional trajectories lying on the α-axis and the vertical line $\alpha = 1$.

(c) N(1, 1) is a nodal point. All trajectories are tangent to the line $(\alpha-1) + (\gamma-1)(\beta_1-1) = 0$ with the exceptional one on $\alpha = 1$.

We recall from 4-16 that regions of physical significance in $(\alpha-\beta_1)$ plane are bounded by $0 \leq \alpha < 1$ and $0 \leq \beta_1 < 1$ for fast waves as well as $\alpha > 1$ and $\beta_1 > 1$ for slow waves. Having the general behavior about singular points, we may construct the entire family of integral curves either by an isocline method or more precisely by a numerical integration method. Their qualitative nature is illustrated in fig. 3-a. Similarly, one may also obtain the general behavior of integral curves in $(\alpha-\beta_2)$ and $(\beta_1-\beta_2)$ planes; they are shown in fig. 3-b and fig. 3-c

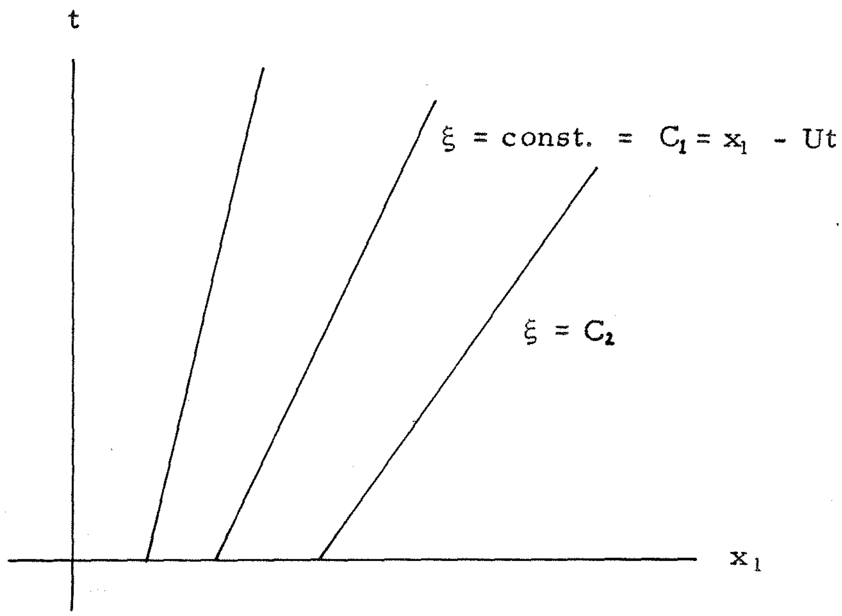


Fig. 1

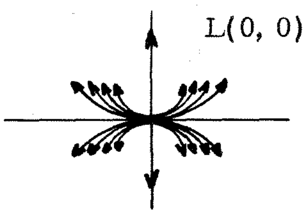


Fig. 2-a

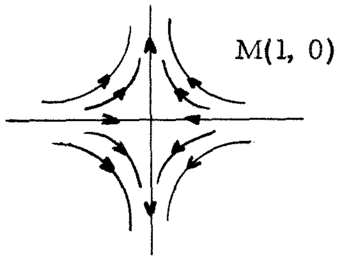


Fig. 2-b

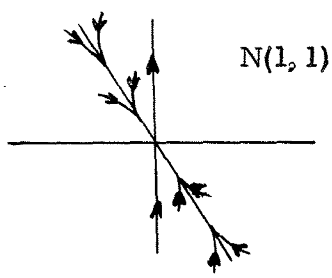


Fig. 2-c

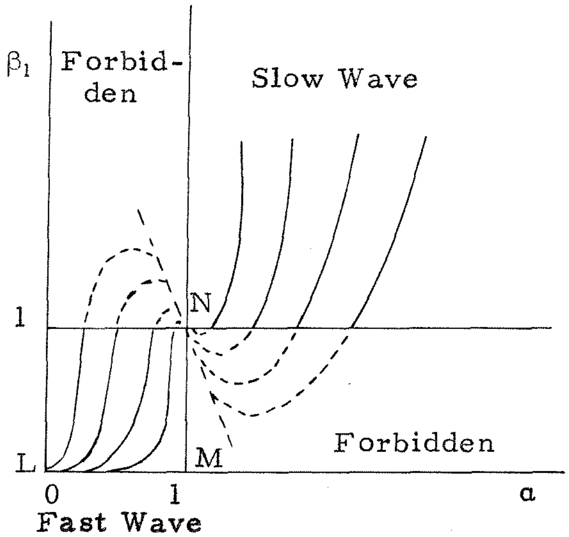


Fig. 3-a

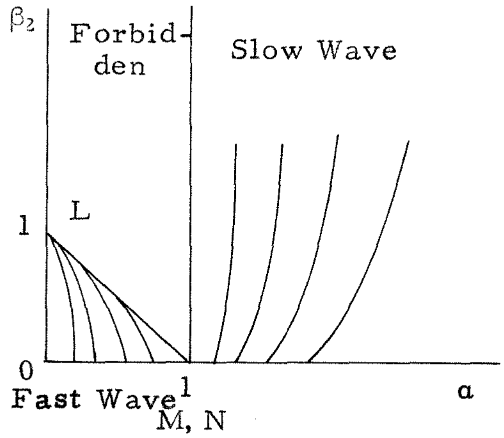


Fig. 3-b

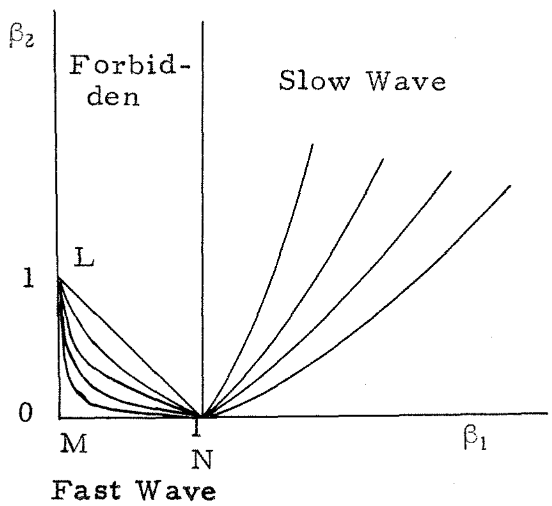


Fig. 3-c

respectively for reference and may prove to be more convenient in describing physical quantities under certain circumstances.

In order to illustrate the general procedure of constructing the graphical solution, a set of precisely calculated integral curves † obtained by applying Adam's method of numerical integration to 4-17a for $\gamma = 5/3$ is given here. They are shown together with constant β_2 lines in fig. 4‡ for slow wave region and in fig. 9 for fast wave region. Incidentally, we may mention here that the analytic expression of integral curves for slow waves in fig. 4 is found from 4-25 and 4-29a as (see Appendix B)

$$\beta_1 = \frac{K_1^2 a\left(\frac{a}{a-1}\right)^5}{\left(\frac{a_I}{a_I-1}\right)^5 - 5K_1^2 D_s} \qquad (4\text{-}42a)$$

where

$$D_s = (a-a_I)+6\ln\frac{a-1}{a_I-1} - \{15\left[\frac{1}{a-1}-\frac{1}{a_I-1}\right]+10\left[\frac{1}{(a-1)^2}-\frac{1}{(a_I-1)^2}\right]$$

$$+5\left[\frac{1}{(a-1)^3}-\frac{1}{(a_I-1)^3}\right]+\frac{3}{2}\left[\frac{1}{(a-1)^4}-\frac{1}{(a_I-1)^4}\right]$$

$$+\frac{1}{5}\left[\frac{1}{(a-1)^5}-\frac{1}{(a_I-1)^5}\right]\}$$

For fast waves in fig. 9, it is found from 4-25 and 4-29b as (see Appendix B)

$$\beta_1 = \frac{K_1^2 a\left(\frac{a}{1-a}\right)^5}{\left(\frac{a_I}{1-a_I}\right)^5 + 5K_1^2 D_f} \qquad (4\text{-}42b)$$

---

† Integral curves are labeled by their value of a at $\beta_1 = 1$ respectively.

‡ They are given here in the part of slow wave region bounded by $1 \leq a \leq 12$ and $1 \leq \beta_1 \leq 12$.

where

$$D_f = (a-a_I) - 6\ln\frac{1-a_I}{1-a} - \{15[\frac{1}{1-a_I} - \frac{1}{1-a}] - 10[\frac{1}{(1-a_I)^2} - \frac{1}{(1-a)^2}]$$

$$+ 5[\frac{1}{(1-a_I)^3} - \frac{1}{(1-a)^3}] - \frac{3}{2}[\frac{1}{(1-a_I)^4} - \frac{1}{(1-a)^4}]$$

$$+ \frac{1}{5}[\frac{1}{(1-a_I)^5} - \frac{1}{(1-a)^5}]\}$$

Physical quantities can be expressed as functions of a and β_1 as shown in 4-20 and 4-22. They (except for $\overline{B_2}$) are seen to depend explicitly on the initial parameter K_1. Furthermore, all velocities have the same multiplicating factor $K_1^{(\gamma-1)/\gamma}$. Thus lines of constant values of these quantities for any given K_1 can be plotted on the $(a-\beta_1)$ plane. For convenience of labeling these lines, we give only those values for $K_1 = 1$ on each curve which are designated by the subscript "*" and assume the following expressions.

$$\overline{\rho}_* = (\frac{a}{\beta_1})^{1/\gamma} \qquad (4-43a)$$

$$\overline{p}_* = \frac{a}{\beta_1} \qquad (4-43b)$$

$$\overline{a}_* = (\frac{a}{\beta_1})^{(\gamma-1)/(2\gamma)} \qquad (4-43c)$$

$$\overline{b}_{1*} = \text{sgn}(B_1)\ (\frac{\beta_1}{a})^{1/2\gamma} \qquad (4-43d)$$

$$\overline{b}_{2*} = \text{sgn}(B_{2\,I})\ [(\frac{\beta_1}{a})^{1/\gamma}(a-1)(1-\frac{1}{\beta_1})]^{1/2} \qquad (4-43e)$$

$$\overline{c}_* = [\frac{1}{a\beta_1^{\gamma-1}}]^{1/2\gamma} \qquad (4-43f)$$

The values for K_1 different from unity are then obtained by multiplying the appropriate scaling factors. Thus

$$\bar{\rho} = K_1^{2/\gamma} \bar{\rho}_* \tag{4-44a}$$

$$\bar{P} = K_1^{2} \bar{P}_* \tag{4-44b}$$

$$\bar{a} = K_1^{(\gamma-1)/\gamma} \bar{a}_* \tag{4-44c}$$

$$\bar{b}_1 = K_1^{(\gamma-1)/\gamma} \bar{b}_{1*} \tag{4-44d}$$

$$\bar{b}_2 = K_1^{(\gamma-1)/\gamma} \bar{b}_{2*} \tag{4-44e}$$

$$\bar{c} = K_1^{(\gamma-1)/\gamma} \bar{c}_* \tag{4-44f}$$

It is apparent from 4-43 that one of the main advantages of choosing $(\alpha-\beta_1)$ plane for our graphical analysis lies in the fact that lines of constant $\bar{\rho}_*$ as well as those of constant $\bar{P}_*$, $\bar{a}_*$ and $\bar{b}_{1*}$ are rays from the origin. This simplifies the graphical representation a great deal and provides a clear picture of the variation of physical quantities. Fig. 3a shows that two different types of coupled simple waves occupy completely separated regions. It can be proved that no direct transition from a fast wave to slow wave or vice versa is possible (see Section IV 4). Any given state of the fluid always corresponds to two points in the $(\alpha-\beta_1)$ plane where one is in the fast wave region and the other in the slow wave region. Physical properties across waves of different type are entirely different. This serves as the basis for the selection of a correct type of coupled waves to satisfy the boundary condition of a given problem. After the particular wave is determined, we may confine our attention only to one specific wave region in which each point corresponds to a unique physical state.

In principle, lines of constant values of various physical quantities can be constructed in fig. 4 and fig. 9 for slow and fast waves respectively; but in order to avoid possible confusion among them, they have been prepared in separate figures. Relations among various lines can however be correlated, for instance, by using transparent paper for graphs. For $\gamma = 5/3$, constant $\bar{\rho}_*$ and $|\bar{B}_2|$ lines are illustrated in fig. 5 for slow waves and in fig. 10 for fast waves respectively; constant $\bar{\rho}_*$ and $\bar{c}_*$ lines are illustrated in fig. 6 for slow waves and in fig. 11 for fast waves respectively. Lines of constant $\bar{P}_*$, $\bar{a}_*$ and $\bar{b}_{1*}$ are centered rays, they amount only to a simple change of the labeling of constant $\bar{\rho}_*$ lines hence are not given separately here. The constant $\bar{b}_{2*}$ lines can also be constructed easily; but they are not of special interest to us usually and do not participate directly in the construction of other solutions; thus they are omitted also.

Let us now proceed to determine the point corresponding to the initial state of the problem in each wave region. We form first the parameters K_1 and $\bar{B}_{2_I}$ from known initial physical quantities

$$K_1 = \frac{|B_1|}{\sqrt{\mu\rho_I}\, a_I} = \frac{|B_1|}{\sqrt{\mu\gamma P_I}}$$

and

$$\bar{B}_{2_I} = \frac{B_2}{B_1}$$

Since $\bar{\rho}_I = 1$ always and from 4-44a $\bar{\rho}_I = K_1^{2/\gamma}\bar{\rho}_{*I} = 1$, we have

$$\overline{\rho}_{*I} = K_1^{-2/\gamma} = \left(\frac{\mu\gamma P_I}{B_1^2}\right)^{1/\gamma}$$

From 4-43a, this corresponds to

$$\frac{\alpha}{\beta_1} = \frac{\mu\gamma P_I}{B_1^2}$$

which can also be obtained directly from

$$\left(\frac{\alpha}{\beta_1}\right)_I = \frac{a_I^2}{b_{1_I}^2} = \frac{\mu\gamma P_I}{B_1^2}$$

This relation gives the slope of a particular ray in the $(\alpha-\beta_1)$ plane on which the two initial state points rest. The specific locations of these two points in slow and fast wave regions respectively are determined by points of intersection of $\overline{\rho}_* = \left(\frac{\mu\gamma P_I}{B_1^2}\right)^{1/\gamma}$ and $\overline{B}_2 = \overline{B}_{2_I}$ curves. Alternatively, they may be obtained by first solving c_I from 4-26 to get two vertical lines with one in each of the two wave regions which intersect the ray $\overline{\rho}_* = \left(\frac{\mu\gamma P_I}{B_1^2}\right)^{1/\gamma}$ at the point of initial state in each wave region respectively. The particular integral curves for slow and fast waves are selected uniquely by these initial state points. Since one of the two types of coupled waves should be excluded by boundary conditions, in any case we have, at our disposal, only one integral curve in a pertinent wave region along which the change of physical state across the wave follows.

So far the flow velocities $\overline{u}_1$, $\overline{u}_2$ are undetermined; they involve first integrals and a further numerical integration along a specific integral curve together with satisfying their initial conditions must be performed to get the solution.

For the longitudinal flow velocity, $\overline{u}_1$, we have from 4-22g

$$d\overline{u}_1 = \overline{c}\,\frac{d\overline{\rho}}{\overline{\rho}}$$

If we define $\overline{u}_1 = K_1^{(\gamma-1)/\gamma}\,\overline{u}_{1*}$, then

$$d\overline{u}_{1*} = \overline{c}_*\,\frac{d\overline{\rho}_*}{\overline{\rho}_*}$$

The differentials may be approximated by finite intervals, i. e.

$$\Delta\overline{u}_{1*} = \overline{c}_*\,\frac{\Delta\overline{\rho}_*}{\overline{\rho}_*} \qquad (4\text{-}45)$$

For each small interval $\Delta\overline{\rho}_*$ along an integral curve, we read off the mean value of $\overline{\rho}_*$ and $\overline{c}_*$ within it and obtain the change $\Delta\overline{u}_{1*}$ across that interval from 4-45. The same procedure can be applied successively along each integral curve and the change of $\overline{u}_{1*}$ as a function of $\overline{\rho}_*$ is obtained accordingly. It is convenient for graphical representation to introduce a notation w_1 which is defined by the following expression

$$\overline{u}_{1*} = \mp\,(w_1 - w_1') \qquad (4\text{-}46a)$$

where w_1' is a constant for each integral curve and its value is determined by the condition that

$$w_1 = 0 \qquad \text{at } \beta_1 = 1 \qquad (4\text{-}46b)$$

always on each integral curve. The (-) and (+) signs in front of the bracket in 4-46a refer to forward and backward facing coupled waves respectively. The plot of w_1 versus $\bar{\rho}_*$ along different integral curves for slow waves is shown in fig. 7 and that for fast waves is shown in fig. 12. Lines of constant $\bar{B}_2$ are given in these graphs also. It is evident from 4-46 that the value of w_1' for each integral curve depends on specific physical condition of a given problem. In most cases, we are interested only in the difference of $\bar{u}_{1*}$ between two states which is equal to that of w_1 and can be obtained readily from fig. 7 or fig. 12. The precise value of w_1 as well as w_1' are usually of no direct concern to us.

Similarly, for the transverse flow velocity, $\bar{u}_2$, we have from 4-22h

$$d\bar{u}_2 = \text{sgn}\,(B_1 B_{2_I})\left[\frac{(\alpha-1)\beta_1}{\beta_1 - 1}\right]^{1/2} d\bar{u}_1$$

We define $\bar{u}_2 = K_1^{(\gamma-1)/\gamma}\,\bar{u}_{2*}$, then

$$d\bar{u}_{2*} = \text{sgn}\,(B_1 B_{2_I})\left[\frac{(\alpha-1)\beta_1}{\beta_1 - 1}\right]^{1/2} d\bar{u}_{1*}$$

$$= \text{sgn}\,(B_1 B_{2_I})\left[\frac{(\alpha-1)\beta_1}{\beta_1 - 1}\right]^{1/2} \bar{c}_* \frac{d\bar{\rho}_*}{\bar{\rho}_*}$$

The differentials may be approximated by finite intervals as

$$\Delta\bar{u}_{2*} = \text{sgn}\,(B_1 B_{2_I})\left[\frac{(\alpha-1)\beta_1}{\beta_1 - 1}\right]^{1/2} \Delta\bar{u}_{1*}$$

$$= \text{sgn}\,(B_1 B_{2_I})\left[\frac{(\alpha-1)\beta_1}{\beta_1 - 1}\right]^{1/2} \bar{c}_* \frac{\Delta\bar{\rho}_*}{\bar{\rho}_*} \qquad (4\text{-}47)$$

A similar procedure of evaluating the change of $\Delta\bar{u}_{2*}$ across each small interval along an integral curve as that for $\Delta\bar{u}_{1*}$ can be applied here. Let us define w_2 by the following expression

$$\overline{u}_{2*} = \mp \operatorname{sgn} (B_1 B_{2I})(w_2 - w_2') \qquad (4\text{-}48a)$$

where w_2 is a constant for each integral curve and its value is determined by the condition that

$$w_2 = 0 \qquad \text{at} \qquad \beta_1 = 1 \qquad (4\text{-}48b)$$

always on each integral curve. The (-) and (+) signs in front of the term on the right-hand side of 4-48a refer to forward and backward facing coupled waves respectively. The plot of w_2 versus $\overline{\rho}_*$ along different integral curves for slow waves is shown in fig. 8 and that for fast waves is shown in fig. 13. The change of $\overline{u}_{2*}$ between two states can be obtained readily from these while the precise value of w_2 as well as w_2' need not be known in most cases.

4. General Behavior of Physical Variables across Coupled Waves

Some basic properties of physical variables associated with the coupled wave problem are investigated. These considerations are helpful in understanding the fundamental mechanism involved as the wave travels and are essential to the choice of a particular type of wave to describe a given boundary value problem. Some of the behavior can be visualized directly from graphs shown in figs. 4 through 13, however, a more precise consideration from the mathematical point of view is undertaken here.

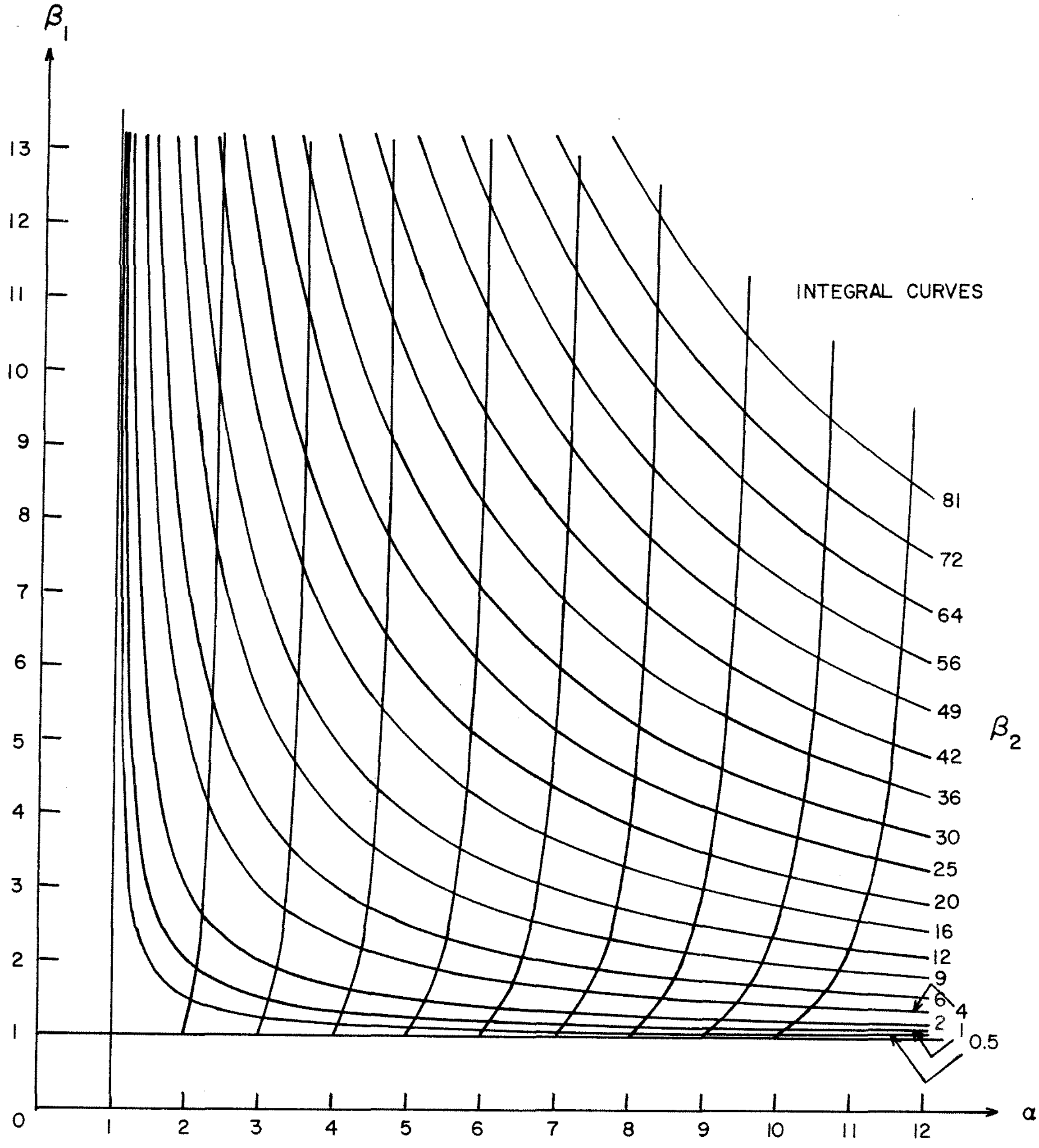

β1
13
12
11
10
9
8
7
6
5
4
3
2
1
0
1
2
3
4
5
6
7
8
9
10
11
12
α
INTEGRAL CURVES
81
72
64
56
49
42
36
30
25
20
16
12
9
6
4
2
1
0.5
β2

FIG. 4

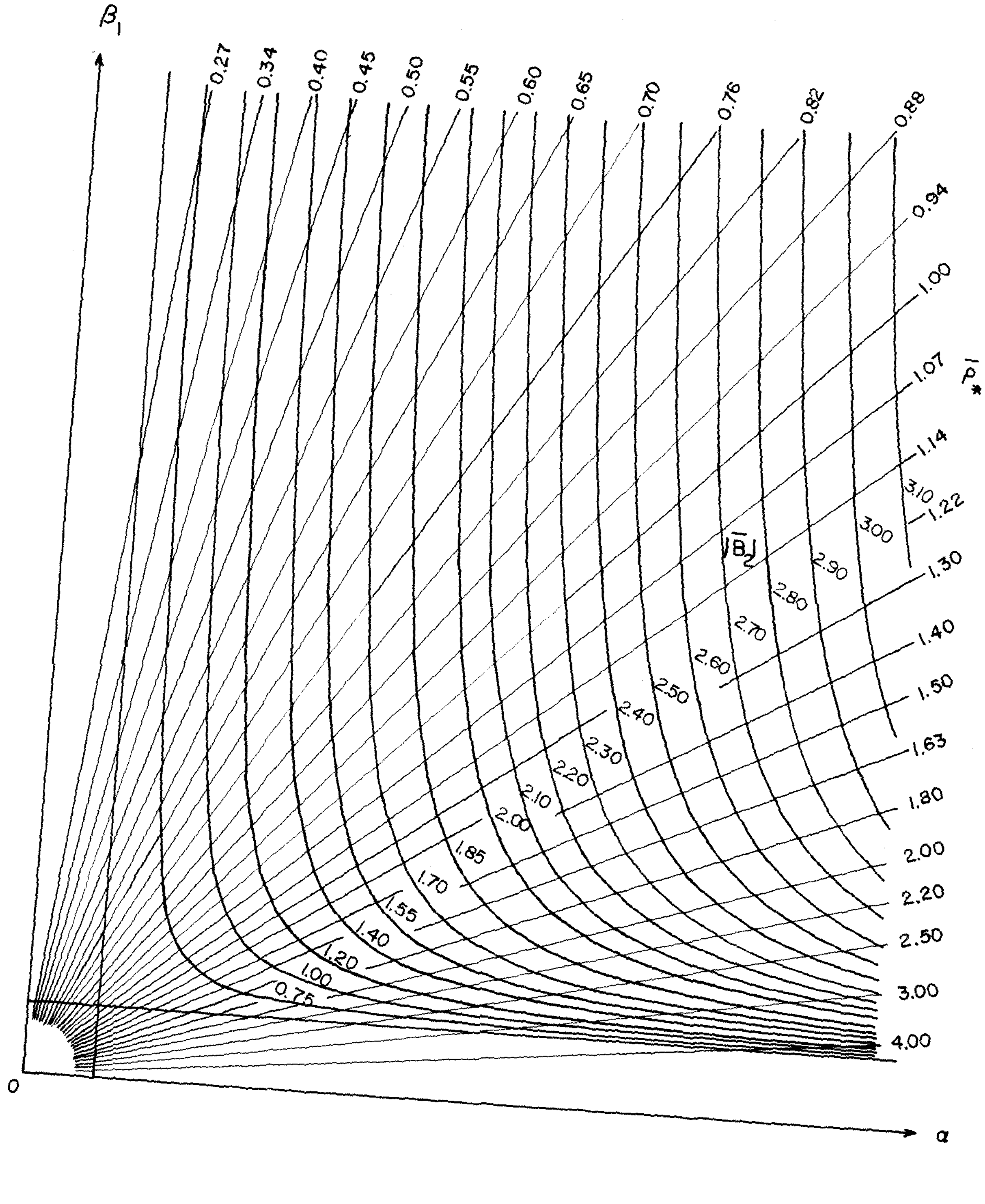
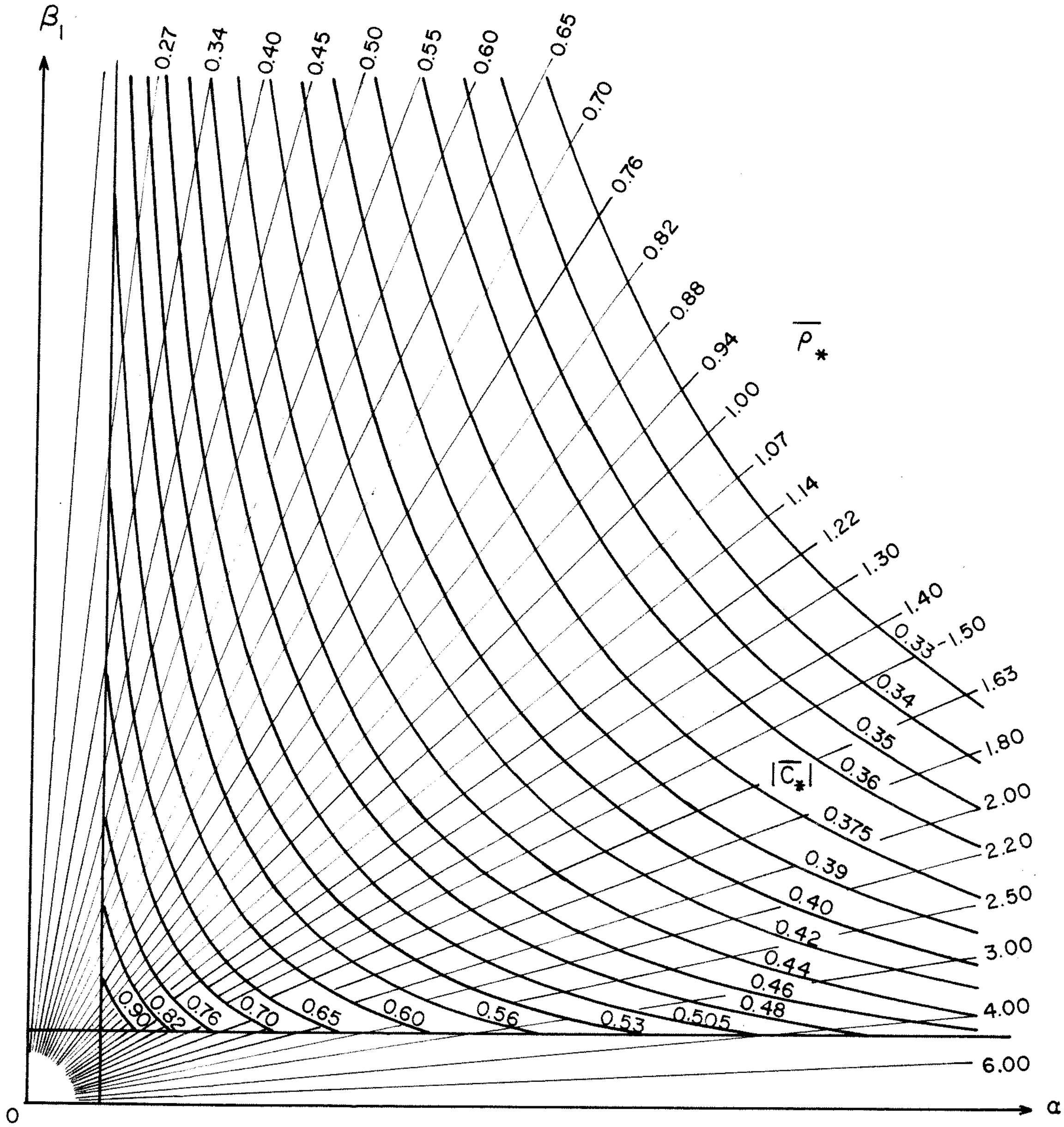
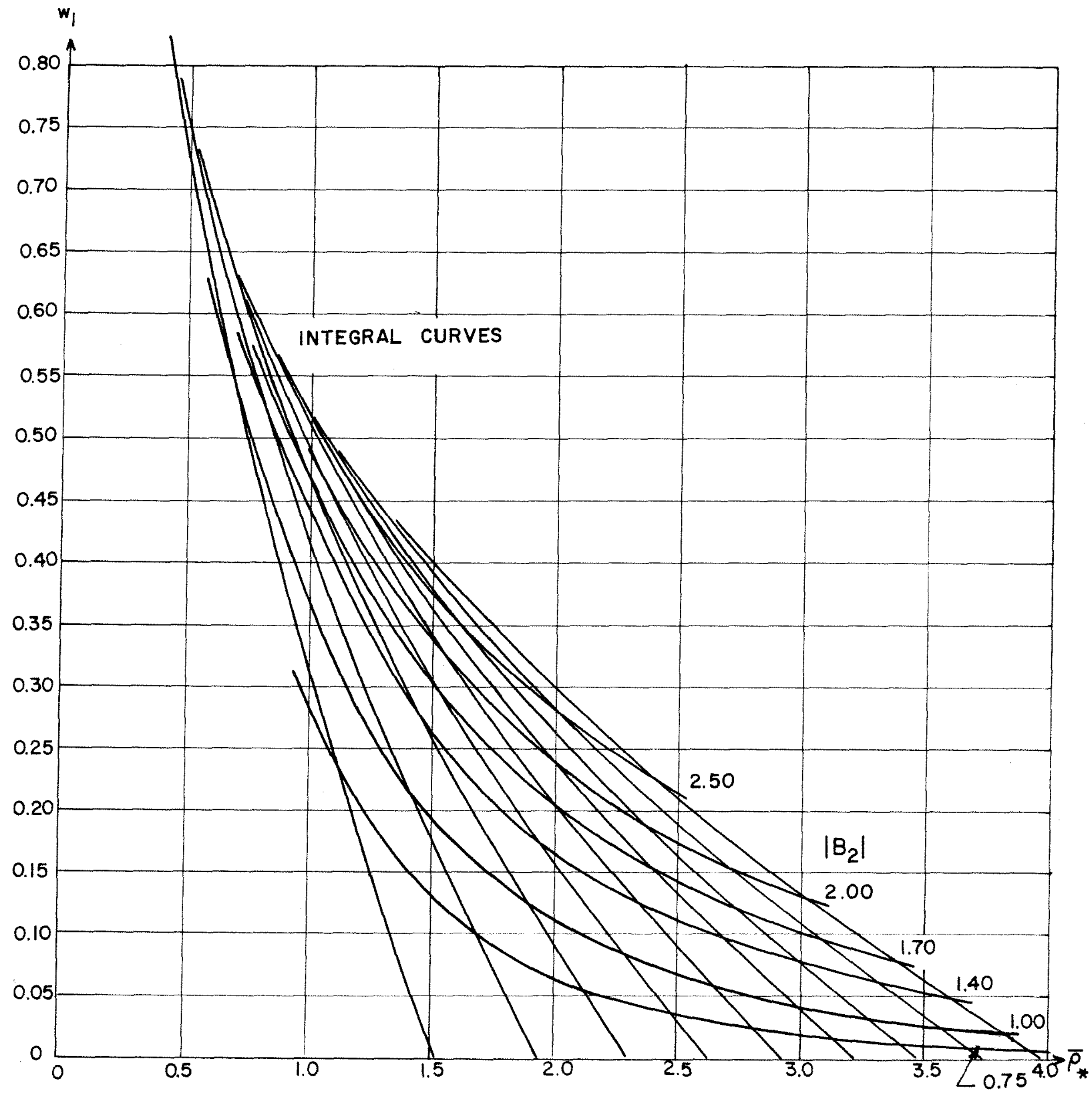

β1
0.27
0.34
0.40
0.45
0.50
0.55
0.60
0.65
0.70
0.76
0.82
0.88
0.94
1.00
1.07
1.14
1.22
1.30
1.40
1.50
1.63
1.80
2.00
2.20
2.50
3.00
4.00
ρ̄*
|B̄2|
3.10
3.00
2.90
2.80
2.70
2.60
2.50
2.40
2.30
2.20
2.10
2.00
1.85
1.70
1.55
1.40
1.20
1.00
0.75
0
α

FIG. 5



β1
0.27
0.34
0.40
0.45
0.50
0.55
0.60
0.65
0.70
0.76
0.82
0.88
0.94
1.00
1.07
1.14
1.22
1.30
1.40
1.50
1.63
1.80
2.00
2.20
2.50
3.00
4.00
6.00
ρ*
|C*|
0.33
0.34
0.35
0.36
0.375
0.39
0.40
0.42
0.44
0.46
0.48
0.505
0.53
0.56
0.60
0.65
0.70
0.76
0.82
0.90
0
α

FIG. 6



w1
0.80
0.75
0.70
0.65
0.60
0.55
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0
INTEGRAL CURVES
2.50
|B2|
2.00
1.70
1.40
1.00
0.75
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0

FIG. 7

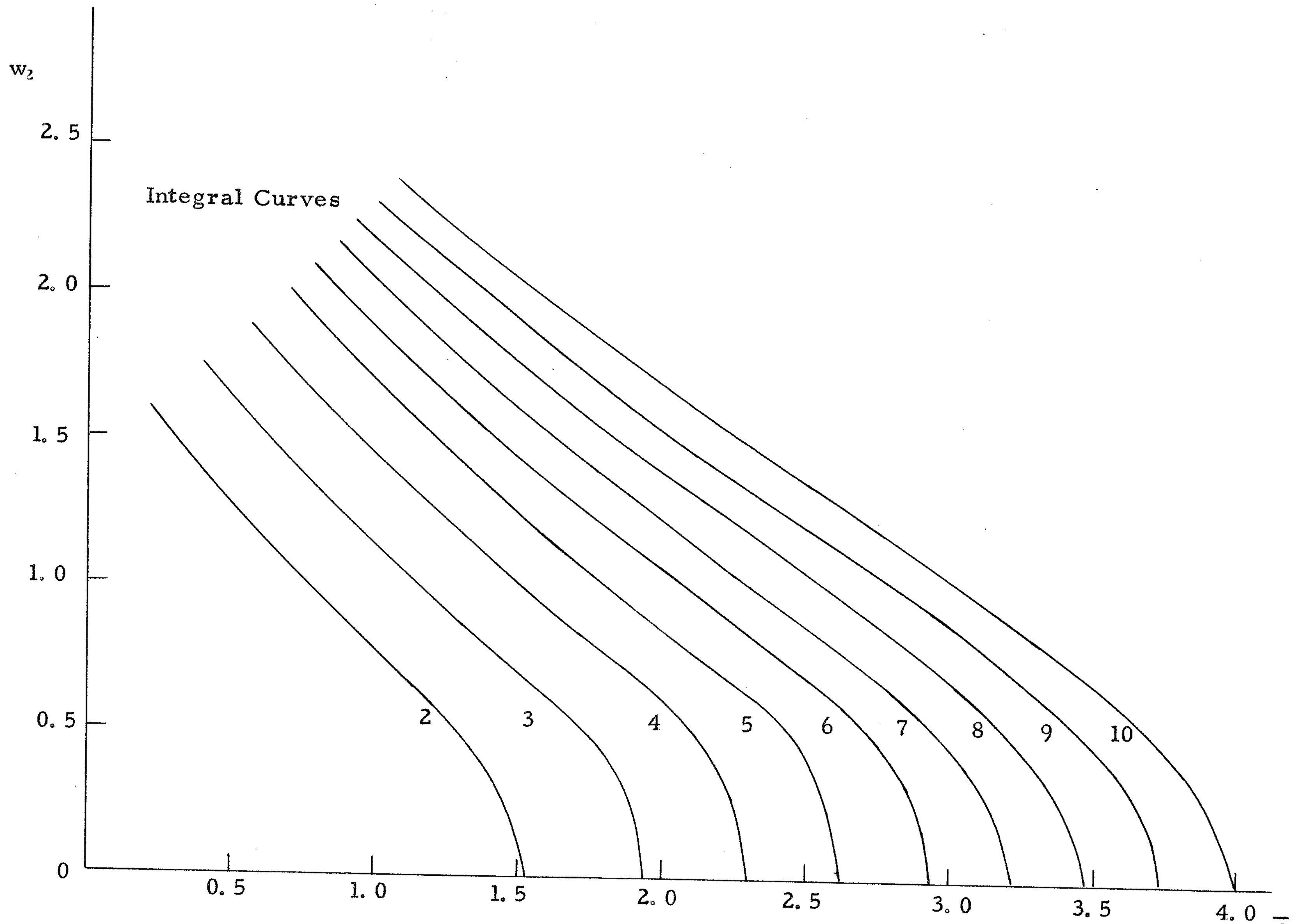

w2
2.5
2.0
1.5
1.0
0.5
0
Integral Curves
2
3
4
5
6
7
8
9
10
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
ρ̄*

Figure 8

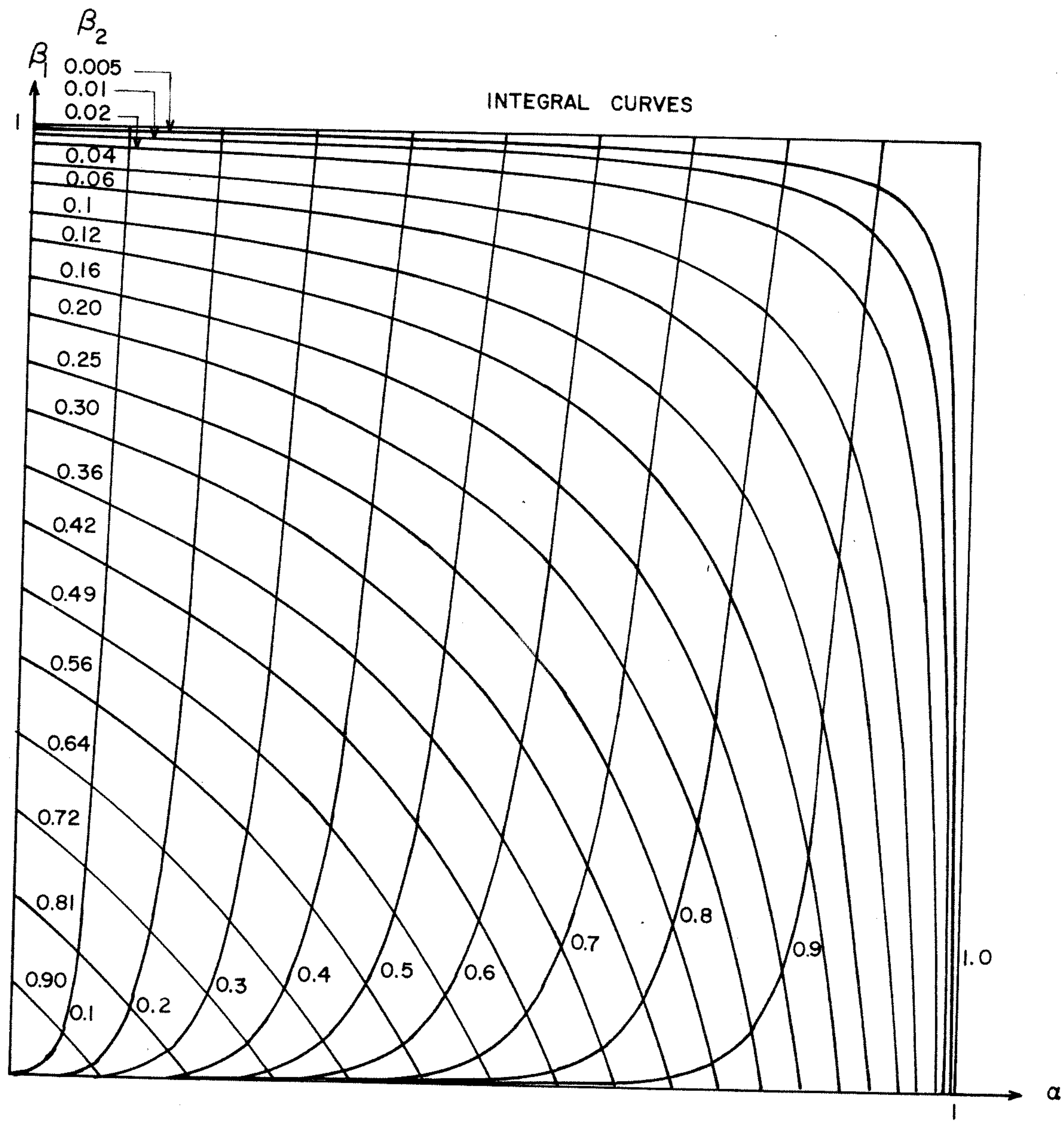
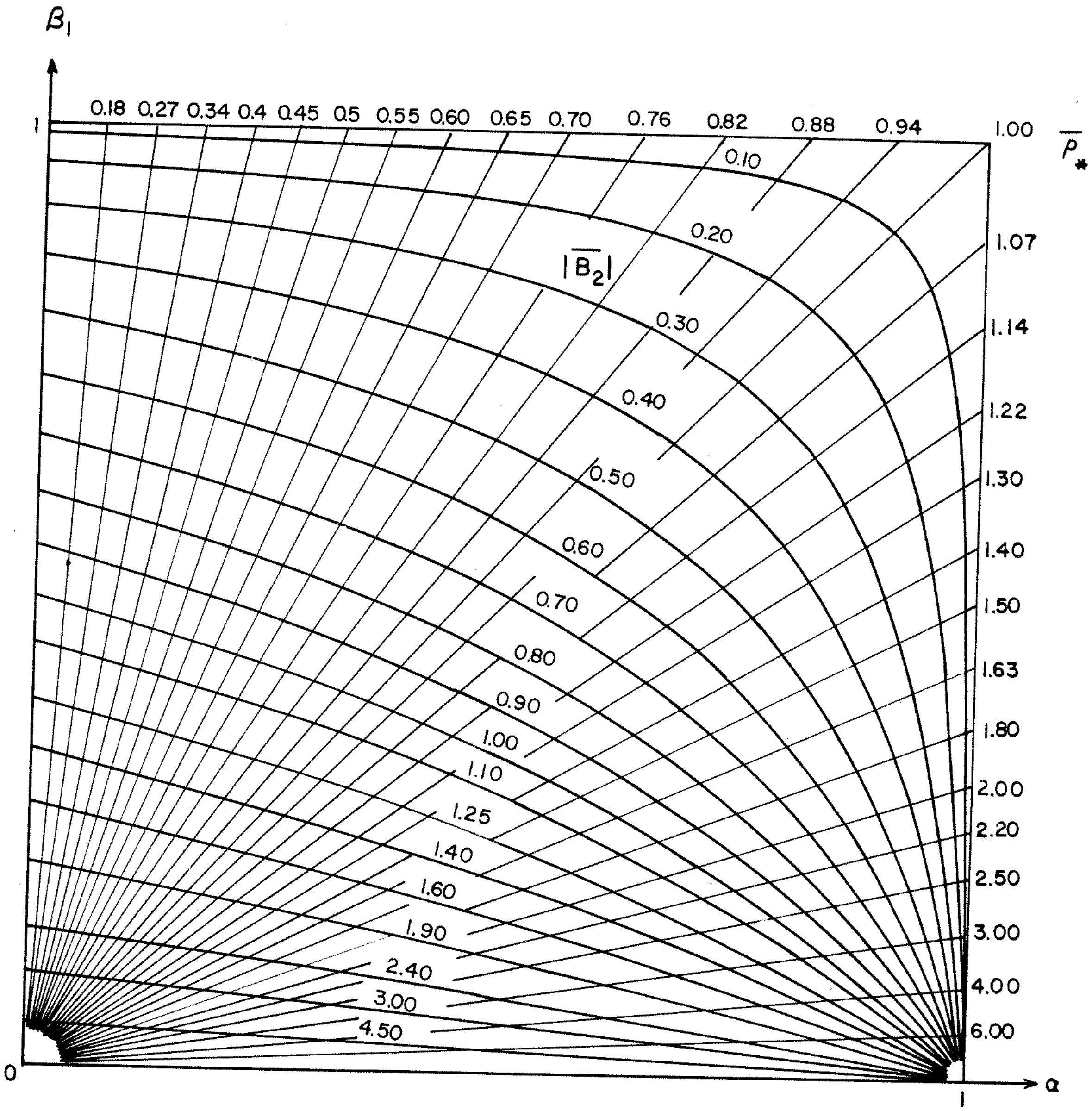
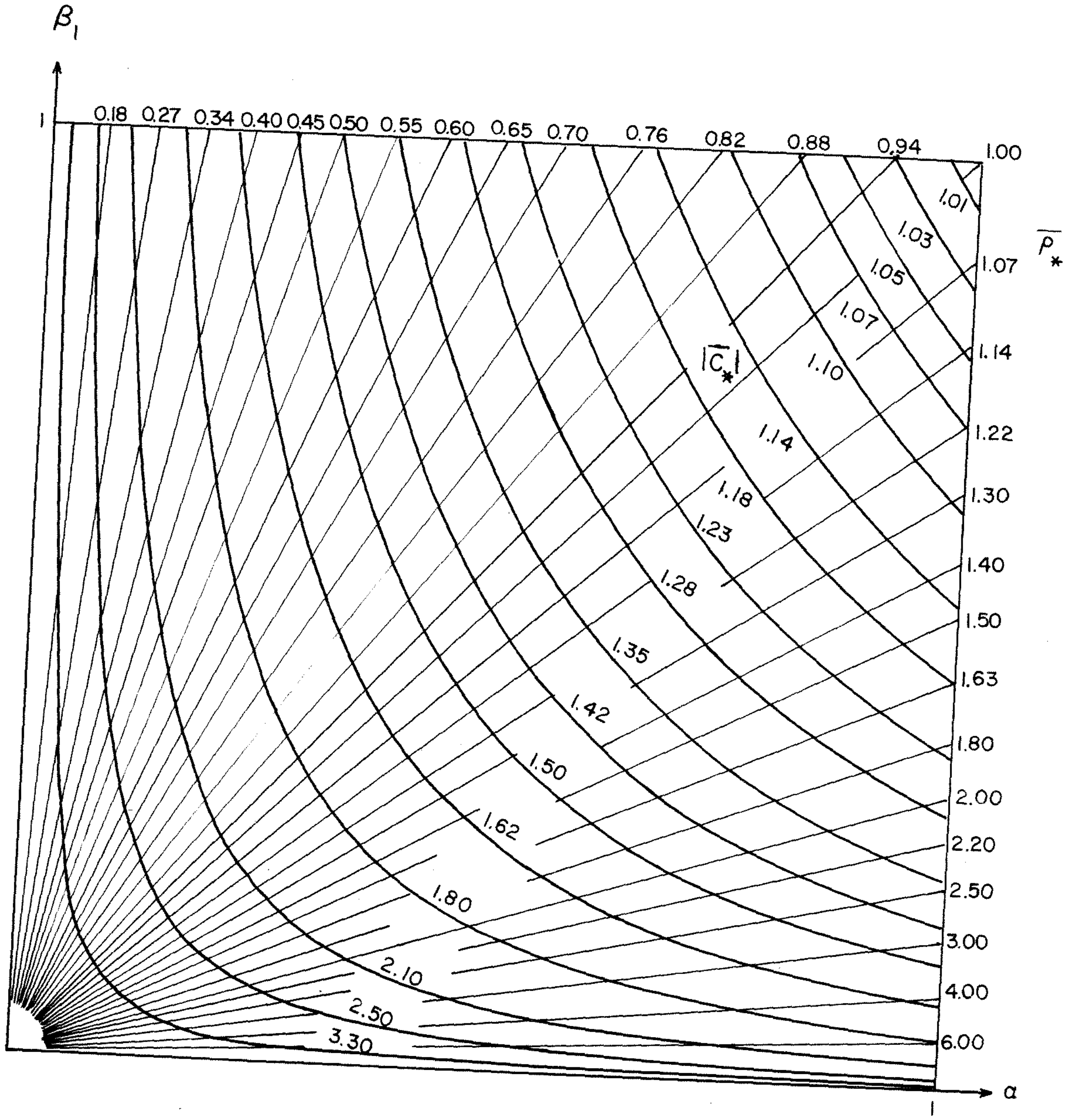

β_2
β_1
0.005
0.01
0.02
1
INTEGRAL CURVES
0.04
0.06
0.1
0.12
0.16
0.20
0.25
0.30
0.36
0.42
0.49
0.56
0.64
0.72
0.81
0.90
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1
α

FIG. 9



β_1
1
0.18 0.27 0.34 0.4 0.45 0.5 0.55 0.60 0.65 0.70 0.76 0.82 0.88 0.94 1.00
$\overline{P}_*$
1.07
1.14
1.22
1.30
1.40
1.50
1.63
1.80
2.00
2.20
2.50
3.00
4.00
6.00
$|\overline{B_2}|$
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
1.10
1.25
1.40
1.60
1.90
2.40
3.00
4.50
0
1
α

FIG. 10



β_1
1
0.18 0.27 0.34 0.40 0.45 0.50 0.55 0.60 0.65 0.70 0.76 0.82 0.88 0.94
1.00
$\overline{P_*}$
1.07
1.14
1.22
1.30
1.40
1.50
1.63
1.80
2.00
2.20
2.50
3.00
4.00
6.00
1
α
$|\overline{C_*}|$
1.01
1.03
1.05
1.07
1.10
1.14
1.18
1.23
1.28
1.35
1.42
1.50
1.62
1.80
2.10
2.50
3.30

FIG. 11

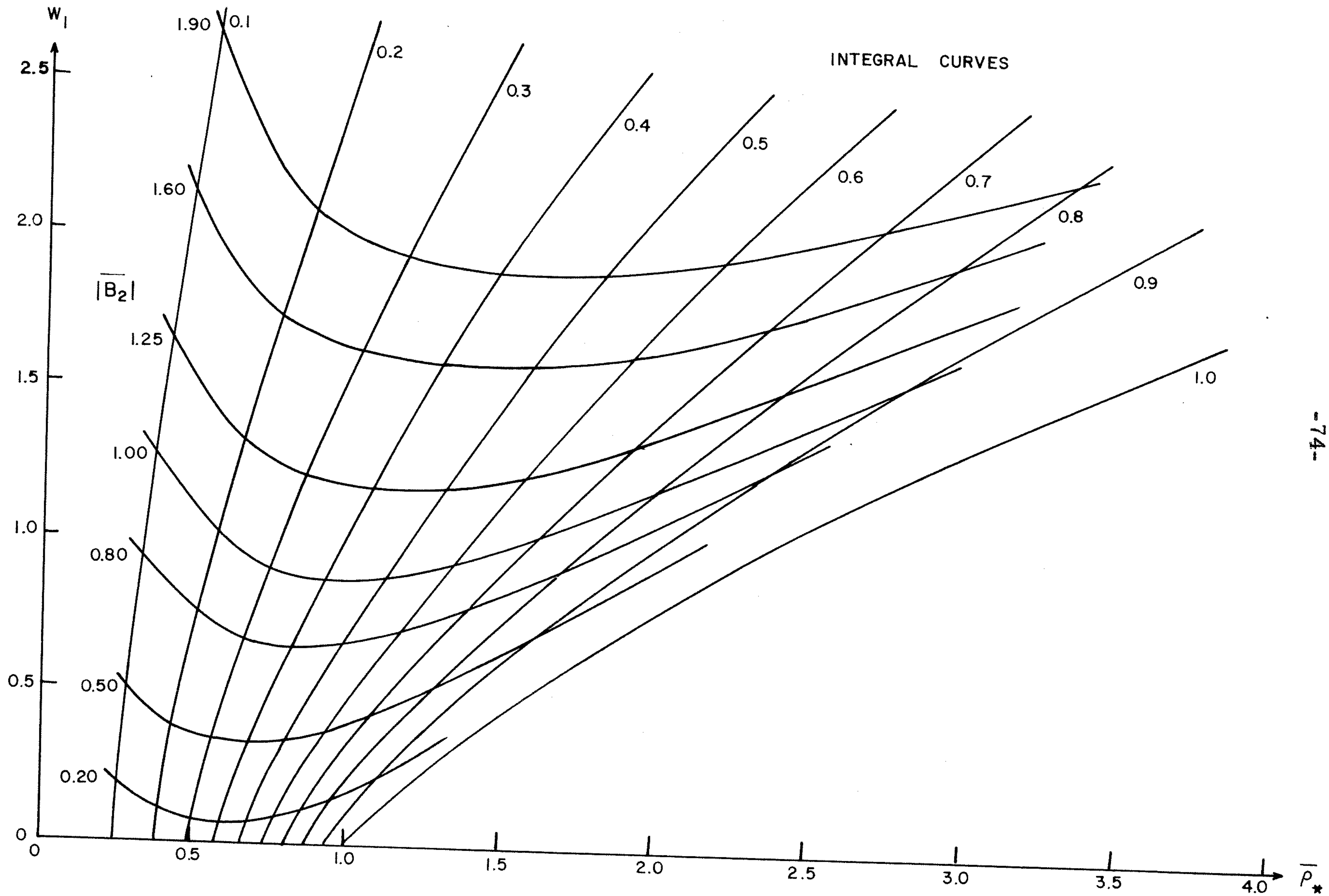
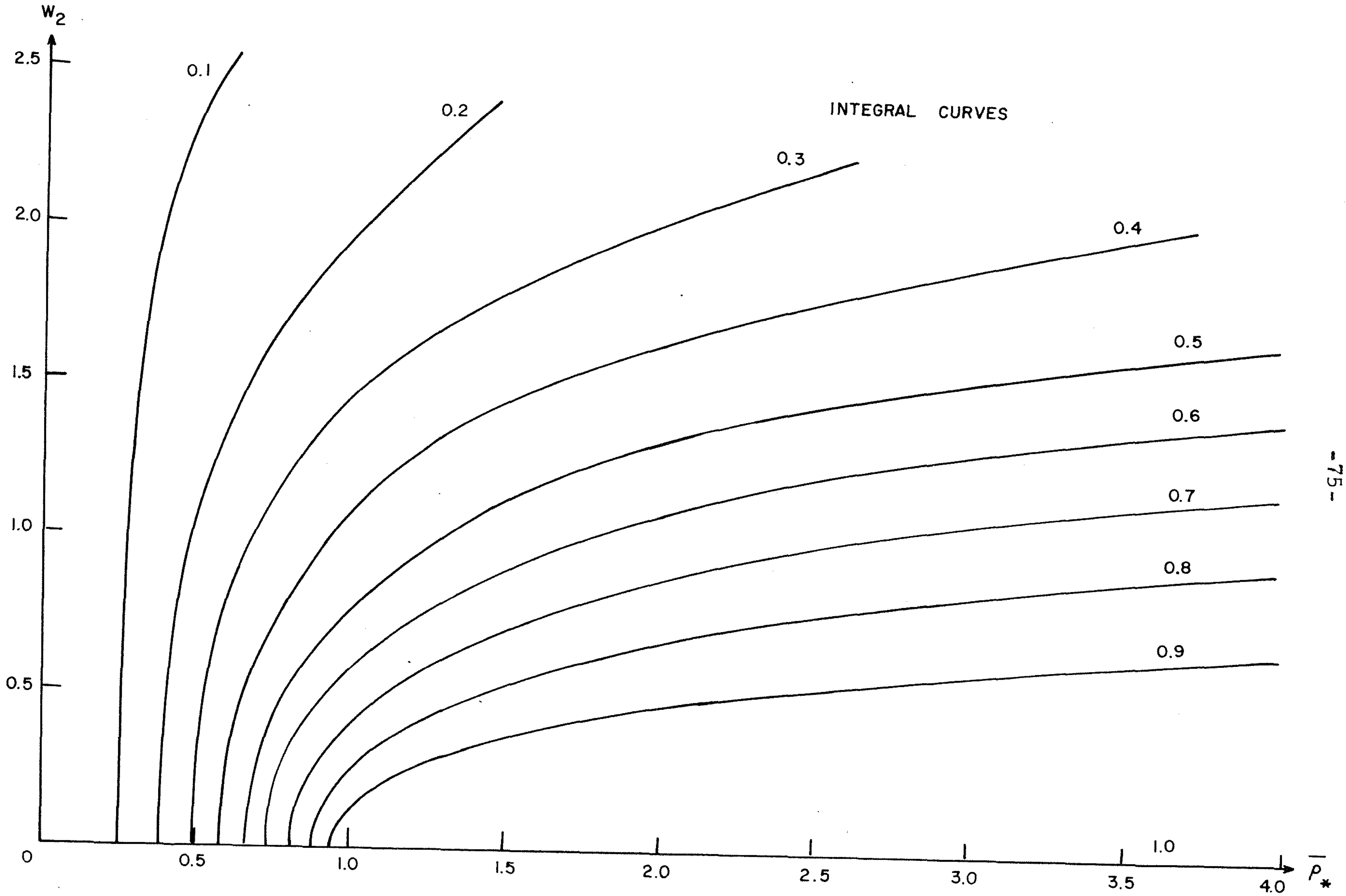

INTEGRAL CURVES
W_1
2.5
2.0
1.5
1.0
0.5
0
$\overline{|B_2|}$
1.90
1.60
1.25
1.00
0.80
0.50
0.20
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
$\overline{\rho}_*$

FIG. 12



INTEGRAL CURVES
W_2
2.5
2.0
1.5
1.0
0.5
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
$\bar{\rho}_*$
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

FIG. 13

A. Properties of Integral Curves in $(\alpha-\beta_1)$ plane

It is seen from 4-17a that for both fast waves (i.e. $1 > \alpha \geq 0$, $1 > \beta_1 \geq 0$) and slow waves (i.e. $\alpha > 1$, $\beta_1 > 1$), $\frac{d\beta_1}{d\alpha} > 0$ always except at $\alpha = 0$, $\beta_1 = 0$ and at $\alpha \to \infty$ for any finite value of β_1 where $\frac{d\beta_1}{d\alpha} = 0$. (As $\beta_1 \to \infty$, α approaches a definite constant as shown in Section IV 2.) Hence β_1 always increases monotonically as α increases in crossing coupled waves.

We shall show next the impossibility of coalescence with the ordinary gasdynamic sound wave from either slow or fast wave. Let us begin with an examination of 4-17a at $\alpha = 1$ (i.e. $c = a$); it shows that $\frac{d\beta_1}{d\alpha} \to \infty$ except at $\beta_1 = 0$ and $\beta_1 = 1$. Thus no integral curve in the $(\alpha-\beta_1)$ plane can pass across $\alpha = 1$ except through two singular points, namely $M(1,0)$ and $N(1,1)$ as shown in fig. 3-a. It is known from Section IV 3 that $M(1,0)$ is a saddle point. The only trajectories able to reach M are $\alpha = 1$ and $\beta_1 = 0$; the former corresponds to the ordinary gasdynamic case and the latter corresponds to a degenerate case of general MHD flow in which the longitudinal magnetic field is absent always. For the general case of coupled waves ($B_1 \neq 0$), there is no integral curve that can reach $\alpha = 1$ through M. Consider now the singular point $N(1,1)$ which was shown to be a nodal point. Except for one of the integral curves lying on $\alpha = 1$ which corresponds to the ordinary gasdynamic case, all other trajectories are shown in Section IV 3 to be tangent to the line $(\alpha-1) + (\gamma-1)(\beta_1 - 1) = 0$. This tangent line has the slope $\frac{d\beta_1}{d\alpha} = -\frac{1}{\gamma-1} < 0$ and it extends to the physically unrealistic region (forbidden region). However, we have just shown that all trajectories should have positive slopes in the slow and fast wave regions.

Therefore, it is obvious that there is no way to connect trajectories in either the fast or slow wave region to $\alpha = 1$ through a physically possible process. This concludes the proof of our statement. As a direct consequence to this, since $\alpha = 1$ is the line that separates fast and slow wave regions, we may also assert that it is not possible to have a direct transition from fast waves to slow waves or vice versa. In other words, the particular type of coupled wave remains unchanged always.

We may remark here that for the special case of purely transverse magnetic field, all integral curves in $(\alpha-\beta_1)$ plane degenerate to a single line LM. Only fast waves exist and they are able to coalesce with ordinary gasdynamic sound waves through the singluar point M(1, 0). The gas expands completely to vacuum and the transverse magnetic field is switched off at this specific point. In this special situation fast and slow waves as well as sound waves coalesce.

B. Variation of Density Across Waves

The density, $\bar{\rho}$, is a monotonic function of α along any integral curve (i. e. for any fixed values of K_1 and α_I). This is shown as follows.

We first obtain a total differential equation describing the change of $\bar{\rho}$ across waves as a function of α by eliminating β_1 from 4-17a and 4-22a. It reads

$$\frac{d\bar{\rho}}{d\alpha} = \frac{1}{2-\gamma} \frac{\bar{\rho}^{\gamma} - K_1^2 \alpha^2}{\alpha(\alpha-1)\bar{\rho}^{\gamma-1}} \tag{4-49a}$$

for both fast and slow waves. This can also be found by direct differentiation of the solution of $\bar{\rho}$ as a function of α given in 4-32a. Since $\beta_1 = \frac{K_1^2}{\bar{\rho}^{\gamma}}\alpha$, 4-49a may be written as

$$\frac{d\bar{\rho}}{d\alpha} = -\frac{1}{2-\gamma}\frac{\bar{\rho}\beta_1}{\alpha}\frac{(\alpha-\frac{1}{\beta_1})}{(\alpha-1)} \qquad (4\text{-}49b)$$

But $\beta_1 > 1$, $\alpha > 1$ for slow waves and $\beta_1 < 1$, $0 \leq \alpha < 1$ for fast waves, 4-39b becomes

$$\frac{d\bar{\rho}}{d\alpha} \leq -\frac{1}{2-\gamma}\frac{\bar{\rho}\beta_1}{\alpha} = -\frac{1}{2-\gamma}\frac{K_1^2}{\bar{\rho}^{\gamma-1}} < 0 \qquad (4\text{-}49c)$$

for all finite values of $\bar{\rho}$. Hence $\bar{\rho}$ decreases monotonically as α increases together with the increase of β_1 along any integral curve. Or one may say that the square of the wave speed ratio, $\alpha = \frac{a^2}{c^2}$, increases across rarefaction waves and decreases across compression waves.

Let us now proceed to consider some limiting values of $\bar{\rho}$ for fast and slow waves.

(a) Fast Waves

It is obvious from 4-32a that $\bar{\rho} \to \infty$ at $\alpha = 0$ always. The density, $\bar{\rho}$, decreases monotonically as α increases until $\beta_1 = 1$ is reached for any integral curve. Thus $\bar{\rho}$ attains its lowest possible value at this state which is characterized by initial parameters of the problem. The corresponding value of α is denoted by α' as defined in Section IV 2 and it can be found analytically from 4-35. The minimum value of density, $\bar{\rho}_{min}$ is determined by

$$\bar{\rho}_{min} = (K_1^2\alpha')^{1/\gamma} \qquad (4\text{-}50)$$

since $\bar{\rho} = \left(\frac{K_1^2 a}{\beta_1}\right)^{1/\gamma}$ and $\beta_1 = 1$ at this state. A remarkable fact to be observed is that the gas cannot generally expand to vacuum in this case. It reaches a final state with the transverse magnetic field switched off entirely. Since $\beta_1 = 1$, the fast wave degenerates to a special case of transverse wave. It is due to the basic property of the latter that no transverse magnetic field can be induced from then on.

(b) Slow Waves

We have shown in Section IV 2 that a tends to a finite value as $\beta_1 \to \infty$. Therefore in this case $\bar{\rho} = 0$ is possible for any integral curve as can be seen from 4-22a. a increases monotonically along a given integral curve as β_1 increases describing a continuous expansion process of the gas. $\bar{\rho}$ attains its minimum value, zero, as $\beta_1 \to \infty$. The finite value of a at this state is denoted by a' and can be found from 4-36. On the other hand, when the gas undergoes a compression process, a decreases monotonically along a given integral curve until it reaches its lowest possible value at $\beta_1 = 1$. We denote this minimum value of a by a_{min} at which $\bar{\rho}$ attains its maximum $\bar{\rho}_{max}$. By putting $\beta_1 = 1$ in 4-25 for the slow wave case, $a_{min} = a_{min}(K_1, a_I)$ is determined from the following expression

$$a_{min}\left(\frac{a_{min}}{a_{min}-1}\right)^{\gamma/(2-\gamma)} + \frac{\gamma}{2-\gamma}\int_{a_I}^{a_{min}}\left(\frac{\zeta}{\zeta-1}\right)^{2/(2-\gamma)} d\zeta$$
$$= \frac{1}{K_1^2}\left(\frac{a_I}{a_I-1}\right)^{\gamma/(2-\gamma)} \qquad (4\text{-}51)$$

Then $\bar{\rho}_{max}$ for the given integral curve is obtained from 4-32a as a function of a_{min}

$$\bar{\rho}_{max} = (K_1^2 \alpha_{min})^{1/2} \tag{4-52}$$

The gas cannot be compressed isentropically to a density higher than this. The transverse magnetic field is switched off at this state and a special case of pure transverse wave can start.

C. Variation of Transverse Magnetic Field across Waves

Let us consider now the behavior of transverse magnetic field strength across coupled waves. We have from 4-20

$$\overline{B_2}^2 = (\alpha-1)(1-\frac{1}{\beta_1}) \tag{4-53a}$$

Differentiating the above equation with respect to α and making use of 4-17a, we arrive at

$$\frac{d\overline{B_2}^2}{d\alpha} = \frac{2}{2-\gamma}(1-\frac{1}{\alpha\beta_1}) \tag{4-53b}$$

For slow waves $\alpha > 1$, $\beta_1 > 1$; so that

$$\frac{d\overline{B_2}^2}{d\alpha} > 0 \tag{4-54a}$$

always. For fast waves $0 \leq \alpha < 1$, $0 \leq \beta_1 < 1$; so that

$$\frac{d\overline{B_2}^2}{d\alpha} < 0 \tag{4-54b}$$

always. Combining 4-54 and 4-49c which shows $\frac{d\bar{\rho}}{d\alpha} < 0$ for all finite values of $\bar{\rho}$, we conclude that across slow waves the magnitude of the transverse magnetic field decreases as the gas compresses and increases as the gas expands. On the contrary, the transverse magnetic field strength increases across fast compression waves and decreases across fast rarefaction waves. Moreover, owing to the monotonic variation of all physical variables across waves, the sense of the transverse magnetic field direction cannot be inverted through either fast or slow waves. A further investigation shows that the allowable value of $|\overline{B_2}|$ is unbounded in fast waves but it has a maximum value in slow waves which is characterized by the initial state of the problem. These values are discussed as follows.

(a) Fast Waves

We have from 4-20 that $\overline{B_2} = 0$ at $\beta_1 = 1$ for all integral curves. $|\overline{B_2}|$ increases continuously as the gas compresses until $\alpha = 0, \beta_1 = 0$ is reached. It is found from 4-20 that $|\overline{B_2}| \to \infty$ at this state. Hence the transverse magnetic field strength is unbounded.†

(b) Slow Waves

It is seen from 4-20 that $\overline{B_2} = 0$ at $\beta_1 = 1$ for all integral curves. $|\overline{B_2}|$ increases as $\overline{\rho}$ increases until a vacuum state is attained that $\overline{\rho} = 0$. This corresponds to $\alpha = \alpha'$ and $\beta_1 \to \infty$. Thus the maximum value of $|\overline{B_2}|$ is attained from 4-20 that

$$|\overline{B_2}|_{max} = (\alpha' - 1)^{1/2} \tag{4-55a}$$

or

$$\overline{B_2}_{\substack{max \\ (min)}} = \operatorname{sgn}(B_1 B_{2_I})(\alpha' - 1)^{1/2} \tag{4-55b}$$

where $\alpha' = \alpha'(K_1, \alpha_I)$ is given in 4-36 as an implicit function of the initial state parameters.

† This is merely a theoretical result for ideal gases. Actually for real gases that we are dealing with in practical cases, a shock wave is formed long before the gas is compressed to $\overline{\rho} \to \infty$. The transverse magnetic field strength can have only a finite value.

D. Distortion of Wave Profile as a Function of Time

By wave profile we mean here the distribution of physical quantities, i. e. velocity, density, magnetic field strength, pressure, etc., in space at a specific instant. The distortion of simple wave profile during the propagation of coupled waves results in the steepening of compression (condensation) waves and the flattening of expansion (rarefaction) waves. Consequently, the former leads ultimately to the formation of MHD shocks, i. e. fast and slow shocks as defined by Friedrichs (1954), which involve a similar mechanism as that of ordinary gasdynamic shocks.

The proof can be given as follows. Let us consider, e. g. the longitudinal velocity profile, $\partial u_1/\partial x_1$, at time t which by making use of 2-28 and 2-34 may be written as

$$\frac{\partial u_1}{\partial x_1} = \frac{u_1'(\xi)}{x_1'(\xi)} = \frac{u_1'(\xi)}{1 + [u_1'(\xi) + c'(\xi)]t} \tag{4-56}$$

where $'$ denotes the total derivative with respect to the phase parameter ξ. We shall first show the following inequality

$$\frac{d(u_1 + c)}{du_1} > 0 \tag{4-57}$$

always for forward or backward facing ($c > 0$ or $c < 0$) coupled waves. From 4-7a we may write

$$\frac{d(u_1 + c)}{du_1} = \frac{d(\overline{u}_1 + \overline{c})}{d\overline{u}_1} = 1 + \frac{\overline{\rho}\, d\overline{c}}{\overline{c}\, d\overline{\rho}} \tag{4-58}$$

where $\overline{\rho}$ and $\overline{c}$ can be obtained from 4-22a and 4-22f respectively as functions of a and β_1. They are substituted into 4-58 together with the use of 4-17a, thus one obtains easily the following result

$$\frac{d(u_1 + c)}{du_1} = \frac{(\gamma+1)\alpha(\beta_1 - 1) + 3(\alpha - 1)}{2(\alpha\beta_1 - 1)} \tag{4-59}$$

for both forward and backward facing waves. Now, it is evident from 4-59 that 4-57 is valid for both fast and slow waves. For a forward facing rarefaction wave that $u_1'(\xi) > 0$, we have from 4-57 $u_1'(\xi) + c'(\xi) > 0$. Returning to 4-56, we see the denominator of $\frac{\partial u_1}{\partial x_1}$ increases with time always, hence the longitudinal velocity profile is flattened out. On the other hand, for a compression wave that $u_1'(\xi) < 0$ and $u_1'(\xi) + c'(\xi) < 0$, the denominator of $\frac{\partial u_1}{\partial x_1}$ decreases with time and the velocity profile steepens up always. The denominator may eventually equal zero and causes an infinite slope of the profile; after that the slope changes sign and we are given generally three values of velocity at a position x_1 which is impossible physically. The actual situation corresponds to the formation of shock waves which propagates with shock speeds into the fluid. Fast shocks are generated by fast simple waves and slow shocks are generated by slow simple waves. The maintenance of a constant profile of shocks may be reasoned by the fact that the steepening up of wave profile due to nonlinear effects is balanced by the diffusion mechanism arising from heat conduction, viscous friction and finite electrical conductivity. The change of wave profile of other physical quantities in the course of propagation of coupled waves may be found in a similar manner.

V. SOLUTION FOR THE CASE OF PURELY TRANSVERSE MAGNETIC FIELD

1. The Arbitrary Flow

As a special case of the general problem, we consider now $B_1 = 0$ initially. It follows from the x_1-component of 2-2a that $B_1 = 0$ at all later times. The structure of the governing equations is greatly simplified and certain boundary value problems of an arbitrary flow can even be formulated in a simple form suitable for a detailed study.

We begin with an investigation of the general system of differential equations 2-8. Since the corresponding case of contact surfaces that entropy is allowed to be different along different fluid particle paths has been discussed in Chapter III, we assume here the entropy is constant throughout initially. It is seen from 2-8g that the entropy remains constant throughout for $t > 0$ and the fluid flow is isentropic. From 2-8c,d since $\rho \neq 0$ in general, u_2 and u_3 are constant along each fluid particle path. If u_2 and u_3 are constant throughout the entire space initially, they remain so afterwards. We get from 2-8a,e,f the following equations

$$\frac{D\rho}{Dt} + \rho \frac{\partial u_1}{\partial x_1} = 0 \tag{5-1a}$$

$$\frac{DB_2}{Dt} + B_2 \frac{\partial u_1}{\partial x_1} = 0 \tag{5-1b}$$

$$\frac{DB_3}{Dt} + B_3 \frac{\partial u_1}{\partial x_1} = 0 \tag{5-1c}$$

After eliminating $\frac{\partial u_1}{\partial x_1}$ among them, two independent relations exist, namely

$$\frac{D}{Dt}\left[\log\left(\frac{B_3}{B_2}\right)\right] = 0 \tag{5-2a}$$

and

$$\frac{D}{Dt}\left[\log\left(\frac{B_2}{\rho}\right)\right] = 0 \tag{5-2b}$$

For this isentropic case, we have

$$\frac{B_3}{B_2} = k = \text{constant} \tag{5-3a}$$

and

$$\frac{B_2}{\rho} = \lambda = \text{constant} \tag{5-3b}$$

throughout the entire flow. However, one may always orient the coordinate system $(x_2 - x_3)$ in transverse plane in such a way that $B_3 = 0$ identically. Thus the problem can be studied in the $(x_1 - x_2)$ plane exclusively. With 5-3b, the whole system of equations 2-8 that governs the fluid flow is reduced to the following two equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_1)}{\partial x_1} = 0 \tag{5-4a}$$

$$\rho\left(\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x_1}\right) + \frac{\partial P}{\partial x_1} + \frac{\lambda^2}{\mu} \rho \frac{\partial \rho}{\partial x_1} = 0 \tag{5-4b}$$

From 2-9a

$$a^2 = \frac{dP}{d\rho} = \gamma A \rho^{\gamma - 1} \tag{5-5}$$

with A now constant. After making use of 5-5, we may write 5-4 in terms of velocity variables, u_1 and a, as follows:

$$\frac{\partial a}{\partial t} + u_1 \frac{\partial a}{\partial x_1} + \frac{\gamma - 1}{2} a \frac{\partial u_1}{\partial x_1} = 0 \tag{5-6a}$$

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x_1} + \frac{2a}{\gamma-1}\left[1 + \Lambda a^{\frac{2(2-\gamma)}{\gamma-1}}\right]\frac{\partial a}{\partial x_1} = 0 \tag{5-6b}$$

where $\Lambda = \frac{\lambda^2}{\mu k^{\frac{1}{\gamma-1}}}$. The system 5-6 belongs to the reducible type of non-linear partial differential equations. For purposes of a general study 5-6 can be put in a linear form in terms of (u_1, a) as the independent variables (hodograph transformation). The transformation is regular provided the Jacobian of the transformation, $j = \frac{\partial u_1}{\partial t}\frac{\partial a}{\partial x_1} - \frac{\partial u_1}{\partial x_1}\frac{\partial a}{\partial t}$, is nowhere zero in the flow region. The transformation relations are

$$\frac{\partial u_1}{\partial t} = j\,\frac{\partial x_1}{\partial a} \qquad , \qquad \frac{\partial u_1}{\partial x_1} = -j\,\frac{\partial t}{\partial a}$$

$$\frac{\partial a}{\partial t} = -j\,\frac{\partial x_1}{\partial u_1} \qquad\qquad \frac{\partial a}{\partial x_1} = j\,\frac{\partial t}{\partial u_1} \tag{5-7}$$

It is important to note here that when $j = 0$, the solution corresponds to a simple wave which cannot be obtained from the hodograph representation. This is a degenerate case of the general solution and is thus sometimes called a "lost" solution. Substituting 5-7 into 5-6, we obtain

$$-\frac{\partial x_1}{\partial u_1} + u_1 \frac{\partial t}{\partial u_1} - \frac{\gamma-1}{2}\, a\, \frac{\partial t}{\partial a} = 0 \tag{5-8a}$$

$$\frac{\partial x_1}{\partial a} - u_1 \frac{\partial t}{\partial a} + \frac{2a}{\gamma-1}\left[1 + \Lambda a^{\frac{2(2-\gamma)}{\gamma-1}}\right]\frac{\partial t}{\partial u_1} = 0 \tag{5-8b}$$

Eliminating x_1 from above by cross differentiation, we arrive at a second order linear differential equation for $t(u_1, a)$

$$\frac{\partial^2 t}{\partial a^2} - \frac{4}{(\gamma-1)^2}\left[1 + A\,a^{\frac{2(2-\gamma)}{\gamma-1}}\right]\frac{\partial^2 t}{\partial u_1{}^2} + \frac{\gamma+1}{\gamma-1}\,\frac{1}{a}\,\frac{\partial t}{\partial a} = 0 \qquad (5\text{-}9a)$$

Since $\gamma = \frac{n+2}{2}$ from 1-2, 5-9a can also be written as

$$\frac{\partial^2 t}{\partial a^2} - n^2\left[1 + A\,a^{n-2}\right]\frac{\partial^2 t}{\partial u_1{}^2} + \frac{n+1}{a}\,\frac{\partial t}{\partial a} = 0 \qquad (5\text{-}9b)$$

This is a hyperbolic equation due to the fact that A and a are always greater than or equal to zero. A typical flow problem demands boundary conditions on characteristics, pistons, or shock waves. We shall content ourselves with the formulation given here and no further study of its solution is attempted.

2. Simple Waves

As to simple wave solutions we see that only coupled waves exist in this case owing to the absence of B_1. The entire family of integral curves in $(\alpha-\beta_1)$ plane as well as those in $(\alpha-\beta_2)$ and $(\beta_1-\beta_2)$ planes degenerates to a single straight line LM as shown in Fig. 3. The situation can be visualized better from the $(\alpha-\beta_2)$ plane where LM is described by $\alpha + \beta_2 = 1$. The same result can also be obtained from 4-15 by setting $\beta_1 = 0$. Hence

$$c^2 = a^2 + b_1{}^2 = \gamma A\,\rho^{\gamma-1} + \frac{\lambda^2}{\mu}\rho \qquad (5\text{-}10)$$

One may see readily from Fig. 3-c that c corresponds to the limiting case of a fast wave. The slow wave has zero speed and it coalesces with a contact surface in this case.

Let us introduce the following dimensionless variables normalized with respect to initial physical quantities as

$$\bar{\rho} = \frac{\rho}{\rho_I}, \qquad \bar{B}_2' = \frac{B_2}{B_{2_I}}, \qquad \bar{u}_1 = \frac{u_1}{a_I}$$
$$\bar{a} = \frac{a}{a_I}, \qquad \bar{b}_2 = \frac{b_2}{a_I}, \qquad \bar{c} = \frac{c}{a_I} \tag{5-11}$$

Here $\bar{B}_2' = \bar{\rho}$ always as is seen from 5-3b.

The fundamental parameter of the initial conditions is[+]

$$K_2 = \frac{b_{2_I}}{a_I} = \frac{1}{a_I} \frac{B_{2_I}}{\sqrt{\mu \rho_I}} \tag{5-12}$$

5-10 becomes

$$\bar{c}^2 = \bar{a}^2 + \bar{b}_2^2 = \bar{\rho}^{\gamma-1} (1 + K_2^2 \bar{\rho}^{2-\gamma}) \tag{5-13a}$$

or

$$\bar{c} = \pm \bar{\rho}^{\frac{\gamma-1}{2}} (1 + K_2^2 \bar{\rho}^{2-\gamma})^{\frac{1}{2}} \tag{5-13b}$$

The gas can expand to vacuum in this case of fast waves. Expressing 4-10a in dimensionless variables

$$d\bar{u}_1 = \frac{2}{\gamma-1} \bar{c} \frac{d\bar{a}}{\bar{a}} = \bar{c} \frac{d\bar{\rho}}{\bar{\rho}}$$

and substituting the value of $\bar{c}$ in 5-13b into the above equation, we get

$$d\bar{u}_1 = \pm \bar{\rho}^{\frac{\gamma-3}{2}} (1 + K_2^2 \bar{\rho}^{2-\gamma})^{\frac{1}{2}} d\bar{\rho}$$

Therefore, the generalized Riemann invariants for the longitudinal flow velocity are

$$-2s_1 = \bar{u}_1 - \int_0^{\bar{\rho}} \omega^{\frac{\gamma-3}{2}} (1 + K_2^2 \omega^{2-\gamma})^{\frac{1}{2}} d\omega \tag{5-14a}$$

[+] We have assumed that $B_{2I} \neq 0$; otherwise the magnetic flux lines can never get into fluid because of its infinite electric conductivity, and no direct MHD effect can take place.

and

$$2r_1 = \bar{u}_1 + \int_0^{\bar{\rho}} \omega^{\frac{\gamma-3}{2}} (1 + K_2^{\,2}\,\omega^{2-\gamma})^{\frac{1}{2}}\, d\omega \qquad (5\text{-}14b)$$

for forward and backward facing waves respectively. These have a formal resemblance to those in ordinary gasdynamics except for the additional term $K_2^{\,2}\bar{\rho}^{\,2-\gamma}$ which take into account the magnetohydrodynamic effect due to transverse magnetic field. An equivalent result to 5-14 has been obtained by Mitchner (1959) and Golitsyn (1959).

So far the solution is obtained from the reduced set of differential equations. We may verify it by taking the limiting process $K_1 \rightarrow 0$ of the general solution given in 4-34 and 4-37. From 4-28 and for having fast waves only

$$H = \left(\frac{\alpha_I}{1-\alpha_I}\right)^{\frac{\gamma}{2-\gamma}} = \text{constant}$$

$$\text{Since } K_2^{\,2} = \frac{b_{2_I}^{\,2}}{a_I^{\,2}} = \frac{\beta_{2_I}}{\alpha_I} = \frac{1-\alpha_I}{\alpha_I} \text{ then}$$

$$H = \left(\frac{1}{K_2}\right)^{\frac{2\gamma}{2-\gamma}}$$

The generalized longitudinal Riemann invariants for forward facing waves is obtained from 4-34a as

$$-2s_1 = \bar{u}_1 + \frac{1}{2-\gamma}\left(\frac{1}{K_2}\right)^{\frac{\gamma-1}{2\gamma}} \int_{\alpha'}^{\alpha} \frac{d\sigma}{\sigma^{\frac{5-2\gamma}{2(2-\gamma)}}(1-\sigma)^{\frac{5-3\gamma}{2(2-\gamma)}}} \qquad (5\text{-}15)$$

The density given by 4-32a is

$$\bar{\rho} = \left(\frac{1-\alpha}{K_2^{\,2}\,\alpha}\right)^{\frac{1}{2-\gamma}}$$

or

$$\alpha = \frac{1}{1 + K_2^{\,2}\bar{\rho}^{\,2-\gamma}}$$

thus $\alpha' = 1$ and it corresponds to $\bar{\rho} = 0$. If we change the variable of integration in 5-15 from σ to ω with ω being defined by

$$\sigma = \frac{1}{1 + K_2^{\,2}\omega^{2-\gamma}}$$

so that

$$d\sigma = \frac{-(2-\gamma)\, K_2^{\,2}\, \omega^{1-\gamma}\, d\omega}{(1 + K_2^{\,2}\omega^{2-\gamma})^2}$$

we have

$$\int_{\alpha'}^{\alpha} \frac{d\sigma}{\sigma^{\frac{5-2\gamma}{2(2-\gamma)}}(1-\sigma)^{\frac{5-3\gamma}{2(2-\gamma)}}} = -(2-\gamma)K_2^{\frac{\gamma-1}{2-\gamma}} \int_0^{\bar{\rho}} \omega^{\frac{\gamma-3}{2}} (1+K_2^{\,2}\omega^{2-\gamma})^{\frac{1}{2}}\, d\omega$$

hence

$$-2s_1 = \bar{u}_1 - \int_0^{\bar{\rho}} \omega^{\frac{\gamma-3}{2}} (1+K_2^{\,2}\, \omega^{2-\gamma})^{\frac{1}{2}}\, d\omega \qquad (5\text{-}16a)$$

Similarly,

$$2r_1 = \bar{u}_1 + \int_0^{\bar{\rho}} \omega^{\frac{\gamma-3}{2}} (1+K_2^{\,2}\, \omega^{2-\gamma})^{\frac{1}{2}}\, d\omega \qquad (5\text{-}16b)$$

for backward facing waves. These agree with the previous result 5-14.

In regard to the generalized transverse Riemann invariants, we find from 4-37 that

$$-2s_2 = \bar{u}_2 \qquad (5\text{-}17a)$$

$$2r_2 = \bar{u}_2 \qquad (5\text{-}17b)$$

for forward and backward facing waves. Hence the transverse velocity remains constant throughout as one may expect from 2-11c by putting $B_1 = 0$.

The transverse magnetic field strength, B_2, is obtained as follows. Since $0 < \alpha < 1$ and $H = (\frac{1}{K_2})^{\frac{2\gamma}{2-\gamma}}$, we have from 4-20

$$\bar{B}_2 = \frac{B_2}{B_1} = \text{sgn}\,(B_1 B_{2_I}) \; \left\{ (1-\alpha)\left[\frac{1}{\alpha} \left(\frac{1-\alpha}{K_2{}^2\alpha}\right)^{\frac{\gamma}{2-\gamma}} \frac{1}{K_1{}^2} - 1\right] \right\}^{\frac{1}{2}} \qquad (5\text{-}18)$$

But

$$\frac{B_2}{B_1} = \frac{B_2}{B_{2_I}} \frac{K_2}{K_1}$$

Substituting this into 5-18 and taking the limit as $K_1 \to 0$, we get

$$\frac{B_2}{B_{2_I}} = \bar{B}_2{}' = \lim_{K_1 \to 0} \left\{ \frac{K_1{}^2}{K_2{}^2} (1-\alpha)\left[\frac{1}{\alpha} \left(\frac{1-\alpha}{K_2{}^2\alpha}\right)^{\frac{\gamma}{2-\gamma}} \frac{1}{K_1{}^2} - 1\right] \right\}^{\frac{1}{2}}$$

$$= \left(\frac{1-\alpha}{K_2{}^2\alpha}\right)^{\frac{1}{2-\gamma}} = \bar{\rho} \qquad (5\text{-}19)$$

which agrees with the result obtained before.

The definite integrals in 5-14 or 5-16 can be expressed in terms of a hypergeometric function (See Appendix C). Then

$$-2s_1 = \bar{u}_1 - \frac{2}{\gamma-1} \bar{\rho}^{\frac{\gamma-1}{2}} F\left[-\frac{1}{2}, \frac{\gamma-1}{2(2-\gamma)}; \frac{3-\gamma}{2(2-\gamma)}; -K_2{}^2 \bar{\rho}^{\,2-\gamma}\right]$$

$$= \bar{u}_1 - \frac{2}{\gamma-1} \bar{a}\, F\left[-\frac{1}{2}, \frac{\gamma-1}{2(2-\gamma)}; \frac{3-\gamma}{2(2-\gamma)}; -K_2{}^2 \bar{a}^{\frac{2(2-\gamma)}{\gamma-1}}\right] \qquad (5\text{-}20a)$$

$$2r_1 = \bar{u}_1 + \frac{2}{\gamma-1}\,\bar{\rho}^{\frac{\gamma-1}{2}}\, F\left[-\frac{1}{2},\ \frac{\gamma-1}{2(2-\gamma)};\ \frac{3-\gamma}{2(2-\gamma)};\ -K_2^{\,2}\,\bar{\rho}^{\,2-\gamma}\right]$$

$$= \bar{u}_1 + \frac{2}{\gamma-1}\,\bar{a}\, F\left[-\frac{1}{2},\ \frac{\gamma-1}{2(2-\gamma)};\ \frac{3-\gamma}{2(2-\gamma)};\ -K_2^{\,2}\,\bar{a}^{\frac{2(2-\gamma)}{\gamma-1}}\right] \quad (5\text{-}20b)$$

For the limiting case of ordinary gasdynamics $K_2 = 0$ identically, $F[-\frac{1}{2},\ \frac{\gamma-1}{2(2-\gamma)};\ \frac{3-\gamma}{2(2-\gamma)};\ 0] = 1$, and we recover the familiar Riemann invariants

$$-2s_1 = \bar{u}_1 - \frac{2}{\gamma-1}\,\bar{\rho}^{\frac{\gamma-1}{2}} = \bar{u}_1 - \frac{2}{\gamma-1}\,\bar{a} \qquad (5\text{-}21a)$$

$$2r_1 = \bar{u}_1 + \frac{2}{\gamma-1}\,\bar{\rho}^{\frac{\gamma-1}{2}} = \bar{u}_1 + \frac{2}{\gamma-1}\,\bar{a} \qquad (5\text{-}21b)$$

A comparison between 5-20 and 5-21 reveals that the entire magnetohydrodynamic effect is contained in the factor of hypergeometric function, F, in this transverse magnetic field case.

For monatomic gas with $\gamma = \frac{5}{3}$, we are able to reduce the hypergeometric function to an elementary function by means of the following identity. (Magnus and Oberhettinger, 1954).

$$F[k, \ell;\ m+1;\ z] = \frac{m}{(k-\ell)z}\left[F(k-1, \ell;\ m; z) - F(k, \ell-1; m; z)\right]$$

Then

$$-2s_1 = \bar{u}_1 - \frac{2}{K_2^{\,2}}\left[\left(1+K_2^{\,2}\,\bar{\rho}^{\frac{1}{3}}\right)^{3/2} - 1\right] = \bar{u}_1 - \frac{2}{K_2^{\,2}}\left[\left(1+K_2^{\,2}\,\bar{a}\right)^{3/2} - 1\right] \quad (5\text{-}22a)$$

$$2r_1 = \bar{u}_1 + \frac{2}{K_2^{\,2}}\left[\left(1+K_2^{\,2}\,\bar{\rho}^{\frac{1}{3}}\right)^{3/2} - 1\right] = \bar{u}_1 + \frac{2}{K_2^{\,2}}\left[\left(1+K_2^{\,2}\,\bar{a}\right)^{3/2} - 1\right] \quad (5\text{-}22b)$$

These can also be obtained from 5-14 or 5-16 by carrying out the integration directly for this special case.

Therefore six Riemann invariants in purely transverse magnetic field case are u-ρ from 5-14, B_2-ρ from 5-36 that $\bar{B}_2' = \bar{\rho}$, together with constant values of S, u_2, u_3 and B_3.

The escape speed can be evaluated easily from 5-14 or 5-20. We note that it occurs only for fast simple waves now through which a complete expansion to vacuum is possible. (The slow wave speed vanishes identically in this case.) If $u_1 = 0$ initially, then

$$-2s_1 = \bar{u}_1 - \frac{2}{\gamma-1}\bar{\rho}^{\frac{\gamma-1}{2}}\left[-\frac{1}{2}, \frac{\gamma-1}{2(2-\gamma)}; \frac{3-\gamma}{2(2-\gamma)}; -K_2{}^2\bar{\rho}^{2-\gamma}\right]$$

$$= \bar{u}_{1_{esc.}} = -\frac{2}{\gamma-1}F\left[-\frac{1}{2}, \frac{\gamma-1}{2(2-\gamma)}; \frac{3-\gamma}{2(2-\gamma)}; -K_2{}^2\right] \quad (5\text{-}23a)$$

$$2r_1 = \bar{u}_1 + \frac{2}{\gamma-1}\bar{\rho}^{\frac{\gamma-1}{2}}F\left[-\frac{1}{2}, \frac{\gamma-1}{2(2-\gamma)}; \frac{3-\gamma}{2(2-\gamma)}; -K_2{}^2\bar{\rho}^{2-\gamma}\right]$$

$$= \bar{u}_{1_{esc.}} = -\frac{2}{\gamma-1}F\left[-\frac{1}{2}, \frac{\gamma-1}{2(2-\gamma)}; \frac{3-\gamma}{2(2-\gamma)}; -K_2{}^2\right] \quad (5\text{-}23b)$$

for forward and backward facing fast waves respectively. Thus

$$|u_{1_{esc.}}| = \frac{2a_I}{\gamma-1}\; F\left[-\frac{1}{2}, \frac{\gamma-1}{2(2-\gamma)}; \frac{3-\gamma}{2(2-\gamma)}; -K_2{}^2\right]. \quad (5\text{-}24)$$

VI. EXAMPLES

The general procedure of the application of methods given in previous sections to solve actual flow problems governed by MHD simple waves is discussed in this section.

Contact surfaces apply to flows separated by different entropies; their occurrence is analogous to ordinary gasdynamics. Transverse simple waves are mainly used to align the magnetic field or flow velocity in a transverse plane with a prescribed orientation. An ordinary change in direction is allowed to take place even through a single wave front which propagates with the Alfven wave speed based on the longitudinal magnetic field. Transverse waves usually appear adjacent to coupled waves and thus lie either in front of a slow or behind a fast wave region. Owing to the simple nature of their application, contact surfaces and transverse simple waves will not be discussed in detail here.

Our attention is now focused on coupled waves. The initial state of fluid given in $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ coordinates is characterized by the following physical variables

$$\tilde{\rho}_I, \quad \tilde{P}_I, \quad \tilde{u}_{1I}, \quad \tilde{u}_{2I}, \quad \tilde{u}_{3I}, \quad \tilde{B}_1, \quad \tilde{B}_{2I}, \quad \tilde{B}_{3I}$$

where $\tilde{P}_I$ instead of $\tilde{S}_I$ is chosen for convenience. First, a transformation of coordinates is preferable. This is done by rotating the transverse plane to orient the new x_2-axis in the direction of $\tilde{B}_{tI}$ and then performing a Galilean transformation with a velocity

$$\tilde{u}_{2I}^2 + \tilde{u}_{3I}^2 \sin\left[\tan^{-1}\frac{\tilde{u}_{3I}}{\tilde{u}_{2I}} - \tan^{-1}\frac{\tilde{B}_{3I}}{\tilde{B}_{2I}}\right]$$

in new x_3-direction. Therefore, the initial physical variables expressed in new coordinate system are

$$\rho_I = \tilde{\rho}_I, \quad P_I = \tilde{P}_I, \quad u_{1I} = \tilde{u}_{1I}, \quad u_{2I} = \sqrt{\tilde{u}_{2I}^2 + \tilde{u}_{3I}^2} \cos\left[\tan^{-1}\frac{\tilde{u}_{3I}}{\tilde{u}_{2I}} - \tan^{-1}\frac{\tilde{B}_{3I}}{\tilde{B}_{2I}}\right]$$

$$u_{3I} = 0, \quad B_1 = \tilde{B}_1, \quad B_{2I} = \sqrt{\tilde{B}_{2I}^2 + \tilde{B}_{3I}^2}, \quad B_{3I} = 0$$

The flow can be considered hereafter to be always in the $(x_1 - x_2)$ plane without loss of generality. The general initial value problem is posed as shown in fig. 14. We are given not only the initial uniform state ahead of disturbances but also the boundary condition on an initial curve $\mathcal{C}$ in $(x_1 - t)$ plane from which the phase lines of a simple wave issue. Certainly, only one of ρ, u_1, u_2 and B_2 can be specified on $\mathcal{C}$ because they are governed by a single parameter (e. g. a) through the Riemann invariants in the entire coupled wave region. It is obvious that $\mathcal{C}$ should not coincide with a phase line of the corresponding wave.

From ρ_I, P_I, B_1 and B_{2I} (together with the constant μ), we are able to form only three speeds and therefore two dimensionless parameters, namely K_1 and $\overline{B}_{2I}$; these suffice to characterize the part of initial state for determining ρ and B_2, or the magnetic field Riemann invariant. Due to the Galilean invariant property of u_1 and u_2, dimensionless parameters $\overline{u}_{1I}$ and $\overline{u}_{2I}$ formed from ρ, P, u_{1I} and u_{2I} are important only in the longitudinal and transverse flow Riemann invariants. They depend on the coordinate system chosen and are thus additive; the physics of the fluid is determined by differences of u_1 and

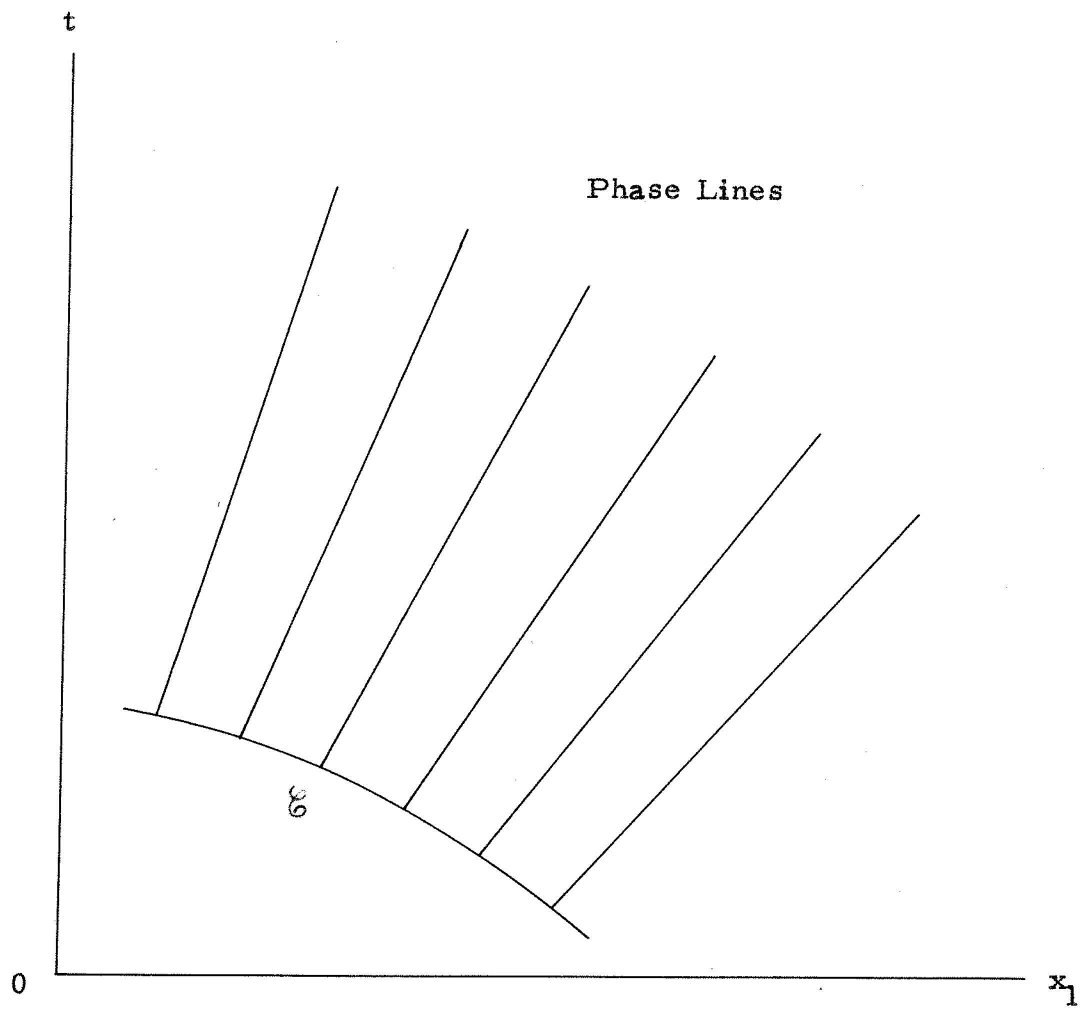


Fig. 14

u_2 between states. As a consequence, constants s_1, r_1, s_2, r_2 need not be known in most cases. The situation may be illustrated by the following simple example. Let us consider the fluid at a given state I initially, and let it undergo an expansion process across forward facing slow waves until finally it reaches a state F characterized by its density $\bar{\rho}_F$. The longitudinal and transverse flow velocities are obtained from 4-34a and 4-37a respectively

$$\bar{u}_{1F} = \bar{u}_{1I} - \frac{1}{2-\gamma}\int_{a_F}^{a_I} H^{(\gamma-1)/2\gamma}\left(\frac{\sigma-1}{\sigma}\right)^{(\gamma-1)/2(2-\gamma)}\left\{\frac{1}{\sigma(\sigma-1)} - \frac{K_1^2}{H}\left(\frac{\sigma}{\sigma-1}\right)^{2/(2-\gamma)}\right\}\frac{d\sigma}{\sigma^{1/2}}$$

and

$$\bar{u}_{2F} = \bar{u}_{2I} - \mathrm{sgn}(B_1 B_{2I})\frac{K_1}{2-\gamma}\int_{a_F}^{a_I}\frac{H^{(\gamma-1)/2\gamma}\left(\frac{\sigma-1}{\sigma}\right)^{1/2(2-\gamma)}}{\left[K_1^2 - \frac{H}{\sigma}\left(\frac{\sigma-1}{\sigma}\right)^{\gamma/(2-\gamma)}\right]^{1/2}}\left\{\frac{1}{\sigma(\sigma-1)} - \frac{K_1^2}{H}\left(\frac{\sigma}{\sigma-1}\right)^{2/(2-\gamma)}\right\}d\sigma$$

where

$$H = H(\sigma; K_1, a_I) = \left(\frac{a_I}{a_I-1}\right)^{\gamma/(2-\gamma)} - \frac{\gamma K_1^2}{2-\gamma}\int_{a_I}^{\sigma}\left(\frac{\zeta}{\zeta-1}\right)^{2/(2-\gamma)} d\zeta$$

a_I is obtained from 4-26 as

$$a_I = \frac{1}{2K_1^2}\left\{1 + K_1^2(1+\bar{B}_{2I}^2) + \sqrt{[1+K_1^2(1+\bar{B}_{2I}^2)]^2 - 4K_1^2}\right\}$$

a_F is determined from 4-32a

$$\bar{\rho}_F = H^{1/\gamma}\left(\frac{a_F-1}{a_F}\right)^{1/(2-\gamma)}$$

$$= \left[\left(\frac{a_I}{a_I-1}\right)^{\gamma/(2-\gamma)} - \frac{\gamma K_1^2}{2-\gamma}\int_{a_I}^{a_F}\left(\frac{\zeta}{\zeta-1}\right)^{2/(2-\gamma)} d\zeta\right]^{1/\gamma}\left(\frac{a_F-1}{a_F}\right)^{1/(2-\gamma)}$$

Knowing a_F, one may get all other physical quantities at the final state from 4-31 and 4-32.

This result can also be seen easily from graphical solutions. We may determine $\overline{\rho}_*$ and $\overline{B}_2$ of a state from $(\alpha-\beta_1)$ plane, but $\overline{u}_1$ and $\overline{u}_2$ must be found from $(\overline{\rho}_*-w_1)$ and $(w_2-\overline{\rho}_*)$ planes respectively where only differences of w_1 and w_2 are important.

We shall discuss in the following several problems describing fluid motion set up by mechanical devices as well as by electromagnetic means.

A. Receding Piston Problem

Let us consider an infinitely conducting gas at rest initially with imposed uniform magnetic field oriented in an arbitrary direction. A piston is attached to one end of the gas that extends to infinity in the x_1-direction at the other end. For this one-dimensional problem, the piston surface can be considered to be perpendicular to the x_1-direction and be situated at $x_1 = 0$ initially. At $t > 0$, the piston moves with a longitudinal velocity $U_1(t)$ in $-x_1$-direction $(U_1(t) < 0)$; thus the gas expands near the surface of the piston. This is a typical example of a fluid motion started mechanically. The physical phenomenon can be described in the (x_1-t) plane as shown in fig. 15a. A wave propagates into the still gas after the motion of the piston and gas particles are disturbed after the arrival of the first wave front. Since the flow region is connected to a constant state at rest, it is described by simple waves.

1. Accelerated Piston Motion

This corresponds to $\frac{dU_1}{dt} < 0$. In view of the relative simplicity of the case having transverse magnetic field only, it is studied first. A scheme of solving the corresponding case in an arbitrarily oriented mag-

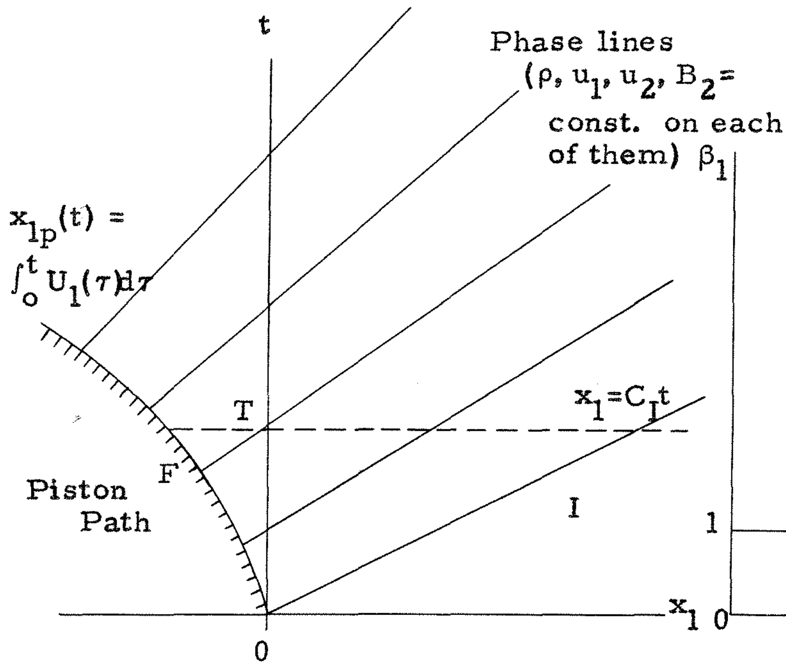


Fig. 15-a

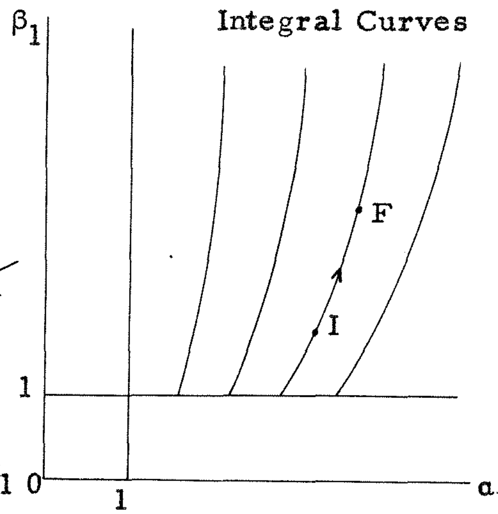


Fig. 15-b

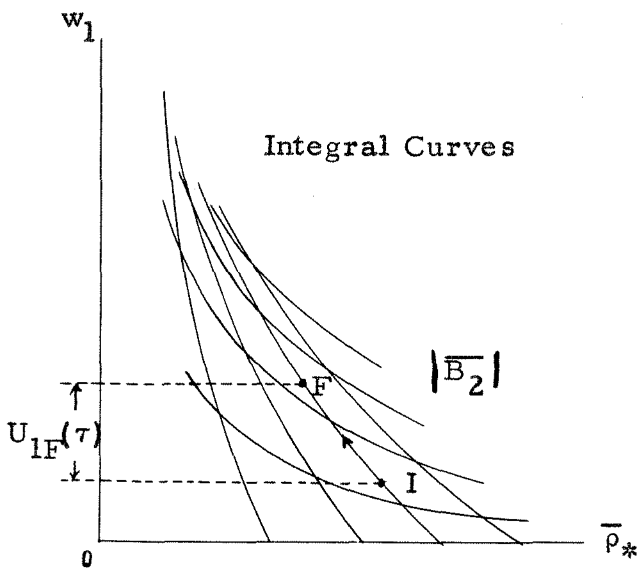


Fig. 15-c

netic field by a graphical method is stated subsequently; no provision is made here to solve it by analytical method because of the complicated nature of the definite integrals involved.

a. Expansion in a transverse magnetic field ($\vec{B}_I = B_{2I}\vec{e}_2$).

The mechanical boundary condition of the problem is specified by the requirement that before cavitation occurs the fluid velocity has always the same value as the piston velocity $U_1(t)$ along the piston path. No magnetic boundary condition need be imposed here owing to the infinite conductivity of the gas. Only fast waves exist in this case, furthermore, they are forward facing ones. The family of phase lines $\frac{dx_1}{dt} = u_1 + \sqrt{a^2+b_1^2}$ issuing from the piston path has the important property that all physical quantities remain constant along each of them. The Riemann invariant 5-20a holds across these phase lines. For $\gamma = \frac{5}{3}$, we have from 5-22a the following simple form

$$-2s_1 = u_1 - \frac{2a_I}{K_2^2}[(1 + K_2^2 \bar{\rho}^{\,1/3})^{3/2} - 1]$$

Here the constant s_1 is determined by the initial condition $\bar{u}_1 = 0$ at $\bar{\rho}_1 = 1$, hence

$$u_1 = \frac{2a_I}{K_2^2}\left[(1 + K_2^2 \bar{\rho}^{\,1/3})^{3/2} - (1 + K_2^2)^{3/2}\right] \qquad (6\text{-}1)$$

Applying this relation to the piston path, we obtain the density variation along it

$$\bar{\rho} = \frac{1}{K_2^6}\left\{\left[(1 + K_2^2)^{3/2} + \frac{K_2^2 U_1(\tau)}{2a_I}\right]^{2/3} - 1\right\}^3 \qquad (6\text{-}2)$$

Thus $\bar{\rho}$ is expressed here as a function of τ, the time elapsed from the start of the motion, which can be regarded as a parameter characterizing the phase lines of the entire flow. Due to the constant speed of different phases, we have for any point (x_1, t) in the region $x_{1p}(t) \leq x_1 \leq c_I t = (1+K_2^2)^{1/2} a_I t$ on physical plane

$$\frac{x_1 - x_{1p}(\tau)}{t - \tau} = U_1(\tau) + c(\tau)$$

But

$$x_{1p}(\tau) = \int_0^\tau U_1(T)\, dT$$

and from 5-13b

$$c(\tau) = a_I \bar{c}(\tau) = a_I \bar{\rho}^{\,1/3}[1 + K_2^2 \bar{\rho}^{\,1/3}]^{1/2}$$

Then

$$x_1 - \int_0^\tau U_1(T)\, dT = (t-\tau)[U_1(\tau) + a_I \bar{\rho}^{\,1/3}(1 + K_2^2 \bar{\rho}^{\,1/3})^{1/2}] \qquad (6\text{-}3)$$

By elimination of τ from 6-2 and 6-3, we are able to get $\bar{\rho} = \bar{\rho}(x_1, t)$ which is usually obtained as an implicit function. $\bar{B}_2(x_1, t)$ is identical with $\bar{\rho}(x_1, t)$; all other physical variables $\bar{P}(x_1, t)$, $\bar{u}_1(x_1, t)$ and $\bar{\varepsilon}(x_1, t)$ can also be found easily from known relations. The strength of the current per unit length, $I(t)$ flowing on the piston surface is obtained as follows:

$$\text{Curl}\, \vec{B} = \mu \vec{J} = \mu I(t)\, \delta(x_1) \vec{e}_3$$

hence

$$\frac{\partial B_2}{\partial x_1} = \mu I(t) \delta(x_1)$$

$$B_2^+(t) - B_2^- = \mu I(t)$$

Or

$$I(t) = \frac{B_{2I}}{\mu}\left[\overline{B}_2'^{+}(t) - 1\right] \qquad (6-4)$$

where B_2^+ is the transverse magnetic field strength at the piston surface.

It is noted that in the region $x_1 > c_I t$, the fluid remains in the initial constant state always.

b. Expansion in an arbitrarily oriented magnetic field.

The problem is more complicated than the previous one and its solution can be found readily by use of the graphical method described in Section IV 4. In order to ensure the possibility of a complete expansion of fluid to vacuum in each case, we consider the fluid motion governed by slow waves (forward facing) only. The procedure is as follows.

(i) At the initial state

$$\overline{u}_{1I} = \overline{u}_{2I} = 0$$

and the fluid is characterized by two parameters,

$$K_1 = \left|\overline{b}_{1I}\right| = \frac{\left|B_I\right|}{\mu\rho_I}\,\frac{1}{a_I}$$

and

$$\overline{B}_{2I} = \frac{B_{2I}}{B_I}$$

We have from 4-44a

$$\overline{\rho}_{*I} = K_1^{\gamma/2}\,\overline{\rho}_I = K_1^{\gamma/2}$$

Knowing the values of $\bar{\rho}_{*I}$ and $\overline{B}_{2I}$, we may determine the point I of the initial state on the $(\alpha-\beta_1)$ plane by making use of fig. 5.

(ii) Read off the coordinates (α_I, β_{1I}) of point I, this selects a particular integral curve in fig. 4 describing the given flow along which the phase of physical states changes continuously. It is illustrated in fig. 15b.

(iii) The point I on $(\bar{\rho}_*-w_1)$ plane can also be determined by $\bar{\rho}_{*I}$ and $\overline{B}_{2I}$ in fig. 7, this in turn selects the corresponding integral curve on the plane. Therefore for each piston velocity U_{1F} corresponding to a state F, we may find the position of F on $(\bar{\rho}_*-w_1)$ plane as illustrated in fig. 15c. The values of $\bar{\rho}_{*F}$ and $\overline{B}_{2F}$ are known from here, and therefore $\bar{u}_{2F}$ can be found from fig. 8. Physical variables at F are now all determined. Incidentally, the point F on the $(\alpha-\beta_1)$ plane may also be located.

(iv) Phase lines (one family of characteristics) are always lines of constant slope issuing from the piston path. For a particular state F on piston path, the slope is given by

$$\left(\frac{dx_1}{dt}\right)_F = U_{1F} + C_F$$

where C_F is found from fig. 6 after knowing other physical variables. Thus we may draw the straight phase line leading from F along which all known physical quantities associated with U_{1F} remain constant and their values can be assigned thereby.

(v) Carrying out the same procedure for several points on the piston path, we get a family of phase lines. At any time T, the distribution of physical state in space is obtained by crossing all phase lines with a horizontal line $t = T$ on the (x_1-t) plane as described in fig. 15a.

2. Constant Motion of the Piston

To show explicitly the MHD effect on the gas flow for receding piston problem, let us consider now the relatively simple case that U_1 = constant. The corresponding solution of the ordinary gasdynamic case is well known and at any instant the velocity distribution is a linear function of space throughout the simple wave region. This serves as a standard of reference with which the solution of an ideal conducting gas in magnetic fields of different directions is compared.

a. The ordinary gasdynamic case (c. f. Courant and Friedrichs, 1948)

Because of the uniform motion of the piston starting impulsively at $t = 0$, the piston path is represented by a straight line through the origin in the (x_1-t) plane. The simple wave zone consists of phase lines which are centered rays as shown in fig. 16. The following equation holds along each ray

$$\frac{dx_1}{dt} = \frac{x_1}{t} = u_1 + a \tag{6-5}$$

The Riemann invariant relation across these forward facing phase lines is

$$\frac{u_1}{2} - \frac{a}{\gamma-1} = - \frac{a_I}{\gamma-1} \tag{6-6}$$

since $u_{1I} = 0$ here. Eliminating a from 6-5 and 6-6 and introducing the

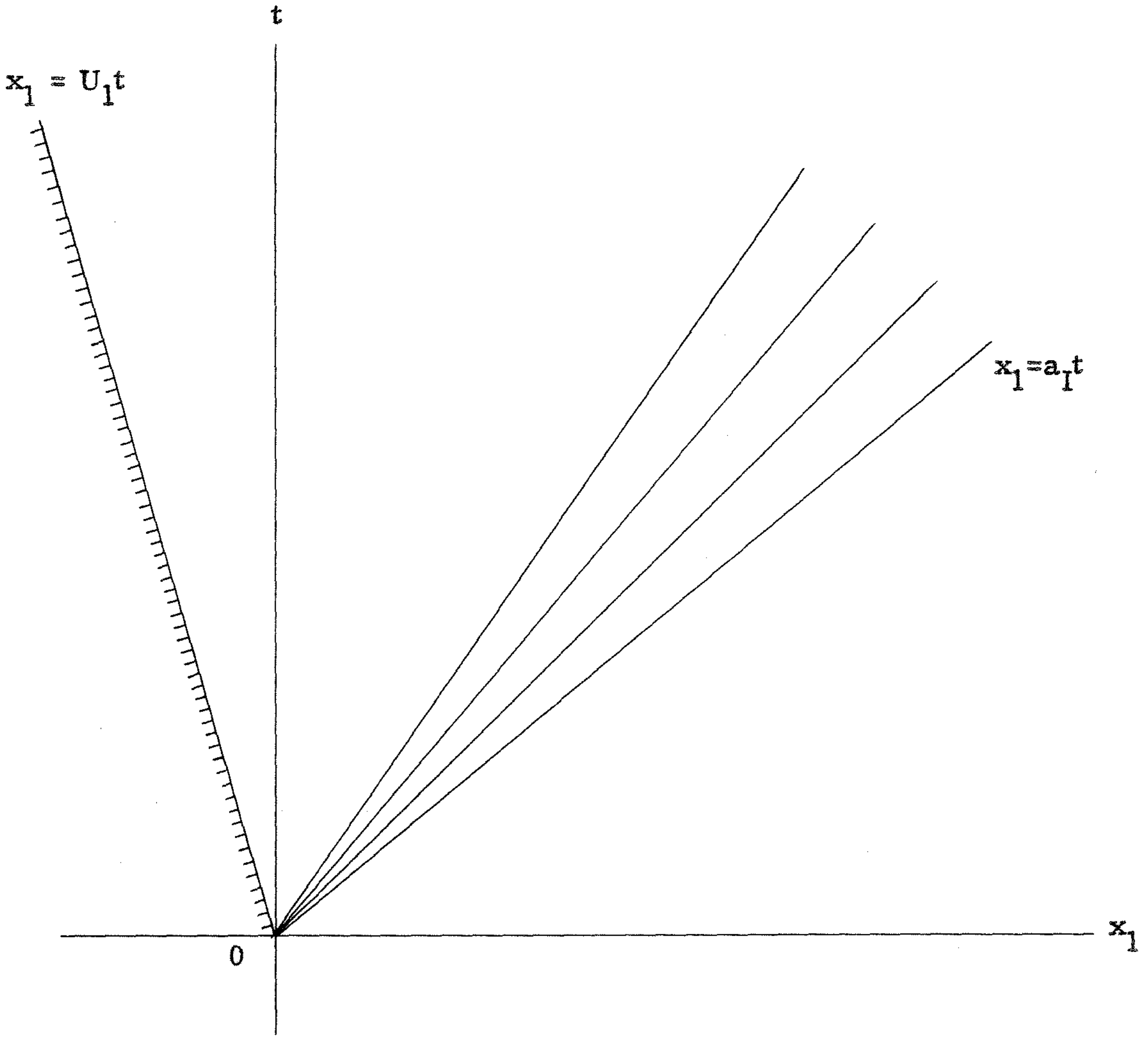


Fig. 16

dimensionless variables $\bar{u}_1 = \frac{u_1}{a_I}$ and $\bar{x}_1 = \frac{x_1}{a_I t}$, we get

$$\bar{u}_1 = \frac{2}{\gamma+1} (\bar{x}_1 - 1) \tag{6-7}$$

Thus the velocity profile at any given instant is always a straight line. The gas at the surface of the piston has always the same longitudinal velocity as that of the piston until a critical value of the latter, escape velocity, is attained which is given by putting $a = 0$ in 6-6, i. e.

$$u_{1\,esc} = - \frac{2a_I}{\gamma-1} \tag{6-8}$$

The gas is at rest in the region $\bar{x}_1 > 1$. When $U_1 > u_{1\,esc}$, simple waves are bounded by $\bar{U}_1 < \bar{x}_1 < 1$. When $U_1 < u_{1\,esc}$, a cavitation zone exists in the region $\bar{U}_1 < \bar{x}_1 < - \frac{2}{\gamma-1}$ which separates the piston and the gas; simple waves are confined to the region $- \frac{2}{\gamma-1} < \bar{x}_1 < 1$.

For the sake of convenience in comparison of cases with or without MHD effects, we shall assume $\gamma = \frac{5}{3}$ and $\bar{U}_1 = - \frac{1}{5}$ for all examples calculated in the following. Hence we have in the absence of magnetic fields

$$\bar{u}_1 = \frac{3}{4} (\bar{x}_1 - 1) \tag{6-9}$$

and simple waves are in the region $\frac{11}{15} < \bar{x} < 1$.

b. In a purely longitudinal magnetic field

The motion must start from $\bar{B}_2 = 0$ and the situation can be easily seen from fig. 4 where $\bar{B}_2 = 0$ is represented by two straight lines $a = 1$ and $\beta_1 = 1$. Since the gas must undergo an expansion, we have to exclude the part of the line $\beta_1 = 1$ for $a < 1$. The exact position on $(a-\beta_1)$ plane representing the initial condition is then determined by the parameter $K_1 = \frac{|b_{1I}|}{a_I} = (\frac{\beta_{1I}}{a_I})^{1/2}$. If $K_1 > 1$, the only solution is $c = a$ and the motion corresponds exactly to an ordinary gasdynamic one. If $K_1 < 1$, two different solutions are possible, either $c = a$ or $c = b_1$. The former corresponds to ordinary gasdynamic case and the latter corresponds to MHD case with a transverse magnetic field being induced during the expansion process of the gas. The choice of one of these solutions depends on boundary conditions (e. g. the magnetic boundary condition at the tail of simple waves), but it is indeterminate in the present problem.

c. In a purely transverse magnetic field .

An analytical solution of the problem is possible in this case. We have only forward facing centered waves and

$$\frac{dx_1}{dt} = \frac{x_1}{t} = u_1 + c \tag{6-10a}$$

or in terms of dimensionless variables

$$\bar{x}_1 = \bar{u}_1 + \bar{c} \tag{6-10b}$$

For $\gamma = \frac{5}{3}$, we get from 5-13a

$$\bar{c} = \bar{\rho}^{\,1/3}(1+K_2^2\bar{\rho}^{\,1/3})^{1/2} = \bar{a}(1 + K_2^2\,\bar{a})^{1/2} \tag{6-11}$$

After satisfying the initial condition that $\bar{u}_1 = 0$, the Riemann invariant relation 5-22a across simple waves assumes the form

$$\bar{u}_1 = \frac{2}{K_2^2}\left[(1 + K_2^2\bar{a})^{3/2} - (1 + K_2^2)^{3/2}\right] \tag{6-12}$$

Eliminating $\bar{a}$ from 6-11 and 6-12 together with the substitution of the corresponding value of $\bar{c}$ into 6-10b, we get

$$\bar{x}_1 = \frac{3\bar{u}_1}{2} + \frac{1}{K_2^2} \{(1 + K_2^2)^{3/2} - [\frac{K_2^2\bar{u}_1}{2} + (1 + K_2^2)^{3/2}]^{1/3}\} \quad (6\text{-}13)$$

This is an implicit solution of the velocity distribution as a function of $\bar{x}_1$. One may easily verify that it reduces to 6-9 in the limit $K_2 \to 0$. As $K_2 >> 1$, 6-13 becomes

$$\bar{u}_1 \cong \frac{2}{3}(\bar{x}_1 - K_2) \quad (6\text{-}14)$$

which is again a linear profile. This result is no accident and can be obtained also by approximating 6-11 at $K_2 >> 1$ as

$$\bar{c} = K_2\bar{\rho}^{1/2} = K_2\bar{a}^{3/2} \quad (6\text{-}15a)$$

If we denote $\tilde{a} = \frac{\bar{c}}{K_2}$, 6-15 becomes

$$\tilde{a} = \bar{\rho}^{1/2} \quad (6\text{-}15b)$$

which corresponds to the sound wave speed in ordinary gasdynamics for a gas of $\gamma = 2$. This is the well known behavior of the conducting gas in a very strong transverse magnetic field.

The escape velocity of the piston is obtained from 6-11 by putting $\bar{\rho} = 0$ (or $\bar{a} = 0$). Then

$$\bar{u}_{1esc} = -\frac{2}{K_2^2}[(1 + K_2^2)^{3/2} - 1] \quad (6\text{-}16)$$

As $K_2 \to 0$,

$$\bar{u}_{1esc} = -3$$

As $K_2 \to \infty$,

$$\bar{u}_{1esc} = -2K_2$$

We have calculated longitudinal velocity profiles for $K_2 = 0.75$, 1.00 and 1.50 respectively which are compared with the ordinary gasdynamic one as shown in fig. 17. They are very nearly straight lines and are displaced to the right of the ordinary gasdynamic profile.

d. In an arbitrarily oriented magnetic field

The solution is obtained by using the graphical method mentioned previously. Some longitudinal velocity profiles at $\overline{E}_2 = 0.75$, 1.50 with $K_1 = 0.75$, 1.00, 1.50 are shown in fig. 18. It is remarkable that they appear also very nearly as straight lines. The dominant effect due to magnetic field consists in displacing them to the left of the ordinary gasdynamic velocity profile, since only slow simple waves are considered in this case.

B. Current Sheet Problem

This is an example of fluid motion initiated by electromagnetic means. A steady current sheet flowing in x_3-direction is suddenly initiated at $x_1 = 0$. $x_1 > 0$ contains a perfectly conducting gas at rest in the presence of an arbitrarily oriented uniform magnetic field. At the surface of the current sheet a tangential magnetic field in the x_2-direction is induced. It exerts a magnetic pressure on the gas next to it which in turn initiates the fluid motion. There is no characteristic length or time in this problem; all physical variables must remain constant along rays issuing from the origin of the $(x_1$-$t)$ plane. As in the one-dimensional piston problem in ordinary gasdynamics, a shock discontinuity is developed but we have here a general hydromagnetic shock instead.

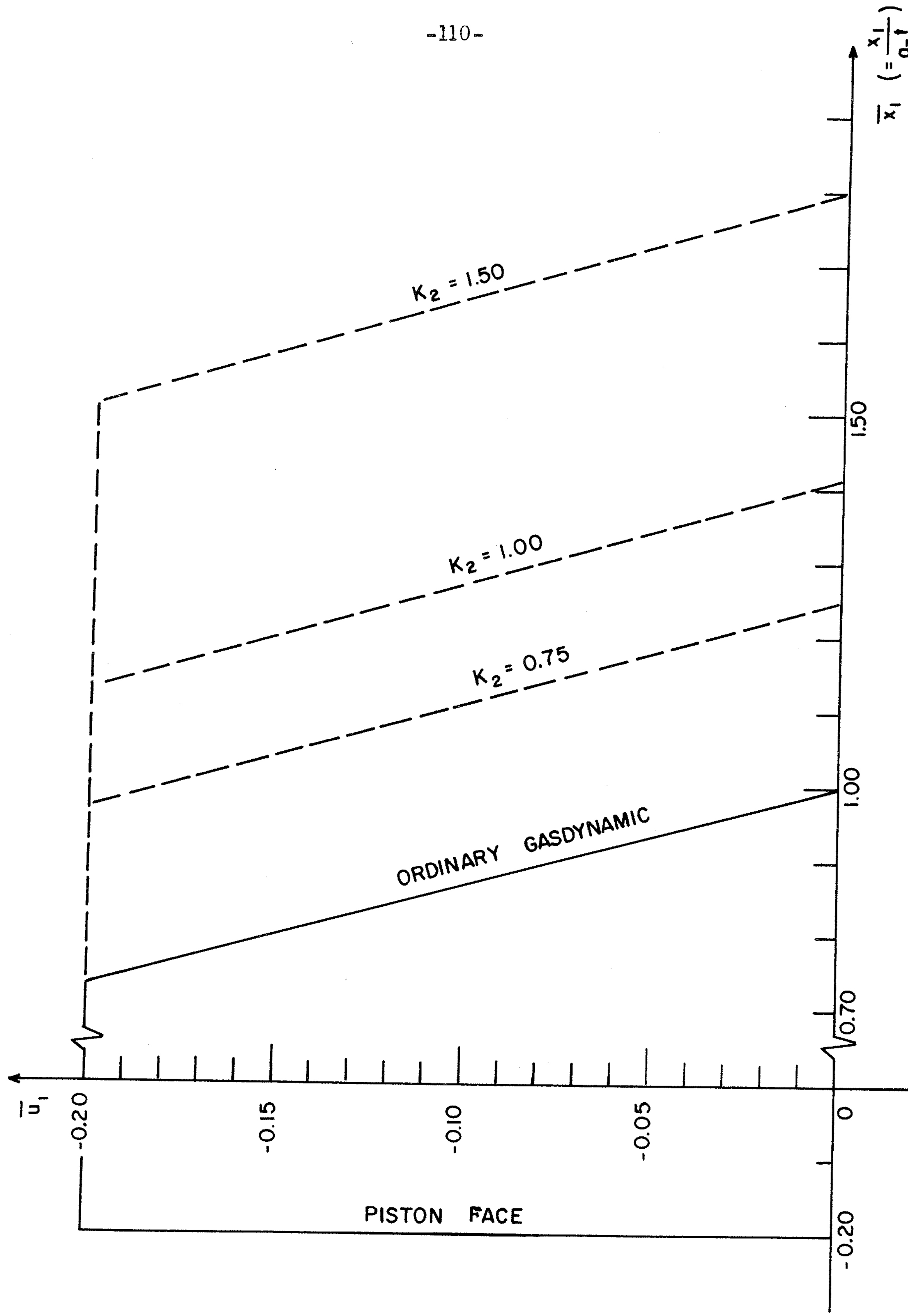

$\overline{x}_1$ $\left(= \frac{x_1}{a_I t}\right)$
$K_2 = 1.50$
$K_2 = 1.00$
$K_2 = 0.75$
ORDINARY GASDYNAMIC
1.50
1.00
0.70
-0.20
$\overline{u}_1$
-0.20
-0.15
-0.10
-0.05
0
PISTON FACE

Fig. 17

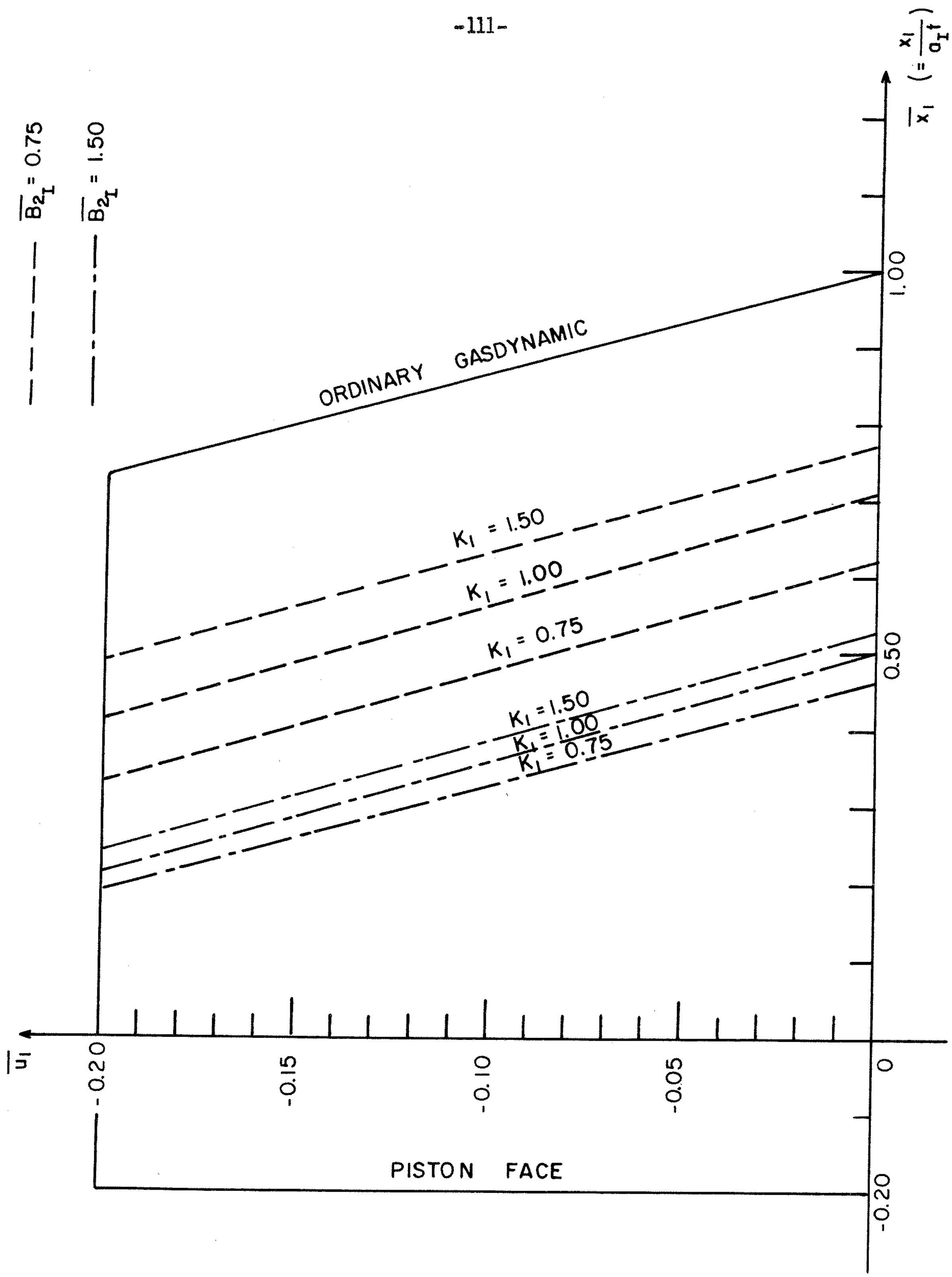

$\overline{B_{2_I}}$ = 0.75
$\overline{B_{2_I}}$ = 1.50
ORDINARY GASDYNAMIC
K_1 = 1.50
K_1 = 1.00
K_1 = 0.75
K_1 = 1.50
K_1 = 1.00
K_1 = 0.75
PISTON FACE
$\overline{u_1}$
$\overline{x_1}$ $\left(= \frac{x_1}{a_I t}\right)$
-0.20
-0.15
-0.10
-0.05
0
0.50
1.00
-0.20

Fig. 18

Since there is no movement of mechanical boundary followed by the pressure jump, the gas behind the shock undergoes an expansion process governed by MHD simple waves rather than remaining in a constant state. If the strength of the current sheet is not too large such that the density of the gas next to it is still finite, the boundary condition at $x_1 = 0$ is simply $u_1 = 0$. For large values of the current density, the gas may expand to vacuum and be displaced from the current sheet by magnetic pressure. In this latter case, the boundary condition on the current sheet should be $\rho = 0$. It was shown in Section 3 that a constant electric field exists always in the vacuum region.

This idea has been used to develop an extremely fast shock wave followed by a very high temperature plasma in a magnetic annular shock tube. [Patrick (1959) and theoretically by Kemp and Petschek (1959)]

VII. CONCLUDING REMARKS

It is important to observe that simple waves can be used to solve only a restricted class of initial value problems. An initially uniform state of fluid together with judiciously posed boundary conditions from which waves are sent out without interactions is essential. Even so the solution may describe physical situations only within a limited period of time beyond which some new facts, e.g. the steepening of compression simple waves to the formation of MHD shocks, should be considered.

For arbitrary flows involving interaction of simple waves, the usual scheme in ordinary gasdynamics of using characteristic coordinates to replace x_1 and t to simplify the original system 2-11 is not appropriate. We have here totally seven distinct families of characteristics while only two independent variables exist. The two degrees of freedom of the physical plane allows only two families of characteristics characterized by two parameters to be chosen freely. Other families of characteristics must be expressed in terms of these two specific parameters. Consequently only two ordinary differential equations can be obtained; the other equations involve quadratic terms of derivatives and do not lead to any simplification.

The wave front of plane simple waves corresponds to a special case of the general three-dimensional characteristic manifold. All physical variables remain constant on these plane wave fronts so that the inner derivatives vanish identically and no attenuation due to geometry will occur. This simplifies greatly the analysis and an exact solution is thus obtainable.

It is of interest to examine the rather peculiar behavior associated with transverse simple waves across which no changes of ρ, s and u_1 are allowed but only the rotation of $\vec{B}_t$ and $\vec{u}_t$. An abrupt change can take place within an infinitely narrow region. An aggregate, or merely a single transverse simple wave can be regarded as a transverse shock through which no entropy change occurs -- a fact quite different from our common understanding of shocks. The absence of a dissipation mechanism inside a transverse shock shows that it is not developed and supported in the usual sense of fast, slow, or ordinary gasdynamic shocks. The transverse simple wave speed, or the transverse shock speed, is b_1 and the mass flow through the wave front can be calculated readily. The somewhat strange property of transverse simple waves may arise from our oversimplification of one-dimensional space variation.

As to coupled waves, the definite integrals appeared in the analytic solution have to be evaluated by computation; an appropriate scheme of calculation is discussed in Appendix D. For graphical solutions, a numerical integration of integral curves by use of electronic computers is also suggestive; this should be combined with carefully prepared graphs of constant values of different physical variables. Due to the successive correlation of various graphs, errors are induced; it is not as accurate as the analytic solution. Besides, the limitation of the $(\alpha-\beta_1)$ plane constitutes a further handicap of graphical solution.

As an opposite case to the receding piston problem consider a piston moving continuously into the gas, the phase lines issuing from the piston path would converge finally. The waves are necessarily compressive and the slowly progressing part in front is overtaken gradually by a faster moving part from behind. The phase lines form a cusp in the (x_1-t) plane eventually. The continuous motion is then terminated and a shock discontinuity is developed. The time and position of the formation of MHD shocks, especially the fast shock generated by certain definite accelerated motion in a purely transverse magnetic field can be predicted by theory.

Since B_1 = constant always, the passage to the limit of ordinary sound waves from MHD simple waves must be carried out by first assuming $B_1 = 0$. The slow wave speed equals zero in this case and we have only fast waves which tend to sound waves in the limit of vanishing B_2.

The coupled wave solution depends explicitly on the adiabatic exponent, γ, of the gas. Transverse simple waves and contact surfaces involve no compression or expansion process; they do not make use of the equation of state and hence are not functions of γ.

When the gas expands to very low densities such that $\frac{B_1}{\sqrt{\mu\rho}} \to \infty$, $\frac{B_2}{\sqrt{\mu\rho}} \to \infty$, etc. and we still consider the problem from the macroscopic point of view, it is obvious that a relativistic correction must be taken into account to prevent the disturbances propagating with a speed higher than that of light.

Whenever there is a variation in transverse magnetic field in space, current sheets occur and their magnitude can be calculated easily. Likewise the energy flux $\vec{s} = \vec{E} \times \vec{H} = \frac{1}{\mu}(\vec{B} \times \vec{u}) \times \vec{B}$ = Poynting vector, across the wave front can also be evaluated. These are straightforward and are not discussed explicitly here.

REFERENCES

Akhiezer, A.I., Liubarsky, G.Ia., and Polovin, R.V.: Simple Waves in Magnetohydrodynamics. Ukr. Fiz. Zh. Vol. 3, No. 4, 1958. pp. 433-438 (Translation by Morris D. Friedman, Inc. P. O. Box 35, W. Newton, Mass. Pap A-139). Also appears in Soviet Physics, Technical Physics Vol. 4. No. 8. Feb. 1960. pp. 849-854.

Akhiezer, A.I., Liubarsky, G. Ia., and Polovin, R. V.: A Theory of Simple and Shock Hydromagnetic Waves. Progress in Nuclear Energy, Series VI. Plasma Physics and Thermonuclear Research, edited by Longmire, C., Tuck, J. L. and Thompson, W. B. Pergamon Press, 1959. pp. 223-231. Also appears in Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy, Vol. 31. Theoretical and Experimental Aspects of Controlled Nuclear Fusion, 1958. pp. 225-229.

Akhiezer, A. I., Polovin, R. V., Tsintsadze, N. L.: Simple Waves in the Chew-Goldberger-Low Approximation. Soviet Physics, J.E.T.P. Vol. 37 (10) No. 3. March 1960. pp. 539-541.

Bazer, J. Resolution of an Initial Shear-Flow Discontinuity in One Dimensional Hydromagnetic Flow. The Astrophysical Journal, Vol. 128. No. 3, Nov. 1958. pp. 686-712.

Bazer, J. and Ericson, W. B. Hydromagnetic Shocks. The Astrophysical Journal, Vol. 129. No. 3, May 1959. pp. 758-785.

Birkhoff, G. Hydrodynamics. Princeton University Press, 1950.

Cole, J. D. Magnetohydrodynamic Waves. Air Force OSR-TN-59-13. January 1959. Also appears in The Magnetodynamics of Conducting Fluids, Edited by Bershader, D., Stanford University

Press, Stanford, California, 1959. pp. 17-36.

Courant, R. and Friedrichs, K.O.: Supersonic Flow and Shock Waves. Interscience Publishers, New York, 1948.

Courant, R. and Hilbert, D.: Methoden der Mathematischen Physik. Vol. II. Springer, Berlin, 1937. (Reprint; Interscience Publishers, New York, 1943.)

Cowling, T. G. Magnetohydrodynamics. Interscience Publishers, New York, 1957.

Friedman, B.: Principles and Techniques of Applied Mathematics. John Wiley and Sons, New York, 1956.

Friedrichs, K. O. Nonlinear Hyperbolic Differential Equations for Functions of Two Independent Variables. American Journal of Mathematics. Vol. 70. 1948. pp. 555-589.

Friedrichs, K. O.: Nonlinear Wave Motion in Magnetohydrodynamics, Los Alamos Scientific Laboratory Technical Report No. LAMS-2105, September 1954. See also, Friedrichs, K. O. and Kranzer, H.: Notes on Magnetohydrodynamics VIII. Nonlinear Wave Motion. Institute of Mathematical Sciences, New York University. July, 1958.

Friedrichs, K. O.: Nichtlineare Differentialgleichungen. Notes of lecture delivered at Göttingen, July 1955.

Golitsyn, G. S.: Plane Problems in Magnetohydrodynamics. Soviet Physics, J.E.T.P. 34(7) No. 3, September 1958. pp.473-477.

Golitsyn, G. S.: Unidimensional Motion in Magnetohydrodynamics. Soviet Physics, J.E.T.P. 35(8) No. 3. March 1959. pp.538-541.

Grad, H.: Propagation of Magnetohydrodynamic Waves without Radial Attenuation. The Magnetodynamics of Conducting Fluids.

Edited by Bershader, D. Stanford University Press, Stanford, California, 1959. pp. 37-60.

Kemp, N. and Petschek, H. Theory of the Flow in the Magnetic Annular Shock Tube. Avco-Everett Research Report 60, July 1959. Also appears in The Physics of Fluids. Vol. 2, No. 6, Nov.-Dec. 1959. pp. 599-608.

Lax, P. D. Hyperbolic Systems of Conservation Laws II. Communications on Pure and Applied Mathematics. Vol. 10. 1957. pp. 537-566.

Lunquist, S. Studies in Magnetohydrodynamics. Arkiv für Fysik, Band 5 Nr15, 1952, pp. 297-347.

Magnus, W. and Oberhettinger, F. Formulas and Theorems for the Functions of Mathematical Physics. Chelsea Publishing Company, New York, 1954.

Mitchner, M. Magnetohydrodynamic Flow in a Shock Tube. The Physics of Fluids. Vol. 2. No. 1. Jan-Feb. 1959.

Mitchner, M. Magnetohydrodynamic Flow with a Transverse Magnetic Field. Magnetodynamics of Conducting Fluids. Edited by Bershader, D. Stanford University Press, Stanford, California, 1959. pp. 61-89.

Patrick, R. M. The Production and Study of High Speed Shock Waves in a Magnetic Annular Shock Tube. Avco-Everett Research Report 59, July 1959. Also appears in The Physics of Fluids, Vol. 2, No. 6. Nov.-Dec. 1959. pp. 589-598.

APPENDIX A

$Y_{ij} = I_{ij}$ = identity matrix in our present problem. We shall find first the eigenvector $\vec{r} = (r_1, r_2, \ldots r_7)$ of the matrix X_{ij} shown in 2-32. Thus from 2-24a

$$(X_{ij} - WI_{ij})\, r_j = 0$$

Or

$$(X_{ij} - UI_{ij}) r_j = 0$$

Since $c = U - u_1$ by definition, the above system can be written explicitly as follows

$$-cr_1 + \rho r_2 = 0 \qquad \text{(A-1a)}$$

$$\frac{a^2}{\rho} r_1 - cr_2 + \frac{B_2}{\mu\rho} r_5 + \frac{B_3}{\mu\rho} r_6 + A'\rho^{\gamma-1} r_7 = 0 \qquad \text{(A-1b)}$$

$$-cr_3 - \frac{B_1}{\mu\rho}\, r_5 = 0 \qquad \text{(A-1c)}$$

$$-cr_4 - \frac{B_1}{\mu\rho} r_6 = 0 \qquad \text{(A-1d)}$$

$$B_2 r_2 - B_1 r_3 - cr_5 = 0 \qquad \text{(A-1e)}$$

$$B_3 r_2 - B_1 r_2 - cr_6 = 0 \qquad \text{(A-1f)}$$

$$-cr_7 = 0 \qquad \text{(A-1g)}$$

Obviously c is determined by the same equation as that given in 2-36.

(a) For coupled waves that $(c^2-a^2)(c^2-b_1{}^2)-c^2b_t{}^2 = 0$ $(B_t \neq 0)$, one may obtain immediately from (A-1) the eigenvector

$$\vec{r} = [\frac{\rho}{c},\ 1,\ \frac{b_1 b_2}{b_1^2 - c^2},\ \frac{b_1 b_3}{b_1^2 - c^2},\ \frac{B_2 c}{c^2 - b_1^2},\ \frac{B_3 c}{c^2 - b_1^2},\ 0]$$

within factor the eigenvector is not normalized. The following relation is obtained according to 2-26

$$c\,\frac{d\rho}{\rho} = du_1 = \frac{b_1^2 - c^2}{b_1 b_2}\,du_2 = \frac{b_1^2 - c^2}{b_1 b_3}\,du_3 = \frac{c^2 - b_1^2}{c}\,\frac{dB_2}{B_2} = \frac{c^2 - b_1^2}{c}\,\frac{dB_3}{B_3}$$

$$= \text{constant} \qquad \text{(A-2)}$$

A-2 contains the entire structure of coupled simple wave solutions. It can be subjected to detailed study readily.

(b) For $c = \pm b_1 \neq 0$ also $B_t \neq 0$, we obtain from A-1 the eigenvector

$$\vec{r} = [0,\ 0,\ b_3,\ -b_2,\ -B_3,\ B_2,\ 0]$$

and from 2-26 the following

$$\frac{du_2}{b_3} = -\frac{du_3}{b_2} = -\frac{dB_2}{B_3} = \frac{dB_3}{B_2} = \text{constant} \qquad \text{(A-3)}$$

A-3 describes completely transverse simple waves.

(c) When $c = 0$, the solutions correspond to contact surfaces which cannot be analyzed by Riemann invariants. Hence the present method does not apply.

APPENDIX B

For $\nu = \frac{2}{2-\gamma}$ = an integer

(a) Slow waves ($\alpha > 1$)

$$\int_{\alpha_I}^{\alpha} \left(\frac{\zeta}{\zeta-1}\right)^{\nu} d\zeta = \int_{\alpha_I-1}^{\alpha-1} \left(\frac{1+\xi}{\xi}\right)^{\nu} d\xi = \int_{\alpha_I-1}^{\alpha-1} \frac{1}{\xi^{\nu}}\left[\sum_{r=0}^{\nu} \frac{\nu!}{(\nu-r)!\,r!}\,\xi^{r}\right] d\xi$$

$$= \int_{\alpha_I-1}^{\alpha-1} \left[1 + \frac{\nu}{\xi} + \sum_{r=0}^{\nu-2} \frac{\nu!}{(\nu-r)!\,r!}\,\xi^{r-\nu}\right] d\xi$$

$$= \left[\xi + \nu\,\ell n\,\xi - \sum_{r=0}^{\nu-2} \frac{\nu!}{(\nu-r)!\,r!}\,\frac{1}{\nu-r-1}\,\frac{1}{\xi^{\nu-r-1}}\right]_{\alpha_I-1}^{\alpha-1}$$

$$= \left[\xi + \nu\,\ell n\,\xi - \sum_{s=1}^{\nu-1} \frac{\nu!}{(1+s)!\,(\nu-s-1)!}\,\frac{1}{s}\,\frac{1}{\xi^{s}}\right]_{\alpha_I-1}^{\alpha-1}$$

$$= (\alpha-\alpha_I) + \nu\,\ell n\,\frac{\alpha-1}{\alpha_I-1} + \sum_{s=1}^{\nu-1} \frac{\nu!}{(1+s)!\,(\nu-s-1)!}\,\frac{1}{s}\left[\frac{1}{(\alpha_I-1)^{s}} - \frac{1}{(\alpha-1)^{s}}\right]$$

For example, $\gamma = \frac{5}{3}$ for monatomic gases, $\nu = 6$

$$\int_{\alpha_I}^{\alpha} \left(\frac{\zeta}{\zeta-1}\right)^{6} d\zeta = (\alpha-\alpha_I) + 6\,\ell n\,\frac{\alpha-1}{\alpha_I-1} + \{15\left[\frac{1}{\alpha_I-1} - \frac{1}{\alpha-1}\right]$$

$$+ 10\left[\frac{1}{(\alpha_I-1)^2} - \frac{1}{(\alpha-1)^2}\right] + 5\left[\frac{1}{(\alpha_I-1)^3} - \frac{1}{(\alpha-1)^3}\right]$$

$$+ \frac{3}{2}\left[\frac{1}{(\alpha_I-1)^4} - \frac{1}{(\alpha-1)^4}\right] + \frac{1}{5}\left[\frac{1}{(\alpha_I-1)^5} - \frac{1}{(\alpha-1)^5}\right]\}$$

(b) Fast waves $(0 \leq \alpha < 1)$

$$\int_{\alpha_I}^{\alpha} \left(\frac{\zeta}{1-\zeta}\right)^{\nu} d\zeta = (-1)^{\nu} \int_{\alpha_I}^{\alpha} \left(\frac{\zeta}{\zeta-1}\right)^{\nu} d\zeta$$

$$= (-1)^{\nu} \{ (\alpha - \alpha_I) + \nu \, \ell n \frac{\alpha - 1}{\alpha_I - 1} + \sum_{s=1}^{\nu-1} \frac{\nu!}{(1+s)!(\nu-s-1)!} \frac{1}{s}\left[\frac{1}{(\alpha_I-1)^s} - \frac{1}{(\alpha-1)^s}\right]\}$$

$$= (-1)^{\nu}\left[(\alpha - \alpha_I) - \nu \, \ell n \frac{1-\alpha_I}{1-\alpha}\right] + \sum_{s=1}^{\nu-1} (-1)^{\nu+s+1} \frac{\nu!}{(1+s)!(\nu-s-1)!} \frac{1}{s}\left[\frac{1}{(1-\alpha)^s} - \frac{1}{(1-\alpha_I)^s}\right]$$

For monatomic gases, $\gamma = \frac{5}{3}$ and $\nu = 6$

$$\int_{\alpha_I}^{\alpha} \left(\frac{\zeta}{1-\zeta}\right)^{6} d\zeta = (\alpha - \alpha_I) - 6\ell n \frac{1-\alpha_I}{1-\alpha} + \{ 15\left[\frac{1}{1-\alpha} - \frac{1}{1-\alpha_I}\right]$$

$$-10\left[\frac{1}{(1-\alpha)^2} - \frac{1}{(1-\alpha_I)^2}\right] + 5\left[\frac{1}{(1-\alpha)^3} - \frac{1}{(1-\alpha_I)^3}\right]$$

$$- \frac{3}{2}\left[\frac{1}{(1-\alpha)^4} - \frac{1}{(1-\alpha_I)^4}\right] + \frac{1}{5}\left[\frac{1}{(1-\alpha)^5} - \frac{1}{(1-\alpha_I)^5}\right]\}$$

It is possible in this fast wave case to express the definite integral explicitly in terms of hypergeometric functions which is valid even for $\nu \neq$ an integer, but the function is shown to consist of divergent series.

The derivation is given as follows

$$\int_{\alpha_I}^{\alpha} \left(\frac{\zeta}{1-\zeta}\right)^{\nu} d\zeta = \int_0^{\alpha} \left(\frac{\zeta}{1-\zeta}\right)^{\nu} d\zeta - \int_0^{\alpha_I} \left(\frac{\zeta}{1-\zeta}\right)^{\nu} d\zeta$$

To evaluate $\int_0^{\alpha} \left(\frac{\zeta}{1-\zeta}\right)^{\nu} d\zeta$, it can be written as

$$\int_0^{\alpha} \left(\frac{\zeta}{1-\zeta}\right)^{\nu} d\zeta = \alpha \int_0^{1} \left(\frac{s}{\frac{1}{\alpha} - s}\right)^{\nu} ds = \alpha^{1+\nu} \int_0^{1} \frac{s^{\nu}}{(1-\alpha s)^{\nu}} ds$$

the integral has now the standard form of a hypergeometric function and it is obtained (Magnus and Oberhettinger, 1954) as

$$\int_0^{1} \frac{s^{\nu}}{(1-\alpha s)^{\nu}} ds = \frac{\Gamma(1+\nu)\Gamma(1)}{\Gamma(2+\nu)} F[\nu, 1+\nu; 2+\nu; \alpha]$$

$$= \frac{1}{1+\nu} F[\nu, 1+\nu; 2+\nu; \alpha]$$

We have finally

$$\int_{\alpha_I}^{\alpha} \left(\frac{\zeta}{\zeta-1}\right)^{\nu} d\zeta = \frac{1}{1+\nu} \{\alpha^{\nu+1} F[\nu, 1+\nu; 2+\nu; \alpha] - \alpha_I^{\nu+1} F[\nu, 1+\nu; 2+\nu; \alpha_I]\}$$

However, a test of convergence shows that

$$\nu + (1+\nu) - (2+\nu) = \nu - 1 = \frac{2}{2-\gamma} - 1 = \frac{\gamma}{2-\gamma} \geq 1$$

always, so that the series expansion about $\alpha = 0$ is divergent in the entire range $0 \leq \alpha < 1$ and consequently is not appropriate for actual calculations. On the other hand, the closed form of the definite integral does have some advantage in formal functional operations.

APPENDIX C

The definite integral is

$$I(\bar{\rho}) = \int_0^{\bar{\rho}} \omega^{\frac{\gamma-3}{2}} (1 + K_2{}^2 \omega^{2-\gamma})^{\frac{1}{2}} \, d\omega$$

Let $s = (\frac{\omega}{\bar{\rho}})^{2-\gamma}$ and $G = K_2{}^2 \bar{\rho}^{2-\gamma}$, we have

$$I(\bar{\rho}) = \frac{1}{2-\gamma} \bar{\rho}^{\frac{\gamma-1}{2}} \int_0^1 s^{\frac{3\gamma-5}{2(2-\gamma)}} (1 + Gs)^{\frac{1}{2}} \, ds$$

This conforms to the standard form of the integral representation of a hypergeometric function (Magnus and Oberhettinger, 1954)

$$F[u, v; w; z] = \frac{\Gamma(w)}{\Gamma(v)\Gamma(w-v)} \int_0^1 t^{v-1} (1-t)^{w-v-1} (1-tz)^{-u} \, dt$$

with $Re(w) > Re(v) > 0$. Here we have

$$u = -\frac{1}{2}$$

$$v = 1 + \frac{3\gamma-5}{2(2-\gamma)} = \frac{\gamma-1}{2(2-\gamma)}$$

$$w = 1 + v = \frac{3-\gamma}{2(2-\gamma)}$$

$$z = -G = -K_2{}^2 \bar{\rho}^{2-\gamma}$$

Hence

$$I(\bar{\rho}) = \frac{2}{\gamma-1} \bar{\rho}^{\frac{\gamma-1}{2}} F\left[-\frac{1}{2}, \frac{\gamma-1}{2(2-\gamma)}; \frac{3-\gamma}{2(2-\gamma)}; -K_2{}^2 \bar{\rho}^{2-\gamma}\right]$$

Since $u+v-w = -\frac{3}{2} < 0$, the hypergeometric series is absolutely convergent.

From 1-2, $\gamma = \frac{n+2}{n}$, the definite integral can also be written as

$$I(\bar{\rho}) = n\,\bar{\rho}^{\frac{1}{n}}\,F\left[-\frac{1}{2},\ \frac{1}{n-2};\ \frac{n-1}{n-2};\ -K_2^{\,2}\,\bar{\rho}^{\,2-\gamma}\right]$$

APPENDIX D

Let us denote the definite integrals by

$$q_{1f}(a) = \frac{1}{2-\gamma} \int_{a_I}^{a} H_f^{(\gamma-1)/2\gamma} \left(\frac{1-\sigma}{\sigma}\right)^{(\gamma-1)/2(2-\gamma)} \left[\frac{1}{\sigma(\sigma-1)} + \frac{K_1^2}{H_f}\left(\frac{\sigma}{1-\sigma}\right)^{2/(2-\gamma)} \right] \frac{d\sigma}{\sigma^{1/2}} \qquad \text{(D-1a)}$$

$$q_{2f}(a) = \frac{K_1}{2-\gamma} \int_{a_I}^{a} \frac{H_f^{(\gamma-1)/2\gamma}\left(\frac{1-\sigma}{\sigma}\right)^{1/2(2-\gamma)}}{\left[\frac{1}{\sigma}\left(\frac{1-\sigma}{\sigma}\right)^{\gamma/(2-\gamma)} H_f - K_1^2\right]^{1/2}} \left\{ \frac{1}{\sigma(\sigma-1)} + \frac{K_1^2}{H_f}\left(\frac{\sigma}{1-\sigma}\right)^{2/(2-\gamma)} \right\} d\sigma \qquad \text{(D-1b)}$$

with

$$H_f(\sigma) = \left(\frac{a_I}{1-a_I}\right)^{\gamma/(2-\gamma)} + \frac{\gamma K_1^2}{2-\gamma} \int_{a_I}^{\sigma} \left(\frac{\zeta}{1-\zeta}\right)^{2/(2-\gamma)} d\zeta \qquad \text{(D-1c)}$$

for fast waves, i.e. $1 > a \geq 0$ and

$$q_{1s}(a) = \frac{1}{2-\gamma} \int_{a_I}^{a} H_s^{(\gamma-1)/2\gamma} \left(\frac{\sigma-1}{\sigma}\right)^{(\gamma-1)/2(2-\gamma)} \left[\frac{1}{\sigma(\sigma-1)} - \frac{K_1^2}{H_s}\left(\frac{\sigma}{\sigma-1}\right)^{2/(2-\gamma)} \right] \frac{d\sigma}{\sigma^{1/2}} \qquad \text{(D-2a)}$$

$$q_{2s}(a) = \frac{K_1}{2-\gamma} \int_{a_I}^{a} \frac{H_s^{(\gamma-1)/(2\gamma)}\left(\frac{\sigma-1}{\sigma}\right)^{1/2(2-\gamma)}}{\left[K_1^2 - \frac{1}{\sigma}\left(\frac{\sigma-1}{\sigma}\right)^{\gamma/(2-\gamma)} H_s\right]^{1/2}} \left\{ \frac{1}{\sigma(\sigma-1)} - \frac{K_1^2}{H_s}\left(\frac{\sigma}{\sigma-1}\right)^{2/(2-\gamma)} \right\} d\sigma \qquad \text{(D-2b)}$$

with

$$H_s(\sigma) = \left(\frac{a_I}{a_I - 1} \right)^{\gamma/(2-\gamma)} - \frac{\gamma K_I^2}{2-\gamma} \int_{a_I}^{\sigma} \left(\frac{\zeta}{1-\zeta} \right)^{2/(2-\gamma)} d\zeta \tag{D-2c}$$

for slow waves, i. e. $a > 1$.

Then, 4-34 can be written explicitly as

$$-2s_{1f} = u_1 - [\, q_{1f}(a) - q_{1f}(a_f')] \tag{D-3a}$$

$$2r_{1f} = u_1 + [\, q_{1f}(a) - q_{1f}(a_f')] \tag{D-3b}$$

across forward and backward facing fast waves respectively and

$$-2s_{1s} = u_1 - [\, q_{1s}(a) - q_{1s}(a_s')] \tag{D-4a}$$

$$2r_{1s} = u_1 + [\, q_{1s}(a) - q_{1s}(a_s')] \tag{D-4b}$$

across forward and backward facing slow waves respectively. Similarly, 4-37 become

$$-2s_{2f} = u_2 - [\, q_{2f}(a) - q_{2f}(a_f')] \tag{D-5a}$$

$$2r_{2f} = u_2 + [\, q_{2f}(a) - q_{2f}(a_f')] \tag{D-5b}$$

across forward and backward facing fast waves respectively and

$$-2s_{2s} = u_2 - [\, q_{2s}(a) - q_{2s}(a_s')] \tag{D-6a}$$

$$2r_{2s} = u_2 + [\, q_{2s}(a) - q_{2s}(a_s')] \tag{D-6b}$$

across forward and backward facing slow waves respectively.

One is usually interested in the difference between two states, say the initial state I and final state F only. Thus, e. g. for forward facing fast waves, we have from D-3a and D-5a

$$u_{1F} = u_{1I} + q_{1f}(\alpha_F)$$

$$u_{2F} = u_{2I} + q_{2f}(\alpha_F)$$

The definite integrals in D-1 and D-2 assume very complicated form and their exact numerical evaluation should be obtained by use of computers. A carefully prepared chart of these integrals is desirable for solving general initial value problems.

It is seen that there are three free parameters, γ, K_1 and α_I, in the integrals. γ is determined by the particular gas and is a fixed value in the problem. K_1 and α_I are found from initial conditions. For each value of K_1, we may prepare in the $(\alpha-\alpha_I)$ plane the curves of constant q_{1f} and q_{2f} for fast waves as well as q_{1s} and q_{2s} for slow waves respectively. A flow process occurs along a constant α_I line always. Curves of constant $H_f(\alpha;K_1,\alpha_I)$ and $H_s(\alpha;K_1,\alpha_I)$ on the $(\alpha-\alpha_I)$ plane for different values of K_1 are also necessary in determining $\overline{\rho}$ and $\overline{B_2}$ from 4-32a and 4-31.